Block Toeplitz matrices and integrable systems.

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Submitted in partial fulfillment of the
requirements for the degree
of Doctor of Philosophy.

September 15th, 2008.

Contents

Introduction.

This thesis deals with the connections between the theory of block Toeplitz matrices and integrable systems. Toeplitz matrices, both in the scalar and in the block case, appears in many different fields of mathematics and physics. Just to mention few aspects we are interested in, these matrices are used to compute some important quantities is physics (e.g. correlation functions) and they are an essential tool in the study of (bi)-orthogonal polynomials on the unit circle; they are connected with some important Riemann-Hilbert problems as well as with the theory of Fredholm determinants. Connection with the theory of integrable hierarchies (in the scalar case) had been established starting from the '90s; both continuous integrable hierarchies (modified KdV) and discrete ones (Ablowitz-Ladik) had been involved. The starting point was the study of the unitary matrix model (see below). Toeplitz matrices are, indeed, very simple objects. Given a function $\gamma(z)$ on the circle we denote $\gamma(z) = \sum_k \gamma^{(k)} z^k$ its Fourier expansion. The N truncated Toeplitz matrix $T_N(\gamma)$ with symbol γ is nothing but

$$T_N(\gamma) := \begin{pmatrix} \gamma^{(0)} & \dots & \dots & \gamma^{(-N)} \\ \gamma^{(1)} & \dots & \dots & \gamma^{(-N+1)} \\ \dots & \dots & \dots & \dots \\ \gamma^{(N)} & \dots & \dots & \gamma^{(0)} \end{pmatrix}. \tag{1}$$

We will denote $D_N(\gamma)$ its determinant. If the function $\gamma(z)$ is matrix valued we will speak about block Toeplitz matrices and block Toeplitz determinants. In recent years it has been shown how to compute (effectively) some physically relevant quantities (e.g. correlation functions) studying asymptotics of some block Toeplitz determinants (see [28],[29],[30]). In this way a connection has been established also with quantum spin chains ([28], [29]) and dimer models ([30]). From a mathematical point of view the study of the limit of Toeplitz and block Toeplitz determinants for large N has a long story. In 1952 Szegö gave a formula for asymptotics of $D_N(\gamma)$ in the scalar case [7]. This result has been generalized by H. Widom in the 70's ([8],[9] and [10]) for the matrix case; namely he proved that under suitable analytical assumptions it exists the limit

$$D_\infty(\gamma) := \lim_{N\to\infty} \frac{D_N(\gamma)}{G(\gamma)^N} = \det(T(\gamma)T(\gamma^{-1}))$$

where $G(\gamma)$ is a normalizing constant and the operator $T(\gamma)T(\gamma^{-1})$ is such that its determinant is well defined as a Fredholm determinant. One important difference between scalar and matrix case should be, nevertheless, underlined. While Szegö's formula is quite explicit its generalization to the matrix case is far from being explicit. The papers we cited ([28],[29] and [30]) are precisely devoted to solve this problem: finding explicit formulas for the Fredholm determinant $D_\infty(\gamma)$ (for some given γ). The difference between scalar case and matrix case arises from the non-commutativity of the algebra of symbols; this fact is evident in the works [28] and [29]. Infact in these articles the problem of computing $D_\infty(\gamma)$ is translated in finding certain Riemann-Hilbert factorizations of γ; i.e. in solving a matrix Riemann-Hilbert problem.
Once the asymptotics had been studied the next quite natural question was to find an expression relating directly $D_N(\gamma)$, and not just its asymptotics, to certain Fredholm determinants. The problem was solved many years later by Borodin and Okounkov in [11] for the scalar case and generalized, in the same year, for matrix case by E. Basor and H.Widom in [12]. For matrix valued case Borodin-Okounkov formula reads

$$D_N(\gamma) = D_\infty(\gamma)\det(I - K_{\gamma,N})$$

(here we assume $G(\gamma) = 1$). The operator $(I - K_{\gamma,N})$ can be written explicitely in coordinates knowing (again) certain Riemann-Hilbert factorizations of γ. Its Fredholm determinant is well defined. Now many proofs of Borodin-Okounkov formula are known (for instance [13] contains another proof of the same formula, see also the earlier paper [14]).

This thesis is based on the results published in [1] and [2]. In the first chapter, based on [1], we use block-Toeplitz determinants to compute the τ function of an (almost) arbitrary solution of Gelfand-Dickey hierarchy

$$\frac{\partial L}{\partial t_j} = [(L^{\frac{j}{n}})_+, L].$$

(L differential operator of order n, $j \neq nk$).
More precisely to a given point

$$W = \mathcal{W}(z) H_+^{(n)}$$

in the big cell of Segal-Wilson vector-valued Grassmannian we associate a $n \times n$ matrix-valued symbol $\mathcal{W}(t; z)$ obtained deforming $\mathcal{W}(z)$ (see formula (1.9)). In this way we define a sequence of N-truncated block Toeplitz determinants $\{\tau_{W,N}(t)\}_{N>0}$ which are shown to be solutions of certain rational reductions of KP; this is our

First result: *Every symbol $\mathcal{W}(t; z)$ defines through its truncated determinants a sequence $\{\tau_{W,N}(t)\}_{N>0}$ of solutions to* KP *such that*

$$\tau_{W,N}(t) \in \mathrm{cKP}_{1,nN} \cap \mathrm{cKP}_{n,n} \quad \forall N > 0.$$

Here we used the notation from [22]; given a τ function for KP with corresponding Lax pseudodifferential operator $\mathcal{L}$ we say that $\tau \in \mathrm{cKP}_{m,n}$ iff $\mathcal{L}^m$ can be written as the ratio of two differential operators of order $m+n$ and n respectively. This sequence admits a *stable limit* which is shown to be equal to the Gelfand-Dickey τ function $\tau_W(t)$ associated to W; this quantity can be computed using Szegö-Widom's theorem. This will give us the remarkable identity

$$\tau_W(\tilde{t}) = \det \left[\mathcal{P}_{\mathcal{W}(\tilde{t};z)} \right] \tag{2}$$

where $\mathcal{P}_{W(\tilde{t};z)}$ is the Fredholm operator appearing in the Szegö-Widom's theorem (here we put $\tilde{t}$ instead of t to remember that, when working with $W \in \mathrm{Gr}^{(n)}$, times t_{nj} multiple of n must be set to 0). Next step is the study of Riemann-Hilbert (also called Wiener-Hopf) factorization of symbol $\mathcal{W}(t; z)$ given by

$$\mathcal{W}(t; z) = T_-(t; z) T_+(t; z) \tag{3}$$

with T_- and T_+ analytical in z outside and inside S^1 respectively and normalized as

$$T_-(\infty) = I.$$

Here we assume that the symbol can be extended to an analytic function in a neighborhood of S^1. Using Plemelj's work [15] we show that $T_-(\tilde{t}; z)$ must satify the integral equation

$$\mathcal{P}^{\mathrm{T}}_{\mathcal{W}(\tilde{t};z)} T_-(\tilde{t}; z) = I \tag{4}$$

and we write a solution of (3) in terms of the wave function $\psi_W(\tilde{t}; z)$ corresponding to W.

In this way we arrive at our second result:

Second result: *Take $W \in \mathrm{Gr}^{(n)}$ in the big cell; denote $\tau_W(\tilde{t})$ its corresponding τ function .*
Then $\tau_W(\tilde{t})$ is equal to the Fredholm determinant of the homogeneous integral equation

$$\mathcal{P}^{\mathrm{T}}_{\mathcal{W}(\tilde{t};z)} T_{-}(\tilde{t};z) = 0 \tag{5}$$

related to Riemann-Hilbert problem (3). The solution of this Riemann-Hilbert problem is unique for every value of parameters $\tilde{t}$ such that

$$\tau_W(\tilde{t}) \neq 0$$

and can be computed by means of related wave function $\psi_W(\tilde{t};z)$.

At the end of the chapter we consider a particular class of symbols $\mathcal{W}(t;z)$ corresponding to algebro-geometric solutions of Gelfand-Dickey hierarchies. We formulate an alternative Riemann-Hilbert problem equivalent to (3) and explain how to solve it using θ-functions. In this way we give concrete formulas for a wide class of symbols that do not have half truncated Fourier series. This is quite remarkable since concrete results for non half truncated symbols were available, till now, only for the concrete cases presented in [28] and [29].
The sections of the first chapter are organized as follows:

- First section states some results about Segal-Wilson Grassmannian and related loop groups we will need in the sequel; proofs can be found in [3] and [4].

- Second section states Szegö-Widom's theorem and related results obtained by Widom in [8],[9] and [10] and the Borodin-Okounkov formula for block Toeplitz determinant [12].

- In the third section we introduce and study the sequence of truncated determinants $\{\tau_{W,N}(t)\}_{N>0}$ and its stable limit $\tau_W(t)$. We want to remark that the property of stability was stated for the first time in [17] (see also [18]). Our sequence is actually a subsequence of the stabilizing chain studied in [19]; nevertheless, to our best knowledge, [1] is the first article in which block Toeplitz determinants enter the game. Also the observation that $\tau_{W,N} \in \mathrm{cKP}_{n,n}$ is something new.

- Fourth section is devoted to establishing the connection between integral equations formulated by Plemelj in [15] and Fredholm operator appearing in Szegö-Widom's theorem.

- In the fifth section we show how to write Riemann-Hilbert factorization of $\mathcal{W}(\tilde{t};z)$ in terms of wave function $\psi_W(\tilde{t};z)$. Of course the relationship between Gelfand-Dickey hierarchy and the factorization problem is something well known; our exposition here is closely related to [16]. Moreover, knowing Riemann-Hilbert factorization of $\mathcal{W}(\tilde{t};z)$, we can apply Borodin-Okounkov formula to give an expression of any $\tau_{W,N}(\tilde{t})$ as Fredholm determinant and a recursion relation to go from $\tau_{W,N}(\tilde{t})$ to $\tau_{W,N+1}(\tilde{t})$.

- Last section gives explicit formulas for symbols and τ functions associated to algebro-geometric rank one solutions of Gelfand-Dickey hierarchies. Also we formulate an alternative Riemann-Hilbert problem equivalent to (3) in analogy with what has been done in [28] and [29]. We explain how to solve it using θ-functions.

The second chapter (based on [2]) extends to the matrix case the link existing between biorthogonal polynomials on the unit circle and Ablowitz-Ladik hierarchy. Before going into details let us recall some basic facts about the unitary matrix model; we will use it to introduce some fondamental notions we used in the second chapter. Unitary matrix model is nothing but the study of the one matrix integral

$$Z_N := \int_{U_N} \exp\Big[-\mathrm{Tr}V(U)\Big]dU.$$

Z_N will be called the partition function of the model. Here we are integrating on the group of $(N \times N)$ unitary matrices, dU is the standard Haar measure on $U(N)$ and $V(U) := \sum_{k\in\mathbb{N}} t_k U^k + s_k U^{-k}$. We will always restrict to the case where all but a finite number of parameters $\{t_k, s_k, k \geq 1\}$ are zero. The standard method for computing this integral is to reduce it to a multiple integral over the eigenvalues of U as explained by Mehta in his book [50]. Namely we have the formula

$$Z_N = K(N) \oint \dots \oint |\Delta(z_j)|^2 \exp\Big[\sum_{j=1}^{N} -\mathrm{Tr}V(z_j)\Big] \prod_{j=1}^{N} \frac{dz_j}{2\pi i z_j}$$

where $K(N)$ depends on the size of the unitary group but does not depend on the potential V. $\Delta(z_j)$ is the standard Vandermonde determinant

$$\Delta(z_j) := \det(z_j^{i-1})_{i,j=0,\dots N}.$$

Here biorthogonal polynomials on the unit circle (BOPUC) enter the game. We start introducing the space V_N of polynomials of order less or equal to

N equipped with the bilinear form $< .,. >$ defined by

$$< P(z), Q(z) >:= \oint P(z)Q(z^{-1}) \exp(-V(z)) \frac{dz}{2\pi i z}.$$

Then we introduce two sets of biorthogonal polynomial $\{p_i^{(1)}(z) : i \geq 0\}$ and $\{p_i^{(2)}(z) : i \geq 0\}$ such that

$$< p_i^{(1)}(z), p_j^{(2)}(z) >= \oint p_i^{(1)}(z) p_j^{(2)}(z^{-1}) \exp(-V(z)) \frac{dz}{2\pi i z} = h_j \delta_{ij}$$

and such that every $p_i^{(1)}(z), p_i^{(2)}(z)$ is monic of order i. It is easy to prove that

$$Z_N = K(N) \prod_{j=0}^{N} h_j.$$

On the other hand $\prod_{j=0}^{N} h_j$ is the determinant of the matrix representing the bilinear form $<,>$ with respect to the bases given by $\{p_i^{(1)}(z), i = 0 \ldots N\}$ and $\{p_i^{(2)}(z), i = 0 \ldots N\}$; i.e.

$$\prod_{j=0}^{N} h_j = \det \Big(< p_i^{(1)}, p_j^{(2)} > \Big)_{i,j=0 \ldots N}.$$

Now, since every $p_i^{(s)}$ is monic of order i, we have the equality

$$\det \Big(< p_i^{(1)}, p_j^{(2)} > \Big)_{i,j=0 \ldots N} = \det \Big(< z^i, z^j > \Big)_{i,j=0 \ldots N}.$$

Denoting $\exp(-V(z)) = \sum_k \mathcal{V}^{(k)} z^k$ the Fourier expansion of $\exp(-V(z))$ we finally obtain

$$Z_N \propto \det \begin{pmatrix} \mathcal{V}(0) & \ldots & \ldots & \mathcal{V}^{(-N)} \\ \mathcal{V}^{(1)} & \ldots & \ldots & \mathcal{V}^{(-N+1)} \\ \ldots & \ldots & \ldots & \ldots \\ \mathcal{V}^{(N)} & \ldots & \ldots & \mathcal{V}^{(0)} \end{pmatrix}$$

which is precisely, by definition, $D_N(\exp(-V(z)))$.
From the point of view of integrable systems both Z_N and its limit for large N are of interest. The double scaling limit of Z_N is related to the string solutions of modified KdV (see for instance [27] and references therein) while,

for finite N, the sequence of Z_N gives a solution for a particular reduction of 2D-Toda lattice, the Toeplitz lattice introduced by Adler and van Moerbeke in [41]. The same hierarchy (which is infact equivalent to the already known Ablowitz-Ladik hierarchy, [33]) has been studied from the point of view of orthogonal polynomials on the unit circle by Irina Nenciu in [45]. Also relations between unitary matrix model, biorthogonal polynomials on the unit circle, Painlevé-type and discrete Painlevé equations should be mentioned (see for instance [42], [43], [41] and references therein.)
AL hierarchy has been introduced in 1975 [33] as a spatial discretization of AKNS hierarchy. As described by Suris in [34] the idea of Ablowitz and Ladik consisted in substituting the celebrated Zakharov-Shabat spectral problem

$$\partial_x \Psi = L\Psi$$
$$\partial_\tau \Psi = M\Psi$$

with a discretized version of it; namely

$$\Psi_{k+1} = L_k \Psi_k$$
$$\partial_\tau \Psi_k = M_k \Psi_k.$$

Here Ψ, Ψ_k are two-component vectors while L, M, L_k and M_k are 2×2 matrices; in particular

$$L := \begin{pmatrix} z & x \\ y & -z \end{pmatrix}$$

and

$$L_k := \begin{pmatrix} z & x_k \\ y_k & z^{-1} \end{pmatrix}.$$

When dealing with Ablowitz-Ladik hierarchy one can consider, as usual, periodic case ($k \in \mathbb{Z}_n$), infinite case ($k \in \mathbb{Z}$) or semi-infinite case ($k \in \mathbb{N}$). Usual Lax equations for AKNS are replaced with semidiscrete zero-curvature equations

$$\partial_\tau L_k = M_{k+1} L_k - L_k M_k. \tag{6}$$

As an example, one of the most important equation of this hierarchy is the discrete complexified version of nonlinear Schrödinger

$$\begin{cases} \partial_\tau x_k = x_{k+1} - 2x_k + x_{k-1} - x_k y_k (x_{k+1} + x_{k-1}) \\ \\ \partial_\tau y_k = -y_{k+1} + 2y_k - y_{k-1} + x_k y_k (y_{k+1} + y_{k-1}). \end{cases} \tag{7}$$

Quite recently different authors (see [41] and [45]) underlined the link between biorthogonal polynomials on the unit circle (BOPUC) and semi-infinite Ablowitz-Ladik hierarchy. This is an analogue of the celebrated link between Toda hierarchy and orthogonal polynomials on the real line (see for instance [40]). In particular the approach of [41] allows to treat the case of Toda and Ablowitz-Ladik hierarchy in a similar way as reductions of 2D-Toda giving, in this way, a clear and unified explanation of the role played by orthogonal polynomials on these hierarchies. As noted by the authors of [41] their approach is quite different from the original one; actually in their paper they always speak about Toeplitz lattice and the coincidence with Ablowitz-Ladik hierarchy is just stated in the introduction. Nevertheless, as explained in the second section of this paper, it is very easy to deduce semidiscrete zero-curvature equations starting from Adler-van Moerbeke's equations. The main point consists in using the recursion relation

$$\begin{pmatrix} p_{n+1}^{(1)}(z) \\ \tilde{p}_{n+1}^{(2)}(z) \end{pmatrix} = \mathfrak{L}_n \begin{pmatrix} p_n^{(1)}(z) \\ \tilde{p}_n^{(2)}(z) \end{pmatrix} = \begin{pmatrix} z & x_{n+1} \\ zy_{n+1} & 1 \end{pmatrix} \begin{pmatrix} p_n^{(1)}(z) \\ \tilde{p}_n^{(2)}(z) \end{pmatrix} \tag{8}$$

for BOPUC (here $\tilde{p}_n^{(2)}(z) := z^n p_n^{(2)}(z^{-1})$) as discrete Lax operator for the Ablowitz-Ladik hierarchy and make these biorthogonal polynomials evolve according to 2D-Toda flow. Indeed $\mathfrak{L}_k$ looks very similar to L_k and, infact, we can go from one to the other as explained in [46]. In this setting the already known relationship between BOPUC and Ablowitz-Ladik hierarchy can be stated as follows. Let's consider a symbol

$$\gamma(z) = \sum_k \gamma^{(k)} z^k$$

without any analytical assumption (i.e. γ is a formal series). We deform it with some parameters $\{t_i, s_i, i \geq 0\}$ in this way:

$$\gamma(t, s; z) := \exp\left(\sum_{i \geq 1} t_i z^i\right) \gamma(z) \exp\left(-\sum_{i \geq 1} s_i z^{-i}\right).$$

As we did for the particular case of the unitary matrix integral we can consider in this more general case the BOPUC associated to the product $< .,. >_\gamma$ defined by

$$< .,. >_\gamma := \oint P(z)\gamma(t, s; z) Q(z^{-1}) \frac{dz}{2\pi i z}$$

(here we consider, as γ is a general formal series, the formal residue, i.e. the right hand side is nothing but the term in z^{-1} of the expression inside the symbol of integral). We will call these biorthogonal polynomials time dependent to underline that they depend on parameters $\{t_i, s_i, i \geq 0\}$. Of course also the related recursion operators $\mathfrak{L}_n$ appearing in (8) will depend on the same parameters. Infact we proved the following

First result: *Consider time-dipendent BOPUC* $\{p_n^{(1)}(z), p_n^{(2)}(z)\}$ *with respect to the pairing*

$$< P(z), Q(z) >_\gamma := \oint P(z)\gamma(t,s;z)Q(z^{-1})\frac{dz}{2\pi i z}$$

with $\gamma(t,s;z) := \exp(\xi(t,z))\gamma(z)\exp(-\xi(s,z^{-1}))$. *Then the related recursion operator (8) evolves according to semidiscrete zero-curvature equations (6) for the Ablowitz-Ladik hierarchy.*

Then, having this approach in mind, we addressed a question arising from the following facts:

- Time evolution for orthogonal polynomials on the real line (OPRL) leads to Toda hierarchy.
- Time evolution for biorthogonal polynomials on the unit circle leads to Ablowitz-Ladik hierarchy.
- Time evolution for matrix orthogonal polynomials on the real line leads to non-abelian Toda hierarchy.

What about time evolution for matrix biorthogonal polynomials on the unit circle?

In other words our goal was to replace the question mark in the table below with the corresponding hierarchy.

	OPRL	BOPUC
scalar case	Toda	Ablowitz-Ladik
matrix case	non-abelian Toda	?

In the article [2] we proved that the relevant hierarchy is the non-abelian version of Ablowitz-Ladik hierarchy. This hierarchy has been already studied by different authors since 1983 (see [35], [36] and [37]) but, at our best knowledge, connection with matrix biorthogonal polynomials on the unit circle was never established before. In our setting this hierarchy appears naturally considering right and left matrix biorthogonal polynomials with respect to a matrix-valued symbol $\gamma(z)$. In the matrix case infact we define two bilinear pairings

$$< P, Q >_r := \oint P^*(z)\gamma(z)Q(z)\frac{dz}{2\pi i z}$$
$$< P, Q >_l := \oint P(z)\gamma(z)Q^*(z)\frac{dz}{2\pi i z}$$

where we define, for a general matrix valued polynomials $P(z)$,

$$P^*(z) := P^T(z^{-1}).$$

Then we have matrix right and left biorthogonal polynomials such that

$$< P_k^{(2)r}, P_j^{(1)r} >_r = \delta_{kj} h_k^r \qquad < P_k^{(1)l}, P_j^{(2)l} >_l = \delta_{kj} h_k^l.$$

Slightly generalizing some already known results (see [47],[48]) we found that these matrix biorthogonal polynomials satisfy block recursion relations

$$\begin{pmatrix} P_{N+1}^{(1)l} \\ \tilde{P}_{N+1}^{(2)r} \end{pmatrix} = \mathcal{L}_N^l \begin{pmatrix} P_N^{(1)l} \\ \tilde{P}_N^{(2)r} \end{pmatrix} \tag{9}$$

$$\begin{pmatrix} P_{N+1}^{(1)r} & \tilde{P}_{N+1}^{(2)l} \end{pmatrix} = \begin{pmatrix} P_N^{(1)r} & \tilde{P}_N^{(2)l} \end{pmatrix} \mathcal{L}_N^r. \tag{10}$$

where, as in the scalar case, for an arbitrary matrix valued polynomial $Q(z)$ of order k we define

$$\tilde{Q}(z) := z^k Q^*(z).$$

Here the recursion operators $\mathcal{L}_N^r$ and $\mathcal{L}_N^l$ are block matrices defined by

$$\mathcal{L}_N^l := \begin{pmatrix} z\mathrm{I} & x_{N+1}^l \\ z y_{N+1}^r & \mathrm{I} \end{pmatrix} \tag{11}$$

$$\mathcal{L}_N^r := \begin{pmatrix} z\mathrm{I} & z y_{N+1}^l \\ x_{N+1}^r & \mathrm{I} \end{pmatrix}. \tag{12}$$

and their coefficients are $(n \times n)$ matrices. Using the Toda flow adapted to the block case we arrive to the original result contained in [2].

