ISAS - INTERNATIONAL SCHOOL FOR ADVANCED STUDIES

Hardy Inequalities and Liouville type Theorems Associated to Degenerate Operators

CANDIDATE
Lorenzo D'Ambrosio

SUPERVISORS
Prof. Enzo Mitidieri

Thesis submitted for the degree of *Doctor Philosophiae*

Academic Year 2001-2002

SISSA - SCUOLA INTERNAZIONALE SUPERIORE DI STUDI AVANZATI
TRIESTE
Via Beirut 2-4

TRIESTE

— Anche le città credono d'essere opera della mente o del caso, ma né l'una né l'altro bastano a tener su le loro mura. D'una città non godi le sette o le settantasette meraviglie, ma la risposta che dà ad una tua domanda.

— O la domanda che ti pone [...]

ITALO CALVINO, *Le città invisibili*

Ringrazio tutti coloro che mi sono stati vicini, mi hanno incoraggiato e sopportato durante la preparazione del Ph.D. Non vorrei citare esplicitamente nessuno, ma è davvero difficile trattenermi dal menzionare (in rigoroso ordine sparso) Marta, Mimmo, Silvia, Lori, Céline, Sandra, Teresa, Antonio, Cristina, Marta, Mario, Giovanni; le mie famiglie: Rosa (la boss), Mimmo (il boss), Angela, Marisa e Biri, Anna, Laura, Luigi; un ringraziamento particolare va al mio supervisor Enzo Mitidieri per quello che mi ha insegnato, per il buon vino, per la buona musica e per l'ottimo rapporto umano.

Ognuno di loro è stato essenziale.

Table of Contents

1 Preface

In this Ph.D. thesis, we shall present two topics related to some degenerate differential operators: Hardy inequalities and Liouville type Theorems for semilinear inequalities. We shall pay particular attention in the cases when the degenerate operators are the Kohn Laplacian or the Grushin type operators.

The role of Hardy inequalities in the study of partial differential equations is well known and a very brief exposition can be found in Section 3.1.

On the other hand, there are several reasons for studying nonexistence theorems for partial differential inequalities. For instance, such results can be used to prove the existence of solutions for Dirichlet problem in a bounded domain or to prove blow-up estimates for parabolic problems (see, for instance, [51] and the large reference therein).

These two different topics are linked. Let us describe the underline idea.

A classical result due to Gidas and Spruck in [37], can be stated as follows. Let $u \in \mathscr{C}^2(\mathbb{R}^N)$ be a solution of

$$-\Delta u = u^q, \ u \geq 0, \quad \text{on } \mathbb{R}^N, q > 1, N \geq 3. \tag{1.1}$$

If $1 < q < q_S := (N+2)/(N-2)$, then $u \equiv 0$. Notice that there is no requirement on the behavior of u at infinity. This result is sharp. Indeed, for $q \geq q_S$, (1.1) has infinitely many solutions [9].

One can consider, instead of (1.1), the inequality

$$-\Delta u \geq u^q, \ u \geq 0, \quad \text{on } \mathbb{R}^N, q > 1, N \geq 3. \tag{1.2}$$

It is clear that if (1.2) has no solutions, then also (1.1) has no solutions. It is well known that (1.2) has a solution if and only if $q > q_c$ where $q_c = N/(N-2)$, the so called *first critical exponent* (Serrin exponent).

A question arises. What are the inequalities which do not admit a first critical exponent? In other words, is there any possibility to classify some non linear partial differential equations of anticoercive type for which the only possible solution is the trivial one?

Ni [53] and Brézis and Cabrè [12] studied, among other things, the problem

$$-\Delta u \geq \frac{u^2}{|x|^2},\ u \geq 0, \quad \text{on } \Omega, \tag{1.3}$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$ containing the origin. They proved that if u is a solution of (1.3) in the distributional sense and $u, u^2/|x|^2 \in L^1_{loc}(\Omega)$, then $u \equiv 0$. Those authors deal also with the inequality

$$-|x|^2 \Delta u \geq u^2,\ u \geq 0, \quad \text{on } \Omega \setminus \{0\}, \tag{1.4}$$

obtaining a stronger result: if $u \in L^2_{loc}(\Omega \setminus \{0\})$ is a solution of (1.3) in distributional sense, then $u \equiv 0$.

On the other hand, Mitidieri and Pohozaev [49, 51] study the inequality

$$-|x|^2 \Delta u \geq |u|^q \quad \text{on } \mathbb{R}^N \setminus \{0\}, q > 1, \tag{1.5}$$

without any assumption on the sign of u. They proved that, for any $q > 1$ the only weak solution (that is, $u \in L^q_{loc}(\mathbb{R}^N \setminus \{0\})$ and (1.5) is satisfied in distributional sense) is the trivial one.

Now, recall the classical Hardy inequality:

$$\int_\Omega |\nabla u|^2 \, dx \geq c \int_\Omega \frac{u^2}{|x|^2} dx, \tag{1.6}$$

where Ω is an open subset of $\mathbb{R}^N$ containing the origin, $N \geq 3$ and $u \in \mathscr{C}^1_0(\Omega)$.

It is clear, at this point, the connection between the inequalities (1.3), (1.4), (1.5) and the Hardy inequality (1.6); indeed we observe that the singularity $|x|^{-2}$ in (1.3) is exactly the weight function in the inequality (1.6). The natural question is then: is it true that the weight function $|x|^{-2}$ is responsible of the non existence of solutions for (1.5) for any $q > 1$? Is this a general fact? In this thesis we shall show that the answer is in the affirmative at least for classes of differential operators considered in this work. Further links between Hardy inequality (1.6), degenerate evolution inequalities and their critical exponents can be seen in Section 5.1.

More precisely, let $\mathcal{L}$ be a degenerate second order partial differential operator. The counterpart of (1.2) for $\mathcal{L}$ is given by

$$-\mathcal{L}u \geq u^q,\ u \geq 0, \quad \text{on } \mathbb{R}^N, q > 1. \tag{1.7}$$

In the last years it has been proved that, for some classes of operators $\mathcal{L}$ the corresponding problem (1.7) has no positive solutions provided $1 < q \leq q_c =$

$Q/(Q-2)$, where Q is the *homogeneous dimension* associated to $\mathcal{L}$ (see [34, 10, 17, 58]).

For a given $\mathcal{L}$, we wish to classify, at least in some particular cases, the possible singular or degenerate version of (1.7) (see (1.3) and (1.4)) such that for any $q > 1$ the only possible solution is $u \equiv 0$.

In order to realize this programme, having in mind similar problems in the Euclidean setting, first we prove some Hardy inequalities related to the degenerate operators involved and then we establish some Liouville type theorems.

The first step, that is the proof of Hardy type inequalities for the Kohn Laplacian and for Grushin type operators, is presented in Part I below. The main tool we use, is introduced in [48] for Euclidean case, and it based on the divergence theorem and on the careful choice of a suitable vector field. The main difficulty is to construct such a vector field. Indeed, it is necessary to take into account the degeneracies involved in the definition of the operator under consideration. The results presented in Chapter 3 are new.

Liouville type theorems are contained in Part II. The techniques we use are developed in the Euclidean setting in [49, 50, 51], and extended for the Heisenberg framework in [58]. The strategy is the following; by a heedful choice of test functions, we find a priori bounds of the solutions of the inequalities that we are dealing with. As a byproduct of these estimates, we derive the uniqueness of the trivial solution. Also the results of the second part are new.

I wish to thank all the people (students, staff and teachers) met at SISSA-ISAS, where the atmosphere is very warm and scientifically exciting at same time. I wish to express my gratitude to professor Enzo Mitidieri for his help and encouragement and for the stimulating conversations about mathematic, wine and music.

Trieste,
6 September 2002

Lorenzo D'Ambrosio

Part I

Hardy Inequalities

2 Preliminary Facts

In this chapter we shall recall some basic facts related to certain classes of degenerate partial differential operator. Special focus will be devoted to the Kohn Laplacian, the Grushin operator and generalized forms of it.

2.1 Notation

We deal with $\mathbb{R}^N$ which can be split in two or more subspaces $\mathbb{R}^N = \mathbb{R}^d \times \mathbb{R}^k$ $(d, k \geq 1)$, whose points are $\xi \in \mathbb{R}^N$, $x \in \mathbb{R}^d$, $y \in \mathbb{R}^k$. If $z \in \mathbb{R}^m$ then $|z|$ stands for the Euclidean norm $|z| := \sqrt{\sum_{i=1}^m z_i^2}$. Given $\alpha \in \mathbb{N}^d$ and $\beta \in \mathbb{N}^k$, we shall denote by (α, β) and $|(\alpha, \beta)|$, the multi-index $(\alpha_1, \ldots, \alpha_d, \beta_1, \ldots, \beta_k)$ and its length $\alpha_1 + \ldots + \alpha_d + \beta_1 + \ldots + \beta_k$, respectively. If $\alpha, \gamma \in \mathbb{N}^m$, then $\alpha \leq \gamma$ means $\alpha_1 \leq \gamma_1, \cdots, \alpha_m \leq \gamma_m$. The scaling operator with respect to x-variable will be denoted by S_λ^I, namely $S_\lambda^I f(x, y) := f(\lambda x, y)$. Similarly $S_\lambda^{II} f(x, y) := f(x, \lambda y)$.

The symbol $|\Omega|$ indicates the Lebesgue measure of a measurable set Ω. Let $B^n(\xi, R)$ be the Euclidean ball of $\mathbb{R}^n$ centered at $\xi \in \mathbb{R}^n$ and radius R. We set $c_n := |\partial B^n(0, 1)|$ for $n \geq 2$, and $c_1 := 1$, if $n = 1$.

Unless otherwise stated Ω stands for an open set of $\mathbb{R}^N$.

Let $\mathscr{C}_0^k(\Omega)$ be the set of functions belonging to $\mathscr{C}^k(\Omega)$ with compact support. With $\mathscr{C}_0^k(\Omega, \mathbb{R}_+)$ we denote the subset of $\mathscr{C}_0^k(\Omega)$ of nonnegative functions. If $w : \mathbb{R}^N \to \mathbb{R}$ is a measurable function, then $L_{loc}^q(\mathbb{R}^N, w)$ stands for the space of measurable functions v such that $|v|^q w \in L_{loc}^1(\mathbb{R}^N)$. Pair of conjugate exponents are written as q, q' where $q > 1$ and $1/q' + 1/q = 1$. We omit to specify the domains of integration in the integrals that follows when no confusion may arise.

The symbols div and ∇ will denote respectively, the usual divergence operator and the gradient operator for functions defined on $\mathbb{R}^N$.

The symbol I_m indicates the square identity matrix of order m. Let X, Y be vector fields. The symbol $[X, Y]$ stands for the Lie brackets $[X, Y] := XY - YX$.

Finally $A(x) \lesssim B(x)$ means that there exists a constant $C > 0$, independent of x, such that $A \leq CB$.

2.2 Kohn Laplacian on the Heisenberg group

In this section we present some basic results concerning the Kohn Laplacian. For more information and proofs we refer the interested reader to [17, 26, 27, 33, 34, 43] and the reference therein.

Let $n \geq 1$ and $N := 2n+1$: $x, y \in \mathbb{R}^n$, $s \in \mathbb{R}$. For $i = 1, \ldots, n$, consider the vector fields

$$X_i := \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial s}, \quad Y_i := \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial s},$$

and the associated Heisenberg gradient as follows

$$\nabla_H := (X_1, \ldots, X_n, Y_1, \ldots, Y_n)^T.$$

The Kohn Laplacian Δ_H is then the operator defined by

$$\Delta_H := \sum_{i=1}^{n} X_i^2 + Y_i^2.$$

Setting

$$\sigma_H := \begin{pmatrix} I_n & 0 & 2y \\ 0 & I_n & -2x \end{pmatrix},$$

for any vector field $h \in \mathscr{C}^1(\Omega, \mathbb{R}^{2n})$, we shall use the following notation

$$\mathrm{div}_H(h) := \mathrm{div}(\sigma_H^T h).$$

We note that

$$\nabla_H = \sigma_H \nabla,$$

and

$$\Delta_H = \mathrm{div}(\sigma_H^T \sigma_H \nabla) = \mathrm{div}_H(\nabla_H).$$

Let δ_λ^H be the dilation defined by

$$\delta_\lambda^H(\xi) := (\lambda x, \lambda y, \lambda^2 s).$$

It is not difficult to check that X_i and Y_i are homogeneous of degree one with respect to dilation δ_λ^H:

$$X_i(\delta_\lambda^H) = \lambda\delta_\lambda^H(X_i), \qquad Y_i(\delta_\lambda^H) = \lambda\delta_\lambda^H(Y_i). \tag{2.1}$$

Let $\xi = (x_1, \ldots, x_n, y_1, \ldots, y_n, s) = (x, y, s) \in \mathbb{R}^{2n+1}$. The Heisenberg group $\mathbb{H}^n$ is the Lie Group whose underline manifold is $\mathbb{R}^{2n+1}$ endowed with the non commutative group law

$$\hat{\xi} \circ \tilde{\xi} := (\hat{x} + \tilde{x}, \hat{y} + \tilde{y}, \hat{s} + \tilde{s} + 2\sum_{i=1}^{n}(\tilde{x}_i\hat{y}_i - \hat{x}_i\tilde{y}_i)).$$

It is important to notice that the vector fields X_i, Y_i are left invariant with respect to the group action. The Lie algebra of left invariant vector fields associated to the Heisenberg group is generated by X_i, Y_i and $S := \partial/\partial s$. It easy to check that $[X_i, Y_j] = -4\delta_{ij}S$, $[X_i, X_j] = [Y_i, Y_j] = 0$ $(i, j \in \{1, \ldots, n\})$. Therefore the vector fields X_i, Y_i and their first order commutators span the whole Lie algebra.

In $\mathbb{H}^n$ we define the norm

$$|\xi|_H := \left(\left(\sum_{i=1}^{n} x_i^2 + y_i^2\right)^2 + s^2\right)^{1/4}$$

and, setting ξ^{-1} the inverse of ξ with respect to $\circ$ (note that $\xi^{-1} = -\xi$), we define the distance

$$d(\xi, \eta) = \left|\eta^{-1} \circ \xi\right|_H.$$

In what follows we shall use the notation

$$z := (x, y), \quad r := (\sum_{i=1}^{n} x_i^2 + y_i^2)^{1/2} = |z|, \quad \rho := |\xi|_H \quad \text{and} \quad \psi_H := |z| / |\xi|_H.$$

It is easy to see that $|\cdot|_H$ is homogeneous of degree one with respect to the dilation δ_λ^H.

The open ball of radius R and centered at ξ will be denoted by

$$B_H(\xi, R) := \{\eta \in \mathbb{H}^n : d_H(\xi, \eta) < R\}.$$

Let $B^{2n+1}(0, R)$ be the Euclidean open ball in $\mathbb{R}^{2n+1}$ of radius R centered at the origin. For $R > 1$ we have,

$$B^{2n+1}(0, R) \subset B_H(0, R) \subset B^{2n+1}(0, R^2).$$

Unless otherwise stated, $\Omega \subset \mathbb{H}^n$ will denote an open set. A function $u : \Omega \to \mathbb{R}$, such that $u(\xi) = u(r, s)$ (u depends only on $r = \sqrt{\sum_i x_i^2 + \sum_i y_i^2}$ and s) is said *cylindrical*, and in particular if $u(\xi) = u(\rho)$, that is u depends only on $|\xi|_H$, then u is said *radial.*

Let $u \in \mathscr{C}(\Omega)$ be a cylindrical function. In order to compute $\int_\Omega u$, as usual we consider the following transformations. Let Ω be the cylindrical open set $B^{2n}(0,1)\times]a,b[$, where $-\infty \le a < b \le +\infty$ and $B^{2n}(0,1)$ is the unitary sphere in $\mathbb{R}^{2n}$. We can consider $\xi = (x,y,s) := \Phi_1(r,\theta_1,\dots,\theta_{2n-1},s)$ defined by

$$\begin{aligned} x_1 &= r\cos\theta_1,\\ y_1 &= r\sin\theta_1\cos\theta_2,\\ &\dots\\ x_n &= r\sin\theta_1\sin\theta_2\dots\cos\theta_{2n-1},\\ y_n &= r\sin\theta_1\sin\theta_2\dots\sin\theta_{2n-1},\\ s &= s, \end{aligned}$$

for $0 \le r \le 1$, $s \in \mathbb{R}$, $\theta_i \in]0,\pi[$ for $i = 1,\dots,2n-2$ and $\theta_{2n-1} \in]0,2\pi[$. Let $J(\Phi_1)$ be the Jacobian of Φ_1. An easy computation shows that

$$\det J(\Phi_1) = r^{2n-1}\sin^{2n-2}\theta_1\dots\sin\theta_{2n-2}.$$

Therefore

$$\int_\Omega u(r,s)d\xi = c_{2n}\int_a^b ds\int_0^1 r^{2n-1}u(r,s)dr, \tag{2.2}$$

where

$$c_{2n} := \int_0^\pi d\theta_1\int_0^\pi d\theta_2\dots\int_0^\pi d\theta_{2n-2}\int_0^{2\pi} d\theta_{2n-1}\sin^{2n-2}\theta_1\dots\sin\theta_{2n-2}$$

is the $2n$-Lebesgue measure of the unit sphere in $\mathbb{R}^{2n}$. If $\Omega = B_H(0,R_2)\setminus\overline{B_H(0,R_1)}$, with $0 \le R_1 < R_2 \le +\infty$, we can consider $\xi = (x,y,s) := \Phi_2(\rho,\phi,\theta_1,\dots,\theta_{2n-1})$ defined by

$$\begin{aligned} x_1 &= \rho(\sin\phi)^{1/2}\cos\theta_1,\\ y_1 &= \rho(\sin\phi)^{1/2}\sin\theta_1\cos\theta_2,\\ &\dots\\ x_n &= \rho(\sin\phi)^{1/2}\sin\theta_1\sin\theta_2\dots\cos\theta_{2n-1},\\ y_n &= \rho(\sin\phi)^{1/2}\sin\theta_1\sin\theta_2\dots\sin\theta_{2n-1},\\ s &= \rho^2\cos\phi, \end{aligned}$$

for $R_1 < \rho < R_2$, $\phi \in]0,\pi[$ $\theta_i \in]0,\pi[$ for $i = 1,\dots,2n-2$ and $\theta_{2n-1} \in]0,2\pi[$. Noting that $r^2 = \rho^2\sin\phi$, acting as before, we have

$$\int_\Omega u(r,s)d\xi = c_{2n}\int_0^\pi d\phi\int_{R_1}^{R_2}\rho^{2n+1}(\sin\phi)^{n-1}u(\rho\sqrt{\sin\phi},\rho^2\cos\phi)d\rho. \tag{2.3}$$

Thereupon, if u has the form $u(\xi) = \psi_H^2 v(|\xi|_H)$, then we have

$$\int_\Omega \psi_H^2 v(|\xi|_H) d\xi = s_n \int_{R_1}^{R_2} \rho^{2n+1} v(\rho) d\rho, \tag{2.4}$$

where we have set $s_n := c_{2n} \int_0^\pi (\sin\theta)^n d\theta$.

Denoting with $|\cdot|$ the Lebesgue measure on $\mathbb{R}^{2n+1}$, it is important to notice that from (2.3) one gets

$$|B_H(\xi, R)| = |B_H(0, R)| = |B_H(0,1)|\, R^Q,$$

where $Q := 2n + 2$ is called the *homogeneous dimension* of $\mathbb{H}^n$.

Remark 1. From (2.3) we can derive the following criteria for the integrability of the function $r^p \rho^q$;

i) if $2n > -p$ and $2n + 2 > -p - q$, then

$$\int_{B_H(0,1)} r^p \rho^q d\xi < +\infty;$$

ii) if $2n > -p$ and $2n + 2 < -p - q$, then

$$\int_{\mathbb{H}^n \setminus B_H(0,1)} r^p \rho^q d\xi < +\infty.$$

Let p be such that $1 \le p < \infty$. We shall denote by $S^{1,p}(\Omega)$ the Banach space of the functions $u \in L^p(\Omega)$ such that the distributional derivatives $X_i u, Y_i u \in L^p(\Omega)$ for $i = 1, \dots, n$. The norm on $S^{1,p}(\Omega)$ is given by

$$\|u\|_{S^{1,p}(\Omega)} := \left(\int_\Omega (|\nabla_H u|^p + |u|^p) d\xi \right)^{1/p}.$$

$S_0^{1,p}(\Omega)$ denotes the closure of $\mathscr{C}_0^\infty(\Omega)$ in the above norm, $D_H^{1,p}(\Omega)$ the closure of $\mathscr{C}_0^\infty(\Omega)$ in the norm $(\int_\Omega |\nabla_H u|^p \, d\xi)^{1/p}$ and $D_H^{2,2}(\Omega)$ the closure of $\mathscr{C}_0^\infty(\Omega)$ in the norm $(\int_\Omega |\Delta_H u|^2 \, d\xi)^{1/2}$. It is well known that if Ω is bounded, the norms of $S_0^{1,p}(\Omega)$ and $D_H^{1,p}(\Omega)$ are equivalent (see Theorem 29). If $w \in L^1_{loc}(\Omega)$ and $w > 0$ a.e. on Ω, $D_H^{1,p}(\Omega, w)$ denotes the closure of $\mathscr{C}_0^\infty(\Omega)$ in the norm $(\int_\Omega |\nabla_H u|^p \, w d\xi)^{1/p}$.

Let $u \in \mathscr{C}^1(\Omega)$. If u is radial, then it is easy to check that

$$|\nabla_H u| = \psi_H \left| u' \right|, \tag{2.5}$$

and if u is cylindrical, we have

$$|\nabla_H u|^2 = u_r^2 + 4r^2 u_s^2. \tag{2.6}$$

Moreover if $u \in \mathscr{C}^2(\Omega)$, we find respectively

$$\Delta_H u = \psi_H^2 \left(u'' + \frac{2n+1}{\rho} u' \right), \tag{2.7}$$

$$\Delta_H u = u_{rr} + \frac{2n-1}{r} u_r + 4r^2 u_{ss} \tag{2.8}$$

(see Remark 17 for related results).

2.3 Grushin type operators

In this section we present some definitions and basic facts concerning Grushin type operators. For more information and proofs on this topic we refer the interested reader to [8, 17, 28, 29, 30, 31, 35, 39] and the references therein. In Section 2.3.1 we shall compute the fundamental solution of the generalized Grushin operator at the origin. We believe that this result is new.

Let γ be a positive real number and let $\xi = (x_1, \ldots, x_d, y_1, \ldots, y_k) = (x, y) \in \mathbb{R}^d \times \mathbb{R}^k = \mathbb{R}^N$ with $d, k \geq 1$ and $N = d+k$. We denote by $|x|$ (resp. $|y|$) the euclidean norm in $\mathbb{R}^d$ (resp. $\mathbb{R}^k$): $|x| := \sqrt{x_1^2 + \cdots + x_d^2}$ (resp. $|y| := \sqrt{y_1^2 + \cdots + y_k^2}$).

The symbols ∇_z and Δ_z stand respectively for the gradient and the Laplace operator for functions defined on $\mathbb{R}^N$ with respect to the z-variable.

For $i = 1, \ldots, d$, and $j = 1, \ldots, k$ consider the vector fields

$$X_i := \frac{\partial}{\partial x_i}, \quad Y_i := |x|^\gamma \frac{\partial}{\partial y_i},$$

and the associated gradient as follows,

$$\nabla_\gamma := (X_1, \ldots, X_d, Y_1, \ldots, Y_k)^T = (\nabla_x, |x|^\gamma \nabla_y)^T,$$

which can be rewritten as $\nabla_\gamma = \sigma^\gamma \nabla$ where

$$\sigma^\gamma := \begin{pmatrix} I_d & 0 \\ 0 & r_\epsilon^\gamma I_k \end{pmatrix}.$$

The Grushin operator Δ_γ is the operator defined by

$$\Delta_\gamma := \sum_{i=1}^{d} X_i^2 + \sum_{j=1}^{k} Y_j^2 = \Delta_x + |x|^{2\gamma} \Delta_y = \nabla_\gamma \cdot \nabla_\gamma.$$

Defining on $\mathbb{R}^N$ the dilation δ_λ^γ as

$$\delta_\lambda^\gamma(x,y) := (\lambda x, \lambda^{1+\gamma} y); \tag{2.9}$$

it is not difficult to check that X_i and Y_i are homogeneous of degree one with respect to the dilation: $X_i(\delta_\lambda^\gamma) = \lambda \delta_\lambda^\gamma(X_i)$, $Y_i(\delta_\lambda^\gamma) = \lambda \delta_\lambda^\gamma(Y_i)$, and hence $\nabla_\gamma(\delta_\lambda^\gamma) = \lambda \delta_\lambda^\gamma(\nabla_\gamma)$.

Notice that it is not possible to endow $\mathbb{R}^N$ with a group law for which the vector fields X_i, Y_i are left invariant (see [4]).

Let $[\![\xi]\!] = [\![(x,y)]\!]$ be the following distance from the origin on $\mathbb{R}^N$:

$$[\![\xi]\!] = [\![(x,y)]\!] := \left((\sum_{i=1}^{d} x_i^2)^{1+\gamma} + (1+\gamma)^2 \sum_{i=1}^{k} y_i^2 \right)^{\frac{1}{2+2\gamma}} .$$

We set $\psi_\gamma := |\nabla_\gamma [\![\xi]\!]| = |x|^\gamma / [\![\xi]\!]^\gamma$.

The function $[\![\cdot]\!]$ is related to the fundamental solution at the origin of Grushin operator Δ_γ (see [21] and Section 2.3.1). Furthermore it is easy to see that $[\![\cdot]\!]$ is homogeneous of degree one with respect to the dilation δ_λ^γ. Let $R > 0$. We shall denote by B_R the set

$$B_R := \{\xi \in \mathbb{R}^N : [\![\xi]\!] < R\}.$$

A function $u : \Omega \to \mathbb{R}$, such that $u(\xi) = u([\![\xi]\!])$ (u depends only on $[\![\xi]\!]$) is said *radial*.

Let $u \in \mathscr{C}^1(\Omega)$. If u is radial, then it is easy to check that

$$|\nabla_\gamma u(\xi)| = \frac{|x|^\gamma}{[\![\xi]\!]^\gamma} \left|u'([\![\xi]\!])\right| = \psi_\gamma \left|u'([\![\xi]\!])\right| . \tag{2.10}$$

Moreover if $u \in \mathscr{C}^2(\Omega)$, we find

$$\Delta_\gamma u = \frac{|x|^{2\gamma}}{[\![\xi]\!]^{2\gamma}} \left(u'' + \frac{d + (1+\gamma)k - 1}{[\![\xi]\!]} u' \right)$$

(see Remark 17 for related results).

Let $\Omega = B_{R_2} \setminus \overline{B_{R_1}}$, with $0 \le R_1 < R_2 \le +\infty$ and $u \in \mathscr{C}(\Omega)$. As we shall see below, in some intermediate inequalities appearing in the proof of our results, we shall need to compute $\int_\Omega u$. For this task we can proceed as follows: we consider the transformation $\xi := \Phi(\rho, \theta, \theta_1, \dots, \theta_{d-1}, \omega_1, \dots, \omega_{k-1})$, introduced in [21], defined by

$$\begin{aligned}
x_1 &= \rho \sin\theta (\sin^2\theta)^{-\frac{\gamma}{2(1+\gamma)}} \cos\omega_1,\\
x_2 &= \rho \sin\theta (\sin^2\theta)^{-\frac{\gamma}{2(1+\gamma)}} \sin\omega_1 \cos\omega_2,\\
&\dots\\
x_{d-1} &= \rho \sin\theta (\sin^2\theta)^{-\frac{\gamma}{2(1+\gamma)}} \sin\omega_1 \sin\omega_2 \dots \cos\omega_{d-1},\\
x_d &= \rho \sin\theta (\sin^2\theta)^{-\frac{\gamma}{2(1+\gamma)}} \sin\omega_1 \sin\omega_2 \dots \sin\omega_{d-1},\\
y_1 &= \tfrac{1}{1+\gamma}\rho^{1+\gamma} \cos\theta \cos\theta_1,\\
y_2 &= \tfrac{1}{1+\gamma}\rho^{1+\gamma} \cos\theta \sin\theta_1 \cos\theta_2,\\
&\dots\\
y_{k-1} &= \tfrac{1}{1+\gamma}\rho^{1+\gamma} \cos\theta \sin\theta_1 \sin\theta_2 \dots \cos\theta_{k-1},\\
y_k &= \tfrac{1}{1+\gamma}\rho^{1+\gamma} \cos\theta \sin\theta_1 \sin\theta_2 \dots \sin\theta_{k-1},,
\end{aligned} \tag{2.11}$$

for $R_1 < \rho < R_2$, $\theta_i, \omega_j \in]0, \pi[$ for $i = 1, \dots, k-2$, $j = 1, \dots, d-2$, $\theta_{k-1}, \omega_{d-1} \in]0, 2\pi[$ and $\theta \in]a_\theta, b_\theta[$ where a_θ and b_θ depend on d and k, that is

- $\theta \in]0, \frac{\pi}{2}[$, if $d, k \geq 2$,
- $\theta \in]0, \pi[$, if $k = 1$ and $d \geq 2$,
- $\theta \in]-\frac{\pi}{2}, \frac{\pi}{2}[$, if $d = 1$ and $k \geq 2$,
- $\theta \in]0, 2\pi[$, if $d = k = 1$.

Let $J(\Phi)$ be the Jacobian of Φ. We have

$$|\det J(\Phi)| = \rho^{Q-1}\Theta(\theta, \theta_1, \dots, \theta_{d-2}, \omega_1, \dots, \omega_{k-2}),$$

where $Q := d + (1+\gamma)k$ and

$$\Theta := (\frac{1}{1+\gamma})^k |\sin\theta|^{\frac{d}{1+\gamma}-1} \cos^{k-1}\theta \sin^{k-2}\theta_1 \dots \sin\theta_{k-2} \sin^{d-2}\omega_1 \dots \sin\omega_{d-2}.$$

Therefore, if $u : \Omega \to \mathbb{R}$ is radial, i.e. $u(\xi) = u([\![\xi]\!])$, then

$$\int_\Omega u([\![\xi]\!]) d\xi = s_n' \int_{R_1}^{R_2} \rho^{Q-1} u(\rho) d\rho, \tag{2.12}$$

where

$$s_n' := \int_{a_\theta}^{b_\theta} d\theta \int_0^\pi d\theta_1 \dots \int_0^\pi d\theta_{k-2} \int_0^{2\pi} d\theta_{k-1} \int_0^\pi d\omega_1 \dots \int_0^\pi d\omega_{d-2} \int_0^{2\pi} d\omega_{d-1} \Theta.$$

It is easy to see that if $d, k \geq 1$, than s_n' can be written as

$$s_n' = c_d c_k (\frac{1}{1+\gamma})^k \int_{a_\theta}^{b_\theta} |\sin\theta|^{\frac{d}{1+\gamma}-1} \cos^{k-1}\theta d\theta.$$

We note it that from (2.12) one gets

$$|B_R| = |B_1|\, R^Q,$$

where $Q := d + (1+\gamma)k$ is called the *homogeneous dimension*.

