Scuola Internazionale Superiore di Studi Avanzati - Trieste

The Worldsheet Corrections to Space-Time Geometries

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Advisers: Loriano Bonora, Martin O'Loughlin

A Dissertation Presented to the Faculty of SISSA
in Candidacy for the Degree of Doctor of Philosophy

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Contents

Chapter 1

Introduction

"String theories were originally developed as a phenomenological model of hadronic interactions. The suggestion that string theory should be used to describe fundamental interaction including gravity was pointed by the realisation that in the limit of infinite string tension the interactions of massless vector particles and massless tensor particles are those of Yang-Mills gauge fields and graviton respectively" [1, 2, 3, 4],"[5]". This suggestion identifies the subleading string corrections to the action of the massless symmetric tensor particle as the quantum corrections to gravity and provides a systematic approach to study the gravitational quantum corrections. In this thesis we investigate the perturbative world-sheet corrections to the classical backgrounds and T-duality. The thesis is organised in the following way,

The second chapter illustrates how the low energy effective action of string theory around a given background can be constructed

- by requiring an exact conformal symmetry in the corresponding sigma model,
- or from string amplitude considerations.

We present the linear and the quadratic α' corrections to backgrounds of dilaton and metric in the Bosonic String Theories. Also we provide the linear α' corrections to backgrounds of metric, NS-two form and dilaton in the critical Heterotic String Theory.

In the third chapter we develop a formalism to study T-duality within the framework of the low energy effective action. We discuss the general form of the α' corrections to the rules of T-duality. We compute the linear and the quadratic α' corrections to the general diagonal Kasner background, two dimensional black hole, Schwarzschild and their T-dual backgrounds in the Bosonic String Theory. We utilise these backgrounds to obtain the linear and the quadratic α' corrections to the rules of T-duality for time-dependent backgrounds of a diagonal metric and dilaton in the Bosonic String Theory. This chapter is the review of the works done in [6, 7].

In the fourth chapter we study the linear α' corrections to null singular backgrounds which represent a wrapped fundamental string in the supergravity approximation to the critical Heterotic String Theory. We shall show that there exist schemes in which

1. the inclusion of the linear α' corrections changes these null singular geometries to black hole geometries with a regular event horizon -to which we often refere as the stretched horizon- for which the modified Hawking-Bekenstein entropy [8, 9, 10] is in agreement with the degeneracy of the states of the wrapped fundamental string,
2. and the higher order α' corrections are perturbative outside the horizon.

This means that there exist schemes in which the α' stretched horizon is small and also there exist schemes where the α' stretched horizon does not exist at all. Note that the modified Hawking-Bekenstein entropy is the same for actions related to each other by field redefinition provided that the α' terms are studied as perturbations around a classical solution [11]. Since the stretched horizon is identified as the exact solution of the truncated equations, the modified Hawking-Bekenstein entropy depends on the field redefinition ambiguity parameters.

We do not know which scheme would be preferred or chosen by the underlying conformal field theory since it is not known what type of conformal field theory (nor if it is a unique one) represents a wrapped fundamental string. Ref. [12, 13] shows that there exists a scheme in which the fields of the fundamental string background retain their forms in the supergravity approximation, thus within this scheme the background remains as a null singular background under the inclusion of all α' corrections. We will conclude from this that the α' expansion series is not an absolutely convergent series on the α' stretched horizon whenever the scheme admits the α' stretched horizon.

We find it disturbing that the thermodynamical entropy is scheme-dependent. The fact that the α' series on the α' stretched horizon is not an absolutely convergent series adds to this problem.

These difficulties may indicate that the thermodynamical properties should be expressed in term of other geometrical properties of null singular geometries rather than requiring the subleading corrections to convert the null singular geometries to black hole geometries with a regular event horizon. We will point out that Mathur and Lunin's description for the entropy [14] may be employed to generate a thermodynamical entropy for a wrapped fundamental string without first requiring the α' corrections to produce an event horizon covering the singularity. This chapter reviews [15].

In the fifth chapter we investigate a toroidal compactification of the critical Heterotic String Theory. We study all the linear α' corrections to dyons which carry arbitrary KK-momentum and winding numbers of a wrapped fundamental string in the presence of a KK-monopole and a H-monopole. We shall compute all the linear α' corrections, excluding however the gravitational Chern-Simons ones, to the entropy of dyonic black holes. We address the problem of how the gravitational Chern-Simons contribution to the entropy could be computed. This chapter reviews [16].

Chapter 2

The Effective Action

String theory is the consistent quantum theory for strings, where strings are one dimensional extended objects. Quantum field theory defines point-like particles as unitary finite dimensional irreducible representations of the little group of the Poincare group. String theory is based on our understanding of quantum field theory for particles, in the sense that whenever we want to compute something we first identify each of the oscillatory modes of the string as a point like particle. Next we apply the quantum field theory techniques on each of the string modes and we define string theory as a consistent union of all the quantum field theories of its oscillations. Thus this understanding of string theory requires accomplishing the following two steps:

1. Finding the string modes around a given background. We refer to the string modes as the spectrum of the string theory. Spectrum of string theory is known only on a few backgrounds including the flat space time.

2. Writing the interactive quantum field theories for all the modes. This means finding the effective action for all the modes. It may not be that easy to write an action for a particle with a given high spin. We can integrate out the massive modes and we can write the explicit covariant action for the massless modes afterwards. We refer to this action as the low energy effective action of the string theory.

The first step is known as the first quantisation and the second step is referred to as the second quantisation. The second quantisation can be done either by requiring an exact conformal symmetry in the corresponding sigma model, or from the string amplitude considerations or by string field theory.

Let A_i represent the set of the fields of a given string background. Let $\mathcal{L}(\mathcal{A})$ be the sigma model Lagrangian density for this string background. As an example for backgrounds of the metric, the NS two form and the dilaton in the closed Bosonic String Theory $\{A_i\} = \{g_{\mu\nu}, \phi, B_{\mu\nu}\}$, then

$$\mathcal{L}(A) \;=\; \sqrt{\det h}\,\Big(h^{\alpha\beta}\,(g_{\mu\nu}(x) + B_{\mu\nu}(x))\,\partial_\alpha x^\mu\,\partial_\beta x^\nu \;+\; \alpha'\,R^{(2)}\,\phi(x)\Big)\;, \tag{2.1}$$

where x^μ stands for target space coordinates and $h_{\alpha\beta}$ is the auxiliary metric on the world sheet and $R^{(2)}$ is the two dimensional Ricci scalar constructed from $h_{\alpha\beta}$. We define the functional integral of the sigma model on a given world-sheet Σ by

$$Z_\Sigma[A,J] \;=\; \frac{1}{Vol(G_\Sigma)}\int Dx\,Dh\,e^{-\frac{1}{2\pi\alpha'}S_\Sigma(A,J)}\,, \tag{2.2}$$

where $S_\Sigma(A,J)$ is the world-sheet action in the presence of the sources on Σ,

$$S_\Sigma(A,J) \;=\; \int_\Sigma d\sigma^2 (L(A) + A_i.J^i) \tag{2.3}$$

and $Vol(G_\Sigma)$ is the volume of the symmetry group on Σ. Note that in $Z_\Sigma[A,J]$ the integration is done on the target space coordinates x^μ and $h_{\alpha\beta}$. The sum over the sigma model functional integral of all allowed world sheets defines the generating functional of the string theory.[1] For the closed oriented string theory the allowed worldsheets are oriented Riemann surfaces with arbitrary numbers of genera. Hence the functional integral of the closed oriented string theory is

$$Z[A,J] \;=\; Z_{\circ}[A,J] \,+\, g_s\,Z_{\circledcirc}[A,J] \,+\, g_s^2\,Z_{\circledcirc\circledcirc}[A,J] \,+\,\cdots\,, \tag{2.4}$$

where $Z_{\circ}[A,J]$, $Z_{\circledcirc}[A,J]$, $Z_{\circledcirc\circledcirc}[A,J]$ and $\cdots$ are the functional integrals of the sigma model on worldsheets with topologies of $\circ$, $\circledcirc$, $\circledcirc\circledcirc$ and $\cdots$. g_s is defined on the right hand side of (2.4) to treat the series as a perturbative series. Naturally g_s is related to the closed string coupling constant. This definition of string theory is perturbative in g_s. We could not compute $e^{-\frac{1}{g_s}}$ corrections to

[1]The generating functions for string theory were first defined in [17, 18, 19].

$Z[A, J]$ based on its definition. Thus we write[2]

$$Z[A, J] \;=\; Z_{\bigcirc}[A, J] + g_s\, Z_{\circledcirc}[A, J] + g_s^2\, Z_{\circledcirc\circledcirc}[A, J] + \cdots + O(e^{-\frac{1}{g_s}}) \,. \tag{2.5}$$

One can compute the functional integral in each of $Z_{\bigcirc}[A, J]$, $Z_{\circledcirc}[A, J]$, $Z_{\circledcirc\circledcirc}[A, J]$ and $\cdots$ by the steepest descent method, i.e. formal expansion series in α'. The ordinary perturbative techniques of quantum field theory are applicable for polynomial actions. In general the Lagrangian density of the sigma model is not polynomial in terms of the target space coordinates. Thus we pick up an arbitrary point in the target space and we choose a neighbourhood of this point. Then we write the Taylor expansion of the background fields A_i on this neighbourhood,

$$A_i \;=\; A_i(x_0) + (x^\mu - x_0^\mu)\partial_\mu A(x_0) + \frac{1}{2!}(x^\mu - x_0^\mu)(x^\nu - x_0^\nu)\partial_\mu\partial_\nu A_i(x_0) + \cdots \,. \tag{2.6}$$

Using this expansion series and similar expansion series for the sources in S_Σ converts S_Σ to a polynomial action. In this polynomial action the derivatives of the background fields at $x = x_0$ play the roles of the coupling constants. Thus for slow varying background fields one can employ the ordinary perturbative techniques of quantum field theory to perform the functional integral in Z_Σ.

One can obtain a set of conditions that A_i should satisfy in order to have an exact conformal symmetry on the worldsheet Σ in Z_Σ. These conditions are a set of α' perturbative equations which A_i should satisfy. Let us represent these conditions for Z_Σ by

$$\beta_{A_i,\Sigma} \;=\; \beta^{(0)}_{A_i,\Sigma} + \alpha'\,\beta^{(1)}_{A_i,\Sigma} + \alpha'^2\,\beta^{(2)}_{A_i,\Sigma} + \cdots + O(e^{-\frac{1}{\alpha'}}) \tag{2.7}$$

If we set $\beta_{A_i,\Sigma} = 0$ then we get exact conformal symmetry for Σ. Requiring that the string theory be consistent on Σ ($\beta_{A_i,\Sigma} = 0$) identifies the dynamics of A_i. In general we expect that there exists an action for A_i , $S_\Sigma[A]$, whose functional derivatives with respect to A_i is

$$\frac{\delta S_\Sigma[A]}{\delta A_i} \;=\; K^\Sigma_{ij}\,\beta_{A_j,\Sigma} \,, \tag{2.8}$$

where K^Σ is a matrix operator which maps $\beta_{A_j,\Sigma}$ to $\frac{\delta S_\Sigma[A]}{\delta A_i}$. We refer to K^Σ_{ij} as the K-matrix on Σ and to $S_\Sigma[A]$ as the effective action for $\{A_i\}$ at the order Σ.

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[2] There should exist an $\mathcal{M}'$-theory, the loop expansion of whose functional integral $Z_\mathcal{M}[A, J] = \int DM\, e^{-\frac{1}{g_s}\, S(M,J)}$ around $g_s = 0$ coincides with eq. (2.5); M stands for the dynamical variable of the $\mathcal{M}'$-theory and $S(M, J)$ is the action of the $\mathcal{M}'$-theory.

String theory requires exact conformal symmetry for $Z[A, J]$, therefore we define the conditions for the exact conformal symmetry of the closed oriented string by

$$\beta_{A_i} \;=\; \beta_{A_i,\bigcirc} + g_s\,\beta_{A_i,\circleddash} + g_s^2\,\beta_{A_i,\circleddash\circleddash} + \cdots + O(e^{-\frac{1}{g_s}})\,. \tag{2.9}$$

where setting $\beta_{A_i,\bigcirc}$, $\beta_{A_i,\circleddash}$, $\beta_{A_i,\circleddash\circleddash}$ and $\cdots$ to zero are the conditions for the exact conformal symmetry on $\bigcirc$, $\circleddash$, $\circleddash\circleddash$ and $\cdots$,

$$\beta_{A_i,\bigcirc} \;=\; \beta^{(0)}_{A_i,\bigcirc} + \alpha'\,\beta^{(1)}_{A_i,\bigcirc} + \alpha'^2\,\beta^{(2)}_{A_i,\bigcirc} + \cdots + O(e^{-\frac{1}{\alpha'}}) \tag{2.10}$$

$$\beta_{A_i,\circleddash} \;=\; \beta^{(0)}_{A_i,\circleddash} + \alpha'\,\beta^{(1)}_{A_i,\circleddash} + \alpha'^2\,\beta^{(2)}_{A_i,\circleddash} + \cdots + O(e^{-\frac{1}{\alpha'}}) \tag{2.11}$$

$$\beta_{A_i,\circleddash\circleddash} \;=\; \beta^{(0)}_{A_i,\circleddash\circleddash} + \alpha'\,\beta^{(1)}_{A_i,\circleddash\circleddash} + \alpha'^2\,\beta^{(2)}_{A_i,\circleddash\circleddash} + \cdots + O(e^{-\frac{1}{\alpha'}}) \tag{2.12}$$

We define the string theory conditions by setting $\boldsymbol{\beta}_{A_i} = 0$. Note that setting (2.9) to zero is a generalisation of the conformal field theory to which we refer as the string conformal theory. For some specific backgrounds like flat space-time it happens that string conformal theory coincides with the conformal theory. Subsequently we define the effective action of the string by

$$\frac{\delta \boldsymbol{S}[A]}{\delta A_i} \;=\; K_{ij}.\boldsymbol{\beta}_{A_j} \tag{2.13}$$

where K is a matrix operator acting on $\boldsymbol{\beta}_{A_j}$. Any string conformal theory is an extremum of the effective action. The K-matrix must be invertible for the reverse to hold true.

The effective action of string by construction has the following expansion series

$$\begin{aligned} \boldsymbol{S}[A] \;=\; & \boldsymbol{S}^{(0)}_{\bigcirc}[A] + \alpha'\,\boldsymbol{S}^{(1)}_{\bigcirc}[A] + \cdots + O(e^{-\frac{1}{\alpha'}}) + \\ & + g_s\left(\boldsymbol{S}^{(0)}_{\circleddash}[A] + \alpha'\,\boldsymbol{S}^{(1)}_{\circleddash}[A] + \cdots + O(e^{-\frac{1}{\alpha'}})\right) + \\ & + g_s^2\left(\boldsymbol{S}^{(0)}_{\circleddash\circleddash}[A] + \alpha'\,\boldsymbol{S}^{(1)}_{\circleddash\circleddash}[A] + \cdots + O(e^{-\frac{1}{\alpha'}})\right) + \\ & + \cdots + O(e^{-\frac{1}{g_s}})\,. \end{aligned} \tag{2.14}$$

We refere to the first line of (2.14) as the α' corrections or the world-sheet corrections to the effective action. The rest of (2.14) will be called the g_s corrections or the string loop corrections to the effective action. More precisely the first line of (2.14) stands for the perturbative α' corrections to the effective action.

Prior to computing the effective action one should choose a regularisation and a renormalisation scheme in the sigma model. The target space coordinates of different schemes of the sigma model are mapped to each other by a perturbative field redefinition in α';

$$x^\mu \to x^\mu + \alpha' \, y_1^\mu(x) + \alpha'^2 \, y_2^\mu(x) + \cdots , \tag{2.15}$$

where $y_1^\mu(x)$, $y_2^\mu(x)$, $\cdots$ are functions of x^μ. Inserting this expansion series into (2.6) implies that the background fields of different schemes should be mapped to each other by a perturbative field redefinition;

$$A_i(x) \quad \to \quad A_i(x) + \alpha' \, F_i^{(1)}(x) + \alpha'^2 \, F_i^{(2)}(x) + \cdots , \tag{2.16}$$

on any given worldsheet. One could choose different schemes to calculate each of $Z_{\bigcirc}[A, J]$, $Z_{\circledcirc}[A, J]$, $Z_{\circledcirc\circledcirc}[A, J]$ and $\cdots$. Therefore the string backgrounds of different schemes generically are mapped to each other by

$$\begin{aligned} A_i(x) \to A_i(x) \; & + \; (\alpha' \, F_i^{(1,0)}(x) + \alpha'^2 \, F_i^{(2,0)}(x) + \cdots) + \\ & + \; g_s \, (\alpha' \, F_i^{(1,1)}(x) + \alpha'^2 \, F_i^{(2,1)}(x) + \cdots) + \\ & + \; \cdots . \end{aligned} \tag{2.17}$$

We refer to this as the field redefinition ambiguities. One may interpret the field redefinition ambiguities as the blurring effects of the quantum mechanics on the classical background fields. Note that different schemes are mapped to each other by a field redefinition but there might not exist schemes for any given field redefinition.

In this thesis we study the perturbative α' corrections to T-duality and the black hole entropy. Thus from this time on we consider only the α' corrections to the effective action

$$S[A] = S^{(0)}_{\bigcirc}[A] + \alpha' \, S^{(1)}_{\bigcirc}[A] + \alpha'^2 \, S^{(2)}_{\bigcirc}[A] + \cdots + O(e^{-\frac{1}{\alpha'}}, g_s, e^{-\frac{1}{g_s}}) . \tag{2.18}$$

Also we study the perturbative α' corrections for backgrounds of the metric, the NS two-form and the dilaton in the closed critical Bosonic String Theory and the Heterotic theory,

$$A \;\in\; \{g_{\mu\nu}(x), \phi(x), B_{\mu\nu}(x)\} . \tag{2.19}$$

Let us have a closer look at the field redefinition ambiguities for the specific backgrounds that we are considering at the order of α' corrections in the Bosonic String Theory. For this part we follow the second section of [20] and we adapt the definitions within this reference. The effective action should be generally covariant and invariant under $B \to B + d\Lambda$,

$$S = \int d^{26}x \sqrt{-g}\, L(g_{\mu\nu}, R_{\mu\nu\lambda\eta}, \nabla_\nu, \phi)\,, \tag{2.20}$$

where throughout this thesis we use

$$\text{Eigenvalues of } g = (-1, 1, \cdots, 1)\,, \tag{2.21}$$

$$R^{\lambda}_{\mu\nu\rho} = \partial_\nu \Gamma^{\lambda}_{\mu\rho} - \cdots\,, \tag{2.22}$$

$$R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu}\,, \tag{2.23}$$

$$\nabla_\nu = \partial_\mu + \Gamma^{\lambda}_{\mu\nu}\,. \tag{2.24}$$

Note that the index of [illegible] in S and L is understood, however we stop writing this index for sake of simplicity. We should consider only the schemes in which the two gauge symmetries of (2.20) are preserved. This implies that we are allowed to consider field redefinitions by

$$g_{\mu\nu} \longrightarrow g_{\mu\nu} + \alpha' T^{(1)}_{\mu\nu} + \alpha'^2 T^{(2)}_{\mu\nu} + \cdots\,, \tag{2.25}$$

$$B_{\mu\nu} \longrightarrow B_{\mu\nu} + \alpha' S^{(1)}_{\mu\nu} + \alpha'^2 S^{(2)}_{\mu\nu} + \cdots\,, \tag{2.26}$$

$$\phi \longrightarrow \phi + \alpha' \Phi^{(1)} + \alpha'^2 \Phi^{(2)} + \cdots\,, \tag{2.27}$$

where $T^{(i)}_{\mu\nu}$, $S^{(i)}_{\mu\nu}$ and $\Phi^{(i)}$ respectively are rank two tensors, two forms and scalars with appropriate mass dimensions constructed from $g_{\mu\nu}$, $B_{\mu\nu}$ and ϕ. It follows from the path integral representation for the generating function of the string theory $Z_{[illegible]}[A, J]$ [21, 22, 23], under proper treatment of the dilaton vertex operator or from the scaling symmetry present in the operator formalism expression for the amplitudes [24, 25, 26], that there exists a subclass of the effective action which have the following dependence on the dilaton

$$S = \int d^{26}x \sqrt{g}\, e^{-2\phi}\, \mathcal{L}(g, R_{\mu\nu\eta\gamma}, H, \nabla_\mu, \partial\phi) \tag{2.28}$$

where $\mathcal{L}$ does not depend on the dilaton but its derivatives. The subclass of field redefinitions that preserve the structure of (2.28) may depend on ϕ through its derivatives. We shall consider only this subclass of the field redefinitions to simplify the computations in the next chapters.

Consider the transformation of the effective action under a general field redefinition which preserves the structure of (2.28). Let $v = \{v_i\}$, $i = 1..N$ be the set of constants which parametrises the effective action to some definite order in α'. We call the coefficients $\lambda = \{\lambda_\alpha\} \subset \{v_i\}$ unambiguous if field redefinitions do not change them. We call the remaining coefficients $\mu = \{\mu_\beta\} = v - \lambda$ a priori ambiguous. Let $\mathcal{V}$ represent the vector space of the a priori ambiguous coefficients. Applying a general field redefinition on a given set of the a priori ambiguous coefficients generically defines a hypersurface in $\mathcal{V}$ which does not necessarily cover the whole of $\mathcal{V}$. We refer to such a hypersurface as an a priori ambiguous hypersurface. The invariant structure of the effective action is presented by unambiguous coefficients and an a priori ambiguous hypersurface which the a priori ambiguous coefficients lay on.

We consider the simplest non-trivial example to illustrate the invariant structure of the effective action, i.e. the linear α' corrections to the effective action for backgrounds of the metric and the dilaton,

$$S = \int d^{26}x \sqrt{-\det g}\, e^{-2\phi} (L_0 + \alpha' L_1) + O(\alpha'^2, g_s, e^{-\frac{1}{\alpha'}}, e^{-\frac{1}{g_s}}), \tag{2.29}$$

$$L_0 = R + 4(\partial\phi)^2, \tag{2.30}$$

$$\begin{aligned} L_1 = {} & \lambda_0 (R_{\mu\nu\lambda\eta}R^{\mu\nu\lambda\eta} + a_1 R_{\mu\nu}R^{\mu\nu} + a_2 R^2 + a_3 R^\mu{}_\nu\, \partial_\mu\phi\, \partial_\nu\phi + \\ & + a_4 R\, |\nabla\phi|^2 + a_5 R\,\Box\phi + a_6(\Box\phi)^2 + a_7\Box\phi\, |\nabla\phi|^2 + a_8\, |\nabla\phi|^4). \end{aligned} \tag{2.31}$$

where in L_1 we have written down all ∂^4 scalar invariant which are not related to each other by integration by parts in the action. The general field redefinition for this example reads

$$\left\{ \begin{aligned} g_{\mu\nu} & \to g_{\mu\nu} + T^{(1)}_{\mu\nu} + O(\alpha'^2), \\ T^{(1)}_{\mu\nu} & = b_1 R_{\mu\nu} + b_2\, \partial_\mu\phi\partial_\nu\phi + g_{\mu\nu}(b_3 R + b_4 |\nabla\phi|^2 + b_5\,\Box\phi), \end{aligned} \right. \tag{2.32}$$

$$\left\{ \begin{aligned} \phi & \to \phi + \alpha'\,\Phi^{(1)} + O(\alpha'^2), \\ \Phi^{(1)} & = (c_1 + \tfrac{1}{4}b_1 + 6b_3)R + (c_2 + \tfrac{1}{4}b_2 + 6b_4)(\partial\phi)^2 + (c_3 + 6b_5)\Box\phi. \end{aligned} \right. \tag{2.33}$$

Using this field redefinition in the effective action we see that λ_0 is an unambiguous coefficient while a_i change in the following way

$$\delta a_1 = -b_1, \tag{2.34}$$

$$\delta a_2 = -2c_1, \tag{2.35}$$

$$\delta a_3 = -4b_1 - b_2, \tag{2.36}$$

$$\delta a_4 = 2b_1 - 4b_3 + 8c_1 - 2c_2, \tag{2.37}$$

$$\delta a_5 = -b_1 + 2b_3 - 8c_1 - 2c_3, \tag{2.38}$$

$$\delta a_6 = 2b_5 - 8c_3\,, \tag{2.39}$$

$$\delta a_7 = b_2 + 2b_4 - 4b_5 - 8c_2 + 8c_3\,, \tag{2.40}$$

$$\delta a_8 = -2b_2 - 4b_4 + 8c_2\,. \tag{2.41}$$

We see that

$$\delta a_8 + 2\delta a_7 + 4\delta a_6 - 8\delta a_5 - 4\delta a_4 + 16\delta a_2 = 0\,, \tag{2.42}$$

thus $\xi = a_8 + 2a_7 + 4a_6 - 8a_5 - 4a_4 + 16a_2$ remains invariant under field redefinition. $\xi = consts$ defines a priori hypersurfaces in the vector space spanned by $(a_1, \cdots, a_8)$. Given an a priori ambiguous hypersurface we may choose any point on it to represent the effective action. We may choose a curve which crosses all the a priori hypersurfaces and crosses each a priori ambiguous hypersurface only once as the representation of the invariant structure of the effective action. The simplest representation of the invariant structure of the example we considered above is

$$a_1 = \cdots = a_7 = 0\,, \tag{2.43}$$

$$L_1 = \lambda_0(R_{\mu\nu\rho\eta}R^{\mu\nu\rho\eta} + a_8|\nabla\phi|^4)\,. \tag{2.44}$$

Perturbative study of the string theory around a given background fixes the invariant structure of the effective action, i.e. gives a definite value for λ_0 and a_8 in the chosen representation of the invariant structure in (2.43). In this example a priori hypersurfaces are flat. However if we go to higher order in α' then the space of the a priori ambiguous coefficients becomes larger and the a priori hypersurfaces would become curved .i.e. they are described by a set of non-linear equations on the a priori ambiguous coefficients.

In the above we reviewed how to construct the effective action from the Weyl anomaly free conditions of the corresponding sigma model. The effective action also can be obtained by using the vertex operator formalism and string S-matrix elements. Any given scattering amplitude of string theory can be pictured as a compact worldsheet with some punctures on it provided that one uses an appropriate limit of the worldsheet conformal symmetry. In this compact picture each puncture represents an incoming or outgoing state of the string and each puncture is described by a local vertex operator determined by the limiting process. Let $\mathcal{V}(s_i)$ represent the vertex of the string state A_i carrying quantum number s_i. In general s_i stands for the target space momentum and the spin of the state, i.e. angular momentum of the string. Then the n-particle connected S-matrix element for the worldsheet with toplogy Σ around a string background $\{B\}$ in the path integral approach is defined by [27]

$$\mathcal{S}_{\Sigma}(s_1,\cdots,s_n;\{B\}) = \frac{1}{Vol(G_{\Sigma})}\int Dx\,Dh\,e^{-S_{\Sigma}[B]}\prod_{i=1}^{n} d^2\sigma_i\,\sqrt{-\det h}\,\mathcal{V}(s_i) \tag{2.45}$$

where $S[B]$ is the action of the sigma model (2.3) and $Vol(G_{\Sigma})$ is the volume of the symmetry group for the worldsheet Σ. The n-particle S-matrix element of the string is defined by summing n-particle S-matrix elements on all allowed worldsheet. Thus for the closed oriented string theory the n-particle scattering element follows

$$\begin{aligned}\mathcal{S}(s_1,\cdots,s_n;\{B\}) &= \mathcal{S}_{\bigcirc}(s_1,\cdots,s_n;\{B\}) + g_s\,\mathcal{S}_{\circledcirc}(s_1,\cdots,s_n;\{B\}) + \\ &\quad + g_s^2\mathcal{S}_{\circledcirc\circledcirc}(s_1,\cdots,s_n;\{B\}) + \cdots\end{aligned} \tag{2.46}$$

The effective action of string theory is a polynomial functional of $\{A_i\}$,

$$\begin{aligned}\Gamma_B(\{A_i\}) &= \sum_{s_1,s_2} c_{ij}(s_1,s_2)A_i(s_1)A_j(s_2) + \\ &\quad + \sum_{s_1,s_2,s_3} c_{ijk}(s_1,s_2,s_3)A_i(s_1)A_j(s_2)A_k(s_3) + \cdots,\end{aligned} \tag{2.47}$$

whose functional derivative respect to $\{A_i\}$ reproduces $\mathcal{S}(s_1,\cdots,s_n;\{B\})$,

$$\left.\frac{\delta\Gamma_B(\{A_i\})}{\delta A_1\cdots\delta A_n}\right|_{A_i=0} = \mathcal{S}(s_1,\cdots,s_n;\{B\})\,. \tag{2.48}$$

We prefer to compute the effective action around the critical flat space-time due to the following reasons

1. The critical flat space-time has an exact conformal symmetry and we know the spectrum of string propagating in the flat space-time.

2. In the flat space-time, $\{B\} = \{g_{\mu\nu}(x) = \eta_{\mu\nu}, B_{\mu\nu} = 0, \phi = 0, \cdots\}$, in the integration in (2.45) $e^{-S_{\Sigma}[B]}$ is a simple Gaussian weight for x_μ.

From this time on we consider the effective action constructed around critical flat space-time. The effective action by construction has the following from

$$\Gamma[A] = \Gamma_{\bigcirc}[A] + g_s\,\Gamma_{\circledcirc}[A] + g_s^2\Gamma_{\circledcirc\circledcirc}[A] + \cdots \tag{2.49}$$

where we have stopped using the index of B on the effective action. $\Gamma_{\bigcirc}[A]$, $\Gamma_{\circledcirc}[A]$ and $\Gamma_{\circledcirc\circledcirc}[A]$ reproduce the S-matrix elements respectively on $\bigcirc$, $\circledcirc$ and $\circledcirc\circledcirc$. Also each of $\Gamma_{\bigcirc}[A]$, $\Gamma_{\circledcirc}[A]$, $\Gamma_{\circledcirc\circledcirc}[A]$ will have an expansion in α',

$$\Gamma_{\bigcirc}[A] = \Gamma^{(0)}_{\bigcirc}[A] + \alpha'\Gamma^{(1)}_{\bigcirc}[A] + \alpha'^2\Gamma^{(2)}_{\bigcirc}[A] + \cdots \tag{2.50}$$

$$\Gamma_{\circledcirc}[A] = \Gamma^{(0)}_{\circledcirc}[A] + \alpha'\Gamma^{(1)}_{\circledcirc}[A] + \alpha'^2\Gamma^{(2)}_{\circledcirc}[A] + \cdots \tag{2.51}$$

$$\Gamma_{\circledcirc\circledcirc}[A] = \Gamma^{(0)}_{\circledcirc\circledcirc}[A] + \alpha'\Gamma^{(1)}_{\circledcirc\circledcirc}[A] + \alpha'^2\Gamma^{(2)}_{\circledcirc\circledcirc}[A] + \cdots \tag{2.52}$$

This is due to the fact that $\mathcal{V}(s_i)$ around the flat space-time is a homogeneous-degree polynomial of the target space coordinates multiplied with the factor of $e^{-k.x}$ where k is the target space momentum of the string. Inserting the expression for $\mathcal{V}(s_i)$ in (2.45) one sees that in general (2.45) is proportional to different powers of α' for different numbers of insertion of string vertex operators.