Second result: *Consider time-dipendent matrix BOPUC* $\{P_n^{(1)r}(z), P_n^{(1)l}(z)\}$ *and* $\{P_n^{(2)r}(z), P_n^{(2)l}(z)\}$ *with respect to the pairings*

$$< P, Q >_r := \oint P^*(z)\gamma(t, s; z)Q(z)\frac{dz}{2\pi i z}$$
$$< P, Q >_l := \oint P(z)\gamma(t, s; z)Q^*(z)\frac{dz}{2\pi i z}$$

with $\gamma(t,s;z) := \exp(\xi(t,z\mathrm{I}))\gamma(z)\exp(-\xi(s,z^{-1}\mathrm{I}))$. *Then the related recursion operators (9) and (10) evolves according to semidiscrete zero-curvature equations (13) and (14) (see below) for the non-abelian Ablowitz-Ladik hierarchy.*

Semidiscrete zero curvature equations are, in this case, given by

$$\partial_\tau \mathcal{L}^l_k = \mathcal{M}^l_{k+1}\mathcal{L}^l_k - \mathcal{L}^l_n\mathcal{M}^l_k \tag{13}$$

$$\partial_\tau \mathcal{L}^r_k = \mathcal{L}^r_k\mathcal{M}^r_{k+1} - \mathcal{M}^r_k\mathcal{L}^r_k. \tag{14}$$

for some block matrices $\mathcal{M}^l_k$ and $\mathcal{M}^r_k$. For instance we have, for this hierarchy, two versions of non-abelian complexified discrete nonlinear Schrödinger that read

$$\begin{cases} \partial_\tau x^l_k = x^l_{k+1} - 2x^l_k + x^l_{k-1} - x^l_{k+1}y^r_k x^l_k - x^l_k y^r_k x^l_{k-1} \\ \\ \partial_\tau y^r_k = -y^r_{k+1} + 2y^r_k - y^r_{k-1} + y^r_{k+1}x^l_k y^r_k + y^r_k x^l_k y^r_{k-1} \end{cases} \tag{15}$$

$$\begin{cases} \partial_\tau x^r_k = x^r_{k+1} - 2x^r_k + x^r_{k-1} - x^r_{k-1}y^l_k x^r_k - x^r_k y^l_k x^r_{k+1} \\ \\ \partial_\tau y^l_k = -y^l_{k+1} + 2y^l_k - y^l_{k-1} + y^l_{k-1}x^r_k y^l_k + y^l_k x^r_k y^l_{k+1} \end{cases} \tag{16}$$

We recall that here coefficients are matrices so that, in general, they do not commute. If they commute (for instance in the scalar case) we return to the standard discrete nonlinear Schrödinger (7) (see [38] for a review about these non abelian equations).
Sections of the second chapter are organized as follows:

- First section gives some preliminary results about 2D-Toda lattice (see [39] and [41]).
- In the second section we deduce semidiscrete zero-curvature equations starting from the Toeplitz lattice.
- In the third section the Toeplitz lattice is extended to the case of block Toeplitz matrices.
- Fourth section gives recursion relations for matrix biorthogonal polynomials on the unit circle; this formulas slightly generalize formulas contained in [47] and [48] for matrix orthogonal polynomials on the unit circle.
- In the fifth section we derive block semidiscrete zero-curvature equations defining non-abelian Ablowitz-Ladik hierarchy. As an example we write the non-abelian analogue of discrete nonlinear Schrödinger.

Chapter 1

Block Toeplitz determinants, constrained KP and Gelfand-Dickey hierarchies.

1.1 Segal-Wilson Grassmannian and related loop groups

Here we recall some definitions and results from [3] and [4] that will be useful in this chapter.

Definition 1.1.1. *Let* $H^{(n)} := L^2(S^1, \mathbb{C}^n)$ *be the space of complex vector-valued square-integrable functions. We choose a orthonormal basis given by*

$$\{e_{\alpha,k} := (0, \dots, z^k, \dots, 0)^{\mathrm{T}} : \alpha = 1 \dots n, k \in \mathbb{Z}\}$$

and the polarization

$$H^{(n)} = H_+^{(n)} \oplus H_-^{(n)}$$

where $H_+^{(n)}$ *and* $H_-^{(n)}$ *are the closed subspaces spanned by elements* $\{e_{\alpha,k}\}$ *with* $k \geq 0$ *and* $k < 0$ *respectively.*

In the sequel in order to avoid cumbersome notations we will write H instead of $H^{(1)}$.

Definition 1.1.2 ([4]). *The Grassmannian* $\mathrm{Gr}(H^{(n)})$ *modeled on* $H^{(n)}$ *consists of the subset of closed subspaces* $W \subseteq H^{(n)}$ *such that:*

- *the orthogonal projection* $\mathrm{pr}_+ : W \to H_+^{(n)}$ *is a Fredholm operator.*

- *the orthogonal projection* $\mathrm{pr}_- : W \to H_-^{(n)}$ *is a Hilbert-Schmidt operator.*

Moreover we will denote $\mathrm{Gr}^{(n)}$ *the subset of* $\mathrm{Gr}(H^{(n)})$ *given by subspaces* W *such that* $zW \subseteq W$.

It's well known [3] that through Segal-Wilson theory we can associate a solution of n^{th} Gelfand-Dickey hierarchy to every element of $\mathrm{Gr}^{(n)}$; this is the reason why we are interested in them.

Lemma 1.1.3 ([3]). *The map*

$$\begin{aligned} \Xi : H^{(n)} &\longrightarrow H \\ (f_0(z), \ldots, f_{n-1}(z))^{\mathrm{T}} &\longmapsto \tilde{f}(z) := f_0(z^n) + \ldots + z^{n-1} f_{n-1}(z^n) \end{aligned}$$

is an isometry. Its inverse is given by

$$f_k(z) = \frac{1}{n} \sum_{\zeta^n = z} \zeta^{-k} \tilde{f}(\zeta)$$

where the sum runs over the n^{th} *roots of* z.

Proposition 1.1.4. *Under the isometry* Ξ *we can identify* $\mathrm{Gr}^{(n)}$ *with the subset*

$$\{W \in \mathrm{Gr}(H) : z^n W \subseteq W\}$$

It is obvious that loop groups act on Hilbert spaces defined above by multiplications. We want to define a certain loop group $L_{\frac{1}{2}}\mathrm{Gl}(n, \mathbb{C})$ with good analytical properties acting transitively on $\mathrm{Gr}^{(n)}$; in such a way we can obtain any $W \in \mathrm{Gr}^{(n)}$ just acting on the reference point $H_+^{(n)}$ with this group. Good analitycal properties will be necessary as we want to construct symbols of some Toeplitz operators out of elements of this group and then apply Widom's results (see below). Given a matrix g we denote with $||g||$ its Hilbert-Schmidt norm

$$||g||^2 = \sum_{i,j=1}^{n} ||g_{i,j}||^2$$

Definition 1.1.5. *Given a measurable matrix-valued loop* γ *we define two norms* $||\gamma||_\infty$ *and* $||\gamma||_{2,\frac{1}{2}}$ *as*

$$||\gamma||_\infty := \operatorname{ess}\sup_{||z||=1} ||\gamma(z)|| \qquad ||\gamma||_{2,\frac{1}{2}} := \sum_k \left(|k| \, ||\gamma^{(k)}||^2 \right)^{\frac{1}{2}}$$

where we have Fourier expansion

$$\gamma(z) = \sum_{k=-\infty}^{\infty} \gamma^{(k)} z^k.$$

Definition 1.1.6. $L_{\frac{1}{2}}\mathrm{Gl}(n, \mathbb{C})$ *is defined as the loop group of invertible measurable loops* γ *such that*

$$\|\gamma\|_\infty + \|\gamma\|_{2,\frac{1}{2}} < \infty.$$

Proposition 1.1.7 ([4]). $L_{\frac{1}{2}}\mathrm{Gl}(n, \mathbb{C})$ *acts transitively on* $\mathrm{Gr}^{(n)}$ *and the isotropy group of* $H_+^{(n)}$ *is the group of constant loops* $\mathrm{Gl}(n, \mathbb{C})$.

Proof can be found in [4], here we just mention the principal steps necessary to arrive to this result.

- We define a subgroup $\mathrm{Gl}_{res}(H^{(n)})$ of invertible linear maps $g : H^{(n)} \to H^{(n)}$ acting on $\mathrm{Gr}(H^{(n)})$ (the restricted general linear group).
- We prove that every element of $\mathrm{Gl}_{res}(H^{(n)})$ commuting with multiplication by z must belong to $L_{\frac{1}{2}}\mathrm{Gl}(n, \mathbb{C})$.
- We take an element $W \in \mathrm{Gr}^{(n)}$ and a basis $\{w_1, ..., w_n\}$ of the orthogonal complement of zW in W.
- Out of this basis, putting vectors side by side, we construct $\mathcal{W}$ and easily check that $W = \mathcal{W}(z)H_+^{(n)}$.
- We verify that multiplication by $\mathcal{W}$ belongs to $\mathrm{Gl}_{res}(H^{(n)})$; since it obviously commutes with multiplication by z we conclude that $\mathcal{W}(z) \in L_{\frac{1}{2}}\mathrm{Gl}(n, \mathbb{C})$.

1.2 Szegö-Widom theorem for block Toeplitz determinants.

In his work ([8],[9] and [10]) H. Widom expressed the limit, for the size going to infinity, of certain block Toeplitz determinants as Fredholm determinants of an operator $\mathcal{P}$ acting on $H_+^{(n)}$. Also he gave two different corollaries that allow us to compute this determinant in some particular cases. In this section we recall, without proofs, these results. Moreover we state Borodin-Okounkov formula as presented in [12] for matrix case.

We begin recalling some notations; given a loop $\gamma \in L_{\frac{1}{2}}\mathrm{Gl}(n, \mathbb{C})$ we denote with $T_N(\gamma)$ the block Toeplitz matrix given by

$$T_N(\gamma) := \begin{pmatrix} \gamma^{(0)} & \dots & \dots & \gamma^{(-N)} \\ \gamma^{(1)} & \dots & \dots & \gamma^{(-N+1)} \\ \dots & \dots & \dots & \dots \\ \gamma^{(N)} & \dots & \dots & \gamma^{(0)} \end{pmatrix}$$

where we have the Fourier expansion $\gamma(z) = \sum_k \gamma^{(k)} z^k$. We denote $D_N(\gamma)$ its determinant. We use the notation $T(\gamma)$ for the $\mathbb{N} \times \mathbb{N}$ matrix obtained letting N go to infinity.

Remark 1.2.1. It's easy to see that, in the base we have chosen above for $H^{(n)}$, $T(\gamma)$ is nothing but the matrix representation of

$$\mathrm{pr}_+ \circ \gamma : H_+^{(n)} \longrightarrow H_+^{(n)}$$

Theorem 1.2.2 (Szegö-Widom theorem,[10]). *Suppose* $\gamma \in L_{\frac{1}{2}}\mathrm{Gl}(n, \mathbb{C})$ *and*

$$\underset{0 \leq \theta \leq 2\pi}{\Delta} \arg\Big(\det\big(\gamma(e^{i\theta})\big)\Big) = 0$$

Then it exists the limit

$$D_\infty(\gamma) := \lim_{N \to \infty} \frac{D_N(\gamma)}{G(\gamma)^N} = \det(T(\gamma)T(\gamma^{-1}))$$

where

$$G(\gamma) = \exp\Big(\frac{1}{2\pi}\int_0^{2\pi} \log\big(\det \gamma(e^{i\theta})\big)\, d\theta\Big)$$

The proof of the theorem is contained in [10]; instead of rewriting it we simply consider the operator $T(\gamma)T(\gamma^{-1})$ and explain the meaning of "det" in this case.

Lemma 1.2.3. *Consider* $\gamma_1, \gamma_2 \in L_{\frac{1}{2}}\mathrm{Gl}_n(n, \mathbb{C})$*; we have*

$$T(\gamma_1\gamma_2) - T(\gamma_1)T(\gamma_2) = \Big[\sum_{k \geq 1} \gamma_1^{(i+k)} \gamma_2^{(-j-k)}\Big]_{i,j \geq 0}.$$

PROOF The (i,j)-entry of left hand side reads

$$\sum_{k=-\infty}^{\infty} \gamma_1^{(i-k)}\gamma_2^{(k-j)} - \sum_{k=0}^{\infty} \gamma_1^{(i-k)}\gamma_2^{(k-j)} = \sum_{k=-\infty}^{-1} \gamma_1^{(i-k)}\gamma_2^{(k-j)} = \sum_{k=0}^{\infty} \gamma_1^{(i+k+1)}\gamma_2^{(-k-j-1)}.$$

□

In particular choosing $\gamma_1 = \gamma$ and $\gamma_2 = \gamma^{-1}$ we obtain

$$I - T(\gamma)T(\gamma^{-1}) = \Big[\sum_{k\geq 1} \gamma^{(i+k)}(\gamma^{-1})^{(-j-k)}\Big]_{i,j\geq 0}$$

Definition 1.2.4.

$$\mathcal{P}_\gamma := T(\gamma)T(\gamma^{-1}) = \left[\delta_i^j - \Big(\sum_{k\geq 1} \gamma^{(i+k)}(\gamma^{-1})^{(-j-k)}\Big)\right]_{i,j\geq 0} \tag{1.1}$$

Thanks to the fact that

$$\sum_{i\geq 0}\sum_{k\geq 1} \|\gamma^{(i+k)}\|^2 = \sum_{k\geq 1} k\|\gamma^{(k)}\|^2 < \infty$$

the product we have written on the right of (1.1) is a product of two Hilbert-Schmidt operators. So $\mathcal{P}_\gamma$ differs from the identity by a nuclear operator. Hence its determinant is well defined (see for instance [31]). In our notation we obtained the equality

$$D_\infty(\gamma) = \det(\mathcal{P}_\gamma) \tag{1.2}$$

We will call $\mathcal{P}_\gamma$ Plemelj's operator as it is related in a clear way with a Riemann-Hilbert factorization problem already considered by Josip Plemelj in 1964 [15]. We will consider this fact later in this chapter.
Unfortunately, in concrete cases, $\det(\mathcal{P}_\gamma)$ turns out to be really hard to compute; nevertheless we can use some shortcuts also provided by Widom in his works ([8],[9] and [10]).

Proposition 1.2.5 ([8]). *Suppose that γ satisfies conditions imposed in Szegö-Widom theorem and, moreover, $\gamma^{(i)} = 0$ for $i \geq j+1$ or $\gamma^{(i)} = 0$ for $i \leq j+1$.*
Then

$$D_\infty(\gamma) = D_j(\gamma^{-1})G(\gamma)^j \tag{1.3}$$

Proposition 1.2.6 ([10]). *Suppose we have a symbol γ satisfying conditions imposed in Szegö-Widom theorem. Suppose moreover that γ depends on a* *parameter x in such a way that the function $x \to \gamma(x)$ is differentiable. If γ^{-1} admits two Riemann-Hilbert factorizations*

$$\gamma^{-1}(z) = t_+(z)t_-(z) = s_-(z)s_+(z)$$

such that

$$t_+(z) := \sum_{k\geq 0} t_+^{(k)} z^k \qquad s_+(z) := \sum_{k\geq 0} s_+^{(k)} z^k$$
$$t_-(z) := \sum_{k\leq 0} t_-^{(k)} z^k \qquad s_-(z) := \sum_{k\leq 0} s_-^{(k)} z^k$$

Then

$$\frac{d}{dx}\log(D_\infty(\gamma)) = \frac{i}{2\pi}\oint \text{trace}\left[\Big((\partial_z t_+)t_- - (\partial_z s_-)s_+\Big)\partial_x \gamma\right] dz. \qquad (1.4)$$

Also $D_N(\gamma)$ can be expressed as a Fredholm determinant as pointed out for the scalar case in [11] and generalized for matrix case in [12].

Theorem 1.2.7 (Borodin-Okounkov formula, [12]). *Suppose that our symbol $\gamma(z)$ satisfying conditions of Szegö-Widom's theorem admits two Riemann-Hilbert factorizations*

$$\gamma(z) = \gamma_+(z)\gamma_-(z) = \theta_-(z)\theta_+(z)$$

such that

$$\gamma_+(z) := \sum_{k\geq 0} \gamma_+^{(k)} z^k \qquad \theta_+(z) := \sum_{k\geq 0} \theta_+^{(k)} z^k$$
$$\gamma_-(z) := \sum_{k\leq 0} \gamma_-^{(k)} z^k \qquad \theta_-(z) := \sum_{k\leq 0} \theta_-^{(k)} z^k$$

and $G(\gamma) = 1$. Then for every N

$$D_N(\gamma) = D_\infty(\gamma)\det(I - K_{\gamma,N}) \qquad (1.5)$$

where, in coordinates, we have

$$(K_{\gamma,N})_{ij} = \begin{cases} 0 & \text{if } \min\{i,j\} < N \\ \sum_{k=1}^{\infty}(\gamma_-\theta_+^{-1})^{(i+k)}(\theta_-^{-1}\gamma_+)^{(-j-k)} & \textit{otherwise.} \end{cases}$$

Remark 1.2.8. One can easily verify that $\theta_-^{-1}\gamma_+$ is the inverse of $\gamma_-\theta_+^{-1}$ so that, again, we deal with operators of type $T(\phi)T(\phi^{-1})$ with $\phi = \gamma_-\theta_+^{-1}$. Also we want to point out that the assumption $G(\gamma) = 1$ is not necessary. The formula for $G(\gamma) \neq 1$ is written in [13]; since in our case we will always have $G(\gamma) = 1$ we wrote the formula as it was given in [12].

1.3 τ functions for constrained KP and Gelfand-Dickey hierarchies as block Toeplitz determinants.

In order to fix notations we state some basic facts about KP hierarchy and some reductions of it. Standard references are [3] and [5]. For cKP reductions we make reference to [20],[21],[22],[23] and [24].
Given the pseudodifferential Lax operator

$$\mathcal{L} := D + \sum_{j=1}^{\infty} u_j D^{-j}$$

KP hierarchy is defined as compatibility conditions of equations

$$\begin{cases} \mathcal{L}\psi = z\psi \\ \frac{\partial}{\partial t_j}\psi = (\mathcal{L}^j)_+\psi \quad j = 1\dots\infty \end{cases} \tag{1.6}$$

where $(\mathcal{L}^j)_+$ denote the differential part of j^{th} power of $\mathcal{L}$. These compatibility conditions are written in Lax form as

$$\frac{\partial}{\partial t_j}\mathcal{L} = \left[(\mathcal{L}^j)_+, \mathcal{L}\right]$$

and should be seen as differential equations for coefficients $\{u_j\}$ with respect to variables $\{t_j\}$. Equivalently one can introduce the dressing operator

$$S = 1 + \sum_{j=1}^{\infty} s_j D^{-j}$$

such that

$$\psi := S(e^{\sum_{j=1}^{\infty} t_i z^i}) = e^{\sum_{j=1}^{\infty} t_i z^i}\left(1 + s_1 z^{-1} + s_2 z^{-2} + \dots\right)$$

is a solution of (1.6). In this way KP hierarchy is rewritten in Sato form as

$$\begin{cases} \mathcal{L} = SDS^{-1} \\ \frac{\partial}{\partial t_j} S = -(\mathcal{L}^j)_- S \end{cases} \tag{1.7}$$

where $(\mathcal{L}^j)_- = \mathcal{L}^j - (\mathcal{L}^j)_+$. The first equation gives expression of $\{u_j\}$ in terms of $\{s_j\}$ and the second one gives time evolution for $\{s_j\}$.
Connection with Grassmannian goes this way: given $W \in \mathrm{Gr}$ one defines

$$W(t) = e^{\sum_{j=1}^{\infty} t_i z^i} W.$$

For every values of parameters $\{t_i\}$ such that the orthogonal projection

$$\mathrm{pr}_+ : W \to H_+$$

is still Fredholm one defines

$$\tau_W(t) := \det\left(\mathrm{pr}_+ : W \to H_+\right)$$

and KP hierarchy can be recast as a set of differential equations for τ_W (Hirota bilinear form). Actually we have the remarkable formula, due to Sato,

$$\psi(t) := \frac{\tau_W(t - \frac{1}{[z]})}{\tau_W(t)} e^{\sum_{j=1}^{\infty} t_i z^i}$$

(here $\tau_W(t - \frac{1}{[z]}) = \tau_W(t_1 - \frac{1}{z}, t_2 - \frac{1}{2z^2}, \ldots)$) that gives ψ (and then S and $\mathcal{L}$) in terms of τ_W.
Given the pseudodifferential symbol $\mathcal{L}$ and related tau function τ_W we say, using the notation of [22], that $\tau \in \mathrm{cKP}_{m,n}$ iff $\mathcal{L}^m$ can be written as the ratio of two differential operators of order $m+n$ and n respectively. For $n = 0$ we recover the usual definition of m^{th} Gelfand-Dickey hierarchy; already Segal and Wilson in [3] noticed that this reduction corresponds to considering points $W \in \mathrm{Gr}$ such that $z^m W \subseteq W$, i.e. $W \in \mathrm{Gr}^{(\mathrm{m})}$.
For n generic these reductions begun to be studied in 1995 by Dickey and Krichever ([20],[21]); a geometric interpretation of corresponding points in the Grassmannian has been given in [23] and [24]. Namely $\tau_W \in \mathrm{cKP}_{m,n}$ iff W contains a subspace W' of codimension n in W such that $z^m W' \subseteq W$.