Remark 2. Using (2.11), we can deduce the following criteria for the integrability of the function $|x|^p [\![\xi]\!]^q$;

i) if $d > -p$ and $Q > -p - q$, then

$$\int_{B_1} |x|^p [\![\xi]\!]^q d\xi < +\infty;$$

ii) if $d > -p$ and $Q < -p - q$, then

$$\int_{\mathbb{R}^N \setminus B_1} |x|^p [\![\xi]\!]^q d\xi < +\infty.$$

Let p be such that $1 \le p < \infty$. We shall denote by $D^{1,p}_\gamma(\Omega)$ the closure of $\mathscr{C}^\infty_0(\Omega)$ in the norm $(\int_\Omega |\nabla_\gamma u|^p\, d\xi)^{1/p}$. If $w \in L^1_{loc}(\Omega)$ and $w > 0$ a.e. on Ω, $D^{1,p}_\gamma(\Omega, w)$ denotes the closure of $\mathscr{C}^\infty_0(\Omega)$ in the norm $(\int_\Omega |\nabla_\gamma u|^p\, w d\xi)^{1/p}$.

2.3.1 Fundamental solution of the Grushin operator

In this section we describe the fundamental solution of the Grushin operator Δ_γ at $(x,y) = (0,0)$. We find a distribution E such that $\Delta_\gamma E = \delta_0$, being δ_0 the Dirac measure at $0 \in \mathbb{R}^N$. The function $[\![\cdot]\!]$ defined above plays the same role of the Euclidean norm for the Laplacian operator.

Let $p > 1$. Let $\Delta_{\gamma,p}$ be the operator defined by

$$\Delta_{\gamma,p} u := \mathrm{div}_\gamma(|\nabla_\gamma u|^{p-2}\, \nabla_\gamma u).$$

This operator is the analogue of the p-Laplacian operator in the Euclidean setting.

Let u_p be the function defined by

$$u_p(\xi) := \begin{cases} [\![\xi]\!]^{\frac{p-Q}{p-1}} & \text{if } p \neq Q, \\ \ln[\![\xi]\!] & \text{if } p = Q. \end{cases}$$

After some computation, we can prove the following.

Lemma 3. *Let $p > 1$. The function u_p is $\Delta_{\gamma,p}$-Harmonic in $\mathbb{R}^N \setminus \{0\}$, that is, $\Delta_{\gamma,p} u = 0$ on $\mathbb{R}^N \setminus \{0\}$.*

Let $\gamma > 0$. For any $(x, y) \neq (0,0)$, we define

$$E(x,y) = C_{d,k,\gamma}\Gamma := C_{d,k,\gamma}\left(|x|^{2+2\gamma} + (1+\gamma)^2|y|^2\right)^{\frac{2-N-\gamma k}{2+2\gamma}} = C_{d,k,\gamma}[\![\xi]\!]^{2-Q},$$

where

$$C_{d,k,\gamma}^{-1} = (N + k\gamma - 2)\int_{[\![\xi]\!]=1} \frac{|x|^{2\gamma}}{(|x|^{2+4\gamma} + (1+\gamma)^2|y|^2)^{1/2}} dS.$$

We remark that $\gamma > 0$ implies $Q := N + k\gamma > 2$ and hence E has the origin $(0,0)$ as singularity.

Theorem 4. *The function E is locally integrable on $\mathbb{R}^N$ and $\Delta_\gamma E = \delta_0$.*

To prove this result, we need of the following:

Lemma 5. *Let $X_1 : \mathbb{R}^N \to \mathbb{R}^d$, $X_2 : \mathbb{R}^N \to \mathbb{R}^k$ be two smooth compactly supported functions and set $X = (X_1, X_2)$. For any $R > 0$, the following relations hold:*

$$\int_{[\![\xi]\!]=R} (X_1, |x|^\gamma X_2)\cdot \mathbf{n}(x,y) dS = \int_{[\![\xi]\!]>R} \nabla_\gamma \cdot X d\xi =$$
$$= R^{Q-1}\int_{[\![\xi]\!]=1} X_1(Rx, R^{1+\gamma}y)\cdot n_x(x.y) + |x|^\gamma X_2(Rx, R^{1+\gamma}y)\cdot n_y(x,y)dS,$$

here $\mathbf{n} = (n_x, n_y)$ is the unit outward normal to $[\![\xi]\!] = R$.

Proof. The first equality is a consequence of the divergence theorem. For the second one we use the change of coordinates $\xi \to \delta_R^\gamma$ and we obtain

$$\int_{[\![\xi]\!]>R} \nabla_\gamma \cdot X d\xi =$$
$$= R^Q \int_{[\![\xi]\!]>1} (\nabla_x \cdot X_1)(Rx, R^{1+\gamma}y) + (\nabla_y \cdot |x|^\gamma X_2)(Rx, R^{1+\gamma}y) d\xi =$$
$$= R^{Q-1}\int_{[\![\xi]\!]>1} \nabla_x \cdot (X_1(Rx, R^{1+\gamma}y)) + |x|^\gamma \nabla_y \cdot (X_2(Rx, R^{1+\gamma}y)) d\xi.$$

Applying the classical Green formula we get the claim. □

Now we come back to the proof of Theorem 4.

Proof (of Theorem 4). The local integrability of E follows from the Remark 2.

We prove that $\int \phi \Delta_\gamma \Gamma d\xi = C_{d,k,\gamma}^{-1}\phi(0)$, for any $\phi \in \mathscr{C}_0^\infty(\mathbb{R}^N)$. By Lemma 3 for $(x,y) \neq (0,0)$, we have $\Delta_\gamma \Gamma = 0$, hence

$$\int_{\mathbb{R}^N} \phi \Delta_\gamma \Gamma d\xi = \int_{\mathbb{R}^N} \Gamma \Delta_\gamma \phi d\xi = \lim_{\varepsilon \to 0} \int_{[\![\xi]\!] > \varepsilon} \Gamma \Delta_\gamma \phi d\xi =$$
$$= \lim_{\varepsilon \to 0} \int_{[\![\xi]\!] > \varepsilon} (\Gamma \Delta_\gamma \phi - \phi \Delta_\gamma \Gamma) d\xi = \lim_{\varepsilon \to 0} \int_{[\![\xi]\!] > \varepsilon} \nabla_\gamma \cdot (\Gamma \nabla_\gamma \phi - \phi \nabla_\gamma \Gamma) d\xi.$$

Combining Lemma 5 with the relation $[\![(\varepsilon x, \varepsilon^{1+\gamma} y)]\!] = \varepsilon [\![\xi]\!]$, we obtain

$$\lim_{\varepsilon \to 0} \int_{[\![\xi]\!] > \varepsilon} \nabla_\gamma \cdot (\Gamma \nabla_\gamma \phi) d\xi = \tag{2.13}$$
$$= \lim_{\varepsilon \to 0} \varepsilon \int_{[\![\xi]\!] = 1} \nabla_x \phi(\varepsilon x, \varepsilon^{1+\gamma} y) \cdot n_x + \varepsilon^\gamma |x|^{2\gamma} \nabla_y \phi(\varepsilon x, \varepsilon^{1+\gamma} y) \cdot n_y dS.$$

For small ε, on $\mathrm{supp}\phi \cap \{[\![z]\!] = 1\}$, we have $\left|(\nabla_x \phi, \varepsilon^\gamma |x|^{2\gamma} \nabla_y \phi) \cdot \mathbf{n}\right| \le C$ with $C > 0$ independent of ε. This implies that the limit in (2.13) vanishes.

We claim that

$$-\int_{[\![\xi]\!] > \varepsilon} \nabla_\gamma \cdot (\phi \nabla_\gamma \Gamma) d\xi \to C_{d,k,\gamma}^{-1} \phi(0) \quad \text{as } \varepsilon \to 0.$$

We observe that $(\nabla_\gamma \Gamma)(\varepsilon x, \varepsilon^{1+\gamma} y) = (2 - Q) \varepsilon^{-Q+1} [\![\xi]\!]^{-Q-\gamma} (|x|^\gamma x, (1+\gamma) y)$. Then applying Lemma 5, we have

$$-\int_{[\![\xi]\!] > \varepsilon} \nabla_\gamma \cdot (\phi \nabla_\gamma \Gamma) d\xi = (Q-2) \int_{[\![\xi]\!] = 1} \phi(\varepsilon x, \varepsilon^{1+\gamma} y) |x|^{2\gamma} (x \cdot n_x + (1+\gamma) y \cdot n_y) dS.$$

Being $x \cdot n_x + (1+\gamma) y \cdot n_y = (|x|^{2+4\gamma} + (1+\gamma)^2 |y|^2)^{-1/2} [\![\xi]\!]^{1+\gamma}$, it follows that

$$-\int_{[\![\xi]\!] > \varepsilon} \nabla_\gamma \cdot (\phi \nabla_\gamma \Gamma) d\xi = (Q-2) \int_{[\![\xi]\!] = 1} \phi(\varepsilon x, \varepsilon^{1+\gamma} y) \frac{|x|^{2\gamma}}{(|x|^{2+4\gamma} + (1+\gamma)^2 |y|^2)^{1/2}} dS.$$

Finally, by using Lebesgue theorem we easily conclude. □

3 Hardy Inequalities

3.1 Introduction

The purpose of this chapter is to present simple proofs of Hardy inequalities for a quite general vector field. We shall pay particular attention to the Kohn Laplacian on the Heisenberg group $\mathbb{H}^n$ and to Grushin type operators.

The well known classical Hardy inequality, written as in [40], is the following

$$c\int_a^\infty \frac{F(x)^p}{(x-a)^p}dx \leq \int_a^\infty f(x)^p dx.$$

Here $f : \mathbb{R} \to \mathbb{R}$ is a positive measurable function, $p > 1$ and for $a \in \mathbb{R}$, $F(x) := \int_a^x f(t)dt$. A higher dimensional generalization of this inequality for function $u : \Omega \to \mathbb{R}$, where Ω is contained in $\mathbb{R}^n$, is given by

$$c\int_\Omega \frac{u^2}{|x|^2}dx \leq \int_\Omega |\nabla u|^2\, dx, \tag{3.1}$$

(see for instance [2, 44] and the references therein). A lot of efforts have been made to give explicit values of the constant c, and even more, to find its best value c_b (see [22, 24]). The preeminent rule of the Hardy inequality in the study of linear and nonlinear partial differential equations is well known. Existence and nonexistence theorems for elliptic, parabolic and also hyperbolic equations in the form

$$\left.\begin{matrix} 0 \\ u_t \\ u_{tt} \end{matrix}\right\} = \Delta u + \lambda\frac{|u|^q}{|x|^2}, \tag{3.2}$$

involve the relationship between λ and the best constant c appearing in (3.1). For instance, let us consider the linear initial value problem

$$\begin{cases} u_t - \Delta u = \lambda\frac{u}{|x|^2},\ x \in \mathbb{R}^n, \quad n \geq 3, \quad t \in]0,T[, \quad \lambda \in \mathbb{R}, \\ u(x,0) = u_0(x),\ \ x \in \mathbb{R}^n, \quad u_0 \in L^2(\mathbb{R}^n), \quad u_0 > 0. \end{cases} \tag{3.3}$$

The problem (3.3) has a solution if and only if $\lambda \leq (\frac{n-2}{2})^2 = c_b$ (see [6] for more details). In the last years this result has been extended in several direction see i.e. [12, 3, 49, 50, 15, 38, 61]. As an example consider the following quasilinear problem,

$$\begin{cases} u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda \frac{|u|^{p-2}u}{|x|^p}, & (x,t) \in \Omega\times]0,T[,\ n > p > 1,\ \lambda > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \quad u_0 > 0, \\ u(x,t) = 0, & x \in \partial\Omega, \quad t \in]0,T[, \end{cases}$$

where u_0 satisfies suitable regularity assumptions and Ω is a bounded domain in $\mathbb{R}^n$ containing the origin. In [3], the authors proved that the above problem with $p \geq \frac{2n}{n+1}$ has a solution if and only if $\lambda \leq c_{n,p} := (\frac{n-p}{p})^p$. Again the result depends on the relation between λ and the best constant $c_{n,p}$ in the inequality of type (3.1) in L^p:

$$c\int_\Omega \frac{|u|^p}{|x|^p}dx \leq \int_\Omega |\nabla u|^p\,dx. \tag{3.4}$$

In the Heisenberg group setting, Garofalo and Lanconelli in [33], Niu, Zhang and Wang in [56] and the author in [19] proved among other results, the following Hardy type inequality related to the Kohn Laplacian operator:

$$c\int_{\mathbb{H}^n} \frac{u^2}{\rho^2}\psi_H^2 d\xi \leq \int_{\mathbb{H}^n} |\nabla_H u|^2\,d\xi, \quad u \in \mathscr{C}_0^1(\mathbb{H}^n\setminus\{0\}) \tag{3.5}$$

where ∇_H denotes the vector field associated to the Kohn Laplacian ($\Delta_H = \nabla_H \cdot \nabla_H$), ρ and ψ_H are respectively a suitable distance from the origin and a weight function such that $0 \leq \psi_H \leq 1$ (see Section 2.2 for precise definitions).

Recently, in [38], it has been pointed out that the analogue problem of (3.3) involving the Kohn Laplacian Δ_H, namely

$$\begin{cases} u_t - \Delta_H u = \lambda\psi_H^2 \frac{u}{\rho^2} \text{ on } \mathbb{R}^{2n+1}\times]0,T[, \quad \lambda \in \mathbb{R}, \\ u(\cdot,0) = u_0(\cdot) \qquad \text{on } \mathbb{R}^{2n+1}, \quad u_0 \in L^2(\mathbb{R}^{2n+1}), \quad u_0 > 0, \end{cases}$$

has positive solution if and only if $\lambda \leq c_{b,H}$, where $c_{b,H}$ is the best constant in (3.5).

Further important connections between the Gelfand problem and Hardy inequality have been pointed out in [12, 15].

Even an estimate in the form $\int_\Omega |\nabla u|^p - c\int_\Omega \frac{|u|^p}{|x|^p} \geq R(u)$, is interesting. When the remainder term $R(u)$ it happens to be a suitable norm of u, then the functionals associated to (3.2) is coercive (see [2, 61] for more details).

Having in mind some extensions of this kind of results in the setting of second order linear degenerate (or singular) partial differential operators, it appears that an important step towards this programme is to establish some fundamental inequalities of Hardy type.

In this chapter we shall prove some inequalities of the type (3.4) associated to a quite general second order linear degenerate (or singular) partial differential operator L. Let ∇_L be a suitable gradient operator (see section 2 for more details), L is defined by $L = -\nabla_L^* \cdot \nabla_L$. Our aim is to prove inequalities of the type

$$c \int_\Omega |u|^p w^p d\xi \leq \int_\Omega |\nabla_L u|^p \, d\xi, \tag{3.6}$$

where w is one of the following functions: $1/|x|$ or $1/d(\xi)$ with $x = (\xi_1, \cdots, \xi_m)$ and d denotes a suitable function (see below for the definition). Furthermore, we give an estimate on the optimal constant in (3.6) and, in some special cases of L, we show its sharp value.

For this goal we shall mainly use a technique developed in [19, 20, 48]. An interesting outcome of this approach is that, in some cases, one can easily obtain the best constant even for higher order generalization of (3.1). We refer to Allegretto and Huang [1] and to Niu, Zhang and Wang [56] for a different and interesting approach based on the Picone identity respectively in the Euclidean and Heisenberg group setting.

We pay particular attention at two special cases of L: the Grushin type operators and the Kohn Laplacian described in the Chapter 1.

In Section 3.3 we shall study some Hardy type inequalities associated to Grushin type operators $\Delta_\gamma := \Delta_x + |x|^{2\gamma} \Delta_y = \nabla_\gamma \cdot \nabla_\gamma$. We prove that the following inequality holds for $u \in \mathscr{C}_0^1(\Omega)$:

$$c \int_\Omega |u|^p w^p d\xi \leq \int_\Omega |\nabla_\gamma u|^p \, d\xi, \tag{3.7}$$

where w is one of the following functions: $1/|x|$, $1/[\![\xi]\!]$ or $|x|^\gamma / [\![\xi]\!]^{1+\gamma}$. In this setting we prove a sharp version of inequality (3.7) with the weight function $w = |x|^\gamma / [\![\xi]\!]^{1+\gamma}$.

In Section 3.4 we shall prove some inequalities of the type (3.4) on the Heisenberg group. Moreover, in this setting, we shall consider an extension of the classical inequality obtained earlier by F. Rellich in the Euclidean setting, that is

$$\frac{n^2(n-4)^2}{16} \int_{\mathbb{R}^n} \frac{u^2}{r^4} dx \leq \int_{\mathbb{R}^n} (\Delta u)^2 dx, \tag{3.8}$$

for $u \in \mathscr{C}_0^2(\mathbb{R}^n)$ and $n > 4$ (see [59]). In [33] and [56], the authors proved (3.5) for $u \in \mathscr{C}_0^\infty(\mathbb{H}^n \setminus \{0\})$. We shall improve (3.5) for powers $p \neq 2$ and for $u \in \mathscr{C}_0^\infty(\Omega)$

with Ω any subset of $\mathbb{H}^n$. Moreover, we present several version of (3.5) depending on the choice of the weights and we obtain the counterpart of (3.8) for $p > 1$ on the Heisenberg setting.

Finally, in Section 3.5, we shall study some version of (3.1) and (3.8) with remainder terms (see Brézis and Vazquez [15] and Gazzola, Grunau and Mitidieri [36] for the Euclidean case).

3.2 General results

The aim of this section is to present some preliminary results and derive some Hardy type inequalities for a quite general vector field.

Let $\mu \in \mathscr{C}(\mathbb{R}^N; \mathbb{R}^l)$ be a matrix with continuous entries, $\mu := (\mu_{ij})$, $i = 1, \ldots, l$, $j = 1, \ldots, N$. Let X_i, $(i = 1, \ldots, l)$ be the vector field defined by

$$X_i := \sum_{j=1}^{N} \mu_{ij}(\xi) \frac{\partial}{\partial \xi_j}$$

and let ∇_L be the vector field defined by

$$\nabla_L := (X_1, \ldots, X_l)^T = \mu \nabla.$$

Let

$$X_i^* := -\sum_{j=1}^{N} \frac{\partial}{\partial \xi_j} \mu_{ij}(\xi) \cdot$$

be the formal adjoint of X_i and set $\nabla_L^* := (X_1^*, \ldots, X_l^*)^T$.

For any vector field $h \in \mathscr{C}^1(\Omega, \mathbb{R}^l)$, we shall use the following notation

$$\mathrm{div}_L(h) := \mathrm{div}(\mu^T h),$$

that is

$$\mathrm{div}_L(h) = -\sum_{i=1}^{l} X_i^* h_i.$$

Let L be the second order differential operator defined by

$$L := \mathrm{div}_L(\nabla_L) = -\sum_{i=1}^{l} X_i^* X_i = -\nabla_L^* \cdot \nabla_L.$$

Example 6. Let ∇_H, ∇_γ be the vector fields described in the previous chapter and let ∇ be the usual gradient operators on $\mathbb{R}^N$. These quantities are associated (respectively) to the following matrices:

$$\sigma_H := \begin{pmatrix} I_n & 0 & 2y \\ 0 & I_n & -2x \end{pmatrix}, \quad \sigma^\gamma := \begin{pmatrix} I_d & 0 \\ 0 & r^\gamma I_k \end{pmatrix}, \quad I_N.$$

The main idea for proving Hardy type inequalities associated to L is to apply the divergence theorem to a suitable vector field.

Definition 7. *We say that a vector field $h \in \mathscr{C}^1(\Omega, \mathbb{R}^l)$, is* μ-compatible *or* ∇_L-compatible, *if the function $g_j(\xi) := \sum_{i=1}^l \mu_{ij}(\xi) h_i(\xi)$ and its derivative $\partial g_j / \partial \xi_j$ are continuous for $j = 1, \dots, N$.*

Example 8. If the coefficients $\mu_{ij} \in \mathscr{C}^1(\Omega)$, then any $h \in \mathscr{C}^1(\Omega, \mathbb{R}^l)$ is μ-compatible. Thus, in particular any smooth vector field $h \in \mathscr{C}^1(\Omega, \mathbb{R}^{2n})$ is σ_H-compatible.

Example 9. Let μ be the matrix σ^γ. In this case, L is the Grushin type operator Δ_γ (see Section 2.3). If $0 < \gamma < 1$, then some entries μ_{ij} does not belong to $\mathscr{C}^1(\Omega)$. However, any $h \in \mathscr{C}^1(\Omega, \mathbb{R}^N)$ is σ^γ-compatible. Indeed, let $h \in \mathscr{C}^1(\Omega, \mathbb{R}^N)$; for $1 \le j \le d$, we have $g_j = h_j$ and $\partial g_j / \partial \xi_j = \partial h_j / \partial \xi_j$, while for $d < j \le N$, $g_j(\xi) = |x|^\gamma h_j /$ and $\partial g_j / \partial \xi_j = |x|^\gamma \partial \xi_j \in \mathscr{C}(\Omega)$. Thus $\partial g_j / \partial \xi_j$ is continuous.

Example 10. Assume that:

$$\frac{\partial}{\partial \xi_j} \mu_{ij}(\xi) = 0, \quad i = 1, \dots, l, j = 1, \dots, N. \tag{3.9}$$

Then any $h \in \mathscr{C}^1(\Omega, \mathbb{R}^N)$ is μ-compatible. Notice that in this case $\nabla_L^* = -\nabla_L$.

Let A be an open subset of $\mathbb{R}^N$ with Lipschitz boundary ∂A and let $\hat{h} \in \mathscr{C}^1(\overline{A}, \mathbb{R}^l)$ be a μ-compatible vector field. By the divergence theorem we have

$$\int_A \mathrm{div}_L \hat{h} d\xi = \int_A \mathrm{div}(\mu^T \hat{h}) d\xi = \int_{\partial A} \hat{h} \cdot \mu\nu d\Sigma = \int_{\partial A} \hat{h} \cdot \nu_L d\Sigma,$$

where $\nu_L := \mu\nu$, and ν denotes the exterior normal at point $\xi \in \partial A$. Let $f \in \mathscr{C}^1(\overline{A})$ and let h be σ-compatible. Then $\hat{h} := f \cdot h$ is σ-compatible and we have

$$\int_A f \mathrm{div}_L h d\xi + \int_A \nabla_L f \cdot h d\xi = \int_{\partial A} f h \cdot \nu_L d\Sigma. \tag{3.10}$$

Moreover, if $h = \nabla_L u$ with $u \in \mathscr{C}^2(\overline{A})$ and $\nabla_L u$ is μ-compatible, then (3.10) yields the Gauss–Green formula

$$\int_A fLud\xi + \int_A \nabla_L f \cdot \nabla_L u d\xi = \int_{\partial A} f \nabla_L u \cdot \nu_L d\Sigma.$$

Let $g \in \mathscr{C}^1(\mathbb{R})$ be such that $g(0) = 0$ and let $\Omega \subset \mathbb{R}^N$ be open. For every μ-compatible vector field h, $u \in \mathscr{C}_0^1(\Omega)$ choosing $f := g(u)$ in (3.10), we obtain

$$\int_\Omega g(u) \mathrm{div}_L h d\xi = - \int_\Omega g'(u) \nabla_L u \cdot h d\xi. \tag{3.11}$$

In particular if $g(t) = |t|^p$ for $p > 1$, then for any $u \in \mathscr{C}_0^1(\Omega)$ we have

$$\int_\Omega |u|^p \, \mathrm{div}_L h d\xi = -p \int_\Omega |u|^{p-2} \, u \nabla_L u \cdot h d\xi. \tag{3.12}$$

Identities (3.11) and (3.12) play an important rule in the proof of the following Hardy type inequalities and the Poincarè inequality too as well as in Rellich type inequalities (see Theorem 18, 31, 32).

Theorem 11. *Let $p > 1$. Let $h \in \mathscr{C}^1(\Omega, \mathbb{R}^l)$ be a μ-compatible vector field such that $\mathrm{div}_L h > 0$. Then for any $u \in \mathscr{C}_0^1(\Omega)$, we have*

$$\int_\Omega |u|^p \, \mathrm{div}_L h d\xi \le p^p \int_\Omega |h|^p \, |\mathrm{div}_L h|^{-(p-1)} \, |\nabla_L u|^p \, d\xi. \tag{3.13}$$

Proof. We note that the right hand side of (3.13) is finite since $u \in \mathscr{C}_0^1(\Omega)$. Using the identity (3.12) and Hölder inequality we obtain

$$\begin{aligned}
\int_\Omega |u|^p \, \mathrm{div}_L h d\xi &\le p \int_\Omega |u|^{p-1} \, |h| \, |\nabla_L u| \, d\xi \\
&= p \int_\Omega |u|^{p-1} \, |\mathrm{div}_L h|^{(p-1)/p} \, \frac{|h|}{|\mathrm{div}_L h|^{(p-1)/p}} \, |\nabla_L u| \, d\xi \\
&\le p \left(\int_\Omega |u|^p \, |\mathrm{div}_L h| \, d\xi \right)^{(p-1)/p} \left(\int_\Omega \frac{|h|^p}{|\mathrm{div}_L h|^{p-1}} \, |\nabla_L u|^p \, d\xi \right)^{1/p}.
\end{aligned}$$

This completes the proof. □

Theorem 12. *Let $p, q > 1$. Let $h \in \mathscr{C}^1(\Omega, \mathbb{R}^l)$ be a μ-compatible vector field such that $\mathrm{div}_L h > 0$. Then for any $u \in \mathscr{C}_0^1(\Omega)$ we have*

$$\int_\Omega |u|^p \, \mathrm{div}_L h d\xi \le p^q \int_\Omega |h|^q \, |\mathrm{div}_L h|^{-(q-1)} \, |\nabla_L u|^q \, |u|^{p-q} \, d\xi.^{1} \tag{3.14}$$

[1] If $p < q$ then ∞ can occur in the right hand side of (3.14).

Proof. Without loosing of generality, we assume that the right hand side of (3.14) is finite. Using the identity (3.12) and Hölder inequality, we obtain

$$\begin{aligned}\int_\Omega |u|^p \operatorname{div}_L h d\xi &\le p \int_\Omega |u|^{p-1} |h| |\nabla_L u| \, d\xi \\ &= p \int_\Omega |u|^{p/q'} |\operatorname{div}_L h|^{1/q'} \frac{|u|^{p-1-p/q} |h|}{|\operatorname{div}_L h|^{1/q'}} |\nabla_L u| \, d\xi \\ &\le p \left(\int_\Omega |u|^p |\operatorname{div}_L h| \, d\xi \right)^{1/q'} \left(\int_\Omega \frac{|h|^q}{|\operatorname{div}_L h|^{q-1}} |\nabla_L u|^q |u|^{p-q} \, d\xi \right)^{1/q},\end{aligned}$$

which gives (3.14). □

Remark 13. In case $p > q > 1$ we can derive Theorem 12 from Theorem 11 by using the identity

$$\left| \nabla_L |u|^{p/q} \right|^q = (\frac{p}{q})^q |u|^{p-q} |\nabla_L u|^q. \tag{3.15}$$

Specializing the vector field h, we shall deduce from (3.13) and (3.14) some concrete inequalities of Hardy type. To this end we shall assume that there exists $m \in \mathbb{N}$, $1 \le m \le l$ such that

$$\frac{\partial \mu_{ij}}{\partial \xi_j} \in \mathscr{C}(\Omega) \text{ and } \mu_{ii}(\xi) \equiv 1, \quad \text{for } i = 1, \dots, m, j = 1, \dots, N. \tag{3.16}$$

Set $x := (\xi_1, \dots, \xi_m)$.

Theorem 14. *Let $m > p > 1$ and assume that (3.16) is satisfied. Then for every $u \in \mathscr{C}_0^1(\Omega)$ the inequality*

$$c_{m,p}^p \int_\Omega \frac{|u|^p}{|x|^p} d\xi \le \int_\Omega |\nabla_L u|^p \, d\xi, \tag{3.17}$$

holds with $c_{m,p} := (m-p)/p$. In particular if $p = 2$ and $m \ge 3$, we have

$$(\frac{m-2}{2})^2 \int_\Omega \frac{u^2}{|x|^2} d\xi \le \int_\Omega |\nabla_L u|^2 \, d\xi.$$

A simple generalization is the following

Theorem 15. *Let $m > q > 1$, $p > 1$ and assume that (3.16) is satisfied. Then for every $u \in \mathscr{C}_0^1(\Omega)$, we have*

$$c_{m,q,p}^q \int_\Omega \frac{|u|^p}{|x|^q} d\xi \le \int_\Omega |\nabla_L u|^q |u|^{p-q} \, d\xi, \tag{3.18}$$

where $c_{m,q,p} := \frac{m-q}{p}$.

Proof. Let $\epsilon \geq 0$. Define

$$r := \left(\sum_{i=1}^{m} \xi_i^2\right)^{1/2}, \quad r_\epsilon := \left(\epsilon^2 + \sum_{i=1}^{m} \xi_i^2\right)^{1/2}, \quad h_\epsilon(\xi) := \frac{1}{r_\epsilon^p}\begin{pmatrix} x \\ 0 \end{pmatrix}. \tag{3.19}$$

The assumption (3.16) guarantees that h_ϵ is μ-compatible. By computation

$$\mathrm{div}_L h_\epsilon = \frac{1}{r_\epsilon^p}\left(m - p\frac{r^2}{r_\epsilon^2}\right),$$

$$|h_\epsilon| = \left|\frac{1}{r_\epsilon^p}\begin{pmatrix} x \\ 0 \end{pmatrix}\right| = \frac{r}{r_\epsilon^p}.$$

If $m > p$ then $\mathrm{div}_H h_\epsilon^2 > 0$. Thus we are in a position to apply Theorem 11 with $h = h_\epsilon$. Indeed, by (3.13) we have

$$\int_\Omega \frac{|u|^p}{r_\epsilon^p}(m - p\frac{r^2}{r_\epsilon^2})d\xi \leq p^p \int_\Omega \frac{r^p}{r_\epsilon^p(m - p\frac{r^2}{r_\epsilon^2})^{p-1}} |\nabla_L u|^p \, d\xi,$$

which implies (3.17) by Lebesgue dominated convergence theorem.

In a similar way, using h_ϵ and (3.14) we prove the estimate (3.18). □

Using the same idea as above, we can prove some other Hardy inequalities of type (3.6).

Assume that (3.9) are fulfilled and there exist a non constant function $d \in \mathscr{C}^2(\Omega)$ and $\alpha \in \mathbb{R} \setminus \{0\}$ such that d^α is L-harmonic in Ω. In other words,

$$L(d^\alpha) = 0, \quad \text{on } \Omega. \tag{3.20}$$

Set $\psi := |\nabla_L d(\xi)|$ and $Z := \{\xi \in \Omega | \psi(\xi)d(\xi) = 0\}$.