The perturbative string S-matrix amplitude does not change under local field redefinitions thus there exists a large class of the effective action which all correspond to the same string S-matrix [28, 29, 30]. The perturbative string S-matrix amplitude fixes only the invariant structure of the effective action. The invariant structure of the effective action constructed from the string S-matrix elements (2.49) should be the same as the invariant structure of the action constructed from the Weyl anomaly free conditions in the corresponding sigma model (2.14). This statement is often referred to as the string equivalence conjecture and it was first put forward in [31, 32, 33, 34, 35]. There is not yet a clear understanding why the string equivalence conjecture is true. The string equivalence conjecture has been verified to be true up to some power in α' for the tree-level effective action [20, 28, 29, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43]. Presumably the string equivalence is valid to all order in α' and g_s.

In this thesis we need the invariant structure of the quadratic α' corrections to backgrounds of the metric and dilaton in Bosonic String Theory and the invariant structure of the linear α' corrections to backgrounds of the metric, the dilaton and the NS two form in the critical Heterotic String Theories. The conditions for exact conformal theory on the worldsheet of the free Bosonic String Theory for backgrounds of the dilaton and the metric in D dimensional space-time at the quadratic order in α' are [44, 45, 46]

$$\begin{aligned}\frac{1}{\alpha'}\beta_{ij} &= R_{ij} + 2\ \nabla_i \nabla_j\ \phi + \frac{1}{2}\ \alpha'\ \boldsymbol{R}_{iklm}\boldsymbol{R}_j{}^{klm} \\ &+\alpha'^2\left\{\frac{1}{8}\nabla_k\,\boldsymbol{R}_{ilmn}\,\nabla^k\,\boldsymbol{R}_j{}^{lmn} - \frac{1}{16}\nabla_i\,\boldsymbol{R}_{klmn}\,\nabla_j\,\boldsymbol{R}^{klmn}\right.\end{aligned} \tag{2.53}$$

$$
\begin{aligned}
& +\frac{1}{2} R_{klmn}\, R_i^{\ mlp}\, R_j^{\ kn}{}_p - \frac{3}{8} R_{iklj}\, R^{kmnp}\, R^l{}_{mnp} + \\
& + \frac{1}{32} \nabla_j \nabla_i (R_{klmn}\, R^{klmn}) \Big\} = 0\,, \\
\frac{1}{\alpha'}\beta_\phi \;=\; & \frac{D-26}{6\alpha'} - \frac{1}{2} \Box\, \phi + \partial_k\phi\, \partial^k\phi + \frac{1}{16}\, \alpha'\, R_{klmn}\, R^{klmn} \\
& +\alpha'^2 \Big\{ -\frac{3}{16} R^{kmnp}\, R^l{}_{mnp} \nabla_k \nabla_l \phi + \frac{1}{32} R_{klmn} R^{mnpq} R_{pq}{}^{kl} \\
& -\frac{1}{24} R_{klmn} R^{qnpl} R_q{}^m{}_p{}^k + \frac{1}{64} \partial_i (R_{klmn}\, R^{klmn}) \partial^i \phi \Big\} = 0\,, \\
& i, \cdots, q \in \{0...25\}\,.
\end{aligned} \tag{2.54}
$$

which are calculated by the dimensional regularisation method [47, 48] and the minimal subtraction scheme. For backgrounds of the metric, the NS two form and the dilaton in the Heterotic String Theories the invariant structure of the linear α' corrections derived from three [5, 49, 50] and four [50, 51, 52, 53] vertex operator insertions of dilaton, NS two form and graviton on the sphere can be represented by

$$
\begin{aligned}
S^{(10)} &= \frac{1}{32\,\pi} \int d^{10}x\, \sqrt{-g}\, e^{-2\phi} \,(L^{(0)} + \frac{\alpha'}{8} L^{(1)}) \\
L^{(0)} &= R_{\mathrm{Ricci}} + 4|\nabla\phi|^2 - \frac{1}{12} H_{ijk} H^{ijk}\,, \\
L^{(1)} &= R_{klmn} R^{klmn} - \frac{1}{2} R_{klmn} H_p{}^{kl} H^{pmn} + \\
& -\frac{1}{8} H_k{}^{mn} H_{lmn} H^{kpq} H^l{}_{pq} + \frac{1}{24} H_{klm} H^k{}_{pq} H_r{}^{lp} H^{rmq}\,.
\end{aligned} \tag{2.55}
$$

where

$$
H = dB + \frac{\alpha'}{4} \omega_{3L}(\Omega)\,, \tag{2.56}
$$

includes the Lorentz Chern-Simons modification to dB. The Lorentz Chern-Simons modification to dB is in agreement with the Green-Schwarz anomaly cancellation mechanism [54] in the low energy effective field theory.[3] Note that the insertion of three vertex operators on sphere gives rise to the Chern-Simons modification to H.

[3] For the definition of the Lorentz Chern-Simons consult page 167 of D-Branes by Clifford Johnson, 2003 Cambridge University Press.

Chapter 3

T-duality

3.1 Introduction and Motivation

Target space duality was first introduced as a symmetry describing the interchange of the momentum and winding modes in the closed string compactified on a torus [55, 56]. Later it was described as the symmetry of the sigma models [57]. The linear α'-corrections to the T-duality rules are obtained in [58, 59, 60]. Further support for the α' expansion of T-duality is presented in [61] where the linear α' corrections to T-duality in the presence of torsion is obtained. Studying the higher order α' corrections to T-duality should provide a better understanding of both the mathematics of string theory in the curved space time and the pre-big bang scenario in string cosmology [62] where T-duality is an essential tool.

In this chapter the three loop α' corrections to T-duality are computed in the critical Bosonic String Theory at the tree level of the string interaction for time dependent backgrounds of a diagonal metric and the dilaton. The chapter is organised in the following way:

In the second section we expound how T-duality is realised in the effective action.

In the third section we review the general diagonal Kasner background in $D = 26$. Since the Kasner background is of interest in Cosmology [63, 64] and particularly in Cosmological Billiard [65] calculating its string corrections should be interesting. we generalise the Kasner metric to a perturbative background in the critical Bosonic string and we calculate the linear (two-loop) and the quadratic (three-loop) α' corrections to this background at the tree level of the string interaction. We write the Kasner metric on a periodic space-like directions and we apply T-duality in one direction to obtain the corresponding T-dual background. Next we add the linear and the quadratic α' corrections to the Kasner background and to its T-dual. We will observe that T-duality fails to relate the α'-corrected Kasner background to its α' corrected T-dual background. We will modify the rules by appropriate α' terms in such a way that the α' modified rules relate the α'-corrected Kasner background to its α' corrected T-dual background. Finally we will rewrite the α'-modifications in a Lorentz invariant form consistent with [59] to obtain the α' corrected T-duality rules for a general time-dependent background with a diagonal metric and the dilaton.

In the fourth section we review the Schwarzschild background in an arbitrary dimension. We introduce the time-dual of the Schwarzschild background by performing T-duality in the time direction of the related Euclidean geometry. We observe that the horizon of the Schwarzschild background changes into an intrinsic singularity under T-duality and the time-dual of the Schwarzschild metric is massless in $D = 4$. We introduce massless geometries in arbitrary dimensions for the low energy gravitational theory of the Bosonic String Theory. We then calculate the linear and the quadratic α' corrections to both the Schwarzschild background and its T-dual background in $D = 4, 5$ in the critical Bosonic String Theory at the tree-level of the string interaction. We observe that when the asymptotic behaviours of the fields are fixed at infinity the α' corrections generically diverge at the horizon of the Schwarzschild black hole. We will discuss this divergence in the fifth chapter. Finally, by requiring that the α' modified T-duality also relates the quadratic α' corrected Schwarzschild background to the quadratic α' corrected time-dual, we are able to identify uniquely the quadratic α' correction to the rule which describes the change of the dilaton under T-duality.

In the fifth section we review the two dimensional black hole. We calculate the quadratic α' corrections to the two dimensional black hole and its T-dual. By requiring that the α' modified T-duality also should relate the quadratic α' corrected two dimensional black hole to its quadratic α' corrected time-dual, we are able to identify uniquely all the quadratic α' corrections to the rules of T-duality.

In the last section we provide a summary and possible generalisations of this work.

3.2 T-duality and α' corrections

Consider a background of the Bosonic string theory in $D+1$ dimensional space-time which is composed of a time-dependent dilaton and a diagonal metric,

$$\phi = \phi(t) \tag{3.1}$$
$$ds^2 = -dt^2 + g_{11}(t)dx_1^2 + \cdots + g_{DD}(t)dx_D^2 \tag{3.2}$$

where

1. x_i are compactified on circle; $x_i \sim x_i + 2\pi$,
2. the space-time approaches the flat space-time at late times,
3. $e^{2\phi(t)}$ remains bounded at late times.

These conditions guarantee that physics at late times is described by free strings propagating in a toroidally compactified flat space-time. The spectrum of free string in the toroidally compactified flat space-time read

$$m^2_{(n_i,w_i)} = \sum_{i=1}^{25}\left(\frac{n_i^2}{R_i^2} + \frac{w_i^2 R_i^2}{\alpha'}\right) + \frac{2}{\alpha'}(N + \tilde{N} - 2) \tag{3.3}$$

where R_i is the radius of circle of x_i and n_i, w_i are the momentum and the winding numbers of string wrapped around x_i. (3.3) is invariant under

$$R_i \rightarrow \frac{\alpha'}{R_i}\ ;\ n_i \leftrightarrow w_i \tag{3.4}$$

for each of $i \in \{1, \cdots, 25\}$. Also the vertex operators are invariant under this equivalence. This equivalence is known as T-duality.

T-duality is a duality connecting different string theories through interchanging the winding and the momentum numbers of the string wrapped around a non-trivial cycle. Thus T-duality should exists for strings propagating in a curved space-time which have some topologically invariant non-trivial cycles. We do not know the spectrum of a free string propagating in such a general background therefore we develop a mechanism to study T-duality in the effective action.

Consider an asymptotically flat space-time background where string theory is perturbative in the flat asymptotic region. It should exist neighbourhoods around the asymptotic infinity in which the spectrum of strings could be approximated by that of flat space-time. We name the union of all these neighbourhoods as the flat-neighbourhood. The flat-neighbourhood necessarily covers whole of the space-time if the background is a consistent string background. Consider a case where in the flat-neighbourhood the low energy effective action of string theory in $D+1$ dimensional space time is described by

$$\begin{aligned} \phi &= \phi(t)\,, && (3.5)\\ ds^2 &= -dt^2 + g_{11}(t)dx_1^2 + \cdots + g_{DD}(t)dx_D^2\,. && (3.6) \end{aligned}$$

Now let T-duality be applied in the direction of x_1. Represent the fields of the T-dual background by

$$\begin{aligned} \tilde{\phi} &= \tilde{\phi}(t)\,, && (3.7)\\ d\tilde{s}^2 &= -dt^2 + \tilde{g}_{11}(t)dx_1^2 + \cdots + \tilde{g}_{DD}(t)dx_D^2 && (3.8) \end{aligned}$$

Note that we have chosen the same coordinate to represent the background and its T-dual, both metrics are written in the co-moving frame and $x_i \equiv x_i + 2\pi$ on both coordinates. (3.4) requires

$$\begin{cases} \dfrac{\tilde{g}_{11}(\infty)}{\alpha'} = \dfrac{\alpha'}{g_{11}(\infty)} \\ \tilde{g}_{jj}(\infty) = g_{jj}(\infty)\ j \neq 1 \end{cases} \qquad (3.9)$$

We generalise the above relations to all points in the flat-neighbourhood by

$$\begin{aligned} \ln\frac{\tilde{g}_{11}(t)}{\alpha'} + \tilde{P}_{\|} &= -(\ln\frac{g_{11}(t)}{\alpha'} + P_{\|})\,, && (3.10)\\ \ln\frac{\tilde{g}_{jj}(t)}{\alpha'} + \tilde{P}_{\perp,j} &= \ln\frac{g_{jj}(t)}{\alpha'} + P_{\perp,j}\ ,\ j \neq 1\,, && (3.11) \end{aligned}$$

where $(\tilde{P}_{\|},\ \tilde{P}_{\perp,j})$ and $(P_{\|},\ P_{\perp,j})$ are functional of $(\tilde{g}_{\mu\nu},\ \tilde{\phi},\partial_t)$ and $(g_{\mu\nu},\ \phi,\partial_t)$. Let them be represented by

$$\begin{aligned} \tilde{P}_{\|} &\equiv \tilde{P}_{\|}(\ln\tilde{g}_{\mu\mu}, \tilde{\phi}, \tilde{\nabla}_\mu)\,, && (3.12)\\ \tilde{P}_{\perp} &\equiv \tilde{P}_{\perp}(\ln\tilde{g}_{\mu\mu}, \tilde{\phi}, \tilde{\nabla}_\mu)\,, && (3.13)\\ P_{\|} &\equiv P_{\|}(\ln g_{\mu\mu}, \phi, \nabla_\mu)\,, && (3.14)\\ P_{\perp} &\equiv P_{\perp}(\ln g_{\mu\mu}, \phi, \nabla_\mu)\,, && (3.15) \end{aligned}$$

where within these expressions the covariant derivatives act on the logarithm of the components of metric as if they were scalars. Then (3.9) requires that

$$\lim_{t\to\infty}\tilde{P}_{\|} = \lim_{t\to\infty}\tilde{P}_{\perp,j} = \lim_{t\to\infty}P_{\|} = \lim_{t\to\infty}P_{\perp,j} = 0 \tag{3.16}$$

We also require that applying T-duality twice in the same direction should not change the background. We satisfy this requirement by choosing

$$\tilde{P}_{\|}(\tilde{g},\tilde{\phi},\tilde{\nabla}) = P_{\|}(\tilde{g},\tilde{\phi},\tilde{\nabla}), \tag{3.17}$$

$$\tilde{P}_{\perp}(\tilde{g},\tilde{\phi},\tilde{\nabla}) = P_{\perp}(\tilde{g},\tilde{\phi},\tilde{\nabla}). \tag{3.18}$$

Therefore we reach to

$$\ln\frac{\tilde{g}_{11}(t)}{\alpha'} + P_{\|}(\ln\tilde{g},\tilde{\phi},\tilde{\nabla}) = -\left(\ln\frac{g_{11}(t)}{\alpha'} + P_{\|}(\ln g,\phi,\nabla)\right), \tag{3.19}$$

$$\ln\frac{\tilde{g}_{jj}(t)}{\alpha'} + \tilde{P}_{\perp,j}(\ln\tilde{g},\tilde{\phi},\tilde{\nabla}) = \ln\frac{g_{jj}(t)}{\alpha'} + P_{\perp,j}(\ln g,\phi,\nabla). \tag{3.20}$$

Both (g,ϕ) and $(\tilde{g},\tilde{\phi})$ solve the equations of motion. The leading equations of motion for (g,ϕ) are

$$R_{\mu\nu} + 2\nabla_\mu\nabla_\nu\phi + O(\alpha',g_s) = 0, \tag{3.21}$$

$$-\frac{1}{2}\Box\phi + |\nabla\phi|^2 + O(\alpha',g_s) = 0, \tag{3.22}$$

and the equations for $(\tilde{g},\tilde{\phi})$ are

$$\tilde{R}_{\mu\nu} + 2\tilde{\nabla}_\mu\tilde{\nabla}_\nu\tilde{\phi} + O(\alpha',g_s) = 0, \tag{3.23}$$

$$-\frac{1}{2}\Box\tilde{\phi} + |\tilde{\nabla}\tilde{\phi}|^2 + O(\alpha',g_s) = 0, \tag{3.24}$$

where $R_{\mu\nu}$ and $\tilde{R}_{\mu\nu}$ are Ricci tensors constructed from g and $\tilde{g}$ respectively. A straightforward calculation shows that if we set

$$P_{\|} = 0 + O(\alpha',g_s), \tag{3.25}$$

$$P_{\perp} = 0 + O(\alpha',g_s), \tag{3.26}$$

then the leading equations (3.21,3.23) of motion are compatible with (3.19,3.20) provided that

$$\tilde{\phi} - \frac{1}{4}\ln\det\tilde{g} = \phi - \frac{1}{4}\ln\det g + c, \tag{3.27}$$

where c is an arbitrary constant real number. $e^{-2\phi}\sqrt{\det g}$ governs the gravitational Newtonian constant. We fix $c=0$ to have the same gravitational Newtonian constant at asymptotic infinity for both backgrounds. Thus the rules describing T-duality in x_1 direction in the leading effective action are

$$\begin{aligned}
\ln\frac{\tilde{g}_{11}(t)}{\alpha'} &= -\ln\frac{g_{11}(t)}{\alpha'}\,, && (3.28)\\
\ln\frac{\tilde{g}_{jj}(t)}{\alpha'} &= \ln\frac{g_{jj}(t)}{\alpha'}\,, && (3.29)\\
\tilde{\phi}-\frac{1}{4}\ln\det\tilde{g} &= \phi-\frac{1}{4}\ln\det g\,. && (3.30)
\end{aligned}$$

When we consider the subleading corrections then these rules should be modified as follows

$$\begin{aligned}
\ln\frac{\tilde{g}_{11}(t)}{\alpha'}+P_{\parallel}(\ln\tilde{g},\tilde{\phi},\tilde{\nabla}) &= -\left(\ln\frac{g_{11}(t)}{\alpha'}+P_{\parallel}(\ln g,\phi,\nabla)\right) && (3.31)\\
\ln\frac{\tilde{g}_{jj}(t)}{\alpha'}+\tilde{P}_{\perp,j}(\ln\tilde{g},\tilde{\phi},\tilde{\nabla}) &= \ln\frac{g_{jj}(t)}{\alpha'}+P_{\perp,j}(\ln g,\phi,\nabla) && (3.32)\\
\tilde{\phi}-\frac{1}{4}\ln\det\tilde{g}+P_{\phi}(\ln\tilde{g},\tilde{\phi},\tilde{\nabla}) &= \phi-\frac{1}{4}\ln\det g+P_{\phi}(\ln g,\phi,\nabla)\,. && (3.33)
\end{aligned}$$

where $P_{\dots}$, similar to the effective action, have a double expansion series in g_s and α'. In the tree-level approximation to the effective action we have

$$\begin{aligned}
P_{\parallel}(\ln g,\phi,\nabla) &= 0+\alpha' P_{\parallel}^{(1)}(\ln g,\phi,\nabla)+\alpha'^2 P_{\parallel}^{(2)}(\ln g,\phi,\nabla)+\cdots\,, && (3.34)\\
P_{\perp}(\ln g,\phi,\nabla) &= 0+\alpha' P_{\perp}^{(1)}(\ln g,\phi,\nabla)+\alpha'^2 P_{\perp}^{(2)}(\ln g,\phi,\nabla)+\cdots\,, && (3.35)\\
P_{\phi}(\ln g,\phi,\nabla) &= 0+\alpha' P_{\phi}^{(1)}(\ln g,\phi,\nabla)+\alpha'^2 P_{\phi}^{(2)}(\ln g,\phi,\nabla)+\cdots\,, && (3.36)
\end{aligned}$$

each of which have an expansion series in ∇, i.e. for example

$$\begin{aligned}
P_{\perp}^{(1)}(\ln g,\phi,\nabla) &= C_{kl}^{\mu\nu}(g,\phi)\nabla_\mu\ln g_{kk}\nabla_\nu\ln g_{ll}+C_{k\phi}^{\mu\nu}(g,\phi)\nabla_\mu\ln g_{kk}\nabla_\nu\phi+\\
&\quad +C^{\mu\nu}(g,\phi)\nabla_\mu\phi\nabla_\nu\phi+H^{\mu\nu}(g,\phi)\nabla_\mu\nabla_\nu\phi+\\
&\quad +H_l^{\mu\nu}(g,\phi)\nabla_\mu\nabla_\nu\ln g_{ll} && (3.37)
\end{aligned}$$

where $C_{kl}^{\mu\nu}(g,\phi)$, $\cdots$ and $H^{\mu\nu}(g,\phi)$ are functions of the metric and the dilaton. We conjecture that

1. The sum over ∇ in each of $P_{\dots}^{(i)}(\ln g,\phi,\nabla)$ is a Lorentz covariant sum, i.e. for example

$$\begin{aligned}
P_{\perp}^{(1)}(\ln g,\phi,\nabla) &= A_{kl}\nabla^\mu\ln g_{kk}\nabla_\mu\ln g_{ll}+B_{k\phi}\nabla^\mu\ln g_{kk}\nabla_\mu\phi+ && (3.38)\\
&\quad +C\nabla^\mu\phi\nabla_\mu\phi+H\Box\phi+H_l\Box\ln g_{ll}
\end{aligned}$$

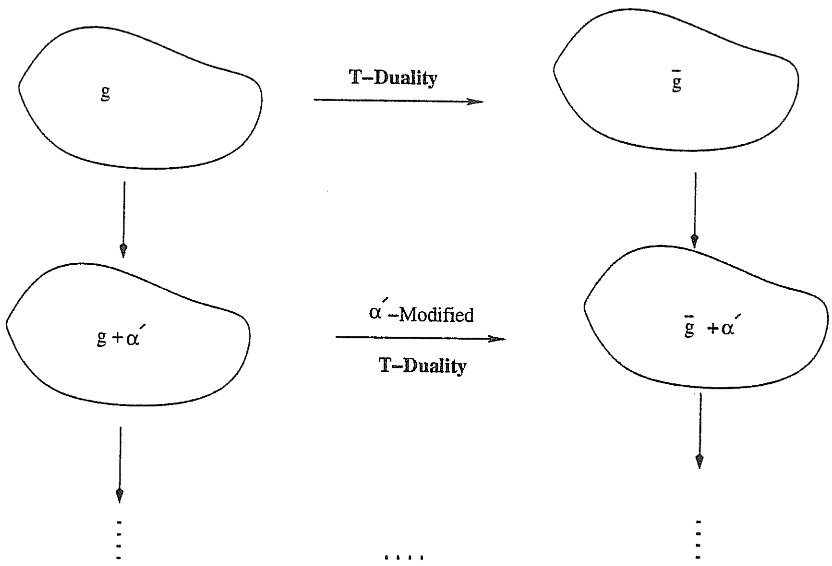

Figure 3.1: ***T-duality and the α' corrections.*** *By requiring that α' modified T-duality maps the α' corrected dual backgrounds to each other the α' modifications to the rules of the T-duality can be identified.*

2. The coefficients in the expansions of $P^{(i)}$ do not depend on the metric and dilaton, i.e. A_{kl}, B_k and $\cdots$ are numbers and they do not depend on the metric and the dilaton.

We refere to these conjectures as the Lorentz-form T-duality conjectures. The Lorentz-form T-duality conjectures shall help us to find the quadratic α' corrections to T-duality rules for time dependent backgrounds of diagonal metric and dilaton by studying the quadratic α' corrections to the diagonal Kasner background, the Schwarzschild metric, two dimensional black holes and their T-duals Fig. (3.1). We will prove that the Lorentz-form T-duality conjectures are correct at the linear order in α' and we discuss their validity at the order α'^2.

3.3 Kasner metrics and α' corrections

In this section we consider the case of the critical Bosonic String Theory where the leading equations for the background composed of the metric and dilaton in the string frame read

$$0 = R_{ij} + 2 \nabla_i \nabla_j \phi \,, \qquad (3.39)$$

$$0 = \Box\phi - (\nabla\phi)^2 + \frac{1}{4} R \,. \qquad (3.40)$$

The stringy Kasner background [66] determined by the set of $\{p_1, \cdots, p_{25}\}$ as a solution of the above equations is

$$ds^2 = -\, dt^2 + \sum_{i=1}^{25} t^{2p_i}\, dx_i^2 \,, \tag{3.41}$$

$$\phi(t) = \frac{\sum p - 1}{2} \ln t \,,$$

$$\sum_{i=1}^{25} p_i^2 = 1 \,, \tag{3.42}$$

where satisfying the equations imposes the constraint (3.42). The intrinsic singularity at $t = 0$ of this metric shows up in the various scalar curvature terms

$$R = \frac{(\sum p - 1)^2}{t^2} \,, \tag{3.43}$$

$$R_{\mu\nu}\, R^{\mu\nu} = 2\, \frac{(\sum p - 1)^2}{t^4} \,, \tag{3.44}$$

$$R_{\mu\nu\lambda\eta}\, R^{\mu\nu\lambda\eta} = \frac{6 + 2 \sum p^4 - 8 \sum p}{t^4} \,, \tag{3.45}$$

$$\nabla_\xi R_{\mu\nu\lambda\eta}\, \nabla^\xi R^{\mu\nu\lambda\eta} = 16\, \frac{-\sum p^4 + 2\sum p^3 + (\sum p^3)^2 - 2}{t^6} \,, \tag{3.46}$$

where R, $R_{\mu\nu}$ and $R_{\mu\nu\lambda\eta}$ stand for the Ricci scalar, Ricci and Riemann tensors respectively.

The usual Kasner metric has $\phi = 0$ as $\sum p - 1 = 0$.[1] Here we allow for a more general configuration with a time dependent string coupling. The string coupling constant given by the local vacuum expectation value of the dilaton reads

$$g_s = g_0\, e^{\phi} = g_0\, t^{(\sum p - 1)/2} \,. \tag{3.47}$$

Therefore for positive values of $\sum p - 1$ the string coupling constant vanishes at the time origin and diverges at infinity. For negative values of $\sum p - 1$ the string coupling constant diverges at the time origin and vanishes at infinity.

In this work we are interested in calculating the α' corrections at the tree level of the string interaction. The calculation at the tree level can be trusted as long as $g_s \ll 1$. For negative values

[1] The subleading β function (2.54) provides a non-zero value for the dilaton even if it vanishes at the leading order

of $\sum p - 1$ this condition is automatically satisfied at large t and for positive values of $\sum p - 1$ we set g_0 close to 0 to get $g_s \ll 1$ at the vicinity of a fixed large value of time where the perturbation in α' is going to be done.

We have defined the Kasner background as the solution to the leading order β-function equations. Before moving to the subleading order corrections, one needs to generalise the Kasner background to a perturbative background in string theory. We implement this generalisation by requiring that

1. String theory admits some time-dependent backgrounds where the metric is globally diagonal and the only non-vanishing field is the dilaton.

2. The above backgrounds admit a perturbative series expansion in α' i.e.

$$g_{\mu\nu}(t) = g^{(0)}_{\mu\nu}(t) + \alpha' \, g^{(1)}_{\mu\nu}(t) + \alpha'^2 \, g^{(2)}_{\mu\nu}(t) + \cdots , \tag{3.48}$$

$$\phi(t) = \phi^{(0)}(t) + \alpha' \, \phi^{(1)}(t) + \alpha'^2 \, \phi^{(2)}(t) + \cdots , \tag{3.49}$$

where $g^{(0)}_{\mu\nu}(t)$ and $\phi^{(0)}(t)$ correspond to the Kasner background. All $g^{(n)}_{\mu\nu}$ become automatically diagonal due to the first assumption.

We expect that for every given Kasner background there exists a string background satisfying the above conditions. In the next sections we are going to compute the linear and the quadratic α' corrections to the general Kasner background.

3.3.1 The linear α' corrections to the Kasner metric

We begin to investigate the α' corrections to the Kasner metric by making the following simple ansatz the generality of which we will verify at the end of this subsection.

$$ds^2 = - \, dt^2 + \sum_{i=1}^{25} t^{2p_i} \, (1 + 2\frac{\alpha'}{t^2} \, b_i) \, dx_i^2 + O(\frac{\alpha'^2}{t^4}) , \tag{3.50}$$

$$\phi(t) = - \, \frac{1}{2}(1 - \sum_{i=1}^{25} p_i) \, \ln t + \frac{\alpha' \, B}{2 \, t^2} + O(\frac{\alpha'^2}{t^4}) , \tag{3.51}$$

$$\sum_{i=1}^{25} p_i^2 = 1 ,$$

where b_i's and B are some unknown constants numbers. Substituting (3.50) and (3.51) in (2.53) and keeping only the linear α' term results in the algebraic equations,

$$2\, b_i \,+\, (-\sum_{j=1}^{25} b_j \,+\, B)\, p_i \,+\, p_i^2 \,-\, p_i^3 \,=\, 0\,, \tag{3.52}$$

$$6\,(-\sum_{i=1}^{25} b_i \,+\, B) \,-\, \sum_{i=1}^{25} p_i^4 \,-\, 1 \,+\, 2\sum_{i=1}^{25} p_i^3 \,+\, 4\sum_{i=1}^{25} p_i\, b_i \,=\, 0\,. \tag{3.53}$$

Multiplying (3.52) by p_i and summing over i gives

$$2\sum_{i=1}^{25} b_i\, p_i \,+\, (-\sum_{i=1}^{25} b_i \,+\, B) \,+\, \sum_{i=1}^{25} p_i^3 \,-\, \sum_{i=1}^{25} p_i^4 \,=\, 0\,. \tag{3.54}$$

(3.54) and (3.53) are solved by

$$B \,-\, \sum_{i=1}^{25} b_i \,=\, \frac{1}{4}(1 \,-\, \sum_{i=1}^{25} p^4)\,. \tag{3.55}$$

Using (3.52) and (3.55) one easily obtains

$$b_i \;=\; -\, p_i \left(\frac{1 \,-\, \sum_{j=1}^{25} p_j^4}{8} \,+\, \frac{1}{2}(p_i \,-\, p_i^2) \right)\,, \tag{3.56}$$

$$B \;=\; (1 \,-\, \sum_{j=1}^{25} p_j^4)\,(\frac{1}{4} \,-\, \frac{\sum_{i=1}^{25} p_i}{8}) \,-\, \frac{1}{2}(1 \,-\, \sum_{i=1}^{25} p_i^3)\,. \tag{3.57}$$

The results obtained for b_i and B satisfies (2.54) as well. Now let us investigate the general solution by writing the corrections in the following form

$$ds^2 \;=\; -\, dt^2 \,+\, \sum_{i=1}^{25} t^{2p_i}\,(1 \,+\, \frac{2\alpha'}{t^2}\,(b_i(t) \,+\, b_i))\, dx_i^2 \,+\, O(\frac{\alpha'^2}{t^4})\,, \tag{3.58}$$

$$\phi(t) \;=\; \frac{1}{2}\,(\sum p \,-\, 1)\,\ln t \,+\, (B \,+\, B(t))\,\frac{\alpha'}{2\,t^2} \,+\, O(\frac{\alpha'^2}{t^4})\,, \tag{3.59}$$

where b_i and B are given respectively in (3.56) and (3.57). In order to find $b_i(t)$ and $B(t)$ we first define the following variables

$$x(t) \;=\; B(t) \,-\, \sum_{i=1}^{25} b_i(t)\,, \tag{3.60}$$

$$y(t) \;=\; \sum_{i=1}^{25} b_i(t)\, p_i\,. \tag{3.61}$$

Inserting (3.58) and (3.59) in (2.53) and (2.54) and keeping only the linear term in α' yields[2]

$$- 2\, x(t) \,+\, \frac{3}{2}\, x'(t)\, t \,-\, \frac{1}{2}\, x''(t)\, t^2 \,-\, y(t) \,+\, \frac{1}{2}\, y'(t)\, t \;=\; 0\,, \tag{3.62}$$

$$6\, x(t) \,-\, 4\, t\, x'(t) \,+\, x''(t)\, t^2 \,+\, 4\, y(t) \,-\, 2\, y'(t)\, t \;=\; 0\,, \tag{3.63}$$

$$2\, b_i(t) \,-\, \frac{3}{2}\, t\, b_i'(t) + \frac{t^2}{2}\, b_i''(t) \,+\, p_i\, (x(t) \,-\, \frac{t}{2}\, x'(t)) \;=\; 0\,. \tag{3.64}$$

The general solution of the above system is

$$b_i(t) \;=\; -\, p_i\, c_1\, t \,+\, c_i^{(1)}\, t^2 \,+\, 2\, t^2\, c_i^{(2)}\, \ln t\,, \tag{3.65}$$

$$B(t) \;=\; c_1\, (1 \,-\, \sum p)\, t \,+\, c_2\, t^2 \,+\, 2\, t^2\, \ln t \sum c_i^{(2)}\,, \tag{3.66}$$

$$\sum c_i^{(2)}\, p_i \;=\; 0\,, \tag{3.67}$$

where c_1, c_2, c_i^1's and c_i^2's are constants of integration. At the first sight the appearance of these constants of integration may seem disappointing, however a closer look shows that

- c_1 corresponds to an infinitesimal time displacement , $t \,\rightarrow\, t \,-\, \alpha'\, c_1$.
- c_2 corresponds to a constant shift in the dilaton field.
- $c_i^{(1)}$'s correspond to proper scaling in the x_i directions.
- $c_i^{(2)}$ describes an infinitesimal change in p_i, $p_i \rightarrow p_i + 2\alpha' c_i^{(2)}$, constrained to $\sum p_i^2 = 1$. (3.67).