Now, given a subspace $W \in \mathrm{Gr}^{(n)}$, we define the corresponding τ_W in a different way from the one used in [3]. Our approach generalizes what has been done by Itzykson and Zuber in the study of Witten-Kontsevich τ function in [17] (see also [18] and [19]). This approach allows us to define not just τ_W but also a sequence of $\{\tau_{W,N}\}_{N>0}$ approximating τ_W and such that

$$\tau_{W,N} \in \mathrm{cKP}_{1,nN} \cap \mathrm{cKP}_{n,n} \quad \forall N.$$

Suppose we have an element $W \in \mathrm{Gr}^{(n)}$; thanks to results stated in Section 1 we can represent this element as

$$W = \begin{pmatrix} w_{11} & \ldots & \ldots & w_{n1} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ w_{1n} & \ldots & \ldots & w_{nn} \end{pmatrix} H_+^{(n)} = \mathcal{W}(z) H_+^{(n)}$$

with $\mathcal{W}(z) \in L_{\frac{1}{2}}\mathrm{Gl}(n, \mathbb{C})$.
Also we assume that the matrix $\mathcal{W}(z) = \{w_{ij}(z)\}_{i,j=1..n}$ satisfies

$$\begin{cases} w_{ii} = 1 + O(\frac{1}{z}) \\ \\ w_{ij} = z(O(\frac{1}{z})), i > j \\ \\ w_{ij} = O(\frac{1}{z}), i < j \end{cases}$$

This means that we restrict to the big cell, i.e. we assume that the orthogonal projection

$$\mathrm{pr}_+ : W \longrightarrow H_+$$

is an isomorphism. Infact we have a base for $W \in \mathrm{Gr}^{(n)}$ given by

$$\{z^s w_j : s \in \mathbb{N}, j = 1 \ldots n\}$$

where w_j is the column vector $(w_{1j}...w_{nj})^{\mathrm{T}}$.
Using the isomorphim $\Xi : H^{(n)} \to H$ the corresponding base for $W \in \mathrm{Gr}$ is given by

$$\{\omega_{ns+j} = z^{ns}\Xi(w_j) : s \in \mathbb{N}, j = 1 \ldots n\}$$

and, as in Section 1, we have

$$[\Xi(w_j)](z) = \sum_{i=1}^{n} z^{i-1} w_{ji}(z^n)$$

This means that we obtain

$$\omega_{ns+j}(z) = z^{ns+j-1}\left(1 + O\left(\frac{1}{z}\right)\right)$$

and from this equation follows that the orthogonal projection onto H_+ is an isomorphism since every ω_k projects to z^{k-1}.
For these points $W \in \mathrm{Gr}^{(n)}$ and vectors spanning them we define the standard time evolution (KP flow) given by

$$\omega_{ns+j}(t; z) := \exp\left(\sum_{i>0} t_i z^i\right)\omega_{ns+j}(z) = \exp(\xi(t, z))\omega_{ns+j}(z)$$

Now we want to define the τ function associated to W as limit for $N \to \infty$ of some block Toeplitz determinants $\tau_{W,N}$.

Definition 1.3.1. *Take $M = Nn$ a multiple of n.*

$$\tau_{W,N}(t) := \det \left[\oint z^{-i}\omega_j(t;z)dz \right]_{1\le i,j\le M=Nn} \tag{1.8}$$

Fist of all we want to prove that $\tau_{W,N}$ is a block Toeplitz determinant and write explicitely the symbol.

Lemma 1.3.2. *For every $j = 1\ldots n$ we have*

$$w_j(t;z) := \Xi^{-1}(\omega_j(t,z)) = \exp(\xi(t,Z))w_j(z)$$

where we denote

$$Z := \begin{pmatrix} 0 & \dots & \dots & \dots & z \\ 1 & 0 & \ddots & \ddots & 0 \\ 0 & 1 & \ddots & \ddots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \ddots & 0 & 1 & 0 \end{pmatrix}$$

PROOF We simply verify that multiplication by z on Gr corresponds to multiplication by Z on $\mathrm{Gr}^{(n)}$ through the isomorphism Ξ^{-1}. □

Proposition 1.3.3. *$\tau_{W,N}$ is the N-truncated $(n\times n)-$block Toeplitz determinant with symbol*

$$\mathcal{W}(t;z) := \exp(\xi(t,Z))\mathcal{W}(z) \tag{1.9}$$

PROOF Take $i,j \le n$ and $s,v \le N$; the $(i+sn, j+vn)$-entry of the matrix in the right hand side of (1.8) is given by

$$\oint z^{-i-sn}\omega_{j+vn}(t;z)dz = \oint z^{-i-sn}z^{vn}\omega_j(t;z)dz =$$
$$\oint z^{-i+(v-s)n} \sum_{k\in\mathbb{Z},l=1..n} w_{jl}(t)^{(k)}z^{nk+l-1}dz = w_{ji}(t)^{(s-v)}$$

so that the right hand side of (1.8) is the transposed of the N-truncated $n\times n$ block Toeplitz matrix with symbol $\mathcal{W}(t;z)$. □

In the sequel of this paper we will call such symbols Gelfand-Dickey (GD) symbols.
Now generalizing what has been done by Itzykson and Zuber in [17] we expand $\tau_{W,N}(t)$ in characters.

Proposition 1.3.4.

$$\tau_{W,N}(t) = \sum_{l_1,\dots,l_{nN}\geq 0} \left(\prod_i \omega_i^{(-l_i+i-1)}\right)\chi_{l_1,\dots,l_{nN}}(X)$$

where $X = \mathrm{diag}(x_1, \dots, x_{nN})$ *is related to times* $\{t_i\}$ *through Miwa's parametrization*

$$t_k := \mathrm{trace}\left(\frac{X^k}{k}\right)$$

and

$$\chi_{l_1,\dots,l_{nN}}(X) := \frac{\det\begin{pmatrix} x_1^{l_1+nN-1} & x_1^{l_2+nN-2} & \dots & x_1^{l_{nN}} \\ \dots & \dots & \dots & \dots \\ x_{nN}^{l_1+nN-1} & x_{nN}^{l_2+nN-2} & \dots & x_{nN}^{l_{nN}} \end{pmatrix}}{\det\begin{pmatrix} x_1^{nN-1} & x_1^{nN-2} & \dots & 1 \\ \dots & \dots & \dots & \dots \\ x_{nN}^{nN-1} & x_{nN}^{nN-2} & \dots & 1 \end{pmatrix}}$$

PROOF We start from determinant representation (1.8). The (i,j)-entry of the matrix will be

$$\sum_n \omega_j^{(-n)} p_{n+i-1}(t)$$

where for every $n \geq 0$

$$p_n(t) := \frac{1}{2\pi i}\oint \frac{\exp(\xi(t,z))}{z^{n+1}}dz$$

are the classical Schur polynomials and $p_n(t) = 0$ for every negative n. Then resumming everything we obtain

$$\tau_{W,N}(t) = \sum_{k_1,\dots,k_{nN}} \left(\prod_j \omega_j^{(-k_j)}\right) \det[p_{k_j+i-1}(t)]_{i,j=1\dots nN}$$

with $k_j \geq 1-j$.
Equivalently we write

$$\tau_{W,N}(t) = \sum_{l_1,\dots,l_{nN}\geq 0} \left(\prod_j \omega_j^{(-l_j+j-1)}\right) \det[p_{l_j-j+i}(t)]_{i,j=1\dots nN}.$$

On the other hand it's well known that under Miwa's parametrization this last determinant can be written as $\chi_{l_1,\dots,l_{nN}}(X)$ (see for instance [17],[18]); this completes the proof. □

We now assign degree 1 to every x_i or, equivalently, degree m to t_m for every m. For every N the function $\tau_{W,N}$ is a formal series belonging to the graded algebra $\mathbb{C}[[t_1, t_2, \ldots]]$. In general given $A \in \mathbb{C}[[t_1, t_2, \ldots]]$ we define its degree as the minimal degree of its terms and we state the following definition of stable limit for sequences in $\mathbb{C}[[t_1, t_2, \ldots]]$.

Definition 1.3.5. *Given a sequence of formal series*

$$\{A_N(t) \in \mathbb{C}[[t_1, t_2, \ldots]], N = 0 \ldots \infty\}$$

we say that the sequence admits a stable limit $A(t)$ iff

$$\lim_{N \to \infty} \deg(A_N(t) - A(t)) = \infty$$

We want to prove that the sequence $\{\tau_{W,N}\}$ admits stable limit. It's easy to see that

$$\deg(\chi_{l_1 \ldots l_{nN}}) = \sum_{i=1}^{M} l_i$$

From this easily verified property we obtain the following

Lemma 1.3.6. *Suppose* $\deg(\chi_{l_1, \ldots, l_{nN}}) = Q \leq nN$*. Then, if the character is different from zero, we have*

$$\chi_{l_1 \ldots l_{nN}} = \chi_{l_1, \ldots, l_Q, 0, \ldots, 0}.$$

Proof Suppose $l_j \neq 0$, $j > Q$ and $l_i = 0 \ \forall \ i > j$.
The j^{th} column of the matrix $[p_{l_j - j + i}(t)]$ has positive subscripts $l_1 + j - 1, l_2 + j - 2, \ldots, l_j$.
On the other hand $\sum l_i = Q$; hence the sum of these subscripts is

$$Q + \sum_{r=0}^{j-1} r \leq \sum_{r=0}^{j} r$$

hence two subscripts must be equal, then two lines of the matrix are equal. □

From this corollary it follows directly the following result.

Proposition 1.3.7. *Up to degree Q the function $\tau_{W,N}(t)$ does not depend on N with $N \geq Q$.*

Thanks to this proposition we deduce that it exists the stable limit

$$\tau_W(t) := \lim_{N\to\infty} \tau_{W,N}(t) \tag{1.10}$$

On the other hand, in the sequel, we will prove that the symbol $\mathcal{W}(t)$ satisfies Szegö-Widom's condition for every values of t_i so that the limit in (1.10) exist pointwise in time parameters and can be written as a Fredholm determinant. Now, following again [17], we write a differential operator $\Delta_{W,N}(t)$ associated to the function $\tau_{W,N}(t)$. In the sequel we will always write D for the partial derivative with respect to t_1. We will prove that for every N the pseudo-differential operator $\Delta_{W,N}(t)D^{-nN}$ satisfies Sato's equations for the dressing and we recover the usual relation between τ and wave functions.

Lemma 1.3.8. *Define*

$$f_{s,N}(t) := \sum_{k>s} \omega_s^{(-k)} p_{k+nN-1}(t)$$

Then we have

$$\tau_{W,N}(t) = \mathrm{Wr}(f_{1,N}(t), \dots, f_{nN,N}(t)) := \det[D^{nN-j} f_{i,N}(t)]_{1\le i,j\le nN}$$

PROOF From definition 1.3.1 the (i,j)-entry of matrix defining $\tau_{W,N}(t)$ is

$$\sum_{k>j} \omega_j^{(-k)} p_{k+i-1}(t)$$

On the other hand we have

$$D^{nN-j} f_{i,N}(t) = D^{nN-j}\left(\sum_{k>i} \omega_i^{(-k)} p_{k+nN-1}(t)\right) = \sum_{k>i} \omega_i^{(-k)} p_{k+j-1}(t)$$

(using the equation $D^s(p_m(t)) = p_{m-s}(t)$). Hence we obtained the proof. □

Definition 1.3.9. *We define the differential operator $\Delta_{W,N}$ of order N in D as*

$$\Delta_{W,N}(f) := \frac{\mathrm{Wr}(f, f_{1,N}(t), \dots, f_{nN,N}(t))}{\mathrm{Wr}(f_{1,N}(t), \dots, f_{nN,N}(t))}$$

where $f \in H$ depends in a differentiable way on $\{t_i\}_{i\ge 1}$.

Proposition 1.3.10. *The following equations for time-derivatves of $\Delta_{W,N}$ holds:*

$$\frac{\partial}{\partial_{t_i}} \Delta_{W,N} = \left(\Delta_{W,N} D^i \Delta_{W,N}^{-1}\right)_+ \Delta_{W,N} - \Delta_{W,N}(t) D^i \tag{1.11}$$

PROOF It is enough to prove the equality of the two differential operators when acting on $f_{1,N}(t), ...f_{nN,N}(t)$ which are nN indipendent solutions of the equation

$$(\Delta_{W,N})(f(t)) = 0$$

But this amounts to proving

$$\left[\frac{\partial}{\partial_{t_i}}(\Delta_{W,N})\right] f_{j,N}(t) + \Delta_{W,N}\frac{\partial^i}{\partial t_1^i} f_{j,N}(t) = 0 \quad \forall j$$

which is true iff

$$\frac{\partial}{\partial t_i}(\Delta_{W,N} f_{j,N}(t)) = 0 \quad \forall j.$$

This equality is obviously satisfied. □

Multiplying $\Delta_{W,N}$ from the right with D^{-nN} we found a pseudodifferential operator that, in fact, gives a solution of KP equations.

Definition 1.3.11.

$$S_{W,N} := \Delta_{W,N} D^{-nN}$$

Proposition 1.3.12. *$S_{W,N}$ is a monic pseudo-differential operator of order 0 satisying Sato's equation*

$$\frac{\partial}{\partial_{t_i}} S_{W,N} = -\left(S_{W,N} D^i S_{W,N}^{-1}\right)_- S_{W,N} \tag{1.12}$$

Hence the monic pseudo-differential operator of order 1

$$\mathcal{L}_{W,N} := (S_{W,N} D S_{W,N}^{-1}) \tag{1.13}$$

satisfies the usual Lax system for KP

$$\frac{\partial \mathcal{L}_{W,N}}{\partial t_k} = [(\mathcal{L}_{W,N}^k)_+, \mathcal{L}_{W,N}] \tag{1.14}$$

PROOF It is obvious that $S_{W,N}$ is a monic pseudo-differential operator of order 0 since $\Delta_{W,N}$, which is of order nN, is normalized so that the leading term is equal to 1. Equation (1.12) follows directly from (1.11). The derivation of Lax system from Sato's equations is well known: one has just to derive the relation

$$\mathcal{L}_{W,N} S_{W,N} = S_{W,N} D$$

for t_k and use the obvious relation $[\mathcal{L}_{W,N}, \mathcal{L}_{W,N}^k] = 0$ □

It remains to prove that $\tau_{W,N}(t)$ is really the τ function for these solutions

$\mathfrak{L}_{W,N}(t)$ of KP equations. We recall the usual relations between the dressing S, the wave function ψ and τ function given by

$$\psi(t;z) = S(t)(\exp(\xi(t,z))) = \exp(\xi(t,z))\frac{\tau(t-\frac{1}{[z]})}{\tau(t)}$$

(we recall that the notation $t - \frac{1}{[z]}$ stands for the vector with i^{th} component equal to $t_i - \frac{1}{iz^i}$) All we have to prove is the following

Proposition 1.3.13.

$$\psi_{W,N}(t;z) := S_{W,N}(\exp(\xi(t,z))) = \exp(\xi(t,z))\frac{\tau_{W,N}(t-\frac{1}{[z]})}{\tau_{W,N}(t)} \qquad (1.15)$$

PROOF Equivalently we prove that

$$(\Delta_{W,N})\exp(\xi(t,z)) = \exp(\xi(t,z))z^{nN}\frac{\tau_{W,N}(t-\frac{1}{[z]})}{\tau_{W,N}(t)}$$

Since we have

$$p_n(t-\frac{1}{[z]}) = p_n(t) - z^{-1}p_{n-1}(t)$$

the right hand side of the equality above can be written as

$$z^{nN}e^{\xi(x,t)}\frac{\det\begin{pmatrix} D^{nN-1}f_1 - z^{-1}D^{nN}f_1 & \dots & f_1 - z^{-1}Df_1 \\ \dots & \dots & \dots \\ D^{nN-1}f_{nN} - z^{-1}D^{nN}f_{nN} & \dots & f_{nN} - z^{-1}Df_{nN}\end{pmatrix}}{\mathrm{Wr}(f_1,\dots,f_{nN})}$$

(here derivative is with respect to t_1, we don't write dependence on f_i on t to avoid heavy notation) The left hand side can be written as

$$\frac{\det\begin{pmatrix} z^{nN}e^{\xi(t,z)} & \dots & \dots & e^{\xi(t,z)} \\ D^{nN}f_1 & \dots & \dots & f_1 \\ \dots & \dots & \dots & \dots \\ D^{nN}f_{nN} & \dots & \dots & f_{nN}\end{pmatrix}}{\mathrm{Wr}(f_1,\dots,f_{nN})}.$$

It is easy to check that these two expressions are equal. □

We want now to study the structure of $\mathfrak{L}_{W,N}$ with more attention; our investigation will lead us to discover that, actually, we are dealing with rational reductions ([21],[20]) of KP.

First of all we recall a useful lemma (proof can be found for instance in [32]).

Lemma 1.3.14. *Let $\{g_1, ..., g_m\}$ be a basis of linearly indipendent solutions of a differential operator K of order m. Then one can factorize K as*

$$K = (D + T_m)(D + T_{m-1}) \cdots (D + T_1)$$

with

$$T_j = \frac{\mathrm{Wr}(g_1, \ldots, g_{m-1})}{\mathrm{Wr}(g_1, \ldots, g_m)}.$$

We will state now properties of symmetry for $f_{s,N}$ that will be useful in the sequel.

Proposition 1.3.15. *The following equalities hold:*

$$f_{s,N+1}(t) = f_{s-n,N}(t) \tag{1.16}$$

$$f_{s,N}(t) = D^n f_{s+n,N}(t) \tag{1.17}$$

PROOF We will use the equality

$$\omega_s^{(-k)} = \omega_{s+n}^{(-k+n)}$$

which follows from the very definition of these coefficients.
Then for (1.16) we have

$$f_{s,N+1}(t) = \sum_k \omega_s^{(-k)} p_{k+n+nN-1}(t) = \sum_k \omega_{s-n}^{(-k-n)} p_{k+n+nN-1}(t) = \sum_k \omega_{s-n}^{(-k)} p_{k+nN-1}(t) = f_{s-n,N}(t)$$

For (1.17) we have

$$D^n(f_{s+n,N}(t)) = D^n \sum_k \omega_{s+n}^{(-k)} p_{k+nN-1}(t) = \sum_k \omega_{s+n}^{(-k)} p_{k+nN-1-n}(t) = \sum_k \omega_s^{(k-n)} p_{k-n+nN-1}(t) = f_{s,N}(t)$$

□

Theorem 1.3.16. *For every $N > 0$ the pseudodifferential operator $\mathcal{L}_{W,N}$ and its n^{th} power*

$$L_{W,N} = \mathcal{L}_{W,N}^n$$

can be factorized as

- $\mathcal{L}_{W,N} = \mathcal{L}_{1,W,N}(\mathcal{L}_{2,W,N})^{-1}$

- $L_{W,N} = M_{1,W,N}(M_{2,W,N})^{-1}$

where all the factors are differential operators and

- $\mathrm{ord}(\mathfrak{L}_{1,W,N}) = nN + 1, \quad \mathrm{ord}(\mathfrak{L}_{2,W,N}) = nN$
- $\mathrm{ord}(M_{1,W,N}) = 2n, \quad \mathrm{ord}(M_{2,W,N}) = n$

Hence, for every N, $\tau_{W,N} \in \mathrm{cKP}_{1,nN} \cap \mathrm{cKP}_{n,n}$.

PROOF The first factorization comes directly from the fact that

$$\mathfrak{L}_{W,N}(t) = \Delta_{W,N}(t) D (\Delta_{W,N}(t))^{-1}$$

For the second factorization we note that we have the factorization

$$L_{W,N}(t) = \Delta_{W,N}(t) D^n (\Delta_{W,N}(t))^{-1}$$

where the first operator $\Delta_{W,N}(t)D^n$ has order $M + n$ while the second (i.e. $\Delta_{W,N}$) has order nN. Moreover as follows from (1.17) we have

- $\Delta_{W,N} f_{i,N} = 0 \quad \forall i = 1, \ldots, nN$
- $\Delta_{W,N} D^n f_{i,N} = 0 \quad \forall i = n+1, \ldots, nN$

hence using lemma 1.3.14 one can simplify factorization above as

$$L_{W,N} = M_{1,W,N}(M_{2,W,N})^{-1}$$

where $M_{2,W,N}$ is given explicitely by the formula

$$M_{2,W,N} = (D + K_{n,N})(D + K_{n-1,N}) \ldots (D + K_{1,N})$$

with

$$K_{j,N} = D\left[\log\left(\frac{\mathrm{Wr}(f_{n+1,N}, \ldots, f_{nN,N}, f_{1,N}, \ldots, f_{j-1,N})}{\mathrm{Wr}(f_{n+1,N}, \ldots, f_{nN,N}, f_{1,N}, \ldots, f_{j,N})}\right)\right]$$

□

Theorem 1.3.17. *The sequence $\{\mathfrak{L}_{W,N}\}_{N \geq 1}$ satisfies recursion relation*

$$\mathfrak{L}_{W,N+1} = \mathfrak{T}_N \mathfrak{L}_{W,N} (\mathfrak{T}_N)^{-1} \tag{1.18}$$

with

$$\mathfrak{T}_N = (D + T_{n,N})(D + T_{n-1,N}) \ldots (D + T_{1,N})$$

$$T_j = D \log\left(\frac{\mathrm{Wr}(f_{1,N}, \ldots, f_{nN,N}, f_{1,N+1}, \ldots f_{j-1,N+1})}{\mathrm{Wr}(f_{1,N}, \ldots, f_{nN,N}, f_{1,N+1}, \ldots f_{j,N+1})}\right).$$

Proof We observe that thanks to (1.16)

- $\Delta_{W,N} f_{i,N} = 0 \quad \forall i = 1, \ldots, nN$
- $\Delta_{W,N+1} f_{i,N} = 0 \quad \forall i = n+1, \ldots, nN$
- $\Delta_{W,N+1} f_{i,N+1} = 0 \quad \forall i = 1, \ldots, n$

Hence using again (1.3.14) we obtain the recursion relation

$$\Delta_{W,N+1} = \mathcal{T}_N \Delta_{W,N}$$

and from this last equation we recover the recursion relation for the Lax operator. □

We want to point out that the first decomposition as well as the recursion formula are already known and, as pointed out in [22], come simply from the fact that we have a truncated dressing. Actually our sequence of $\{\tau_{W,N}\}_{N\geq 1}$ is a part of a sequence already studied by Dickey in [19] under the name of stabilizing chain; in that article Dickey already provided the recursion formula written above as well as some differential equations for coefficients of $\mathcal{T}_N$. Nevertheless, to our best knowledge, connection with block Toeplitz determinants never appeared before our article [1]. Also the fact that $\tau_{W,N} \in \mathrm{cKP_{n,n}}$ is something new. Now we want to go one step further and see what happens for $N \to \infty$. Obviously thanks to the property of stabilization stated in proposition 1.3.7 we can define a pseudodifferential operator $\mathcal{L}_W$ and a wave function ψ_W related to τ_W in the same way as for finite N and we will obtain a solution of KP as well. Actually a stronger statement holds.