Theorem 16. *For any function $u \in \mathscr{C}_0^1(\Omega \setminus Z)$, we have*

$$\frac{\alpha^2}{4}\int_\Omega \psi^2 \frac{u^2}{d(\xi)^2} d\xi \leq \int_\Omega |\nabla_L u|^2 \, d\xi. \tag{3.21}$$

Remark 17. Let us derive some consequences of identity (3.20). We have

$$0 = L(d(\xi)^\alpha) = \alpha d(\xi)^{\alpha-2}\left((\alpha - 1)\psi^2 + d(\xi)L(d(\xi))\right) \quad (\xi \in \Omega).$$

Thus, we infer $L(d(\xi)) = -(\alpha - 1)\psi^2/d(\xi)$ for any $\xi \in \Omega$ such that $d(\xi) \neq 0$. Therefore, if $\phi(\xi) := \varphi(d(\xi))$ with $\varphi \in \mathscr{C}^2(\Omega)$, we obtain

$$L(\phi) = \psi^2\left(\varphi''(d(\xi)) + \frac{1 - \alpha}{d(\xi)}\varphi'(d(\xi))\right).$$

Proof. Let h be the vector field defined by

$$h := \frac{1}{d(\xi)} \nabla_L d(\xi).$$

Since the conditions (3.9) hold, h is μ-compatible and

$$|h| = \frac{\psi}{|d(\xi)|}, \quad \mathrm{div}_L h = -\alpha \frac{\psi^2}{d^2(\xi)}.$$

Without loss of generality we assume $\alpha < 0$ (otherwise, we can consider the vector field $-h$). From Theorem 11 with $p = 2$, we complete the proof. □

We end this section proving a Poincarè inequality for the vector field ∇_L on domains Ω contained in a slab.

Theorem 18. *Let (3.16) be fulfilled with $m \geq 1$ and let Ω be an open subset of $\mathbb{R}^N$. We suppose that there exist $R > 0$, a real number s such that for any $\xi \in \Omega$, there holds $|\xi_1 - s| \leq R$.*
Then for every $u \in \mathscr{C}_0^1(\Omega)$, we have

$$c \int_\Omega |u|^p \, d\xi \leq \int_\Omega |\nabla_L u|^p \, d\xi,$$

with $c = (\frac{1}{pR})^p$.

The claim follows from Theorem 11 by using the vector field defined by

$$h := \begin{pmatrix} \xi_1 - s \\ 0 \end{pmatrix}$$

and $|h| \leq R$.

3.3 Hardy inequalities related to Grushin type operators

In this section we shall prove some Hardy type inequalities associated to the Grushin operator Δ_γ (see Section 2.3) improving some results of the previous section.

With the notation of sections 2.3 and 3.2, it follows that $\mu = \sigma^\gamma$ and any smooth vector field is σ^γ-compatible. The assumptions (3.9), (3.16) and (3.20) are satisfied with $m = d$, $d(\xi) = [\![\xi]\!]$ and $\alpha = 2 - Q = 2 - d - (1+\gamma)k$, hence $\psi = \frac{|x|^\gamma}{[\![\xi]\!]^\gamma}$.

Theorem 19. *Let $p>1$, $d,k\geq 1$ and let $\alpha,\beta\in\mathbb{R}$ be such that $d+(1+\gamma)k>\alpha-\beta-p$ and $d>\gamma p-\beta$. Then, for every $u\in D^{1,p}_\gamma(\Omega,|x|^{\beta-\gamma p}[\![\xi]\!]^{(1+\gamma)p-\alpha})$, we have*

$$c^p_{Q,p,\alpha,\beta}\int_\Omega |u|^p\,\frac{|x|^\beta}{[\![\xi]\!]^\alpha}d\xi\leq\int_\Omega|\nabla_\gamma u|^p\,|x|^{\beta-\gamma p}[\![\xi]\!]^{(1+\gamma)p-\alpha}d\xi, \tag{3.22}$$

where $c_{Q,p,\alpha,\beta}:=\frac{d+(1+\gamma)k+\beta-\alpha}{p}$.

If $0\in\Omega$ then the constant $c^p_{Q,p,\alpha,\beta}$ in (3.22) is sharp.

In particular if $Q:=d+(1+\gamma)k>p>1$, then

$$\left(\frac{Q-p}{p}\right)^p\int_\Omega\frac{|u|^p}{[\![\xi]\!]^p}\frac{|x|^{\gamma p}}{[\![\xi]\!]^{\gamma p}}d\xi\leq\int_\Omega|\nabla_\gamma u|^p\,d\xi,\qquad u\in D^{1,p}_\gamma(\Omega), \tag{3.23}$$

$$\left(\frac{Q-p}{p}\right)^p\int_\Omega\frac{|u|^p}{[\![\xi]\!]^p}d\xi\leq\int_\Omega|\nabla_\gamma u|^p\,\frac{[\![\xi]\!]^{\gamma p}}{|x|^{\gamma p}}d\xi,\ d>\gamma p, u\in D^{1,p}_\gamma(\Omega,\frac{[\![\xi]\!]^{\gamma p}}{|x|^{\gamma p}}), \tag{3.24}$$

$$\left(\frac{Q-p}{p}\right)^p\int_\Omega\frac{|u|^p}{|x|^p}d\xi\leq\int_\Omega|\nabla_\gamma u|^p\,\frac{[\![\xi]\!]^{(1+\gamma)p}}{|x|^{(1+\gamma)p}}d\xi, \tag{3.25}$$

$$d>(1+\gamma)p, u\in D^{1,p}_\gamma(\Omega,\frac{[\![\xi]\!]^{(1+\gamma)p}}{|x|^{(1+\gamma)p}}).$$

Remark 20. If $\gamma=0$, then the operator Δ_γ is the standard Laplacian operator acting on functions defined on $\mathbb{R}^N$ and (3.23) is the classical Hardy inequality (see (3.4)).

Theorem 21. *Let $d>p>1$. Then for every $u\in D^{1,p}_\gamma(\Omega)$ the inequalities*

$$b^p_{d,p}\int_\Omega\frac{|u|^p}{|x|^p}d\xi\leq\int_\Omega|\nabla_\gamma u|^p\,d\xi, \tag{3.26}$$

$$b^p_{d,p}\int_\Omega\frac{|u|^p}{[\![\xi]\!]^p}d\xi\leq\int_\Omega|\nabla_\gamma u|^p\,d\xi, \tag{3.27}$$

hold with $b_{d,p}:=\frac{d-p}{p}$.

In particular if $p=2$ and $d\geq 3$, then we have

$$(\frac{d-2}{2})^2\int_\Omega\frac{u^2}{[\![\xi]\!]^2}d\xi\leq(\frac{d-2}{2})^2\int_\Omega\frac{u^2}{|x|^2}d\xi\leq\int_\Omega|\nabla_\gamma u|^2\,d\xi.$$

Remark 22. From the above results it follows that the best constants in (3.26) and (3.27) lie in $[(\frac{d-p}{p})^p,(\frac{Q-p}{p})^p]$.

Theorem 21 follows directly from Theorem 14 and the fact that $[\![\xi]\!]\geq|x|$.

Proof (of Theorem 19). Without loss of generality we shall consider smooth functions $u \in \mathscr{C}_0^\infty(\Omega)$. The general case will follow by density argument.

Let $\epsilon \geq 0$. Define r_ϵ as in (3.19) and

$$\sigma_\epsilon := \begin{pmatrix} I_d & 0 \\ 0 & r_\epsilon^\gamma I_k \end{pmatrix}.$$

We use the notation of the previous sections. Let $\nabla_\gamma^\epsilon := \sigma_\epsilon \nabla$, we shall write $\mathrm{div}_\gamma^\epsilon(h) := \mathrm{div}(\sigma_\epsilon h)$ where $h \in \mathscr{C}^1(\Omega, \mathbb{R}^N)$. Let L_ϵ be defined by $L_\epsilon := \Delta_x + (\epsilon^2 + |x|^2)^\gamma \Delta_y$, a sort of regularization of Δ_γ.

Clearly, if $\epsilon = 0$, then $r_0 = |x|$ and

$$\nabla_\gamma = \nabla_\gamma^0 = \sigma_0 \nabla, \quad \Delta_\gamma = \mathrm{div}_\gamma^0(\nabla_\gamma).$$

Let $\epsilon > 0$. Define

$$\rho_\epsilon := \left(r_\epsilon^{2+2\gamma} + (1+\gamma)^2 \sum_{i=1}^k y_i^2 \right)^{\frac{1}{2(1+\gamma)}}$$

and the vector field h_ϵ as

$$h_\epsilon(\xi) := \frac{1}{\rho_\epsilon^\alpha} \begin{pmatrix} x r_\epsilon^\beta \\ (1+\gamma) y r_\epsilon^{\beta-\gamma-2} |x|^2 \end{pmatrix}. \tag{3.28}$$

The vector field h_ϵ is σ_ϵ-compatible. A simple computation shows that

$$\begin{aligned} \mathrm{div}_\gamma^\epsilon h_\epsilon &= \mathrm{div} \frac{1}{\rho_\epsilon^\alpha} \begin{pmatrix} x r_\epsilon^\beta \\ (1+\gamma) y r_\epsilon^{\beta-2} |x|^2 \end{pmatrix} \\ &= \frac{r_\epsilon^\beta}{\rho_\epsilon^\alpha} \left(d + ((1+\gamma)k + \beta - \alpha) \frac{|x|^2}{r_\epsilon^2} \right), \\ |h_\epsilon| &= \frac{r_\epsilon^{\beta-\gamma-2} |x|^2}{\rho_\epsilon^\alpha} \left(r_\epsilon^{2\gamma} |x|^2 + (1+\gamma)^2 |y|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Let f_ϵ ($\epsilon > 0$) be defined by $f_\epsilon(r) := d + ((1+\gamma)k + \beta - \alpha)\frac{r^2}{\epsilon^2 + r^2}$, $r \geq 0$. It is not difficult to see that

$$f_\epsilon(r) \geq \begin{cases} d & \text{if } (1+\gamma)k + \beta - \alpha \geq 0, \\ d + (1+\gamma)k + \beta - \alpha & \text{if } (1+\gamma)k + \beta - \alpha < 0, \end{cases} \tag{3.29}$$

for every $r \geq 0$ and $\epsilon > 0$. Since $r_\epsilon \geq \epsilon$, if $d + (1+\gamma)k > \alpha - \beta$, it follows that $\mathrm{div}_\gamma^\epsilon h_\epsilon > 0$.

Thus we are in a position to apply the Theorem 11 with $h = h_\epsilon$. Indeed from (3.13) we obtain,

$$\int_\Omega |u|^p \frac{r_\epsilon^\beta}{\rho_\epsilon^\alpha} f_\epsilon(|x|) d\xi \le p^p \int_\Omega |\nabla_\gamma^\epsilon u|^p \frac{|x|^{2p} r_\epsilon^{\beta-2p-\gamma p}(r_\epsilon^{2\gamma}|x|^2 + (1+\gamma)|y|^2)^{p/2}}{\rho_\epsilon^\alpha f_\epsilon(|x|)^{p-1}} d\xi. \tag{3.30}$$

Let $m := \min\{d, d + (1+\gamma)k + \beta - \alpha\}$. By (3.29) and $r < r_\epsilon$, the integrand on the right hand side of (3.30) can be estimated as follows;

$$|\nabla_\gamma^\epsilon u|^p \frac{|x|^{2p}(r_\epsilon^{2\gamma}|x|^2 + (1+\gamma)|y|^2)^{p/2}}{r_\epsilon^{2p+\gamma p-\beta} \rho_\epsilon^\alpha f_\epsilon(|x|)^{p-1}} \le \frac{|\nabla_\gamma^\epsilon u|^p}{m^{p-1}} r_\epsilon^{\beta-\gamma p} \rho_\epsilon^{(1+\gamma)p-\alpha} \in L^1(\Omega).$$

Therefore, by our assumption $d + (1+\gamma)k > \alpha - \beta$ and $d > p - \beta$, we can apply the Lebesgue dominated convergence theorem to (3.30), and letting $\epsilon \to 0$, we obtain the claim.

The choices $(\alpha, \beta) = ((1+\gamma)p, \gamma p)$, $(\alpha, \beta) = (p, 0)$ and $(\alpha, \beta) = (0, -p)$, in (3.22), yield the inequalities (3.23), (3.24) and (3.25) respectively.

It remains to show that the constant $c_{Q,p,\alpha,\beta}^p$ appearing in (3.22) is sharp.

First we consider the case $\Omega = \mathbb{R}^N$. In doing so we shall adapt the original idea of Hardy (see [41]) for the one dimensional Euclidean case.

Given $\epsilon > 0$, consider the function

$$u(\rho) := \begin{cases} C_\epsilon & \text{if } \rho \in [0,1], \\ C_\epsilon \rho^{-c_{Q,p,\alpha,\beta}-\epsilon} & \text{if } \rho > 1, \end{cases}$$

where $C_\epsilon := (c_{Q,p,\alpha,\beta} + \epsilon)^{-1}$. We have

$$u'(\rho) = \begin{cases} 0 & \text{if } \rho \in]0,1[, \\ -\rho^{-\frac{Q+\beta-\alpha+p}{p}-\epsilon} & \text{if } \rho > 1, \end{cases}$$

and by computation

$$\begin{aligned} \int_{\mathbb{R}^N} |u([\![\xi]\!])|^p \frac{|x|^\beta}{[\![\xi]\!]^\alpha} d\xi &= C_\epsilon^p \left(\int_{B_1} \frac{|x|^\beta}{[\![\xi]\!]^\alpha} d\xi + \int_{\mathbb{R}^N \setminus B_1} |x|^\beta [\![\xi]\!]^{-Q-\beta-\epsilon p} d\xi \right) \\ &= C_\epsilon^p \left(\int_{B_1} \frac{|x|^\beta}{[\![\xi]\!]^\alpha} d\xi + \int_{\mathbb{R}^N \setminus B_1} \frac{|x|^{\gamma p}}{[\![\xi]\!]^{\gamma p}} \left| [\![\xi]\!]^{-\frac{Q+\beta-\alpha+p}{p}-\epsilon} \right|^p |x|^{\beta-\gamma p} [\![\xi]\!]^{(1+\gamma)p-\alpha} d\xi \right) \\ &= C_\epsilon^p \left(\int_{B_1} \frac{|x|^\beta}{[\![\xi]\!]^\alpha} d\xi + \int_{\mathbb{R}^N} |\nabla_\gamma u|^p |x|^{\beta-\gamma p} [\![\xi]\!]^{(1+\gamma)p-\alpha} d\xi \right), \end{aligned} \tag{3.31}$$

where, in the last identity, we have used the relation (2.10) and the fact that u' vanish on B_1. Since the addenda in right hand side of (3.31) are integrable (see remark 2), by letting $\epsilon \to 0$, we easily get the claim.

In order to conclude in the general case we proceed as follows: let $c_b(\Omega)$ be the best constant in (3.22). By invariance of (3.22) under the dilation δ_λ^γ defined in (2.9) we have,

$$c_b(B_R) = c_b(B_1) \quad \text{for any } R > 0.$$

We note that if $B_R \subset \Omega \subset \mathbb{R}^N$ then,

$$c^p_{Q,p,\alpha,\beta} = c_b(\mathbb{R}^N) \leq c_b(\Omega) \leq c_b(B_R) = c_b(B_1). \tag{3.32}$$

Let $\phi \in \mathscr{C}_0^\infty(\mathbb{R}^N)$. Since the support of ϕ is compact, then (3.22) holds for ϕ with $\Omega = B_R$, R large enough and $c = c_b(B_R) = c_b(B_1)$. Therefore $c_b(B_1) \leq c_b(\mathbb{R}^N)$, and from (3.32) we conclude the proof. □

Remark 23. Let $\phi \in \mathscr{C}^2(\Omega)$ and $\epsilon = 0$. Setting $h := \nabla_\gamma \phi$, we see that Theorem 11 can be reformulated as follows: if $\Delta_\gamma \phi > 0$, then for any $p > 1$ and $u \in \mathscr{C}_0^1(\Omega)$ we have

$$\int_\Omega |u|^p \, \Delta_\gamma \phi d\xi \leq p^p \int_\Omega |\nabla_\gamma \phi|^p \, |\Delta_\gamma \phi|^{-(p-1)} \, |\nabla_\gamma u|^p \, d\xi.$$

Following this idea, one can easily realize that the main problem is to decide whether such a function exists. Indeed, via a standard regularization argument, we see that (3.23) can be proved if there exists $\phi \in \mathscr{C}^2(\Omega)$ such that $\nabla_\gamma \phi = h_0$, where h_0 is the vector field defined in (3.28) with $\epsilon = 0$, $\alpha = (1+\gamma)p$ and $\beta = \gamma p$. A simple computation shows that this is possible only when $\gamma = 0$ or $p = 2$.

In the case $\gamma = 0$ and $p > 1$, we obtain the classical Hardy inequality associated to the p-Laplacian operator on $\mathbb{R}^N$ (see [24]), while if $\gamma > 0$ and $p = 2$, the function ϕ is given by $\phi(\xi) := \ln [\![\xi]\!]$.

This show that in the framework of Hardy type inequalities associated to second order elliptic operator with degenerate or singular coefficients, the method based on the choice of a general vector field is more efficient.

Let us to state explicitly a Poincarè inequality for the vector field ∇_γ on domains Ω contained in a slab.

Theorem 24. *Let Ω be an open subset of $\mathbb{R}^N$. Suppose that there exist $R > 0$, a real number s and an integer $1 \leq j \leq d$ such that for any $\xi = (x, y) \in \Omega$, there holds $|x_j - s| \leq R$.*
Then for every $u \in \mathscr{C}_0^1(\Omega)$, we have

$$c \int_\Omega |u|^p \, d\xi \leq \int_\Omega |\nabla_\gamma u|^p \, d\xi,$$

with $c = (\frac{1}{pR})^p$.

3.4 Hardy inequalities related to the Kohn Laplace operator

In this section we shall present some Hardy inequalities related to the Kohn Laplacian operator Δ_H (see Section 2.2). We shall improve some results of Section 3.2 and we shall prove some inequalities of higher order.

With the notation of sections 2.2 and 3.2, it follows that $\mu = \sigma^H$ and any smooth vector field is σ^H-compatible. The assumption (3.9), (3.16) and (3.20) are satisfied with $m = 2n$, $d(\xi) = |\xi|_H$ and $\alpha = 2 - Q = -2n$, hence $\psi = \frac{|z|}{|\xi|_H}$.

Let $\Omega \subset \mathbb{H}^n$ be an open set.

Theorem 25. *Let $p > 1$, $n \geq 1$ and let $\alpha, \beta \in \mathbb{R}$ be such that $2n + 2 > \alpha - \beta$ and $2n > p - \beta$. Then for every $u \in D^{1,p}_H(\Omega, r^{\beta-p}\rho^{2p-\alpha})$ we have,*

$$c^p_{n,p,\alpha,\beta} \int_\Omega |u|^p \frac{r^\beta}{\rho^\alpha} d\xi \leq \int_\Omega |\nabla_H u|^p \, r^{\beta-p} \rho^{2p-\alpha} d\xi, \tag{3.33}$$

where $c_{n,p,\alpha,\beta} := \frac{2n+2+\beta-\alpha}{p}$.
In particular for $Q := 2n + 2 > p > 1$ we get,

$$\left(\frac{Q-p}{p}\right)^p \int_\Omega \frac{|u|^p}{\rho^p} \psi^p_H d\xi \leq \int_\Omega |\nabla_H u|^p \, d\xi, \quad u \in D^{1,p}_H(\Omega), \tag{3.34}$$

$$\left(\frac{Q-p}{p}\right)^p \int_\Omega \frac{|u|^p}{\rho^p} d\xi \leq \int_\Omega |\nabla_H u|^p \frac{\rho^p}{r^p} d\xi, \quad 2n > p, u \in D^{1,p}_H(\Omega, \frac{\rho^p}{r^p}), \tag{3.35}$$

$$\left(\frac{Q-p}{p}\right)^p \int_\Omega \frac{|u|^p}{r^p} d\xi \leq \int_\Omega |\nabla_H u|^p \frac{\rho^{2p}}{r^{2p}} d\xi, \quad n > p, u \in D^{1,p}_H(\Omega, \frac{\rho^{2p}}{r^{2p}}). \tag{3.36}$$

Moreover, if $0 \in \Omega$, then the constant $c^p_{n,p,\alpha,\beta}$ in (3.33)(and hence in (3.34), (3.35) and (3.36)) is sharp.

Theorem 26. *Let $2n > p > 1$. Then for every $u \in D^{1,p}_H(\Omega)$ the inequalities*

$$d^p_{n,p} \int_\Omega \frac{|u|^p}{r^p} d\xi \leq \int_\Omega |\nabla_H u|^p \, d\xi, \tag{3.37}$$

$$d^p_{n,p} \int_\Omega \frac{|u|^p}{\rho^p} d\xi \leq \int_\Omega |\nabla_H u|^p \, d\xi, \tag{3.38}$$

hold with $d_{n,p} := \frac{2n-p}{p}$.
In particular if $p = 2$ and $n \geq 2$, we have

$$(n-1)^2 \int_\Omega \frac{u^2}{\rho^2} d\xi \leq (n-1)^2 \int_\Omega \frac{u^2}{r^2} d\xi \leq \int_\Omega |\nabla_H u|^2 \, d\xi.$$

From the previous theorems, we note that the best constants in (3.37) and (3.38) lie in $[(\frac{2n-p}{p})^p, (\frac{2n+2-p}{p})^p]$.

Before presenting the proof of Theorem 25 and 26, we indicate some simple generalizations of them.

Theorem 27. *Let $p, q > 1$, $n \geq 1$ and let $\alpha, \beta \in \mathbb{R}$ be such that $2n + 2 > \alpha - \beta$ and $2n > q - \beta$. Then for every $u \in \mathscr{C}_0^1(\Omega)$ we have,*

$$c^q_{n,p,\alpha,\beta} \int_\Omega |u|^p \frac{r^\beta}{\rho^\alpha} d\xi \leq \int_\Omega |\nabla_H u|^q \, |u|^{p-q} \, r^{\beta-q} \rho^{2q-\alpha} d\xi. \tag{3.39}$$

Theorem 28. *Let $2n > q > 1$ and $p > 1$. Then for every $u \in \mathscr{C}_0^1(\Omega)$ we have,*

$$d^q_{n,q,p} \int_\Omega \frac{|u|^p}{r^q} d\xi \leq \int_\Omega |\nabla_H u|^q \, |u|^{p-q} \, d\xi, \tag{3.40}$$

$$d^q_{n,q,p} \int_\Omega \frac{|u|^p}{\rho^q} d\xi \leq \int_\Omega |\nabla_H u|^q \, |u|^{p-q} \, d\xi, \tag{3.41}$$

where $d_{n,q,p} := \frac{2n-q}{p}$.

Theorems 26 and 28 follows from Theorem 14 and 15 respectively.

Proof. Without loss of generality we can consider smooth functions $u \in \mathscr{C}_0^\infty(\Omega)$. The general case will follow by density argument.

For $\epsilon > 0$, we define

$$r_\epsilon := \left(\epsilon^2 + \sum_{i=1}^n x_i^2 + y_i^2\right)^{1/2},$$

$$\rho_\epsilon := \left(\left(\epsilon^2 + \sum_{i=1}^n (x_i^2 + y_i^2)\right)^2 + t^2\right)^{1/4} = (r_\epsilon^4 + t^2)^{1/4},$$

and the vector field h_ϵ as

$$h_\epsilon(\xi) \frac{1}{\rho_\epsilon^\alpha} \begin{pmatrix} x r_\epsilon^\beta + y t r_\epsilon^{\beta-2} \\ y r_\epsilon^\beta - x t r_\epsilon^{\beta-2} \end{pmatrix}.$$

A simple computation shows that

$$\begin{aligned} \mathrm{div}_H h_\epsilon &= \mathrm{div} \frac{1}{\rho_\epsilon^\alpha} \begin{pmatrix} x r_\epsilon^\beta + y t r_\epsilon^{\beta-2} \\ y r_\epsilon^\beta - x t r_\epsilon^{\beta-2} \\ 2 t r^2 r_\epsilon^{\beta-2} \end{pmatrix} \\ &= \frac{r_\epsilon^\beta}{\rho_\epsilon^\alpha} \left(2n + (2 + \beta - \alpha) \frac{r^2}{r_\epsilon^2}\right), \\ |h_\epsilon| &= \left| \frac{1}{\rho_\epsilon^\alpha} \begin{pmatrix} x r_\epsilon^\beta + y t r_\epsilon^{\beta-2} \\ y r_\epsilon^\beta - x t r_\epsilon^{\beta-2} \end{pmatrix} \right| = \frac{r r_\epsilon^{\beta-2}}{\rho_\epsilon^{\alpha-2}}. \end{aligned}$$

Introducing the function $f_\epsilon(r) := 2n + (2+\beta-\alpha)\frac{r^2}{r_\epsilon^2}$ $(r \geq 0, \epsilon > 0)$, it is not difficult to see that

$$f_\epsilon(r) \geq \begin{cases} 2n & \text{if } 2+\beta-\alpha \geq 0, \\ 2n+2+\beta-\alpha & \text{if } 2+\beta-\alpha < 0, \end{cases} \quad \text{for every } r \geq 0, \epsilon > 0. \tag{3.42}$$

Therefore if $2n+2 > \alpha - \beta$, from $r_\epsilon \geq \epsilon$, we have $\mathrm{div}_H h_\epsilon^1 > 0$. Thus we are in a position to apply the Theorem 11 with $h = h_\epsilon$. Indeed from (3.13) we obtain

$$\int_\Omega |u|^p \frac{r_\epsilon^\beta}{\rho_\epsilon^\alpha} f_\epsilon(r) d\xi \leq p^p \int_\Omega |\nabla_H u|^p \frac{r^p r_\epsilon^{\beta-2p} \rho_\epsilon^{2p-\alpha}}{f_\epsilon(r)^{p-1}} d\xi. \tag{3.43}$$

Let $m := \min\{2n, 2n+2+\beta-\alpha\}$. By (3.42) and $r < r_\epsilon$, the integrand appearing in the right hand side of (3.43) satisfies the estimate

$$|\nabla_H u|^p \frac{r^p r_\epsilon^{\beta-2p} \rho_\epsilon^{2p-\alpha}}{(2n + (2+\beta-\alpha)\frac{r^2}{r_\epsilon^2})^{p-1}} \leq \frac{|\nabla_H u|^p}{m^{p-1}} r_\epsilon^{\beta-p} \rho_\epsilon^{2p-\alpha} \in L^1(\Omega).$$

Under the hypotheses $2n+2 > \alpha - \beta$ and $2n > p - \beta$, we can apply the Lebesgue dominated convergence theorem to (3.43), and letting $\epsilon \to 0$, we obtain the claim.

The choices $(\alpha, \beta) = (2p, p)$, $(\alpha, \beta) = (p, 0)$ and $(\alpha, \beta) = (0, -p)$, in (3.33), yield the inequalities (3.34), (3.35) and (3.36) respectively.

Analogously, by means of (3.14) and h_ϵ, we get (3.39).

Now we shall prove that the constants appearing in the previous estimates (3.33), (3.34), (3.35) and (3.36) are sharp.

First we consider the case $\Omega = \mathbb{H}^n$. We shall adapt the idea of Hardy [41] for one dimensional Euclidean case. In the sequel we shall use the notation $B := B_H(0,1)$. Given $\epsilon > 0$, consider the function

$$u(\rho) := \begin{cases} C_\epsilon & \text{if } \rho \in [0,1], \\ C_\epsilon \rho^{-c_{n,p,\alpha,\beta}-\epsilon} & \text{if } \rho > 1, \end{cases}$$

where $C_\epsilon := (c_{n,p,\alpha,\beta} + \epsilon)^{-1}$. We have

$$u'(\rho) = \begin{cases} 0 & \text{if } \rho \in]0,1[, \\ -\rho^{-\frac{2n+2+\beta-\alpha+p}{p}-\epsilon} & \text{if } \rho > 1, \end{cases}$$

and by computation

$$\int_{\mathbb{H}^n} |u|^p \frac{r^\beta}{\rho^\alpha} d\xi = C_\epsilon^p \left(\int_B \frac{r^\beta}{\rho^\alpha} d\xi + \int_{\mathbb{H}^n \setminus B} r^\beta \rho^{-2n-2-\beta-\epsilon p} d\xi \right)$$

$$= C_\epsilon^p \left(\int_B \frac{r^\beta}{\rho^\alpha} d\xi + \int_{\mathbb{H}^n \setminus B} \frac{r^p}{\rho^p} \left| \rho^{-\frac{2n+2+\beta-\alpha+p}{p} - \epsilon} \right|^p r^{\beta-p} \rho^{2p-\alpha} d\xi \right)$$

$$= C_\epsilon^p \left(\int_B \frac{r^\beta}{\rho^\alpha} d\xi + \int_{\mathbb{H}^n} |\nabla_H u|^p \, r^{\beta-p} \rho^{2p-\alpha} d\xi \right), \tag{3.44}$$

where, in the last identity, we have used the relation (2.5) and the fact that u' vanish on B. The addenda in right hand side of (3.44) are integrable (see Remark 1). By letting $\epsilon \to 0$, we easily get the claim for (3.33), and hence for (3.34), (3.35) and (3.36).

In order to get the claim for the general case, let us denote with $c_b(\Omega)$ the best constant in (3.33). We have

$$c_b(\Omega) = \inf\{ \frac{\int_\Omega |\nabla_H u|^p \, r^{\beta-p} \rho^{2p-\alpha} d\xi}{\int_\Omega |u|^p \, r^\beta \rho^{-\alpha} d\xi}, u \in \mathscr{C}_0^\infty(\Omega), u \neq 0 \}.$$

By invariance of (3.33) under the dilation δ_R^H, we obtain $c_b(B_H(0,R)) = c_b(B_H(0,1))$ for any $R > 0$. We note that, if $B_H(0,R) \subset \Omega \subset \mathbb{H}^n$, then

$$c_{n,p,\alpha,\beta}^p = c_b(\mathbb{H}^n) \leq c_b(\Omega) \leq c_b(B_H(0,R)) = c_b(B_H(0,1)). \tag{3.45}$$

Finally, let us consider $\phi \in \mathscr{C}_0^\infty(\mathbb{H}^n)$. Since the support of ϕ is compact, it follows that (3.33) holds for ϕ with $\Omega = B_H(0,R)$, R large enough and $c_b(B_H(0,R)) = c_b(B_H(0,1))$. Therefore $c_b(B_H(0,1)) \leq c_b(\mathbb{H}^n)$ and from (3.45), we conclude the proof. □

A Poincarè inequality on the Heisenberg group for domains Ω contained in a slab is given by the following

Theorem 29. *Let Ω be an open subset of $\mathbb{H}^n$. Suppose that there exist $R > 0$, a real number s and an integer $1 \leq j \leq n$ such that for any $\xi = (x, y, t) \in \Omega$, there holds $|x_j - s| \leq R$ [resp. $|y_j - s| \leq R$].*
Then for every $u \in S_0^{1,p}(\Omega)$, we have

$$c \int_\Omega |u|^p \, d\xi \leq \int_\Omega |\nabla_H u|^p \, d\xi,$$

with $c = (\frac{1}{pR})^p$.

Remark 30. If Ω is the cylinder $B^{2n}(0,R) \times \mathbb{R}$, with $B^{2n}(0,R)$ the ball in $\mathbb{R}^{2n}$ of radius R centered at the origin, a value of c is $(\frac{2n}{pR})^p$. Indeed it is sufficient to repeat the proof of Theorem 18 with the vector field

$$h := \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}.$$

The divergence theorem, in its forms (3.11) and (3.12), allows us to obtain also inequality involving the Kohn Laplacian operator.