Therefore all the arbitrary constants in (3.65) and (3.66) are infinitesimal redefinitions of the variables. We fix the definition of the variables and set all of the arbitrary constants to zero. Doing so we obtain (3.56) and (3.57) as the values of the linear α' corrections to the metric and the dilaton.[3]

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[2] (3.62) is obtained from $\frac{1}{4}\, g^{ij}\, \beta_{ij} \,-\, \beta_\phi$

[3] In the Heterotic String Theory the β functions at two-loop in α' are the same as the ones of the Bosonic String Theory by replacing α' with $\frac{\alpha'}{2}$. Therefore the results obtained in (3.56) and (3.57) trivially can be extended to the Heterotic String Theory.

3.3.2 The α'^2 correction to the Kasner metric

Similar to what was done in the previous section the quadratic α' correction to the metric and the dilaton may be written as

$$ds^2 = -dt^2 + \sum_{i=1}^{25} t^{2p_i} (1 + 2\frac{\alpha'}{t^2} b_i + 2\frac{\alpha'^2}{t^4} c_i) + O(\frac{\alpha'^3}{t^6}) , \quad (3.68)$$

$$\phi(t) = \frac{\sum p - 1}{2} \ln t + \frac{\alpha'}{2\, t^2} B + \frac{\alpha'^2}{t^4} W + O(\frac{\alpha'^3}{t^6}) , \quad (3.69)$$

where c_i's and W are some unknown constant numbers and b_i's and B are written explicitly in the (3.56) and (3.57). The equation that comes from the time-time component of (2.53) reads

$$\begin{aligned} 0 = & -\frac{17}{4}\sum p^3 \sum p^4 - \sum p^5 \sum p^4 - \frac{\sum p^3 (\sum p^4)^2}{8} \\ & + \sum(-\frac{45}{8} p^3 + \frac{17}{4} p^4 - 19\, p^5 + 11\, p^6 - 2\, p^7 + 14\, p^4) \\ & + \frac{11}{4} + 8 \sum c\, p + 20\, (2\, W - \sum c) . \end{aligned} \quad (3.70)$$

The remaining equations generated by (2.53) are

$$\begin{aligned} 0 = & \frac{3}{16} p_i - \frac{1}{4} p_i^2 + 7\, p_i^3 - 18\, p_i^4 + 16\, p_i^5 - 4\, p_i^6 \\ & + p_i \sum(p^6 - 2\, p^5 + \frac{5}{2} p^3 - \frac{15}{8} p^4) + \frac{11}{16} p_i (\sum p^4)^2 \\ & - p_i^2 (\frac{7}{2} \sum p^4 + \frac{1}{4}(\sum p^4)^2) + 5\, p_i^3 \sum p^4 - 2\, p_i^4 \sum p^4 \\ & - \frac{p_i}{2} \sum p^3 \sum p^5 + 4\, (2\, W - \sum c)\, p_i + 16\, c_i . \end{aligned} \quad (3.71)$$

Multiplying (3.71) with p_i and summing over i gives

$$\begin{aligned} 0 = & -4 \sum p^3 \sum p^4 - 2 \sum p^5 \sum p^4 - \frac{1}{4} \sum p^3 (\sum p^4)^2 \\ & + \sum(-20\, p^5 + 17\, p^6 + \frac{9}{4} p^3 + \frac{41}{8} p^4 - 4\, p^7) \\ & + \frac{3}{16} + \frac{91}{16}(\sum p^4)^2 + 16 \sum c\, p + 4\, (2\, W - \sum c) . \end{aligned} \quad (3.72)$$

From (3.71) and (3.72) one obtains

$$\begin{aligned} 2\, W - \sum c = & \sum(\frac{3}{8} p^3 - \frac{61}{96} p^4 + \frac{1}{2} p^5 - \frac{5}{36} p^6) + \\ & + \frac{1}{8} \sum p^4 \sum p^3 - \frac{5}{64} (\sum p^4)^2 - \frac{85}{576} . \end{aligned} \quad (3.73)$$

Substituting (3.73) in (3.71) identifies the c_i giving

$$\begin{aligned} c_i &= p_i \left\{ \frac{1}{4}\, p_i^5 \;-\; p_i^4 \;+\; \frac{9}{8}\, p_i^3 \;-\; \frac{7}{16}\, p_i^2 \;+\; \frac{1}{64}\, p_i \;+\; \frac{29}{1152} \right. \\ &\quad + p_i \left(\frac{7}{32} \;+\; \frac{1}{8}\, p_i^2 \;-\; \frac{5}{16}\, p_i\right) \sum p^4 \;+\; \frac{1}{64}\, p_i \left(\sum p^4\right)^2 \\ &\quad \left. + \sum\left(\frac{53}{192}\, p^4 \;-\; \frac{1}{36}\, p^6\right) \;-\; \frac{3}{128} \left(\sum p^4\right)^2 \underbrace{-\frac{1}{4} \sum p^3}\right\}, \end{aligned} \tag{3.74}$$

$$\begin{aligned} W &= -\frac{\sum p^3}{32} + \frac{17}{48}\sum p^4 - \frac{\sum p^5}{4} + \frac{\sum p^6}{18} + \frac{1}{32}\left(\sum p^4\right)^2 - \frac{3}{32}\sum p^3 \sum p^4 \\ &\quad + \left\{\frac{53\sum p^4}{384} - \frac{\sum p^3}{8} - \frac{3(\sum p^4)^2}{256} + \frac{29}{2304} - \frac{\sum p^6}{72}\right\}\sum p - \frac{19}{288}\,. \end{aligned} \tag{3.75}$$

Having obtained W and c_i, it is not difficult to find the general α'^2 corrections to the metric and the dilaton. Let the quadratic α' corrections be written in the following way

$$\begin{aligned} ds^2 &= -dt^2 + \sum t^{2p_i} \left(1 + \frac{2\,\alpha'}{t^2}\, b_i + \frac{2\,\alpha'^2}{t^4}\,(c_i + c_i(t))\right) dx_i^2 + O\left(\frac{\alpha'^3}{t^6}\right), \\ \phi(t) &= \frac{\sum p \;-\; 1}{2}\, \ln t \;+\; \frac{\alpha'}{2\,t^2}\, B \;+\; \frac{\alpha'^2}{t^4}\,(W \;+\; W(t)) \;+\; O\left(\frac{\alpha'^3}{t^6}\right). \end{aligned} \tag{3.76}$$

We get the following equations for the auxiliary variables $x(t)$ and $y(t)$ -defined below- as the result of substituting (3.76) in (2.53) and (2.54)[4],

$$x(t) \;=\; 2\, W(t) \;-\; \sum_{i=1}^{25} c_i(t), \tag{3.77}$$

$$y(t) \;=\; \sum_{i=1}^{25} c_i(t)\; p_i \qquad , \tag{3.78}$$

$$20\, x(t) \;-\; 8\, x'(t)\, t \;+\; x''(t)\, t^2 \;-\; 2\, y'(t)\, t \;+\; 8\, y(t) \;=\; 0\,, \tag{3.79}$$

$$16\, c_i(t) \;-\; 7\, c_i'(t)\, t \;+\; c_i''(t)\, t^2 \;+\; p_i\, (4\, x(t) \;-\; x'(t)\, t) \;=\; 0\,, \tag{3.80}$$

$$-\, 8\, x(t) \;+\; \frac{7}{2}\, x'(t)\, t \;-\; \frac{t^2}{2}\, x''(t) \;-\; 2\, y(t) \;+\; \frac{t}{2}\, y'(t) \;=\; 0\,. \tag{3.81}$$

The general solution of the above ordinary system of differentiable equations in terms of the integration constants $c_1, c_2, c_i^{(1)}$ and $c_i^{(2)}$ is as follows

$$c_i(t) \;=\; p_i\, t^3\, c_2 \;+\; t^4\, c_i^{(1)} \;+\; t^4\, \ln t\, c_i^{(2)}\,, \tag{3.82}$$

$$W(t) \;=\; c_1\, t^4 \;+\frac{\sum p \;-\; 1}{2}\, t^3\, c_2 \;+\; t^4\, \ln t\, \frac{\sum c_i^{(2)}}{2}\,, \tag{3.83}$$

[4](3.81) is obtained from $\frac{1}{4}\, g^{ij}\, \beta_{ij} \;-\; \beta_\phi$

$$\sum_{i=1}^{25} c_i^{(2)} \, p_i \;=\; 0 \, .$$

Again all of the above constants of integration can be eliminated by redefining the variables appropriately. We set all of these constants of integration to zero and obtain (3.75) and (3.74) as the quadratic α' corrections to the dilaton and the metric. Let us emphasise that setting the constants of integration to zero is the same as fixing the asymptotic behaviour of the metric and dilaton at infinity as we saw already at the end of the previous subsection.

3.3.3 T-Duality, the α' corrections and the Kasner

In this section we are going to obtain the quadratic α' modifications to the T-duality rules for a time-dependent background composed of a diagonal metric and dilaton, consistent with the results of the previous sections on the α' corrections to the Kasner metric and its T-dual.

In the following we write the Kasner metric on a periodic space-like direction and we apply T-duality in x_{25} direction to obtain the corresponding T-dual background. Next we add the α' corrections to the Kasner background and to its T-dual. We will observe that T-duality fails to relate the α'-corrected Kasner background to its α' corrected T-dual background. We will modify the rules by appropriate α' terms in such a way that the α' modified rules relate the α'-corrected Kasner background to its T-dual. We see that the Lorentz T-duality conjectures give rise to corrections which are consistent with those of [59]. [59] gives the linear α' corrected T-duality rules for a general time-dependent background with diagonal metric and dilaton

Let us start the calculation by writing the Kasner background on periodic space directions

$$ds^2 \;=\; -\,dt^2 \,+\, \sum_{i=1}^{25} t^{2p_i} \left(\frac{r_i}{\sqrt{\alpha'}}\right)^2 dx_i^2 \, , \tag{3.84}$$

$$\phi(t) \;=\; \frac{\sum p \,-\, 1}{2} \, \ln t \, , \tag{3.85}$$

$$\sum_{i=1}^{25} p_i^2 \;=\; 1 \, , \tag{3.86}$$

$$x_i \;\equiv\; x_i \,+\, 2\,\pi \, , \tag{3.87}$$

where each x_i is compactified on a circle with time dependent radius $r_i(t) \;=\; r_i \; t^{p_i}$.

For the Kasner background the rules that describe T-duality in x_{25} read [5]

$$\ln \tilde{g}_{25\,25} \;=\; -\ln g_{25\,25} \,, \tag{3.88}$$

$$\ln \tilde{g}_{ii} \;=\; \ln g_{ii} \qquad i \in \{1, \cdots, 24\} \,, \tag{3.89}$$

$$\tilde{\phi} - \frac{1}{4} \ln \det \tilde{g} \;=\; \phi - \frac{1}{4} \ln \det g \,. \tag{3.90}$$

Applying the above rules on the Kasner background returns its T-dual background,

$$d\tilde{s}^2 = -\,dt^2 + \sum_{i=1}^{24} t^{2p_i} \left(\frac{r_i}{\sqrt{\alpha'}}\right)^2 d\tilde{x}_i^2 + t^{-2p_{25}} \left(\frac{r_{25}}{\sqrt{\alpha'}}\right)^{-2} d\tilde{x}_{25}^2 \,, \tag{3.91}$$

$$\tilde{\phi}(t) = (\sum_{i=1}^{24} p_i - p_{25} - 1) \frac{\ln t}{2} \,, \tag{3.92}$$

$$\tilde{x}_i \equiv \tilde{x}_i + 2\pi \,.$$

The dual background is still a Kasner background where x_{25} is compactified on a circle with radius $\tilde{r}_{25}(t) = \frac{\alpha'}{r_{25}} t^{-p_{25}}$. Now let the α' corrections be added to the Kasner background

$$ds^2 = -dt^2 + \sum_{i=1}^{25} t^{2p_i} \left(\frac{r_i}{\sqrt{\alpha'}}\right)^2 \{1 + \frac{2\alpha'}{t^2} b_i + \frac{2\alpha'^2}{t^4} c_i\} \, dx_i^2 + O(\frac{\alpha'^3}{t^6}) \tag{3.93}$$

$$\phi(t) = (\sum_{i=1}^{24} p_i + p_{25} - 1) \frac{\ln t}{2} + \frac{\alpha'}{2\,t^2} B + \frac{\alpha'^2}{t^4} W + O(\frac{\alpha'^3}{t^6}) \,, \tag{3.94}$$

where b_i, B, c_i and W respectively are identified by (3.56), (3.57), (3.74) and (3.75) for the set of $(p_1, \cdots, p_{24}, p_{25})$. Adding the corresponding α' corrections to the dual background gives

$$d\tilde{s}^2 = -dt^2 + \sum_{i=1}^{24} t^{2p_i} \left(\frac{r_i}{\sqrt{\alpha'}}\right)^2 \{1 + \frac{2\alpha'}{t^2} \tilde{b}_i + \frac{2\alpha'^2}{t^4} \tilde{c}_i\} \, dx_i^2 \tag{3.95}$$

$$+\; t^{-2p_{25}} \left(\frac{r_{25}}{\sqrt{\alpha'}}\right)^{-2} \{1 + \frac{2\alpha'}{t^2} \tilde{b}_{25} + \frac{2\alpha'^2}{t^4} \tilde{c}_{25}\} \, dx_{25}^2 \,, \tag{3.96}$$

$$\tilde{\phi}(t) = (\sum_{i=1}^{24} p_i - p_{25} - 1) \frac{\ln t}{2} + \frac{\alpha'}{2\,t^2} \tilde{B} + \frac{\alpha'^2}{t^4} \tilde{W} + O(\frac{\alpha'^3}{t^6}) \,, \tag{3.97}$$

where $\tilde{b}_i$, $\tilde{B}$, $\tilde{c}_i$ and $\tilde{W}$ respectively are provided by (3.56), (3.57), (3.74) and (3.75) for the set of $(p_1, \cdots, p_{24}, -p_{25})$. It should be noticed that for $1 \leq i \leq 24$ we have $b_i = \tilde{b}_i$ but due to the presence of the under-braced term in (3.74) $c_i \neq \tilde{c}_i$.

[5] These rules are written in such a way that it is manifest that $T^2 = 1$, where T represents the T-duality

Now let us check whether or not the T-duality rules (3.88, 3.89, 3.90) map the α' corrected backgrounds (3.95, 3.93) to each other. To perform this check it is better to write the rules describing the T-duality in the x_{25} direction in the following way

$$\ln \tilde{g}_{25\ 25} + \ln g_{25\ 25} = 0\,, \qquad (3.98)$$

$$\ln \tilde{g}_{ii} - \ln g_{ii} = 0\,, \qquad (3.99)$$

$$\tilde{\phi} - \phi + \frac{1}{4}\,(\ln \det g - \ln \det \tilde{g}) = 0\,. \qquad (3.100)$$

Substituting the α' corrected backgrounds in the l.h.s. of the above formulae gives

$$\ln \tilde{g}_{25\ 25} + \ln g_{25\ 25} = \frac{2\alpha'}{t^2}\,(b_{25} + \tilde{b}_{25}) + \frac{2\alpha'^2}{t^4}\,(c_{25} + \tilde{c}_{25} - b^2_{25} - \tilde{b}^2_{25}) + O(\frac{\alpha'^3}{t^6})\,, \qquad (3.101)$$

$$\ln \tilde{g}_{ii} - \ln g_{ii} = \frac{2\alpha'^2}{t^4}(c_i - \tilde{c}_i) + O(\frac{\alpha'^3}{t^6})\,, \qquad (3.102)$$

$$\begin{aligned}\tilde{\phi} - \phi + \frac{1}{4}\ln\frac{\det g}{\det \tilde{g}} &= \frac{\alpha'}{2t^2}(\tilde{B} - B + \sum b - \sum \tilde{b}) \\ &+ \frac{\alpha'^2}{2t^2}(2\tilde{W} - \sum \tilde{c} + \sum \tilde{b}^2 - 2W + \sum c - \sum b^2) + O(\frac{\alpha'^3}{t^6})\,.\end{aligned} \qquad (3.103)$$

Keeping only the leading non-vanishing α' terms and expressing the r.h.s. of the above formulae in term of $p_1, \cdots, p_{25}$ gives[6]

$$\ln \tilde{g}_{25\ 25} + \ln g_{25\ 25} = -\frac{2\,\alpha'}{t^2}\,p^2_{25} + O(\frac{\alpha'^2}{t^4})\,, \qquad (3.104)$$

$$\ln \tilde{g}_{ii} - \ln g_{ii} = \frac{\alpha'^2}{t^4}\,p_i\,p^3_{25} + O(\frac{\alpha'^3}{t^6})\,, \qquad (3.105)$$

$$\tilde{\phi} - \phi + \frac{1}{4}\,\ln\frac{\det g}{\det \tilde{g}} = -\frac{\alpha'^2}{2\,t^4}\,p^3_{25} + O(\frac{\alpha'^3}{t^6})\,. \qquad (3.106)$$

The above relations indicate that none of the T-duality rules are satisfied and they all must be modified by appropriate α' terms i.e. $P_{\|}, P_{\perp}, P_{\phi}$ are not zero. In order to find the modifications let (3.104), (3.105) and (3.106) be written in the following forms

$$\ln \tilde{g}_{25\,25} + \frac{\alpha'}{t^2}\tilde{p}^2_{25} = -\,(\ln g_{25\,25} + \frac{\alpha'}{t^2}p^2_{25}) + O(\frac{\alpha'^2}{t^4})\,, \qquad (3.107)$$

[6]Only the linear α' term in (3.104) is kept. After finding the linear α' term in (3.126) the quadratic term is fixed in (3.129) .

$$\ln \tilde{g}_{ii} + \frac{\alpha'^2}{2\,t^4}\,\tilde{p}_i\,\tilde{p}_{25}^3 = \ln g_{ii} + \frac{\alpha'^2}{2\,t^4}\,p_i\,p_{25}^3 + O(\frac{\alpha'^3}{t^6})\,, \tag{3.108}$$

$$\tilde{\phi} - \frac{1}{4}\ln\det\tilde{g} - \frac{\alpha'^2}{4\,t^4}\,\tilde{p}_{25}^3 = \phi - \frac{1}{4}\ln\det g - \frac{\alpha'^2}{4\,t^4}\,p_{25}^3 + O(\frac{\alpha'^3}{t^6})\,, \tag{3.109}$$

where $\tilde{p}_i = p_i$ and $\tilde{p}_{25} = -p_{25}$. Comparing these relations with (3.31), (3.32) and (3.33) we conclude that

$$P_{\|}(\ln g_{kk},\phi,\nabla) = \frac{\alpha'}{t^2}p_{25}^2 + P_{\|}^{\star}(\ln g_{kk},\phi,\nabla) + O(\frac{\alpha'^2}{t^4})\,, \tag{3.110}$$

$$P_{\perp}(\ln g_{kk},\phi,\nabla) = \frac{\alpha'^2}{2\,t^4}\,p_i\,p_{25}^3 + P_{\perp}^{\star}(\ln g_{kk},\phi,\nabla) + O(\frac{\alpha'^3}{t^6})\,, \tag{3.111}$$

$$P_{\phi}(\ln g_{kk},\phi,\nabla) = -\frac{\alpha'^2}{4\,t^4}\,p_{25}^3 + P_{\phi}^{\star}(\ln g_{kk},\phi,\nabla) + O(\frac{\alpha'^3}{t^6})\,, \tag{3.112}$$

where $P_{\perp}^{\star}$ and $P_{\phi}^{\star}$ are even under $\ln g_{25\ 25} \to -\ln g_{25\ 25}$

$$\ln g_{25\ 15} \to -\ln g_{25\ 25} \tag{3.113}$$

$$P_{\perp}^{\star} \to P_{\perp}^{\star}\,, \tag{3.114}$$

$$P_{\phi}^{\star} \to P_{\phi}^{\star}\,, \tag{3.115}$$

and $P_{\|}^{\star}$ is odd under $\ln g_{25\ 25} \to -\ln g_{25\ 25}$

$$P_{\|}^{\star} \to -P_{\|}^{\star}\,. \tag{3.116}$$

We presume that $P_{\perp}^{\star} = P_{\phi}^{\star} = P_{\|}^{\star} = 0$ and we get

$$P_{\|}(\ln g_{kk},\phi,\nabla) = \frac{\alpha'}{t^2}p_{25}^2 + O(\frac{\alpha'^2}{t^4}) \tag{3.117}$$

$$P_{\perp}(\ln g_{kk},\phi,\nabla) = \frac{\alpha'^2}{2\,t^4}\,p_i\,p_{25}^3 + O(\frac{\alpha'^3}{t^6}) \tag{3.118}$$

$$P_{\phi}(\ln g_{kk},\phi,\nabla) = -\frac{\alpha'^2}{4\,t^4}\,p_{25}^3\,. \tag{3.119}$$

Later we will provide evidence for the validity of this assumption by applying the α' corrected T-duality rules on other backgrounds including the Schwarzschild metric and the two-dimensional black hole. The rules of T-duality follow

$$\ln \tilde{g}_{25\,25} + P_{\|}(\ln\tilde{g}_{kk},\tilde{\phi},\tilde{\nabla}) = -\,(\ln g_{25\,25} + P_{\|}(\ln g_{kk},\phi,\nabla)) + O(\frac{\alpha'^2}{t^4})\,, \tag{3.120}$$

$$\ln \tilde{g}_{ii} \; + \; \tilde{P}_{\perp}(\ln \tilde{g}_{kk}, \phi, \nabla) \;\; = \;\; \ln g_{ii} \; + \; P_{\perp}(\ln g_{kk}, \phi, \nabla) \; + \; O(\frac{\alpha'^3}{t^6}) \; , \qquad (3.121)$$

$$\tilde{\phi} \; - \; \frac{1}{4} \ln \det \tilde{g} \; + \; P_{\phi}(\ln \tilde{g}_{kk}, \tilde{\phi}, \tilde{\nabla}) \;\; = \;\; \phi \; - \; \frac{1}{4} \ln \det g \; + \; P_{\phi}(\ln g_{kk}, \phi, \nabla) \; + \; O(\frac{\alpha'^3}{t^6}) \; . \qquad (3.122)$$

According to the Lorentz form T-duality conjectures we could write the α' corrections to T-duality as polynomials in derivatives of the metric and dilaton, in the following covariant forms[7]

$$P_{\|}(\ln g_{kk}, \phi, \nabla) \;\; = \;\; \frac{1}{16} \; \nabla_{\mu} \ln g_{ii} . \nabla^{\mu} \ln g_{25\,25} \;\; \nabla_{\nu} \ln g_{25\,25} . \nabla^{\nu} \ln g_{25\,25} \; + \; O(\alpha') \; , \qquad (3.123)$$

$$P_{\perp}(\ln g_{kk}, \phi, \nabla) \;\; = \;\; -\frac{1}{4} \nabla_{\mu} \ln g_{25\,25} . \nabla^{\mu} \ln g_{25\,25} \; + \; O(\alpha') \; , \qquad (3.124)$$

$$\begin{aligned} P_{\phi}(\ln g_{kk}, \phi, \nabla) \;\; = \;\; & \frac{A}{32} \; \nabla_{\mu} \ln g_{25\,25} \;\; \nabla_{\nu} \ln g_{25\,25} \;\; \nabla^{\mu} \nabla^{\nu} \ln g_{25\,25} \\ & + \frac{B}{16} \; \nabla_{\mu} \ln g_{25\,25} . \nabla^{\mu} \ln g_{25\,25} \;\; \nabla_{\nu} \ln g_{25\,25} . \nabla^{\nu} (\phi - \frac{1}{4} \ln \det g) \\ & + \frac{C}{32} \; \nabla_{\mu} \ln g_{25\,25} . \nabla^{\mu} \ln g_{25\,25} \; \Box \ln g_{25\,25} \; + \; O(\alpha') \; , \end{aligned} \qquad (3.125)$$

where A, B and C are real numbers satisfying $A + B + C = 1$. Using the above identities gives the leading α' modified T-duality rules on the metric

$$\begin{aligned} \ln \tilde{g}_{25\,25} \;\; - \;\; & \frac{\alpha'}{4} \tilde{\nabla}_{\mu} \ln \tilde{g}_{25\,25} . \tilde{\nabla}^{\mu} \ln \tilde{g}_{25\,25} \\ = \;\; & - \left\{ \ln g_{25\,25} - \frac{\alpha'}{4} \nabla_{\mu} \ln g_{25\,25} . \nabla^{\mu} \ln g_{25\,25} \right\} + O(\alpha'^2 \nabla^4) \; , \end{aligned} \qquad (3.126)$$

$$\begin{aligned} \ln \tilde{g}_{ii} \;\; + \;\; & \frac{\alpha'^2}{32} \tilde{\nabla}_{\mu} \ln \tilde{g}_{25\,25} . \tilde{\nabla}^{\mu} \ln \tilde{g}_{ii} \;\; \tilde{\nabla}_{\nu} \ln \tilde{g}_{25\,25} . \tilde{\nabla}^{\nu} \ln \tilde{g}_{25\,25} \\ = \;\; & \ln g_{ii} + \frac{\alpha'^2}{32} \nabla_{\mu} \ln g_{ii} . \nabla^{\mu} \ln g_{25\,25} \;\; \nabla_{\nu} \ln g_{25\,25} . \nabla^{\nu} \ln g_{25\,25} + O(\alpha'^3 \nabla^6) \; . \end{aligned} \qquad (3.127)$$

The T-duality rule which describes the change in the dilaton reads

$$\begin{aligned} \tilde{\phi} - \frac{1}{4} \ln \det \tilde{g} \;\; + \;\; & \frac{\alpha'^2 A}{32} \; \tilde{\nabla}_{\mu} \ln \tilde{g}_{25\,25} \;\; \tilde{\nabla}_{\nu} \ln \tilde{g}_{25\,25} \;\; \tilde{\nabla}^{\mu} \tilde{\nabla}^{\nu} \ln \tilde{g}_{25\,25} \\ + \;\; & \frac{\alpha'^2 B}{16} \; \tilde{\nabla}_{\mu} \ln \tilde{g}_{25\,25} . \tilde{\nabla}^{\mu} \ln \tilde{g}_{25\,25} \;\; \tilde{\nabla}_{\nu} \ln \tilde{g}_{25\,25} . \tilde{\nabla}^{\nu} (\tilde{\phi} - \frac{1}{4} \ln \det \tilde{g}) \end{aligned} \qquad (3.128)$$

[7]In this notation, the covariant derivative acts on the logarithm of the metric as if it were a scalar.

$$
\begin{aligned}
&+ \frac{\alpha'^2 C}{32} \tilde{\nabla}_\mu \ln \tilde{g}_{25\,25} . \tilde{\nabla}^\mu \ln \tilde{g}_{25\,25} \, \Box \ln \tilde{g}_{25\,25} \\
= \phi &- \frac{1}{4} \ln \det g + \frac{\alpha'^2 A}{32} \nabla_\mu \ln g_{25\,25} \, \nabla_\nu \ln g_{25\,25} \, \nabla^\mu \nabla^\nu \ln g_{25\,25} \\
&+ \frac{\alpha'^2 B}{16} \nabla_\mu \ln g_{25\,25} . \nabla^\mu \ln g_{25\,25} \, \nabla_\nu \ln g_{25\,25} . \nabla^\nu (\phi - \frac{1}{4} \ln \det g) \\
&+ \frac{\alpha'^2 C}{32} \nabla_\mu \ln g_{25\,25} . \nabla^\mu \ln g_{25\,25} \, \Box \ln g_{25\,25} + O(\alpha'^3 \nabla^6) ,
\end{aligned}
$$

where $A+B+C=1$. In a similar way one notices that (3.126) should be modified to

$$
\begin{aligned}
\ln \tilde{g}_{25\,25} &- \frac{\alpha'}{4} \tilde{\nabla}_\mu \ln \tilde{g}_{25\,25} \tilde{\nabla}^\mu \ln \tilde{g}_{25\,25} \\
&= - \left\{ \ln g_{25\,25} - \frac{\alpha'}{4} \nabla_\mu \ln g_{25\,25} \nabla^\mu \ln g_{25\,25} + \frac{\alpha'^2 p_{25}^4}{t^4} - \frac{\alpha'^2 p_{25}^2}{t^4} \right\} ,
\end{aligned} \tag{3.129}
$$

There exists only one way to express $\frac{p_{25}^4}{t^4}$ as the "covariant" derivative of the dilaton and the logarithm of the Kasner metric:

$$
\frac{p_{25}^4}{t^4} = \frac{1}{16} (\nabla_\mu \ln g_{25\,25} \, \nabla^\mu \ln g_{25\,25})^2 , \tag{3.130}
$$

while there are the following five candidates for $\frac{p_{25}^2}{t^4}$,

$$
E^\star = \nabla_\mu \ln g_{25\,25} \, \nabla^\mu \ln g_{25\,25} \, \nabla_\nu (\phi - \frac{1}{4} \ln \det g) \, \nabla^\nu (\phi - \frac{1}{4} \ln \det g) , \tag{3.131}
$$

$$
E = \nabla_\mu \ln g_{25\,25} \, \nabla_\nu \ln g_{25\,25} \, \nabla^\mu (\phi - \frac{1}{4} \ln \det g) \, \nabla^\nu (\phi - \frac{1}{4} \ln \det g) , \tag{3.132}
$$

$$
F = \frac{1}{2} \nabla_\mu \ln g_{25\,25} \, \nabla_\nu \ln g_{25\,25} \, \nabla^\nu \nabla^\mu (\phi - \frac{1}{4} \ln \det g) , \tag{3.133}
$$

$$
G = \frac{1}{2} \nabla_\mu \ln g_{25\,25} \, \nabla_\nu (\phi - \frac{1}{4} \ln \det g) \, \nabla^\nu \nabla^\mu \ln g_{25\,25} , \tag{3.134}
$$

$$
H = \frac{1}{8} \nabla_\mu \nabla_\nu \ln g_{25\,25} \, \nabla^\mu \nabla^\nu \ln g_{25\,25} . \tag{3.135}
$$

Therefore we conclude that

$$
P_{\parallel}^{(2)} (\ln g_{kk}, \phi, \nabla) = \frac{\alpha'^2}{16} (\nabla_\mu \ln g_{25\,25} \nabla^\mu \ln g_{25\,25})^2 - \frac{\alpha'^2}{2} (a\,E + b\,F + c\,G + e\,H) \tag{3.136}
$$

where "a, b, c, e" are real numbers satisfying "$a+b+c+e=1$". In direct multiplication of spaces $|\nabla(\phi - \frac{1}{4} \ln \det g)|^2$ can depend on the coordinates of the individual spaces. We assume that T-duality rules for direct multiplications of curved spaces are given by the T-duality rules of each individual space. Therefore we exclude the possibility of writing $\frac{p_{25}^2}{t^4}$ by (3.131). Note that the rest of possibilities respect this assumption and the general form of "(3.129)" must be a linear combination of them,

$$\begin{aligned}\ln \tilde{g}_{25\,25} &- \frac{\alpha'}{4}\tilde{\nabla}_\mu \ln \tilde{g}_{25\,25}\tilde{\nabla}^\mu \ln \tilde{g}_{25\,25} - \frac{\alpha'^2}{2}(a\,\tilde{E} + b\,\tilde{F} + c\,\tilde{G} + e\,\tilde{H}) \\ &= -\Big\{ \ln g_{25\,25} - \frac{\alpha'}{4}\nabla_\mu \ln g_{25\,25}\nabla^\mu \ln g_{25\,25} + \frac{\alpha'^2}{16}(\nabla_\mu \ln g_{25\,25}\nabla^\mu \ln g_{25\,25})^2 \\ &\quad -\frac{\alpha'^2}{2}(a\,E + b\,F + c\,G + e\,H)\Big\}.\end{aligned} \tag{3.137}$$

where $\tilde{E}, \cdots, \tilde{H}$ are $(E, \cdots, H)$ written for the T-dual backgrounds. The α'^2 corrections to other backgrounds should be calculated to fix these numerical coefficients.