Proposition 1.3.18. *Given $W \in \mathrm{Gr}^{(n)}$ the functions τ_W, ψ_W and $L_W := (\mathcal{L}_W)^n$ are respectively the τ function, the wave function and the differential operator of order n corresponding to a solution of n^{th} Gelfand-Dickey hierarchy.*

Proof It is known [3] that subspaces satisfying $z^n W \subseteq W$ correspond to solutions of n^{th} Gelfand-Dickey hierarchy. What we have to prove is that $L_W(t) = (L_W(t))_+$.
From the usual relation

$$\frac{\partial \psi_W}{\partial t_n}(t;z) = (L_W)_+ \psi_W(t;z) \tag{1.19}$$

we obtain immediately

$$\frac{\partial S_W}{\partial t_n} + S_W D^n = (L_W)_+ S_W$$

so that we have to prove that

$$\frac{\partial S_W}{\partial t_n} = 0$$

On the other hand

$$\psi_W(t;z) = \exp(\xi(t,z))(1 + \sum_{1=1}^{\infty} s_i(t) z^{-i})$$

where

$$S_W = 1 + \sum_{1=1}^{\infty} s_i(t) D^{-i}$$

Using this explicit expression for the wave function and substituting in (1.19) we obtain

$$(L_W)_+ \psi_W(t;z) - z^n \psi_W(t;z) = \exp\left(\xi(t,z)\right) \sum_{i=1}^{\infty} \frac{\partial s_i(t)}{\partial t_n} z^{-i}$$

The left hand side of this equation lies on $W(t) = \exp(\xi(t,z))W$ for every t so that multiplying both terms for $\exp(-\xi(t,z))$ one obtains that they belong to subspaces transverse one to the other (W and H_-), hence both of them vanish. This means that $\frac{\partial s_i}{\partial t_n} = 0$ for every i. □

In virtue of this proposition, when computing τ_W associated to $W \in \mathrm{Gr}^{(n)}$, we will always omit times t_{jn} multiple of n. Setting $\{t_{jn} = 0, j \in \mathbb{N}\}$ will be important in order to be able to apply Szegö-Widom's theorem; in this case we will write $\tilde{t}$ instead of t.

Proposition 1.3.19. *Take any $W \in \mathrm{Gr}^{(n)}$ in the big cell of $\mathrm{Gr}^{(n)}$ and a corresponding GD symbol $\mathcal{W}(t;z)$.*
Then

$$\tau_W(\tilde{t}) = \det(\mathcal{P}_{\mathcal{W}(\tilde{t};z)}). \tag{1.20}$$

PROOF All we have to prove is that conditions of Szegö-Widom's theorem are satisfied and $G(\mathcal{W}(\tilde{t};z)) = 1$. We observe that

$$\mathcal{W}(\tilde{t};z) \in L_{\frac{1}{2}}\mathrm{Gl}(n,\mathbb{C}) \quad \forall \tilde{t}$$

since we can always find $\mathcal{W}(z) \in L_{\frac{1}{2}}\mathrm{Gl}(n,\mathbb{C})$ such that $W = \mathcal{W}(z)H_+^{(n)}$ and $\exp(\xi(\tilde{t},Z))$ is continuously differentiable (obviously when restricted to a finite number of times).
Moreover

$$\det[\exp(\xi(\tilde{t},Z))] = 1$$

since we deleted times multiple of n and $\det(\mathcal{W}(z)) = 1 + O(z^{-1})$ by big cell assumption.
This implies that we have

$$\underset{0\le\theta\le 2\pi}{\Delta} \det\left(\mathcal{W}(\tilde{t}; e^{i\theta})\right) = 0$$

and

$$G(\mathcal{W}(\tilde{t}; z)) = 1.$$

□

We are now in the position to state the main result of this paper.

Theorem 1.3.20. *Given any point*

$$\mathcal{W}(z)H_+^{(n)} = W \in \mathrm{Gr}^{(n)}$$

and corresponding GD symbol

$$\mathcal{W}(t; z) = \exp\left(\sum_{i=1}^{\infty} t_i Z^i\right)\mathcal{W}(z)$$

the following facts hold true:

- $\{\tau_{W,N}(t) := D_N(\mathcal{W}(t; z))\}_{0\le N<\infty}$
  is a sequence of τ *functions for* KP *associated to wave function*

  $$\psi_{W,N}(t; z) = S_{W,N}(e^{\sum_{i=1}^{\infty} t_i z^i})$$

  and pseudodifferential Lax operator

  $$\mathcal{L}_{W,N} = S_{W,N} D S_{W,N}^{-1}.$$

  The dressing is given by the formula

  $$S_{W,N} = \Delta_{W,N} D^{-nN}.$$

- *For every* $N > 0$ *we have* $\tau_{W,N} \in \mathrm{cKP}_{1,nN} \cap \mathrm{cKP}_{n,n}$.

- *The sequence admits stable limit*

  $$\tau_W(\tilde{t}) = \lim_{N\to\infty} \tau_{W,N}(\tilde{t}).$$

  $\tau_W(\tilde{t})$ *is a solution of the* n^{th} *Gelfand-Dickey hierarchy and can be written as the Fredholm determinant*

  $$\tau_W(\tilde{t}) = \det\left(\mathcal{P}_{\mathcal{W}(\tilde{t};z)}\right).$$

PROOF The expression of the dressing as well as the expression of $\mathcal{L}_{W,N}$ are given in Proposition 1.3.12. Proposition 1.3.13 gives the expression of $\psi_{W,N}$ and proves at the same time that $\tau_{W,N}$ is the corresponding τ function. The fact that $\tau_{W,N} \in \mathrm{cKP}_{1,nN} \cap \mathrm{cKP}_{n,n}$ is proven in Theorem 1.3.16 while the existence of the stable limit $\tau_W(\tilde{t})$ is given by Propostion 1.3.7; Proposition 1.3.18 and Proposition 1.3.19 prove respectively that $\tau_W(\tilde{t})$ is a solution of the n^{th} Gelfand-Dickey hierarchy and that it can be written as a Fredholm determinant. □

Remark 1.3.21. Also all the $\tau_{W,N}(\tilde{t})$ can be expressed as Fredholm determinants; in order to give explicit expressions we need a certain Riemann-Hilbert factorization of symbol $\mathcal{W}(\tilde{t};z)$. This factorization will be obtained in section 5 and it will be exploited to express $\tau_{W,N}(\tilde{t})$ as a Fredholm determinant.

1.4 Riemann-Hilbert problem and Plemelj's integral formula.

It is evident from proposition 1.2.6 that Riemann-Hilbert decompositions of symbol γ for a block Toeplitz operator plays an important role in computing $D_\infty(\gamma)$.
Here we will show that actually Plemelj's operator itself enters in a integral equation (see [15]) giving solutions of Riemann-Hilbert problem

$$\varphi_+(z) = \gamma^{\mathrm{T}}(z)\varphi_-(z). \tag{1.21}$$

Here $\varphi_+(z)$ and $\varphi_-(z)$ are respectively analytical functions defined inside and outside the circle. In this section we consider a smaller class of loops; $\gamma(z)$ will be a matrix-valued function that extends analytically on a neighborhood of S^1. For convenience of the reader we recall here the main steps to arrive to Plemelj's integral formula [15].

Lemma 1.4.1. *Suppose that $f_+(z), f_-(z)$ are functions on S^1 satisfying*

$$|f(\zeta_2) - f(\zeta_1)| < |\zeta_2 - \zeta_1|^\mu C$$

for some positive constants μ, C and for every $\zeta_1, \zeta_2 \in S^1$. Necessary and sufficient conditions for $f_+(z)$ and $f_-(z)$ to be boundary values of analytic functions regular inside or outside $S^1 \subseteq \mathbb{C}$ and with value c at infinity are respectively

$$\frac{1}{2\pi i}\oint \frac{f_+(\zeta) - f_+(z)}{\zeta - z} d\zeta = 0 \tag{1.22}$$

$$\frac{1}{2\pi i}\oint \frac{f_-(\zeta) - f_-(z)}{\zeta - z} d\zeta + f_-(z) - c = 0 \tag{1.23}$$

We have to point out that here both ζ and z lies on S^1 so that one has to be careful and define (1.22) and (1.23) as appropriate limits. Namely one proves that taking ζ slightly inside or outside S^1 along the normal and making it approach to the circle we obtain the same result which will be, by definition, the value of our integral. Now suppose we want to find solutions of (1.21); we normalize the problem requiring φ_- taking value C at infinity. Taking an appropriate linear combination of (1.22) and (1.23) and using (1.21) we find that $\varphi_-(z)$ must satisfy the equation

$$C = \varphi(z) - \frac{1}{2\pi i}\oint \frac{(\gamma^{\mathrm{T}})^{-1}(z)\gamma^{\mathrm{T}}(\zeta) - I}{\zeta - z}\varphi(\zeta)d\zeta \tag{1.24}$$

Note that here we do not have to take any limit since the integrand is well defined for every point of S^1. We also want to consider the associate homogeneous equation

$$0 = \varphi(z) - \frac{1}{2\pi i}\oint \frac{(\gamma^{\mathrm{T}})^{-1}(z)\gamma^{\mathrm{T}}(\zeta) - I}{\zeta - z}\varphi(\zeta)d\zeta \tag{1.25}$$

as well as its adjoint

$$0 = \psi(z) + \frac{1}{2\pi i}\oint \frac{\gamma(z)\gamma^{-1}(\zeta) - I}{\zeta - z}\psi(\zeta)d\zeta \tag{1.26}$$

Obviously, as usual in Fredholm's theory, the equations (1.25) and (1.26) either have only trivial solution or they have the same number of linearly indipendent solutions.

Lemma 1.4.2. *Consider two adjoint RH problems*

$$\varphi_+(z) = \gamma(z)^{\mathrm{T}}\varphi_-(z) \tag{1.27}$$

$$\psi_+(z) = \gamma(z)\psi_-(z) \tag{1.28}$$

normalized as $\psi_-(\infty) = \varphi_-(\infty) = 0$.
Any solution φ_- of (1.27) is a solution of (1.25) as well as any solution ψ_+ of (1.28) is a solution of (1.26).

PROOF We just repeat computations made for non-homogeneous case. □

Now we introduce a new integrable operator acting on $H_+^{(n)}$ and prove that it is actually equal to the Plemelj's operator.

Definition 1.4.3. *For every $f \in H_+^{(n)}$ we define*

$$[\tilde{\mathcal{P}}_\gamma\psi](z) := \mathrm{pr}_+\left(\psi(z) + \frac{1}{2\pi i}\oint \frac{\gamma(z)\gamma^{-1}(\zeta) - I}{\zeta - z}\psi(\zeta)d\zeta\right) \tag{1.29}$$

where pr_+ *denote the projection onto* $H_+^{(n)}$.

Proposition 1.4.4.
$$\tilde{\mathcal{P}}_\gamma = \mathcal{P}_\gamma.$$

Proof We write $\tilde{\mathcal{P}}_\gamma$ in coordinates and verify we obtain the same as in (1.1). To do so as in the definition of integrals (1.22) and (1.23) we compute (1.29) imposing $|\zeta| < |z|$; the formula will hold when ζ approach to S^1 in the same way as in (1.22) and (1.23). For a consistency check we will prove we obtain the same result imposing $|\zeta| > |z|$. Let's start with $|\zeta| < |z|$; we have

$$\psi(z) + \frac{1}{2\pi i}\oint \frac{\gamma(z)\gamma^{-1}(\zeta) - I}{\zeta - z}\psi(\zeta)d\zeta =$$

$$\psi(z) + \frac{1}{2\pi i}\oint \sum_{k\geq 1}\frac{\zeta^k}{z^k}\Big(I - \sum_{p,q\in\mathbb{Z}}\gamma^{(p)}(\gamma^{-1})^{(q)}z^p\zeta^q\Big)\sum_{s\geq 0}\psi^{(s)}\zeta^s\frac{d\zeta}{\zeta}$$

Imposing $k+q+s=0$ we get that this is equal to

$$\psi(z) + \sum_{p\in\mathbb{Z}}\sum_{k\geq 1}\sum_{s\geq 0}\gamma^{(p)}(\gamma^{-1})^{(-k-s)}\psi^{(s)}z^{p-k} = \psi(z) + \sum_{t\in\mathbb{Z}}\sum_{k\geq 1}\sum_{s\geq 0}\gamma^{(t+k)}(\gamma^{-1})^{(-k-s)}\psi^{(s)}z^t$$

Taking the projection on $H_+^{(n)}$ we obtain exactly formula (1.1). Now for $|\zeta| > |z|$ we have

$$\psi(z) + \frac{1}{2\pi i}\oint \frac{\gamma(z)\gamma^{-1}(\zeta) - I}{\zeta - z}\psi(\zeta)d\zeta =$$

$$\psi(z) + \frac{1}{2\pi i}\oint \sum_{k\geq 0}\frac{\zeta^k}{z^k}\Big(\sum_{p,q\in\mathbb{Z}}\gamma^{(p)}(\gamma^{-1})^{(q)}z^p\zeta^q - I\Big)\sum_{s\geq 0}\psi^{(s)}\zeta^s\frac{d\zeta}{\zeta}$$

Imposing $q+s=k$ we arrive to

$$\sum_{k,s\geq 0}\sum_{p\in\mathbb{Z}}\gamma^{(p)}(\gamma^{-1})^{(k-s)}\psi^{(s)}z^{k+p} = \sum_{k,s\geq 0}\sum_{t\in\mathbb{Z}}\gamma^{(t-k)}(\gamma^{-1})^{(k-s)}\psi^{(s)}z^t$$

Taking the projection on $H_+^{(n)}$ we obtain that this is equal to $T(\gamma)T(\gamma^{-1})$ so that the two computations for $|\zeta| < |z|$ and for $|\zeta| > |z|$ coincide in virtue of lemma 1.2.3 □

Theorem 1.4.5. *Suppose we are given a symbol $\gamma(z)$ analytic in a neighborhood of S^1 and such that*
$$D_\infty(\gamma) \neq 0$$

Then the Riemann-Hilbert problem

$$\varphi_+(z) = \gamma(z)^{\mathrm{T}}\varphi_-(z)$$

normalized as $\varphi_-(\infty) = C$ admits (if existing) a unique solution.

PROOF Suppose we have two distinct solutions $(\varphi_{1-}, \varphi_{1+})$ and $(\varphi_{2-}, \varphi_{2+})$; taking the difference we obtain a non-trivial solution of (1.27). Then also (1.28) admits non trivial solutions and the same holds for (1.26). But this means that we have a non zero $\psi(z) \in H_+^{(n)}$ such that $[\mathcal{P}_\gamma\psi](z) = 0$ which is impossible since

$$\det(\mathcal{P}_\gamma) = D_\infty(\gamma) \neq 0$$

□

Existence of factorization will be treated in the next section for the specific case of Gelfand-Dickey symbols. For a general treatment of the problem of existence see [15].

1.5 Factorization for Gelfand-Dickey symbols

Here we will prove that for Gelfand-Dickey symbols we can write the unique solution of factorization (1.21) in terms of data $L_W(\tilde{t}), \psi_W(\tilde{t}; z)$. We recall that $L_W(\tilde{t})$ and $\psi_W(\tilde{t}; z)$ are the stable limits of $L_{W,N}(\tilde{t})$ and $\psi_{W,N}(\tilde{t}; z)$. They represent the differential operator and the wave function associated to the solution $\tau_W(\tilde{t})$. Our exposition here is closely related to [16]. At the end of the section we will use the factorization obtained to express any $\tau_{W,N}(\tilde{t})$ as a Fredholm determinant. As we have written before in the proof of proposition 1.3.18 we have the relation

$$L_W(\tilde{t})\psi_W(\tilde{t}; z) = z^n\psi_W(\tilde{t}; z) \tag{1.30}$$

where $\psi_W(\tilde{t}; z)$ admits asymptotic expansion

$$\psi_W(\tilde{t}; z) = \exp(\xi(\tilde{t}, z))(1 + O(z^{-1}))$$

Now out of ψ_W we construct n time-dependent functions

$$\psi_{W,i}(\tilde{t}; z) := D^i(\psi_W(\tilde{t}; z)) : i = 0, \ldots, n-1$$

belonging to the subspace $W \in \mathrm{Gr}$.

Definition 1.5.1.

$$\Psi_W(\tilde{t};z) := \begin{pmatrix} 1 & \zeta_1 & \dots & \zeta_1^{n-1} \\ 1 & \zeta_2 & \dots & \zeta_2^{n-1} \\ \dots & \dots & \dots & \dots \\ 1 & \zeta_n & \dots & \zeta_n^{n-1} \end{pmatrix}^{-1} \begin{pmatrix} \psi_{W,0}(\tilde{t};\zeta_1) & \psi_{W,1}(\tilde{t};\zeta_1) & \dots & \psi_{W,n-1}(\tilde{t};\zeta_1) \\ \psi_{W,0}(\tilde{t};\zeta_2) & \psi_{W,1}(\tilde{t};\zeta_2) & \dots & \psi_{W,n-1}(\tilde{t};\zeta_2) \\ \dots & \dots & \dots & \dots \\ \psi_{W,0}(\tilde{t};\zeta_n) & \psi_{W,1}(\tilde{t};\zeta_n) & \dots & \psi_{W,n-1}(\tilde{t};\zeta_n) \end{pmatrix}$$

where ζ_i is the i^{th} root of z.

Proposition 1.5.2. *The matrix $\Psi_W(\tilde{t};z)$ admits asympotic expansion*

$$\Psi_W(\tilde{t};Z) = \exp(\xi(\tilde{t};Z))(I + O(z^{-1}))$$

Moreover under the isomorphism $\Xi^{-1} : H \to H^{(n)}$ we can write $W \in \mathrm{Gr}^{(n)}$ as

$$W = \Psi_W(0,z)H_+^{(n)} \tag{1.31}$$

PROOF One has to note that the i^{th} column of matrix $\Psi_W(\tilde{t};z)$ is nothing but $\Xi^{-1}(\psi_{W,i}(\tilde{t},z))$ so that asymptotic expansion follows easily. Equation (1.31) corresponds to the fact that $\{z^{ns}\psi_{W,i}(0,z) : s \in \mathbb{Z}\}$ is a basis for W. □

Observe that, since we also have

$$W = \mathcal{W}(z)H_+^{(n)}$$

we obtain

$$\Psi_W(0,z) = \mathcal{W}(z)(I + O(z^{-1}).$$

From this equation and from lemma 1.2.3 it follows that for every $N > 0$ we have

$$T_N\Big(\mathcal{W}(\tilde{t};z)(I + O(z^{-1}))\Big) = T_N(\mathcal{W}(\tilde{t};z))T_N(I + O(z^{-1})).$$

Now since for every N

$$\det(T_N(I + O(z^{-1}))) = 1$$

we will assume, without loss of generality, that

$$\Psi_W(0,z) = \mathcal{W}(z)$$

since this is true modulo an irrelevant term that does not affect values of determinants we want to compute. We now want to define a matrix $\Phi_W(\tilde{t};z)$ analytic in z near 0 and with similar properties as $\Psi_W(\tilde{t};z)$.

Definition 1.5.3. *Let $\phi_W(\tilde{t};z)$ be the unique solution of*

$$L_W(\tilde{t})\phi_W(\tilde{t};z) = z^n\phi_W(\tilde{t};z)$$

analytic in $z=0$ and such that

$$(D^i\phi)(0,z) = z^i : i = 0,\ldots,n-1$$

We define

$$\Phi_W(\tilde{t};z) := \begin{pmatrix} 1 & \zeta_1 & \ldots & \zeta_1^{n-1} \\ 1 & \zeta_2 & \ldots & \zeta_2^{n-1} \\ \ldots & \ldots & \ldots & \ldots \\ 1 & \zeta_n & \ldots & \zeta_n^{n-1} \end{pmatrix}^{-1} \begin{pmatrix} \phi_{W,0}(\tilde{t};\zeta_1) & \phi_{W,1}(\tilde{t};\zeta_1) & \ldots & \phi_{W,n-1}(\tilde{t};\zeta_1) \\ \phi_{W,0}(\tilde{t};\zeta_2) & \phi_{W,1}(\tilde{t};\zeta_2) & \ldots & \phi_{W,n-1}(\tilde{t};\zeta_2) \\ \ldots & \ldots & \ldots & \ldots \\ \phi_{W,0}(\tilde{t};\zeta_n) & \phi_{W,1}(\tilde{t};\zeta_n) & \ldots & \phi_{W,n-1}(\tilde{t};\zeta_n) \end{pmatrix}$$

where as before ζ_i is the i^{th} root of z and

$$\phi_{W,i}(\tilde{t}) := D^i(\phi_W(\tilde{t};z)) : i = 0,\ldots,n-1.$$

Remark 1.5.4. $\Phi_W(\tilde{t};z)$ admits regular expansion in $z=0$ and Cauchy initial values we imposed on ϕ_W imply

$$\Phi_W(0;z) = I.$$

Proposition 1.5.5. *$\Psi_W(\tilde{t};z)\Phi_W^{-1}(\tilde{t};z)$ does not depend on t_i for any i.*

PROOF It is well known that equations

$$\frac{\partial}{\partial t_i}f = (L_W^{\frac{i}{n}})_+ f$$

satisfied by ϕ_W and ψ_W can be translated into matrix equations

$$\frac{\partial}{\partial t_i}F = FM$$

satisfied by $\Psi_W(\tilde{t};z)$ and $\Phi_W(\tilde{t};z)$ (one can write explicitely M in terms of coefficients of $(L_W^{\frac{i}{n}})_+$). Hence we have

$$\frac{\partial}{\partial t_i}(\Psi_W(\tilde{t};z)\Phi_W^{-1}(\tilde{t};z)) = \Psi_W(\tilde{t};z)M\Phi_W^{-1}(\tilde{t};z) - \\ \Psi_W(\tilde{t};z)\Phi_W^{-1}(\tilde{t};z)\Phi_W(\tilde{t};z)M\Phi_W^{-1}(\tilde{t};z) = 0$$

□

Theorem 1.5.6. *Given a Gelfand-Dickey symbol*

$$\mathcal{W}(\tilde{t}; z) = \exp\Big(\xi(\tilde{t}, Z)\Big)\mathcal{W}(z)$$

one can factorize it as

$$\mathcal{W}(\tilde{t}; z) = \left[\exp\Big(\xi(\tilde{t}, Z)\Big)\Psi_W(-\tilde{t}, z)\right]\Phi_W^{-1}(-\tilde{t}; z)$$

where the term inside the square bracket is analytic around $z = \infty$ and the other is analytic around $z = 0$. For assigned values of $\tilde{t}$ for which

$$\tau_{\mathrm{W}}(\tilde{t}) \neq 0$$

this is the unique solution of the factorization problem (1.21) normalized at infinity to the identity.