Theorem 31. *Let $\phi \in \mathscr{C}^2(\Omega)$ be a positive function and let $\theta > 0$. Suppose that $-\Delta_H \phi \geq \theta \frac{|\nabla_H \phi|^2}{\phi}$. Then for $p > 1$ and every $u \in \mathscr{C}_0^2(\Omega)$, there holds*

$$\left(\frac{p+\theta(p-1)}{p^2}\right)^p \int_\Omega |\Delta_H \phi| \, |u|^p \, d\xi \leq \int_\Omega \frac{\phi^p}{|\Delta_H \phi|^{p-1}} |\Delta_H u|^p \, d\xi. \tag{3.46}$$

Proof. Choosing $h = -\nabla_H \phi$ in (3.12) we get,

$$\int_\Omega |u|^p (-\Delta_H \phi) d\xi = p \int_\Omega |u|^{p-2} u \nabla_H u \cdot \nabla_H \phi d\xi. \tag{3.47}$$

Using (3.47) and hypotheses on ϕ we get,

$$\begin{aligned} \int_\Omega |u|^p (-\Delta_H \phi) d\xi &\leq p \int_\Omega |u|^{p-1} |\nabla_H u| \, |\nabla_H \phi| \, d\xi \\ &\leq p \left(\int_\Omega |u|^p \frac{|\nabla_H \phi|^2}{\phi} d\xi \right)^{1/2} \left(\int_\Omega \phi \, |u|^{p-2} \, |\nabla_H u|^2 \, d\xi \right)^{1/2} \\ &\leq p \left(\frac{1}{\theta} \int_\Omega |u|^p (-\Delta_H \phi) d\xi \right)^{1/2} \left(\int_\Omega \phi \, |u|^{p-2} \, |\nabla_H u|^2 \, d\xi \right)^{1/2}. \end{aligned}$$

In other words

$$\frac{\theta}{p^2} \int_\Omega |u|^p (-\Delta_H \phi) d\xi \leq \int_\Omega \phi \, |u|^{p-2} \, |\nabla_H u|^2 \, d\xi. \tag{3.48}$$

Choosing $g(u) := (u^2 + \epsilon^2)^{p/2-1} u$ and $h := \phi \nabla_H u$ in (3.11) and letting $\epsilon \to 0$, we have

$$\int_\Omega |u|^{p-2} u \phi \Delta_H u d\xi + \int_\Omega |u|^{p-2} u \nabla_H \phi \cdot \nabla_H u d\xi = -(p-1) \int_\Omega |u|^{p-2} \phi \, |\nabla_H u|^2 \, d\xi.$$

Taking into account (3.47), this last identity can be rewritten as

$$\int_\Omega |u|^p (-\Delta_H \phi) d\xi + p(p-1) \int_\Omega \phi \, |u|^{p-2} \, |\nabla_H u|^2 \, d\xi = -p \int_\Omega \phi \, |u|^{p-2} u \Delta_H u d\xi. \tag{3.49}$$

Using the relations (3.48), (3.49) and Hölder inequality, we get

$$\begin{aligned}
&\left(1+\theta\frac{p-1}{p}\right)\int_{\Omega}|u|^p\,(-\Delta_H\phi)d\xi \leq \\
&\leq \int_{\Omega}|u|^p\,(-\Delta_H\phi)d\xi + p(p-1)\int_{\Omega}\phi\,|u|^{p-2}\,|\nabla_H u|^2\,d\xi \\
&\leq p\int_{\Omega}\phi\,|u|^{p-1}\,|\Delta_H u|\,d\xi \\
&\leq p\left(\int_{\Omega}\frac{\phi^p}{|\Delta_H\phi|^{p-1}}\,|\Delta_H u|^p\,d\xi\right)^{1/p}\left(\int_{\Omega}|\Delta_H\phi|\,|u|^p\,d\xi\right)^{(p-1)/p}.
\end{aligned}$$

This completes the proof. □

By specializing the function ϕ in (3.46), we obtain:

Theorem 32. *Let $p>1$ and $n>\alpha>1$. For every $u\in\mathscr{C}_0^2(\Omega)$ we have,*

$$b_{n,p,\alpha}^p\int_{\Omega}\frac{|u|^p}{r^{2\alpha}}d\xi \leq \int_{\Omega}\frac{|\Delta_H u|^p}{r^{2\alpha-2p}}d\xi,$$

where $b_{n,p,\alpha} := 4(n-\alpha)\frac{p(\alpha-1)+(n-\alpha)(p-1)}{p^2}$. In particular if $n>2$, then we get,

$$n^2(n-2)^2\int_{\Omega}\frac{u^2}{r^4}d\xi \leq \int_{\Omega}(\Delta_H u)^2 d\xi. \tag{3.50}$$

Proof. Let $\epsilon>0$ and define the function ϕ as $\phi(\xi) := r_\epsilon^{-2\alpha+2}$. By computation it follows that

$$\begin{aligned}
-\Delta_H\phi &= 4\frac{\alpha-1}{r_\epsilon^{2\alpha}}(n-\alpha\frac{r^2}{r_\epsilon^2}),\\
|\nabla_H\phi|^2 &= 4(\alpha-1)^2\frac{r^2}{r_\epsilon^{4\alpha}} = 4(\alpha-1)^2\frac{r^2}{r_\epsilon^{2\alpha+2}}\phi.
\end{aligned}$$

That is ϕ satisfies the hypotheses of Theorem (31) with $\theta=\frac{n-\alpha}{\alpha-1}$. Thus, from (3.46) and letting $\epsilon\to 0$, we easily conclude. □

Remark 33. From inequality (3.50) it follows that

$$b\int_{\Omega}u^2\frac{r^4}{\rho^8}d\xi \leq \int_{\Omega}(\Delta_H u)^2 d\xi, \tag{3.51}$$

for $n>2$ and $u\in\mathscr{C}_0^2(\Omega)$ with $b:=n^2(n-2)^2$. We expect that (3.51) is true for $n>1$ with $b=\overline{b}:=(n^2-1)^2$ and $\overline{b}$ is sharp for $u\in D_H^{2,2}(\mathbb{H}^n)$. See Theorem (40) and inequality (3.53) in the next section.

3.5 Hardy inequalities with remainder terms in the Heisenberg group setting

In this section we show that some Hardy inequalities can be improved by adding some remainder terms.

We deal with cylindrical and radial functions u defined respectively on the cylinder $\Omega = B^{2n}(0,R)\times]a,b[$, with $-\infty \le a < b \le +\infty$ and on $B_H(0,R)$. Set

$$X^H := \{w \in \mathscr{C}^1([0,1]\times]a,b[)|\ w \neq 0, \lim_{s\to a,b} w(r,s) = w(1,s) = 0 = w_r(0,s)\}$$

and $X := \{w \in \mathscr{C}^1([0,1])|\ w \neq 0, w(1) = 0 = w'(0)\}$. Consider

$$\Lambda_p^H := \inf\left\{\frac{\int_a^b ds \int_0^1 r^{p-1}\left|\nabla_H w\right|^p dr}{\int_a^b ds \int_0^1 r^{p-1}\left|w\right|^p dr}, w \in X^H\right\},$$

$$\Lambda_p^* := \inf\left\{\frac{\int_a^b ds \int_0^1 r^{p-1}\left|w_r(r,s)\right|^p dr}{\int_a^b ds \int_0^1 r^{p-1}\left|w(r,s)\right|^p dr}, w \in X^H\right\},$$

$$\Lambda_p := \inf\left\{\frac{\int_0^1 r^{p-1}\left|w'\right|^p dr}{\int_0^1 r^{p-1}\left|w\right|^p dr}, w \in X\right\}.$$

It easy to see that these quantities are positive and the following relations hold

$$0 < \Lambda_p \le \Lambda_p^* \le \Lambda_p^H.$$

Indeed, if $w \in X^H$ then $w(\cdot,s) \in X$ for all $t \in]a,b[$. Thus, $\Lambda_p \le \Lambda_p^*$ is verified. From $|v_r|^p \le |\nabla_H v|^p$, we derive the remain inequality.

Remark 34. Λ_p does not depend on a and b. However (a priori) Λ_p^* and Λ_p^H may depend on a and b. Actually, Λ_p^* is independent of the value of a and b: for instance when $-\infty < a < b < +\infty$, it is sufficient to use the change of variable $s = a+(b-a)\tau$ ($\tau \in [0,1]$) into the quotient $\frac{\int_a^b ds \int_0^1 r^{p-1}|w_r(r,s)|^p dr}{\int_a^b ds \int_0^1 r^{p-1}|w(r,s)|^p dr}$.

Theorem 35. *Let $n \ge 1$, $\Omega = B_H(0,R) \subset \mathbb{H}^n$, $\alpha,\beta \in \mathbb{R}$ be such that $2n+2 > \alpha-\beta$ and $2n+\beta > 2$. For every radial function $u \in D_0^{1,2}(\Omega, r^{\beta-2}\rho^{4-\alpha})$, the following inequality holds*

$$\left(\frac{2n+2+\beta-\alpha}{2}\right)^2 \int_\Omega u^2 \frac{r^\beta}{\rho^\alpha} d\xi + \frac{\Lambda_2}{R^2}\int_\Omega u^2 \frac{r^\beta}{\rho^{\alpha-2}} d\xi \le \int_\Omega \left|\nabla_H u\right|^2 r^{\beta-2}\rho^{4-\alpha} d\xi.$$

In particular

$$n^2 \int_\Omega u^2 \frac{r^2}{\rho^4} d\xi + \frac{\Lambda_2}{R^2}\int_\Omega u^2 \frac{r^2}{\rho^2} d\xi \le \int_\Omega \left|\nabla_H u\right|^2 d\xi.$$

Theorem 36. *Let $\Omega = B^{2n}(0,R)\times]-\infty,+\infty[$. For every cylindrical function $u \in S_0^{1,2}(\Omega)$ the following inequality holds*

$$(n-1)^2 \int_\Omega \frac{|u|^2}{r^2} d\xi + \frac{\Lambda_2^H}{R^2} \int_\Omega |u|^2 \, d\xi \le \int_\Omega |\nabla_H u|^2 \, d\xi.$$

Remark 37. Λ_2^H is the best Poincarè constant for cylindrical functions on the cylinder $B^2(0,1)\times]-\infty,+\infty[$ in $\mathbb{H}^1$.

Proof. First we prove Theorem 36. Without loss of generality, we shall proceed considering smooth cylindrical functions $u \in \mathscr{C}_0^\infty(\Omega)$ and $R = 1$. The general case will follow by rescaling and density arguments.

We set

$$I := \int_\Omega |\nabla_H u|^2 \, d\xi - (n-1)^2 \int_\Omega \frac{u^2}{r^2} d\xi.$$

Putting $v(r,s) := r^{n-1} u(r,s)$ it follows that

$$u_r = -(n-1)\frac{v}{r^n} + \frac{v_r}{r^{n-1}}, \qquad u_s = \frac{v_s}{r^{n-1}},$$

which, according to (2.6), yields

$$\begin{aligned} I &= \int_\Omega u_r^2 + 4r^2 u_s^2 d\xi - (n-1)^2 \int_\Omega \frac{u^2}{r^2} d\xi \\ &= \int_\Omega \left(\frac{v_r^2}{r^{2n-2}} - 2(n-1)\frac{v v_r}{r^{2n-1}} + 4r^2 \frac{v_s^2}{r^{2n-2}} \right) d\xi. \end{aligned}$$

Using (2.2), and the fact that $v(1,s) = 0 = v(0,s)$, we obtain

$$\int_\Omega \frac{v v_r}{r^{2n-1}} d\xi = c_{2n} \int_{-\infty}^{+\infty} ds \int_0^1 v v_r dr = 0.$$

Therefore, by the definition of Λ_2^H, (2.2) and (2.6), we get

$$\begin{aligned} I &= \int_\Omega \frac{v_r^2 + 4r^2 v_s^2}{r^{2n-2}} d\xi = c_{2n} \int_{-\infty}^{+\infty} ds \int_0^1 r \, |\nabla_H v|^2 \, dr \\ &\ge \Lambda_2^H c_{2n} \int_{-\infty}^{+\infty} ds \int_0^1 r v^2 dr = \Lambda_2^H \int_\Omega u^2 d\xi. \end{aligned}$$

This completes the proof of Theorem 36.

The proof of Theorem 35 can be obtained by miming the previous proof and using the change of variable $v(\rho) := \rho^{(2n+2+\beta-\alpha)/2} u(\rho)$ and (2.3). □

Extensions of the previous results for powers $p \neq 2$ in the Euclidean setting are contained in [36]. Here, we shall consider only the case $p > 2$.

Theorem 38. *Let $2 < p < 2n$. Let $u = u(r,s) \in S_0^{1,p}(\Omega)$ be a positive cylindrical function which is non increasing with respect to r. Then*

$$d_{n,p}^p \int_\Omega \frac{|u|^p}{r^p} d\xi + \frac{\Lambda_p^*}{R^p} \int_\Omega |u|^p \, d\xi \leq \int_\Omega |\nabla_H u|^p \, d\xi,$$

with $d_{n,p} = \frac{2n-p}{p}$.

Proof. It is enough to consider functions $u \in \mathscr{C}_0^\infty(B^{2n}(0,1)\times]a,b[)$. The claim will follow by the rescaling (2.1) and a density argument.

Setting

$$I_p := \int_\Omega \left(|\nabla_H u|^p \, d\xi - d_{n,p}^p \frac{|u|^p}{r^p} \right) d\xi,$$

from (2.6), we obtain

$$I_p \geq \int_\Omega \left(|u_r|^p \, d\xi - d_{n,p}^p \frac{|u|^p}{r^p} \right) d\xi.$$

At this point the proof shadows the one given in [36]. So we shall be brief.

In [36, Lemma 1] is stated that for $p \geq 2$, $t \geq 0$ and $w \leq t$, the following inequality holds

$$(t-w)^p \geq t^p + |w|^p - pt^{p-1}w.$$

Putting $v(r,s) := r^{d_{n,p}} u(r,s)$, $t := d_{n,p}\frac{v}{r}$ and $w := v_r$, we obtain

$$I_p \geq \int_\Omega \frac{|v_r|^p}{r^{2n-p}} d\xi - p d_{n,p}^{p-1} \int_\Omega \frac{v^{p-1} v_r}{r^{2n-1}} d\xi.$$

The second integral in the right side of the above inequality vanishes. Indeed, as in the proof of the previous theorem, we can transform the integral with (2.2), and then use the boundary condition $v(1,s) = 0 = v(0,s)$. Finally, by (2.2), the definition of Λ_p^*, and the relation between v and u, we easily get the claim. □

In order to prove the estimate of remainder terms for inequality of type (3.40), we can use the identity (3.15). Indeed by Theorem 36 and (3.15), we obtain the following:

Theorem 39. *Let $\Omega = B^{2n}(0,R)\times]-\infty,+\infty[$ and $p > 2$. For every cylindrical function $u \in S_0^{1,2}(\Omega)$, the following inequality holds*

$$\left(\frac{2n-2}{p}\right)^2 \int_\Omega \frac{|u|^p}{r^2} d\xi + 4\frac{\Lambda_2^H}{p^2 R^2} \int_\Omega |u|^p \, d\xi \leq \int_\Omega |\nabla_H u|^2 \, |u|^{p-2} \, d\xi.$$

In order to show an inequality with remainder terms for the Kohn Laplacian, we set

$$\Gamma := \inf \left\{ \frac{\int_0^1 r^3 (M(w))^2 dr}{\int_0^1 r^3 w^2 dr}, w \in X \cap \mathscr{C}^2(]0,1[) \right\},$$

where M is the operator defined by

$$M(w) := w'' + \frac{3}{r} w'.$$

Notice that Γ can be read as the infimum of $\frac{\int_\Omega (\Delta_H w)^2}{\int_\Omega w^2 r^2/\rho^4}$ over the radial functions $w \in \mathscr{C}_0^2(\Omega)$ defined on $\Omega = B_H(0,1) \subset \mathbb{H}^1$.

Theorem 40. *Let $n \geq 2$. Set $\Omega = B_H(0,R)$. Then for every radial function $u \in \mathscr{C}_0^2(\Omega)$, we have*

$$l_{n,2}^2 \int_\Omega u^2 \frac{r^4}{\rho^8} d\xi + 2 l_{n,2} \frac{A_2}{R^2} \int_\Omega u^2 \frac{r^4}{\rho^6} d\xi + \frac{\Gamma}{R^4} \int_\Omega u^2 \frac{r^4}{\rho^4} d\xi \leq \int_\Omega (\Delta_H u)^2 d\xi, \tag{3.52}$$

where $l_{n,2} = n^2 - 1$.

Let $\Omega = \mathbb{H}^n$. Then for every radial function $u \in D_H^{2,2}(\mathbb{H}^n)$ there holds

$$l_{n,2}^2 \int_{\mathbb{H}^n} u^2 \frac{r^4}{\rho^8} d\xi \leq \int_{\mathbb{H}^n} (\Delta_H u)^2 d\xi, \tag{3.53}$$

and $l_{n,2}^2$ is sharp.

Proof. Without loss of generality we shall prove (3.52) for radial functions $u \in \mathscr{C}_0^\infty(\Omega)$ and $R = 1$. As usual the claim will follow by rescaling and density argument.

Setting $v(\rho) := \rho^{n-1} u(\rho)$, we have

$$u' = -(n-1)\frac{v}{r^n} + \frac{v'}{r^{n-1}}, \quad u'' = n(n-1)\frac{v}{r^{n+1}} + \frac{v''}{r^{n-1}} - 2(n-1)\frac{v'}{r^n},$$

and by (2.7), it follows that

$$\begin{aligned} \Delta_H u &= \frac{r^2}{\rho^2}\left(u'' + \frac{2n+1}{\rho} u'\right) \\ &= \frac{r^2}{\rho^2}\left(\frac{v''}{\rho^{n-1}} + 3\frac{v'}{\rho^n} - (n^2-1)\frac{v}{\rho^{n+1}}\right) \\ &= \frac{r^2}{\rho^{n+3}}\left(\rho^2 M(v) - (n^2-1)v\right). \end{aligned}$$

A simple computation using (2.3), gives

$$
\begin{aligned}
I &:= \int_{\Omega} (\Delta_H u)^2 d\xi - (n^2-1)^2 \int_{\Omega} u^2 \frac{r^4}{\rho^8} d\xi \\
&= \int_{\Omega} \frac{r^4}{\rho^{2n+6}} \left(\rho^4 (M(v))^2 - 2(n^2-1)\rho^2 v M(v)) \right) d\xi \\
&= c_{2n} \int_0^{\pi} (\sin\phi)^{n+1} d\phi \int_0^1 \left(\rho^3 (M(v))^2 - 2(n^2-1)\rho v M(v) \right) d\rho.
\end{aligned}
$$

Since $v(1) = v(0) = 0$, an integration by part shows that

$$
\int_0^1 \rho v M(v) d\rho = -\int_0^1 \rho (v')^2 d\rho.
$$

Finally, by definition of Γ and Λ_2, we obtain

$$
\begin{aligned}
I =& c_{2n} \int_0^{\pi} (\sin\phi)^{n+1} d\phi \int_0^1 \left(\rho^3 (M(v))^2 + 2(n^2-1)\rho (v')^2 \right) d\rho \\
\geq& \Gamma c_{2n} \int_0^{\pi} (\sin\phi)^{n+1} d\phi \int_0^1 \rho^3 (v)^2 d\rho + 2(n^2-1)\Lambda_2 c_{2n} \int_0^{\pi} (\sin\phi)^{n+1} d\phi \int_0^1 \rho v^2 d\rho,
\end{aligned}
$$

which concludes the proof of (3.52).

The inequality (3.53) holds for all $u \in \mathscr{C}_0^{\infty}(\mathbb{H}^n)$. Indeed, let $u \in \mathscr{C}_0^{\infty}(\mathbb{H}^n)$ and consider R large enough. Clearly (3.52) holds. This imply the validity of (3.53). The general case follows by density argument.

The proof of the sharpness of $l_{n,2}^2$ is similar to the proof of optimality of $c_{n,p,\alpha,\beta}^p$ in (3.33). More explicitly, for $0 < \epsilon < 1$ we can consider the function

$$
u(\rho) := \begin{cases} D_\epsilon & \text{if } \rho \in [0,1], \\ D_\epsilon \rho^{-(n-1)-\epsilon} & \text{if } \rho > 1, \end{cases}
$$

where $D_\epsilon := (l_{n,2} + 2\epsilon - \epsilon^2)^{-1}$. Arguing as in the previous proofs, we get the claim.
□

In order to show a remainder term for (3.50), we set

$$
\Gamma^H := \inf \left\{ \frac{\int_a^b ds \int_0^1 r^3 (L(w))^2 dr}{\int_a^b ds \int_0^1 r^3 w^2 dr}, w \in X^H \cap \mathscr{C}^2(]0,1[\times]a,b[) \right\},
$$

where L is the operator defined by

$$
L(w) := w_{rr} + \frac{3}{r} w_r + 4r^2 w_{ss}.
$$

Notice that Γ^H can be seen as the infimum of $\frac{\int_{\Omega} (\Delta_H w)^2}{\int_{\Omega} w^2}$ over the cylindrical functions $w \in \mathscr{C}_0^2(\Omega)$ defined on $\Omega = B^4(0,1) \times]a,b[\subset \mathbb{H}^2$.

Theorem 41. *Let $n > 2$. Set $\Omega = B^{2n}(0,R)\times]-\infty,+\infty[$. Then for every cylindrical function $u \in D_H^{2,2}(\Omega)$ we have,*

$$h_{n,2}^2 \int_\Omega \frac{u^2}{r^4} d\xi + 2h_{n,2} \frac{\Lambda_2^H}{R^2} \int_\Omega \frac{u^2}{r^2} d\xi + \frac{\Gamma^H}{R^4} \int_\Omega u^2 d\xi \le \int_\Omega (\Delta_H u)^2 d\xi, \tag{3.54}$$

where $h_{n,2} = n(n-2)$.

Proof. Without loss of generality we shall prove (3.54) for cylindrical functions $u \in \mathscr{C}_0^\infty(\Omega)$ and $R = 1$. As usual the claim will follow by rescaling and density argument.

Setting $v(r,s) := r^{n-2}u(r,s)$, we have

$$u_r = -(n-2)\tfrac{v}{r^{n-1}} + \tfrac{v_r}{r^{n-2}}, \qquad u_s = \tfrac{v_s}{r^{n-2}},$$
$$u_{rr} = (n-2)(n-1)\tfrac{v}{r^n} + \tfrac{v_{rr}}{r^{n-2}} - 2(n-2)\tfrac{v_r}{r^{n-1}}, \qquad u_{ss} = \tfrac{v_{ss}}{r^{n-2}},$$

and by (2.8), it follows that

$$\begin{aligned}\Delta_H u &= u_{rr} + \frac{2n-1}{r} u_r + 4r^2 u_{ss} \\ &= \frac{v_{rr}}{r^{n-2}} + 3\frac{v_r}{r^{n-1}} - n(n-2)\frac{v}{r^n} + 4r^2 \frac{v_{ss}}{r^{n-2}} \\ &= \frac{1}{r^n}(r^2 L(v) - n(n-2)v).\end{aligned}$$

A simple computation, using (2.2), gives

$$\begin{aligned} I &:= \int_\Omega (\Delta_H u)^2 d\xi - n^2(n-2)^2 \int_\Omega \frac{u^2}{r^4} d\xi \\ &= c_{2n} \int_a^b ds \int_0^1 \left[\frac{r^{2n-1}}{r^{2n}} (r^2 L(v) - n(n-2)v)^2 - n^2(n-2)^2 \frac{r^{2n-1}}{r^4} \frac{v^2}{r^{2n-4}} \right] dr \\ &= c_{2n} \int_a^b ds \int_0^1 \left[r^3 (L(v))^2 - 2n(n-2) rvL(v) \right] dr. \end{aligned}$$

Next we evaluate,

$$\int_a^b ds \int_0^1 rvL(v)dr = \int_a^b ds \int_0^1 [rvv_{rr} + 3vv_r + 4r^3 vv_{ss}]dr,$$

that is

$$\int_a^b ds \int_0^1 vv_r dr = 1/2 \int_a^b [v^2(1,s) - v^2(0,s)] = 0,$$
$$\int_a^b ds \int_0^1 rvv_{rr} dr = \int_a^b \left\{ [rv(r,s)v_r(r,s)]_{r=0}^{r=1} - \int_0^1 [vv_r + rv_r^2] dr \right\} ds$$

$$= -\int_a^b ds \int_0^1 r v_r^2 dr,$$

$$\int_a^b ds \int_0^1 r^3 v v_{ss} dr = \int_0^1 r^3 \left\{ [v(r,s)v_s(r,s)]_{s=a}^{s=b} - \int_a^b v_s^2 ds \right\} dr$$
$$= -\int_a^b ds \int_0^1 r^3 v_s^2 dr,$$

where we have used the fact that $v(1,s) = v(0,s) = 0 = v(r,a) = v(r,b)$.

Finally, by (2.6), the definition of Γ^H and Λ_2^H, we obtain

$$I = c_{2n} \int_a^b ds \int_0^1 r^3 (L(v))^2 dr + 2n(n-2)c_{2n} \int_a^b ds \int_0^1 r(v_r^2 + 4r^2 v_s^2) dr$$
$$\geq \Gamma^H c_{2n} \int_a^b ds \int_0^1 r^3 v^2 dr + 2n(n-2)\Lambda_2^H c_{2n} \int_a^b ds \int_0^1 r v^2 dr,$$

which concludes the proof. □

Remark 42. We expect that the results of this section hold for any function $u \in \mathscr{C}_0^\infty(\Omega)$ without any other special assumption.

Part II

Nonlinear Liouville Theorems

4 Subcritical Degeneracies

4.1 Introduction

In this chapter we shall investigate the Liouville property for some quasilinear inequalities associated to a degenerate partial differential operator.

In the last years a lot of efforts have been made to study nonexistence results, or, in other words, necessary condition for the existence of solutions, for inequalities of the type

$$-\mathcal{L}u \geq a(\xi)u^q, \ u \geq 0, \quad \text{on } \mathbb{R}^N, q > 1, \tag{4.1}$$

where the operator $\mathcal{L}$ is a degenerate partial differential operator. In [34, 10, 58], the authors deal with the case $\mathcal{L} = \Delta_H$ and $a(\xi) \geq \psi_H^2/|\xi|_H^\theta$, $\theta < 2$ (actually, in [58] the positivity of the solutions is not required). In [17] a more general case is investigated: $\mathcal{L}$ is a sublaplacian in Hörmander form on a stratified group and $a(\xi) \geq 1/d(\xi)^\theta$ where d is the *homogeneous norm* (see [17]) and $\theta < 2$. The authors also show that the employed technique can be apply also when $\mathcal{L}$ is the Grushin operator $\Delta_x + |x|^{2r} \Delta_y$ with $r > 1$ integer and $a \equiv 1$.

In Grushin's original paper [39], he considers a class of differential operators $L(y, D_x, D_y)$, which satisfy a suitable quasi-homogeneity condition. On the other hand, Deng and Levine in [25] suggest to investigate non negative solutions of the heat equation with a nonlinear term of the form $|x_1|^{\theta_1} \cdots |x_s|^{\theta_s} u^q$ where $x_i \in \mathbb{R}^{d_i}$ and $\mathbb{R}^{d_1} \times \ldots \times \mathbb{R}^{d_s} = \mathbb{R}^N$, namely

$$u_t - \Delta_x = |x_1|^{\theta_1} \cdots |x_s|^{\theta_s} u^q \quad \text{on } \mathbb{R}^N \times \mathbb{R}_+.$$

Motivated by these papers, in this chapter we shall study necessary conditions for the solvability of the inequality

$$L(x, y, D_x, D_y)u \geq |x|^{-\theta_1}|y|^{-\theta_2}|u|^q, \quad x \in \mathbb{R}^d, y \in \mathbb{R}^k, \tag{4.2}$$

where the class of such differential operators L contains, as a particular cases, Grushin type operators, generalizations to $\mathbb{R}^N$ of Tricomi operator and operators of higher order like $|x|^\sigma(-\Delta_y)^p + |y|^\gamma(-\Delta_x)^m$. Our results give a partial answer for the stationary case to the question posed by Deng and Levine (see Remark 55).

Our basic requirement on the operator L is the quasi-homogeneity property, that is

$$L(f(\lambda^{\delta_1}\cdot, \lambda^{\delta_2}\cdot))(x,y) = \lambda^h (Lf)(\lambda^{\delta_1}x, \lambda^{\delta_2}y), \quad \text{for any } \lambda > 0 \text{ and } f \in D(L),$$

where δ_1, δ_2 and h are positive real numbers. Under some suitable assumptions on δ_i, θ_i and h, there exists a (first) critical exponent $q_c = \frac{\delta_1 d + \delta_2 k - \delta_1\theta_1 - \delta_2\theta_2}{\delta_1 d + \delta_2 k - h}$ such that for $1 < q \le q_c$ there are no nontrivial solutions of (4.2).

The main technique that will be employed throughout this part is the so called "test functions method" (see [51]). Roughly speaking, this approach is based on the derivation of suitable a priori bounds of the weak solutions by careful selection of special test functions, which takes into account the structure of the operator involved, that is, the different behavior of the operator in x-variable and in y-variable. As a byproduct of our estimates on the solutions (see for instance (5.45) or (5.57) below), we derive some nonexistence theorems for the problems under consideration. This method, developed in the Euclidean framework (see e.g. [49, 50]), has been recently applied on the Heisenberg group setting (see [18], [58]).

We note that we avoid the use of comparison or maximum principle arguments and the properties of the fundamental solution of the operator under consideration. In general, the classes of operators considered here cannot be written in Hörmander form.

We remark that no assumption on the sign of the solution u is required and that the coefficients of the operator can be singular.

The plan of this chapter is the following. In Section 2 we present a non-existence theorem for quasi-homogeneous operators. Section 3 is devoted to the applications of our results to some remarkable operators starting with Tricomi and Grushin operators. Section 4 contains some generalizations of the Theorem 45 stated in Section 2 including a quasilinear case.

In the sequel we shall use function $\varphi_0 \in \mathscr{C}_0^2(\mathbb{R})$ satisfying the property

$$0 \le \varphi_0 \le 1 \quad \text{and} \quad \varphi_0(y) = \begin{cases} 1 \text{ if } |y| \le 1, \\ 0 \text{ if } |y| \ge 2, \end{cases} \tag{4.3}$$

and we shall meet quantities as

$$\int_{\mathbb{R}} \frac{|\varphi_0'(\tau)|^q}{\varphi_0^{q-1}(\tau)} d\tau,$$

or

$$\int_{\mathbb{R}} \frac{\left|\varphi_0^{(k)}(\tau)\right|^q}{\varphi_0^{q-1}(\tau)} d\tau,$$

with $q > 1$. When we shall claim that these quantities are finite, it means that it is possible to choose a suitable φ_0, with the property (4.3) such that the integrals are finite (see [49, 50]). Indeed, it suffices to take a power of φ_0, instead of φ_0. More precisely, we choose that $\varphi_0 = \psi_0^\beta$, where ψ_0 satisfies the condition (4.3), and β is an integer sufficiently large. A function φ_0 satisfying above hypotheses is called *admissible function.*

4.2 Main results

In this section we consider the differential inequality

$$Lu \geq |x|^{-\theta_1}|y|^{-\theta_2}|u|^q \text{ on } \mathbb{R}^N, \tag{4.4}$$

where $\theta_1, \theta_2 \in \mathbb{R}$, $q > 1$ and L is a linear differential operator of order $m \geq 1$ of the form

$$L(x, y, D_x, D_y) = \sum_{1 \leq |(\alpha,\beta)| \leq m} l_{\alpha,\beta}(x, y) D_x^\alpha D_y^\beta. \tag{4.5}$$

The adjoint of L, denoted by L^*, satisfies

$$\int (Lf) g dx dy = \int f L^* g dx dy,$$

for any $f \in D(L)$, $g \in D(L^*)$. Clearly, the domains $D(L)$ and $D(L^*)$ depends on the regularity of the coefficients $l_{\alpha,\beta}$.