The above rules (3.126), (3.127) and (3.137) are written in Lorentz invariant forms compatible with the Lorentz-form T-duality conjectures and they describe T-duality on backgrounds composed of diagonal metric and dilaton given that the fields are in the string frame and the co-moving frame. These rules are in agreement with those of [58] and [59] where only the linear α' modifications were considered. This agreement proves the the Lorentz-form T-duality conjectures at the linear order in α'.

One observes that redefining the metrics g_{ii} and $\tilde{g}_{ii}$ to g^*_{ii} and $\tilde{g}^*_{ii}$ in the following way

$$g^*_{ii} = g_{ii}\exp\left(\frac{\alpha'^2}{32}\nabla_\mu \ln g_{ii}.\sum_{k=1}^{25}\nabla^\mu \ln g_{kk}\ \nabla_\nu \ln g_{kk}.\nabla^\nu \ln g_{kk}\right), \tag{3.138}$$

$$\tilde{g}^*_{ii} = \tilde{g}_{ii}\exp\left(\frac{\alpha'^2}{32}\tilde{\nabla}_\mu \ln \tilde{g}_{ii}.\sum_{k=1}^{25}\tilde{\nabla}^\mu \ln \tilde{g}_{kk}\ \tilde{\nabla}_\nu \ln \tilde{g}_{kk}.\tilde{\nabla}^\nu \ln \tilde{g}_{kk}\right), \tag{3.139}$$

compensates the α' corrections to the rules describing the change of the metric in the transverse directions under T-duality in any direction. In general this transformation implies that one really needs a field redefinition to rewrite the higher order T-duality rules in the same form as the leading order T-duality rules.

Jack and Parson in [67] have calculated the same corrections to T-duality and proved that $O(d,d)$ invariance of the conformal invariance condition, observed at one loop [68, 69] and two loops [70], can be preserved also at three loops given an appropriate field redefinition and coordinate transformation (either on the background or its T-dual background but not both). They did not explicitly provide the modification to T-duality but rewriting their results for the case of the Kasner background reproduces (3.127).

In this work *we do not redefine the metric and the dilaton* and we maintain the convention given by the dimensional regularisation method in the minimal substraction scheme which is the same as fixing the definition of the metric and dilaton in such a way that the corresponding β functions are provided by (2.53) and (2.54). This convention implies (3.127) which means that applying T-duality in one direction alters the metric in all directions.

Sometimes writing the fields in the co-moving frame is not easy. Thus we are going to write the α' T-duality rules in an alternative frame. As a simple generalisation of the co-moving frame let us introduce an "*almost co-moving frame*" with a "*characteristic function (f)*" in which the metrics read

$$ds^2 = -f(T)\,dT^2 + ds_\perp^2 \quad , \quad \phi = \phi(T) \tag{3.140}$$

$$d\tilde{s}^2 = -f(T)\,dT^2 + d\tilde{s}_\perp^2 \quad , \quad \tilde{\phi} = \tilde{\phi}(T) \tag{3.141}$$

where $f(T)$ can be an arbitrary function of time. The α' modified T-duality rules in the almost co-moving frame for a general characteristic function are provided by those of the co-moving frame if within (3.128) and (3.137) we replace $\det g$ and $\det \tilde{g}$ respectively by $\det^* g$ and $\det^* \tilde{g}$ given below

$$\det{}^* g = \frac{\det g}{f(T)} , \tag{3.142}$$

$$\det{}^* \tilde{g} = \frac{\det \tilde{g}}{f(T)} . \tag{3.143}$$

For a specific background, one may choose a conveniant characteristic function to simplify the computations.

3.4 The Schwarzschild black hole

In this section the Schwarzschild background in an arbitrary dimension is reviewed. The time-dual of Schwarzschild background is introduced by performing a non-compact T-duality in the time direction of the Schwarzschild metric in the region outside the black hole horizon. We must note the this non-compact T-duality in a time-like direction is not on the same footing as the usual

T-duality, but it has been studied in [71]. The linear and the quadratic α' corrections to the Schwarzschild background and its time-dual in $D=4$ are computed. Requiring (3.128) to relate the quadratic α' corrected Schwarzschild background to the quadratic α' corrected its time-dual we are able to identify the unknown coefficients in (3.128) with the values $A=1$ and $B=C=0$.

The Schwarzschild background in D dimensions, ($D>3$), is given by

$$\begin{aligned} ds^2 &= -(1-\frac{1}{r^{D-3}})\,dt^2 + \frac{dr^2}{1-\frac{1}{r^{D-3}}} + r^2\,d\Omega_{D-2}\,, \\ \phi(r) &= 0\,, \end{aligned} \tag{3.144}$$

where the mass has been chosen to give simply a coefficient of one in the metric and we will maintain this convention in the following sections.

The intrinsic singularity of the Schwarzschild metric shows itself in various scalar curvatures

$$R_{\mu\nu\lambda\eta}R^{\mu\nu\lambda\eta} = \frac{(D-1)\,(D-2)^2\,(D-3)}{r^{2D-2}}\,, \tag{3.145}$$

$$\nabla_\xi R_{\mu\nu\lambda\eta}\,\nabla^\xi R^{\mu\nu\lambda\eta} = \frac{(D+1)\,(D-1)^2\,(D-2)^2\,(D-3)}{r^{3\,D-3}}\,(r^{D-3}-1)\,. \tag{3.146}$$

Applying T-duality to the time direction, on the metric outside the horizon, we get the T-dual background (denoted from hereon by a tilde) is,

$$\begin{aligned} d\tilde{s}^2 &= -\frac{dt^2}{1-\frac{1}{r^{D-3}}} + \frac{dr^2}{1-\frac{1}{r^{D-3}}} + r^2\,d\Omega_{D-2}\,, \\ \tilde{\phi}(r) &= -\frac{1}{2}\,\ln(1-\frac{1}{r^{D-3}})\,. \end{aligned} \tag{3.147}$$

The above background solves (3.39) and (3.40). We refer to this background as the "*time-dual of the Schwarzschild background*". The singularities of the time-dual of the Schwarzschild metric can be seen in the following scalar curvatures,

$$\tilde{R} = -\frac{(D-3)^2}{r^{D-1}\,(r^{D-3}-1)}\,, \tag{3.148}$$

$$\begin{aligned} \tilde{R}_{\mu\nu\lambda\eta}\tilde{R}^{\mu\nu\lambda\eta} &= \frac{D-3}{r^{2D-2}(r^{D-3}-1)^2}(-2(D-2)(3D-7)r^{D-3}+ \\ &\quad +(D-1)(2D-5+(D-2)^2r^{2D-6}))\,. \end{aligned} \tag{3.149}$$

Therefore the time-dual of the Schwarzschild metric has two intrinsic singularities at $r=0,1$ which means in particular that the coordinate singularity of the Schwarzschild metric at the horizon has changed to an intrinsic singularity in its time-dual metric. This behaviour is similar to what have been observed in [58] and in [72].

In order to get a better understanding of the time-dual of the Schwarzschild background let us write it in the Einstein frame[8],

$$\begin{aligned} d\tilde{s}_E^2 &= \left(1 - \frac{1}{r^{D-3}}\right)^{\frac{4-D}{D-2}} (-\,dt^2 + dr^2) + r^2 \left(1 - \frac{1}{r^{D-3}}\right)^{\frac{2}{D-2}} d\Omega_{D-2}\,, \\ \tilde{\phi}(r) &= -\frac{1}{2}\ln(1 - \frac{1}{r^{D-3}})\,. \end{aligned} \tag{3.150}$$

The singularities of the time-dual of the Schwarzschild metric in the Einstein frame are the same as the ones of the string frame because the Ricci scalar of (3.150) reads

$$\tilde{R}_E = \frac{(D-3)^2}{D-2}\frac{1}{r^{2D-4}}\left(\frac{r^{D-3}}{r^{D-3}-1}\right)^{\frac{D}{D-2}}\,. \tag{3.151}$$

The time-dual of the Schwarzschild metric in the Einstein frame for $D > 4$ describes a geometry with two singularities and a Newtonian mass proportional to $(4-D)/(D-2)$. In $D = 4$ the time-dual of the Schwarzschild metric is massless and in the Einstein frame reads

$$\begin{aligned} ds^2 &= -\,dt^2 + dr^2 + r\,|r-1|\,(d\theta^2 + \sin^2\theta\, d\phi^2)\,, \\ \phi(r) &= -\frac{1}{2}\ln|1 - \frac{1}{r}|\,. \end{aligned} \tag{3.152}$$

The above metric (3.152), describes a geometry with a naked singularity at $r = 1$ and a vanishing Newtonian mass. In supergravity similar geometries are studied and named massless black(white) holes [73, 74, 75, 76]. Here (3.152) represents the corresponding object in the low energy gravitational theory of the Bosonic String Theory in $D = 4$. Other similar objects in higher dimensions within the low energy theory of the Bosonic String Theory can be found. For example the background provided below

$$ds^2 = -\,dt^2 + \frac{dr^2}{1 + (\frac{q}{r^{D-3}})^2} + r^2\, d\Omega_{D-2}\,, \tag{3.153}$$

$$\phi = \pm\frac{D-2}{2\sqrt{D-3}}\, ArcSinh(\frac{q}{r^{D-3}})\,, \tag{3.154}$$

solves the leading order equations of motion in the Einstein frame.

$$R_{\mu\nu} - \frac{4}{D-2}\nabla_\mu\phi\nabla_\nu\phi = 0\,, \tag{3.155}$$

[8]The Einstein frame is obtained by $g_{\mu\nu} \rightarrow g_{\mu\nu}\, e^{-\frac{4\phi}{D-2}}$

$$\Box\phi \;=\; 0\,. \tag{3.156}$$

This background (3.153), represents a geometry with a vanishing Newtonian mass and an intrinsic singularity at $r = 0$ as is clear from the corresponding scalar curvature terms

$$R \;=\; (D-2)(D-3)\left(\frac{q}{r^{D-3}}\right)^2\,, \tag{3.157}$$

$$R_{\mu\nu\eta\xi}\,R^{\mu\nu\eta\xi} \;=\; 2(D-2)(D-3)(2D-5)\left(\frac{q}{r^{D-2}}\right)^4\,. \tag{3.158}$$

To ensure that the dilaton is real we must choose q to also be real. We then see that there is a naked singularity at $r = 0$.

The massless black holes are stationary and they should not be thought as massless particles but new vacua of the theory. In superstring it turns out that massless black holes play quite important roles [73], a modification of these roles is expected to persist to the Bosonic String Theory.

3.4.1 The linear and the quadratic α' corrections to the Schwarzschild background in $D = 4$

The Schwarzschild metric in $D = 3 + 1$ reads

$$ds^2 \;=\; -\,(1-\frac{1}{r})\,dt^2 + \frac{1}{1-\frac{1}{r}}\,dr^2 + r^2\,(d\theta^2 + \sin^2\theta\,d\phi^2)\,, \tag{3.159}$$

where its intrinsic singularity can be seen in the following scalar curvatures

$$R_{\mu\nu\lambda\eta}\,R^{\mu\nu\lambda\eta} \;=\; \frac{12}{r^6}\,, \tag{3.160}$$

$$\nabla_\xi R_{\mu\nu\lambda\eta}\,\nabla^\xi R^{\mu\nu\lambda\eta} \;=\; \frac{180\,(r-1)}{r^9}\,, \tag{3.161}$$

$$\nabla_{\xi_1}\nabla_{\xi_2}R_{\mu\nu\lambda\eta}\,\nabla^{\xi_1}\nabla^{\xi_2}R^{\mu\nu\lambda\eta} \;=\; \frac{90}{r^{12}}(56\,r^2 - 120\,r + 65)\,, \tag{3.162}$$

$$\nabla_{\xi_1}\nabla_{\xi_2}\nabla_{\xi_3}R_{\mu\nu\lambda\eta}\,\nabla^{\xi_1}\nabla^{\xi_2}\nabla^{\xi_3}R^{\mu\nu\lambda\eta} \;=\; \frac{540\,(r-1)}{r^{15}}(420\,r^2 - 1000\,r + 609)\,, \tag{3.163}$$

$$\nabla_{\xi_1}\cdots\nabla_{\xi_4}R_{\mu\nu\lambda\eta}\,\nabla^{\xi_1}\ldots\nabla^{\xi_4}R^{\mu\nu\lambda\eta} \;=\; \frac{270}{r^{18}}(55440\,r^4 - 259920\,r^3 + 457898\,r^2 - 358522\,r + 105133)\,, \tag{3.164}$$

$$\nabla_{\xi_1}\cdots\nabla_{\xi_5}R_{\mu\nu\lambda\eta}\,\nabla^{\xi_1}\ldots\nabla^{\xi_5}R^{\mu\nu\lambda\eta} \;=\; \frac{540\,(r-1)}{r^{21}}\,(2522520\,r^4 - 12736080\,r^3 + 24176940\,r^2 - 20406448\,r + 6454623)\,, \tag{3.165}$$

The Schwarzschild metric can be generalised to a perturbative consistent background of the Bosonic String Theory in the critical dimension by assuming that

1. The critical Bosonic String Theory admits the following background

$$ds^2 = - g_{tt}(r)\, dt^2 + g_{rr}(r)\, dr^2 + g_\Omega(r)\,(d\theta^2 + \sin^2\theta\, d\phi^2) + dx_\perp^2\,, \tag{3.166}$$

$$\phi = \phi(r)\,, \tag{3.167}$$

where $dx_\perp^2$ and ϕ respectively represent the 22-dimensional flat space and the dilaton.

2. Within the above background the metric and the dilaton admit the following perturbative series in α'

$$g_{tt}(r) = (1 - \frac{1}{r})\,(1 + \alpha'\, g_{tt}^{(1)}(r) + \alpha'^2\, g_{tt}^{(2)}(r) + \cdots)\,, \tag{3.168}$$

$$g_{rr}(r) = \frac{1}{1-\frac{1}{r}}\,(1 + \alpha'\, g_{rr}^{(1)}(r) + \alpha'^2\, g_{rr}^{(2)}(r) + \cdots)\,, \tag{3.169}$$

$$g_\Omega(r) = r^2\,, \tag{3.170}$$

$$\phi(r) = 0 + \alpha'\, \phi^{(1)}(r) + \alpha'^2\, \phi^{(2)}(r) + \cdots\,. \tag{3.171}$$

Using the β-functions (2.53),(2.54) gives the general solution for the linear α' corrections

$$\phi^{(1)}(r) = -\frac{2 + 3r + 6r^2}{12\,r^3} - (c_3 + \frac{1}{2})\,\ln(1 - \frac{1}{r}) + c_1\,, \tag{3.172}$$

$$g_{rr}^{(1)}(r) = \frac{10 - 3r - 6r^2 + 12\,(c_2 + 4\,c_3)\,r^3}{12\,r^3\,(r-1)} - (c_3 + \frac{1}{2})\frac{\ln(1-\frac{1}{r})}{r-1}\,, \tag{3.173}$$

$$g_{tt}^{(1)}(r) = \frac{6 + 5r + 12r^2 - 12r^3 - 12\,(c_2 + 2\,c_3)\,r^3 + c_4\,(r^4 - r^3)}{12\,r^3\,(r-1)} + \\ + (3 - 2r)\,(c_3 + \frac{1}{2})\,\frac{\ln(1-\frac{1}{r})}{r-1}\,. \tag{3.174}$$

In [77] and many following works a particular boundary conditions were chosen for the metric and dilaton. In these works it was assumed that after choosing (3.168) and (3.169) as the perturbative series for the metric then the α' corrections to the metric are finite at the horizon of the black hole and the dilaton vanishes at infinity, [9]

$$\phi^{\star(1)}(r)|_{r=\infty} = 0\,, \tag{3.175}$$

[9] One star is used upon the metric and dilaton with these boundary conditions. We only review these metrics and dilaton and we are not going to use them.

$$g_{tt}^{\star(1)}(r)|_{r=1} \quad < \quad \infty \,,$$
$$g_{rr}^{\star(1)}(r)|_{r=1} \quad < \quad \infty \,.$$

The above boundary conditions fix the constants of the integration to values of $c_1 = 0$, $c_2 = \frac{23}{12}$, $c_3 = -\frac{1}{2}$, $c_4 = 0$ giving

$$g_{tt}^{\star(1)}(r) = -\frac{23\,r^2 + 11\,r + 6}{12\,r^3}\,, \tag{3.176}$$
$$g_{rr}^{\star(1)}(r) = -\frac{r^2 + 7\,r + 10}{12\,r^3}\,, \tag{3.177}$$
$$\phi^{\star(1)}(r) = -\frac{2 + 3\,r + 6\,r^2}{12\,r^3}\,. \tag{3.178}$$

The boundary conditions imposed in (3.175) produce finite corrections to the Hawking temperature and the entropy of the black-hole. In addition it produces corrections to the Newtonian mass which is provided by the asymptotic behaviour of $g_{tt}(r)$ at large r.

In the next section we first calculate the α' corrections to the time-dual of the Schwarzschild metric and then we will use the α' modified T-duality rules. We obtained these rules by studying the α' corrections to the Kasner background. Within the α'-calculations we fixed the asymptotic behaviours at infinity both for the Kasner background and its dual. To be consistent with those calculations and due to the intrinsic singularity of the time-dual of the Schwarzschild metric we use the following boundary conditions on the field contents of the Schwarzschild background (as opposed to the boundary condition in (3.175)),

$$\begin{aligned} \phi(r)|_{r=\infty} &= 0\,, \\ g_{tt}(r)|_{r\sim\infty} &= 1 - \frac{1}{r} + O(\frac{1}{r^2})\,, \\ g_{rr}(r)|_{r\sim\infty} &= 1 + \frac{1}{r} + O(\frac{1}{r^2})\,. \end{aligned} \tag{3.179}$$

Choosing the above boundary conditions and using the β functions (2.53),(2.54) identifies the linear α' corrections

$$\begin{aligned} g_{tt}^{(1)}(r) &= \frac{6 + 5\,r + 12\,r^2 - 12\,r^3 + (18\,r^3 - 12\,r^4)\ln(1-\frac{1}{r})}{12\,r^3\,(r-1)}\,, \\ g_{rr}^{(1)}(r) &= \frac{10 - 3\,r - 6\,r^2 - 6\,r^3\ln(1-\frac{1}{r})}{12\,r^3\,(r-1)}\,, \\ \phi^{(1)}(r) &= -\frac{1 + \frac{3}{2}\,r + 3\,r^2 + 3\,r^3\ln(1-\frac{1}{r})}{6\,r^3}\,. \end{aligned} \tag{3.180}$$

as well as the quadratic α' corrections

$$\begin{aligned} g_{tt}^{(2)}(r) &= \frac{7050 - 5758r + 8125r^2 - 2757r^3 + 1940r^4 + 8140r^5 - 19680r^6 + 6240r^7}{7200\, r^6\,(r-1)^2} \\ &+ \ln(1-\frac{1}{r})\,\frac{120 - 45\,r + 45\,r^2 + 126\,r^3 - 320\,r^4 + 104\,r^5}{120\,r^3\,(r-1)^2} \\ &+ \{\ln(1-\frac{1}{r})\}^2\,\frac{4\,r - 9}{8\,(r-1)^2}\,, \end{aligned} \tag{3.181}$$

$$\begin{aligned} g_{rr}^{(2)}(r) &= \frac{6250 - 10154\,r + 5049\,r^2 - 5135\,r^3 + 1880\,r^4 - 2460\,r^5 - 480\,r^6}{7200\,r^6\,(r-1)^2} \\ &- \ln(1-\frac{1}{r})\,\frac{85 - 75\,r + 22\,r^2 + 8\,r^3}{120\,r^2\,(r-1)^2} + \{\ln(1-\frac{1}{r})\}^2\,\frac{1+r}{8\,(r-1)^2}\,, \end{aligned} \tag{3.182}$$

$$\begin{aligned} \phi^{(2)}(r) &= \frac{-225 - 327\,r - 513\,r^2 - 205\,r^3 - 710\,r^4 - 3930\,r^5 + 4260\,r^6}{3600\,r^6\,(r-1)} \\ &+ \ln(1-\frac{1}{r})\,\frac{-5 + 5\,r + 15\,r^2 - 101\,r^3 + 71\,r^4}{60\,r^3\,(r-1)}\,. \end{aligned} \tag{3.183}$$

Inserting the above linear and quadratic α' corrections in (3.168,3.169,3.171) identifies the quadratic α' corrected Schwarzschild background. The asymptotic behaviours of the fields of the quadratic α' corrected Schwarzschild background at large r follow

$$g_{rr}(r)\,(1-\frac{1}{r}) = 1 + \frac{40\,r^2 + 45\,r + 49}{40\,r^6}\,\alpha' + \frac{9}{8\,r^6}\alpha'^2 + O(\frac{1}{r^7})\,, \tag{3.184}$$

$$g_{tt}(r)\,\frac{1}{1-\frac{1}{r}} = 1 + \frac{30\,r^2 + 9\,r - 7}{120\,r^6}\,\alpha' + \frac{3}{8\,r^6}\alpha'^2 + O(\frac{1}{r^7})\,, \tag{3.185}$$

$$\phi(r) = \frac{105\,r^3 + 84\,r^2 + 70\,r + 60}{840\,r^7}\,\alpha' + \frac{1}{168\,r^7}\,\alpha'^2 + O(\frac{1}{r^8}). \tag{3.186}$$

Looking carefully at these expressions one notices that at an α' dependent location outside what was the horizon at $r = 1$ the component g_{tt} of the metric passes through zero. At this point both g_{rr} and $\phi(r)$ remain finite. If the zero in g_{tt} happens at $r = r_0$ then defining a new coordinate $\rho = r - r_0$ the metric near this zero takes the form,

$$ds^2 = -\rho dt^2 + d\rho^2 + c^2 d\omega^2, \tag{3.187}$$

where c is a real constant. This metric has a singular Ricci scalar at $\rho = 0$ and thus the α' corrections to the Schwarzschild metric appear to give rise to a naked singularity. The fact that the generic α' corrections to the Schwarzschild metric have singularities outside what was the horizon at $r = 1$ was already noted in [77].

3.4.2 The linear and the quadratic α' corrections to the time-dual of the Schwarzschild background in $D = 4$

The time dual of the Schwarzschild metric in $D = 4$ is

$$ds^2 = -\frac{1}{1-\frac{1}{r}}\,dt^2 + \frac{1}{1-\frac{1}{r}}\,dr^2 + r^2\,(d\theta^2 + \sin^2\theta\,d\phi^2)\,, \tag{3.188}$$

$$\phi = -\frac{1}{2}\ln(1-\frac{1}{r})\,. \tag{3.189}$$

The various scalar curvatures show intrinsic singularities at $r = 0$ and at $r = 1$,

$$R = \frac{1}{r^2\,(-r+1)}\,, \tag{3.190}$$

$$R_{\mu\nu}R^{\mu\nu} = \frac{9-20\,r+12\,r^2}{2\,r^6\,(-r+1)^2}\,, \tag{3.191}$$

$$R_{\mu\nu\eta\xi}R^{\mu\nu\eta\xi} = \frac{9-20\,r+12\,r^2}{r^6\,(-r+1)^2}\,, \tag{3.192}$$

$$\nabla_\gamma R_{\mu\nu\eta\xi}\,\nabla^\gamma\,R^{\mu\nu\eta\xi} = \frac{180\,r^4-648\,r^3+900\,r^2-568\,r+137}{(r-1)^3\,r^9}\,. \tag{3.193}$$

The local string coupling constant in this background reads

$$g_s = e^{+\phi} = \frac{g_0}{\sqrt{1-\frac{1}{r}}}\,. \tag{3.194}$$

Therefore far away from $r = 1$ both the string theory is perturbative and the space-time is asymptotically flat. Thus within this regime the time-dual of the Schwarzschild metric can be extended to the following perturbative background in the critical Bosonic String Theory

$$ds^2 = -G_{tt}(r)\,dt^2 + G_{rr}(r)\,dr^2 + G_\Omega(r)\,(d\theta^2 + \sin^2\theta\,d\phi^2) + dx_\perp^2\,,$$

$$\phi = \Phi(r)\,,$$

where the metric and the dilaton admit the following perturbative series in α'

$$G_{tt}(r) = \frac{1}{1-\frac{1}{r}}\,(1 + \alpha'\,G_{tt}^{(1)}(r) + \alpha'^2\,G_{tt}^{(2)}(r) + \cdots)\,, \tag{3.195}$$

$$G_{rr}(r) = \frac{1}{1-\frac{1}{r}}\,(1 + \alpha'\, G^{(1)}_{rr}(1) + \alpha'^2\, G^{(2)}_{rr}(r) + \cdots)\,, \tag{3.196}$$

$$G_{\Omega}(r) = r^2\,, \tag{3.197}$$

$$\Phi(r) = -\frac{1}{2}\ln(1-\frac{1}{r}) + \alpha'\,\Phi^{(1)}(r) + \alpha'^2\,\Phi^{(2)}(r) + \cdots\,. \tag{3.198}$$

We fix the asymptotic behaviours of the metric and the dilaton at large r by

$$\Phi(r)|_{r\sim\infty} = \frac{1}{2}\frac{1}{r} + O(\frac{1}{r^2})\,, \tag{3.199}$$

$$G_{tt}(r)|_{r\sim\infty} = 1 + \frac{1}{r} + O(\frac{1}{r^2})\,, \tag{3.200}$$

$$G_{rr}(r)|_{r\sim\infty} = 1 + \frac{1}{r} + O(\frac{1}{r^2})\,. \tag{3.201}$$

In [78] the linear α' corrected Schwarzschild metric computed in [77] with the boundary condition provided in (3.175) and the linear α' modified T-duality is used to obtain the linear α' corrections to the time-dual of the Schwarzschild metric in $D = 4$. This procedure means choosing a specific boundary condition for the time-dual of the Schwarzschild metric at $r = 1$. However we think that due to the intrinsic singularity at $r = 1$, it is not reasonable to set any boundary condition at $r = 1$ in the time-dual of the Schwarzschild background.

Using the β-functions (2.53), (2.54) and the above asymptotic behaviours identifies the linear α' corrections

$$G^{(1)}_{tt}(r) = \frac{\frac{5}{4} + 3\,r - 3\,r^2 - (3\,r^3 - \frac{9}{2}\,r^2)\,\ln(1-\frac{1}{r})}{3\,r^2\,(-r+1)}\,, \tag{3.202}$$

$$G^{(1)}_{rr}(r) = \frac{10 - 3\,r - 6\,r^2 - 6\,r^3\,\ln(1-\frac{1}{r})}{12\,r^3\,(r-1)}\,, \tag{3.203}$$

$$\Phi^{(1)}(r) = \frac{1 - 3\,r - 6\,r^2 - 6\,r^3\,\ln(1-\frac{1}{r})}{24\,r^3\,(r-1)}\,, \tag{3.204}$$

as well as the quadratic α' corrections

$$\begin{aligned} G^{(2)}_{tt}(r) = {} & \frac{300 - 2042\,r + 2125\,r^2 + 1557\,r^3 - 740\,r^4 - 22540\,r^5 + 26880\,r^6 - 6240\,r^7}{7200\,r^6\,(r-1)^2} + \\ & + \ln(1-\frac{1}{r})\,\frac{75 + 215r - 726r^2 + 560r^3 - 104r^4}{120\,r^2\,(r-1)^2} \\ & + \{\ln(1-\frac{1}{r})\}^2\,\frac{9 - 11r + 4r^2}{8\,(r-1)^2}\,, \end{aligned} \tag{3.205}$$

$$G^{(2)}_{rr}(r) = \frac{2200 - 4754\,r + 5049\,r^2 - 5135\,r^3 + 1880\,r^4 - 2460\,r^5 - 480\,r^6}{7200\;r^6\;(r\;-\;1)^2}$$
$$+\;\ln(1-\frac{1}{r})\;\frac{-85+75r-22r^2-8r^3}{120\;r^2\;(r-1)^2}\;+\;\{\ln(1-\frac{1}{r})\}^2\;\frac{r+1}{8\;(r-1)^2}\,, \qquad (3.206)$$

$$\Phi^{(2)}(r) = \frac{-1350 + 2116\,r - 456\,r^2 - 875\,r^3 + 680\,r^4 - 2460\,r^5 - 9480\,r^6 + 10800\,r^7}{14400\,r^6\,(r-1)^2}$$
$$+\ln(1-\frac{1}{r})\,\frac{5 - 10\,r + 45\,r^2 + 38\,r^3 - 248\,r^4 + 180\,r^5}{240\,r^3\,(r-1)^2} \qquad (3.207)$$
$$+\{\ln(1-\frac{1}{r})\}^2 \frac{r}{16(r\;-\;1)^2}\,.$$

3.4.3 Applying the quadratic α' modified T-duality on the α'^2 corrected Schwarzschild background and its dual in $D = 4$

Earlier we obtained the α' modified T-duality rules for time-dependent geometries. These same rules should also describe T-duality in the Schwarzschild metric, but now the T-duality acts in the direction of the time-like Killing vector outside the black hole horizon. These include the Euclidean geometry of the Schwarzschild metric. Consequently the α' modified T-duality rules can be legitimately applied to the Schwarzschild background. In order to do this we first introduce the analog of the co-moving and the almost co-moving frame for the metrics of the Schwarzschild background and its time-dual.