Proof Using the previous proposition we have

$$\mathcal{W}(\tilde{t}; z) = \exp \xi(\tilde{t}, Z)\mathcal{W}(z) = \exp(\xi(\tilde{t}, Z))\Psi_W(0, z) =$$
$$\exp(\xi(\tilde{t}, Z))\Psi(-\tilde{t}; z)\Phi_W^{-1}(-\tilde{t}; z)\Phi_W(0; z) = \exp(\xi(\tilde{t}, Z))\Psi(-\tilde{t}; z)\Phi_W^{-1}(-\tilde{t}; z)$$

Unicity of the factorization follows from section 4. □

Corollary 1.5.7. *For every $N > 0$*

$$\tau_{W,N}(\tilde{t}) = \tau_W(\tilde{t}) \det(I - K_{\mathcal{W}(\tilde{t};z),N})$$

with

$$(K_{\mathcal{W}(\tilde{t};z),N})_{ij} = \begin{cases} 0 & \text{if } \min\{i, j\} < N \\ \sum_{k=1}^{\infty}(\Psi_W(-\tilde{t}; z))^{(i+k)}(\Psi_W(-\tilde{t}; z)^{-1})^{(-j-k)} & \text{otherwise.} \end{cases}$$

Proof It is enough to apply Borodin-Okounkov formula using factorization obtained above. □

Corollary 1.5.8. *For every $N > 0$*

$$\frac{\tau_{W,N}(\tilde{t})}{\tau_{W,N+1}(\tilde{t})} = \det\Big(T\big(\Psi_W(-\tilde{t}; z)\big)T\big(\Psi_W(-\tilde{t}; z)^{-1}\big)\Big)_{N,N}$$

(observe that the right hand side of this equation is an ordinary $n \times n$ determinant, not a Fredholm determinant).

PROOF

$$\frac{\tau_{W,N}(\tilde{t})}{\tau_{W,N+1}(\tilde{t})} = \frac{\det(I - K_{\mathcal{W}(\tilde{t};z),N})}{\det(I - K_{\mathcal{W}(\tilde{t};z),N+1})}.$$

On the other hand the operator $(I - K_{\mathcal{W}(\tilde{t};z),N+1})^{-1}(I - K_{\mathcal{W}(\tilde{t};z),N})$ can be written as a block matrix obtained taking the identity matrix and replacing the N^{th} block column by the N^{th} block column of the matrix with (i,j)-entry equal to

$$\sum_{k=1}^{\infty}(\Psi_W(-\tilde{t};z))^{(i+k)}(\Psi_W(-\tilde{t};z)^{-1})^{(-j-k)}.$$

Hence proof is obtained applying lemma 1.2.3 □

1.6 Rank one stationary reductions and corresponding Gelfand-Dickey symbols

We want to describe, more explicitely, GD symbols corresponding to solutions of Gelfand-Dickey hierarchies obtained by rank one stationary reductions. In order to emphasize that we are dealing with rank-one generic case instead of the standard expression *Krichever locus* we will speak about *Burchnall-Chaundy locus.*

Definition 1.6.1. *Given a point $W \in Gr^{(n)}$ we say that W stays in Burchnall-Chaundy locus iff the Lax operator L_W of the corresponding solution satisfies*

$$[L_W, M_W] = 0$$

for some differential operator M_W of order m coprime with n. Without loss of generality we also assume $m > n$.

The name we use is due to the fact that, already in 1923, Burchnall and Chaundy were the first to study algebras of commuting differential operators in [25] where they stated this important proposition we will use in the sequel.

Proposition 1.6.2 ([25]). *Given a pair of commuting differential operator L, M with relatively prime orders it exists an irreducible polynomial $F(x,y)$ such that*

$$F(x,y) = x^m + ... \pm y^n$$

and $F(L,M) = 0$.

This proposition in particular allows us to associate to every Burchnall-Chaundy solution a spectral curve defined by polynomial relation existing between the pair of commuting differential operators. From the Grassmannian point of view one can define an action $\mathcal{A}$ of pseudodifferential operators in variable t_1 on H by

$$\mathcal{A} : \Psi\mathrm{DO} \times H \longrightarrow H$$
$$\left((t_1)^m \frac{\partial^n}{\partial t_1^n}, \varphi(z)\right) \longmapsto \left(\frac{\partial^n}{\partial z^n}\right)\left(z^n\right)\varphi(z)$$

and, using this action, prove the following propostitition

Proposition 1.6.3 ([6]). *Given a point W in the Burchnall-Chaundy locus one has*

$$z^n W \subseteq W \tag{1.32}$$
$$b(z) W \subseteq W \tag{1.33}$$

where L_W and M_W are of order n and m respectively and $b(z)$ is a series in z whose leading term is z^m. Conversely, if W satisfies above properties, it stays in the Burchnall-Chaundy locus.

PROOF We just sketch the proof and make reference to Mulase's article [6]. Suppose we are given L_W and M_W; under conjugation with the dressing $S_W(\tilde{t})$ we have

$$S_W^{-1}(\tilde{t}) L_W(\tilde{t}) S_W(\tilde{t}) = \frac{\partial^n}{\partial t_1^n}$$

Under the action $\mathcal{A}$ this gives invariance of W with respect to z^n while invariance with respect to $b(z)$ is obtained acting with

$$S_W^{-1}(\tilde{t}) M_W(\tilde{t}) S_W(\tilde{t})$$

Viceversa given W we reconstruct the dressing $S_W(\tilde{t})$; using it we define $L_W(\tilde{t})$ and $M_W(\tilde{t})$ conjugating pseudodifferential operators corresponding to z^n and $b(z)$. In particular observe that also z^n and $b(z)$ will satisfy the same polynomial relation as $L_W(\tilde{t})$ and $M_W(\tilde{t})$. □

Remark 1.6.4. Without loss of generality we can assume

$$\frac{1}{2\pi i} \oint \frac{b(z)}{z^{ns+1}} dz = 0 \quad \forall s \in \mathbb{Z}. \tag{1.34}$$

Now suppose we are given an element $W = \mathcal{W}(z)H_+^{(n)} \in Gr^{(n)}$ in the Burchnall-Chaundy locus. Using the explicit isomorphism Ξ we can construct a matrix $B(z) := b(Z)$ such that

$$B(z)W \subseteq W. \tag{1.35}$$

Proposition 1.6.5.

$$C(z) := \mathcal{W}^{-1}(z)B(z)\mathcal{W}(z)$$

has the following properties:

- *$C(z)$ is polynomial in z.*
- $\mathrm{trace}(C(z)) = 0$
- $m = \max_i(j - i + n \deg C_{ij}(z)) \quad \forall j = 1 \dots n$
- *The characteristic polynomial $p_{C(z)}(Z)$ of $C(z)$ defines the spectral curve of the solution.*

PROOF Equation (1.35) can be equivalently written as

$$\mathcal{W}^{-1}(z)B(z)\mathcal{W}(z)H_+^{(n)} \subseteq H_+^{(n)}$$

and this means precisely that $C(z)$ can't have terms in z^{-k} for any $k > 0$. The other properties are satisfied if and only if they are equally satisfied by $B(z)$ so that we will prove them for $B(z)$ instead of $C(z)$. $B(z)$ is traceless thanks to equation (1.34) and thanks to the fact that

$$\mathrm{trace}(Z^k) = 0 \quad \forall k \neq sn$$

The third properties is satisfied as $B(z) = b(\Lambda)$ represents in H multiplication by a series whose leading term is equal to m. For the last property we observe that if $F(x, y)$ is the polynomial defining the spectral curve, i.e. $F(L_W, M_W) = 0$, then we will have

$$F(\mathrm{diag}(z, z, \dots, z), B(z)) = 0$$

as well; on the other hand thanks to Cayley-Hamilton theorem we have

$$p_{B(z)}(B(z)) = 0.$$

Since F is irreducible and $p_{B(z)}(\lambda)$ has the same form

$$p_{B(z)}(\lambda) = \lambda^n + \dots \pm z^m$$

we conclude that they are equal. □

Observe that since $\mathcal{W}(z)$ is defined modulo multiplication on the left by invertible triangular matrices also $C(z)$ is defined modulo conjugation by elements of the group Δ of upper trianguar invertible matrices. It was a remarkable observation of Schwarz [26] that actually Burchnall-Chaundy locus can be described by means of matrices with properties as in proposition 1.6.5 modulo the action of Δ. Here we adapt the results of [26] to our situation. Namely we explain how, given $C(z)$, one can recover $\mathcal{W}(z)$ and the corresponding spectral curve.

Proposition 1.6.6. *Given a matrix $C(z)$ such that:*

- *$C(z)$ is polynomial in z.*
- $\mathrm{trace}(C(z)) = 0$
- $m = \max_i(j - i + n \deg C_{ij}(z)) \quad \forall j = 1 \dots n$

it exists a unique $W = \mathcal{W}(z)H_+^{(n)}$ in Burchnall-Chaundy locus such that its spectral curve is defined by $p_{C(z)}(\lambda)$.

In order to prove this proposition we need two lemmas.

Lemma 1.6.7. *Given a polynomial matrix $C(z)$ such that*

$$m = \max_i(j - i + n \deg C_{ij}(z)) \quad \forall j = 1 \dots n$$

(with m and n coprime) coefficients of characteristic polynomial

$$p_{C(z)}(\lambda) := \lambda^n + c_1(z)\lambda^{n-1} + \dots + c_n(z)$$

satisfy

$$n \deg c_s \leq ms \quad \forall s = 1, \dots, n-1$$
$$\deg c_n = m$$

PROOF From

$$n \deg C_{i,j} \leq m - j + i$$

and definition of determinant follows immediately that

$$n \deg c_s \leq ms \quad \forall s = 1, \dots, n.$$

Strict inequality for $s < n$ follows from the fact that m and n are coprime. For the equality

$$\deg c_n = \deg\Big(\det(C(z))\Big) = m$$

we observe that in every line there is a unique element $C_{ij}(z)$ such that $m = j - i + n \deg C_{ij}(z)$; taking this unique element for every line and multiplying them we will obtain the leading term of determinant which will be of order m. $\square$

Lemma 1.6.8. *The equation*

$$\lambda^n + c_1(z)\lambda^{n-1} + \ldots + c_n(z) = 0 \tag{1.36}$$

with

$$n \deg c_s \leq ms \quad \forall s = 1, \ldots, n-1$$
$$\deg c_n = m$$

and n, m coprime has n distinct solutions $\{\lambda_i = b(\zeta_i), \quad i = 1 \ldots n\}$ with

$$b(\zeta) = \zeta^m(1 + O(\zeta^{-1}))$$

(as usual ζ_i is the i^{th} root of z).

Proof Imposing $\lambda_i = \zeta_i^m$ we have a solution of the equation

$$(\zeta_i^m)^n + c_1(\zeta_i^n)(\zeta_i^m)^{n-1} + \ldots + c_n(\zeta_i^n) = 0$$

at the leading order mn. Then imposing $\lambda_i = \zeta_i^m(1 + l_1\zeta_i^{-1})$ and plugging it into the equation (1.36) one obtains

$$(\zeta_i^m + l_1\zeta_i^{m-1})^n + c_1(\zeta_i^n)(\zeta_i^m + l_1\zeta_i^{m-1})^{n-1} + \ldots + c_n(\zeta_i^n) = O(\zeta_i^{mn})$$

l_1 can be found so that terms of order $nm - 1$ in the equation vanish; going on solving the equation term by term we obtain

$$\lambda_i = \zeta_i^m\Big(1 + \sum_{j<0} l_j\zeta_i^{-j}\Big)$$

Clearly coefficients l_j do not depend on the choice of the root ζ_i so that it exists $b(\lambda)$ with stated properties. $\square$

Now we can prove proposition 1.6.6.
Proof We start computing the characteristic polynomial $p_{C(z)}(\lambda)$; thanks to lemmas 1.6.7 and 1.6.8 we find n distinct roots $b(\zeta_1), \ldots, b(\zeta_n)$ with properties stated above.
The aim is to find $\mathcal{W}(z)$ such that

$$\mathcal{W}(z)b(Z)\mathcal{W}^{-1}(z) = C(z)$$

Since we have n distinct solutions $\{b(\zeta_i), \quad i = 1, \dots, n\}$ of the equation

$$p_{C(z)}(\lambda) = 0$$

it exists a matrix $\Upsilon(\zeta_1, \dots, \zeta_n)$ such that

$$\Upsilon(\zeta_i)C(z)\Upsilon^{-1}(\zeta_i) = \begin{pmatrix} b(\zeta_1) & 0 & \dots & 0 \\ 0 & b(\zeta_2) & \dots & 0 \\ 0 & \dots & \ddots & 0 \\ 0 & \dots & \dots & b(\zeta_n) \end{pmatrix}$$

On the other hand it's easy to observe that the matrix Z can be diagonalized as

$$Z = \begin{pmatrix} 1 & \zeta_1 & \dots & \zeta_1^{n-1} \\ 1 & \zeta_2 & \dots & \zeta_2^{n-1} \\ \dots & \dots & \dots & \dots \\ 1 & \zeta_n & \dots & \zeta_n^{n-1} \end{pmatrix}^{-1} \begin{pmatrix} \zeta_1 & 0 & \dots & 0 \\ 0 & \zeta_2 & \dots & 0 \\ 0 & \dots & \ddots & 0 \\ 0 & \dots & \dots & \zeta_n \end{pmatrix} \begin{pmatrix} 1 & \zeta_1 & \dots & \zeta_1^{n-1} \\ 1 & \zeta_2 & \dots & \zeta_2^{n-1} \\ \dots & \dots & \dots & \dots \\ 1 & \zeta_n & \dots & \zeta_n^{n-1} \end{pmatrix}$$

and this means that multiplication by $b(z)$ can be written in $H_+^{(n)}$ as multiplication by

$$\begin{pmatrix} 1 & \zeta_1 & \dots & \zeta_1^{n-1} \\ 1 & \zeta_2 & \dots & \zeta_2^{n-1} \\ \dots & \dots & \dots & \dots \\ 1 & \zeta_n & \dots & \zeta_n^{n-1} \end{pmatrix}^{-1} \begin{pmatrix} b(\zeta_1) & 0 & \dots & 0 \\ 0 & b(\zeta_2) & \dots & 0 \\ 0 & \dots & \ddots & 0 \\ 0 & \dots & \dots & b(\zeta_n) \end{pmatrix} \begin{pmatrix} 1 & \zeta_1 & \dots & \zeta_1^{n-1} \\ 1 & \zeta_2 & \dots & \zeta_2^{n-1} \\ \dots & \dots & \dots & \dots \\ 1 & \zeta_n & \dots & \zeta_n^{n-1} \end{pmatrix}$$

Hence we have

$$\mathcal{W}(z) = \Upsilon^{-1}(\zeta_i) \begin{pmatrix} 1 & \zeta_1 & \dots & \zeta_1^{n-1} \\ 1 & \zeta_2 & \dots & \zeta_2^{n-1} \\ \dots & \dots & \dots & \dots \\ 1 & \zeta_n & \dots & \zeta_n^{n-1} \end{pmatrix}$$

Note that $\mathcal{W}(z)$ is defined modulo the action of Δ so that, by construction, $C(z)$ corresponds to a unique $W \in \mathrm{Gr}^{(n)}$ such that

$$W = \mathcal{W}(z)H_+^{(n)}.$$

□

Remark 1.6.9. As it was pointed out by Schwarz [26], matrices $C(z)$ with properties stated above can be used to describe points in the Grassmannian describing string solutions of Gelfand-Dickey hierarchies, i.e. solutions associated to reduction of type

$$[L, M] = 1$$

This class of solutions has not been treated here since they do not live in Segal-Wilson Grassmannian but just on Sato's Grassmannian constructed on the space of formal series; this means that we cannot use any more Szegö-Widom theorem as the analytical requirements are not satisfied. Nevertheless some results obtained in section 3 still hold since the property of stability for $\{\tau_{W,N}(t)\}$ does not depend on analytical properties of the symbol $\mathcal{W}(z)$.

Example 1.6.10 *(Symmetric n-coverings).* Take a symmetric n-covering $\mathcal{C}$ of $\mathbb{P}^1$ given by equation

$$\lambda^n = P(z) = \prod_{j=1}^{nk+1} (z - a_j) \tag{1.37}$$

For this particular type of curves, choosing in a appropriate way the divisor on the curve, we can write explicitely $\mathcal{W}(z), B(z)$ and $C(z)$. We start to observe that for any W corresponding to this spectral curve we have $b(z)W \subseteq W$ with

$$b(z) = P(z^n)^{\frac{1}{n}}$$

Then it's easy to prove that the corresponding $B(z) = b(\Lambda)$ can be written as

$$B(z) = \begin{pmatrix} 0 & 0 & \dots & 0 & z^{\frac{n-1}{n}} P(z)^{\frac{1}{n}} \\ z^{-\frac{1}{n}} P(z)^{\frac{1}{n}} & 0 & \dots & 0 & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & z^{-\frac{1}{n}} P(z)^{\frac{1}{n}} & 0 \end{pmatrix}$$

Now we define n functions

$$w_i(z) := \left(\frac{P(z)}{z}\right)^{\frac{i-1}{n}} \frac{1}{\prod_{j=1}^{(i-1)k} (z - a_j)}, \quad i = 1, \dots, n.$$

We take

$$\mathcal{W} := \operatorname{diag}(w_1(z), \dots, w_n(z))$$

It is easy to verify that the matrix

$$C(z) = \mathcal{W}^{-1}(z)B(z)\mathcal{W}(z) =$$

$$\begin{pmatrix} 0 & 0 & \dots & 0 & z^{\frac{n-1}{n}}P(z)^{\frac{1}{n}}\frac{w_n(z)}{w_1(z)} \\ z^{-\frac{1}{n}}P(z)^{\frac{1}{n}}\frac{w_1(z)}{w_2(z)} & 0 & \dots & 0 & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & z^{-\frac{1}{n}}P(z)^{\frac{1}{n}}\frac{w_{n-1}(z)}{w_n(z)} & 0 \end{pmatrix}$$

is polynomial in z. It is worth noticing that this example already gives all possible double coverings; hence for any (possibly singular) hyperelliptc surface we found (assigning a particular divisor) the GD symbol of the corresponding algebro-geometric rank one solution of KdV.

Example 1.6.11 *(Rational solutions).* As pointed out by Segal and Wilson [3], subspace of Burchnall-Chaundy locus corresponding to rational curves are given by $W = \mathcal{W}(z)H_+^{(n)}$ with $\mathcal{W}(z)$ rational in z. In particular the corresponding Gelfand-Dickey symbol will satisfy hypothesis given in proposition 1.2.5 so that we recover the following (known) result.

Proposition 1.6.12. *Every rational solution of Gelfand-Dickey hierarchies can be written as a finite-size determinant.*

For instance, for $n = 2$, taking

$$\mathcal{W}(z) = \begin{pmatrix} 1 - d^2z^{-1} & 0 \\ 0 & 1 - c^2z^{-1} \end{pmatrix} H_+^2$$

the inverse of Gelfand-Dickey symbol is equal to

$$\mathcal{W}^{-1}(\tilde{t}; z) =$$

$$\begin{pmatrix} \cosh(z^{\frac{1}{2}}\left(\sum_{i\geq 0} t_{2i+1}z^{2i}\right)) & -z^{\frac{1}{2}}\sinh(z^{\frac{1}{2}}\left(\sum_{i\geq 0} t_{2i+1}z^{2i}\right)) \\ -z^{-\frac{1}{2}}\sinh(z^{\frac{1}{2}}\left(\sum_{i\geq 0} t_{2i+1}z^{2i}\right)) & \cosh(z^{\frac{1}{2}}\left(\sum_{i\geq 0} t_{2i+1}z^{2i}\right)) \end{pmatrix} \begin{pmatrix} \frac{z}{z-d^2} & 0 \\ 0 & \frac{z}{z-c^2} \end{pmatrix}$$

Simply taking the residue one obtains that the corresponding τ function will be equal to

$$\tau_W(t_1, t_3, \ldots) = \det \begin{pmatrix} \cosh\left(\sum_{i\geq 0} t_{2i+1}d^{2i+1}\right) & -d\sinh\left(\sum_{i\geq 0} t_{2i+1}d^{2i+1}\right) \\ -c^{-1}\sinh\left(\sum_{i\geq 0} t_{2i+1}c^{2i+1}\right) & \cosh\left(\sum_{i\geq 0} t_{2i+1}c^{2i+1}\right) \end{pmatrix}$$

and recover 2-solitons solution for KdV.