We assume that

$$|x|^{\theta_1}|y|^{\theta_2} D_x^{\alpha_1} D_y^{\beta_1} l_{\alpha,\beta}(x, y) \in L_{loc}^{q'}(\mathbb{R}^N, |x|^{-\theta_1}|y|^{-\theta_2}) \quad \text{if } \alpha_1 \leq \alpha,\ \beta_1 \leq \beta; \tag{4.6}$$

$$D_x^\alpha D_y^\beta l_{\alpha,\beta}(x, y) = 0. \tag{4.7}$$

Definition 43. *Let $q > 1$. A function $u \in L_{loc}^q(\mathbb{R}^N, |x|^{-\theta_1}|y|^{-\theta_2})$ is called weak solution of (4.4) if*

$$\int_{\mathbb{R}^N} |x|^{-\theta_1}|y|^{-\theta_2}|u|^q \varphi dx dy \leq \int_{\mathbb{R}^N} u L^* \varphi dx dy, \tag{4.8}$$

for any $\varphi \in \mathscr{C}_0^\infty(\mathbb{R}^N, \mathbb{R}_+)$.

Notice that (4.6) implies that the right hand side of (4.8) is finite (see the proof of Theorem 45 for details) and that the symbol of the operator (4.5) is given by

$$L(x,y,\zeta,\eta)=\sum_{1\le|(\alpha,\beta)|\le m}(-1)^{|\alpha|+|\beta|}l_{\alpha,\beta}(x,y)\zeta^\alpha\eta^\beta.$$

Following [39] and [46], we assume that L is quasi-homogeneous in the following sense:

Definition 44. *Let $\delta_1,\delta_2>0$ and $h\in\mathbb{R}$. An operator $L(x,y,D_x,D_y)$ is called quasi-homogeneous of type (h,δ_1,δ_2) if, for any $\lambda>0$, $(x,y),(\zeta,\eta)\in\mathbb{R}^N$, there holds*

$$L(\lambda^{-\delta_1}x,\lambda^{-\delta_2}y,\lambda^{\delta_1}\zeta,\lambda^{\delta_2}\eta)=\lambda^hL(x,y,\zeta,\eta). \tag{4.9}$$

It can be proved (see Appendix at the end of this chapter) that for a quasi-homogeneous operator L of type (h,δ_1,δ_2), one has

$$LS^I_{\lambda^{\delta_1}}S^{II}_{\lambda^{\delta_2}}f=\lambda^hS^I_{\lambda^{\delta_1}}S^{II}_{\lambda^{\delta_2}}Lf\quad \text{for } f\in D(L), \tag{4.10}$$

and

$$L^*S^I_{\lambda^{\delta_1}}S^{II}_{\lambda^{\delta_2}}g=\lambda^hS^I_{\lambda^{\delta_1}}S^{II}_{\lambda^{\delta_2}}L^*g\quad \text{for } g\in D(L^*). \tag{4.11}$$

The main result of this section is the following.

Theorem 45. *Let $q>1$, $\theta_1,\theta_2\in\mathbb{R}$. Suppose that L is quasi-homogeneous of type (h,δ_1,δ_2), with $h,\delta_1,\delta_2>0$ and that (4.6), (4.7) hold. If*

$$(\delta_1d+\delta_2k-h)q\le\delta_1d+\delta_2k-\delta_1\theta_1-\delta_2\theta_2, \tag{4.12}$$

then (4.4) has no nontrivial weak solutions.

Proof. We shall prove the claim arguing by contradiction. Let u be a nontrivial weak solution of (4.4) and $\varphi\in\mathscr{C}_0^\infty(\mathbb{R}^N,\mathbb{R}_+)$. Since $q>1$ and u satisfies (4.8), by Hölder inequality we get

$$\int_{\mathbb{R}^N}|x|^{-\theta_1}|y|^{-\theta_2}|u|^q\varphi\,dxdy\le\int_{\mathbb{R}^N}|x|^{\theta_1(q'-1)}|y|^{\theta_2(q'-1)}\frac{|L^*\varphi|^{q'}}{\varphi^{q'-1}}dxdy. \tag{4.13}$$

We point out, as we shall see below, that the integral appearing on the right hand side of (4.13) is finite for admissible functions φ. Let $\varphi_0\in\mathscr{C}_0^\infty(\mathbb{R})$ defined as

$$\varphi_0(s)=\begin{cases}1 \text{ if } |s|\le1,\\ 0 \text{ if } |s|\ge2,\end{cases}\qquad 0\le\varphi_0\le1. \tag{4.14}$$

Choose $\phi(x,y) = \varphi_0(|x|)\varphi_0(|y|)$ and $\varphi(x,y) = \varphi_R(x,y) = \varphi_0(R^{-\delta_1}|x|)\varphi_0(R^{-\delta_2}|y|)$ with $R > 1$. From (4.11), it follows that

$$L^*\varphi = L^*\varphi_R = L^* S^I_{R^{-\delta_1}} S^{II}_{R^{-\delta_2}}\phi = R^{-h} S^I_{R^{-\delta_1}} S^{II}_{R^{-\delta_2}} L^*\phi.$$

Using the change of variable $x' = R^{-\delta_1}x$, $y' = R^{-\delta_2}y$, (4.13) becomes

$$\int |x|^{-\theta_1}|y|^{-\theta_2}|u|^q\varphi_R dxdy \leq I\ R^{\delta_1 d+\delta_2 k+\delta_1\theta_1(q'-1)+\delta_2\theta_2(q'-1)-hq'} \tag{4.15}$$

where

$$I := \int |x|^{\theta_1(q'-1)}|y|^{\theta_2(q'-1)}\frac{|L^*\phi|^{q'}}{\phi^{q'-1}}dxdy. \tag{4.16}$$

Let us to prove the existence of $\phi(x,y) = \varphi_0(|x|)\varphi_0(|y|)$ such that I is finite. In turn this will imply that (4.13) is meaningful. Since

$$L^*\phi = \sum_{1\leq|(\alpha,\beta)|\leq m}\ \sum_{\substack{\alpha_1+\alpha_2=\alpha\\ \beta_1+\beta_2=\beta}} C(\alpha_1,\alpha_2,\beta_1,\beta_2)D_x^{\alpha_1}D_y^{\beta_1}l_{\alpha,\beta}D_x^{\alpha_2}D_y^{\beta_2}\phi, \tag{4.17}$$

for a suitable constant C, we have

$$(|x|^{\theta_1}|y|^{\theta_2})^{q'-1}|L^*\phi|^{q'} \lesssim \sum_{1\leq|(\alpha,\beta)|\leq m}\ \sum_{\substack{\alpha_1+\alpha_2=\alpha\\ \beta_1+\beta_2=\beta}} (|x|^{\theta_1}|y|^{\theta_2})^{q'-1}|D_x^{\alpha_1}D_y^{\beta_1}l_{\alpha,\beta}|^{q'}|D_x^{\alpha_2}D_y^{\beta_2}\phi|^{q'}.$$

Following [49], we take $\phi(x,y) = \psi_0^\sigma(|x|)\psi_0^\sigma(|y|)$, with $\psi_0 \in \mathscr{C}_0^\infty(\mathbb{R})$ satisfying (4.14) and $\sigma > mq'$. A simple computation shows that there exists a function $b \in \mathscr{C}_0^\infty(\mathbb{R}^N,\mathbb{R}_+)$ such that $|D_x^{\alpha_2}D_y^{\beta_2}\phi(x,y)|^{q'} \leq b(x,y)|\psi_0^\sigma(|x|)\psi_0^\sigma(|y|)|^{q'-1}$ for any multi-index α_2 and β_2 involved in the sum in (4.17). Combining these estimates and hypothesis (4.6), the claim follows.

In order to complete the proof, we shall distinguish two cases.

i) $(\delta_1 d+\delta_2 k-h)q < \delta_1 d+\delta_2 k-\delta_1\theta_1-\delta_2\theta_2$;
ii) $(\delta_1 d+\delta_2 k-h)q = \delta_1 d+\delta_2 k-\delta_1\theta_1-\delta_2\theta_2$.

If i) holds then we have $\delta_1 d+\delta_2 k+\delta_1\theta_1(q'-1)+\delta_2\theta_2(q'-1)-hq' < 0$. By letting $R \to +\infty$ in (4.15), it follows that

$$\int_{\mathbb{R}^N}|x|^{-\theta_1}|y|^{-\theta_2}|u|^q dxdy = 0.$$

a contradiction with the assumption $u \neq 0$. This completes the proof in the case i).

If ii) is satisfied, then (4.15) reduces to

$$\int |x|^{-\theta_1}|y|^{-\theta_2}|u|^q\varphi_R dxdy \leq \int |x|^{\theta_1(q'-1)}|y|^{\theta_2(q'-1)}\frac{|L^*\phi|^{q'}}{\phi^{q'-1}}dxdy. \tag{4.18}$$

Since the right hand side of (4.18) is finite and independent of R, by letting $R \to +\infty$ in (4.18), we deduce that $u \in L^q(\mathbb{R}^N, |x|^{-\theta_1}|y|^{-\theta_2})$. On the other hand, using (4.8) and Hölder inequality, we have

$$\int_{\mathbb{R}^N} |x|^{-\theta_1}|y|^{-\theta_2}|u|^q\varphi_R dxdy \leq \Big(\int_{\text{supp}(L^*\varphi_R)} \frac{|u|^q}{|x|^{\theta_1}|y|^{\theta_2}}\varphi_R dxdy\Big)^{\frac{1}{q}} \times \Big(\int_{\text{supp}(L^*\varphi_R)} (|x|^{\theta_1}|y|^{\theta_2})^{(q'-1)}\frac{|L^*\varphi_R|^{q'}}{\varphi_R^{q'-1}}dxdy.\Big)^{\frac{1}{q'}}.$$

Arguing as before and using the change of variable $x' = R^{-\delta_1}x$, $y' = R^{-\delta_2}y$, we deduce that the second integral in right hand side of the above inequality is finite and independent of R. We claim that

$$\int_{\text{supp}(L^*\varphi_R)} \frac{|u|^q}{|x|^{\theta_1}|y|^{\theta_2}}dxdy \to 0 \quad \text{as } R \to +\infty. \tag{4.19}$$

Indeed by (4.17) and the assumption (4.7), the characteristic function of $\text{supp}(L^*\varphi_R)$ converges pointwise to zero when $R \to +\infty$. Since $u \in L^q(\mathbb{R}^N, |x|^{-\theta_1}|y|^{-\theta_2})$, by Lebesgue theorem we get (4.19). The proof is complete. □

Remark 46. As we shall see below, for some classes of differential operators, the integrability condition (4.6) can be relaxed (see Theorem 57).

Remark 47. Notice that if $\delta_1\theta_1 + \delta_2\theta_2 \leq \delta_1 d + \delta_2 k \leq h$, then for any $q > 1$ the only weak solution of (4.4) is the trivial one. On the other hand if $\delta_1 d + \delta_2 k \neq h$, then by setting

$$q_c := \frac{\delta_1 d + \delta_2 k - \delta_1\theta_1 - \delta_2\theta_2}{\delta_1 d + \delta_2 k - h},$$

it follows that

i) if $\delta_1\theta_1 + \delta_2\theta_2 < h < \delta_1 d + \delta_2 k$, then (4.4) has no nontrivial weak solutions whenever $1 < q \leq q_c$.
ii) if $\delta_1 d + \delta_2 k < \min\{h, \delta_1\theta_1 + \delta_2\theta_2\}$, then (4.4) has no nontrivial weak solutions for $q > \max\{1, q_c\}$.

Remark 48. Let L be satisfying the assumptions of Theorem 45. Consider

$$Lu \geq |z|^{-\theta}|u|^q, \quad \text{with } z = (x, y) \in \mathbb{R}^N \tag{4.20}$$

where $\theta \geq 0$ and $q > 1$. Since $|z|^\theta \leq C_\theta(|x|^\theta + |y|^\theta)$, proceeding as in the proof of Theorem 45, we can easily deduce that (4.20) has only the trivial solution provided both inequalities

$$Lu \geq |x|^{-\theta}|u|^q \text{ and } Lu \geq |y|^{-\theta}|u|^q \quad \text{on } \mathbb{R}^N,$$

have no nontrivial weak solutions. For instance, if $\delta_1 d + \delta_2 k > h$, then (4.20) does not admit nontrivial weak solutions whenever

$$1 < q \leq 1 + \frac{h - \theta \max\{\delta_1, \delta_2\}}{\delta_1 d + \delta_2 k - h}, \quad \delta_1\theta < h, \ \delta_2\theta < h.$$

Remark 49. Let us consider the operator $u_t - \Delta_x$ on $\mathbb{R}^d \times \mathbb{R}$. This operator is quasi-homogeneous of type $(2, 1, 2)$. According to Theorem 45, the critical exponent for the problem

$$u_t - \Delta_x u \geq |u|^q \quad \text{on } \mathbb{R}^d \times \mathbb{R}$$

is given by $q_c = 1 + \frac{2}{n}$. Note that Fujita in [32], found the same critical exponent for positive solutions of the evolution equation

$$u_t - \Delta_x u = u^q, \quad \text{on } \mathbb{R}^d \times [0, \infty)$$

with positive initial condition $u(\cdot, 0) = u_0 \geq 0$ on $\mathbb{R}^d$.

4.3 Applications

In this section we shall consider some classes of differential inequalities for which our results apply.

4.3.1 Tricomi-type operators

Let $T_\gamma := \partial_{xx} + g(x)\Delta_y$, with $x \in \mathbb{R}$, $y \in \mathbb{R}^k$ and let g be a homogeneous function of order $2\gamma \in \mathbb{R}$. It is easy to check that for $\gamma > -1$, T_γ is self-adjoint and quasi-homogeneous of type $(2, 1, 1 + \gamma)$. A consequences of Theorem 45 is the following:

Theorem 50. *Let $q > 1$, $\gamma > -1$, $\theta_1 > -(q-1) + \max\{0, -2\gamma q\}$, $\theta_2 > -k(q-1)$. Then the differential inequalities*

$$\pm T_\gamma u \geq |x|^{-\theta_1}|y|^{-\theta_2}|u|^q \quad \text{on } \mathbb{R}^{k+1},$$

have no nontrivial weak solutions provided

$$((1+\gamma)k - 1)q \leq 1 + (1+\gamma)k - \theta_2\gamma - (\theta_1 + \theta_2).$$

In particular, for $g(x) = x$ and $k = 1$, T_γ is the classical Tricomi operator $T := \frac{\partial^2}{\partial x^2} + x\frac{\partial^2}{\partial y^2}$ (see [7, 45, 60] and references therein), and for $1 < q \leq 5$, the inequality $Tu \geq |u|^q$ on $\mathbb{R}^2$ admits only the trivial solution.

We emphasize that our results allow to consider also negative γ. Indeed, non-existence results can be proved for the differential inequalities of the type,

$$u_{tt} \pm t^{-1}\Delta_x u \geq |x|^{1/2}t^{-2}|u|^q, \quad x \in \mathbb{R}^2,\, t \in \mathbb{R},$$

for any $q > 1$.

4.3.2 Grushin-type operators

Let $\Delta_\gamma = \Delta_x + |x|^{2\gamma}\Delta_y$ be a Grushin type operator. This operator is a special case of

$$\tilde{G}_\gamma := \Delta_x + g(x)\Delta_y,$$

where g is a homogeneous function of order $2\gamma \in \mathbb{R}$. Since in the case $d = 1$, $\tilde{G}_\gamma$ coincides with T_γ, we shall need to study only the case $d \geq 2$.

Theorem 51. *Let $d \geq 2$, $\gamma > -1$, $\theta_1 > -d(q-1) + \max\{0, -2\gamma q\}$, $\theta_2 > -k(q-1)$ and $\theta_1 + (1+\gamma)\theta_2 < 2$. Let u be a weak solution of*

$$-\tilde{G}_\gamma u \geq |x|^{-\theta_1}|y|^{-\theta_2}|u|^q \qquad \text{on } \mathbb{R}^N. \tag{4.21}$$

If

$$1 < q \leq 1 + \frac{2 - \theta_1 - (1+\gamma)\theta_2}{N + k\gamma - 2}, \tag{4.22}$$

then $u \equiv 0$.

Remark 52. The above theorem contains some results proved in [17]. In that paper, the authors consider positive solutions of (4.21) in the case $\theta_1 = \theta_2 = 0$ and with smooth coefficients $g(y) = |y|^{2r}$, $r \in \mathbb{N}$, $r > 1$.

The following non-existence theorems for inequalities of the form

$$-\tilde{G}_\gamma(au) \geq \frac{|y|^{2\gamma}}{[\![\xi]\!]^{2\gamma}} \frac{|u|^q}{[\![\xi]\!]^\theta} \quad \text{on } \mathbb{R}^N, \tag{4.23}$$

can be proved by using the same technique of the proof of Theorem 45.

Theorem 53. *Let* $\theta < 2$ *and* $\gamma > -1$. *Let* u *be a weak solution of (4.23). If* $1 < q \leq q_c := 1 + \frac{2-\theta}{N+k\gamma-2}$, *then* $u \equiv 0$.

We point out that in some cases the exponent q_c is optimal. This means that for $q > q_{CC} := 1 + \frac{2}{N+k\gamma-2}$ the inequality

$$-\Delta_\gamma u \geq \frac{|y|^{2\gamma}}{[\![\xi]\!]^{2\gamma}} u^q \quad \text{on } \mathbb{R}^N \tag{4.24}$$

has a positive solution. Indeed, let $\alpha \geq 0$ and define $u(z) := C(1 + [\![\xi]\!]^2)^{-\alpha}$. It follows that

$$\begin{aligned} -\Delta_\gamma u &= 2C\alpha \frac{|y|^{2\gamma}}{[\![\xi]\!]^{2\gamma}} (1 + [\![\xi]\!]^2)^{-\alpha-2} \left((N + k\gamma - 2\alpha - 2)[\![\xi]\!]^2 + N + k\gamma\right) \geq \\ &\geq 2C\alpha \frac{|y|^{2\gamma}}{[\![\xi]\!]^{2\gamma}} (N + k\gamma - 2\alpha - 2)(1 + [\![\xi]\!]^2)^{-\alpha-1}. \end{aligned}$$

For $q > q_{CC}$, with the choice $\alpha = 1/(q-1)$ and $C = (2\alpha(N + k\gamma - 2\alpha - 2))^{1/(q-1)}$, we easily check that u is a solution of (4.24).

Remark 54. Notice that for a suitable choice of the positive constant A, the function

$$u(x,y) := \frac{A}{((1 + |x|^2)^2 + 4|y|^2)^{1/(q-1)}}$$

is a solution of

$$-(\Delta_x + |x|^2 \Delta_y)u > u^q, \quad u > 0, \quad \text{on } \mathbb{R}^N, \tag{4.25}$$

for any $q > q_G$, where $q_G = \frac{N+k+2}{N+k-2}$. If $q = q_G$, then u solves the equation $-(\Delta_x + |x|^2\Delta_y)u = u^q$ on $\mathbb{R}^N$. From Theorem 51 we find that the (first) critical exponent for (4.25) is given by $q_{CC} := \frac{N+k}{N+k-2}$. Observe that $q_G > q_{CC}$.

Other inequalities involving the operator $\Delta_\gamma = \Delta_x + |x|^{2\gamma}\Delta_y$ will be studied in Section 5.2 below.

Remark 55. If $g(x) = 1$ (hence $\gamma = 0$), then $\tilde{G}_0 = \Delta_\xi = \Delta$ is the standard Laplace operator. Applying Theorem 51, we find that if one of the following conditions holds,

1. $N \geq 3$, $\theta_1 + \theta_2 < 2$ and $1 < q \leq \frac{N-(\theta_1+\theta_2)}{N-2}$;
2. $N = 2$, $\theta_1 + \theta_2 \leq 2$ and $q > 1$;

then the differential inequality,

$$-\Delta u \geq |x|^{-\theta_1}|y|^{-\theta_2}|u|^q \quad \text{on } \mathbb{R}^N,$$

has no nontrivial weak solutions. This result gives an answer, for the stationary case, to a question posed by Deng and Levine [25].

From Remark 48, we obtain the known conditions for the non-existence of solutions of $-\Delta u \geq |\xi|^{-\theta}|u|^q$ on $\mathbb{R}^N$ (see i.e. [49]).

Another extension of Grushin operator is given by the self-adjoint operator

$$L_{\sigma,\gamma} = a|x|^\sigma \Delta_y + b|y|^\gamma \Delta_x,$$

where a and b are real constants. For $\sigma, \gamma > -2$, the operator $L_{\sigma,\gamma}$ is quasi-homogeneous of type $(4 - \gamma\sigma, 2 + \gamma, 2 + \sigma)$. Applying Theorem 45 we obtain the following result.

Theorem 56. *Let $q > 1$, $\sigma, \gamma > -2$, $\sigma\gamma < 4$, $\theta_1 > -d(q-1) + \max\{0, -\sigma q\}$ and $\theta_2 > -k(q-1) + \max\{0, -\gamma q\}$. If*

$$(2N + \gamma d + \sigma k - 4 + \sigma\gamma)q \leq 2N + \gamma d + \sigma k - \theta_1\gamma - \theta_2\sigma - 2(\theta_1 + \theta_2),$$

then the problem

$$-L_{\sigma,\gamma}u \geq |x|^{-\theta_1}|y|^{-\theta_2}|u|^q \quad \text{on } \mathbb{R}^N$$

has no nontrivial weak solutions.

Now let us discuss the case $b = 0, \sigma = 0$. Theorem 56 can be improved as follows.

Theorem 57. *Let $q > 1$, $\theta_1 > -d(q-1)$ and $\theta_2 \in \mathbb{R}$. Let u be a weak solution of*

$$-\Delta_y u \geq |x|^{-\theta_1}|y|^{-\theta_2}|u|^q, \quad x \in \mathbb{R}^d \ y \in \mathbb{R}^k. \tag{4.26}$$

If

$$(k-2)q \leq k - \theta_2, \tag{4.27}$$

then $u \equiv 0$.

Proof. We begin by proving that if there exists $\delta_1 > 0$ such that

$$(k + \delta_1 d - 2)q \leq k + \delta_1 d - \delta_1\theta_1 - \theta_2, \tag{4.28}$$

then (4.26) has no nontrivial weak solutions. Indeed, since the operator $L = -\Delta_y$ is quasi-homogeneous of type $(2, \delta_1, 1)$ for any $\delta_1 > 0$, then, from the proof of Theorem 45 we obtain the claim.

Let us consider the assumption (4.27). For this matter we shall distinguish two cases.

i) If $(k-2)q < k - \theta_2$, then (4.28) holds for a suitable choice of δ_1, and then the claim follows.

ii) Let $(k-2)q = k - \theta_2$. Let u be a nontrivial weak solution of (4.26). Choose $\varphi_0 \in \mathscr{C}_0^\infty(\mathbb{R})$ as in (4.14) and $\psi \in \mathscr{C}_0^\infty(\mathbb{R}^N, \mathbb{R}_+)$. Proceeding as in the proof of Theorem 45, with $\varphi(x,y) = \varphi_R(x,y) = \psi(x)\varphi_0(R^{-1}|y|)$, we have

$$\int |x|^{-\theta_1}|y|^{-\theta_2}|u|^q\varphi_R dxdy \leq \int (|x|^{\theta_1}|y|^{\theta_2})^{(q'-1)}\psi(x)\frac{|(\Delta_y\varphi_0)(R^{-1}|y|)|^{q'}}{\varphi_0^{q'-1}(R^{-1}|y|)}dxdy. \tag{4.29}$$

Set $\Gamma(y) := |y|^{-\theta_2}\int_{\mathbb{R}^d}|x|^{-\theta_1}|u(x,y)|^q\psi(x)dx$. $\Gamma(y)$ is well defined for a.e. $y \in \mathbb{R}^k$. Arguing as in the proof of Theorem 45, we see that (4.29) implies $\Gamma \in L^1(\mathbb{R}^k)$ and we find

$$\int_{\mathbb{R}^k}\Gamma(y)\varphi_0(R^{-1}|y|)dy \leq \Big(\int_{S_R}\Gamma(y)\varphi_0(R^{-1}|y|)dy\Big)^{1/q} \times \Big(\int_{S_R}(|x|^{\theta_1}|y|^{\theta_2})^{(q'-1)}\psi(x)\frac{|\Delta_y\varphi_0(R^{-1}|y|)|^{q'}}{\varphi_0^{q'-1}(R^{-1}|y|)}dxdy.\Big)^{1/q'},$$

where $S_R = \text{supp}\Delta_y\varphi_0(\frac{y}{R})$. Since the last integral is finite and $\int_{S_R}\Gamma(y)\varphi_0(R^{-1}|y|)dy$ vanishes as $R \to +\infty$, we have $\Gamma(y) = 0$ for a.e. $y \in \mathbb{R}^k$. By definition of Γ and the arbitrary choice of ψ we deduce that $u \equiv 0$. This concludes the proof. □

Remark 58. If we wish to apply directly Theorem 45 or Theorem 56 for studying (4.26), then we need to assume $\theta_2 > -k(q-1)$. However, this condition is not necessary. Indeed, for any $\varphi \in \mathscr{C}_0^\infty(\mathbb{R}^N)$, the support of $\Delta_y\varphi$ does not contain the axes $y = 0$, hence, the assumption $\theta_1 > -d(q-1)$ guarantees that I (see (4.16) of the proof of Theorem 45) is finite.

Remark 59. A particular case of (4.26) has been studied in [17]. In that paper the authors found that $-\Delta_y u \geq u^q$, $u \geq 0$, on $\mathbb{R}^N$, has no nontrivial solutions provided $k \geq 2$ and $(k-2)q < k$.

Remark 60. We notice that the critical exponent given by (4.27) does not depend on θ_1.

A higher order version of the differential inequality associate to Grushin operator is given by

$$(a\,|x|^{\sigma}\,\Delta_y^p + b|y|^{\gamma}\Delta_x^m)u \geq |x|^{-\theta_1}|y|^{-\theta_2}|u|^q \quad \text{on } \mathbb{R}^N, \tag{4.30}$$

where $a, b \in \mathbb{R}$ and the symbol Δ^r stands for the operator Δ iterated r times. We have

Theorem 61. *Let $\gamma > -2p$, $\sigma > -2m$, $4mp > \sigma\gamma$, $\theta_1 > -d(q-1) + \max\{0, -\sigma q\}$ and $\theta_2 > -k(q-1) + \max\{0, -\gamma q\}$. If*

$$((2p+\gamma)d + (2m+\sigma)k - 4mp + \sigma\gamma))q \leq (2p+\gamma)(d-\theta_1) + (2m+\sigma)(k-\theta_2),$$

then (4.30) has no nontrivial weak solutions.

4.3.3 The Kohn Laplacian

Another example of operator for which our result apply, is the Kohn Laplacian. Let $z := (x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ and $s \in \mathbb{R}$. Let Δ_H be the operator defined in Section 2.2, which can be explicitly rewritten as

$$\Delta_H := \Delta_x + \Delta_y + 4(|x|^2 + |y|^2)D_{ss} + 4(y \cdot \nabla_x - x \cdot \nabla_y)D_s. \tag{4.31}$$

It is easy to see that the operator $\Delta_H = L(z, s, D_z, D_s)$ is quasi-homogeneous of type (2,1,2).

Theorem 62. *Let $\theta_1 > -2n(q-1)$, $\theta_2 > 1-q$ and $\theta_1 + 2\theta_2 < 2$. Let u be a weak solution of*

$$-\Delta_H u \geq \frac{|u|^q}{|z|^{\theta_1}|s|^{\theta_2}} \quad \text{on } \mathbb{R}^{2n+1}.$$

If

$$1 < q \leq 1 + \frac{2-\theta_1-2\theta_2}{2n},$$

then $u \equiv 0$.

Related results have been proved in [58].

4.4 Remarks and generalizations

The purpose of this section is to prove some other non existence results under weaker assumptions on the operator L.

Consider the inequality

$$L'u \geq |x|^{-\theta_1}|y|^{-\theta_2}|u|^q \quad \text{on } \mathbb{R}^N, \tag{4.32}$$

where L' is defined by

$$L'u(x,y) = \sum_{1\leq|(\alpha,\beta)|\leq m} l_{\alpha,\beta}(x,y)D_x^\alpha D_y^\beta \left(c_{\alpha,\beta}(x,y)u(x,y)\right).$$

Here $c_{\alpha,\beta} \in L^\infty(\mathbb{R}^N)$, the operator $\sum_{1\leq|(\alpha,\beta)|\leq m} l_{\alpha,\beta}(x,y)D_x^\alpha D_y^\beta$ is quasi-homogeneous and satisfies (4.6) and (4.7). As we shall see below the claim stated in Theorem 45 holds also for (4.32). Indeed, this result will follow from a general one involving a quasilinear operator (see Theorem 64).

As an example of quasilinear generalization of L', we consider the operator M defined by

$$Mu = \sum_{1\leq|(\alpha,\beta)|\leq m} l_{\alpha,\beta}(x,y)D_x^\alpha D_y^\beta a(x,y,u), \tag{4.33}$$

where $a : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function.

Let $\theta_1, \theta_2 \in \mathbb{R}$. Consider the inequality

$$Mu \geq |x|^{-\theta_1}|y|^{-\theta_2}|u|^q, \quad x \in \mathbb{R}^d, y \in \mathbb{R}^k. \tag{4.34}$$

Using the notation of Section 4.2, we can rewrite the operator M as $M(u) = L(a(\cdot,u))$.