The quadratic α' corrected Schwarzschild metric and its time-dual are spherically symmetric. A spherically symmetric metric can be written in a coordinate where the radial component of the metric is the identity. This coordinate is the analog of the co-moving frame which we refer to as the *"radial co-moving frame"*. On the other hand the *"almost radial co-moving frame"* is defined as a coordinate that it can be transformed to the radial co-moving frame by a *single* re-parametrisation of the radial direction . In the almost radial co-moving frame the metric reads

$$ds^2 \;=\; f(r)\;dr^2 \;+\; ds^2_{\perp}\;, \qquad (3.208)$$

where $f(r)$ is called the *"characteristic function"* of the almost radial co-moving frame. The quadratic α' Schwarzschild metric in (3.166) and its time-dual in (3.195) are already in the almost radial co-moving frame respectively with $g_{rr}(r)$ and $G_{rr}(r)$ as their characteristic functions. These two characteristic functions are not equal:

$$G_{rr}(r) - g_{rr}(r) = \frac{3\,(4r - 3)\,\alpha'^2}{16\,(r-1)^3\,r^5}\,, \tag{3.209}$$

In order to apply the α' modified T-duality rules to the α' corrected Schwarzschild metric and its α' corrected time-dual we should first write them in the almost radial co-moving frame with the same characteristic functions. In the following, we write the quadratic α' corrected time-dual of the Schwarzschild metric in the almost radial co-moving frame with the characteristic function of the quadratic α' corrected Schwarzschild metric.

The α' corrected time-dual of the Schwarzschild metric in the new coordinate provided by

$$r \;\to\; r - \frac{\alpha'^2}{16\,(r-1)\,r^5}\,, \tag{3.210}$$

reads

$$\begin{aligned} ds^2 &= -\,\tilde{g}_{tt}(r)\,dt^2 + g_{rr}(r)\,dr^2 + \tilde{g}_{\Omega}(r)\,(d\theta^2 + \sin^2\theta\,d\phi^2)\,, \\ \phi &= \tilde{\phi}(r)\,, \end{aligned} \tag{3.211}$$

where $g_{rr}(r)$ is given by (3.169). Other components of the metric and the dilaton read

$$\tilde{g}_{tt}(r) = \frac{1}{1-\frac{1}{r}}\,(1 + \alpha'\,\tilde{g}^{(1)}_{tt}(r) + \alpha'^2\,\tilde{g}^{(2)}_{tt}(r))\,, \tag{3.212}$$

$$\tilde{g}_{\Omega}(r) = r^2 + \frac{\alpha'^2}{8\,r^4\,(r-1)}\,, \tag{3.213}$$

$$\tilde{\phi}(r) = -\frac{1}{2}\ln(1-\frac{1}{r}) + \alpha'\,\tilde{\phi}^{(1)}(r) + \alpha'^2\,\tilde{\phi}^{(2)}(r)\,. \tag{3.214}$$

The α' coefficients to the dilaton $\phi(r)$ are

$$\tilde{\phi}^{(1)}(r) = \frac{1 - 3r - 6r^2 - 6r^3\ln(1-\frac{1}{r})}{24\,r^3\,(r-1)}\,, \tag{3.215}$$

$$\begin{aligned} \tilde{\phi}^{(2)}(r) &= \frac{-1800 + 2116r - 456r^2 + 10800r^7 - 875r^3 + 680r^4 - 2460r^5 - 9480r^6}{14400\,r^6\,(r-1)^2} \\ &\quad + \ln(1-\frac{1}{r})\,\frac{-248\,r^4 + 180\,r^5 + 38\,r^3 - 10\,r + 45\,r^2 + 5}{240\,r^3\,(r-1)^2} \\ &\quad + \{\ln(1-\frac{1}{r})\}^2\frac{r}{16\,(r-1)^2}\,. \end{aligned} \tag{3.216}$$

The α' coefficients to the $\tilde{g}_{tt}(r)$ read

$$\tilde{g}^{(1)}_{tt}(r) = \frac{-5 - 12r + 12r^2 + (12r^3 - 18r^2)\ln(1-\frac{1}{r})}{12r^2(r-1)}, \tag{3.217}$$

$$\begin{aligned}\tilde{g}^{(2)}_{tt}(r) &= -\frac{150 + 2042r - 2125r^2 - 1557r^3 + 740r^4 + 22540r^5 - 26880r^6 + 6240r^7}{7200r^5(r-1)^3} \\ &+\ln(1-\frac{1}{r})\,\frac{215r + 75 - 104r^4 - 726r^2 + 560r^3}{120r(r-1)^3} \\ &+\{\ln(1-\frac{1}{r})\}^2\frac{r(-11r + 9 + 4r^2)}{8(r-1)^3}\,.\end{aligned} \tag{3.218}$$

This new coordinate is the almost radial co-moving frame with the characteristic function $g_{rr}(r)$. Now the α' modified T-duality rules provided by (3.126,3.127,3.128) can be applied to the quadratic α' corrected Schwarzschild background (3.166) and its time-dual (3.211).

The tt, $\phi\phi$ and $\theta\theta$ components of the quadratic α' corrected Schwarzschild background (3.166) and its time dual (3.211) satisfy the T-duality rules given by (3.126) and (3.127)

$$\begin{aligned}\ln\tilde{g}_{tt} &- \frac{\alpha'}{4}\,\tilde{\nabla}_\mu\ln\tilde{g}_{tt}\,.\,\tilde{\nabla}^\mu\ln\tilde{g}_{tt} \\ &= -\{\ln g_{tt} - \frac{\alpha'}{4}\,\nabla_\mu\ln g_{tt}\,.\,\nabla^\mu\ln g_{tt}\} + O(\alpha'^2\nabla^4)\,,\end{aligned} \tag{3.219}$$

$$\begin{aligned}\ln\tilde{g}_{ii} &+ \frac{\alpha'^2}{32}\tilde{\nabla}_\mu\ln\tilde{g}_{tt}.\tilde{\nabla}^\mu\ln\tilde{g}_{ii}\;\tilde{\nabla}_\nu\ln\tilde{g}_{tt}.\tilde{\nabla}^\nu\ln\tilde{g}_{tt} \\ &= \ln g_{ii} + \frac{\alpha'^2}{32}\nabla_\mu\ln g_{ii}.\nabla^\mu\ln g_{tt}\;\nabla_\nu\ln g_{tt}.\nabla^\nu\ln g_{tt} + O(\alpha'^3\nabla^6)\,.\end{aligned} \tag{3.220}$$

Note that in (3.128) we should substitute $\det g$ and $\det\tilde{g}$ respectively with $\det^* g$ and $\det^*\tilde{g}$ given by

$$det^* g = \frac{\det g}{g_{rr}(r)}, \tag{3.221}$$

$$det^*\tilde{g} = \frac{\det\tilde{g}}{g_{rr}(r)}. \tag{3.222}$$

In order to check (3.128) we first calculate

$$\tilde{\phi}(r) - \frac{1}{4}\ln\det\tilde{g} - \phi(r) + \frac{1}{4}\ln\det g = \frac{3 - 4r}{32\,r^6\,(r-1)^2}\,\alpha'^2\,. \tag{3.223}$$

Also the α'^2 terms in the l.h.s of (3.128)

$$\tilde{\nabla}_\mu\ln\tilde{g}_{tt}\;\tilde{\nabla}_\nu\ln\tilde{g}_{tt}\;\tilde{\nabla}^\mu\tilde{\nabla}^\nu\ln\tilde{g}_{tt} = \frac{4r - 3}{2\,r^6\,(r-1)^2} + O(\alpha')\,, \tag{3.224}$$

$$\tilde{\nabla}_\mu \ln \tilde{g}_{tt}.\tilde{\nabla}^\mu \ln \tilde{g}_{tt}\ \tilde{\nabla}_\nu \ln \tilde{g}_{tt}.\tilde{\nabla}^\nu(\tilde{\phi} - \frac{1}{4}\ln \det{}^* \tilde{g}) = \frac{4r - 5}{4r^6\,(r - 1)^2} + O(\alpha')\,, \tag{3.225}$$

$$\tilde{\nabla}_\mu \ln \tilde{g}_{tt}.\tilde{\nabla}^\mu \ln \tilde{g}_{tt}\ \Box \ln \tilde{g}_{tt} = \frac{1}{r^6\,(r - 1)^2} + O(\alpha')\,, \tag{3.226}$$

and the α'^2 terms in the r.h.s of (3.128)

$$\nabla_\mu \ln g_{tt}\ \nabla_\nu \ln g_{tt}\ \nabla^\mu \nabla^\nu \ln g_{tt} = -\frac{4r - 3}{2\,r^6\,(r - 1)^2} + O(\alpha')\,, \tag{3.227}$$

$$\nabla_\mu \ln g_{tt}.\nabla^\mu \ln g_{tt}\ \nabla_\nu \ln g_{tt}.\nabla^\nu(\phi - \frac{1}{4}\ln \det{}^* g) = -\frac{4r - 3}{4r^6\,(r - 1)^2} + O(\alpha')\,, \tag{3.228}$$

$$\nabla_\mu \ln g_{tt}.\nabla^\mu \ln g_{tt}\ \Box \ln g_{tt} = 0 + O(\alpha')\,. \tag{3.229}$$

Inserting the above expressions in (3.128) results the following equations for A, B and C

$$\left.\begin{array}{rcl} A + B + C &=& 1 \\ A + B &=& 1 \\ 3A + 4B - C &=& 3 \end{array}\right\} \Longrightarrow \left\{\begin{array}{rcl} A &=& 1 \\ B &=& 0 \\ C &=& 0 \end{array}\right. \tag{3.230}$$

which identifies $A = 1$ and $B = C = 0$. Substituting these values in (3.128) results

$$\begin{aligned} \tilde{\phi} - \frac{1}{4}\ln \det \tilde{g} &+ \frac{\alpha'^2}{32}\,\tilde{\nabla}_\mu \ln \tilde{g}_{25\,25}\ \tilde{\nabla}_\nu \ln \tilde{g}_{25\,25}\ \tilde{\nabla}^\mu \tilde{\nabla}^\nu \ln \tilde{g}_{25\,25} \\ &= \phi - \frac{1}{4}\ln \det g + \frac{\alpha'^2}{32}\,\nabla_\mu \ln g_{25\,25}\ \nabla_\nu \ln g_{25\,25}\ \nabla^\mu \nabla^\nu \ln g_{25\,25} + O(\alpha'^3 \nabla^6) \end{aligned} \tag{3.231}$$

The fact that we can find a consistent assignment of the constants A, B, C and satisfy (3.219) and (3.220) is a nontrivial check on the consistency of our procedure.

3.4.4 The α'^2 corrections to the Schwarzschild metric in $D = 5$

The five dimensional Schwarzschild black hole reads

$$ds^2 = -(1 - \frac{1}{r^2})\,dt^2 + \frac{dr^2}{1 - \frac{1}{r^2}} + r^2\,d\Omega_3\,, \tag{3.232}$$

where the mass of the black hole is appropriately chosen to give a factor of one in $\frac{1}{r^2}$. For very large r this metric can be generalised to a perturbative background of free critical Bosonic String Theory,[10]

$$ds^2 = -f_5(r)\,dt^2 + g_5(r)\,dr^2 + r^2\,d\Omega_3 + \overrightarrow{dx}_{21}{}^2\,, \tag{3.233}$$

[10] The linear α' corrections to the four dimensional black hole were computed in [77].

$$
\begin{aligned}
f_5(r) &= (1-\frac{1}{r^2})\,(1+\alpha'\,f^{(1)}(r)+\alpha'^2\,f^{(2)}(r)+\cdots)\,, &(3.234)\\
g_5(r) &= \frac{1}{1-\frac{1}{r^2}}(1+\alpha'\,g_5^{(1)}(r)+\alpha'^2\,g_5^{(2)}(r)+\cdots)\,, &(3.235)\\
\phi_5(r) &= 0+\alpha'\,\phi_5^{(1)}(r)+\alpha'^2\,\phi_5^{(2)}(r)+\cdots\,, &(3.236)
\end{aligned}
$$

where $\overrightarrow{dx}_{21}$ represents the 21 flat directions compactified on a torus and $\phi(r)$ is the dilaton. Using the beta functions and assuming that there is no correction to the mass of the black hole or to the fall off of the dilaton at infinity, one finds

$$
\begin{aligned}
\phi_5^{(1)}(r) &= \frac{25}{8}\ln(\frac{r^2+1}{r^2-1})-\frac{25}{12\,r^6}-\frac{25}{4\,r^2}\,, &(3.237)\\
f_5^{(1)}(r) &= \frac{1}{6\,r^6\,(r^4-1)}\left(48+80r^4-30r^8+15r^6(r^4-3)\ln(\frac{r^2+1}{r^2-1})\right)\,, &(3.238)\\
g_5^{(1)}(r) &= \frac{1}{3r^6(r^4-1)}\left(11-30r^4+15\ln(\frac{r^2+1}{r^2-1})\right)\,, &(3.239)
\end{aligned}
$$

and

$$
\begin{aligned}
\phi_5^{(2)}(r) &= \frac{2364r^8-2478r^6-394r^4-953r^2-915}{768\,r^8\,(r^2-1)}+ &(3.240)\\
&+\frac{\ln(1-\frac{1}{r^2})}{64r^4(r^2-1)}(197r^6-305r^4+54r^2-54)\,,\\
f_5^{(2)}(r) &= \frac{1068\,r^{10}-3966r^8+1214\,r^6-435\,r^4-1553r^2+6048}{576\,(r^2-1)\,r^{10}}+ &(3.241)\\
&+\frac{\ln(1-\frac{1}{r^2})}{48\,r^{12}\,(r^2-1)^2}\,(89\,r^8-321\,r^6+16\,r^4-108\,r^2+432)+\\
&+\frac{9\,(r^2-4)}{8}(\ln(1-\frac{1}{r^2}))^2\,,\\
g_5^{(2)}(r) &= \frac{876\,r^8+210\,r^6+574\,r^4+6449\,r^2-4473}{576\,r^6\,(r^2-1)^3}+ &(3.242)\\
&-\frac{(73\,r^4-73\,r^2+144)}{48\,(r^2-1)^3}\ln(1-\frac{1}{r^2})+\frac{9\,r^2\,(r^2+1)}{8\,(r^2-1)^3}\ln(1-\frac{1}{r^2})^2\,.
\end{aligned}
$$

Note that the α' corrected metric has a singularity outside the horizon. This is reminiscent of what happens in the case of four dimensional Schwarzschild black hole. Now let us apply T-duality in the direction of the time-like Killing vector outside the horizon

$$
\begin{aligned}
d\tilde{s}^2 &= -\frac{dt^2}{1-\frac{1}{r^2}}+\frac{dr^2}{1-\frac{1}{r^2}}+r^2\,d\Omega_3\,, &(3.243)\\
\tilde{\phi}(r) &= 0\,. &(3.244)
\end{aligned}
$$

The T-dual of the Schwarzschild metric is singular at "$r = 1$" which means that the coordinate singularity of the Schwarzschild metric has changed to an intrinsic singularity. Since T-duality relates the singular time-dual geometry to the Schwarzschild metric then one expects that the time-dual metric to be as stable as the Schwarzschild metric [79]. This argument should not sound strange because the classical stability of a naked singularity recently has been explored in [80]. At large r T-dual background can be generalised to a perturbative background of string theory whose quadratic α' corrections follow [11]

$$\begin{aligned} d\tilde{s}^2 &= -\tilde{f}_5(r)\,dt^2 + g_5(r)\,dr^2 + r^2\,(1 + \frac{\alpha'^2}{r^8\,(r^2-1)})\,d\Omega_3 + \overrightarrow{dx}_{21}{}^2\,, & (3.245)\\ \tilde{f}_5(r) &= \frac{1}{1-\frac{1}{r^2}}\,(1 + \alpha'\,f_5^{(1)}(r) + \alpha'^2\,f_5^{(2)}(r) + \cdots)\,, & (3.246)\\ \tilde{\phi}_5(r) &= 0 + \alpha'\,\tilde{\phi}_5^{(1)}(r) + \alpha'^2\,\tilde{\phi}_5^{(2)}(r) + \cdots\,, & (3.247)\end{aligned}$$

where we are going to assume that there is no α'-correction to the fall off of $f(r)$ at infinity. Note that $g_5(r)$ is provided by (3.235). The linear and the quadratic terms in α' in $\tilde{f}_5(r)$ follow

$$\begin{aligned} \tilde{f}_5^{(1)}(r) &= \frac{6\,r^2-9}{4\,(r^2-1)\,r^2} + \frac{3\,(r^2-2)}{2\,(r^2-1)}\ln(1-\frac{1}{r^2})\,, & (3.248)\\ \tilde{f}_5^{(2)}(r) &= -\frac{1068\,r^{10} - 5262\,r^8 + 5102\,r^6 - 1623\,r^4 + 1039\,r^2 + 576}{576\,r^8\,(r^2-1)^2} + & (3.249)\\ &\quad -\frac{89\,r^4 - 448\,r^2 + 324}{48\,r^2\,(r^2-1)}\ln(1-\frac{1}{r^2}) + \frac{9(r^4-3r^2+4)}{8\,(r^2-1)^2}(\ln(1-\frac{1}{r^2}))^2\,. \end{aligned}$$

And the coefficients of α' in $\tilde{\phi}(r)$ read

$$\begin{aligned} \tilde{\phi}_5^{(1)}(r) &= -\frac{6\,r^4 + 9\,r^2 - 1}{16\,r^4\,(r^2-1)} - \frac{3\,(r^2+1)}{8\,(r^2-1)}\ln(1-\frac{1}{r^2})\,, & (3.250)\\ \tilde{\phi}_5^{(2)}(r) &= \frac{4956r^{10} - 5298r^8 - 64r^6 - 1347r^4 + 1492r^2 - 2151}{2304(r^2-1)^2r^8} + & (3.251)\\ &\quad + \frac{413r^8 - 648r^6 + 289r^4 - 36r^2 + 18}{194\,r^4\,(r^2-1)}\ln(1-\frac{1}{r^2}) + \\ &\quad + \frac{9\,r^2}{16\,(r^2-1)^2}(\ln(1-\frac{1}{r^2}))^2\,. \end{aligned}$$

Instead of writing the α' corrected Schwarzschild black hole and its T-dual metric in the "radial" co-moving frame, it is easier to rewrite the rules of T-duality for metrics which can be transform to the "radial" co-moving frame by the same coordinate transformation. Here the α' corrected

[11] We have chosen this specific coordinate since it is easier to apply T-duality in this coordinate.

Schwarzschild black hole and its T-dual can be transformed to the "radial" co-moving frame by the same coordinate transformation. Luckily this transformation does not alter the rules of T-duality. It is a straightforward computation to check that the α' corrected T-duality rules (3.126,3.127,3.128) relate the α' corrected black-hole to its T-dual background.

The α' corrected metric is singular outside the horizon. If one insists on having a smooth geometry then one can choose the boundary conditions by requiring finite corrections at $r = 1$,

$$\begin{aligned} ds^2 &= -f_5^\star(r)\,(1-\frac{1}{r^2})\,dt^2 + \frac{g_5^\star(r)}{1-\frac{1}{r^2}}\,dr^2 + r^2\,d\Omega_3^2\,, \\ f_5^\star(r) &= 1 + c_1\alpha' + c_2\alpha'^2 - \alpha'\frac{17r^2+8}{4r^4} - \alpha'^2\frac{1039\,r^6 - 4811\,r^4 - 10543\,r^2 - 6048}{576\,r^8}\,, \\ g_5^\star(r) &= 1 - \alpha'\,\frac{r^2+7}{4\,r^4} + \alpha'^2\,\frac{355\,r^6 - 1565\,r^4 + 2497\,r^2 + 4473}{576\,r^8}\,, \\ \phi_5^\star(r) &= 0 - \alpha'\frac{9\,(1+2\,r^2)}{16\,r^4} - \alpha'^2\frac{228\,r^6 + 6\,r^4 - 1868\,r^2 - 915}{768\,r^8}\,, \end{aligned} \tag{3.252}$$

where c_1 and c_2 are numerical constants. These boundary conditions also give finite corrections to the string coupling constant. One may fix "$c_1 = c_2 = 0$" assuming no correction exists at infinity. The price of having a smooth geometry is to change the fall off of the metric at infinity. The asymptotic behaviour of the linear α' corrected metric at large r in the Einstein frame is

$$ds_E^{\star 2} = (1 + 2c_1\alpha' - \frac{1+\frac{11}{4}\alpha'}{r^2})\,dt^2 + (1 + \frac{1+\frac{5}{4}\alpha'}{r^2})dr^2 + r^2\,d\Omega_3^2 + O(\alpha'^2) + O(\frac{1}{r^4})\,,$$

The corrections to the T-dual background are

$$\begin{aligned} d\tilde{s}_5^{\star 2} &= -\frac{\tilde{f}_5^\star(r)}{1-\frac{1}{r^2}}\,dt^2 + \frac{g_5^\star(r)}{1-\frac{1}{r^2}}\,dr^2 + r^2(1 + \frac{\alpha'^2}{r^8\,(r^2-1)})\,d\Omega_3^2\,, \\ \tilde{f}_5^\star(r) &= 1 - \alpha'\frac{17\,r^2-9}{4\,r^2\,(r^2-1)} + \\ &\quad - \alpha'^2\frac{-576 + 1623\,r^4 - 1039\,r^2 + 3515\,r^8 - 1106\,r^6 + 1039\,r^{10}}{576\,r^8\,(r^2-1)^2}\,, \\ \tilde{\phi}_5^\star(r) &= -\frac{1}{2}\ln(1-\frac{1}{r^2}) + \alpha'\frac{16\,r^4 - 9\,r^2 + 1}{16\,r^4\,(r^2-1)} + \\ &\quad + \alpha'^2\,\frac{-1527\,r^4 + 1492\,r^2 - 2024\,r^8 + 3968\,r^6 + 1394\,r^{10} - 2151}{2304\,r^8\,(r^2-1)^2}\,. \end{aligned} \tag{3.253}$$

These expressions are consistent with α' corrected T-duality. We could have used T-duality to find the corrections, by the same method that [78] finds the linear α' corrections to the T-dual of the four dimensional black hole.

As mentioned earlier a general proof of the validity of T-duality at up to third order in α' is presented in [67]. This implies that once we have fully fixed the constants A, B and C, there is no need to check further that this form is consistent with the T-duality for other metrics. One may of course argue that applying T-duality to the Schwarzschild metric is not completely conventional. To resolve this possible ambiguity one could follow the above algorithm applied to another homogeneous cosmology for example of the type studied in [81] where T-duality and cosmology is studied in some detail or the two dimensional black hole as we will study in the next section.

3.5 Two dimensional black hole

The two-dimensional black hole is as a solution of the leading beta-functions of the non-critical string theory :

$$\begin{aligned} R_{\mu\nu} + 2\nabla_\mu\nabla_\nu\phi &= 0\,, && (3.254)\\ \frac{d-26}{6\alpha'} - \frac{1}{2}\Box\phi + |\nabla\phi|^2 &= 0\,. && (3.255)\end{aligned}$$

Using the convention "$\alpha' = \frac{26-d}{6}$" in "d" dimensional space-time this solution reads [82, 83]

$$\begin{aligned} ds^2 &= dt^2 + \tanh(t)^2\, dr^2 + dx_{d-2}^2\,, && (3.256)\\ \phi(t) &= -\ln(\cosh(t))\,, \end{aligned}$$

where dx_{d-2} is "$d-2$" flat directions. This solution with an appropriate periodicity ($r \sim r + 2\pi$) is a fair candidate for a two-dimensional black hole with a geometry of semi-infinite cigar [83, 84]. We assume that the non-critical string theory has the following perturbative[12] background

$$\begin{aligned} ds^2 &= dt^2 + f(t)\, dr^2 + dx_{d-2}^2\,, && (3.257)\\ f(t) &= \tanh(t)^2\,(1 + \alpha' f^{(1)}(t) + \alpha'^2 f^{(2)}(t) + \cdots)\,, && (3.258)\\ \phi(t) &= -\ln(\cosh(t)) + \alpha'\,\phi^{(1)}(t) + \alpha'^2\,\phi^{(2)}(t) + \cdots\,. && (3.259)\end{aligned}$$

[12]To make it perturbative, first we analytically extend the two-dimensional black hole to $d = 26 - \epsilon$ where ϵ is a sufficiently small positive number. And at the end of the calculation we analytically extend the solution back to arbitrary d.

Inserting these perturbative series in the beta functions gives a set of ordinary linear differential equations for "$f^{(1)}(t), f^{(2)}(t), \phi^{(1)}(t), \phi^{(2)}(t)$". These equations have unique answer for any given boundary condition. Therefore the assumptions do not contradict themselves in the sense that the set of differential equations for the α' terms is not overdetermined. We fix the boundary condition assuming that there exists no correction at infinity to the fields

$$f^{(1)}(\infty) = f^{(2)}(\infty) = \phi^{(1)}(\infty) = \phi^{(2)}(\infty) = 0\,, \tag{3.260}$$

and their fall off at asymptotic infinity

$$\frac{df^{(1)}(t)}{dt}|_\infty = \frac{df^{(2)}(t)}{dt}|_\infty = \frac{d\phi^{(1)}(t)}{dt}|_\infty = \frac{d\phi^{(2)}(t)}{dt}|_\infty = 0\,. \tag{3.261}$$

The α'^2 corrected background constraint to these boundary conditions read

$$f(t) = \tanh(t)^2\,(1 - \frac{2\,\alpha'}{\cosh(t)^2} + \frac{\alpha'^2\,(5 - \cosh(2t))}{\cosh(t)^4} + \cdots)\,, \tag{3.262}$$

$$\phi(t) = -\ln(\cosh(t)) - \frac{\alpha'}{2\,\cosh(t)^2} - \frac{\alpha'^2\sinh(t)^2}{2\cosh(t)^4} + \cdots\,. \tag{3.263}$$

We are interested in the T-duality so we consider the following background

$$d\tilde{s}^2 = dt^2 + \frac{1}{\tanh(t)^2}\,dr^2 + dx_{d-2}^2\,, \tag{3.264}$$

$$\tilde{\phi}(t) = -\ln(\sinh(t))\,, \tag{3.265}$$

in the supergravity approximation which is related to the two-dimensional black-hole by T-duality in the direction of r. Its α'^2 corrections follow

$$d\tilde{s}^2 = dt^2 + \tilde{f}(t)\,dr^2\,, \tag{3.266}$$

$$\tilde{f}(t) = \frac{1}{\tanh(t)^2}\,(1 + \frac{2\alpha'}{\sinh(t)^2} + \frac{\alpha'^2\,(5 + \cosh(2t))}{\sinh(t)^4})\,, \tag{3.267}$$

$$\tilde{\phi}(t) = -\ln(\sinh(t)) + \frac{\alpha'}{2\,\sinh(t)^2} + \frac{\alpha'^2\,\cosh(t)^2}{2\,\sinh(t)^4}\,, \tag{3.268}$$

where it is assumed that the corrections to the fields at their fall off vanish at infinity. The covariant form of The T-duality rule (3.137) maps the α'^2 corrected two-dimensional black hole to its T-dual if

$$a = -c\,, \tag{3.269}$$

$$\begin{aligned} b &= 0, \\ e &= 1. \end{aligned}$$

We set a, b and c to the above values. Furthermore the leading β functions for a time-dependent background[13] implies

$$cG + aE = c(G - E) \sim \beta_{25\,25} = 0. \tag{3.270}$$

Using "(3.269)" and "(3.270)" simplifies the T-duality rule (3.137)

$$\begin{aligned} \ln \tilde{g}_{25\,25} &- \frac{\alpha'}{4}\tilde{\nabla}_\mu \ln \tilde{g}_{25\,25}\, \tilde{\nabla}^\mu \ln \tilde{g}_{25\,25} + \frac{\alpha'^2}{32}(\tilde{\nabla}_\mu \ln \tilde{g}_{25\,25}\, \tilde{\nabla}^\mu \ln \tilde{g}_{25\,25})^2 \\ &- \frac{\alpha'^2}{16}\tilde{\nabla}_\mu \tilde{\nabla}_\nu \ln \tilde{g}_{25\,25}\, \tilde{\nabla}^\mu \tilde{\nabla}^\nu \ln \tilde{g}_{25\,25} \\ = &- \Big(\ln g_{25\,25} - \frac{\alpha'}{4}\nabla_\mu \ln g_{25\,25}\, \nabla^\mu \ln g_{25\,25} + \frac{\alpha'^2}{32}(\nabla_\mu \ln g_{25\,25}\, \nabla^\mu \ln g_{25\,25})^2 \\ &- \frac{\alpha'^2}{16}\nabla_\mu \nabla_\nu \ln g_{25\,25}\, \nabla^\mu \nabla^\nu \ln g_{25\,25} \Big) . \end{aligned} \tag{3.271}$$

This rule beside "(3.126)" and "(3.127)" provides the quadratic α' corrections to T-duality on time-dependent backgrounds of diagonal metric and dilaton. As a check of the consistency of our procedure these rules have been checked to be true for the Schwarzschild metric in "$D = 4, 5$" dimensions and T-duality in the direction of the time-like Killing vector outside the horizon.