We want to point out that, for algebro geometric solutions treated in this section, the problem of factorization for Gelfand Dickey symbol can be easily translated into a Riemann-Hilbert problem on some cuts on the plane with constant jumps. For simplicity we reduce to the case $n = 2$; the procedure used here is equivalent to the one used by Its, Jin and Korepin in [28] and generalized by Its, Mezzadri and Mo in [29]. Suppose we want to solve the factorization problem

$$\mathcal{W}(\tilde{t}; z) := \exp\Big(\xi(\tilde{t}, \Lambda)\Big)\mathcal{W}(z) = T_-(\tilde{t}; z)T_+(\tilde{t}; z)$$

for our GD symbol with $\mathcal{W}(z) = \operatorname{diag}(w_1(z), w_2(z))$ as in example 1.6.10; since it will appear many times we denote A the matrix

$$A := \begin{pmatrix} 1 & \sqrt{z} \\ 1 & -\sqrt{z} \end{pmatrix}$$

Also we impose

$$P(z) := \prod_{j=1}^{2g+1}(z - a_j)$$

with all a_j having modulo less then 1 and

$$\|a_1\| < \|a_2\| < \ldots < \|a_{2g+1}\|$$

We denote $l_1, ...l_{g+1}$ the oriented intervals $(a_1, a_2), (a_3, a_4), ...(a_{2g+1}, \infty)$. Instead of looking for $T_-(\tilde{t}; z)$ and $T_+(\tilde{t}; z)$ we define a new matrix $S(\tilde{t}; z)$ imposing

$$\begin{cases} S(\tilde{t}; z) := A\exp\big(-\xi(\tilde{t}, \Lambda)\big)T_-(\tilde{t}; z) & z \geq 1 \\ \\ S(\tilde{t}; z) := A\mathcal{W}(z)T_+^{-1}(\tilde{t}; z) & z \leq 1 \end{cases}$$

Proposition 1.6.13. *$S(\tilde{t}; z)$ has the following properties:*

- *It has no jumps on S^1*
- *It has jumps on intervals l_j; precisely calling $S_L(\tilde{t}; z)$ and $S_R(\tilde{t}; z)$ the values of $S(\tilde{t}; z)$ approaching from the left and approching from the right the interval we have*

$$S_L(\tilde{t}; z) := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} S_R(\tilde{t}; z)$$

- *It is invertible in any points but a_j; there it has singular behaviour of type*

$$S(\tilde{t};z) \sim \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} (z-a_j)^{\begin{pmatrix} 1 & 0 \\ 0 & \pm\frac{1}{2} \end{pmatrix}} S_j(\tilde{t};z)$$

with $S_j(\tilde{t};z)$ invertible in a_j; minus is for $a_1,\ldots,a_g$, plus for the others.

- *At infinity it behaves as*

$$S(\tilde{t};z) \sim \begin{pmatrix} \exp\Big(-\sqrt{z}(t_1 z + t_3 z + \ldots)\Big) & \sqrt{z}\exp\Big(-\sqrt{z}(t_1 z + t_3 z + \ldots)\Big) \\ \exp\Big(\sqrt{z}(t_1 z + t_3 z + \ldots)\Big) & -\sqrt{z}\exp\Big(\sqrt{z}(t_1 z + t_3 z + \ldots)\Big) \end{pmatrix}$$

PROOF Let's call $S_+(\tilde{t};z)$ and $S_-(\tilde{t};z)$ the limiting values of $S(\tilde{t};z)$ approaching the unit circle from inside and outside; we have

$$\begin{gathered} S_-(\tilde{t};z)S_+^{-1}(\tilde{t};z) = A\exp(-\xi(\tilde{t},z))T_-(\tilde{t};z)T_+(\tilde{t};z)\mathcal{W}^{-1}(z)A^{-1} = \\ A\exp(-\xi(\tilde{t},z))\exp(\xi(\tilde{t},z))\mathcal{W}(z)\mathcal{W}^{-1}(z)A^{-1} = I \end{gathered}$$

and this proves we haven't any jumps on S^1.
Writing explicitely $S(\tilde{t};z)$ as

$$\begin{cases} S(\tilde{t};z) = \begin{pmatrix} \exp\Big(-\sqrt{z}(t_1 z + t_3 z + \ldots)\Big) & \sqrt{z}\exp\Big(-\sqrt{z}(t_1 z + t_3 z + \ldots)\Big) \\ \\ \exp\Big(\sqrt{z}(t_1 z + t_3 z + \ldots)\Big) & -\sqrt{z}\exp\Big(\sqrt{z}(t_1 z + t_3 z + \ldots)\Big) \end{pmatrix} T_-(\tilde{t};z) \quad z \geq 1 \\ \\ S(\tilde{t};z) = \begin{pmatrix} 1 & \dfrac{(P(z))^{\frac{1}{2}}}{\prod_{j=0}^{g}(z-a_j)} \\ \\ 1 & -\dfrac{(P(z))^{\frac{1}{2}}}{\prod_{j=0}^{g}(z-a_j)} \end{pmatrix} T_+^{-1}(\tilde{t};z) \quad z \leq 1 \end{cases}$$

we obtain almost immediately the other points of the proposition; the only thing we have to observe is that both $T_+(\tilde{t};z)$ and $T_-(\tilde{t};z)$ are invertible inside and outside the circle respectively. This is because we have

$$\det \mathcal{W}(\tilde{t};z) = \frac{(P(z))^{\frac{1}{2}}}{\prod_{j=0}^{g}(z-a_j)} = \det(T_+(\tilde{t};z))\det(T_-(\tilde{t};z))$$

This condition combined with

$$\lim_{z\to\infty}\det(T_-(\tilde{t};z)) = 1$$

gives

$$\det(T_+(\tilde{t};z)) = 1$$
$$\det(T_-(\tilde{t};z)) = \frac{(P(z))^{\frac{1}{2}}}{\prod_{j=0}^{g}(z-a_j)}.$$

□

The Riemann-Hilbert problem given by propostion 1.6.13 is equivalent to the one proposed in section 5. What can be done is to write explicitely the solution $S(\tilde{t};z)$ using θ functions associated to the curve; this is what has been done in [28] and [29]. Actually comparing previous proposition with results obtained in section 5 we immediately realize that

$$S(\tilde{t};z) = \begin{pmatrix} \psi_{W,0}(\tilde{t};\sqrt{z}) & \psi_{W,1}(\tilde{t};\sqrt{z}) \\ \psi_{W,0}(\tilde{t};-\sqrt{z}) & \psi_{W,1}(\tilde{t};-\sqrt{z}) \end{pmatrix}$$

so that all we have to do in our case is to write down Baker-Akhiezer function in terms of special functions. We can carry on the same procedure for n arbitary; the only difference will be that the jump matrices will remain constant but more complicated; in any case the solution of this Riemann-Hilbert problem with constant jumps will be

$$S(\tilde{t};z) = \begin{pmatrix} \psi_{W,0}(\tilde{t};\zeta_1) & \psi_{W,1}(\tilde{t};\zeta_1) & \dots & \psi_{W,n-1}(\tilde{t};\zeta_1) \\ \psi_{W,0}(\tilde{t};\zeta_2) & \psi_{W,1}(\tilde{t};\zeta_2) & \dots & \psi_{W,n-1}(\tilde{t};\zeta_2) \\ \dots & \dots & \dots & \dots \\ \psi_{W,0}(\tilde{t};\zeta_n) & \psi_{W,1}(\tilde{t};\zeta_n) & \dots & \psi_{W,n-1}(\tilde{t};\zeta_n) \end{pmatrix}$$

Explicit formulas involving special functions can be used here to apply proposition 1.2.6 to our case. For instance taking the elliptic curve $\mathcal{C}$ given by equation

$$w^2 = 4z^3 - g_2 z - g_3$$

with uniformization given by the Weierstass $\wp$ function

$$(z,w) = (\wp(u), \wp'(u))$$

one can write wave function as

$$\psi(x,t,u) := \frac{\sigma(u-c-x)\sigma(c)}{\sigma(u-c)\sigma(x+c)} \exp\left(x\zeta(u) - \frac{1}{2}t\wp'(u)\right)$$

(here ζ and σ are Weierstrass ζ and σ function respectively, x and t correspond to the first and the third time). With some tedious computations, making the change of variables $u = u(z)$, the right hand side of equation (1.4) can be obtained. It turns out that the only relevant factorization is the one given by

$$\mathcal{W}^{-1}(x,t;u) = \left[\mathcal{W}^{-1}(u)\Psi(-x,-t;u)\right]\left[\Psi^{-1}(-x,-t;u)\exp(-x\Lambda - t\Lambda^3)\right]$$

where as before (we just wrote z as a function of u) we have

$$\Psi(x,t;u) := \begin{pmatrix} 1 & (\wp(u))^{\frac{1}{2}} \\ 1 & -(\wp(u))^{\frac{1}{2}} \end{pmatrix}^{-1} \begin{pmatrix} \psi(x,t,u) & \partial_x\psi(x,t,u) \\ \psi(x,t,-u) & \partial_x\psi(x,t,-u) \end{pmatrix}$$

Plugging into equation (1.4) we obtain

$$\frac{d}{dx}\tau(x,t) = Kt + 2\zeta(-c) - 2\zeta(x-c)$$

(here K is some constant); taking another derivative we obtain elliptic solution of KdV as expected.

Chapter 2

Matrix biorthogonal polynomials on the unit circle and non-abelian Ablowitz-Ladik hierarchy.

2.1 2D-Toda; linearization and biorthogonal polynomials.

In this section we recall some basic facts about 2D-Toda hierarchy as presented in [39]. Moreover we describe the connection with biorthogonal polynomials as originally explained in [41].
We are interested in the semi-infinite case; we start denoting with Λ the shift matrix

$$\Lambda := (\delta_{ij})_{i,j\geq 0}.$$

For the transpose we use the notation $\Lambda^T = \Lambda^{-1}$. Then we define two Lax matrices

$$\begin{cases} L_1 := \Lambda + \sum_{i\leq 0} a_i^{(1)}\Lambda^i \\ \\ L_2 := a_{-1}^{(2)}\Lambda^{-1} + \sum_{i\geq 0} a_i^{(2)}\Lambda^i \end{cases}$$

where $\{a_i^{(s)}, s = 1,2\}$ are some diagonal matrices. 2D-Toda equations, expressed in Lax form, arises as compatibility conditions for the following

Zakarov-Shabat spectral problem:

$$\begin{cases} L_1\Psi_1 = z\Psi_1 \\ L_2^T\Psi_2^* = z^{-1}\Psi_2^* \\ \partial_{t_n}\Psi_1 = (L_1^n)_+\Psi_1 \\ \partial_{t_n}\Psi_2^* = -(L_1^n)_+^T\Psi_2^* \\ \partial_{s_n}\Psi_1 = (L_2^n)_-\Psi_1 \\ \partial_{s_n}\Psi_2^* = -(L_2^n)_-^T\Psi_2^* \end{cases}$$

Here we introduced two infinite sets of times $\{t_i, i \geq 0\}$ and $\{s_i, i \geq 0\}$. We denoted with N_+ the upper triangular part of a matrix N (including the main diagonal) and with N_- the lower triangular part (excluding the main diagonal). Ψ_1 and Ψ_2^* are semi-infinte column vectors of type

$$\Psi_1(z) = (\Psi_{1,0}(z), \Psi_{1,1}(z), \dots)^T$$
$$\Psi_2^*(z) = (\Psi_{2,0}^*(z), \Psi_{2,1}^*(z), \dots)^T.$$

For every k the two expressions $e^{-\xi(t,z)}\Psi_{1,k}(z)$ and $e^{-\xi(s,z)}\Psi_{2,k}^*(z^{-1})$ are polynomials in z of order k. Lax equations are written as

$$\partial_{t_n}L_i = \left[(L_1^n)_+, L_i\right] \quad \partial_{s_n}L_i = \left[(L_2^n)_-, L_i\right], \quad i = 1, 2.$$

It should be noted that, while in the first section we had Lax equations for (pseudo-)differential operators, here we have Lax equations for matrices. Nevertheless we use the same letter L (plus different subscripts) since this is the standard notation used in almost every article. The same holds for the letters ψ and Ψ which are used for wave functions in the first secion and for wave vectors in the second.

2D-Toda equations can be linearized with a procedure very similar to the one used with Grassmannians in the first section. We describe it as presented in [39]. We start with an initial value matrix $M(0,0) = \{M_{ij}(0,0)\}_{i,j\geq 0}$ and we define its time evolution through the equation

$$M(t;s) := \exp\Big(\xi(t,\Lambda)\Big)M(0,0)\exp\Big(-\xi(s,\Lambda^{-1})\Big).$$

We assume that there exist a factorization

$$M(0,0) = S_1(0,0)^{-1}S_2(0,0).$$

Here S_1 is lower triangular while S_2 is upper triangular. We assume that both S_1 and S_2 have non zero elements on the main diagonal and we normalize them in such a way that every element on the main diagonal of S_1 is equal to 1. Moreover we consider values of t and s for which we can write

$$M(t,s) = S_1(t,s)^{-1}S_2(t,s) \tag{2.1}$$

with S_1 and S_2 having the same properties as above. Now we denote with $\chi(z)$ the infinite vector $\chi(z) := \begin{pmatrix} 1 & z & z^2 & \ldots \end{pmatrix}^T$. Wave vectors for 2D-Toda and Lax matrices are constructed in the following way.

Theorem 2.1.1 ([39]). *The wave vectors*

$$\Psi_1(z) := \exp(\xi(t,z))S_1\chi(z)$$
$$\Psi_2^*(z) := \exp(-\xi(s,z^{-1}))(S_2^{-1})^T\chi(z^{-1}).$$

and the two Lax operators $L_1 := S_1\Lambda S_1^{-1}$ and $L_2 := S_2\Lambda^{-1}S_2^{-1}$ satisfy 2D-*Toda Zakharov-Shabat spectral problem.*

PROOF We just sketch the proof and make reference to the article [39]. It is clear that the matrix $M(t,s)$ satisfies differential equations

$$\partial_{t_i}M = \Lambda^i M$$
$$\partial_{s_i}M = -M\Lambda^{-i}.$$

Then it is easy to deduce Sato's equations

$$\partial_{t_n}S_1 = -(L_1^n)_-S_1$$
$$\partial_{t_n}S_2 = (L_1^n)_+S_2$$
$$\partial_{s_n}S_1 = (L_2^n)_-S_1$$
$$\partial_{s_n}S_2 = -(L_2^n)_+S_2.$$

and Zakharov-Shabat's equations can be deduced from the expression of wave vectors in terms of S_1 and S_2. □

The last thing we need is the link between factorization of M and biorthogonal polynomials; we introduce a bilinear pairing on the space of polynomials in z defining

$$< z^i, z^j >_M := M_{ij}.$$

The following proposition is a direct consequence of (2.1).

Proposition 2.1.2 ([41]).

$$q^{(1)} = (q_i^{(1)})_{i\geq 0} := S_1\chi(z)$$
$$q^{(2)} = (q_i^{(2)})_{i\geq 0} := (S_2^{-1})^T\chi(z)$$

are biorthonormal polynomials with respect to the pairing $<,>_M$; i.e.

$$< q_i^{(1)}, q_j^{(2)} >_M = \delta_{ij} \quad \forall\, i,j \in \mathbb{N}.$$

2.2 From Toeplitz lattice hierarchy to semidiscrete zero-curvature equations for Ablowitz-Ladik hierarchy.

In this section we briefly recall the reduction from 2D-Toda to Toeplitz lattice as described in [41]. Then we will show how Ablowitz-Ladik equations are easily obtained from Toeplitz lattice.
Suppose that our initial value $M(0,0)$ is a Toeplitz matrix; i.e. we have

$$M(0,0) = T(\gamma) = \begin{pmatrix} \gamma^{(0)} & \gamma^{(-1)} & \gamma^{(-2)} & \dots \\ \gamma^{(1)} & \gamma^{(0)} & \gamma^{(-1)} & \dots \\ \gamma^{(2)} & \gamma^{(1)} & \gamma^{(0)} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

for some formal power series $\gamma(z) = \sum_{n\in\mathbb{Z}} \gamma^{(n)} z^n$. Now since $\Lambda = T(z^{-1})$ is an upper triangular Toeplitz matrix it follows easily (see for instance [49]) that

$$M(t,s) = \exp\Big(\xi(t,\Lambda)\Big) M(0,0) \exp\Big(-\xi(s,\Lambda^{-1})\Big) = \\ T\Big(\exp\big(\xi(t,z^{-1})\big)\gamma(z)\exp\big(-\xi(s,z)\big)\Big).$$

This means that Toeplitz form in conserved along 2D-Toda flow, hence we are dealing with a reduction of it. This reduction is called Toeplitz lattice in [41]. In that article the authors noticed, in the introduction, that this is nothing but Ablowitz-Ladik hierarchy. Now we will describe how to obtain the original formulation of Ablowitz-Ladik equations starting from Adler-van Moerbeke's formulation.
The key observation is that, in this case, the bilinear pairing $< p, q >_M$ between two arbitrary polynomials is given by

$$< p, q >_M = \oint p(z)\gamma(z)q^*(z)\frac{dz}{2\pi i z}.$$

Here the symbol of integration means that we are taking the residue of the formal series $p(z)\gamma(z)q^*(z)$ and $q^*(z) = q(z^{-1})$. In other words $q_i^{(1)}$ and $q_j^{(2)}$ are nothing but orthonormal polynomials on the unit circle. We also define monic biorthogonal polynomials

$$p^{(1)} = (p_i^{(1)})_{i\geq 0} := S_1\chi(z)$$
$$p^{(2)} = (p_i^{(2)})_{i\geq 0} := h(S_2^{-1})^T\chi(z)$$

with $h = \operatorname{diag}(h_0, h_1, h_2, \ldots)$ some diagonal matrix.
Now given an arbitrary polynomial $q(z)$ we define its reversed polynomial $\tilde{q}(z) := z^n q^*(z)$ and reflection coefficients

$$x_n := p_n^{(1)}(0) \quad y_n := p_n^{(2)}(0).$$

We can state the standard recursion relation associated to biorthogonal polynomials on the unit circle.

Proposition 2.2.1 ([42]). *The following recursion relation holds:*

$$\begin{pmatrix} p_{n+1}^{(1)}(z) \\ \tilde{p}_{n+1}^{(2)}(z) \end{pmatrix} = \mathcal{L}_n \begin{pmatrix} p_n^{(1)}(z) \\ \tilde{p}_n^{(2)}(z) \end{pmatrix} = \begin{pmatrix} z & x_{n+1} \\ zy_{n+1} & 1 \end{pmatrix} \begin{pmatrix} p_n^{(1)}(z) \\ \tilde{p}_n^{(2)}(z) \end{pmatrix} \tag{2.2}$$

Using this recursion relation Adler and van Moerbeke in [41] wrote the peculiar form of Lax operators for the Toeplitz reduction.

Proposition 2.2.2 ([41]). *Lax operators of Toeplitz lattice are of the following form:*

$$h^{-1}L_1h = \begin{pmatrix} -x_1y_0 & 1-x_1y_1 & 0 & \ldots & \ldots \\ -x_2y_0 & -x_2y_1 & 1-x_2y_2 & 0 & \ldots \\ -x_3y_0 & -x_3y_1 & -x_3y_2 & 1-x_3y_3 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

$$L_2 = \begin{pmatrix} -x_0y_1 & -x_0y_2 & -x_0y_3 & \ldots & \ldots \\ 1-x_1y_1 & -x_1y_2 & -x_1y_3 & \ldots & \ldots \\ 0 & 1-x_2y_2 & -x_2y_3 & \ldots & \ldots \\ \vdots & 0 & 1-x_3y_3 & \ldots & \ldots \\ \vdots & \vdots & 0 & \ldots & \ldots \end{pmatrix}.$$

Corollary 2.2.3. *In particular we obtain*

$$\frac{h_{n+1}}{h_n} = 1 - x_{n+1}y_{n+1}.$$

We can now state the theorem that relate Toeplitz lattice to the original form of Ablowitz-Ladik hierarchy.

Theorem 2.2.4. *Toeplitz lattice flow can be written in the form*

$$\partial_{t_i}\begin{pmatrix} p_n^{(1)}(z) \\ \tilde{p}_n^{(2)}(z)\end{pmatrix} = \mathcal{M}_{t_i,n}\begin{pmatrix} p_n^{(1)}(z) \\ \tilde{p}_n^{(2)}(z)\end{pmatrix} \tag{2.3}$$

$$\partial_{s_i}\begin{pmatrix} p_n^{(1)}(z) \\ \tilde{p}_n^{(2)}(z)\end{pmatrix} = \mathcal{M}_{s_i,n}\begin{pmatrix} p_n^{(1)}(z) \\ \tilde{p}_n^{(2)}(z)\end{pmatrix} \tag{2.4}$$

for some matrices $\mathcal{M}_{t_i,n}, \mathcal{M}_{s_i,n}$ depending on $\{x_j, y_j, z\}$.

Proof We prove it for times t.
We denote $d([z]) = \operatorname{diag}(1, z, z^2, z^3, \ldots)$. We have the identities

$$\Psi_1(z) = \exp(\xi(t,z))p^{(1)}(z)$$
$$\Psi_2^*(z) = h^{-1}d([z^{-1}])\exp(-\xi(s,z^{-1}))\tilde{p}^{(2)}(z)$$

that gives the following time evolution for orthogonal polynomials

$$\partial_{t_i}p^{(1)}(z) = -z^i p^{(1)}(z) + (L_1^i)_+ p^{(1)}(z)$$
$$\partial_{t_i}\tilde{p}^{(2)}(z) = -hd([z])(L_1^i)_{++}^T h^{-1}d([z^{-1}])\tilde{p}^{(2)}(z).$$

Here $(L_1^i)_{++}$ denotes the strictly upper diagonal part of L_1^n. Formulas above are obtained from a straightforward computation and using the fact, proven in [41], that

$$\partial_{t_i}\log(h_n) = (L_1^i)_{nn}.$$

Hence we have that, for every k, $\partial_{t_i}p_k^{(1)}$ is a linear combination of $\{p_k^{(1)}, p_{k+1}^{(1)}, p_{k+2}^{(1)}, \ldots\}$ with coefficients in $\mathbb{C}[x_j, y_j]$. In the same way, for every k, $\partial_{t_i}\tilde{p}_k^{(2)}$ is a linear combination of $\{\tilde{p}_k^{(2)}, \tilde{p}_{k-1}^{(2)}, \tilde{p}_{k-2}^{(2)}, \ldots\}$ with coefficients in $\mathbb{C}[x_j, y_j]$. Now using the recursion relation (2.2) and its inverse

$$\begin{pmatrix} p_n^{(1)}(z) \\ \tilde{p}_n^{(2)}(z)\end{pmatrix} = \mathcal{L}_n^{-1}\begin{pmatrix} p_{n+1}^{(1)}(z) \\ \tilde{p}_{n+1}^{(2)}(z)\end{pmatrix} = \frac{h_n}{h_{n+1}}\begin{pmatrix} z^{-1} & -z^{-1}x_{n+1} \\ -y_{n+1} & 1\end{pmatrix}\begin{pmatrix} p_{n+1}^{(1)}(z) \\ \tilde{p}_{n+1}^{(2)}(z)\end{pmatrix}$$

we can obtain the desired matrices $\mathcal{M}_{t_i,n}$. □

Corollary 2.2.5 (Ablowitz-Ladik semidiscrete zero-curvature equations). *Matrices $\mathcal{L}_n$ satisfy the following time evolution*

$$\partial_{t_i}\mathcal{L}_n = \mathcal{M}_{t_i,n+1}\mathcal{L}_n - \mathcal{L}_n\mathcal{M}_{t_i,n} \tag{2.5}$$

$$\partial_{s_i}\mathcal{L}_n = \mathcal{M}_{s_i,n+1}\mathcal{L}_n - \mathcal{L}_n\mathcal{M}_{s_i,n}. \tag{2.6}$$

PROOF These equations are nothing but compatibility conditions of recursion relation (2.2) with time evolution (2.3) and (2.4). □

Remark 2.2.6. Actually our Lax operator $\mathcal{L}_n$ is slightly different from the Lax operator L_n used in [33] by Ablowitz and Ladik and written in the introduction above. Nevertheless, as shown in [46], these two Lax operators are linked through a simple change of spectral parameter.