Throughout this section we shall assume that there exist $C, p > 0$, $\mu_1, \mu_2 \in \mathbb{R}$, such that

$$a(x,y,s) \leq C|x|^{\mu_1}|y|^{\mu_2}|s|^p \quad x \in \mathbb{R}^d, y \in \mathbb{R}^k, s \in \mathbb{R}, \tag{4.35}$$

and that for any $\alpha_1 \leq \alpha$, $\beta_1 \leq \beta$, we have $D_x^{\alpha_1}D_y^{\beta_1}l_{\alpha,\beta}$ such that satisfy

$$|x|^{\theta_1+\mu_1}|y|^{\theta_2+\mu_2}D_x^{\alpha_1}D_y^{\beta_1}l_{\alpha,\beta}(x,y) \in L_{loc}^{\frac{q}{q-p}}(\mathbb{R}^N, |x|^{-\theta_1}|y|^{-\theta_2}); \tag{4.36}$$
$$D_x^\alpha D_y^\beta l_{\alpha,\beta}(x,y) = 0.$$

Definition 63. *Let $q > p > 0$. We say that $u : \mathbb{R}^N \to \mathbb{R}$ is a weak solution of (4.34) if $u \in L_{loc}^q(\mathbb{R}^N, |x|^{-\theta_1}|y|^{-\theta_2})$ and for any $\varphi \in \mathscr{C}_0^\infty(\mathbb{R}^N, \mathbb{R}_+)$, there holds*

$$\int |x|^{-\theta_1}|y|^{-\theta_2}|u|^q\varphi\,dxdy \leq \int a(x,y,u)L^*\varphi\,dxdy.$$

Theorem 64. *Let $q > p > 0$ and $a : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be a Caratheodory function verifying* (4.35)*. Consider $M = L(a(\cdot))$ as in* (4.33)*, where L is quasi-homogeneous of type (h, δ_1, δ_2) with $h, \delta_1, \delta_2 > 0$ and satisfies* (4.36)*. If*

$$(\delta_1 d + \delta_2 k + \delta_1\mu_1 + \delta_2\mu_2 - h)q \leq p(\delta_1 d + \delta_2 k - \delta_1\theta_1 - \delta_2\theta_2),$$

then the differential inequality (4.34) *has no nontrivial weak solutions.*

Proof. Let u be a nontrivial weak solution of (4.34). Since a satisfies (4.35), for any $\varphi \in \mathscr{C}_0^\infty(\mathbb{R}^N, \mathbb{R}_+)$ we have

$$\int |x|^{-\theta_1}|y|^{-\theta_2}|u|^q\varphi dxdy \lesssim \int |u|^p|x|^{\mu_1}|y|^{\mu_2}|L^*\varphi|dxdy.$$

By using Hölder inequality with exponents q/p, we obtain

$$\int |x|^{-\theta_1}|y|^{-\theta_2}|u|^q\varphi dxdy \lesssim \int |x|^{\frac{\theta_1 p+\mu_1 q}{q-p}}|y|^{\frac{\theta_2 p+\mu_2 q}{q-p}}\frac{|L^*\varphi|^{q/(q-p)}}{\varphi^{p/(q-p)}}dxdy.$$

Arguing as in the proof of Theorem 45 we get the claim. □

It is easy to see that our results can be extended also when we split $\mathbb{R}^N$ in more than two subspaces: $\mathbb{R}^N = \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_r}$.

Let L be defined by

$$L = \sum_{1\leq|(\alpha_1,\ldots,\alpha_r)|\leq m} l_{\alpha_1,\ldots,\alpha_r}(x_1,\ldots,x_r)D_{x_1}^{\alpha_1}\cdots D_{x_r}^{\alpha_r}.$$

The corresponding quasi-homogeneity condition on L becomes

$$L(\lambda^{-\delta_1}x_1,\cdots,\lambda^{-\delta_r}x_r,\lambda^{\delta_1}\zeta_1,\cdots,\lambda^{\delta_r}\zeta_r) = \lambda^h L(\zeta_1,\cdots,x_r,\zeta_1,\cdots,\zeta_r),$$

for any $\lambda > 0$, $x_i, \zeta_i \in \mathbb{R}^{d_i}$ $(i = 1, \ldots, r)$, where $\delta_1, \ldots, \delta_r > 0$ and $h \in \mathbb{R}$ are fixed. In this setting, we can obtain non-existence results for inequalities of the type,

$$Lu \geq |x_1|^{-\theta_1}\cdots|x_r|^{-\theta_r}|u|^q \quad \text{on } \mathbb{R}^N,$$

provided $h > 0$ and

$$\Big(\sum_{i=1}^r \delta_i d_i - h\Big)q \leq \sum_{i=1}^r \delta_i d_i - \sum_{i=1}^r \delta_i\theta_i,$$

with

$$|x_1|^{\theta_1}\cdots|x_r|^{\theta_r}D_{x_1}^{\alpha_1'}\cdots D_{x_r}^{\alpha_r'}l_{\alpha_1,\ldots,\alpha_r}(x_1,\ldots,x_r)\in L^{q'}(\mathbb{R}^N,|x_1|^{-\theta_1}\cdots|x_r|^{-\theta_r})$$

for any $\alpha_i'\leq\alpha_i$ and $D_{x_1}^{\alpha_1}\cdots D_{x_r}^{\alpha_r}l_{\alpha_1,\ldots,\alpha_r}(x_1,\ldots,x_r)=0$.

The previous modification allow us to establish a result on non-existence of solutions for

$$-\Delta_H u\geq\frac{|u|^q}{|x|^{\theta_1}|y|^{\theta_2}|s|^{\theta_3}},\quad \theta_1+\theta_2+2\theta_3<2.$$

where Δ_H the Kohn Laplacian.

We end this section with the following results related to an operator which is not quasi-homogeneous.

Theorem 65. *Let $\theta_1,\theta_2\in\mathbb{R}$ and $\delta_1,\delta_2>0$. Let L_i $(i=1,\ldots,s)$ be a differential operator of the form* (4.5). *Suppose L_i $(i=1,\ldots,s)$ quasi-homogeneous of type (h_i,δ_1,δ_2) with $h_i>0$ and the coefficients $l^i_{\alpha,\beta}$ satisfy* (4.6) *and* (4.7). *Let u be a weak solution of*

$$Lu:=\sum_{i=1}^{s}L_iu\geq|x|^{-\theta_1}|y|^{-\theta_2}|u|^q\qquad\text{on }\mathbb{R}^N.$$

If

$$(\delta_1d+\delta_2k-\min_{i=1,\ldots,s}h_i)q\leq\delta_1d+\delta_2k-\delta_1\theta_1-\delta_2\theta_2,$$

then $u\equiv0$.

A special case of above result is the following. Consider

$$-\Delta_y(au)-(x_1+|y|^{\gamma})\Delta_x(bu)\geq|x|^{-\theta_1}|y|^{-\theta_2}|u|^q\text{ on }\mathbb{R}^{\mathrm{N}},\tag{4.37}$$

where $a,b\in L^\infty(\mathbb{R}^N)$, $\theta_1,\theta_2\geq0$ and $\gamma>0$. Clearly, the operators Δ_y and $(x_1+|y|^\gamma)\Delta_x$ are quasi-homogeneous of type $(2,\delta_1,1)$ for any $\delta_1>0$ and $(\gamma,\gamma,1)$ respectively. Therefore, (4.37) has no nontrivial weak solutions provided

$$(\gamma d+k-\min\{\gamma,2\})q\leq\gamma d+k-\gamma\theta_1-\theta_2.$$

Remark 66. We notice that the condition (4.9) is used in the proof of Theorem 45 for estimating $|L^*\varphi|$. Therefore, we can replace the quasi-homogeneity assumption on L with a weaker one. Our results can be improved up to include a differential operator L such that

$$|L^*\varphi|\leq\sum_{i=1}^{r}|L_i\varphi|\quad\text{for any }\varphi\in\mathscr{C}_0^\infty(\mathbb{R}^N,\mathbb{R}_+),$$

where L_i are quasi-homogeneous operators satisfying the hypotheses of Theorem 65.

4.5 Appendix

Let $L(x, y, D_x, D_y)$ be defined as in (4.5). We assume that L is quasi-homogeneous of type (δ_1, δ_2, h) (see Definition 44).

In this appendix we shall prove that (4.10) and (4.11) hold.

The symbol of L is given by

$$L(x, y, \zeta, \eta) = \sum_{1 \leq |(\alpha,\beta)| \leq m} (-1)^{|\alpha|+|\beta|} l_{\alpha,\beta}(x, y) \zeta^\alpha \eta^\beta,$$

indeed,

$$L(x, y, D_x, D_y) f(x, y) = (2\pi)^{-n/2} \int e^{ix\cdot\zeta} e^{iy\cdot\eta} L(x, y, \zeta, \eta) \mathcal{F}_{x,y} f(\zeta, \eta) d\zeta d\eta =$$
$$= (2\pi)^{-n} \int e^{i(x-\zeta)\cdot\zeta} e^{i(y-\chi)\cdot\eta} L(x, y, \zeta, \eta) f(\zeta, \chi) d\zeta d\eta d\zeta d\chi,$$

where $\mathcal{F}_{x,y}$ denotes the Fourier transform on $\mathbb{R}^N$.

Lemma 67. *Let L be a quasi-homogeneous operator of type (h, δ_1, δ_2), then*

$$L S^I_{\lambda^{\delta_1}} S^{II}_{\lambda^{\delta_2}} f = \lambda^h S^I_{\lambda^{\delta_1}} S^{II}_{\lambda^{\delta_2}} L f, \quad \text{for any } \lambda > 0, f \in D(L).$$

Proof. Since

$$\mathcal{F}_{x,y} S^I_\alpha S^{II}_\beta f(\zeta, \eta) = \alpha^{-d} \beta^{-k} (\mathcal{F}_{x,y} f)(\alpha^{-1}\zeta, \beta^{-1}\eta),$$

by using the change of variable $\zeta' = \lambda^{\delta_1}\zeta$, $\eta' = \lambda^{\delta_2}\eta$, we obtain

$$(L S^I_{\lambda^{\delta_1}} S^{II}_{\lambda^{\delta_2}}) f(x, y) = (2\pi)^{-n/2} \int e^{ix\cdot\zeta} e^{iy\cdot\eta} L(x, y, \zeta, \eta) \mathcal{F}_{x,y}(S^I_{\lambda^{\delta_1}} S^{II}_{\lambda^{\delta_2}} f)(\zeta, \eta) d\zeta d\eta$$
$$= (2\pi)^{-n/2} \int e^{ix\cdot\zeta} e^{iy\cdot\eta} L(x, y, \zeta, \eta) \lambda^{-\delta_1 d} \lambda^{-\delta_2 k} \mathcal{F}_{x,y} f(\lambda^{-\delta_1}\zeta, \lambda^{-\delta_2}\eta) d\zeta d\eta$$
$$= (2\pi)^{-n/2} \int e^{ix\cdot\lambda^{\delta_1}\zeta} e^{iy\cdot\lambda^{\delta_2}\eta} L(x, y, \lambda^{\delta_1}\zeta, \lambda^{\delta_2}\eta) \mathcal{F}_{x,y} f(\zeta, \eta) d\zeta d\eta$$
$$= (2\pi)^{-n/2} \int e^{i\lambda^{\delta_1} x\cdot\zeta} e^{i\lambda^{\delta_2} y\cdot\eta} \lambda^h L(\lambda^{\delta_1} x, \lambda^{\delta_2} y, \zeta, \eta) \mathcal{F}_{x,y} f(\zeta, \eta) d\zeta d\eta$$
$$= \lambda^h S^I_{\lambda^\delta_1} S^{II}_{\lambda^\delta_2} L f(x, y).$$

This completes the proof. □

Lemma 68. *Let L be a quasi-homogeneous operator of type (h, δ_1, δ_2), then*

$$L^* S^I_{\lambda^{\delta_1}} S^{II}_{\lambda^{\delta_2}} g = \lambda^h S^I_{\lambda^{\delta_1}} S^{II}_{\lambda^{\delta_2}} L^* g, \quad \text{for any } \lambda > 0, g \in D(L^*).$$

Proof. For any $g \in D(L^*)$, we have

$$L^* g(\zeta, \chi) = (2\pi)^{-n} \int K(\zeta, \chi, x, y) g(x, y) dx dy$$

where

$$K(\zeta, \chi, x, y) = \int \mathrm{e}^{\mathrm{i}(x-\zeta)\cdot\zeta} \mathrm{e}^{\mathrm{i}(y-\chi)\cdot\eta} L(x, y, \zeta, \eta) d\zeta d\eta.$$

Using the change of variable, we get

$$L^* S^I_{\lambda^{\delta_1}} S^{II}_{\lambda^{\delta_2}} g(\zeta, \chi) = (2\pi)^{-n} \lambda^{-\delta_1 d} \lambda^{-\delta_2 k} \int K(\zeta, \chi, \lambda^{-\delta_1} x, \lambda^{-\delta_2} y) g(x, y) dx dy.$$

In order to conclude the proof, it suffices to check that

$$\lambda^{-\delta_1 d} \lambda^{-\delta_2 k} K(\zeta, \chi, \lambda^{-\delta_1} x, \lambda^{-\delta_2} y) = \lambda^h K(\lambda^{\delta_1} \zeta, \lambda^{\delta_2} \chi, x, y).$$

Since this runs as in the proof of Lemma 67 the proof is complete. □

5 Critical Degeneracies

5.1 Introduction

Nonexistence theorems of positive solutions for singular differential inequalities

$$-\Delta u \geq \frac{|u|^q}{|x|^\sigma} \qquad \text{on } \mathbb{R}^n, \quad q>1, \tag{5.1}$$
$$-|x|^\sigma \Delta u \geq |u|^q \qquad \text{on } \mathbb{R}^n, \quad q>1,$$

has been widely studied by many authors. The analogue problems for parabolic and hyperbolic inequalities have been also largely studied:

$$u_t - \Delta u \geq \frac{|u|^q}{|x|^\sigma} \qquad \text{on } \mathbb{R}^n \times]0,+\infty[, \tag{5.2}$$
$$u_{tt} - \Delta u \geq \frac{|u|^q}{|x|^\sigma} \qquad \text{on } \mathbb{R}^n \times]0,+\infty[. \tag{5.3}$$

A typical result can be stated as follows: for $1 < q \leq q_0$ (or $1 < q < q_0$), the problem has no nontrivial positive solutions, where q_0 depends on n, σ and the equation. For instance, let $q > 1$ and $\sigma < 2$, then (5.1) has no nontrivial weak solutions if and only if $q \leq q_0$, where $q_0 = \frac{n-\sigma}{n-2}$ (see Mitidieri and Pohozaev [49]).

In the case $\sigma = 2$, it results $q_0 = +\infty$, that is, for any $q > 1$ (5.1) has no nontrivial weak solutions (see [49]). In this case q_0 does not depend on the dimension n. For this reason $\sigma = 2$ is often referred as the *critical case*. Brézis and Cabrè in [12] treat the equation (5.1) on a bounded set $\Omega \subset \mathbb{R}^n$ with $\sigma = 2$, $q = 2$.

For parabolic and hyperbolic problems the same phenomena appear. Under a suitable assumption on initial condition, the problems (5.2) and (5.3) have no nontrivial weak solutions for $\sigma < 2$ and $1 < q \leq q_0$, where $q_0 = \frac{n+2-\sigma}{n}$ for the parabolic case and $q_0 = \frac{n+1-\sigma}{n-1}$ for the hyperbolic case. When $\sigma = 2$, the inequalities

$$u_t - |x|^2 \Delta u \geq |u|^q \qquad \text{on } \mathbb{R}^n \setminus \{0\} \times]0,+\infty[, \quad q>1, \tag{5.4}$$
$$u_{tt} - |x|^2 \Delta u \geq |u|^q \qquad \text{on } \mathbb{R}^n \setminus \{0\} \times]0.+\infty[, \quad q>1, \tag{5.5}$$

have no solutions for $1 < q \leq q_0 = 3$ (in both problems), provided some suitable assumptions on initial conditions are satisfied. Actually, even the higher order inequality

$$(-1)^m |x|^{2m} \Delta^m u \geq |u|^q \tag{5.6}$$

has no nontrivial weak solutions on $\mathbb{R}^N \setminus \{0\}$ for any $q > 1$ and the evolution problems

$$u_t + (-1)^m |x|^{2m} \Delta^m u \geq |u|^q, \qquad u(\cdot,0) = u_0 \qquad x \in \mathbb{R}^N \setminus \{0\}, t > 0, (5.7)$$

$$u_{tt} + (-1)^m |x|^{2m} \Delta^m u \geq |u|^q, \, u(\cdot,0) = u_1, u_t(\cdot,0) = u_0 \; x \in \mathbb{R}^N \setminus \{0\}, t > 0, (5.8)$$

have no weak solutions for $1 < q \leq q_0$ with $q_0 = 3$ provided $\int_{\mathbb{R}^N} u_0 \geq 0$. See Mitidieri and Pohozaev [49, 50, 51] for more details, references and further generalizations. Observe that the critical exponents of previous examples (5.4-5.8) do not depend on the dimension of the space $\mathbb{R}^N$ and the operators involved are quasi-homogeneous in the space variable with $h = 0$. Notice that if L is quasi-homogeneous with $h = 0$, then it is scaling invariant with respect to the dilation $(x, y) \to (\lambda^{\delta_1} x, \lambda^{\delta_2} y)$.

A Liouville property related to the Grushin operator for the inequality

$$-(\Delta_x + |x|^{2r} \Delta_y) u \geq u^q \qquad x \in \mathbb{R}^d, y \in \mathbb{R}^k, \tag{5.9}$$

was studied in [17] for positive solutions and $r > 1$ integer. In that papers is showed that for $1 < q \leq 1 + 2/(N + rk - 2)$ (5.9) has no positive solutions.

The counterpart of (5.1) in Heisenberg setting is given by

$$-\Delta_H u \geq \psi_H^2 \frac{|u|^q}{|\xi|_H^\sigma} \qquad \text{on } \mathbb{H}^n. \tag{5.10}$$

Nonexistence results for positive solutions of (5.10) were studied by Garofalo and Lanconelli [34] under some assumptions on u and later by Birindelli, Capuzzo Dolcetta and Cutrì [10] under less restrictive assumptions. In [10] the authors proved that for $\sigma < 2$ and $1 < q \leq q_0^e := 1 + \frac{2-\sigma}{2n}$, there exists no positive solutions to (5.10). The papers [34, 10] require the positivity of the solution u.

Recently, Pohozaev and Veron in [58] studied the inequality

$$-\Delta_H (au) \geq \frac{|u|^q}{|\xi|_H^\sigma} \qquad \text{on } \mathbb{H}^n, \tag{5.11}$$

where a is a measurable and bounded function, and without any hypothesis on sign of u. They proved that for $\sigma < 2$ and $1 < q \leq q_0^e$ there exists no weak solutions of (5.11). Then they extend their result to parabolic and hyperbolic case:

$$u_t - \Delta_H(au) \geq \frac{|u|^q}{|\xi|_H^\sigma} \quad \text{on } \mathbb{H}^n \times [0,+\infty[, \quad u(\cdot,0) = u_0, \tag{5.12}$$

$$u_{tt} - \Delta_H(au) \geq \frac{|u|^q}{|\xi|_H^\sigma} \quad \text{on } \mathbb{H}^n \times [0,+\infty[, \quad u(\cdot,0) = u_0,\ u_t(\cdot,0) = u_1. \tag{5.13}$$

Pohozaev and Veron prove that for $\sigma < 2$, if

$$\int_{\mathbb{H}^n} u_0(\xi)d\xi \geq 0 \quad \text{and} \quad 1 < q \leq 1 + \frac{2-\sigma}{2n+2},$$

then no weak solutions of (5.12) exist, and if

$$\int_{\mathbb{H}^n} u_1(\xi)d\xi \geq 0 \quad \text{and} \quad 1 < q \leq 1 + \frac{2-\sigma}{2n+1},$$

then no weak solutions of (5.13) exist.

In all these cases $\sigma \neq 2$. Aim of this chapter is to study the *critical case* $\sigma = 2$. More precisely, we deal with the following degenerate inequalities:

$$-\frac{|\xi|_H^2}{\psi_H^2}\Delta_H(au) \geq |u|^q \text{on } \mathbb{H}^n\setminus\{0\}, \tag{5.14}$$

$$u_t - \frac{|\xi|_H^2}{\psi_H^2}\Delta_H(au) \geq |u|^q \text{on } \mathbb{H}^n\setminus\{0\}\times]0,+\infty[,\ u(\cdot,0) = u_0, \tag{5.15}$$

$$u_{tt} - \frac{|\xi|_H^2}{\psi_H^2}\Delta_H(au) \geq |u|^q \text{on } \mathbb{H}^n\setminus\{0\}\times]0,+\infty[,\ u(\cdot,0) = u_0,\ u_t(\cdot,0) = u_1, \tag{5.16}$$

$$-[\![\xi]\!]^2\frac{[\![\xi]\!]^{2\gamma}}{|x|^{2\gamma}}\Delta_\gamma(au) \geq |u|^q \text{on } \mathbb{R}^N \setminus \{0\}. \tag{5.17}$$

We emphasize that we do not assume that the possible solutions are non negative.

We find that these are *critical cases* too. As well as in Euclidean case, the exponent q_0 is $q_0 = +\infty$ for (5.14) and (5.17) and $q_0 = 3$ for (5.15) and (5.16).

Our approach does not need the knowledge of the behavior of the fundamental solution of the differential operators appearing on left hand side of the problems (5.14)–(5.17).

As in the previous chapter, the main technique that will be employed is based on the so called "test functions method" (see Section 4.1 for a roughly description and [51]). Another interesting aspect of our approach is that for instance, the result on inequalities (5.14), (5.17), can be stated as follows: let $q > 1$, then (5.14) and (5.17) has no nontrivial weak solutions. For the evolutionary inequalities (5.15) and (5.16), we can easily find sufficient integral conditions on the behavior of the initial data, for the nonexistence of global solutions.

The stationary inequalities (5.14) and (5.17) are studied in the next two sections. Other cases with critical degeneracies are presented in Section 5.4. The last two sections are devoted to study the evolution inequalities (5.15) and (5.16).

5.2 Grushin type operators

In this section we shall assume that $q > 1$, $\gamma, \theta_1, \theta_2, \geq 0$ and $a \in L^\infty(\mathbb{R}^N)$. Let Δ_γ be the Grushin-type operator defined as $\Delta_\gamma := \Delta_x + |x|^{2\gamma}\Delta_y$.

In Section 4.3 we saw that the inequality,

$$-\Delta_\gamma(au) \geq \frac{|u|^q}{|x|^{\theta_1}\,|y|^{\theta_2}}, \tag{5.18}$$

has no nontrivial weak solutions on $\mathbb{R}^N$ if $1 < q \leq q_c = 1 + \frac{2-(1+\gamma)\theta_2-\theta_2}{N+k\gamma-2}$ and $(1+\gamma)\theta_2 + \theta_1 < 2$. When $(1+\gamma)\theta_2 + \theta_1 = 2$ a stronger non-existence result holds. Namely, inequality (5.18) has no nontrivial weak solutions for any $q > 1$. Our aim is to study nonexistence results for a generalized version of inequality (5.18).

Since $\frac{|x|}{[\![\xi]\!]} \leq 1$, in order to study non-existence results for the differential inequality $-[\![\xi]\!]^2\Delta_\gamma(au) \geq |u|^q$, it suffices to consider

$$-[\![\xi]\!]^2\Delta_\gamma(au) \geq \frac{|y|^{2\gamma}}{[\![\xi]\!]^{2\gamma}}|u|^q \quad \text{on } \mathbb{R}^N \setminus \{0\}. \tag{5.19}$$

Definition 69. *Let $q > 1$ and $a \in L^\infty(\mathbb{R}^N)$. A function $u \in L^q_{loc}(\mathbb{R}^N \setminus \{0\})$ is a weak solution of (5.19), if*

$$\int_{\mathbb{R}^N} \frac{|x|^{2\gamma}}{[\![\xi]\!]^{2\gamma}}|u|^q\varphi d\xi \leq -\int_{\mathbb{R}^N} ua\Delta_\gamma([\![\xi]\!]^2\varphi)d\xi, \tag{5.20}$$

for any $\varphi \in \mathscr{C}_0^2(\mathbb{R}^N \setminus \{0\}, \mathbb{R}_+)$.

Theorem 70. *Let $q > 1$. Then the inequality* (5.19) *has no nontrivial weak solutions.*

Proof. In the sequel we set $\psi := |x|^\gamma / [\![\xi]\!]^\gamma$. Let u be a nontrivial weak solution of (5.19). Since u satisfies (5.20), using Hölder inequality, we get

$$\int_{\mathbb{R}^N} \frac{|x|^{2\gamma}}{[\![\xi]\!]^{2\gamma}}|u|^q\varphi(z)d\xi \leq \|a\|_\infty \left(\int_{\mathbb{R}^N}\psi^2|u|^q\varphi d\xi\right)^{1/q}\left(\int_{\mathbb{R}^N}\psi^{2(1-q')}\frac{|\Delta_\gamma([\![\xi]\!]^2\varphi)|^{q'}}{\varphi^{q'-1}}d\xi\right)^{1/q'}.$$

This implies

$$\int_{\mathbb{R}^N} \psi^2 |u|^q \varphi(z) d\xi \leq \|a\|_\infty^{q'} \int_{\mathbb{R}^N} \psi^{2(1-q')} \frac{|\varDelta_\gamma([\![\xi]\!]^2 \varphi)|^{q'}}{\varphi^{q'-1}} d\xi. \tag{5.21}$$

For any $\phi \in \mathscr{C}_0^2(\mathbb{R} \setminus \{0\}, \mathbb{R}_+)$, the function φ defined by $\varphi(z) := [\![\xi]\!]^{-Q} \phi([\![\xi]\!])$ belongs to $\mathscr{C}_0^2(\mathbb{R}^N \setminus \{0\}, \mathbb{R}_+)$, hence, this function is admissible for testing (5.20). We have

$$\begin{aligned} \varDelta_\gamma([\![\xi]\!]^{2-Q} \phi([\![\xi]\!])) &= [\![\xi]\!]^{-Q} \left([\![\xi]\!]^2 \varDelta_\gamma \phi([\![\xi]\!]) + 2(2-Q)[\![\xi]\!]\phi'([\![\xi]\!]) \,|\nabla_\gamma [\![\xi]\!]|^2\right) \\ &= \psi^2 [\![\xi]\!]^{-Q} \left([\![\xi]\!]^2 \phi''([\![\xi]\!]) + [\![\xi]\!](3-Q)\phi'([\![\xi]\!])\right). \end{aligned}$$

Hence, from (5.21), we obtain

$$\int_{\mathbb{R}^N} \psi^2 \frac{|u|^q}{[\![\xi]\!]^Q} \phi([\![\xi]\!]) d\xi \leq \|a\|_\infty^{q'} \int_{\mathbb{R}^N} \psi^2 \frac{\left|[\![\xi]\!]^2 \phi''([\![\xi]\!]) + [\![\xi]\!](3-Q)\phi'([\![\xi]\!])\right|^{q'}}{[\![\xi]\!]^Q \phi([\![\xi]\!])^{q'-1}} d\xi. \tag{5.22}$$

Combining (2.12) and $\psi = \frac{|y|}{[\![\xi]\!]} \leq 1$, it follows that

$$\int_{\mathbb{R}^N} \psi^2 \frac{|u|^q}{[\![\xi]\!]^Q} \phi([\![\xi]\!]) d\xi \lesssim \|a\|_\infty^{q'} \int_0^\infty \frac{\left|\rho^2 \phi''(\rho) + \rho(3-Q)\phi'(\rho)\right|^{q'}}{\rho \phi(\rho)^{q'-1}} d\rho =: \|a\|_\infty^{q'} I_1. \tag{5.23}$$

From now on the proof proceeds as in the Heisenberg settings (see next section), hence we shall be brief. Let $\varphi_0 \in \mathscr{C}_0^2(\mathbb{R})$ be as in (4.14). We choose $\phi(\rho) := \varphi_0(\frac{\ln \rho}{R})$ with $R > 1$. Thus the integral I_1 becomes

$$I_1 = \int_{R \leq |s| \leq 2R} \frac{\left|\frac{\varphi_0''(s/R)}{R^2} + (2-Q)\frac{\varphi_0'(s/R)}{R}\right|^{q'}}{\varphi_0^{q'-1}(s/R)} ds. \tag{5.24}$$

Since we have $I_1 = R^{1-q'} I_2$, where

$$I_2 := \int_{1 \leq |\tau| \leq 2} \frac{|\varphi_0''(\tau)/R + (2-Q)\varphi_0'(\tau)|^{q'}}{\varphi_0^{q'-1}(\tau)} d\tau \leq M < +\infty$$

with M independent of R, it follows that

$$\int_{e^{-R} \leq [\![\xi]\!] \leq e^R} \psi^2 \frac{|u|^q}{[\![\xi]\!]^Q} d\xi \leq \|a\|_\infty^{q'} s_n I_2 R^{1-q'}.$$

Letting $R \to +\infty$ in the above inequality, we deduce that $u \equiv 0$, thereby concluding the proof. □

Our next result is related to singular inequalities associated to Grushin type operators $\varDelta_\gamma$.

Definition 71. *Let $q > 1$, $a \in L^\infty(\mathbb{R}^N)$ and $s \in L^\infty_{loc}(\mathbb{R}^N)$. A function u is a weak solution of*

$$-\Delta_\gamma(au) \geq \frac{|x|^{2\gamma}}{[\![\xi]\!]^{2\gamma}} \frac{|u|^q}{s(x,y)} \quad \text{on } \mathbb{R}^N \setminus \{0\}, \tag{5.25}$$

if $\frac{|u|^q}{s} \in L^1_{loc}(\mathbb{R}^N \setminus \{0\})$ and

$$\int_{\mathbb{R}^N} \frac{|x|^{2\gamma}}{[\![\xi]\!]^{2\gamma}} \frac{|u|^q}{s(x,y)} \varphi d\xi \leq -\int_{\mathbb{R}^N} ua\Delta_\gamma \varphi d\xi \tag{5.26}$$

for any $\varphi \in \mathscr{C}^2_0(\mathbb{R}^N \setminus \{0\}, \mathbb{R}_+)$.

Two canonical cases are $s(x,y) = [\![\xi]\!]^\theta$ and $s(x,y) = |x|^{\theta_1} |y|^{\theta_2}$.

Corollary 72. *Let $q > 1$. Then the problem*

$$-\Delta_\gamma(au) \geq \frac{|x|^{2\gamma}}{[\![\xi]\!]^{2\gamma}} \frac{|u|^q}{[\![\xi]\!]^2} \quad \text{on } \mathbb{R}^N \setminus \{0\} \tag{5.27}$$

has no nontrivial weak solutions.

Proof. It suffices to prove that any solution of (5.27) is a solution of (5.19). Let u be a solution of (5.25) with $s(x,y) = [\![\xi]\!]^2$. For any $\phi \in \mathscr{C}^2_0(\mathbb{R}^N \setminus \{0\}, \mathbb{R}_+)$, we choose $\varphi(z) := [\![\xi]\!]^2 \phi(z)$. Now, from (5.26), we deduce that (5.20) holds, hence the claim follows. □

Corollary 73. *Let $q > 1$. Let $\theta_1, \theta_2 \geq 0$ be such that $(1+\gamma)\theta_2 + \theta_1 = 2$. Then the problem*

$$-\Delta_\gamma(au) \geq \frac{|x|^{2\gamma}}{[\![\xi]\!]^{2\gamma}} \frac{|u|^q}{|x|^{\theta_1} |y|^{\theta_2}} \quad \text{on } \mathbb{R}^N \setminus \{0\} \tag{5.28}$$

has no nontrivial weak solutions.

Being $|x| \leq [\![\xi]\!]$ and $|y| \leq [\![\xi]\!]^{\gamma+1}$, every solution of (5.28) is a solution of (5.27). The conclusion follows from Corollary 72.