3.6 Summary, conclusion and discussions of the chapter

We have considered a time dependent background of Bosonic String Theory composed of the dilaton and a diagonal metric

$$\begin{aligned} ds^2 &= -dt^2 + g_{11}(t)dx_1^2 + \cdots + g_{dd}(t)dx_d^2 \\ \phi &= \phi(t) \end{aligned} \tag{3.272}$$

in $D = d + 1$ dimensional space time where x_1 is compactified on a circle $x_1 \equiv x_1 + 1$ and the late time geometry is flat. This background has a dual

$$ds^2 = -dt^2 + \tilde{g}_{11}(t)dx_1^2 + \cdots + \tilde{g}_{dd}(t)dx_d^2 \tag{3.273}$$

[13] When the components of the metric depend only on time.

$$\phi = \tilde{\phi}(t)$$

which is related to (3.272) by applying T-duality in the x_1 direction. In the low energy effective action we have realised T-duality as a set of rules which map (3.272) to (3.273)

$$\begin{aligned}
\ln\frac{\tilde{g}_{11}(t)}{\alpha'} + P_{\parallel}(\ln\tilde{g}, \tilde{\phi}, \tilde{\nabla}) &= -\left(\ln\frac{g_{11}(t)}{\alpha'} + P_{\parallel}(\ln g, \phi, \nabla)\right), &(3.274)\\
\ln\frac{\tilde{g}_{jj}(t)}{\alpha'} + \tilde{P}_{\perp,j}(\ln\tilde{g}, \tilde{\phi}, \tilde{\nabla}) &= \ln\frac{g_{jj}(t)}{\alpha'} + P_{\perp,j}(\ln g, \phi, \nabla), &(3.275)\\
\tilde{\phi} - \frac{1}{4}\ln\det\tilde{g} + P_{\phi}(\ln\tilde{g}, \tilde{\phi}, \tilde{\nabla}) &= \phi - \frac{1}{4}\ln\det g + P_{\phi}(\ln g, \phi, \nabla), &(3.276)
\end{aligned}$$

where

$$\begin{aligned}
P_{\parallel}(\ln g, \phi, \nabla) &= 0 + \alpha' P_{\parallel}^{(1)}(\ln g, \phi, \nabla) + \alpha'^2 P_{\parallel}^{(2)}(\ln g, \phi, \nabla) + \cdots + O(g_s, e^{-\frac{1}{\alpha'}}, e^{-\frac{1}{g_s}}) &(3.277)\\
P_{\perp}(\ln g, \phi, \nabla) &= 0 + \alpha' P_{\perp}^{(1)}(\ln g, \phi, \nabla) + \alpha'^2 P_{\perp}^{(2)}(\ln g, \phi, \nabla) + \cdots + O(g_s, e^{-\frac{1}{\alpha'}}, e^{-\frac{1}{g_s}}) &(3.278)\\
P_{\phi}(\ln g, \phi, \nabla) &= 0 + \alpha' P_{\phi}^{(1)}(\ln g, \phi, \nabla) + \alpha'^2 P_{\phi}^{(2)}(\ln g, \phi, \nabla) + \cdots + O(g_s, e^{-\frac{1}{\alpha'}}, e^{-\frac{1}{g_s}}) &(3.279)
\end{aligned}$$

where $P_{\parallel}^{i}(\ln g, \phi, \nabla)$, $P_{\perp}^{i}(\ln g, \phi, \nabla)$ and $P_{\phi}^{i}(\ln g, \phi, \nabla)$ are identified by demanding that T-duality rules should commute with the equations of motion. The explicit computation of [67] supports the expansion of T-duality rules at order α'^2.

In the previous chapter we saw that the perturbative studies of string theory around flat space-time identifies only the invariant structure of the effective action. This implies that T-duality rules could be identified up to such a perturbative field redefinition. However once the rules are obtained in a given scheme then the rules of any other scheme can be generated by an appropriate field redefinition. We have chosen the definition of the metric and the dilaton in such a way that the equations of motions are given by (2.53) and (2.54). We have realised that the quadratic α' corrections to diagonal Kaser background, the four and five dimensional Schwarzschild black hole, a general two dimensional black hole and their T-duals beside the Lorentz-form T-duality conjectures enable one to write the quadratic α' corrections to the rules of T-duality for a general time-dependent background composed of a diagonal metric and the dilaton. The quadratic α' corrections to the rules of T-duality in x_1 direction of (3.273) reads

$$\begin{aligned}
\ln\tilde{g}_{11} &- \frac{\alpha'}{4}\tilde{\nabla}_{\mu}\ln\tilde{g}_{11}\,\tilde{\nabla}^{\mu}\ln\tilde{g}_{11} + \frac{\alpha'^2}{32}(\tilde{\nabla}_{\mu}\ln\tilde{g}_{11}\,\tilde{\nabla}^{\mu}\ln\tilde{g}_{11})^2 &(3.280)\\
&- \frac{\alpha'^2}{16}\tilde{\nabla}_{\mu}\tilde{\nabla}_{\nu}\ln\tilde{g}_{11}\,\tilde{\nabla}^{\mu}\tilde{\nabla}^{\nu}\ln\tilde{g}_{11}
\end{aligned}$$

$$= - \left(\ln g_{11} - \frac{\alpha'}{4} \nabla_\mu \ln g_{11} \, \nabla^\mu \ln g_{11} + \frac{\alpha'^2}{32} (\nabla_\mu \ln g_{11} \, \nabla^\mu \ln g_{11})^2 - \frac{\alpha'^2}{16} \nabla_\mu \nabla_\nu \ln g_{11} \, \nabla^\mu \nabla^\nu \ln g_{11} \right) ,$$

$$\ln \tilde{g}_{ii} + \frac{\alpha'^2}{32} \tilde{\nabla}_\mu \ln \tilde{g}_{11} . \tilde{\nabla}^\mu \ln \tilde{g}_{ii} \; \tilde{\nabla}_\nu \ln \tilde{g}_{11} . \tilde{\nabla}^\nu \ln \tilde{g}_{11} = \ln g_{ii} + \frac{\alpha'^2}{32} \nabla_\mu \ln g_{ii} . \nabla^\mu \ln g_{11} \; \nabla_\nu \ln g_{11} . \nabla^\nu \ln g_{11} + O(\alpha'^3 \nabla^6) \; , \tag{3.281}$$

$$\tilde{\phi} - \frac{1}{4} \ln \det \tilde{g} + \frac{\alpha'^2}{32} \; \tilde{\nabla}_\mu \ln \tilde{g}_{11} \; \tilde{\nabla}_\nu \ln \tilde{g}_{11} \; \tilde{\nabla}^\mu \tilde{\nabla}^\nu \ln \tilde{g}_{11} = \phi - \frac{1}{4} \ln \det g + \frac{\alpha'^2}{32} \; \nabla_\mu \ln g_{11} \; \nabla_\nu \ln g_{11} \; \nabla^\mu \nabla^\nu \ln g_{11} \; + \; O(\alpha'^3 \nabla^6) , \tag{3.282}$$

where covariant derivative acts on the logarithm of the components of metric as if they were scalars. These are in agreement with the linear α' corrections to T-duality rules computed in [58] and [59]. This agreement proves the validity of the Lorentz-form T-duality conjectures at the linear order in α'. The Lorentz-form T-duality conjectures hold true for a general diagonal Kasner background in $D = 26$, the Schwarzschild black hole in $D = 4, 5$ and a general two dimensional black hole at quadratic order in α'. Thus it is decent to accept that the Lorentz-form T-duality conjecture is valid at the quadratic order in α' for time-dependent backgrounds composed of diagonal metric and dilaton.

At this stage it is natural to ask if the α' corrections to T-duality can be cancelled by an appropriate field redefinition. In other words, is there any appropriate renormalisation scheme and regularisation method which gives no correction to the tree level rules of T-duality? The answer is negative. One can check that even the linear α' corrections to T-duality can not be compensated by a field redefinition which leaves the tensor property of the metric intact. In the former works [70, 67] either the redefined metric had not been a tensor or the metric definitions had not been the same in both spaces. Choosing different definitions for the metrics means choosing different schemes for the background and for its T-dual. If we do so then the corrections in the space can not be directly mapped to the corrections in the T-dual space. Choosing a scheme in which the metric is not a tensor means choosing a regularisation method which breaks the general covariance of the theory. In such a regularisation method one must be extremely careful about interpreting the results. It is preferable to work in a scheme which respects the fundamental symmetries of the theory. In such a scheme T-duality must be modified. This argument is supported by the fact that T-duality rules are modified for the conjectured α' exact backgrounds [85].

We have found intersting, as follows from (3.281), that applying T-duality in one direction alters the components of the metric in all the directions not only of the direction which the T-duality is applied in.

Closed Bosonic String Theories are not free from closed tachyon instability. However since T-duality is common in both Bosonic and supersymmetric string theories then we expect that some features, like the one we highlighted in the previous paragraph, happen in the superstring theories.

As a generalisation of the work reviewed within this chapter one may consider the four-loop α' corrections in the critical Bosonic String Theory where the corresponding β-functions are computed in [86, 87] or the four-loop α' corrections in the superstring theory [88, 89, 90, 91, 92, 93].

We have computed the linear and the quadratic α' corrections to the general diagonal Kasner metric. Kasner metric has a big-bang like singularity. It would be intersting to explore if there exists any scheme in which a natural extrapolation of the α'^2 metric toward the singularity of the supergravity approximation admits no big bang singularity. For example it would be nice to find a scheme in which the quadratic α' corrected Kasner metric can be represented as large time expansion of

$$ds^2 = -dt^2 + \sum_{i=1}^{25}(\alpha'^2 b_i^{(2)} + \alpha' b_i^{(1)} t^2 + b_i^{(0)} t^4)^{\frac{p_i}{2}} dx_i^2 \tag{3.283}$$

$$\phi = \frac{\sum p - 1}{8} \ln(\alpha'^2 \phi^{(2)} + \alpha' \phi^{(1)} t^2 + \phi^{(0)} t^4) \tag{3.284}$$

where $b_i^{(\cdots)}$ and $\phi^{(\cdots)}$ are numbers and neither the metric nor the dilaton has singularity in the interval of $t \in (-\infty, +\infty)$ and the maximum of the curvature of the space-time is bigger than square root of α'. If such a scheme exists than it may be claimed that α' corrections within such a scheme can smoothen the big bang singularity.

In this chapter we studied the α' corrections to T-duality. Other dualities of string theories may receive subleading corrections. Among these dualities are AdS/CFT correspondence. One may compute the α'^3 corrections to the LLM solutions [94] to study if (and how) the AdS/CFT correspondence needs to be modified upon the inclusion of the subleading α' corrections in the string side. Also it would be intersting to explore the possibility that the subleading α' corrections might change the null singular LLM solutions [95] to either regular black holes or smooth geometries.

S-duality maps the type II string theory compactified on Calabi-Yau manifold to the Heterotic String Theory compactified on T^6. In the supergravity approximation the rules of S-duality are known. The perturbative world-sheet corrections of the Heterotic String Theory starts at the linear order in α' while the perturbative word-sheet corrections of the superstring starts at cubic order in α'. It would be intersting to explore how the S-duality rules should be modified upon the inclusion of different α' corrections in the Heterotic and the type II sides. This modification may require mapping the perturbative α' corrections of one side to the non-perturbative α' corrections of the other side.

Chapter 4

A wrapped F-String

4.1 Introduction

The massless field of helicity two in the spectrum of string theory is identified as the gravitational field since its low energy effective action around flat space-time coincides with the Einstein-Hilbert action [1, 2, 3, 4]. This identification sets the subleading string corrections as the quantum corrections to gravity and allows one to ask if and how quantum corrections preserve or change the properties of the classical backgrounds. In particular one may ask if the subleading string corrections induce a regular horizon on the singular classical geometries which have an entropy associated to them.

Amongst these singular classical geometries are the half BPS null singular ones which represent a wrapped fundamental string with general momentum and winding numbers [96]. These null singular geometries have a statistical entropy associated to them since string states with given momentum and winding numbers are degenerate [97]. It is conjectured that quantum effects convert these singular geometries to black holes with a regular horizon.

The leading world-sheet corrections of the Heterotic string includes the square of the Riemann tensor -Eq. (2.55). Ref [98], motivated by [99], observed that the inclusion of the square of the Riemann tensor and its supersymmetric completion in $D = 4$ [100, 101, 102, 103, 104, 105, 106, 107, 108, 109] induces a local horizon with geometry $AdS_2 \times S^2$ on these backgrounds and for which the modified Hawking-Bekenstein entropy [8, 9, 10] is in agreement with the statistical entropy. This observation renewed interest in the subject [110, 111, 112, 113, 114, 115, 116, 117, 118, 119, 116]. Ref. [113, 120, 121] introduced the entropy formalism and concluded that the inclusion of the Gauss-Bonnet action as a part of the linear α' corrections in an arbitrary dimension induces a local horizon with geometry $AdS_2 \times S^{D-2}$ for which the modified Hawking-Bekenstein entropy is in agreement with the statistical entropy up to a numerical constant factor.

In this chapter we present a way to calculate all the linear α' corrections in an arbitrary dimension and we study how these corrections may change these null singular backgrounds to black holes. The chapter is organised in the following way;

In the second section we review the classical solutions representing a wrapped fundamental string on a cycle. We realise them as ten dimensional backgrounds composed of the metric, the NS two form and the dilaton first compacted on a torus of appropriate dimensionality to $D+1$ dimensional space-time and then through KK compactification on a circle to a D dimensional space-time.

In the third section we study the field redefinition ambiguities. We require that the generalisation of the Einstein tensor is covariantly divergence free. This requirement fixes the curvature squared terms to the Gauss-Bonnet Lagrangian keeping some of the field redefinition ambiguity parameters untouched.

In the fourth section we discuss how the singularity could be modified by the inclusion of the α' corrections. We employ the compactification process of the second section to account for all the linear α' corrections in lower dimensions using the corrections in ten dimensions. We compute the local horizon configuration parameters for all field redefinitions compatible with ten dimensional diffeomorphism group. Note that the modified Hawking-Bekenstein entropy is the same for actions related to each other by field redefinition provided that the α' terms are studied as perturbations around a classical solution [11]. Since the stretched horizon is the exact solution of the truncated equations then the modified Hawking-Bekenstein entropy depends on the field redefinition ambiguity parameters. We show that there exist schemes in which the inclusion of all the linear α' corrections in an arbitrary dimension gives rise to a local horizon with geometry $AdS_2 \times S^{D-2}$ for which the modified Hawking-Bekenstein entropy is in agreement with the statistical entropy and outside which the higher order α' corrections are perturbative. We also discuss on the existence of a smooth solution connecting the local horizon to asymptotic infinity.

In the fifth section the conclusions are presented.

4.2 The induced action

We consider a ten-dimensional Riemannian manifold $\boldsymbol{M}_{10}$ homeomorphic to $M_D \times T^{10-D}$ whose metric has $10-D$ killing vectors in T^{10-D} and admits an asymptotically flat region. We represent the coordinate patch that covers the asymptotic flat region of $\boldsymbol{M}_{10}$ by $\boldsymbol{x}^i = (x^1, \cdots, x^D, z^1, \cdots, z^{10-D})$ where $x^\mu = (x^1, \cdots, x^D)$ and $z^m = (z^1, z^2, \cdots, z^{10-D})$ are coordinates respectively on M_D and T^{10-D} and $(dz^1, \cdots, dz^{10-D})$ are the killing vectors. We refer to x^μ and z^m respectively as the D dimensions and the compactified space. The string perturbations can be studied at the vicinity of the asymptotic region of $\boldsymbol{M}_{10}$ which is covered by $\boldsymbol{x}^i$. We realise this neighbourhood as a background of the Heterotic string theory composed of the metric, the NS two-form and the dilaton whose field configuration follows

$$\begin{aligned} \boldsymbol{ds}^2 &= \sum_{\mu,\nu=1}^{D} \boldsymbol{g}_{\mu\nu}(x) dx^\mu dx^\nu + \sum_{m=1}^{10-D} \{2\boldsymbol{g}_{z^m\mu}(x)\, dz^m\, dx^\mu + \boldsymbol{g}_{z^m z^m}(x)(dz^m)^2\}, \\ \boldsymbol{B} &= \boldsymbol{B}_{\mu\nu}(x) dx^\mu \wedge dx^\nu + \sum_{m=1}^{10-D} \boldsymbol{B}_{z^m\mu}(x) dx^\mu \wedge dz^m, \qquad (4.1) \\ \boldsymbol{\phi} &= \boldsymbol{\phi}(x), \qquad (4.2) \end{aligned}$$

We use the bold symbols to represent the fields in ten dimensions. We rewrite the metric in the following form

$$\boldsymbol{ds}^2 = g_{\mu\nu}(x)\, dx^\mu dx^\nu + \sum_{m=1}^{10-D} T_m(x)^2 (dz^m + 2A^m_\mu(x) dx^\mu)^2, \qquad (4.3)$$

where $g_{\mu\nu}$, T_m and A^m_μ are used to re-express the components of the ten dimensional metric in a way that we shall see is more convenient. The metric (4.3) is invariant under the following transformations

$$z^m \to z^m - 2\Lambda^m(x) \qquad (4.4)$$

$$A^m_\mu(x) \to A^m_\mu(x) + \partial_\mu \Lambda^m(x) \qquad (4.5)$$

where $\Lambda^m(x)$ are arbitrary scalars. These symmetries are remnants of the ten-dimensional diffeomorphism. We interpret $A^1_\mu(x)$, $\cdots$ and $A^{10-D}_\mu(x)$ as $10-D$ distinct $U(1)$ gauge connections in the D dimensions because they are vectors and they have $U(1)$ symmetries associated to them.

We rewrite the NS two-form in the following way

$$\boldsymbol{B} = B_{\mu\nu} dx^\mu \wedge dx^\nu + 2 \sum_{m=1}^{10-D} A^{10-D+m}_\mu dx^\mu \wedge (dz^m + 2A^m_\nu dx^\nu) \tag{4.6}$$

where $B_{\mu\nu}$, A^{11-D}_μ, $\cdots$ and A^{20-2D}_μ are used to re-express the components of the ten dimensional NS two-form in a way that we shall see is more convenient. Note that the $U(1)$ transformations associated to $A^1_\mu(x), \cdots, A^{10-D}_\mu(x)$ leave intact $B_{\mu\nu}(x)$, $A^{11-D}(x), \cdots, A^{20-2D}(x)$ since $dz^m + 2A^m_\nu dx^\nu$ is gauge invariant. The sigma model for the background we are considering is invariant under altering $\boldsymbol{B}$ by any exact two-form, i.e.

$$\boldsymbol{B} \to \boldsymbol{B} + \boldsymbol{d\Lambda}\,, \tag{4.7}$$

and thus the low energy action is invariant under $\boldsymbol{B} \to \boldsymbol{B} + \boldsymbol{d\Lambda}$. Amongst these Λ's we consider the ones given by

$$\boldsymbol{\Lambda} = \Lambda_\mu(x) dx^\mu - \sum_{m=1}^{6} \Lambda^{m+10-D}(x) dz^m\,,$$

which imply that the LEEA is invariant under

$$B \to B + d\Lambda\,, \tag{4.8}$$
$$A^{11-D}_\mu(x) \to A^{11-D}_\mu(x) + \partial_\mu \Lambda^{11-D}(x) \tag{4.9}$$
$$\vdots \quad \vdots \quad \vdots$$
$$A^{20-2D}_\mu(x) \to A^{20-2D}_\mu(x) + \partial_\mu \Lambda^{20-2D}(x)\,. \tag{4.10}$$

We see that independent $U(1)$ symmetries are associated to $A^{11-D}_\mu(x), \cdots, A^{20-2D}_\mu(x)$. These $U(1)$ symmetries are remnants of the gauge symmetries in ten-dimensions. We interpret $A^{11-D}_\mu(x)$, $\cdots$, $A^{20-2D}_\mu(x)$ as $10-D$ distinct gauge connections in the D dimensions.

Due to the symmetries of the metric we can choose a sufficiently large volume for any non-trivial cycle[1] in the compactified space in the patch of $\boldsymbol{x}^i$. Thus we ignore the world-sheet or target space

[1] A cycle which does not shrink to a point under any given homeomorphism.

instantons corrections to the LEEA. The rest of the string corrections respect the ten dimensional diffeomorphism symmetry group. Therefore the ten-dimensional low energy effective action reads

$$S = \frac{1}{32\,\pi}\int d^{10}x\sqrt{-\det g}\, e^{-2\phi}\, L(B,g,\phi)\,, \tag{4.11}$$

where $L(B,g,\phi)$ includes all the perturbative string corrections and it is invariant under ten-dimensional diffeomorphism and $B\to B+d\Lambda$. Inserting (4.3) and (4.6) into the action (4.11) we obtain

$$S = \frac{1}{32\,\pi}\int d^{D}x\,\sqrt{-\det g}\, e^{-2\phi}\, L(g_{\mu\nu},B_{\mu\nu},A^1_\mu,\cdots,A^{20-2D}_\mu,T_1,\cdots,T_{10-D},\phi)\,, \tag{4.12}$$

where the integration on the compactified space is performed and we have defined

$$2\phi = 2\phi - \sum_{m=1}^{10-D}\ln T_m - \ln V\,, \tag{4.13}$$

where V is the volume of the compactified space. (4.12) is the pullback of the action into the D dimensions and we refer to it as the induced action. The induced action inherits the remnants symmetries of the ten dimensional action. Thus it is invariant under D dimensional diffeomorphism group, $U(1)$ symmetries associated to $A^1_\mu,\cdots,A^{20-2D}_\mu$ and $B\to B+d\Lambda$. This means that the induced Lagrangian is expressible in a covariant form in terms of the Riemann tensor constructed from $g_{\mu\nu}$, the form fields, T_1, , $\cdots$, T_{10-D}, ϕ and their covariant derivatives with respect to $g_{\mu\nu}$,

$$L = L(R_{\mu\nu\lambda\eta},B_{\mu\nu},A^1_\mu,\cdots,A^{20-2D}_\mu,T_1,\cdots,T_{10-D},\phi,g_{\mu\nu},\nabla_\mu)\,. \tag{4.14}$$

It is not a hard task to obtain the explicit form of the induced action at the level of supergravity approximation,

$$\begin{aligned} S^{(0)} &= \frac{1}{32\pi}\int d^{D}x\,\sqrt{-\det g}\, e^{-2\phi}\Big(R-\frac{|dB|^2}{12}+4|\nabla\phi|^2- \\ &\qquad \sum_{m=1}^{10-D}(|\nabla\ln T_m|^2-|T_m\,dA^m|^2-|\frac{dA^{m+10-D}}{T_m}|^2)\Big)\,, \end{aligned} \tag{4.15}$$

where R denotes the Ricci scalar of $g_{\mu\nu}$ and integrations by parts are understood. We do not obtain the explicit form of the linear α' corrections to the induced action. We suffice to present the linear α' corrections to the induced action by

$$S^{(1)} = \frac{1}{32\,\pi}\int d^{D}x\,\sqrt{-\det g}\, e^{-2\phi}\, L^{(1)}\,, \tag{4.16}$$

and we know that $L^{(1)}$ is a functional of the D dimensional Riemann tensor, the gauge fields and their covariant derivatives,

$$L^{(1)} = L^{(1)}(R_{\mu\nu\lambda\eta}, B_{\mu\nu}, A^1_\mu, \cdots, A^{20-2D}_\mu, T_1, \cdots, T_{10-D}, \phi, g_{\mu\nu}, \nabla_\mu)\,. \tag{4.17}$$

We divide $L^{(1)}$ to a part which is the pull back of the gravitational Chern-Simons terms in ten dimension $L^{(1)}_{CS}$ and a part which is the pull back of the rest of the corrections $L^{(1)}_{NCS}$

$$L^{(1)} = L^{(1)}_{CS} + L^{(1)}_{NCS}\,. \tag{4.18}$$

$L^{(1)}_{NCS}$ is a functional of the D dimensional Riemann tensor, the exterior derivatives of the gauge fields and their covariant derivatives,

$$L^{(1)}_{NCS} = L^{(1)}_{NCS}(R_{\mu\nu\lambda\eta}, dB, dA^1, \cdots, dA^{20-2D}, T_1, \cdots, T_{10-D}, \phi, g_{\mu\nu}, \nabla_\mu)\,, \tag{4.19}$$

while $L^{(1)}_{CS}$ depends on the gauge fields. In this chapter and the next chapter we calculate the corrections given by $L^{(1)}_{NCS}$ to the black hole entropy and we discuss on the contribution of $L^{(1)}_{CS}$ to black holes entropy.

In this chapter we consider the backgrounds in which

$$A^1 = A^{(1)}\,, \tag{4.20}$$

$$A^{11-D} = A^{(2)}\,, \tag{4.21}$$

$$A^2_\mu = A^3_\mu = \cdots = A^{10-D}_\mu = A^{12-D}_\mu = \cdots = A^{20-2D}_\mu = 0\,, \tag{4.22}$$

$$T^2 = T^3 = \cdots = T^{10-D} = \text{Constants}\,. \tag{4.23}$$

A family of the extrema of the compactified action in the supergravity approximation is given by

$$ds^2_{string} = -\,e^{4\phi(r)}\,dt^2 + dr^2 + r^2\,d\Omega^2_{D-2}\,, \tag{4.24}$$

$$e^{-4\phi(r)} = \frac{(r^{D-3}+2W)\,(r^{D-3}+2N)}{r^{2(D-3)}}\,, \qquad T(r) = \sqrt{\frac{r^{D-3}+2N}{r^{D-3}+2W}}\,, \tag{4.25}$$

$$A^{(1)}_\tau(r) = -\frac{N}{r^{D-3}+2N}\,, \qquad A^{(2)}_\tau(r) = -\frac{W}{r^{D-3}+2W}\,, \tag{4.26}$$

where N and W are two arbitrary numbers labelling the solution. We only consider the case where N and W are both positive. These backgrounds are constructed in [96] as singular limits of regular black-holes obtained by applying a solution generating transformation [122, 123] on a higher dimensional Kerr metric. Here we use the notation of [124]. Ref. [96] proved that they break half of the ten dimensional supersymmetries leaving eight unbroken supersymmetry parameters. These backgrounds are null-singular, i.e. the horizon coincides with the singularity. They represent BPS states of an elementary string carrying n units of momentum and w units of winding charges along one cycle where [124]

$$n = \frac{(D-3)\Omega_{D-2}}{4\pi} N\,, \tag{4.27}$$

$$w = \frac{(D-3)\Omega_{D-2}}{4\pi} W\,, \tag{4.28}$$

and the unit of $\alpha' = 16$ is used.[2] For general values of N and W a tachyon instability may exist around the singularity, reminiscent of the tachyon instability outside the horizon of Euclidean black holes presented in [125, 126]. We focus on the cases where $N \sim W$ and this instability is not present.

An entropy may be associated to these backgrounds since in general there exists more than one state of the Heterotic string carrying w units of winding and n units of momentum. For large n and w the degeneracy of these states grows as $e^{4\pi\sqrt{nw}}$ [127]. Thus the entropy, defined by the logarithm of the degeneracy of the states, is given by:

$$S_{\text{statistical}} = 4\pi\sqrt{nw}\,, \tag{4.29}$$

when n and w are large. We refer to this entropy as the statistical entropy. A dilemma will arise as soon as the statistical entropy is associated to these tree-level backgrounds since they are singular and do not possess a regular event horizon to which the thermodynamical properties can be connected. This dilemma can be resolved in either of the following ways,

I. Statistical entropy should not be associated to these backgrounds.

II. Thermodynamical properties should be expressed in term of other geometrical properties of the null singular geometries.

[2]We have chosen a specific value for the radius of the compactification because the α' perturbative corrections to (4.24) do not depend on the radius of the compactification. The solution which represents KK-compactification on a circle with an arbitrary radius can be generated by rescaling the compact direction and applying the compactification process. This solution is written in [124].

III. The subleading string corrections will induce an event horizon and the horizon cloaks the singularity.

Of the above possibilities, the first seems unnatural since the statistical entropy is associated to regular black holes [128, 129, 130] and these singular backgrounds are a limit of regular black holes. The fact that both the Euclidean path integral approach[3] [132] and the Noether current method [8, 9] express the entropy of a given black hole in term of its event horizon is not sufficient to conclude that entropy could not be associated to geometries without the event horizon. We would like to point out that Mathur and Lunin's description of the entropy [14] may resolve the dilemma in the second way. It is intersting that for the case of singular backgrounds representing D1-D5 branes, which have an entropy associated to them, both Mathur-Lunin's description [133] and the subleading string corrections [119] can generate the entropy. In this chapter we study if the inclusion of subleading corrections can generate a horizon for the backgrounds representing a fundamental string.

4.3 The α' corrections

The string coupling constant of (4.24), $g_s^2 = g_0^2 e^{2\phi}$, is

$$g_s^2 = g_0^2 \frac{r^{D-3}}{\sqrt{(r^{D-3}+2W)(r^{D-3}+2N)}} \leq g_0^2 \,, \tag{4.30}$$

where g_0 is an arbitrary parameter. We choose a sufficiently small value for g_0. Thus we ignore the string loop corrections. The α' corrections to the Lagrangian read

$$L = L^{(0)} + \alpha' L^{(1)} + \alpha'^2 L^{(2)} + \cdots \,, \tag{4.31}$$

where $L^{(0)}$ stands for the tree-level Lagrangian and the rest is its successive subleading corrections. This series may not make sense for (4.24) since each term of the α' series diverges at its singularity. However note that the α' corrections change the background itself

$$g \to g = g^{(0)} + \alpha' g^{(1)} + \alpha'^2 g^{(2)} + \cdots \,, \tag{4.32}$$

[3]Note that in string theory the presence of the tachyon-like winding modes of the tachyon wrapped around the Euclidean time which survive GSO projection [125, 126] adds to the known disturbing aspect [131] of the Euclidean approach.

and the α'-corrected metric, possibly, can have a horizon outside which the α' expansion makes sense. Also the α' corrections to the string coupling constant may remain finite outside the horizon and the string loop corrections could be ignored consistently. In order to check this possibility we truncate the equations of motion at $O(\alpha'^2)$. Then we study if a exact solution of the truncated equations is a black hole with a regular horizon outside which the higher order α' corrections are perturbative.