Example 2.2.7 *(The first flows; discrete nonlinear Schrödinger.).* First matrices $\mathcal{M}_{t_i,n}$ and $\mathcal{M}_{s_i,n}$ are easily computed. We have

$$\partial_{t_1} p_k^{(1)} = -z p_k^{(1)} - x_{k+1} y_k p_k^{(1)} + p_{k+1}^{(1)} = -x_{k+1} y_k p_k^{(1)} + x_{k+1} \tilde{p}_k^{(2)}$$
$$\partial_{t_1} \tilde{p}_k^{(2)} = -z \frac{h_{k+1}}{h_k} \tilde{p}_{k-1}^{(2)} = z y_k p_k^{(1)} - z \tilde{p}_k^{(2)}$$

that gives immediately

$$\mathcal{M}_{t_1,k} = \begin{pmatrix} -x_{k+1} y_k & x_{k+1} \\ z y_k & -z \end{pmatrix}$$

One can do an analogue computation for s_1 or even skip it using some symmetry considerations between t-times and s-times as in [41]. In this way we obtain

$$\mathcal{M}_{s_1,k} = \begin{pmatrix} z^{-1} & -z^{-1} x_k \\ -y_{k+1} & x_k y_{k+1} \end{pmatrix}.$$

Already with these two times plus the introduction of two trivial rescaling times we can write the well known integrable discretization of nonlinear Schrödinger. Trivial rescaling times are introduced with substitutions

$$p^{(1)} \longmapsto \exp(t_0) p^{(1)}$$
$$\tilde{p}^{(2)} \longmapsto \exp(-s_0) \tilde{p}^{(2)}$$

and correspond to matrices

$$\mathcal{M}_{t_0,k} := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$
$$\mathcal{M}_{s_0,k} := \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}.$$

Now we can construct the matrix

$$\mathcal{M}_{\tau,k} = \mathcal{M}_{t_1,k} + \mathcal{M}_{s_1,k} - \mathcal{M}_{t_0,k} - \mathcal{M}_{s_0,k} =$$

$$\begin{pmatrix} z^{-1} - 1 - x_{k+1} y_k & x_{k+1} - z^{-1} x_k \\ z y_k - y_{k+1} & x_k y_{k+1} + 1 - z \end{pmatrix}$$

associated to the time $\tau = t_1 + s_1 - t_0 - s_0$ so that semidiscrete zero-curvature equation

$$\partial_\tau \mathcal{L}_k = \mathcal{M}_{\tau,k+1}\mathcal{L}_k - \mathcal{L}_k\mathcal{M}_{\tau,k}$$

is equivalent to the system

$$\begin{cases} \partial_\tau x_k = x_{k+1} - 2x_k + x_{k-1} - x_k y_k (x_{k+1} + x_{k-1}) \\ \\ \partial_\tau y_k = -y_{k+1} + 2y_k - y_{k-1} + x_k y_k (y_{k+1} + y_{k-1}) \end{cases} \tag{2.7}$$

i.e. exactly the complexified version of discrete nonlinear Schrödinger. Rescaling $\tau \mapsto i\tau$ and imposing $y_k = \pm x_k^*$ we obtain

$$-i\partial_\tau x_k = x_{k+1} - 2x_k + x_{k-1} \mp \|x_k\|^2 (x_{k+1} + x_{k-1}). \tag{2.8}$$

2.3 Toda flow for block Toeplitz matrices and related Lax operators.

We want now to generalize Toeplitz lattice to the block case. This means that we start with a matrix-valued formal series

$$\gamma(z) = \sum_{k \in \mathbb{Z}} \gamma^{(k)} z^k.$$

Here every element $\gamma^{(k)}$ is a $n \times n$ matrix. Then we define its time evolution as

$$\gamma(t, s; z) := \exp\big(-\xi(s, z^{-1}\mathrm{I})\big)\gamma(z)\exp\big(\xi(t, z\mathrm{I})\big).$$

where I is the $n \times n$ identity matrix. Differently from the scalar case (and also from the first section) we don't consider just one Toeplitz matrix but the two block Toeplitz matrices, right and left, given by

$$T^r(\gamma) := \begin{pmatrix} \gamma^{(0)} & \gamma^{(-1)} & \gamma^{(-2)} & \dots \\ \gamma^{(1)} & \gamma^{(0)} & \gamma^{(-1)} & \dots \\ \gamma^{(2)} & \gamma^{(1)} & \gamma^{(0)} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$T^l(\gamma) := \begin{pmatrix} \gamma^{(0)} & \gamma^{(1)} & \gamma^{(2)} & \dots \\ \gamma^{(-1)} & \gamma^{(0)} & \gamma^{(1)} & \dots \\ \gamma^{(-2)} & \gamma^{(-1)} & \gamma^{(0)} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

In this way we obtain the following linear time evolution for our block Toeplitz matrices (in the following we will omit the symbol γ):

$$\partial_{t_i} T^l = \Lambda^i T^l \quad \partial_{t_i} T^r = T^r \Lambda^{-i} \tag{2.9}$$

$$\partial_{s_i} T^l = -T^l \Lambda^{-i} \quad \partial_{s_i} T^r = -\Lambda^i T^r \tag{2.10}$$

where, in this case, we have $\Lambda = T^r(z^{-1}\mathrm{I})$. Then we assume that there exist two factorizations

$$T^l = S_1^{-1} S_2 \quad T^r = Z_2 Z_1^{-1}.$$

Here S_1, Z_2 are block-lower triangular while S_2, Z_1 are block-upper triangular. We assume that all these matrices have non degenerate blocks on the main diagonal (i.e. these blocks must have non zero determinants) and we normalize them in such a way that every element on the main block-diagonal of S_1 and Z_2 is equal to the identity matrix I (as we did before we assume that these conditions hold when every time is equal to 0 and the we consider just values of t and s for which these conditions still hold). In the matrix case we can define two bilinear pairings given by the following definition.

Definition 2.3.1.

$$< P, Q >_r := \oint P^*(z)\gamma(z)Q(z)\frac{dz}{2\pi i z} \quad < P, Q >_l := \oint P(z)\gamma(z)Q^*(z)\frac{dz}{2\pi i z}$$

where P and Q are two arbitrary matrix polynomials and $P^*(z) := (P(z^{-1}))^T$

Our two factorizations give exactly biorthonormal polynomials for $<,>_r$ and $<,>_l$. In the following we denote $\chi(z) := (\mathrm{I}, z\mathrm{I}, z^2\mathrm{I}, z^3\mathrm{I}, \dots)^T$.

Proposition 2.3.2.

$$Q^{(1)l} := \begin{pmatrix} Q_0^{(1)l} \\ Q_1^{(1)l} \\ \vdots \end{pmatrix} = S_1\chi(z) \tag{2.11}$$

$$Q^{(2)l} := \begin{pmatrix} Q_0^{(2)l} \\ Q_1^{(2)l} \\ \vdots \end{pmatrix} = (S_2^{-1})^T\chi(z) \tag{2.12}$$

$$Q^{(1)r} := \begin{pmatrix} Q_0^{(1)r} & Q_1^{(1)r} & \dots \end{pmatrix} = \chi(z)^T Z_1 \tag{2.13}$$

$$Q^{(2)r} := \begin{pmatrix} Q_0^{(2)r} & Q_1^{(2)r} & \dots \end{pmatrix} = \chi(z)^T (Z_2^{-1})^T \tag{2.14}$$

are the biorthonormal polynomials associated to the pairing $<,>_l$ and $<,>_r$. This means that for every i, j we have

$$< Q_i^{(1)l}, Q_j^{(2)l} >_l = \delta_{ij} \qquad < Q_i^{(2)r}, Q_j^{(1)r} >_r = \delta_{ij}$$

PROOF We just prove, as an example, the proposition for the right polynomials; on the other hand the one for left polynomials is identical to the usual proof for 2D-Toda. We have

$$\left(< Q_i^{(2)r}, Q_j^{(1)r} >_r \right)_{i,j\geq 0} = \left(\sum_{k,l\geq 0} (Z_2^{-1})_{ki} < z^k\mathrm{I}, z^l\mathrm{I} >_r (Z_1)_{lj} \right) =$$

$$Z_2^{-1} T^l Z_1 = \mathrm{I} \Longleftrightarrow T^l = Z_2 Z^{-1}.$$

(it should be noted that, in this case, subscripts of type $(Z_1)_{ij}$ denote the block in position (i, j) and not the element (i, j).) □

We are now in the position to write the corresponding Sato's equations for S_i and Z_i. It is convenient to introduce the following Lax operators.

Definition 2.3.3.

$$L_1 := S_1 \Lambda S_1^{-1} \quad L_2 := S_2 \Lambda^{-1} S_2^{-1} \tag{2.15}$$

$$R_1 := Z_1^{-1} \Lambda^{-1} Z_1 \quad R_2 := Z_2^{-1} \Lambda Z_2. \tag{2.16}$$

Proposition 2.3.4. *The following Sato's equations are satisfied.*

$$\partial_{t_n} S_1 = -(L_1^n)_- S_1 \quad \partial_{t_n} Z_1 = -Z_1(R_1^n)_+ \tag{2.17}$$

$$\partial_{t_n} S_2 = (L_1^n)_+ S_2 \quad \partial_{t_n} Z_2 = Z_2(R_1^n)_- \tag{2.18}$$

$$\partial_{s_n} S_1 = (L_2^n)_- S_1 \quad \partial_{s_n} Z_1 = Z_1(R_2^n)_+ \tag{2.19}$$

$$\partial_{s_n} S_2 = -(L_2^n)_+ S_2 \quad \partial_{s_n} Z_2 = -Z_2(R_2^n)_- \tag{2.20}$$

Proof We will just prove, as an example, the equations involving t-derivative of Z_1 and Z_2.
We assume as an ansatz that we have

$$\partial_{t_n} Z_1 = Z_1 A$$
$$\partial_{t_n} Z_2 = Z_2 B.$$

for some matrices A and B. Then exploiting time evolution of T^r we can write

$$T^r \Lambda^{-n} = \partial_{t_n} T^r = \partial_{t_n}(Z_2 Z_1^{-1}) =$$
$$Z_2 B Z_1^{-1} - Z_2 Z_1^{-1} Z_1 A Z_1^{-1} = Z_2(B-A)Z_1^{-1}$$

hence we must have $(B-A)Z_1^{-1} = Z_1^{-1}\Lambda^{-n}$; keeping in mind that B must be lower triangular and A upper triangular we arrive to the proof. □

Now it's just a matter of trivial computations to write down the corresponding Lax equations for L_i and R_i.

Proposition 2.3.5. *The following Lax equations are satisfied:*

$$\partial_{t_n} L_i = \left[(L_1^n)_+, L_i\right] \quad \partial_{t_n} R_i = \left[R_i, (R_1^n)_-\right] \tag{2.21}$$

$$\partial_{s_n} L_i = \left[(L_2^n)_-, L_i\right] \quad \partial_{s_n} R_i = \left[R_i, (R_2^n)_+\right]. \tag{2.22}$$

The definition of our Lax operators will give us eigenvalue equations for suitably defined wave vectors.

Definition 2.3.6.

$$\Psi_1(z) := \exp(\xi(t, z\mathrm{I})) S_1 \chi(z) \tag{2.23}$$

$$\Phi_1(z) := \exp(\xi(t, z\mathrm{I})) \left[\chi(z)\right]^T Z_1 \tag{2.24}$$

$$\Psi_2^*(z) := \exp(-\xi(s, z^{-1}\mathrm{I}))(S_2^{-1})^T \chi(z^{-1}) \tag{2.25}$$

$$\Phi_2^*(z) := \exp(-\xi(s, z^{-1}\mathrm{I}))\chi(z^{-1})^T (Z_2^{-1})^T. \tag{2.26}$$

Proposition 2.3.7. *The following equations hold true:*

$$L_1\Psi_1(z) = z\mathrm{I}\Psi_1(z) \quad \Phi_1(z)R_1 = z\mathrm{I}\Phi_1(z) \tag{2.27}$$

$$L_2^T\Psi_2^*(z) = z^{-1}\mathrm{I}\Psi_2^*(z) \quad \Phi_2^*(z)R_2^T = z^{-1}\mathrm{I}\Phi_2^*(z). \tag{2.28}$$

PROOF We will just prove the last equation, all the other ones are proved in a similar way. From the very definition we have

$$\Phi_2^*(z)R_2^T = z^{-1}\mathrm{I}\Phi_2^*(z) \Longleftrightarrow [\chi(z^{-1})]^T(Z_2^{-1})^T R_2^T = z^{-1}[\chi(z^{-1})]^T(Z_2^{-1})^T =$$
$$[\chi(z^{-1})]^T\Lambda^{-1}(Z_2^{-1})^T \Longleftrightarrow R_2^T = Z_2^T\Lambda^{-1}(Z_2^{-1})^T \Longleftrightarrow R_2 = Z_2^{-1}\Lambda Z_2$$

□

The proof of the following proposition is straightforward.

Proposition 2.3.8. *Lax equations (2.21) and (2.22) are compatibility conditions of eigenvalue equations (2.27) and (2.28) with the following equations:*

$$\partial_{t_n}\Psi_1 = (L_1^n)_+\Psi_1 \quad \partial_{t_n}\Phi_1 = \Phi_1(R_1^n)_- \tag{2.29}$$

$$\partial_{s_n}\Psi_1 = (L_2^n)_-\Psi_1 \quad \partial_{s_n}\Phi_1 = \Phi_1(R_2^n)_+ \tag{2.30}$$

$$\partial_{t_n}\Psi_2^* = -(L_{1+}^n)^T\Psi_2^* \quad \partial_{t_n}\Phi_2^* = -\Phi_2^*(R_{1-}^n)^T \tag{2.31}$$

$$\partial_{s_n}\Psi_2^* = -(L_{2-}^n)^T\Psi_2^* \quad \partial_{s_n}\Phi_2^* = -\Phi_2^*(R_{2+}^n)^T. \tag{2.32}$$

2.4 Recursion relations for matrix biorthogonal polynomials on the unit circle.

In order to generalize scalar theory we have to construct an analogue of recursion relation given by proposition 2.2.1. Recursion relations for matrix orthogonal polynomial on the unit circle are already known, see [47] and [48]. Here we slightly generalize to the case of matrix biorthogonal polynomials on the unit circle.

We define the following important $n \times n$ matrices:

Definition 2.4.1.

$$h_N^r := \mathrm{SC}(T_{N+1}^r) \quad h_N^l := \mathrm{SC}(T_{N+1}^l)$$

where SC denote the $n \times n$ Schur complement of a block matrix with respect to the upper left block; for instance

$$\mathrm{SC}(T_{N+1}^r) = \gamma^{(0)} - \begin{pmatrix}\gamma^{(N)} & \dots & \dots & \gamma^{(1)}\end{pmatrix} T_N^{-r} \begin{pmatrix}\gamma^{(-N)} \\ \dots \\ \dots \\ \gamma^{(-1)}\end{pmatrix}$$

(here and below $T_N^{-r} := (T_N^r)^{-1}$ and similarly for h_N^r, h_N^l and T_N^l).

Proposition 2.4.2. *Monic biorthogonal polynomials such that*

$$< P_k^{(2)r}, P_j^{(1)r} >_r = \delta_{kj} h_k^r \qquad < P_k^{(1)l}, P_j^{(2)l} >_l = \delta_{kj} h_k^l.$$

are given by the following formulas:

$$P_N^{(1)r} = \mathrm{SC}\begin{pmatrix} \gamma^{(0)} & \dots & \dots & \gamma^{(-N+1)} & \gamma^{(-N)} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \gamma^{(N-1)} & \dots & \dots & \gamma^{(0)} & \gamma^{(-1)} \\ \mathrm{I} & z\mathrm{I} & \dots & z^{N-1}\mathrm{I} & z^N\mathrm{I} \end{pmatrix}$$

$$(P_N^{(2)r})^T = \mathrm{SC}\begin{pmatrix} \gamma^{(0)} & \dots & \dots & \gamma^{(-N+1)} & \mathrm{I} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \gamma^{(N-1)} & \dots & \dots & \gamma^{(0)} & z^{N-1}\mathrm{I} \\ \gamma^{(N)} & \dots & \dots & \gamma^{(1)} & z^N\mathrm{I} \end{pmatrix}$$

$$P^{(1)l_N} = \mathrm{SC}\begin{pmatrix} \gamma^{(0)} & \dots & \dots & \gamma^{(N-1)} & \mathrm{I} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \gamma^{(-N+1)} & \dots & \dots & \gamma^{(0)} & z^{N-1}\mathrm{I} \\ \gamma^{(-N)} & \dots & \dots & \gamma^{(-1)} & z^N\mathrm{I} \end{pmatrix}$$

$$(P_N^{(2)l})^T = \mathrm{SC}\begin{pmatrix} \gamma^{(0)} & \dots & \dots & \gamma^{(N-1)} & \gamma^{(1)} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \gamma^{(-N+1)} & \dots & \dots & \gamma^{(0)} & \gamma^{(N-1)} \\ \mathrm{I} & z\mathrm{I} & \dots & z^{N-1}\mathrm{I} & z^N\mathrm{I} \end{pmatrix}.$$

Proof We will just prove the first formula, the second one is proved similarly. First of all we have $\forall\, 0 \le m \le N-1$

$$< z^m\mathrm{I}, P_N^{(1)r}(z) >_r = \oint z^{-m}\gamma(z)\left(z^N\mathrm{I} - \begin{pmatrix}\mathrm{I} & \dots & \dots & z^{N-1}\mathrm{I}\end{pmatrix} T_N^{-r}\begin{pmatrix}\gamma^{(-N)} \\ \dots \\ \dots \\ \gamma^{(-1)}\end{pmatrix}\right)\frac{dz}{2\pi i z}$$

$$= \gamma^{(m-N)} - \gamma^{(m-N)} = 0.$$

In the same way $\forall\, 0 \le m \le N-1$

$$< P_N^{(2)r}(z), z^m\mathrm{I} >_r = \oint \left(z^{-N}\mathrm{I} - \begin{pmatrix}\gamma^{(N)} & \dots & \dots & \gamma^{(1)}\end{pmatrix} T_N^{-r}\begin{pmatrix}\mathrm{I} \\ \dots \\ \dots \\ z^{-N+1}\mathrm{I}\end{pmatrix}\right)\gamma(z) z^m \frac{dz}{2\pi i z}$$

$$= \gamma^{(N-m)} - \gamma^{(N-m)} = 0.$$

Finally

$$< P_N^{(2)r}(z), P_N^{(1)r}(z) >_r = < z^N \mathrm{I}, P_N^{(1)r} >_r =$$

$$\gamma^{(0)} - \begin{pmatrix} \gamma^{(N)} & \dots & \dots & \gamma^{(1)} \end{pmatrix} T_N^{-r} \begin{pmatrix} \gamma^{(-N)} \\ \dots \\ \dots \\ \gamma^{(-1)} \end{pmatrix} = h_N^r.$$

This completes the proof of the first formula, the second one is proved similarly. □

Remark 2.4.3. Note that, imposing

$$\begin{array}{ll} Q_k^{(1)l} := P_k^{(1)l} & Q_k^{(2)l} := (h_k^{-l})^T P_k^{(2)l} \\ Q_k^{(1)r} := P_k^{(1)r}(h_k^{-r}) & Q_k^{(2)r} := P_k^{(2)r}. \end{array}$$

we obtain biorthonormal polynomials.

Now we will write a long list of relations among this polynomials and reflection coefficients. First of all, as before, given any matrix polynomial $Q(z)$ of degree n we define the associated reversed polynomial as

$$\tilde{Q}(z) = z^n Q^*(z).$$

The reflection coefficients are the following:

$$\begin{array}{ll} x_N^l := P_N^{(1)l}(0) & x_N^r := P_N^{(1)r}(0) \\ y_N^l := (P_N^{(2)l}(0))^T & y_N^r := (P_N^{(2)r}(0))^T. \end{array}$$

Proposition 2.4.4. *The following formulas hold true:*

$$P_{N+1}^{(1)l} - zP_N^{(1)l} = x_{N+1}^l \tilde{P}_N^{(2)r} \tag{2.33}$$

$$\tilde{P}_{N+1}^{(2)r} - \tilde{P}_N^{(2)r} = zy_{N+1}^r P_N^{(1)l} \tag{2.34}$$

$$P_{N+1}^{(1)r} - zP_N^{(1)r} = \tilde{P}_N^{(2)l} x_{N+1}^r \tag{2.35}$$

$$\tilde{P}_{N+1}^{(2)l} - \tilde{P}_N^{(2)l} = zP_N^{(1)r} y_{N+1}^l \tag{2.36}$$

$$P_{N+1}^{(1)r} = zP_N^{(1)r}(\mathrm{I} - y_{N+1}^l x_{N+1}^r) + \tilde{P}_{N+1}^{(2)l} x_{N+1}^r \tag{2.37}$$

$$P_{N+1}^{(1)l} = z(\mathrm{I} - x_{N+1}^l y_{N+1}^r) P_N^{(1)l} + x_{N+1}^l \tilde{P}_{N+1}^{(2)l} \tag{2.38}$$

$$\tilde{P}_{N+1}^{(2)r} = (\mathrm{I} - y_{N+1}^r x_{N+1}^l)\tilde{P}_N^{(2)r} + y_{N+1}^r P_{N+1}^{(1)l} \tag{2.39}$$

$$\tilde{P}_{N+1}^{(2)l} = \tilde{P}_N^{(2)l}(\mathrm{I} - x_{N+1}^r y_{N+1}^l) + P_{N+1}^{(1)r} y_{N+1}^l \tag{2.40}$$

$$x_N^l h_N^r = h_N^l x_N^r \tag{2.41}$$

$$y_N^r h_N^l = h_N^r y_N^l \tag{2.42}$$

$$h_N^{-r} h_{N+1}^r = \mathrm{I} - y_{N+1}^l x_{N+1}^r \tag{2.43}$$

$$h_{N+1}^l h_N^{-l} = \mathrm{I} - x_{N+1}^l y_{N+1}^r. \tag{2.44}$$

PROOF The first four formulas are proved observing, for instance for the first case, that $\forall\, 1 \leq i \leq N$ we have

$$0 =< P_{N+1}^{(1)l} - zP_N^{(1)l}, z^i\mathrm{I} >_l =< \tilde{P}_N^{(2)r}, z^i\mathrm{I} >_l$$

so that $P_{N+1}^{(1)l} - zP_N^{(1)l}$ and $\tilde{P}_N^{(2)r}$ must be proportional. Setting $z = 0$ you also find the constant of proportionality. In particular, when proving (2.34) and (2.36), we find a formula and then we have to take the reversed one. (2.37) is proved substituting (2.36) into (2.35) and similarly for (2.38),(2.39),(2.40). (2.41) and (2.42) are proven respectively observing that we have

$$< \tilde{P}_N^{(1)l}, P_N^{(1)r} >_r =< P_N^{(1)l}, \tilde{P}_N^{(1)r} >_l$$

and

$$< P_N^{(2)r}, \tilde{P}_N^{(2)l} >_r =< \tilde{P}_N^{(2)r}, P_N^{(2)l} >_l$$

and then doing explicit computations. Finally (2.43) is obtained rewriting (2.37) as

$$\frac{P_{N+1}^{(1)r}}{z^{N+1}} = \frac{P_N^{(1)r}}{z^N}(\mathrm{I} - y_{N+1}^l x_{N+1}^r) + (P_{N+1}^{(2)l})^* x_{N+1}^r,$$

multiplying from the left for $P_N^{(1)l}\gamma$ and then taking the residue. (2.44) is proved similarly. □

Now we define two sets of block matrices $\{\mathcal{L}_N^r\}_{N\geq 0}$ and $\{\mathcal{L}_N^l\}_{N\geq 0}$. They will have in the matrix case the same role played by $\{\mathcal{L}_n\}_{n\geq 0}$ in the scalar case.