5.3 The stationary inequality in the Heisenberg setting

In this section we consider the inequality

$$-\frac{|\xi|^2_H}{\psi^2_H} \Delta_H(au) \geq |u|^q \qquad \text{on } \mathbb{H}^n \setminus \{0\}, \tag{5.29}$$

where a is a fixed function belonging to $L^\infty(\mathbb{H}^n)$.

Definition 74. *Let $q \geq 1$. We say that u is a* weak solution of (5.29) *if $u \in L^q_{loc}(\mathbb{H}^n \setminus \{0\})$ and*

$$\int_{\mathbb{H}^n} \frac{|u|^q}{|\xi|_H^Q} \psi_H^2 \varphi(\xi) d\xi \leq - \int_{\mathbb{H}^n} a(\xi) u(\xi) \Delta_H (|\xi|_H^{-2n} \varphi(\xi)) d\xi, \tag{5.30}$$

for any non negative $\varphi \in \mathscr{C}_0^2(\mathbb{H}^n \setminus \{0\})$.

Theorem 75. *For any $q > 1$, (5.29) has no nontrivial weak solutions.*

Proof. For sake of simplicity, here we shall write ψ instead of ψ_H. Let u be a nontrivial weak solution of (5.29) and $\varphi \in \mathscr{C}_0^2(\mathbb{H}^n \setminus \{0\})$, $\varphi \geq 0$. We shall specialize φ later in order to have a contradiction. Set

$$\Gamma_1 := |\xi|_H^2 \, \Delta_H \varphi - 4n \, |\xi|_H \, (\nabla_H \, |\xi|_H \, , \nabla_H \varphi).$$

Since u satisfies (5.30), using Hölder inequality, we get

$$\begin{aligned} \int_{\mathbb{H}^n} \frac{|u|^q}{|\xi|_H^Q} \psi^2 \varphi(\xi) d\xi &\leq \|a\|_\infty \int_{\mathbb{H}^n} \frac{|u| \, |\Gamma_1|}{|\xi|_H^Q} d\xi \\ &\leq \|a\|_\infty \left(\int_{\mathbb{H}^n} \frac{|u|^q}{|\xi|_H^Q} \psi^2 \varphi d\xi \right)^{1/q} \left(\int_{\mathbb{H}^n} \frac{|\Gamma_1|^{q'}}{|\xi|_H^Q \, \psi^{2q'-2} \varphi^{q'-1}} d\xi \right)^{1/q'}, \end{aligned}$$

and therefore

$$\int_{\mathbb{H}^n} \frac{|u|^q}{|\xi|_H^Q} \psi^2 \varphi(\xi) d\xi \leq \|a\|_\infty^{q'} \int_{\mathbb{H}^n} \frac{|\Gamma_1|^{q'}}{|\xi|_H^Q \, \psi^{2q'-2} \varphi^{q'-1}} d\xi. \tag{5.31}$$

Choosing φ radial, i.e. $\varphi = \varphi(|\xi|_H)$ with $\varphi \in \mathscr{C}_0^2(\mathbb{R} \setminus \{0\})$, by (2.5) and (2.7) Γ_1 becomes

$$\Gamma_1 = \psi^2 \left[|\xi|_H^2 \, \varphi''(\rho) + (1 - 2n) \, |\xi|_H \, \varphi'(\rho) \right].$$

Thus, using (2.12), the quantity $I_1 := \int_{\mathbb{H}^n} \frac{|\Gamma_1|^{q'}}{|\xi|_H^Q \psi^{2q'-2} \varphi^{q'-1}} d\xi$ can be rewritten as

$$I_1 = s_n \int_0^{+\infty} \frac{\left|\rho^2 \varphi''(\rho) + (1-2n) \rho \varphi'(\rho)\right|^{q'}}{\varphi^{q'-1} \rho} d\theta d\rho.$$

The transformation $\tilde{\varphi}(s) = \varphi(\rho)$, with $s = \ln \rho (= \ln |\xi|_H)$, yields

$$I_1 = s_n \int_{-\infty}^{+\infty} \frac{|\tilde{\varphi}''(s) - 2n \tilde{\varphi}'(s)|^{q'}}{\tilde{\varphi}^{q'-1}(s)} ds.$$

Taking $\tilde{\varphi}(s) = \varphi_0(\frac{s}{R})$, with φ_0 as in (4.3), we obtain

$$I_1 = s_n \int_{R \le |s| \le 2R} \frac{\left| \frac{\varphi_0''(s/R)}{R^2} - 2n \frac{\varphi_0'(s/R)}{R} \right|^{q'}}{\varphi_0^{q'-1}(s/R)} ds = s_n R^{1-q'} I_2, \tag{5.32}$$

where

$$I_2 = \int_{1 \le |\tau| \le 2} \frac{|\varphi_0''(\tau)/R - 2n\varphi_0'(\tau)|^{q'}}{\varphi_0^{q'-1}(\tau)} d\tau.$$

Let φ_0 an admissible function. For $R > 1$, it follows that $I_2 \le M < +\infty$, with M independent of R.

Merging (5.31) and (5.32) and taking into account the choice on $\varphi(\xi) = \tilde{\varphi}(\ln|\xi|_H) = \varphi_0(\frac{\ln|\xi|_H}{R})$, we have

$$\int_{e^{-R} \le |\xi|_H \le e^R} \frac{|u|^q}{|\xi|_H^Q} \psi^2 d\xi \le \|a\|_\infty^{q'} s_n I_2 R^{1-q'}.$$

Letting $R \to +\infty$, we deduce $u = 0$. □

Remark 76. Let $q = 1$ and $a \in \mathbb{R} \setminus \{0\}$. In this setting the inequality (5.29) admits solutions of the form $u = |\xi|_H^\alpha$. For instance, if $a = 1$, then $|\xi|_H^\alpha$ is a solution of (5.29) whenever $\alpha \in [\alpha_1, \alpha_2]$, where $\alpha_1 = -n - \sqrt{n^2 - 1}$ and $\alpha_2 = -n + \sqrt{n^2 - 1}$, and moreover for $u = |\xi|_H^{\alpha_i}$ (i=1,2), u realizes the equality.

The results presented in the previous section can be proved for the Kohn Laplacian in the Heisenberg group setting. Let $\xi = (z, s) = (x, y, s) \in \mathbb{H}^n$, $a \in L^\infty(\mathbb{H}^n)$. Arguing as in the previous section, we can prove that the inequalities,

$$-|\xi|_H^2 \Delta_H(au) \ge \frac{|z|^2}{|\xi|_H^2}|u|^q, \quad -\Delta_H(au) \ge \frac{|z|^2}{|\xi|_H^2}\frac{|u|^q}{|\xi|_H^2},$$

$$-\Delta_H(au) \ge \frac{|z|^2}{|\xi|_H^2}\frac{|u|^q}{|z|^{\theta_1}|s|^{\theta_2}}, \quad \theta_1 + 2\theta_2 = 2,$$

$$-\Delta_H(au) \ge \frac{|z|^2}{|\xi|_H^2}\frac{|u|^q}{|x|^{\theta_1}|y|^{\theta_2}|s|^{\theta_3}}, \quad \theta_1 + \theta_2 + 2\theta_3 = 2,$$

have no nontrivial weak solutions for every $q > 1$.

5.4 Some inequalities with critical degeneracy

In this section we shall study other problems related to degenerate operators with critical degeneracy. Let $\mathbb{R}^N = \mathbb{R}^d \times \mathbb{R}^k$ and consider a differential operator $L(x, y, D_x, D_y)$ of type (4.5). We expect that if L is sum of a term with critical degeneracy in the variable $x \in \mathbb{R}^d$ and other terms are independent of x, then the critical exponent will not depend on d. As a model operator we first consider $L(x, y, D_x, D_y) := \Delta_y + |x|^2 \Delta_x$.

Definition 77. *Let $q > 1$ and $a, b \in L^\infty(\mathbb{R}^N)$. We say that u is a* weak solution *of*

$$Pu := -\Delta_y(a\,u) - |x|^2\Delta_x(b\,u) \geq |u|^q \text{ on } \mathbb{R}^d \setminus \{0\} \times \mathbb{R}^k, \tag{5.33}$$

if $u \in L^q_{loc}(\mathbb{R}^d \setminus \{0\} \times \mathbb{R}^k)$ and

$$\int |u|^q \varphi dx dy \leq \int u P^* \varphi dx dy, \tag{5.34}$$

for any $\varphi \in \mathscr{C}^2_0(\mathbb{R}^d \setminus \{0\} \times \mathbb{R}^k, \mathbb{R}_+)$.

Theorem 78. *Let one of the following conditions be satisfied:*

1. *$d \neq 2$ and $1 < q \leq 1 + 2/k$;*
2. *$d = 2$, $k \neq 1$ and $1 < q \leq 1 + 2/(k-1)$;*
3. *$d = 2$, $k = 1$ and $q > 1$;*
4. *$a \equiv 0$ and $q > 1$.*

Then (5.33) *has no nontrivial weak solutions.*

Proof. Let u be a nontrivial weak solution of (5.33). Since u satisfies (5.34), then by Hölder inequality we have,

$$\int |u|^q \varphi dx dy \leq \int \frac{|P^*\varphi|^{q'}}{\varphi^{q'-1}} dx dy \lesssim \|a\|^{q'}_\infty I_1 + \|b\|^{q'}_\infty I_2, \tag{5.35}$$

where

$$I_1 := \int \frac{|\Delta_y \varphi|^{q'}}{\varphi^{q'-1}} dx dy \quad \text{and} \quad I_2 := \int \frac{|\Delta_x(|x|^2\varphi)|^{q'}}{\varphi^{q'-1}} dx dy.$$

Let $\varphi(x, y) := |x|^{-d}\varphi_1(|x|)\varphi_2(|y|)$ where $\varphi_1 \in \mathscr{C}^2{}_0(\mathbb{R} \setminus \{0\}, \mathbb{R}_+)$ and $\varphi_2 \in \mathscr{C}^2{}_0(\mathbb{R}, \mathbb{R}_+)$. The crucial point is that $\Delta_x(|x|^{2-d}\varphi_1(|x|))$ does not involve φ_1, but only φ_1'' and φ_1'; indeed

$$\Delta_x(|x|^{2-d}\varphi_1(|x|)) = |x|^{2-d}\varphi_1''(|x|) + (3-d)|x|^{1-d}\varphi_1'(|x|). \tag{5.36}$$

Let $\delta_1, \delta_2 > 0$ and $R > 1$. Let $\varphi_1(|x|) := \varphi_0(\frac{\lg|x|}{R^{\delta_1}})$ and $\varphi_2(|x|) := \varphi_0(\frac{|y|}{R^{\delta_2}})$, where $\varphi_0 \in \mathscr{C}^2{}_0(\mathbb{R})$ is defined as in (4.14). We can estimate I_2 as follows:

$$I_2 \leq \int_{\mathbb{R}^k} \varphi_0(\frac{|y|}{R^{\delta_2}}) dy \int_{\mathbb{R}^d} |x|^d \frac{\left|R^{-2\delta_1}\varphi_0''(\frac{\lg|x|}{R^{\delta_1}}) + (2-d)R^{-\delta_1}\varphi_0'(\frac{\lg|x|}{R^{\delta_1}})\right|^{q'}}{\varphi_0(\frac{\lg|x|}{R^{\delta_1}})^{q'-1}} dx.$$

Using the change of variable $y' = \frac{y}{R^{\delta_2}}$ and $s = \frac{\lg|x|}{R^{\delta_1}}$, it follows that

$$I_2 \lesssim R^{\delta_2 k - \delta_1(q'-1)} \int_{\mathbb{R}^k} \varphi_0(|y|)dy \int_{-\infty}^{+\infty} \frac{\left|\frac{\varphi_0''(s)}{R^{\delta_1}} + (2-d)\varphi_0'(s)\right|^{q'}}{\varphi_0(s)^{q'-1}} ds =: R^{\delta_2 k - \delta_1(q'-1)} A. \tag{5.37}$$

Similarly we obtain,

$$\begin{aligned} I_1 &= \int_{\mathbb{R}^d} |x|^{-d} \varphi_0\left(\frac{\lg|x|}{R^{\delta_1}}\right)dx \int_{\mathbb{R}^k} \frac{|\Delta_y \varphi_0(\frac{|y|}{R^{\delta_2}})|^{q'}}{\varphi_0(\frac{|y|}{R^{\delta_2}})^{q'-1}} dy \lesssim \\ &\lesssim R^{\delta_1 - \delta_2(2q'-k)} \int_{-\infty}^{+\infty} \varphi_0(s)ds \int_{\mathbb{R}^k} \frac{|\Delta_y \varphi_0(|y|)|^{q'}}{\varphi_0(|y|)^{q'-1}} dy =: R^{\delta_1 - \delta_2(2q'-k)} B. \end{aligned} \tag{5.38}$$

Now, as shown in the proof of Theorem 45, we can can choose φ_0 such that there exists a positive constant M, independent of $R > 1$ such that $A, B \leq M$. Thus from (5.35), we deduce that

$$\int_{\Omega_R} \frac{|u|^q}{|x|^d} dxdy \lesssim \|a\|_\infty^{q'} R^{\delta_1 - \delta_2(2q'-k)} + \|b\|_\infty^{q'} R^{\delta_2 k - \delta_1(q'-1)}, \tag{5.39}$$

where $\Omega_R = \{\exp(-R^{\delta_1}) < |x| < \exp(R^{\delta_1})\} \times \{|y| \leq R^{\delta_2}\}$.

Choose $\delta_1 = 2$ and $\delta_2 = 1$. We have $\delta_1 - \delta_2(2q'-k) = \delta_2 k - \delta_1(q'-1) = 2 + k - 2q' < 0$ provided $q < 1 + 2/k$. Letting $R \to +\infty$ in (5.39), it follows that $u = 0$. This contradicts our assumption, so that we get the claim. In the case $q = 1 + 2/k$, from (5.39), we obtain $|x|^{-d}|u|^q \in L^1(\mathbb{R}^d \setminus \{0\} \times \mathbb{R}^k)$. Hence, by an argument as in the proof of Theorem 45, we easily conclude.

Further, if $d = 2$, then (5.37) implies that $I_2 \lesssim R^{\delta_2 k - \delta_1(2q'-1)}$. With the choice $\delta_1 = \delta_2 = 1$ and arguing as before we get the claim.

Finally, if $a \equiv 0$ for $q > 1$, $\delta_1 = 1$ and δ_2 small enough, we have $\delta_2 k - \delta_1(q'-1) < 0$ and then the claim follows from (5.39) by letting $R \to +\infty$. □

Remark 79. The previous result can be generalized in several directions. As an example, consider the following differential inequality with a singularity in y-variable, namely:

$$-(\Delta_y + |x|^2 \Delta_x)u \geq |y|^{-\theta}|u|^q \quad \text{on } \mathbb{R}^d \setminus \{0\} \times \mathbb{R}^k. \tag{5.40}$$

Using a slight modification of the proof of Theorem 78, we can prove that if $2 > \theta > -k(q-1)$ and $1 < q \leq 1 + (2-\theta)/k$, then (5.40) has no nontrivial solutions.

Let $\mathbb{R}^N = \Pi_{i=1}^r \mathbb{R}^{d_i}$ and $z = (x^1, x^2, \ldots, x^r) \in \mathbb{R}^N$ with $x^i \in \mathbb{R}^{d_i}$ $(i = 1 \ldots r)$.

Definition 80. *Let $q > 1$ and $a_i \in L^\infty(\mathbb{R}^N)$, $i = 1, \ldots, r$. We say that u is a* weak solution *of*

$$P_r u := -\sum_{i=1}^r |x^i|^2 \Delta_{x^i}(a_i u) \geq |u|^q \quad \text{on } \Pi_{i=1}^r(\mathbb{R}^{d_i} \setminus \{0\}), \tag{5.41}$$

if $u \in L^q_{loc}(\Pi_{i=1}^r(\mathbb{R}^{d_i} \setminus \{0\}))$ and

$$\int |u|^q \varphi dxdy \leq \int u P_r^* \varphi dxdy,$$

for any $\varphi \in \mathscr{C}_0^2(\Pi_{i=1}^r(\mathbb{R}^{d_i} \setminus \{0\}), \mathbb{R}_+)$.

Theorem 81. *Let $q > 1$ and m be the number of two-dimensional subspaces in the splitting $\Pi_{i=1}^r(\mathbb{R}^{d_i} \setminus \{0\})$, that is $d_1 = \ldots = d_m = 2$, $d_{m+1} \neq 2, \ldots, d_r \neq 2$. If*

$$(2r - m - 2)q \leq (2r - m),$$

then (5.33) has no nontrivial weak solutions.

Proof. Without loosing generality, we only discuss the case $r = 2$ and set $d := d_1$, $k := d_2$. Let u be a nontrivial weak solution of (5.33). Choosing

$$\varphi_R(x, y) = |x|^{-d}|y|^{-k}\varphi_0(\frac{\lg|x|}{R^{\delta_1}})\varphi_0(\frac{\lg|y|}{R^{\delta_2}}),$$

where $\varphi_0 \in \mathscr{C}_0^2(\mathbb{R})$ satisfies (4.14), $\delta_1, \delta_2 > 0$ and $R > 1$, from a computation similar to (5.39) it follows that:

i) for $d, k \geq 1$, we have

$$\int |u|^q \varphi_R dxdy \lesssim R^{\delta_1+\delta_2-\delta_1 q'} + R^{\delta_1+\delta_2-\delta_2 q'}; \tag{5.42}$$

ii) if $k = 2$, then

$$\int |u|^q \varphi_R dxdy \lesssim R^{\delta_1+\delta_2-2\delta_1 q'} + R^{\delta_1+\delta_2-\delta_2 q'};$$

iii) if $k = d = 2$, then

$$\int |u|^q \varphi_R dxdy \lesssim R^{\delta_1+\delta_2-2\delta_1 q'} + R^{\delta_1+\delta_2-2\delta_2 q'}.$$

The choices $\delta_1 = \delta_2 = 1$ in cases i) and iii), and $\delta_1 = 1$, $\delta_2 = 2$ in ii), imply $\int_{\mathbb{R}^N} |u|^q dxdy = 0$. This contradiction concludes the proof. □

In the statement of the above theorem, the critical exponent depends only on the number of splits and it is independent of the comprehensive space dimension N. Hence, the scaling invariant operator P_r contains a set of critical degeneracies.

The previous result can be generalized up to include some polyharmonic operators. As an example, it is possible to prove that also the critical exponent for the inequality

$$\sum_{i=1}^{r} |x^i|^{2m_i}(-\Delta_{x^i})^{m_i}(a_i u) \geq |u|^q \quad \text{on } \Pi_{i=1}^{r}(\mathbb{R}^{d_i} \setminus \{0\}),$$

does not depend on the space dimension N.

5.5 A first order evolution inequality for the Kohn Laplacian

The aim of this section is to study the evolution inequality

$$\begin{cases} u_t - \frac{|\xi|_H^2}{\psi_H^2}\Delta_H(au) \geq |u|^q \text{ on } \mathbb{H}^n \setminus \{0\}\times]0,+\infty[, \\ u(\xi,0) = u_0(\xi) \qquad\qquad \text{on } \mathbb{H}^n\setminus\{0\}, \end{cases} \tag{5.43}$$

where $a \in \mathbb{R}$.

Definition 82. *Let $q \geq 1$. We say that u is a* weak solution of (5.43) *if $u_0 \in L^1_{loc}(\mathbb{H}^n\setminus\{0\})$, $u \in L^q_{loc}(\mathbb{H}^n\setminus\{0\} \times [0,+\infty[)$ and*

$$\int_0^{+\infty}\int_{\mathbb{H}^n} \frac{|u|^q}{|\xi|_H^Q}\psi_H^2\varphi d\xi dt \leq -\int_0^{+\infty}\int_{\mathbb{H}^n} au\Delta_H(|\xi|_H^{-2n}\varphi)d\xi dt + \\ -\int_0^{+\infty}\int_{\mathbb{H}^n} \frac{u}{|\xi|_H^Q}\psi_H^2\varphi_t d\xi dt - \int_{\mathbb{H}^n} \frac{u_0}{|\xi|_H^Q}\psi_H^2\varphi(\xi,0)d\xi, \tag{5.44}$$

for any non negative $\varphi \in \mathscr{C}_0^2(\mathbb{H}^n\setminus\{0\} \times [0,+\infty[)$.

Lemma 83. *Let $q > 1$. Let u be a weak solution of (5.43). For any admissible function $\varphi_0 \in \mathscr{C}_0^2(\mathbb{R})$ (see end of section 4.1) and $0 < \epsilon < q^{\frac{1}{q}}$, there exists a positive constant C_2 such that the following estimate holds*

$$C_1\int_{A_R} \frac{|u|^q}{|\xi|_H^Q}\psi_H^2 dtd\xi + \int_{A_{2R}^0} \frac{u_0}{|\xi|_H^Q}\psi_H^2\varphi_0(\frac{\ln|\xi|_H}{R})d\xi \leq C_2 R^{3-2q'}, \tag{5.45}$$

where $C_1 := 1 - \epsilon^q/q$, and

$$A_R := \{(\xi,t)|0 \le t \le R^2,\ |\ln|\xi|_H + 2\,n\,a\,t| \le R\},$$
$$A^0_{2R} := \{\xi|\,|\ln|\xi|_H| \le 2R\}.$$

Moreover, the constant C_2 *has the form* $C_2 = \frac{s_n}{\varepsilon^{q'}q'}C_q(\varphi_0,\varphi_1)$*, where*

$$C_q(\varphi_0,\varphi_1) := \int_0^2\int_{-2}^{+2}\frac{|a\varphi_1(\tau)\varphi_0''(v)+\varphi_1'(\tau)\varphi_0(v)|^{q'}}{(\varphi_0(v)\varphi_1(\tau))^{q'-1}}d\tau dv, \tag{5.46}$$

and $\varphi_1 \in \mathscr{C}^2_0([0,+\infty[)$ *is any admissible function.*

The estimate (5.45) allows us to get the following nonexistence results.

Theorem 84. *Let* $1 < q \le 3$*. Let* u *be a weak solution of (5.43), then*

$$\int_0^{+\infty}\int_{\mathbb{H}^n}\frac{|u|^q}{|\xi|_H^Q}\psi_H^2 d\xi dt \le -U_0,$$

where

$$U_0 := \liminf_{R\to+\infty}\int_{R^{-1}\le|\xi|_H\le R}\frac{u_0(\xi)}{|\xi|_H^Q}\psi_H^2 d\xi \qquad \textit{(possibly infinite)}.$$

Therefore, if $U_0 \ge 0$*, then (5.43) has no nontrivial weak solutions.*

Remark 85. From the above result, as particular case, we have that if $u_0 \ge 0$, then (5.43) has no weak solutions for $1 < q \le 3$.

In the following theorem, in order to get nonexistence results for $q > 3$, we shall analyze the asymptotic behavior of the initial condition u_0 at infinity and at the origin.

Theorem 86. *Let* $q > 3$ *and* $u_0 \ne 0$*. Let* $\varphi_0 \in \mathscr{C}^2_0(\mathbb{R})$ *be an admissible function. Set, for* $R > 0$*,* $F_0(R)$ *as follows*

$$F_0(R) := \int_{A^0_{2R}}\frac{u_0}{|\xi|_H^Q}\psi_H^2\varphi_0(\frac{\ln|\xi|_H}{R})d\xi.$$

Let C_2 *be the constant defined in Lemma (83). We suppose that the following conditions hold:*

i)

$$\liminf_{R\to+\infty}F_0(R) \ge 0;$$

ii)

$$\liminf_{R\to+\infty}\frac{R^{\frac{q-3}{q-1}}}{F_0(R)}<\frac{1}{C_2}.$$

There exists no weak solutions of the problem (5.43).

As simple corollary of previous theorem in case of non negative initial data is the following.

Corollary 87. *Let $u_0 \geq 0$ and there exist $c_0 > 0$, $0 < \alpha < 1$ and $R_0 > 0$, such that*

$$u_0(\xi) \geq \frac{c_0}{|\ln|\xi|_H|^\alpha} \tag{5.47}$$

for $0 < |\xi|_H < R_0$ or $|\xi|_H > R_0$. Then (5.43) has no weak solutions for $1 < q < 1+\frac{2}{\alpha}$. If $q = 1+\frac{2}{\alpha}$, then there exists a positive constant c_α given by

$$c_\alpha := \frac{(1-\alpha)(q-1)}{q^{q'}}\inf\{C_q(\varphi_0,\varphi_1)|\ \varphi_0,\varphi_1 \text{ admissible functions}\},$$

where $C_q(\varphi_0,\varphi_1)$ is defined in (5.46), such that if $c_0 > c_\alpha$, then (5.43) has no weak solutions.

Remark 88. We note that in Corollary 87, like in Theorem 84, the exponent $q_0 = 1+\frac{2}{\alpha}$ does not depend on the dimension of $\mathbb{H}^n$.

Proof (of Lemma 83). Here and in the following proofs we shall write ψ instead of ψ_H. Let u be a nontrivial weak solution of (5.43) and $\varphi \in \mathscr{C}_0^2(\mathbb{H}^n\backslash\{0\}\times[0,+\infty[)$, $\varphi \geq 0$. We set

$$\Gamma_2 := a\left(|\xi|_H^2\,\Delta_H\varphi - 4n\,|\xi|_H\left(\nabla_H|\xi|_H\,,\nabla_H\varphi\right)\right)+\psi^2\varphi_t.$$

Since u satisfies (5.44), using Hölder inequality, we have

$$\begin{aligned}\int_0^{+\infty}\int_{\mathbb{H}^n}\frac{|u|^q}{|\xi|_H^Q}\psi^2\varphi(\xi)dtd\xi+\int_{\mathbb{H}^n}\frac{u_0}{|\xi|_H^Q}\psi^2\varphi(\xi,0)d\xi &\leq \int_0^{+\infty}\int_{\mathbb{H}^n}\frac{|u|\,|\Gamma_2|}{|\xi|_H^Q}dtd\xi\\ &\leq \left(\int_0^{+\infty}\int_{\mathbb{H}^n}\frac{|u|^q}{|\xi|_H^Q}\psi^2\varphi(\xi)dtd\xi\right)^{1/q} I_2^{q'},\end{aligned}$$

where $I_2 := \int_0^{+\infty}\int_{\mathbb{H}^n}\frac{|\Gamma_2|^{q'}}{|\xi|_H^Q\psi^{2q'-2}\varphi^{q'-1}}dtd\xi$. Applying Young inequality, for $0 < \epsilon < q^{\frac{1}{q}}$, we get

$$C_1 \int_0^{+\infty} \int_{\mathbb{H}^n} \frac{|u|^q}{|\xi|_H^Q} \psi^2 \varphi(\xi) dt d\xi + \int_{\mathbb{H}^n} \frac{u_0}{|\xi|_H^Q} \psi^2 \varphi(\xi, 0) d\xi \leq C_3 I_2, \tag{5.48}$$

with $C_1 := 1 - \epsilon^q/q$ and $C_3 := 1/(\epsilon^{q'} q')$.

In order to obtain the estimate (5.45), we shall specialize the function φ. We begin requiring that φ be radial in the ξ variable, that is $\varphi = \varphi(|\xi|_H, t) = \varphi(\rho, t)$. With this assumption we have

$$\Gamma_2 = \psi^2 \left[a \left(\rho^2 \varphi_{\rho\rho} + (1 - 2n) \rho \varphi_\rho \right) + \varphi_t \right],$$

and then, using (2.12), we obtain

$$I_2 = s_n \int_0^{+\infty} \int_0^{+\infty} \frac{\left| a \left(\rho^2 \varphi_{\rho\rho} + (1 - 2n) \rho \varphi_\rho \right) + \varphi_t \right|^{q'}}{\varphi^{q'-1} \rho} dt d\rho.$$

Introducing the change of variable $s = \ln \rho$, and setting $\tilde{\varphi}(s, t) = \varphi(\rho, t)$, we get

$$I_2 = s_n \int_0^{+\infty} \int_{-\infty}^{+\infty} \frac{|a(\tilde{\varphi}_{ss} - 2n\tilde{\varphi}_s) + \tilde{\varphi}_t|^{q'}}{\tilde{\varphi}^{q'-1}} dt ds.$$

We perform our choice of φ by taking

$$\tilde{\varphi} := \varphi_0 (\frac{s + 2\,n\,a\,t}{R}) \varphi_1 (\frac{t}{R^2}),$$

where φ_1 is such that

$$0 \leq \varphi_1 \leq 1 \quad \text{and} \quad \varphi_1(y) = \begin{cases} 1 \text{ if } 0 \leq y \leq 1, \\ 0 \text{ if } y \geq 2, \end{cases} \tag{5.49}$$

and φ_0 is chosen as in (4.3). Now, the change of variable $s = Rx$ and $t = R^2 \tau$, yields

$$I_2 = s_n R^{3-2q'} \int_0^2 \int_{-\infty}^{+\infty} \frac{|a\varphi_1(\tau)\varphi_0''(x + 2\,n\,a\,R\,\tau) + \varphi_1'(\tau)\varphi_0(x + 2\,n\,a\,R\,\tau)|^{q'}}{(\varphi_0(x + 2\,n\,a\,R\,\tau)\varphi_1(\tau))^{q'-1}} d\tau dx. \tag{5.50}$$

Finally, choosing

$$\varphi(\xi, t) = \varphi_0 (\frac{\ln |\xi|_H + 2\,n\,a\,t}{R}) \varphi_1 (\frac{t}{R^2}), \tag{5.51}$$

with φ_0 and φ_1 admissible, the integral in the right hand side of (5.50) is finite and independent of R.

Being $\varphi = 1$ on A_R, from (5.48) and (5.50), we obtain the estimate (5.45). □

Proof (of Theorem 84). Let $\varphi_0 \in \mathscr{C}_0^2(\mathbb{R})$ an even, non increasing on $\mathbb{R}_+$, admissible function. Under these assumptions it follows that

$$\liminf_{R\to+\infty} \int_{A^0_{2R}} \frac{u_0}{|\xi|_H^Q} \psi^2 \varphi_0(\frac{\ln|\xi|_H}{R}) d\xi \geq \liminf_{R\to+\infty} \int_{A^0_R} \frac{u_0}{|\xi|_H^Q} \psi^2 d\xi = U_0$$

(see remark 2.2 in [50]).

If $1 < q < 3$, taking the inferior limit as $R \to +\infty$ in (5.45), we have

$$(1 - \frac{\epsilon^q}{q}) \int_0^{+\infty} \int_{\mathbb{H}^n} \frac{|u|^q}{|\xi|_H^Q} \psi^2 dt d\xi + U_0 \leq 0,$$

and then passing to the limit as $\epsilon \to 0$, we obtain the claim.