The exact solutions of the truncated equations depend on the a priori ambiguous parameters of the effective action. We study the exact solutions for a set of the a priori ambiguous parameters. We consider a general field redefinition

$$g_{ij} \rightarrow g_{ij} + \alpha' T_{ij}\,, \tag{4.33}$$

$$B_{ij} \rightarrow B_{ij} + \alpha' S_{ij}\,, \tag{4.34}$$

$$\phi \rightarrow \phi - \alpha' \frac{X}{2}\,, \tag{4.35}$$

which induces a change in (2.55) of the form [134]

$$\begin{aligned} \Delta L &= -T^{ij}(R_{ij} - \frac{1}{4} H_{ikl} H_j{}^{kl} + 2\nabla_i \nabla_j \phi) + \\ &+(\frac{1}{2} T_i{}^i + X)(R - \frac{1}{12} H^2 + 4\nabla^2 \phi - 4(\nabla\phi)^2) - \frac{1}{2}\nabla_k S_{lm} H^{klm}\,. \end{aligned} \tag{4.36}$$

where X, S_{ij} and T_{ij} are tensors with appropriate properties and are polynomials of g_{ij}, B_{ij}, ϕ and their derivatives.[4] We consider only a class of the a priori ambiguities parameters given by

$$T_{ij} = aR_{ij} + \frac{b}{8} H_{ikl} H_j{}^{kl} + (e-12f)\, g_{ij} R + f g_{ij} H_{klm} H^{klm}\,, \tag{4.37}$$

$$X + \frac{1}{2} T^i{}_i = (c - 12f) R + (\frac{d}{12} + 3f) H_{ijk} H^{ijk}\,, \tag{4.38}$$

$$S_{ij} = 0\,, \tag{4.39}$$

where a, b, c, d, e and f are real numbers representing some of the ambiguous parameters. This class of field redefinition alters the linear α' corrected action by

$$\begin{aligned} \frac{1}{\alpha'} \Delta L &= -aR_{ij}R^{ij} + (c-e)R^2 + (\frac{d}{12} - \frac{c}{12} + \frac{e}{4}) R H^2 - \frac{d}{144}(H^2)^2 \\ &+(\frac{a}{4} - \frac{b}{8}) H^2_{ij} R^{ij} + \frac{b}{32} H^2_{ij} H^{2ij} + O(\nabla\phi)\,, \end{aligned} \tag{4.40}$$

[4]To compute ΔL it is enough to remember that $g^{ij}\delta R_{ij} = (\nabla^i\nabla^j - g^{ij}\Box)\delta g_{ij}$.[135]

where

$$\begin{aligned} \boldsymbol{H}^2_{ij} &= \boldsymbol{H}_{ikl}\boldsymbol{H}_j^{\ kl}\,, & (4.41)\\ \boldsymbol{H}^2 &= \boldsymbol{H}_{ijk}\boldsymbol{H}^{ijk}\,, & (4.42)\end{aligned}$$

and the derivatives of the dilaton are not written to save space. In the forthcoming computations we do not need them. We require the generalisation of the Einstein tensor to be covariantly divergence free for a trivial dilaton. Adding this requirement to the linear α' corrections changes it to the first order Lovelock gravity [136] where $(a, c-e) = (\frac{1}{2}, \frac{1}{8})$.[5] Thus we set $(a, c) = (\frac{1}{2}, \frac{1}{8}+e)$ for which the linear α' corrected action reads

$$\begin{aligned} S &= \frac{1}{32\pi}\int d^{10}x\,\sqrt{-\det \boldsymbol{g}}\,e^{-2\phi}\,\boldsymbol{L} & (4.43)\\ \boldsymbol{L} &= \boldsymbol{R} - \frac{1}{12}\boldsymbol{H}^2 + 4|\boldsymbol{\nabla}\phi|^2 + \alpha' \boldsymbol{L}^{(1)}_{NCS} + \alpha' O(\boldsymbol{\nabla}\phi) + O(\alpha'^2) & (4.44)\\ \boldsymbol{L}^{(1)}_{NCS} &= \frac{1}{8}\boldsymbol{L}_{GB} + \frac{1}{192}\boldsymbol{H}_{klm}\boldsymbol{H}^k_{\ pq}\boldsymbol{H}_r^{\ lp}\boldsymbol{H}^{rmq} - \frac{1}{16}\boldsymbol{R}_{klmn}\boldsymbol{H}_p^{\ kl}\boldsymbol{H}^{pmn} + \\ &\quad +(\frac{b}{32}-\frac{1}{64})\boldsymbol{H}^2_{ij}\boldsymbol{H}^{2ij} + (\frac{d}{12}-\frac{e}{6}-\frac{1}{96})\boldsymbol{R}\,\boldsymbol{H}^2 - \frac{d}{144}(\boldsymbol{H}^2)^2 + (\frac{1}{8}-\frac{b}{8})\boldsymbol{H}^2_{ij}\boldsymbol{R}^{ij} \end{aligned}$$

where $\boldsymbol{L}_{GB} = \boldsymbol{R}_{ijkl}\boldsymbol{R}^{ijkl} - 4\boldsymbol{R}_{ij}\boldsymbol{R}^{ij} + \boldsymbol{R}^2$ is the Gauss-Bonnet term. In the work [137] the α' corrections were required not to produce new extrema for the bi-linear part of the action describing deviation from flat Minkowski space. This criterion, the no-ghost criterion, is questionable since the new extrema are not perturbative in α'. The criterion we used produces the same results and is independent of the perturbative behaviour of the α' series. However both of these criteria fail to identify a unique action. Mavromates and Miramontes have suggested exploiting the field redefinition ambiguities to demand that the K-matrix operator -(2.8) and (2.13)- should contain no derivative operators acting on the β equations and have claimed that this criterion, the MM-criterion, is essential to ensure perturbative invertibility of the K-matrix [134, 140]. It is intriguing that for the Bosonic string the action resulting from the application of the MM-criterion is uniquely specified and is manifestly free of ghosts. It is intersting to apply the MM-criterion at the linear order in α' in the presence of the gravitational Chern-Simons terms and then to study if the

[5]Lovelock gravity [136] is a generalisation of Einstein-Hilbert action where the generalisation of Einstein tensor G_{ij}: (1) is symmetric in its indices, (2) is a function of the metric and its first two derivatives, (3) is covariantly divergence free. The linear α' corrections can be chosen to satisfy all these conditions [137]. However the higher order α' corrections include also higher derivatives of the metric and can not be rewritten as higher order [138] Lovelock gravity [139].

resulting action converts the singular backgrounds representing the F-string to a regular black hole.

4.4 Modification of the singularity

We presume that there exists an exact α' background in the large dimensions which in the string frame reads

$$ds_{\text{exact}} = -f(r)dt^2 + dr^2 + g(r)d\Omega_{D-2}^2 \tag{4.45}$$

$$\phi = \phi(r)\,, \qquad T = T(r)\,, \tag{4.46}$$

$$A_t^{(1)} = A_t^{(1)}(r)\,, \qquad A_t^{(2)} = A_t^{(2)}(r)\,, \tag{4.47}$$

the large r limits of which are (4.24), (4.25) and (4.26). The number of the modified supersymmetry charges[6] of this α' exact background should be the same as the number of SUSY charges of the tree-level background. It is conjectured [124] that this α' exact background has a regular event horizon with isometry group of $AdS_2 \times S^{D-2}$ whose fields in the vicinity of its horizon can be approximated by

$$ds^2 = v_1(-\rho^2 d\tau^2 + \frac{d\rho^2}{\rho^2}) + v_2 d\Omega_{D-2}^2\,, \tag{4.48}$$

$$e^{-2\phi(\rho)} = s\,, \tag{4.49}$$

$$T(\rho) = T\,, \tag{4.50}$$

$$F_{t\rho}^{(1)} = e_1\,, \tag{4.51}$$

$$F_{t\rho}^{(2)} = e_2\,, \tag{4.52}$$

where v_1, v_2, s, T, e_1 and e_2 are constant real (s, T are positive) numbers to be fixed by the equations of motion and the behaviour of the fields at infinity. We often refer to this horizon as the stretched horizon. A concrete proof or refutal of this conjecture requires knowing all the α' corrections. Neither the string scattering amplitudes nor the sigma model techniques nor CSFT are practically useful to compute the infinite terms of the α'-expansion series. There exists no

[6] In LEEA the supersymmetry is realised as the symmetry of the action therefore, at least, the on shell SUSY constraints needs modification upon the inclusion of the subleading corrections.

other known method capable of producing the full α'-corrected action.[7] Currently the conjecture is supported by

I. Inclusion of only the Gauss-Bonnet action in the induced action allows for the existence of a local horizon geometry whose modified thermodynamical entropy [8, 9, 10][8] is in agreement with the statistical entropy up to a numerical constant [124].

II. Inclusion of $R_{ijkl}R^{ijkl}$ and the terms needed by SUSY [100, 101, 102, 103, 104, 105, 106, 107, 108, 109] in the four dimensional induced action allows for a local horizon whose modified thermodynamical entropy is in agreement with the statistical entropy [98]. In higher dimensions it is not known which terms should be added to $R_{ijkl}R^{ijkl}$ to maintain SUSY.

The conjecture may be contradicted by :

I. The fundamental string is a special case of the null sigma models [12, 13]. It means that there exists a scheme in which the background fields retain their forms in the supergravity approximation. Thus within this scheme the fundamental string remains as a null singular background even after the inclusion of all the α' corrections. Does this contradict the appearance of a horizon due to the inclusion of the α' corrections?

II. The value of the Wald entropy is invariant under field redefinition provided that the α' terms are studied as perturbations around a classical background [11]. Here since Wald's formula is applied on the local horizon which is the exact solution of the truncated equations of motion then the Wald entropy depends on the a priori ambiguous parameters. Therefore

[7]There have been attempts to guess a compact form for the α' expansion series of the metric [141, 142].

[8]The Wald's formula for the entropy of a D dimensional static spherical black hole, g_{ij}, is

$$S = \frac{1}{4}\frac{\delta L}{\delta R_{rtrt}}\, g_{tt}\, g_{rr}\, A_{D-2}|_{r=0} \, ,$$

where L stands for the Lagrangian not including $\sqrt{\det g}$, and A_{D-2} is the area of the horizon, and the radial coordinate is chosen in such a way that the horizon is at $r=0$. Note that $\frac{\delta L}{\delta R_{rtrt}}$ is simply the functional derivative of L with respect to R_{rtrt} holding g_{ij} (and ∇_i) fixed,

$$\frac{\delta L}{\delta R_{rtrt}} = \frac{\partial L}{\partial R_{rtrt}} - \nabla_i \frac{\partial L}{\partial \nabla_i R_{rtrt}} + \cdots \, .$$

which values should be chosen for the a priori ambiguous parameters s to calculate the Wald entropy?

III. The Gauss-Bonnet action or the supersymmetric version of curvature squared terms are not all the linear α' corrections. Does the inclusion of all the linear α' corrections allow for the existence of the horizon?

IV. Is there a smooth interpolating solution from the horizon toward the asymptotic infinity?

V. Could the higher order α' corrections be consistently neglected?

Let us consider the α' expansion series for the Lagrangian density,

$$\mathcal{L}(p) = \sum_{n=0}^{\infty} \alpha'^n L_n(p) \tag{4.53}$$

where p represents a point in the space-time on which the Lagrangian density is evaluated and $L_0(p)$ is the Lagrangian density in the supergravity approximation and $L_n(p)$ is the n^{th} order α' corrections to the Lagrangian density in the supergravity approximation. There exist neighbourhoods around the asymptotic infinity where $\sum \alpha'^n L_n(p)$ is an absolute convergent series. We call the union of all these neighbourhoods as the $\mathcal{C}$-neighbourhood. We refer to the boundary of the $\mathcal{C}$-neighbourhood as the $\mathcal{C}$-horizon. The $\mathcal{C}$-neighbourhood defines a subset of the space-time in which $\sum \alpha'^n L_n(p)$ is defined unambiguously in the sense that the rearrangements of terms in $\sum \alpha'^n L_n(p)$ does not change the series sum, $\mathcal{L}(p)$. The singularity of the supergravity approximation is outside of the $\mathcal{C}$-neighbourhood. In general the α' corrections could be positive or negative. This means that there exist neighbourhoods in which $\sum \alpha'^n L_n(p)$ is a conditionally convergent series. We refere to the union of all these neighbourhoods as the $\mathcal{NC}$-neighbourhood. The $\mathcal{NC}$-neighbourhood has two boundaries, the $\mathcal{C}$-horizon is one of them and we call the other boundary as the $\mathcal{NC}$-horizon[9]. The Lagrangian density on the singularity should be defined as the extrapolation of $\sum \alpha'^n L_n(p)$ from $\mathcal{NC}$-neighbourhood toward the singularity. In the $\mathcal{NC}$-neighbourhood

[9]The $\mathcal{C}$-horizon and $\mathcal{NC}$-horizon are scheme dependent. The $\mathcal{C}$-horizon could be pushed toward infinity by a field redefinition but the $\mathcal{NC}$-horizon might not shrink to a point under any field redefinition. It is tempting either to identify the boundary of the union of the $\mathcal{NC}$-neighbourhoods of all the schemes as a mathematical description for the "stretched horizon" defined in [14] applied to the case of a wrapped fundamental string or to choose the schemes in which the $\mathcal{NC}$-horizon coincides with the $\mathcal{C}$-horizon and then to identify the boundary of the union of the $\mathcal{C}$-neighbourhoods of all such schemes as the "stretched horizon".

by a suitable rearrangement of terms, $\mathcal{L}(p)$ may be made to converge to any desired values or even diverge. In the number theory this statement sometimes is referred to as the Riemann theorem. The field redefinition can be thought of as a tool to "rearrange" the α' series. Thus the Lagrangian density before reaching the singularity of the supergravity approximation depends on the rearrangements of the terms or almost equivalently on the field redefinition ambiguities. We do not know which of these rearrangements would be preferred or chosen by the underlying conformal field theory since it is not known what a conformal field theory (and if a unique one) represents a wrapped fundamental string. Ref. [12, 13] shows that there exists a scheme in which the background fields retain there forms in the supergravity approximation. This does not mean that we could not rearrange the α' expansion series in the $\mathcal{NC}$-neighbourhood and then extrapolate the Lagrangian density toward the singularity in such a way that the singularity is covered by an α' stretched horizon. The consistency will requires that the α' stretched horizon should be at least outside the $\mathcal{C}$-neighbourhood. Thus the α' series on the α' stretched horizon are not absolutely convergent series.

We do not know all the α' series. Therefore we could not identify the $\mathcal{C}$-neighbourhood and the $\mathcal{NC}$-neighbourhood in order to compare them with the stretched horizon. In the following we include all the linear α' corrections in a general scheme. We truncate the α' series at $O(\alpha'^2)$. We will show that the local horizon exists upon the inclusion of all linear α' corrections. We illustrate that in general the modified Hawking-Bekenstein entropy associated to the local horizon is not the same for actions related to each other by field redefinitions. Amongst these actions, the choices for which the modified Hawking-Bekenstein entropy is in agreement with the statistical entropy would be preferred. We provide convincing arguments that the interpolating solution exists and we show that in some schemes the higher order corrections are perturbative outside the α' stretched horizon.

We obtain the linear α' corrections to the induced action by applying the compactification process to the linear α' corrected action in ten dimensions (4.43). We consider the linear α' corrected action in (4.43) for all values of the field redefinition parameters, (b, d, e, f). We have seen that the pull back of (4.36) to the four dimensional space time has a covariant representation. Thus we can employ the Wald formula to calculate the entropy. For the time being we exclude the gravitational Chern-Simons terms to use the entropy formalism [113, 121]. We discuss on the contribution of the gravitational Chern-Simons terms to the entropy at the end of this section. For the sake of simplicity from this time on we set $D = 4$ and we study the four dimensional background. The entropy formalism utilises the entropy function defined by

$$f(\vec{v}, T, \vec{e}) = \frac{1}{32\pi} \int d\theta \, d\phi \sqrt{-\det g} \, s \, L(\vec{v}, T, \vec{e}) \tag{4.54}$$

where $L(\vec{v}, T, \vec{e})$ is the induced Lagrangian evaluated on the horizon configuration,

$$S = \frac{1}{32\pi} \int d^4x \sqrt{-\det g} \, e^{-2\phi} \, L(\vec{v}, T, \vec{e}). \tag{4.55}$$

Then the equations of motions are equivalent to

$$\frac{\partial f}{\partial v_i} = 0, \tag{4.56}$$
$$\frac{\partial f}{\partial s} = 0, \tag{4.57}$$
$$\frac{\partial f}{\partial T} = 0, \tag{4.58}$$
$$\frac{\partial f}{\partial e_1} = \frac{N}{2}, \tag{4.59}$$
$$\frac{\partial f}{\partial e_2} = \frac{W}{2}, \tag{4.60}$$

where we have used the notation of Appendix A of [121] for the normalisation of the charges. To evaluate the induced action near the horizon we reconstruct the horizon configuration in ten dimensions from (4.48)-(4.52)[10]

$$\begin{aligned} ds^2 &= ds^2 + T^2(dy + 2\, e_1 \, r \, d\tau)^2 + \sum dz_i^2, \\ e^{-2\phi} &= \frac{s}{T}, \\ B &= -2\, e_2 \, r \, d\tau \wedge dy \, . \end{aligned} \tag{4.61}$$

where the gauges are fixed by

$$A_1 = (e_1 \, r, 0, 0, 0), \tag{4.62}$$

$$A_2 = (e_2 \, r, 0, 0, 0). \tag{4.63}$$

Note that the class of field redefinitions considered in (4.40) includes any field redefinition which produces non-zero terms in the action near the horizon (4.61) and whose metric and NS two-form

[10] The compactification of the Gauss-Bonnet action has been done in [143].

equations of motion are second order differential equations. Using the ten dimensional background near the horizon (4.61) one finds that

$$L_0 = \boldsymbol{R} - \frac{1}{12}\boldsymbol{H}^2 = -\frac{2}{v_1} + \frac{2}{v_1} + \frac{2\,{e_1}^2\,T^2}{{v_1}^2} + \frac{2\,{e_1}^2}{{v_1}^2\,T^2} \tag{4.64}$$

$$L_1 = \frac{1}{8}\boldsymbol{L}_{GB} = -\frac{1}{v_1\,v_2} + \frac{T^2\,{e_1}^2}{{v_1}^2\,v_2} \tag{4.65}$$

$$L_2 = \frac{1}{192}\boldsymbol{H}_{klm}\boldsymbol{H}^k{}_{pq}\boldsymbol{H}_r{}^{lp}\boldsymbol{H}^{rmq} = \frac{{e_2}^4}{2\,{v_1}^4\,T^4} \tag{4.66}$$

$$L_3 = -\frac{1}{16}\boldsymbol{R}_{klmn}\boldsymbol{H}_p{}^{kl}\boldsymbol{H}^{pmn} = \frac{{e_1}^2\,{e_2}^2}{{v_1}^4} - \frac{{e_2}^2}{{v_1}^3\,T^2} \tag{4.67}$$

$$L_4 = (\frac{b}{32} - \frac{1}{64})\,\boldsymbol{H}^2_{ij}\boldsymbol{H}^{2ij} = 6\,(b - \frac{1}{2})\frac{{e_2}^4}{{v_1}^4\,T^4} \tag{4.68}$$

$$L_5 = (\frac{1}{8} - \frac{b}{8})\boldsymbol{H}^2_{ij}\boldsymbol{R}^{ij} = 2\,(b-1)\,(\frac{{e_1}^2\,{e_2}^2}{{v_1}^4} - \frac{{e_2}^2}{{v_1}^3\,T^2}) \tag{4.69}$$

$$L_6 = (\frac{d}{12} - \frac{e}{6} - \frac{1}{96})\boldsymbol{R}\,\boldsymbol{H}^2 = h\,{e_2}^2\,(\frac{1}{v_1^3\,T^2} - \frac{{e_1}^2}{{v_1}^4} - \frac{1}{{v_1}^2\,v_2\,T^2}) \tag{4.70}$$

$$L_7 = \frac{d}{144}\,(\boldsymbol{H}^2)^2 = 4\,d\,\frac{{e_2}^4}{v_1^4\,T^4} \tag{4.71}$$

where we used h defined by $h = 4d - 8e - \frac{1}{2}$ to represent L_6 in a more convenient way. Inserting the above expressions in ten dimensional action we get

$$S = \boldsymbol{S} = \frac{1}{32\,\pi}\int dt\,dr\,d\phi\,d\cos\theta\;s\;v_1\;v_2\,(L_0 + \alpha'\sum_{i=1}^{7} L_i) + O(\alpha'^2, \alpha' CS), \tag{4.72}$$

where the integration over the compact space is understood. Then the entropy function follows

$$f(\vec{v}, \vec{e}, s, T) = \frac{1}{8}\,s\,v_1\,v_2(L_0 + \alpha'\sum_{i=1}^{7} L_i) \tag{4.73}$$

where we have truncated the α' series. Using (4.73) in (4.56)-(4.60) gives the equations of motion. The solution of the equations of motion identifies the horizon parameters. The identification of the near horizon geometry of half BPS backgrounds is an example of the supersymmetric attractor mechanism [144, 145], where the explicit equations of motion are solved rather than the supersymmetric constraints. Solving the equations of motion was first carried out by Ashoke Sen in [124] where only the Gauss-Bonnet Lagrangian was included in the induced action. The Gauss-Bonnet Lagrangian in four dimensions reads

$$\frac{1}{8}(R_{ijkl}R^{ijkl} - 4R_{ij}R^{ij} + R^2) = -\frac{1}{v_1\,v_2} \tag{4.74}$$

which coincided with the first term in L_1. We see that in total five terms in the the summation of $L_1 + \cdots + L_7$ are not reproduced by the inclusion of the four-dimensional Gauss-Bonnet Lagrangian.

A linear combination of the equations of motion of T and of v_1 factorises

$$\frac{\partial f}{\partial s} = 0 \rightarrow f = 0, \tag{4.75}$$

$$\left(\frac{1}{T}\frac{\partial f}{\partial T} - 4{e_1}^2\frac{\partial f}{\partial v_1}\right)\Big|_{f=0} = \left(T^2{e_1}^2 - \frac{{v_1}^2}{4}\right)(\ldots)\,. \tag{4.76}$$

Eq. (4.76) implies that some of the solutions may be given by

$$e_1 = \frac{\sqrt{v_1}}{2T}\,. \tag{4.77}$$

Eq. (4.77) simplifies the equations of motion of v_1, v_2, s and T and enables one to solve them,

$$v_1 = (3 + h\,x^2)\,\frac{\alpha'}{8}\,, \tag{4.78}$$

$$\frac{v_2}{v_1} = \frac{4(1 + h\,x^2)}{-h\,x^4 + (3h + 4b - 5)x^2 + 15}\,, \tag{4.79}$$

$$s = \sqrt{\frac{x\,N\,W}{v_1}}\,\frac{h\,x^4 + 1}{3 + (b-1)\,x^2}\,\frac{v_1}{v_2} \tag{4.80}$$

$$T = \sqrt{\frac{N}{W\,x}} \tag{4.81}$$

$$e_2 = \frac{1}{2}\sqrt{v_1}\,x\,T\,, \tag{4.82}$$

where x is a root of

$$\left(-4\,d - 6\,b - h + \frac{5}{2}\right)x^4 - 6\,(1 - b)\,x^2 + 9 = 0\,, \tag{4.83}$$

Note that we used x as a different parametrisation of b, d, h to express the near horizon configuration in a more convenient way. Eq's (4.77)-(4.82) identify the near horizon configuration. We use the entropy formula of entropy formalism [113, 120, 121] to calculate the Wald entropy associated to the local horizon. The entropy formalism expresses the Wald entropy, S_{BH}, by

$$S_{BH} = 2\pi\left(\frac{\partial f}{\partial e_1}e_1 + \frac{\partial f}{\partial e_2}e_2 - f\right), \tag{4.84}$$

which is evaluated on the horizon. We can use (4.56)-(4.60) to write

$$S_{BH} = 2\,\pi\left(\frac{N}{2}\,e_1 + \frac{W}{2}\,e_2\right) = \pi\,\sqrt{N\,W\,x\,v_1} \tag{4.85}$$

where we used the local horizon parameters (4.77), (4.81) and (4.82). We see that both the local horizon parameters and the entropy depend on the a priori ambiguous parameters. We have expected this dependence since we have applied the Wald entropy formula on the exact solution of the truncated action. The equality of the statistical entropy (4.29) and the Wald entropy (4.85) happens in the schemes where

$$x\, v_1 \;=\; \alpha' \tag{4.86}$$

provided that the gravitational Chern-Simons terms can be excluded. In order to elaborate the local horizon in more details we exclude the gravitational Chern-Simons terms and we select the schemes given by (4.86). There exist a set of ranges for the parameters of the field redefinition ambiguity where v_1, v_2, T, s are all positive. It is straightforward to identify these ranges. Here we focus on the subset of the parameters where identity is a root of (4.83) or equivalently $h = -4d+\frac{11}{2}$. In this subset T-duality in the compactified direction remains trivial in the sense that interchanging N and W describes T-duality both at asymptotic infinity and near the horizon. Then using (4.86) for $x = 1$ fixes d to $d = \frac{1}{8}$ for which the near horizon configuration is simplified to

$$\begin{aligned}
v_1 &= 16, && (4.87)\\
\frac{v_2}{v_1} &= \frac{6}{5}, && (4.88)\\
T &= \sqrt{\frac{N}{W}}, && (4.89)\\
e_1 &= 2\sqrt{\frac{W}{N}}, && (4.90)\\
e_2 &= 2\sqrt{\frac{N}{W}}, && (4.91)\\
s &= \frac{5}{8}\sqrt{N\,W}, && (4.92)
\end{aligned}$$

and we have chosen $b = 0$ and used the unit of $\alpha' = 16$. We see that $(\frac{v_1}{\alpha'}, \frac{v_2}{\alpha'}) \sim (1,1)$, and the stretched horizon is not larger than α'. We can choose other values for the field redefinition ambiguity parameters to make the local horizon arbitrarily large. For example we can choose $x = \frac{1}{2}, b = 0, h = 52, d = \frac{141}{8}$ to get

$$\begin{aligned}
v_1 &= 2\,\alpha', && (4.93)\\
v_2 &= \frac{224}{99}\,\alpha', && (4.94)
\end{aligned}$$

$$T = \sqrt{2\frac{N}{W}}, \tag{4.95}$$

$$e_1 = \frac{1}{2}\sqrt{\frac{\alpha' W}{N}}, \tag{4.96}$$

$$e_2 = \frac{1}{2}\sqrt{\frac{\alpha' N}{W}}, \tag{4.97}$$

$$s = \frac{9}{4}\sqrt{\frac{N W}{\alpha'}}, \tag{4.98}$$

for which one can argue that the higher order α' corrections are suppressed outside the horizon and the higher order α' corrections only provide perturbations around the "black hole". This shows that there exist schemes in which the Wald entropy for a black hole is in agreement with the statistical entropy and $(\frac{v_1}{\alpha'}, \frac{v_2}{\alpha'}) >> (1,1)$, therefore the higher order α' corrections could be ignored outside the stretched horizon within these schemes. Also within these schemes the gravitational Chern-Simons terms could be studied as perturbations outside the black hole. However we notice that the values of the field redefinition parameters are not small in these schemes. For the case of the WZW models where the exact conformal theory is known the values of the field redefinition ambiguity in which the background fields retain their forms are of order one [146]. Thus it is unlikely that very large values for the field redefinition ambiguity parameters are going to be chosen by the underlying conformal field theory. This suggests that higher order α' corrections can not be totally ignored outside the stretched horizon in physically acceptable schemes, however it allows for some "physically acceptable" schemes in which the higher order α' corrections contribute to the thermodynamical entropy in a perturbative way.

Note that there exist field redefinition ambiguities which vanish near the horizon and infinity. The class of field redefinitions that leave the equations of the metric and NS two-form as second order differential equations is

$$T_{ij} = c_1 \nabla_i \nabla_j \phi + c_2 g_{ij} \Box\phi + c_3 \nabla_i\phi \nabla_j\phi + c_4 g_{ij} |\nabla\phi|^2 \tag{4.99}$$

$$X = c_5 \Box\phi + c_6 |\nabla\phi|^2 \tag{4.100}$$

where $c_1, c_2, \cdots, c_6$ are arbitrary real numbers. Ref. [124, 147] have looked for a numerical interpolating solution in one single set of the a priori ambiguity parameters. One should study if there exists any set of values for $b, d, e, f, c_1, \cdots, c_6$ for which a smooth solution interpolates from the near horizon geometry to infinity. This question needs further investigation, however due to the large number of free parameters it is tempting to argue that the interpolating solution exists in general. If we knew the interpolating solution in an arbitrary scheme then we would treated the gravitational Chern-Simons terms as perturbation around the interpolating solution to compute how the gravitational Chern-Simons contributions to the entropy. Finding the interpolating solution requires a further investigation.

4.5 Summary: geometry of a wrapped F-string?

We have studied the linear α' corrections and the field redefinition ambiguities in the critical Heterotic String Theory for the backgrounds representing a fundamental string wrapped around a two cycle.

We have required the α' corrections to the Einstein tensor to be covariantly divergence free. This requirement has enabled us to rewrite the square of the Riemann tensor as the Gauss-Bonnet Lagrangian keeping some of the field redefinition ambiguity parameters untouched. One may ask if this requirement, similar to the ghost-freedom criterion [148], could be applied to all orders in α'. This question needs further investigation.

Having excluded the gravitational Chern-Simons terms, we have shown that there exist schemes in which the α' stretched horizon is large and the Wald entropy is comparable with the statistical entropy. Thus the higher order α' corrections are perturbative outside the stretched horizon within these schemes. Also we have argued that a smooth solution connects the α' stretched horizon to the fall off of the fields at asymptotic infinity. The gravitational Chern-Simons terms outside the stretched horizon can be studied as perturbation around the interpolating solution. If we knew the interpolating solution in a general scheme then we could have computed the Chern-Simons contributions to the entropy and we could have preferred the schemes in which the statistical entropy is in agreement with the thermodynamical entropy.

This means that there exist schemes in which the α' stretched horizon is small and also there exist schemes where the α' stretched horizon does not exist at all. We do not know which scheme would be preferred or chosen by the underlying conformal field theory since it is not known what type of a conformal field theory (nor if a unique one) represents a wrapped fundamental string. Ref. [12, 13]

shows that there exists a scheme in which the fields of the fundamental string background retain their forms in the supergravity approximation, thus within this scheme the background remains as a null singular background under the inclusion of all α' corrections. We have concluded from this that the α' expansion series is not an absolutely convergent series on the α' stretched horizon whenever the scheme admits the α' stretched horizon.

It would be intersting to apply the MM-criterion on the linear α' corrected action in the presence of the gravitational Chern-Simons terms and to study if the MM-criterion allows for a solution for which the statistical entropy is in agreement with the thermodynamical entropy.

Although we have argued on the existence of the schemes in which the α' stretched horizon is larger than the string length and for which the statistical entropy is in agreement with the Wald entropy, still we find it disturbing that the the thermodynamical entropy is scheme-dependent. The fact that the α' series on the α' stretched horizon is not an absolutely convergent series adds to this disturbing problem. These difficulties indicate that the thermodynamical properties should be expressed in term of other geometrical properties of the null singular geometries rather than requiring the subleading corrections to convert the null singular backgrounds to black holes with a regular event horizon.

We would like to point out that Mathur and Lunin's description for the entropy [14] may be employed to generate a thermodynamical entropy for a wrapped fundamental string without first requiring the α' corrections to produce an event horizon covering the singularity.

Chapter 5

Dyons

5.1 Introduction

Dyons carry both electric and magnetic charges. Dyonic black holes are black holes which carry electric charges and magnetic charges of some gauge fields. Some of the dyonic black holes can be realised as the solutions of the supergravity approximation to the critical Heterotic String Theory compactified on T^6. There exists a proposal for the exact degeneracy of microsates of dyons in torodially compactified critical Heterotic String Theory [149, 150, 151, 152, 153, 154, 155]. The logarithm of the degeneracy of dyons defines the statistical entropy.