Definition 2.4.5.

$$\mathcal{L}_N^l := \begin{pmatrix} z\mathrm{I} & x_{N+1}^l \\ zy_{N+1}^r & \mathrm{I} \end{pmatrix} \tag{2.45}$$

$$\mathcal{L}_N^r := \begin{pmatrix} z\mathrm{I} & zy_{N+1}^l \\ x_{N+1}^r & \mathrm{I} \end{pmatrix}. \tag{2.46}$$

Corollary 2.4.6. *The following block matrices recursion relations are satisfied*

$$\begin{pmatrix} P_{N+1}^{(1)l} \\ \tilde{P}_{N+1}^{(2)r} \end{pmatrix} = \mathcal{L}_N^l \begin{pmatrix} P_N^{(1)l} \\ \tilde{P}_N^{(2)r} \end{pmatrix} \tag{2.47}$$

$$\begin{pmatrix} P_{N+1}^{(1)r} & \tilde{P}_{N+1}^{(2)l} \end{pmatrix} = \begin{pmatrix} P_N^{(1)r} & \tilde{P}_N^{(2)l} \end{pmatrix} \mathcal{L}_N^r. \tag{2.48}$$

PROOF These are nothing but (2.33),(2.34),(2.35) and (2.36). □

2.5 Explicit expressions for Lax operators and related semidiscrete zero-curvature equations.

Using our recursion relations we want to find explicit expression for L_i and R_i in terms of our reflection coefficients $x_k^l, x_k^r, y_k^l, y_k^r$. Before doing this we underline a remarkable symmetry that will allow us to reduce the amount of our computations. Doing the following three substitutions

$$z \mapsto z^{-1}$$
$$t \mapsto -s$$
$$s \mapsto -t$$

we obtain immediately the following proposition.

Proposition 2.5.1. *Under the symmetry above dressings, orthogonal poly-*

nomials, Lax operators and reflection coefficients change as follows:

$$
\begin{gathered}
T^r \mapsto T^l \quad T^l \mapsto T^r \\
S_1 \mapsto Z_2^{-1} \quad S_2 \mapsto Z_1^{-1} \\
L_1 \mapsto R_2 \quad L_2 \mapsto R_1 \\
Q^{(1)l} \mapsto (Q^{(2)r})^* \quad Q^{(2)l} \mapsto (Q^{(1)r})^* \\
P^{(1)l} \mapsto (P^{(2)r})^* \quad P^{(2)l} \mapsto (P^{(2)r})^* \\
x^l \mapsto y^r \quad y^l \mapsto x^r \\
h_k^l \mapsto h_k^r.
\end{gathered}
$$

This means in particular that we can write just the left theory and then we will have the right one as well; actually every computation made above for right theory can be deduced from left theory and this symmetry which will be called in the sequel $t-s$ symmetry. In the theorem below the symbol $\prod_{j=N+2}^{M-}$ means that the terms in the product must be taken in decrescent order from left to right while $\prod_{j=N+2}^{M+}$ means that the product must be taken in the opposite direction.

Theorem 2.5.2 (Lax operators for block Toeplitz lattice). *Lax operators L_i and R_i are expressed in terms of reflection coefficients according to the following formulas:*
$\forall\, N > M \geq -1$

$$(L_1)_{N,M+1} = -x_{N+1}^l \Big(\prod_{j=N+2}^{M-} (\mathrm{I} - y_j^r x_j^l) \Big) y_{M+1}^r \tag{2.49}$$

$$(R_2)_{N,M+1} = -y_{N+1}^r \Big(\prod_{j=N+2}^{M-} (\mathrm{I} - x_j^l y_j^r) \Big) x_{M+1}^l \tag{2.50}$$

$$(L_2)_{M+1,N} = -h_{M+1}^{-l} x_{M+1}^r \Big(\prod_{j=N+2}^{M+} (\mathrm{I} - y_j^l x_j^r) \Big) y_{N+1}^l h_N^l \tag{2.51}$$

$$(R_1)_{M+1,N} = -h_{M+1}^{-r} y_{M+1}^l \Big(\prod_{j=N+2}^{M+} (\mathrm{I} - x_j^r y_j^l) \Big) x_{N+1}^r h_N^r. \tag{2.52}$$

Moreover

$$(L_1)_{N,N+1} = (R_2)_{N,N+1} = \mathrm{I} \tag{2.53}$$

$$(L_2)_{N+1,N} = h_{N+1}^l h_N^{-l} \quad (R_1)_{N+1,N} = h_{N+1}^r h_N^{-r}. \tag{2.54}$$

PROOF (2.53) and (2.54) follow trivially from the expressions of dressings S_i, Z_i and normalization of biorthonormal polynomials. Now let's begin with (2.49) and (2.50); the important point is that we have

$$\Psi_1 = \exp(\xi(t, z\mathrm{I}))P^{(1)l}.$$

Hence as done in [41] we can find that $\forall\, N > M \geq -1$ we have

$$(L_1)_{N,M+1} = -x^l_{N+1}h^r_N h^{-r}_{M+1}y^r_{M+1}. \tag{2.55}$$

Infact $\forall\, N > M \geq -1$

$$< P^{(1)l}_{N+1} - zP^{(1)l}_N, P^{(2)l}_{M+1} - zP^{(2)l}_M >_l = - < zP^{(1)l}_N, P^{(2)l}_{M+1} >_l =$$
$$- < P^{(1)l}_{N+1} + \ldots + (L_1)_{N,M+1}P^{(1)l}_{M+1} + \ldots, P^{(2)l}_{M+1} >_l = -(L_1)_{N,M+1}h^l_{M+1}.$$

On the other hand using recursion relations I also have $\forall N \geq M \geq -1$

$$< P^{(1)l}_{N+1} - zP^{(1)l}_N, P^{(2)l}_{M+1} - zP^{(2)l}_M >_l =< x^l_{N+1}\tilde{P}^{(2)r}_N, (y^l_{M+1})^T\tilde{P}^{(1)r}_M >_l =$$
$$x^l_{N+1}\left(\oint z^{N-M}(P^{(2)r}_N)^*\gamma(z)P^{(1)r}_M\frac{dz}{2\pi iz}\right)y^l_{M+1} =$$
$$x^l_{N+1} < P^{(2)r}_N, z^{N-M}P^{(1)r}_M >_r y^l_{M+1} = x^l_{N+1}h^r_N y^l_{M+1}$$

and comparing them we find (2.55). Now we use $t-s$ symmetry to simplify this expression. We obtain

$$(R_2)_{N,M+1} = -y^r_{N+1}h^l_N h^{-l}_{M+1}x^l_{M+1}.$$

and thanks to recursion (2.44) we get (2.50). (2.49) is obtained using $t-s$ symmetry. For (2.51) and (2.52) we start defining $\tilde{R}_1$ such that $zP^{(1)r} = P^{(1)r}\tilde{R}_1$; then we will have $R_1 = h^r\tilde{R}_1h^{-r}$ and computations for $\tilde{R}_1$ is carried on similarly as for L_1. □

Remark 2.5.3. Our equations (2.49),(2.50),(2.51) and (2.52) extend to the matrix biorthogonal setting the equations written in [48] for matrix orthogonal polynomials (see equation (4.2),(4.3)). In that article properties of M are applied to study some problems in computational mathematics (multivariate time series analysis and multichannel signal processing) and no relation is established with Lax theory and integrable systems.

Theorem above describe completely the block-analogue of Toeplitz lattice; we are now in the position to prove the analogue of theorem 2.2.4.

Theorem 2.5.4. *Block Toeplitz lattice flow can be written in the form*

$$\partial_{t_i/s_i}\begin{pmatrix} P_N^{(1)l}(z) \\ \tilde{P}_N^{(2)r}(z) \end{pmatrix} = \mathcal{M}^l_{t_i/s_i,N}\begin{pmatrix} P_N^{(1)l}(z) \\ \tilde{P}_N^{(2)r}(z) \end{pmatrix} \tag{2.56}$$

$$\partial_{t_i/s_i}\begin{pmatrix} P_N^{(1)r}(z) & \tilde{P}_N^{(2)l}(z) \end{pmatrix} = \begin{pmatrix} P_N^{(1)r}(z) & \tilde{P}_N^{(2)l}(z) \end{pmatrix}\mathcal{M}^r_{t_i/s_i,N} \tag{2.57}$$

for some block matrices $\mathcal{M}^r_{t_i,N}, \mathcal{M}^r_{s_i,N}, \mathcal{M}^l_{t_i,N}, \mathcal{M}^l_{s_i,N}$ depending on the matrices $\{x_j^l, y_j^l, x_j^r, y_j^r\}$ and the spectral parameter z.

Proof As we did for the scalar case we prove it just for t times. The relevant equations linking biorthogonal polynomials with wave vectors are

$$\begin{aligned}
\Psi_1(z) &= \exp(\xi(t, z\mathrm{I}))P^{(1)l}(z) \\
\Phi_2^*(z) &= \exp(-\xi(s, z^{-1}\mathrm{I}))(\tilde{P}^{(2)r})^T d([z^{-1}]) \\
\Phi_1(z) &= \exp(\xi(t, z\mathrm{I}))P^{(1)r}h^{-r} \\
\Psi_2^*(z) &= \exp(-\xi(s, z^{-1}\mathrm{I}))(h^{-l})^T d([z^{-1}])(\tilde{P}^{(2)l})^T.
\end{aligned}$$

Then trivial computations give the following time evolution:

$$\begin{aligned}
\partial_{t_n}P^{(1)l} &= (L_1^n)_+P^{(1)l} - z^n\mathrm{I}P^{(1)l} \\
\partial_{t_n}\tilde{P}^{(2)r} &= -d([z])(R_1^n)_- d([z^{-1}])\tilde{P}^{(2)r} \\
\partial_{t_n}P^{(1)r} &= P^{(1)r}(h^{-r}(R_1^n)_- h^r + h^{-r}(\partial_{t_n}h^r) - z^n\mathrm{I}) \\
\partial_{t_n}\tilde{P}^{(2)l} &= \tilde{P}^{(2)l}(-h^{-l}(L_1^n)_+h^l + h^{-l}(\partial_{t_n}h^l)).
\end{aligned}$$

the last two can be simplified giving

$$\partial_{t_n}P^{(1)l} = (L_1^n)_+P^{(1)l} - z^n\mathrm{I}P^{(1)l} \tag{2.58}$$

$$\partial_{t_n}\tilde{P}^{(2)r} = -d([z])(R_1^n)_- d([z^{-1}])\tilde{P}^{(2)r} \tag{2.59}$$

$$\partial_{t_n}P^{(1)r} = P^{(1)r}(h^{-r}(R_1^n)_{--}h^r - z^n\mathrm{I}) \tag{2.60}$$

$$\partial_{t_n}\tilde{P}^{(2)l} = -\tilde{P}^{(2)l}\Big(d([z^{-1}])h^{-l}(L_1^n)_{++}h^l d([z])\Big). \tag{2.61}$$

where $(R_1^n)_{--}$ means the lower triangular part including the main diagonal and $(L_1^n)_{++}$ means the strictly upper diagonal part. This simplification can be obtained evaluating the terms $h^{-r}(\partial_{t_n}h^r)$ and $h^{-l}(\partial_{t_n}h^l))$ using Sato's equations or, equivalently, observing that $P_N^{(1)r}$ and $P_N^{(2)l}$ are monic so that the derivative of the leading term is equal to 0. Then the proof is obtained as we did in the scalar case using forward and backward recursion relations (2.33),(2.35),(2.39) and (2.40). □

Corollary 2.5.5 (Non-abelian AL semidiscrete zero-curvature equations). *Matrices $\mathcal{L}_n^r$ and $\mathcal{L}_n^l$ satisfy the following time evolution*

$$\partial_{t_i/s_i}\mathcal{L}_n^l = \mathcal{M}_{t_i/s_i,n+1}^l\mathcal{L}_n^l - \mathcal{L}_n^l\mathcal{M}_{t_i/s_i,n}^l \tag{2.62}$$

$$\partial_{t_i/s_i}\mathcal{L}_n^r = \mathcal{L}_n^r\mathcal{M}_{t_i/s_i,n+1}^r - \mathcal{M}_{t_i/s_i,n}^r\mathcal{L}_n^r. \tag{2.63}$$

PROOF These equations are nothing but compatibility conditions of recursion relations (2.47) and (2.48) with time evolution (2.56) and (2.57). □

Remark 2.5.6. It should be noticed that, with respect to the equations originally written in [35], here we have two coupled non-abelian Ablowitz-Ladik equations.

Example 2.5.7 *(The first flows; non-abelian analogue of discrete nonlinear Schrödinger).* As we did for the scalar case we will compute the first matrices $\mathcal{M}_{t_1/s_1,k}^{r/l}$ and use them to construct the non-abelian version of discrete nonlinear Schrödinger. We start with $\mathcal{M}_{t_1,k}^l$; (2.58) gives us immediately

$$\partial_{t_1}P_k^{(1)l} = P_{k+1}^{(1)l} - x_{k+1}^l y_k^r P_k^{(1)l} - zP_k^{(1)l} =$$
$$zP_k^{(1)l} + x_{k+1}^l\tilde{P}_k^{(2)r} - x_{k+1}^l y_k^r P_k^{(1)l} - zP_k^{(1)l} = -x_{k+1}^l y_k^r P_k^{(1)l} + x_{k+1}^l\tilde{P}_k^{(2)r}$$

while we obtain immediately from (2.59) that

$$\partial_{t_1}\tilde{P}_k^{(2)r} = -z\mathrm{I}h_k^r h_{k-1}^{-r}\tilde{P}_{k-1}^{(2)r}.$$

Then we use recursion relation (2.39) combined with

$$h_k^r h_{k-1}^{-r} = (\mathrm{I} - y_k^r x_k^l)$$

(this one comes from recursion relation (2.41) combined with $t-s$ symmetry) to arrive to

$$\partial_{t_1}\tilde{P}_k^{(2)r} = zy_k^r P_k^{(1)l} - zP_k^{(2)r}.$$

These computations give us

$$\mathcal{M}_{t_1,k}^l = \begin{pmatrix} -x_{k+1}^l y_k^r & x_{k+1}^l \\ zy_k^r & -z\mathrm{I} \end{pmatrix}. \tag{2.64}$$

Also exploiting $t-s$ symmetry we can write immediately

$$\mathcal{M}_{s_1,k}^l = \begin{pmatrix} z^{-1}\mathrm{I} & -z^{-1}x_k^l \\ -y_{k+1}^r & y_{k+1}^r x_k^l \end{pmatrix}. \tag{2.65}$$

Analogue computations for $\mathcal{M}^r_{t_1}$ gives us

$$\partial_{t_1} P^{(1)r}_k = P^{(1)r}_{k+1} - P^{(1)r}_k y^l_k x^r_{k+1} - zP^{1(r)}_k =$$
$$zP^{1(r)}_k + \tilde{P}^{(2)l}_k x^r_{k+1} - P^{(1)r}_k y^l_k x^r_{k+1} - zP^{1(r)}_k = \tilde{P}^{(2)l}_k x^r_{k+1} - P^{(1)r}_k y^l_k x^r_{k+1}$$

and

$$\partial_{t_1} \tilde{P}^{(2)l}_k = -z\tilde{P}^{(2)l}_{k-1}(h^{-l}_{k-1} h^l_k) = -\tilde{P}^{(2)l}_k z + P^{(1)r}_k z y^l_k$$

(here we started from (2.60) and (2.61) and we used recursion relations (2.37),(2.40),(2.42), the last one combined with $t-s$ symmetry). Then we arrive to

$$\mathcal{M}^r_{t_1,k} = \begin{pmatrix} -y^l_k x^r_{k+1} & z y^l_k \\ x^r_{k+1} & -z\mathrm{I} \end{pmatrix} \tag{2.66}$$

and using again $t-s$ symmetry we also get

$$\mathcal{M}^r_{s_1,k} = \begin{pmatrix} z^{-1}\mathrm{I} & -y^l_{k+1} \\ -z^{-1}x^r_k & x^r_k y^l_{k+1} \end{pmatrix}. \tag{2.67}$$

As we did for the scalar case we introduce times t_0 and s_0 that give matrices

$$\mathcal{M}^{r/l}_{t_0,k} = \begin{pmatrix} \mathrm{I} & 0 \\ 0 & 0 \end{pmatrix} \tag{2.68}$$

$$\mathcal{M}^{r/l}_{s_0,k} = \begin{pmatrix} 0 & 0 \\ 0 & -\mathrm{I} \end{pmatrix}. \tag{2.69}$$

Then we construct the matrices

$$\mathcal{M}^l_{\tau,k} = \mathcal{M}^l_{t_1,k} + \mathcal{M}^l_{s_1,k} - \mathcal{M}^l_{t_0,k} - \mathcal{M}^l_{s_0,k} =$$

$$\begin{pmatrix} z^{-1}\mathrm{I} - \mathrm{I} - x^l_{k+1} y^r_k & x^l_{k+1} - z^{-1}x^l_k \\ z y^r_k - y^r_{k+1} & y^r_{k+1} x^l_k + \mathrm{I} - z\mathrm{I} \end{pmatrix}$$

$$\mathcal{M}^r_{\tau,k} = \mathcal{M}^r_{t_1,k} + \mathcal{M}^r_{s_1,k} - \mathcal{M}^r_{t_0,k} - \mathcal{M}^r_{s_0,k} =$$

$$\begin{pmatrix} z^{-1}\mathrm{I} - \mathrm{I} - y^l_k x^r_{k+1} & z y^l_k - y^l_{k+1} \\ x^r_{k+1} - z^{-1}x^r_k & x^r_k y^l_{k+1} - z\mathrm{I} + \mathrm{I} \end{pmatrix}$$

associate to the time $\tau = t_1 + s_1 - t_0 - s_0$. Semidiscrete zero-curvature equations

$$\partial_\tau \mathcal{L}^l_k = \mathcal{M}^l_{\tau,k+1}\mathcal{L}^l_k - \mathcal{L}^l_k\mathcal{M}^l_{\tau,k}$$
$$\partial_\tau \mathcal{L}^r_k = \mathcal{L}^r_k\mathcal{M}^r_{\tau,k+1} - \mathcal{M}^r_{\tau,k}\mathcal{L}^r_k$$

are equivalent to the systems

$$\begin{cases} \partial_\tau x^l_k = x^l_{k+1} - 2x^l_k + x^l_{k-1} - x^l_{k+1}y^r_k x^l_k - x^l_k y^r_k x^l_{k-1} \\ \\ \partial_\tau y^r_k = -y^r_{k+1} + 2y^r_k - y^r_{k-1} + y^r_{k+1}x^l_k y^r_k + y^r_k x^l_k y^r_{k-1} \end{cases} \tag{2.70}$$

$$\begin{cases} \partial_\tau x^r_k = x^r_{k+1} - 2x^r_k + x^r_{k-1} - x^r_{k-1}y^l_k x^r_k - x^r_k y^l_k x^r_{k+1} \\ \\ \partial_\tau y^l_k = -y^l_{k+1} + 2y^l_k - y^l_{k-1} + y^l_{k-1}x^r_k y^l_k + y^l_k x^r_k y^l_{k+1}. \end{cases} \tag{2.71}$$

Note that both of them are equivalent to the discrete matrix NLS as written, for instance, in [38]. Using (2.70) and (2.71) together we perform the reduction to the hermitian case in a different way from [38]. First of all we rescale $\tau \mapsto i\tau$ and then we impose

$$y^r_k = \pm(x^r_k)^*$$
$$y^l_k = \pm(x^l_k)^*$$

Note that this reduction (with the sign plus) corresponds to studying the theory of matrix orthogonal polynomials on the unit circle as described in [47] and [48], hence it is very natural. This reduction gives us the two coupled equations

$$\begin{cases} -i\partial_\tau x^l_k = x^l_{k+1} - 2x^l_k + x^l_{k-1} \mp x^l_{k+1}(x^r_k)^* x^l_k \mp x^l_k(x^r_k)^* x^l_{k-1} \\ \\ -i\partial_\tau x^r_k = x^r_{k+1} - 2x^r_k + x^r_{k-1} \mp x^r_{k-1}(x^l_k)^* x^r_k \mp x^r_k(x^l_k)^* x^r_{k+1} \end{cases} \tag{2.72}$$

already studied in [36] and generalized in [37].

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