Let $q = 3$. Taking the inferior limit as $R \to +\infty$ in (5.45), we get

$$C_2 \geq C_1 \int_0^{+\infty} \int_{\mathbb{H}^n} \frac{|u|^q}{|\xi|_H^Q} \psi^2 dt d\xi + U_0. \tag{5.52}$$

Without loss of generality, we assume $|U_0| < +\infty$ (otherwise there is nothing to prove), it follows that inequality (5.52) implies

$$\int_0^{+\infty} \int_{\mathbb{H}^n} \frac{|u|^q}{|\xi|_H^Q} \psi^2 dt d\xi < +\infty. \tag{5.53}$$

Using the notation of previous proof, our choice (5.51) on φ, allows us to rewrite Γ_2 as $\Gamma_2 = \frac{\psi^2}{R^2}(\Gamma_2^1 + \Gamma_2^2)$, where

$$\Gamma_2^1 := a\varphi_1(\frac{t}{R^2})\varphi_0''(\frac{\ln|\xi|_H + 2\,n\,a\,t}{R}),$$
$$\Gamma_2^2 := a\varphi_1'(\frac{t}{R^2})\varphi_0(\frac{\ln|\xi|_H + 2\,n\,a\,t}{R}),$$

which vanish respectively outside the sets

$$B_R^1 := \{(\xi,t) | 0 \leq t \leq 2R^2,\ R \leq |\ln|\xi|_H + 2\,n\,a\,t| \leq 2R\},$$
$$B_R^2 := \{(\xi,t) | R^2 \leq t \leq 2R^2,\ |\ln|\xi|_H + 2\,n\,a\,t| \leq 2R\}.$$

Next from (5.44), by Hölder inequality, we have

$$\int_{A_R} \frac{|u|^q}{|\xi|_H^q} \psi^2 dt d\xi + \int_{A^0_{2R}} \frac{u_0}{|\xi|_H^Q} \psi^2 \varphi_0(\frac{\ln|\xi|_H}{R}) d\xi \leq$$
$$\leq - \int_{B_R^1} \frac{u\Gamma_2^1 \psi^2}{|\xi|_H^Q R^2} dt d\xi - \int_{B_R^2} \frac{u\Gamma_2^2 \psi^2}{|\xi|_H^Q R^2} dt d\xi \leq I_{1,R}^{1/q} I_{2,R}^{1/q'} + I_{3,R}^{1/q} I_{4,R}^{1/q'}, \tag{5.54}$$

where

$$I_{1,R} := \int_{B_R^1} \frac{|u|^q}{|\xi|_H^Q} \psi^2 dtd\xi, \quad I_{2,R} := \int_{B_R^1} \frac{|\Gamma_2^1|^{q'}}{|\xi|_H^Q} \psi^2 R^{-2q'} dtd\xi,$$

$$I_{3,R} := \int_{B_R^2} \frac{|u|^q}{|\xi|_H^Q} \psi^2 dtd\xi, \quad I_{4,R} := \int_{B_R^2} \frac{|\Gamma_2^2|^{q'}}{|\xi|_H^Q} \psi^2 R^{-2q'} dtd\xi.$$

Standard argument shows that indeed $I_{2,R} < M$ and $I_{4,R} < M$ with $M < +\infty$ independent of R. From (5.53), we deduce that for $R \to +\infty$, $I_{1,R} \to 0$ and $I_{3,R} \to 0$. Hence, by taking the inferior limit of (5.54), we get the claim. □

Proof (of Theorem 86). Let u be a weak solution of (5.43). Let $R_n \to +\infty$ be a sequence, such that

$$\liminf_{R \to +\infty} \frac{R^{\frac{q-3}{q-1}}}{F_0(R)} = \lim_n \frac{R_n^{\frac{q-3}{q-1}}}{F_0(R_n)}.$$

Fixed $\epsilon > 0$ small and for n large enough, we have

$$C_2 \frac{R_n^{\frac{q-3}{q-1}}}{F_0(R_n)} - 1 < -\epsilon C_2.$$

Thus, the estimate (5.45) can be rewritten as

$$C_1 \int_{A_{R_n}} \frac{|u|^q}{|\xi|_H^Q} \psi^2 dtd\xi \leq C_2 R_n^{\frac{q-3}{q-1}} - F_0(R_n) = F_0(R_n) \left(C_2 \frac{R_n^{\frac{q-3}{q-1}}}{F_0(R_n)} - 1 \right),$$

hence

$$\int_0^{+\infty} \int_{\mathbb{H}^n} \frac{|u|^q}{|\xi|_H^Q} \psi^2 dtd\xi \leq 0.$$

This contradiction completes the proof. □

Proof (of Corollary 87). Let u be a weak solution of (5.43) with initial condition u_0. Assume that for $|\xi|_H > R_0 > 0$, (5.47) holds for $c_0 > 0$ and $0 < \alpha < 1$. The proof when the behavior (5.47) occurs near the origin, that is for $0 < |\xi|_H < R_0$, is similar.

Choose $0 < \epsilon < q^{\frac{1}{q}}$, φ_0 and φ_1 admissible functions and moreover if $q = 1 + \frac{2}{\alpha}$, we require then they are such that $c_0 > \frac{1-\alpha}{\epsilon^{q'} q'} C_q(\varphi_0, \varphi_1) \geq c_\alpha$, which from Lemma 83, implies that $\frac{c_0 s_n}{1-\alpha} > C_2$.

Since $u_0 \geq 0$, assumptions *i)* and *ii)* of Theorem 86 hold. Therefore we have a contradiction. Indeed, in this case, for $R > R_0$, we have

$$F_0(R) \geq \int_{A_R^0} \frac{u_0}{|\xi|_H^Q}\psi^2 d\xi = I_R + \int_{R_0 \leq \ln|\xi|_H \leq R} \frac{|\xi|_H^{-Q} c_0}{(\ln|\xi|_H)^\alpha}\psi^2 d\xi \geq \frac{c_0 s_n}{1-\alpha} R^{1-\alpha} - C(R_0),$$

where $I_R := \int_{-R \leq \ln|\xi|_H \leq R_0} \frac{u_0}{|\xi|_H^Q}\psi^2 d\xi$ and $C(R_0)$ is a non negative constant. This proves the claim. □

Remark 89. If in Corollary 87 we fix $q > 3$, then (5.43) has no weak solutions if the initial condition $u_0 \geq 0$ satisfy (5.47) with $0 < \alpha < \frac{2}{q-1}$. Actually, from Theorem 86, it is possible to see that Corollary 87 still holds if u_0 is an infinitesimal of order smaller than $(\ln|\xi|_H)^{\frac{2}{1-q}}$ at infinity or at the origin. For instance, the function $u_0 = \frac{|\ln|\ln|\xi|_H||}{(\ln|\xi|_H)^{\frac{2}{q-1}}}$ does not satisfy (5.47) with no $0 < \alpha < \frac{2}{q-1}$, but it belongs to the blow-up case.

Remark 90. The hypothesis $a \in \mathbb{R}$ can be weakened requiring a suitable assumption of asymptotic behavior of a at infinity. Let us consider a simple case. Let $a \in L^\infty(\mathbb{H}^n \times [0, +\infty[)$, $c \in \mathbb{R}$ such that $|c - a(\xi, t)| \leq \frac{M}{f(|\xi|_H, t)}$, where f is a function such that

$$f(e^{Rs}, R^2\tau) \geq R\alpha(s, \tau, R) \geq Rc_0 > 0,$$

for $-2 \leq s \leq 2$, $0 \leq \tau \leq 2$, $R \geq R_0 > 0$, and a suitable function α. With these assumptions Theorem 84 still holds.

5.6 A second order evolution inequality for the Khon Laplacian

In this section we study the following second order evolution inequality

$$\begin{cases} u_{tt} - \frac{|\xi|_H^2}{\psi_H^2}\Delta_H(au) \geq |u|^q & \text{on } \mathbb{H}^n \setminus \{0\} \times]0, +\infty[, \\ u(\xi, 0) = u_0(\xi) & \text{on } \mathbb{H}^n \setminus \{0\}, \\ u_t(\xi, 0) = u_1(\xi) & \text{on } \mathbb{H}^n \setminus \{0\}, \end{cases} \tag{5.55}$$

where $a \in L^\infty(\mathbb{H}^n \times [0, +\infty[)$.

Definition 91. *Let $q \geq 1$. We say that u is a* weak solution of (5.43) *if $u_0, u_1 \in L^1_{loc}(\mathbb{H}^n \setminus \{0\})$, $u \in L^q_{loc}(\mathbb{H}^n \setminus \{0\} \times [0, +\infty[)$ and*

$$\int_0^{+\infty}\int_{\mathbb{H}^n}\frac{|u|^q}{|\xi|_H^Q}\psi_H^2\varphi d\xi dt \le -\int_0^{+\infty}\int_{\mathbb{H}^n} au\Delta_H(|\xi|_H^{-2n}\varphi)d\xi dt$$
$$+\int_0^{+\infty}\int_{\mathbb{H}^n}\frac{u}{|\xi|_H^Q}\psi_H^2\varphi_{tt}d\xi dt+\int_{\mathbb{H}^n}\frac{\psi_H^2}{|\xi|_H^Q}\left[u_0(\xi)\varphi_t(\xi,0)-u_1(\xi)\varphi(\xi,0)\right]d\xi, \quad (5.56)$$

for any non negative $\varphi \in \mathscr{C}_0^2(\mathbb{H}^n\setminus\{0\}\times[0,+\infty[)$.

Lemma 92. *Let* $q>1$. *Let* u *be a weak solution of (5.55). For any admissible function* $\varphi_0\in\mathscr{C}^2(\mathbb{R})$, $R_0>0$ *and* $0<\epsilon<q^{\frac{1}{q}}$, *there exists a positive constant* C_2 *such that for* $R>R_0$, *the following estimate holds*

$$C_1\int_{A_R}\frac{|u|^q}{|\xi|_H^Q}\psi_H^2 dt d\xi+\int_{A^0_{2R^2}}\frac{u_1}{|\xi|_H^Q}\psi_H^2\varphi_0\left(\frac{\ln|\xi|_H}{R^2}\right)d\xi\le C_2R^{3-2q'}, \quad (5.57)$$

where $C_1:=1-\epsilon^q/q$ *and*

$$A_R:=\{(\xi,t)|0\le t\le R,\ |\ln|\xi|_H|\le R^2\},$$
$$A^0_{2R^2}:=\{\xi|\,|\ln|\xi|_H|\le 2R^2\}.$$

Moreover, the constant C_2 *has the form* $C_2=\frac{1}{\epsilon^{q'}q'}\sup_{R>R_0}K_q(\varphi_0,\varphi_1,R)$, *where*

$$K_q(\varphi_0,\varphi_1,R):=\int_0^2 d\tau\int_{-2}^{+2}dv\int_0^{\pi}d\theta\int_0^{\pi}d\theta_1\cdots\int_0^{\pi}d\theta_{2n-2}\int_0^{2\pi}d\theta_{2n-1}\sin^n\theta\sin^{2n-2}\theta_1\cdots$$
$$\sin\theta_{2n-2}\frac{\left|\tilde a(e^{R^2s},\theta,\ldots,\theta_{2n-1},R\tau)\varphi_1(\tau)(\frac{\varphi_0''(v)}{R^2}-2n\varphi_0'(v))+\varphi_1''(\tau)\varphi_0(v)\right|^{q'}}{(\varphi_0(v)\varphi_1(\tau))^{q'-1}}, \quad (5.58)$$

$\varphi_1\in\mathscr{C}_0^2([0,+\infty[)$ *is any admissible function and by the change of variable (2.11), we have set* $\tilde a(\rho,\theta,\ldots,\theta_{2n-1},t)=a(\xi,t)$.

Estimate (5.57) allows us to get the following nonexistence results.

Theorem 93. *Let* $1<q\le 3$. *Let* u *be a weak solution of (5.55), then*

$$\int_0^{+\infty}\int_{\mathbb{H}^n}\frac{|u|^q}{|\xi|_H^Q}\psi_H^2 d\xi dt\le -U_1,$$

where

$$U_1:=\liminf_{R\to+\infty}\int_{R^{-1}\le|\xi|_H\le R}\frac{u_1(\xi)}{|\xi|_H^Q}\psi_H^2 d\xi \qquad (possibly\ infinite).$$

Therefore, if $U_1\ge 0$, *then (5.55) has no nontrivial weak solutions.*

Remark 94. From the above result, it follows that if $u_1 \geq 0$, then (5.55) has no weak solutions for $1 < q \leq 3$. Furthermore, no assumptions is made on the initial condition u_0.

In the following theorem, in order to get nonexistence results for $q > 3$, we shall analyze the asymptotic behavior of the initial condition u_1 at infinity and at the origin.

Theorem 95. *Let $q > 3$ and $u_1 \neq 0$. Let $\varphi_0 \in \mathscr{C}_0^2(\mathbb{R})$ be an admissible function. Let $R > 0$, and set $F_1(\cdot)$ as follows:*

$$F_1(R) := \int_{A^0_{2R^2}} \frac{u_1}{|\xi|_H^Q} \psi_H^2 \varphi_0(\frac{\ln|\xi|_H}{R^2}) d\xi.$$

Let C_2 be the constant defined in Lemma (92). Suppose that

i)

$$\liminf_{R \to +\infty} F_1(R) \geq 0,$$

ii)

$$\liminf_{R \to +\infty} \frac{R^{\frac{q-3}{q-1}}}{F_1(R)} < \frac{1}{C_2},$$

then problem (5.55) has no weak solutions.

As simple corollary of previous theorem in case of non negative initial data u_1 is the following.

Corollary 96. *Let $u_1 \neq 0$. Moreover, suppose that there exist $c_1 > 0$, $1/2 < \alpha < 1$ and $R_0 > 0$, such that*

$$u_1(\xi) \geq \frac{c_1}{|\ln|\xi|_H|^\alpha} \tag{5.59}$$

for $0 < |\xi|_H < R_0$ or $|\xi|_H > R_0$. Then (5.55) has no weak solutions for $1 < q < 1 + \frac{2}{2\alpha-1}$. If $q = 1 + \frac{2}{2\alpha-1}$, then there exists a positive constant c'_α such that, if $c_1 > c'_\alpha$, (5.55) has no weak solutions, and the value of c'_α is given by

$$c'_\alpha := \frac{(1-\alpha)(q-1)}{s_n q^{q'}} \inf\{\left(\sup_{R > R_0} K_q(\varphi_0, \varphi_1, R)\right) \mid \varphi_0, \varphi_1 \text{ admissible functions}\},$$

where $K_q(\varphi_0, \varphi_1, R)$ is defined in (5.58).

Remark 97. We note that in Corollary 96, like in Theorem 93, the exponent $q_0 = 1 + \frac{2}{2\alpha-1}$ does not depend on the dimension of $\mathbb{H}^n$.

Proof (of Lemma 92). As in the previous proofs we set $\psi = \psi_H$. Let u be a non-trivial weak solution of (5.55) and $\varphi \in \mathscr{C}_0^2(\mathbb{H}^n\backslash\{0\} \times [0, +\infty[)$, $\varphi \geq 0$. We set

$$\Gamma_3 := a\left(|\xi|_H^2 \,\Delta_H \varphi - 4n\,|\xi|_H \left(\nabla_H \,|\xi|_H \,, \nabla_H \varphi\right)\right) + \psi^2 \varphi_{tt}.$$

Since u satisfies (5.56), as in the previous sections, by Hölder and Young inequality, we obtain

$$C_1 \int_0^{+\infty} \int_{\mathbb{H}^n} \frac{|u|^q}{|\xi|_H^Q} \psi^2 \varphi(\xi) dt d\xi + \int_{\mathbb{H}^n} \frac{\psi^2}{|\xi|_H^Q} \left[u_1(\xi)\varphi(\xi,0) - u_0(\xi)\varphi_t(\xi,0)\right] d\xi \leq C_3 I_2, \tag{5.60}$$

where $I_2 := \int_0^{+\infty} \int_{\mathbb{H}^n} \frac{|\Gamma_3|^{q'}}{|\xi|_H^Q \psi^{2q'-2} \varphi^{q'-1}} dt d\xi$, $C_1 := 1 - \epsilon^q/q$ and $C_3 := 1/(\epsilon^{q'} q')$.

Assuming that the function φ is radial in the variable ξ, that is $\varphi = \varphi(|\xi|_H\,, t) = \varphi(\rho, t)$, we have

$$\Gamma_3 = \psi^2 \left(a(\rho^2 \varphi_{\rho\rho} + (1-2n)\rho\varphi_\rho) + \varphi_{tt}\right).$$

The change of variable $\xi = \Phi(\rho, \theta, \theta_1, \ldots, \theta_{2n-1})$, defined in (2.11), allows us to write

$$I_2 = \int_0^{+\infty} dt \int_0^{+\infty} d\rho \int_0^{\pi} d\theta \cdots \int_0^{\pi} d\theta_{2n-2} \int_0^{2\pi} d\theta_{2n-1} \sin^n \theta \sin^{2n-2} \theta_1 \cdots$$
$$\cdots \sin\theta_{2n-2} \frac{|\tilde{a}(\rho^2 \varphi_{\rho\rho} + (1-2n)\rho\varphi_\rho) + \varphi_{tt}|^{q'}}{\rho\varphi^{q'-1}}.$$

Setting $\tilde{\varphi}(s,t) = \varphi(\rho,t)$, with $s = \ln \rho$, we get

$$I_2 = \int_0^{+\infty} dt \int_{-\infty}^{+\infty} ds \int_0^{\pi} d\theta \cdots \int_0^{\pi} d\theta_{2n-2} \int_0^{2\pi} d\theta_{2n-1} \sin^n \theta \sin^{2n-2} \theta_1 \cdots$$
$$\cdots \sin\theta_{2n-2} \frac{|\tilde{a}(e^s, \theta, \ldots, t)(\tilde{\varphi}_{ss} - 2n\tilde{\varphi}_s) + \tilde{\varphi}_{tt}|^{q'}}{\tilde{\varphi}^{q'-1}}.$$

Next we choose $\tilde{\varphi}$ as follows:

$$\tilde{\varphi} := \varphi_0(\frac{s}{R^2})\varphi_1(\frac{t}{R}),$$

where φ_0 and φ_1 are admissible functions as in (4.3) and (5.49) respectively. The change of variable $s = R^2 x$ and $t = R\tau$, yields

$$I_2 = R^{3-2q'} K_q(\varphi_0, \varphi_1, R), \tag{5.61}$$

where $K_q(\varphi_0, \varphi_1, R)$ is defined in (5.58). Setting $c = \|a\|_\infty^{q'}$, since $(\cdot)^{q'}$ is convex, we obtain

$$K_q(\varphi_0,\varphi_1,R) \le Cs_n \left\{ \int_0^2 \int_{-1}^{+1} c\varphi_1(\tau) \frac{|\varphi_0''(x)/R^2|^{q'} + |2n\varphi_0'(x)|^{q'}}{\varphi_0^{q'-1}(x)} d\tau dx \right. \\ \left. + \int_1^2 \int_{-2}^{+2} \varphi_0(x) \frac{|\varphi_1''(\tau)|^{q'}}{\varphi_1^{q'-1}(\tau)} d\tau dx \right\}.$$

This estimate furnishes an upper bound for $\sup_{R>R_0} K_q(\varphi_0,\varphi_1,R)$, consequently the quantity C_2 is well defined.

Therefore, the final choice on φ is given by

$$\varphi(\xi,t) = \varphi_0(\frac{\ln|\xi|_H}{R^2})\varphi_1(\frac{t}{R}).$$

Being $\varphi = 1$ on A_R, from (5.60) and (5.61), we obtain the claim. □

Proof (of Theorem 93). Let u be a weak solution of (5.55) and let $\varphi_0 \in \mathscr{C}_0^2(\mathbb{R})$ be an even, non increasing on $\mathbb{R}_+$, admissible function.

If $1 < q < 3$, taking the inferior limit as $R \to +\infty$ in (5.57), we get the claim.

When $q = 3$ and $|U_1| < +\infty$, (5.57) implies

$$\int_0^{+\infty} \int_{\mathbb{H}^n} \frac{|u|^q}{|\xi|_H^Q} \psi^2 dt d\xi < +\infty,$$

and arguing as in the proof of Theorem 84, we complete the proof. □

For sake of brevity, we omit the proofs of Theorem 95 and Corollary 96 since they are very similar respectively to the proofs of Theorem 86 and Corollary 87.

Remark 98. If in Corollary 96 we fix $q > 3$, then (5.55) has no weak solutions if the initial condition $u_1 \ge 0$ satisfy (5.59) with $\frac{1}{2} < \alpha < \frac{1}{2} + \frac{1}{q-1}$. Actually, from Theorem 95, it is possible to see that Corollary 96 still holds if u_1 is an infinitesimal of order smaller than $(\ln|\xi|_H)^{\frac{1}{1-q}-\frac{1}{2}}$ at infinity or at the origin. For instance, the function $u_1 = \frac{|\ln|\ln|\xi|_H||}{(\ln|\xi|_H)^{\frac{1}{2}+\frac{1}{q-1}}}$ does not satisfy (5.59) with no $\frac{1}{2} < \alpha < \frac{1}{2} + \frac{1}{q-1}$, but it belongs to the blow-up case.

References

1. W. Allegretto, Y.X. Huang, *A Picone's identity for the p-Laplacian and applications*, Nonlinear Anal. **32** (1998), 819–830.
2. A. Ancona, *On strong barriers and an inequality of Hardy for domains in* $\mathbb{R}^n$, J. London Math. Soc. **34** (1986), 274–290.
3. J.P. García Azorero, I. Peral Alonso, *Hardy Inequalities and Some Critical Elliptic and Parabolic Problems*, J. Diff. Eq. **144** (1998), 441–476.
4. Z.M. Balogh, J.T. Tyson, *Polar coordinates in Carnot Groups*, preprint (2001).
5. M. S. Baouendi, *Sur une Classe d'Operateurs Elliptiques Degeneres*, Bull. Soc. Math. France **95** (1967), 45–87.
6. P. Baras, J.A. Goldstein, *The heat equation with a singular potential*, Trans. Amer. Math. Soc. **284** (1984), n. 1, 121–139.
7. J. Barros-Neto, F. Cardoso, *Bessel Integrals and Fundamental Solutions for a Generalized Tricomi Operator*, J. Funct. Anal. **183** (2001), 472–497.
8. W. Beckner, *On the Grushin Operator and Hyperbolic Symmetry*, Proc. Am. Math. Soc. **129** (2001), 1233–1246.
9. H. Berestycki, P.-L. Lions, L.A. Peletier, *An ODE approach to the existence of positive solutions for semilinear problems in* $\mathbb{R}^N$, Indiana Univ. Math. J. **30** (1981), 141–157.
10. I. Birindelli, I. Capuzzo Dolcetta, A. Cutrì, *Liouville theorems for semilinear equations on the Heisenberg group*, Ann. Inst. Henri Poincaré **14** (1997), 295–308.
11. J.M. Bony, *Principle du maximum, inégalité de Harnack et unicité du problem de Cauchy pour les opérateurs elliptiques dégénérés*, Ann. Inst. Fourier, Grenoble **19** (1969), 277–304.
12. H. Brézis, X. Cabré, *Some Simple Nonlinear PDE's Without Solutions*, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8) **1** (1998), 223–262.
13. H. Brézis, M. Marcus, *Hardy's inequalities revisited*, Ann. Sc. Norm. Pisa **25** (1997), n. 1-2, 217–237.
14. H. Brézis, M. Marcus, I. Shafrir, *Extremal Functions for Hardy's Inequality with Weight*, J. Funct. Anal. **171** (2000), n. 1, 177–191.
15. H. Brézis, J.L. Vázquez, *Blow-up solutions of some nonlinear elliptic problem*, Rev. Mat. Univ. Complut. Madrid **10** (1997), n. 2, 443–469.
16. L. Capogna, *Regularity of Quasi-Linear Equations on the Heisenberg group*, Comm. Pure Appl. Math. **L** (1997), 867–889.
17. I. Capuzzo Dolcetta, A. Cutrì, *On the Liouville Property for Sublaplacian*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **25** (1997), 239–256.
18. L. D'Ambrosio, *Critical Degenerate Inequalities on the Heisenberg Group*, Manuscripta Math. **106** (2001), 519-536.

19. L. D'Ambrosio, *Some Hardy Inequalities on the Heisenberg Group*, Preprint SISSA ref. 2/2001/M (2001).
20. L. D'Ambrosio, *Hardy Inequalities related to Grushin type operators*, Preprint SISSA ref. 54/2002/M (2002)
21. L. D'Ambrosio, S. Lucente, *Nonlinear Liouville Theorems for Grushin and Tricomi Operators*, Preprint SISSA ref. 53/2002/M (2002)
22. E.B. Davies, *The Hardy constant*, Quart. J. Math. Oxford Ser. **46** (1995), n. 184, 417–431.
23. E.B. Davies, *A review of Hardy inequalities*, in Operator Theory: Advances and Applications, Vol.**110** (1999), Birkhäuser Verlag Basel, 55–67.
24. E.B. Davies and A.M. Hinz, *Explicit constants for Rellich inequalities in $L_p(\Omega)$*, Math. Z. **227** (1998), 511–523.
25. K. Deng, H.A. Levine, *The Role of Critical Exponents in Blow-Up Theorems: The Sequel*, J. Math. Anal. Appl. **243** (2000), 85–126.
26. G.B. Folland, *Subelliptic estimates and function spaces on nilpotent Lie groups*, Arkiv gör Mat. **13** (1975), 161–207.
27. G.B. Folland, E.M. Stein, *Estimates for the $\overline{\partial}_b$ Complex and Analysis on the Heisenberg Group*, Comm. Pure Appl. Math. **27** (1974), 429–522.
28. B. Franchi, E. Lanconelli, *Hölder Regularity Theorem for a Class of Linear Nonuniformly Elliptic Operators with Measurable Coefficient*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **10** (1983), 523–541.
29. B. Franchi, E. Lanconelli, *Une métrique associéte à une classe d'opérateurs elliptiques dégénérés* Linear partial and pseudo differential operators, Conv. Torino/Italy 1982, Rend. Semin. Mat., Fasc. Spec., (1983) 105-114 in French.
30. B. Franchi, E. Lanconelli, *An embedding theorem for Sobolev spaces related to non-smooth vector fields and Harnack inequality*, Commun. Partial Differ. Equations **9** (1984), 1237-1264.
31. B. Franchi, M.C. Tesi, *A finite element approximation for a class of degenerate elliptic equations*, Math. Comp. **69** (1999), 41-63.
32. H. Fujita,*On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$*, J. Fac. Sci. Univ. Tokyo, Sec. I **13** (1966), 109–124.
33. N. Garofalo, E. Lanconelli, *Frequency Functions on the Heisenberg Group, the Uncertainty Principle and Unique Continuation*, Ann. Inst. Fourier, Grenoble **40** (1990), 313–356.
34. N. Garofalo, E. Lanconelli, *Existence and Nonexistence Results for Semilinear Equations on the Heisenberg Group*, Indiana Univ. Math. J. **41** (1992), 71–98.
35. N. Garofalo, Z. Shen, *Calerman estimates for a subelliptic operator and unique continuation*, Ann. Inst. Fourier, Grenoble **44** (1994), 129–166.
36. F. Gazzola, H.-C. Grunau, E. Mitidieri, *Hardy inequalities with optimal constants and remainder terms*, preprint (2000).
37. B. Gidas, J. Spruck, *Global and local behavior of positive solutions of nonlinear elliptic equations*, Commun. Pure Appl. Math. **34** (1981), 525-598.
38. J.A. Goldstein, Qi S. Zhang, *On a Degenerate Heat Equation with a Singular Potential*, J. Func. Anal. **186** (2001), 342–359.
39. V. V. Grushin, *On a Class of Hypoelliptic Operators*, Math. USSR Sbornik **12** (1970), 458–476.
40. G.H. Hardy, *Note on a Theorem of Hilbert*, Math. Z. **6** (1920), 314–317.
41. G.H. Hardy, J.E. Littlewood, G. Polya, Inequalities, Cambridge Univ. Press, UK, 1934.

42. L. Hörmander, *Hypoelliptic Second Order Differential Equations*, Acta Math. **119** (1968), 147–171.
43. E. Lanconelli, F. Uguzzoni, *Non-Existence Results for Semilinear Kohn-Laplace Equations in Unbounded Domains*, Comm. Par. Diff. Eq. **25** (2000), 1703–1739.
44. J.-L. Lions, E. Magenes, Problèmes aux limites non homogènes et applications. Vol. 3, Dunod, Paris, 1970.
45. A.R. Manwell, The Tricomi Equation with Applications to the Theory of Plane Transonic Flow, Pitman, Research Notes in Mathematics, 35, (1979).
46. T. Matsuzawa, *Gevrey Hypoellipticity for Grushin Operators*, Publ. RIMS. Kyoto Univ. **33** (1997), 775–799.
47. V.G. Maz'ya, Sobolev Spaces, Springer-Verlag, Berlin 1985.
48. E. Mitidieri, *A simple approach to Hardy's inequalities* Mat. Zametki **67** (2000), no. 4, 563-572.
49. E. Mitidieri, S.I. Pohozaev, *Nonexistence of weak solutions for some degenerate elliptic and parabolic problems on* $\mathbb{R}^n$, Journal of Evolution Equations **1** (2001), 189–220.
50. E. Mitidieri, S.I. Pohozaev, *Nonexistence of weak solutions for some degenerate and singular hyperbolic problems on* $\mathbb{R}^n$, Proc. Steklov Institute of Mathematics **232** (2001), 240–259. Translated from Trudy Matematicheskogo Instituta imeni V.A. Steklova **232** (2001), 248–267.
51. E. Mitidieri, S.I. Pohozaev, *A Priori Estimates of Solutions to Nonlinear Partial Differential Equations and Inequalities and Applications*, Proc. Steklov Institute of Mathematics **234** (2001), 1–375.
52. A. Nagel, E.M. Stein, S. Wainger, *Balls and metrics defined by vector fields I: basic properties*, Acta Math. **155** (1986), 103–147.
53. W.-M. Ni, *On a singular elliptic equation*, Proc. Amer. Math. Soc. **88** (1983), 614–616.
54. W.-M. Ni, J. Serrin, *Non-existence theorems for quasilinear partial differential equations*, Rend. Circ. Mat. Palermo, Ser. 2, Suppl. **8** (1985), 171–185.
55. W.-M. Ni, J. Serrin, *Existence and nonexistence theorems for ground states of quasilinear partial differential equations. The anomalous case*, Atti Convegni Lincei **77** (1986), 231–257.
56. P. Niu, H. Zhang, Y. Wang, *Hardy Type and Rellich Type Inequalities on the Heisenberg Group*, Proc. Amer. Math. Soc. **129** (2001), 3623–3630.
57. B. Opic, A. Kufner, Hardy-type Inequalities, Pitman Research Notes in Math., Vol. 219, Longman, 1990.
58. S. Pohozaev, L. Veron, *Nonexistence results of solutions of semilinear differential inequalities on the Heisenberg group*, Manuscripta Math. **102** (2000), 85–99.
59. F. Rellich, *Halbbeschränkte Differentialoperatoren höherer Ordnung*, in Proceedings of the International Congress of Mathematicians 1954, Groningen Noorthoff, Vol. III (1956), 243–250.
60. F.G. Tricomi, *Sulle Equazioni Lineari alle Derivate Parziali di 2° Ordine di Tipo Misto*, Memorie Lincei **14** (1923), 133–247.
61. J.L. Vazquez, E. Zuazua, *The Hardy Inequality and the Asymptotic Behavior of the Heat Equation with an inverse-Square Potential*, J. Func. Anal. **173** (2000), 103–153.
62. C.-J. Xu, *The Harnack's Inequality for Second Order Degenerate Elliptic Operators*, Chinese Ann. Math. Ser. A **10** (1989), 359–365, in Chinese.