In the supergravity approximation the Hawking-Bekenstein entropy is in agreement with the large-charge-limit of the statistical entropy. The dominant string corrections to the dyons are the α' corrections. Thus the α' corrections to the thermodynamical entropy for the dyonic black hole should be in agreement with the large charge expansion series of the statistical entropy. Ref [156, 103, 104, 157, 106, 107, 158, 159] observed that upon the inclusion of the square of the Riemann tensor and a supersymmetric completion of that, the modified Hawking-Bekenstein entropy [8, 9] is in agreement with the statistical entropy. Ref [160] showed that the inclusion of the Gauss-Bonnet action gives the same corrections in modified Hawking-Bekenstein entropy as those given by the inclusion of supersymmetric version of the square of the Riemann tensor.

The Gauss-Bonnet Lagrangian or the supersymmetric version of square of the Riemann tensor are not all the linear α' corrections to the dyonic black holes. It remains unanswered why other linear α' terms should not contribute to the modified Hawking-Bekenstein entropy. In this chapter we consider a BPS static spherical four dimensional dyonic black hole representing a wrapped fundamental string carrying arbitrary winding and momentum charges along one cycle in the presence of KK-monopole and H-monopole charges associated to another cycle. Then we compute all the linear α' corrections in the modified Hawking-Bekenstein entropy [8, 9] for this dyon. This chapter is organised in the following way:

In the second section we consider the Low Energy Effective Action of the Heterotic String Theory. We study a KK-compactification of the Heterotic String Theory on T^6 relevent for a BPS static spherical four dimensional dyonic black hole representing a wrapped fundamental string carrying arbitrary winding and momentum charges along one cycle in the presence of KK-monopole and H-monopole charges of another cycle [161].

In the third section we apply the compactification process of the second section to account for all the linear α' corrections in the induced action. We study the α' corrections as perturbations outside the horizon. We notice that for a general black hole requiring a smooth α' perturbation on the horizon may alter the charges of the black hole. The attractor equations and the entropy formalism do not answer if (and how much) the charges are corrected. Therefore generically the attractor equations [144, 145] and the entropy formalism [113, 121] do not suffice to express the parameters of the horizon configuration in terms of the values of the charges in the supergravity approximation. We show that the charges of a dyonic black hole retain their values in the supergravity approximation because there exists a scheme in which the fields in the supergravity approximation do not receive any α' corrections [13, 12].

In the fourth section we divide the induced action to the gravitational Chern-Simons terms and the rest of the terms. We evaluate the induced action near the horizon configuration. We employ the attractor mechanism [144, 145] and the entropy formalism [113, 121] to calculate the modified Hawking-Bekenstein entropy when the gravitational Chern-Simons terms are excluded. Then we will see that agreement between the statistical entropy and thermodynamical entropy requires taking into account the gravitational Chern-Simons terms.

In the last section we summarise and discuss the results.

5.2 Dyonic black holes and the α' corrections

In the second section of the previous chapter we have studied the low energy effective action for the compactification of the Heterotic string backgrounds composed of the metric, the NS two form and dilaton on T^{10-D}. In this chapter we consider such a compactification on T^6 where only the $U(1)$ gauge fields associated to two cycles are not trivial. Let y_1 and y_2 represent cycles then the ten dimensional background reads

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu + \sum_{m=1}^{2} T_m(x)(dy^m + 2A_\mu^m dx^\mu)^2 + dz^2 \tag{5.1}$$

$$B = B_{\mu\nu}(x)dx^\mu \wedge dx^\nu + 2\sum_{m=1}^{2} A_\mu^{2+m} dx^\mu \wedge (dy^m + 2A_\nu^m dx^\nu) \tag{5.2}$$

$$\phi = \phi(x) \tag{5.3}$$

where A_μ^1, A_μ^2, A_μ^3 and A_μ^4 represents four distinct gauge connections in the four dimensions $x^\mu = (t, r, \theta, \phi)$. At the level of the four dimensional supergravity approximation we consider a static spherical dyonic black hole which carries electric charges of A^1 and A^3 and magnetic charges of A^2 and A^4 in an asymptotically flat space-time. The electric charges of A^1 and A^3 represent respectively the KK-momentum and winding numbers of a fundamental string wrapped around the cycle of y_1. The magnetic charges of A^2 and A^4 represent respectively the KK-monopole and the H-monopole charges associated to the cycle of y_2. The explicit forms of the fields for this dyonic black hole are presented in [161]. When none of the charges is zero then the dyonic black hole has a regular horizon with geometry of $AdS_2 \times S^2$ outside which the string loop corrections can be ignored. In the supergravity approximation the $SO(6,22) \times SL(2,Z)$ duality transformations can be applied on the dyonic black hole to obtain a general dyonic black hole [162]. Recalling the α' correction to T-duality of third chapter, we expect that the duality transformations themselves get modified by the α' corrections. We do not use the duality transformations. We consider a dyonic black hole with large momentum, winding, KK-monopole and H-monopole charges and we study the α' corrections as perturbations on and outside its horizon.

We use Ψ_i to represent all the fields of the dyonic black hole in a collective fashion[1],

$$\Psi_i \in \{g_{\mu\nu}(x), T_1(x), T_2(x), A_\mu^1(x), \cdots, A_\mu^4(x), \phi(x)\} . \tag{5.4}$$

[1]This collective notation is constructed in analogy with the compact notation used in [163].

The action for this collective notation follows

$$S[\Psi] \;=\; S^0[\Psi] + \alpha' S^1[\Psi] + O(\alpha'^2)\,, \qquad (5.5)$$

the equations of motion of which read

$$0 = \frac{\delta S[\Psi]}{\delta \Psi_i} \;=\; \frac{\delta S^0[\Psi]}{\delta \Psi_i} + \alpha' \frac{\delta S^1[\Psi]}{\delta \Psi_i} + O(\alpha'^2)\,, \qquad (5.6)$$

where $\frac{\delta}{\delta \Psi_i}$ stands for the functional derivative respect to Ψ_i. We write an α' expansion series for Ψ_i and we solve (5.6) perturbatively,

$$\Psi_i \;=\; \Psi_i^0 + \alpha' \Psi_i^1 + O(\alpha'^2)\,. \qquad (5.7)$$

Inserting this perturbative expansion in (5.6) gives

$$\frac{\delta S^0[\Psi]}{\delta \Psi_i}\Big|_{\alpha'=0} + \alpha' \Big(\frac{\delta^2 S^0[\Psi]}{\delta \Psi_i \delta \Psi_j}\Big|_{\alpha'=0} \Psi_j^1 + \frac{\delta S^1[\Psi]}{\delta \Psi_i}\Big|_{\alpha'=0} \Big) + O(\alpha'^2) \;=\; 0\,, \qquad (5.8)$$

which implies

$$\frac{\delta S^0[\Psi]}{\delta \Psi_i}\Big|_{\alpha'=0} \;=\; 0\,, \qquad (5.9)$$

$$\frac{\delta^2 S^0[\Psi]}{\delta \Psi_i \delta \Psi_j}\Big|_{\alpha'=0} \Psi_j^1 \;=\; -\frac{\delta S^1[\Psi]}{\delta \Psi_i}\Big|_{\alpha'=0}\,. \qquad (5.10)$$

Note that (5.9) stands for the equations of motion in the supergravity approximation and (5.10) gives a set of non-homogeneous linear second order differential equations for $\{\Psi_i^1\}$ for any given solution in the supergravity approximation $\{\Psi_i^0\}$.

Let us first study the solutions to the homogeneous equations which correspond to (5.10),

$$\frac{\delta^2 S^0[\Psi]}{\delta \Psi_i \delta \Psi_j}\Big|_{\alpha'=0} \Psi_{j,B}^1 \;=\; 0\,. \qquad (5.11)$$

These are the equations describing the fluctuations around $\Psi = \Psi_0$ in the supergravity approximation. For the dyonic black hole, the equation for a static spherical fluctuation of the dilaton in the canonical frame is simplified to

$$\partial_r((r - r_H)^2 \partial \phi_B^1) \;=\; 0\,, \qquad (5.12)$$

where r_H is the radius of the horizon and ϕ_B^1 is the fluctuation of the dilaton and we have used the explicit form of the background fields presented in [161]. The general solution of (5.12) is diverging on the horizon,

$$\phi_B^1(r) \;=\; \frac{c_1}{r - r_H} + c_2\,. \qquad (5.13)$$

The diverging mode of the dilaton fluctuations plays the role of the diverging source for the fluctuations of all other fields through their couplings to the dilaton. Thus the fluctuations of all other fields admit modes which diverge on the horizon. Therefore we conclude that

Lemma 1: The general solutions to the homogeneous equations (5.11) diverge on the horizon.

The diverging solutions on the horizon should be excluded by the boundary conditions. We impose the following boundary conditions on the solutions of (5.10)

$$\begin{cases} \Psi_i^1(x)\big|_{x=\infty} &= 0\,, \\ \Psi_i^1(x)\big|_{x \text{ on the Horizon}} &< \infty\,, \end{cases} \tag{5.14}$$

we refer to which as the H-boundary conditions. The first condition of the H-boundary conditions set the α' corrections to zero at infinity and its second condition excludes the diverging modes on the horizon. Depending on how we decide to represent the metric, some of the components of the metric may diverge on the horizon in the supergravity approximation. For these components of the metric we substitute the second condition of the H-boundary conditions by

$$\lim_{x\to\Sigma_h} \frac{\Psi_i^1(x)}{\Psi_i^0(x)} < \infty\,, \tag{5.15}$$

where Σ_h represents any point on the horizon. Because the α' corrections reaches their largest values on the horizon then having fixed the symmetries the H-boundary conditions guaranty that $\Psi_i^1(x)$ is bounded outside the horizon,

$$\Psi_i^1(x) \;<\; \infty\,,\; \forall |x| \in [r_H, \infty)\,. \tag{5.16}$$

Second order linear differential equations have two solutions. In general the H-boundary conditions exclude one of the solutions and identify the other one. There exists no further freedom to impose more constraints on the solutions. Thus we conclude that:

Lemma 2: The H-boundary conditions do not necessarily retain the fall off of the fields at asymptotic infinity.

Note that these lemmas are not in contradiction with supersymmetry. If we knew the α' corrections to the supersymmetric constraints then we could have used the supersymmetric constraints rather the equations of motions to obtain a set of non-homogeneous first order linear differential equations for the α' corrections to the background fields. Requiring the α' corrections to vanish at infinity fixes all the boundary conditions for these first order equations. Thus again we conclude that the fall off of the fields at asymptotic infinity might receive α' corrections. In addition we learn that the diverging modes on the horizon are non-supersymmetric fluctuations on the supersymmetric background.

For the Schwarzschild black hole, as has been showed in the second chapter, imposing the H-boundary conditions produces corrections to the Newtonian mass of the black hole which is given by the fall off of the time-time component of the canonical metric at asymptotic infinity. The fall off of the fields identifies the charges of the dyonic black hole. Thus the second lemma implies that the charges of the dyonic black hole might get modified by the α' corrections.

In the perturbative study of the string scattering amplitudes one is allowed to redefine the fields,

$$\tilde{\Psi}_i \;=\; \Psi_i + \alpha' R_i + O(\alpha'^2) \tag{5.17}$$

where R_i are tensors of appropriate degree and dimension constructed from polynomials of Ψ_i^0 and their derivatives. The field redefinition alters the induced action and subsequently the equations of motion derived from the action. For example a general field redefinition given by (5.17) changes the equations for the linear α' corrections (5.10) to,

$$\frac{\delta^2 S^0[\Psi]}{\delta\Psi_i\delta\Psi_j}\Big|_{\alpha'=0}\tilde{\Psi}_j^1 \;=\; -\frac{\delta S^1[\Psi]}{\delta\Psi_i}\Big|_{\alpha'=0} - \frac{\delta^2 S^0[\Psi]}{\delta\Psi_i\delta\Psi_j}\Big|_{\alpha'=0} R_i \,. \tag{5.18}$$

The field redefinition ambiguity is related to the freedom in choosing different renormalisation and regularisation schemes in the sigma model. Ref. [13] has considered the dyonic black as a generalisation of the null chiral sigma models [12] and has proved that there exists a scheme in which Ψ_i^0 does not receive any α' corrections. This means that there exists $R_i = R_i{}^\star$ for which the right hand side of (5.18) vanishes and $\tilde{\Psi}_j^1 = 0$ is the solution to (5.18). Thus the solution in the scheme where the α' corrections are given by (5.10) reads

$$\Psi_i \;=\; \Psi_i^0 - \alpha' R_i{}^\star + O(\alpha'^2)\,, \tag{5.19}$$

Since any field redefinition should contain two derivatives then the fall off of Ψ_i at infinity relevant for the charge identification is the same as the one of Ψ_i^0. We conclude that

Lemma 3: There exists no α' correction to the charges of the dyonic black hole.

5.3 The α' corrections to the entropy of dyons

The near horizon configuration of the dyonic black hole in the supergravity approximation is $AdS_2 \times S^2$. When the horizon is large the α' corrections do not change the geometry of the horizon. Therefore the near horizon configuration of the α' corrected dyonic black hole can be written in the following way

$$ds^2 = v_1(-r^2 d\tau^2 + \frac{dr^2}{r^2}) + v_2(d\theta^2 + \sin^2\theta d\phi^2), \quad (5.20)$$

$$T_1 = T_1 \, , \, T_2 = T_2 \, , \, e^{-2\Phi} = s \, , \quad (5.21)$$

$$F^1_{r\tau} = e_1 \, , \, F^3_{r\tau} = e_3 \, , \, F^2_{\theta\phi} = \frac{p^0_2}{4\pi}\sin\theta \, , \, F^4_{\theta\phi} = \frac{p^0_4}{4\pi}\sin\theta \, , \quad (5.22)$$

where the horizon is located at $r = 0$ and $v_1, \cdots, p^0_4$ are constant parameters labelling the horizon. Note that v_1 and v_2 are constant due to the geometry of the horizon and T_1, T_2, s are constant since they represent the limit $r \to 0$ of the scalars. e_1, e_3, p^0_2 and p^0_4 are constant due to the coordinates chosen to represent the background and in accordance with the supergravity approximation.

In the second section of the previous chapter we saw that the induced action can be partition the part which is the pull back of the ten-dimensional gravitational Chern-Simons terms into four dimension, S_{CS}, and the rest of the α' corrections S_{NCS}. Each of S_{CS} and S_{NCS} contributes to the entropy of a dyonic black hole. In this chapter we compute the contribution of S_{NCS} to the entropy of the black hole and we postpone computing the contribution of S_{CS} to future studies.

S_{NCS} is a functional of the gauge field strengths but not of the gauge fields themselves. Thus the entropy formalism techniques [113, 121] can be employed to express the parameters of the near horizon configuration in terms of the charges of the dyonic black hole. The entropy formalism utilises the entropy function defined by

$$f(\vec{v}, \vec{T}, \vec{e}, \vec{p}, s) = \frac{1}{32\pi}\int d\theta d\phi \sqrt{-\det g}\, s\, L(\vec{v}, \vec{T}, \vec{e}, \vec{p}) \, , \quad (5.23)$$

where $L(\vec{v}, \vec{T}, \vec{e}, \vec{p})$ is the induced Lagrangian (4.14) evaluated on the horizon configuration when

$\boldsymbol{H} = dB$. The equations of motions are equivalent to

$$\begin{aligned}\frac{\partial f(\vec{v},\vec{T},\vec{e},\vec{p},s)}{\partial v_i} &= 0\,,\\ \frac{\partial f(\vec{v},\vec{T},\vec{e},\vec{p},s)}{\partial T_i} &= 0\,,\\ \frac{\partial f(\vec{v},\vec{T},\vec{e},\vec{p},s)}{\partial s} &= 0\,,\\ \frac{\partial f(\vec{v},\vec{T},\vec{e},\vec{p},s)}{\partial e_i} &= q_i\,,\ i\in\{1,3\}\end{aligned} \tag{5.24}$$

where q_1 and q_3 are the electric charges of the dyonic black hole and p_2 and p_4 are identified as the magnetic charges. In the following we first obtain these equations and next we find their α' perturbative solutions.

To evaluate the induced action on the horizon configuration we use (4.3), (4.6) and (4.13) to write the ten dimensional fields for the near horizon configuration,

$$ds^2 = ds^2 + T_1(dy_1 + 2\,e_1\,r\,dt)^2 + T_2(dy_2 - \frac{p_2}{2\pi}\cos\theta\,d\phi)^2 + dz_m^2\,, \tag{5.25}$$

$$\boldsymbol{B} = 2\,e_3\,r\,dt\wedge dy_1 - \frac{p_4}{2\pi}\cos\theta\,d\phi\wedge dy_2\,, \tag{5.26}$$

$$e^{-2\phi} = \frac{s}{V\,T_1\,T_2}\,, \tag{5.27}$$

where V is the volume of the compactified space and the gauges (5.22) are chosen by

$$A^1_\mu = [e_1\,r,0,0,0]\ ,\ A^3_\mu = [e_3\,r,0,0,0]\,, \tag{5.28}$$

$$A^2_\mu = [0,0,0,-\frac{p^0_2}{4\pi}\cos\theta]\ ,\ A^4_\mu = [0,0,0,-\frac{p^0_4}{4\pi}\cos\theta]\,. \tag{5.29}$$

In ten dimensions using the ten dimensional fields (5.25) and (5.26) one finds that

$$\begin{aligned} L_0 &= \boldsymbol{R}_{\text{Ricci}} - \frac{1}{12}\boldsymbol{H}_{ijk}\boldsymbol{H}^{ijk} = \\ &= -\frac{2}{v_1} + \frac{2}{v_2} + \frac{2{e_1}^2{T_1}^2}{{v_1}^2} + \frac{2{e_3}^2}{{v_1}^2{T_1}^2} - \frac{{p_2}^2{T_2}^2}{8{v_2}^2\pi^2} - \frac{{p_4}^2}{8{v_2}^2\pi^2{T_2}^2}\,,\end{aligned} \tag{5.30}$$

$$\begin{aligned} L_1 &= \frac{1}{8}\boldsymbol{R}_{klmn}\boldsymbol{R}^{klmn} = \\ &= +\frac{1}{2\,{v_1}^2} + \frac{1}{2\,{v_2}^2} - \frac{3{e_1}^2{T_1}^2}{{v_1}^3} - \frac{3{p_2}^2{T_2}^2}{16\,{v_2}^3\pi^2} + \frac{11{T_1}^4{e_1}^4}{2\,{v_1}^4} + \frac{11{p_2}^4{T_2}^4}{512\,{v_2}^4\pi^4}\,,\end{aligned} \tag{5.31}$$

$$\begin{aligned} L_2 &= -\frac{1}{16} R_{klmn} H_p{}^{kl} H^{pmn} = \\ &= -\frac{e_3{}^2}{v_1{}^3\, T_1{}^2} - \frac{p_4{}^2}{16\,\pi^2\, v_2{}^3\, T_2{}^2} + \frac{e_3{}^2\, e_1{}^2}{v_1{}^4} + \frac{p_4{}^2\, p_2{}^2}{256\,\pi^4\, v_2{}^4}\,, \end{aligned} \tag{5.32}$$

$$L_3 = -\frac{1}{64} H_k{}^{mn} H_{lmn} H^{kpq} H^l{}_{pq} = -\frac{3\,e_3{}^4}{v_1{}^4\, T_1{}^4} - \frac{3\,p_4{}^4}{256\,\pi^4\, v_2{}^4\, T_2{}^4}\,, \tag{5.33}$$

$$L_4 = \frac{1}{192} H_{klm} H^k{}_{pq} H_r{}^{lp} H^{rmq} = \frac{e_3{}^4}{2\,v_1{}^4\, T_1{}^4} + \frac{p_4{}^4}{512\,\pi^4\, v_2{}^4\, T_2{}^4}\,. \tag{5.34}$$

Inserting the above expressions in the ten dimensional action in the supergravity approximation and its linear α' corrections (2.55) gives

$$S = S = \frac{1}{32\pi} \int dt\, dr\, d\phi\, d\cos\theta \;\, s\, v_1\, v_2\, (L_0 + \alpha'(L_1 + L_2 + L_3 + L_4)) + O(\alpha'^2)\,, \tag{5.35}$$

where the integration over the compactified space has been done. Then the entropy function reads

$$f(\vec{v}, \vec{T}, \vec{e}, \vec{p}, s) = \frac{1}{8} s\, v_1\, v_2\, (L_0 + \alpha'(L_1 + L_2 + L_3 + L_4)) + O(\alpha'^2)\,, \tag{5.36}$$

and inserting this in (5.24) gives the equations of motion near the horizon. These equations should be solved perturbatively. Thus we write α' expansion series for the constant parameters labelling the horizon configuration,

$$\begin{aligned} v_i &= v_i{}^0\,(1 + \alpha'\, \tilde{v}_i + O(\alpha'^2))\,, \\ T_i &= T_i{}^0\,(1 + \alpha'\, \tilde{T}_i + O(\alpha'^2))\,, \\ e_i &= e_i{}^0\,(1 + \alpha'\, \tilde{e}_i + O(\alpha'^2))\,, \\ s &= s^0\,(1 + \alpha'\, \tilde{s} + O(\alpha'^2))\,. \end{aligned} \tag{5.37}$$

Note that the electric and magnetic charges retain their values if we add not only S_{NSC} but also S_{SC}. We are not including the gravitational Chern-Simons terms thus we should write α' expansion series for the electric charges

$$q_1 = q_1{}^0\,(1 + \frac{\alpha'}{p_2^0 p_4^0}\, \tilde{q}_1 + O(\alpha'^2))\,, \tag{5.38}$$

$$q_3 = q_3{}^0\,(1 + \frac{\alpha'}{p_2^0 p_4^0}\, \tilde{q}_3 + O(\alpha'^2))\,, \tag{5.39}$$

$$p_2 = p_2^0\,(1 + \frac{\alpha'}{p_2^0 p_4^0}\, \tilde{p}_2 + O(\alpha'^2))\,, \tag{5.40}$$

$$p_4 = p_4^0\,(1 + \frac{\alpha'}{p_2^0 p_4^0}\,\tilde{p_4} + O(\alpha'^2))\,, \tag{5.41}$$

where $\tilde{q_1}$, $\tilde{q_3}$, $\tilde{p_2}$, and $\tilde{p_4}$ are the corrections to the charges due to imposing the H-boundary conditions on the solutions. Note that we have to include the corrections to the charges since we are not including all the linear α' corrections and we are excluding the gravitational Chern-Simons corrections. The equations of motion (5.24) in the supergravity approximation ($\alpha' = 0$) are solved by

$$\begin{aligned} {v_1}^0 &= {v_2}^0 = \frac{p_2^0\,p_4^0}{4\,\pi^2}\,, \\ {T_1}^0 &= \sqrt{\frac{p_4^0}{p_2^0}}\,,\; {T_2}^0 = \sqrt{\frac{q_1^0}{q_3^0}}\,, \\ {e_1}^0 &= \frac{1}{4\pi}\sqrt{\frac{q_3^0\,p_2^0\,p_4^0}{q_1^0}}\,,\; {e_3}^0 = \frac{1}{4\pi}\sqrt{\frac{q_1^0\,p_2^0\,p_4^0}{q_3^0}}\,, \\ s^0 &= 8\,\pi\sqrt{\frac{q_1^0\,q_3^0}{p_2^0\,p_4^0}}\,. \end{aligned} \tag{5.42}$$

These are the horizon configuration parameters in the supergravity approximation. Inserting (5.37) and (5.42) in (5.24) gives a set of linear algebraic equations for the linear α' corrections (5.37) to the supergravity approximation (5.42). These linear equations are solved by

$$\tilde{v}_1 = 0\,, \tag{5.43}$$

$$\frac{1}{2}\tilde{v}_2 = \tilde{T}_2 = -\,\tilde{T}_1 = -\,\tilde{s} = \tilde{e}_1 = \tilde{e}_3 = \frac{\pi^2}{p_2^0\,p_4^0}\,. \tag{5.44}$$

The modified Hawking-Bekenstein (Wald) entropy is expressed by the Legendre transformation of the entropy function

$$S_{BH} = 2\,\pi\,(e_1\,\frac{\partial f(\vec{v},\vec{T},\vec{e},\vec{p},s)}{\partial e_1} + e_3\,\frac{\partial f(\vec{v},\vec{T},\vec{e},\vec{p},s)}{\partial e_3} - f(\vec{v},\vec{T},\vec{e},\vec{p},s))\,, \tag{5.45}$$

evaluated on the horizon configuration [113, 121]. Inserting (5.37) and (5.42) in (5.45) we get

$$S_{BH} = \sqrt{p_2^0\,p_4^0\,q_1^0\,q_3^0}\,(1 + \frac{\pi^2\,\alpha'}{p_2^0\,p_4^0} + \frac{\alpha'}{2p_2^0 p_4^0}\,(\tilde{q}_1 + \tilde{q}_3 + \tilde{p}_2 + \tilde{p}_4) + O(\alpha'^2)\,. \tag{5.46}$$

Note that (5.46) is computed by doing perturbation in $\frac{\alpha'}{p_2^0 p_4^0}$. Therefore we could not extrapolate (5.46) to the case of a wrapped fundamental string as an extremal dyon with zero-magnetic charges.

Therefore (5.46) being valid for large p_2^0 and p_4^0 is not necessarily in contradiction with the limit of vanishing p_2^0 and p_4^0 studied in the previous chapter.

In the following we would like to compare all the linear α' corrections to the entropy (5.46) with the corrections given by only the inclusion of the Gauss-Bonnet Lagrangian in the induced action [160]. In this part we do add the α' corrections to the charges. The square of the Riemann tensor of the four dimensional metric is

$$\frac{1}{8}\,R_{ijkl}\,R^{ijkl} \;=\; \frac{1}{2\,{v_1}^2} + \frac{1}{2\,{v_2}^2}\,. \tag{5.47}$$

We see that (5.47) coincides with the first two terms in L_1. In total ten terms in $L_1+L_2+L_3+L_4$ are not given by the square of the Riemann tensor. The Gauss-Bonnet Lagrangian in the four dimensions is

$$L_{GB} \;=\; R_{ijkl}\,R^{ijkl} - 4\,R_{ij}\,R^{ij} + R^2 = -\frac{8}{v_1\,v_2}\,. \tag{5.48}$$

Including the Gauss-Bonnet action to the induced action in the supergravity approximation is equal to including (5.47) and performing a field redefinition. Thus ten terms in the linear α' corrections to the induced action are not produced by the inclusion of the Gauss-Bonnet action. The inclusion of the Gauss-Bonnet action in the induced action in the supergravity approximation gives the following entropy function

$$f^\star \;=\; \frac{1}{8}\,s^\star\,{v_1}^\star\,{v_2}^\star\,(L_0 + \frac{\alpha'}{8}\,L_{GB}) = \frac{1}{8}\,s^\star\,{v_1}^\star\,{v_2}^\star\,L_0 - \frac{\alpha'}{8}\,s^\star + O(\alpha')\,, \tag{5.49}$$

where we used * to distinguish the near horizon parameters identified by (5.49) with those identified by (5.36). This entropy function (5.49) identifies the horizon configuration parameters to

$$\frac{{v_i}^\star}{{v_i}^0} = \frac{{e_i}^\star}{{e_i}^0} = 1 + \frac{2\,\pi^2\,\alpha'}{p_2^0\,p_4^0}\,,\ \frac{{T_i}^\star}{{T_i}^0} = 1\,,\ \frac{s^\star}{s^0} = 1 - \frac{2\pi^2\,\alpha'}{p_2^0\,p_4^0} \tag{5.50}$$

where $v_1^0, \cdots, s^0$ are given by (5.42) and for which the entropy reads

$$S_{GB} \;=\; \sqrt{p_2^0\,p_4^0\,q_1^0\,q_3^0}\,(1 + \frac{2\,\pi^2\,\alpha'}{p_2^0\,p_4^0}) + O(\alpha')\,, \tag{5.51}$$

Ref. [160] has included the Gauss-Bonnet action in the induced action in the supergravity approximation and has solved the corresponding truncated α'-corrected equations of motion exactly. We note that (5.51) is in agreement with the large charge expansion of eq. (3.13) of ref. [160] after setting $n = 2\,q_1^0$, $w = 2\,q_3^0$, $\tilde{N} = \frac{p_2^0}{4\pi}$, $\tilde{W} = \frac{p_4^0}{4\pi}$ and using the unit of $\alpha' = 16$.

We notice that (5.51) is not in agreement with (5.46) therefore we see that as long as we are excluding the gravitational Chern-Simons contribution to the entropy then the corrections are not in agreement with those reproduced by inclusion of only the Gauss-Bonnet Lagrangian to the Lagrangian density in the supergravity approximation.

The contribution of the gravitational Chern-Simons terms to the entropy within the framework of the entropy formalism or attractor equations. Also employing H-boundary conditions on the Chern-Simons α' corrections may alter the values of the charges of the dyonic black hole

$$q_1 = {q_1}^0 (1 + \frac{\alpha'}{p_2^0 p_4^0} \tilde{q_1}^\star + O(\alpha'^2)), \tag{5.52}$$

$$q_3 = {q_3}^0 (1 + \frac{\alpha'}{p_2^0 p_4^0} \tilde{q_3}^\star + O(\alpha'^2)), \tag{5.53}$$

$$p_2 = p_2^0 (1 + \frac{\alpha'}{p_2^0 p_4^0} \tilde{p_2}^\star + O(\alpha'^2)), \tag{5.54}$$

$$p_4 = p_4^0 (1 + \frac{\alpha'}{p_2^0 p_4^0} \tilde{p_4}^\star + O(\alpha'^2)), \tag{5.55}$$

where $\tilde{q_1}^\star$,...$\tilde{p_4}^\star$ are numbers and could be computed. Then the third lemma of the previous section implies that

$$\tilde{q_1} = -\tilde{q_1}^\star, \tag{5.56}$$

$$\tilde{q_3} = -\tilde{q_3}^\star, \tag{5.57}$$

$$\tilde{p_2} = -\tilde{p_2}^\star, \tag{5.58}$$

$$\tilde{p_4} = -\tilde{p_4}^\star, \tag{5.59}$$

Computing the Chern-Simons contributions to the entropy requires further investigation.

5.4 Conclusions

We studied all the linear α' corrections to the thermodynamical entropy for a four dimensional dyonic black hole carrying arbitrary momentum, winding, KK-monopole and H-monopole charges in the toroidal compactification of the Heterotic String Theory.

We have computed all the linear α' corrections, excluding however the gravitational Chern-Simons ones, to the entropy of a dyon. We have seen that the Chern-Simons gravitational contribution to the entropy should not vanish if the statistical entropy [149]-[155] is in agreement with thermodynamical entropy. Thus the gravitational Chern-Simons correction to the entropy of the dyonic black hole must be computed.

We have studied the α' corrections as perturbation on a given black hole geometry in an asymptotically flat space time. We have shown that in general the existence of smooth α' corrections on and outside the horizon requires a modification of the fall off of the fields at asymptotic infinity. Thus the charges may receive α' corrections. The attractor equations and the entropy formalism do not answer if (and how much) the charges are corrected. Therefore generically the attractor equations [144, 145] and the entropy formalism [113, 121] do not suffice to express the parameters of the horizon configuration in terms of the values of the charges in the supergravity approximation. We have shown that the charges of the dyonic black retain their values in the supergravity approximation.

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