Emergent Gravity:

The Analogue Models Perspective

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To my family

EMERGENT GRAVITY: The Analogue Models Perspective

Lorenzo Sindoni — *Ph.D. Thesis*

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Abstract

This thesis is devoted to the study of some aspects of emergent gravity scenarios, *i.e.* of non-gravitational systems which exhibit, under suitable conditions, the emergence of effective spacetime metrics and the associated gravitational dynamics. While this area of research is rather broad, we will assume a particular perspective: the starting point is the discussion of analogue models for gravity, which have so far provided precious insights on aspects of physics on curved spacetimes, on extensions of Riemannian geometry and on the possible role of high energy Lorentz symmetry violations in low energy physics. The first part is devoted to the study of the most relevant kinematical features of emergent spacetimes typical of condensed matter analogue models. In particular, it will be shown that a natural framework for the description of the geometrical properties of emergent spacetimes is Finsler geometry, which thus represent an interesting candidate for extensions of special (and eventually general) relativity. We will present the basic concepts of this particular generalization of Riemannian geometry, and hence we will pass to the careful analysis of the main issues to be solved and of the features that can be of major interest for physical applications, in particular the geometrical interpretation of modified dispersion relations and the fate of Lorentz symmetry at the Planck scale. Some emphasis will be given to the conditions under which pseudo-Riemannian geometries are selected among the larger class of Finsler geometries. The second part is devoted to the discussion of some features of the dynamics of the emergent spacetimes. In particular, we will discuss the lessons that can be learned from a Bose–Einstein condensate analogue model. It is shown that, in an appropriate regime, a modified non-relativistic gravitational theory does appear to be the effective description of quasiparticles and their interactions with the condensate. While this is by no means an attempt to realize a perfect analogue of a realistic gravitational theory, this particular example gives the possibility of discussing a number of interesting features (*e.g.* locality and the nature of the cosmological constant). These features could be used as an inspiration to understand some of the problems encountered in understanding gravity at both classical and quantum levels. Finally, a generalization to the relativistic case will give the possibility of discussing more refined questions, like the possibility of emerging diffeomorphism invariance and time out of a system defined in flat Euclidean space. Despite the particular perspective and the very specific form of the toy models considered, we believe that all the observations and hints that can be collected from these investigations, in particular the role of symmetries, can shed some light onto some of the most challenging problems in the formulation of a satisfactory theory of emergent gravity.

List of papers

The research presented in this thesis is part of the outcome of the scientific collaborations stated below, as well as of the author's own work. The research activity has led to the following research papers published in refereed Journals.

- Planck-scale modified dispersion relations and Finsler geometry.
  Florian Girelli, Stefano Liberati, Lorenzo Sindoni (SISSA, Trieste & INFN, Trieste) .
  Published in Phys.Rev.**D75**:064015, 2007. [e-Print: gr-qc/0611024]
  (Ref. [234], discussed in chapter 5)
- Shell-mediated tunnelling between (anti-)de Sitter vacua.
  Stefano Ansoldi (MIT, LNS & MIT & ICRA, Rome), Lorenzo Sindoni (SISSA, Trieste & INFN, Trieste) .
  Published in Phys.Rev.**D76**:064020,2007. [e-Print: arXiv:0704.1073 [gr-qc]]
- The Higgs mechanism in Finsler spacetimes.
  Lorenzo Sindoni (SISSA, Trieste & INFN, Trieste) .
  Published in Phys.Rev.**D77**:124009,2008. [e-Print: arXiv:0712.3518 [gr-qc]]
  (Ref. [241], discussed in chapter 5)
- On the emergence of Lorentzian signature and scalar gravity.
  Florian Girelli, Stefano Liberati, Lorenzo Sindoni (SISSA, Trieste& INFN, Trieste) .
  Published in Phys.Rev.**D79**:044019,2009. [e-Print: arXiv:0806.4239 [gr-qc]]
  (Ref. [295], discussed in chapter 8)
- Gravitational dynamics in Bose Einstein condensates.
  Florian Girelli, Stefano Liberati, Lorenzo Sindoni (SISSA, Trieste & INFN, Sezione di Trieste).
  Published in Phys.Rev.**D78**:084013,2008. [e-Print: arXiv:0807.4910 [gr-qc]]
  (Ref. [283], discussed in chapter 7)
- Reconciling MOND and dark matter?
  Jean Philippe Bruneton, Stefano Liberati, Lorenzo Sindoni and Benoit Famaey
  Published in JCAP 0903:021,2009. [e-Print arXiv:0811.3143 [astro-ph]]
  (Ref. [315], mentioned in chapter 9)
- Linking the trans-Planckian and the information loss problems in black hole physics.
  Stefano Liberati, Lorenzo Sindoni (SISSA, Trieste & INFN, Trieste) Sebastiano Sonego (Università di Udine)
  e-Print: arXiv:0904.0815 [gr-qc]
  Submitted to Annals of Physics
  (Ref. [82], mentioned in chapter 2)

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Lorenzo Sindoni

Contents

Chapter 1

Motivations and overview

Where we stand: General Relativity

The gravitational interaction is the fundamental force of Nature which we experience more commonly in our life. Not only: we do have one of the most compelling theoretical description to account for its phenomenology, general relativity (GR) [1, 2, 3].

The present status of GR is pretty encouraging. Indeed, it shows excellent agreement with experimental tests (see for instance [4, 5, 6, 7]). Also, in recent years it has been shown how it is successfully describing many aspects of Cosmology: many details of the life of universe are correctly grasped by a theory where gravity is described by GR (with the addition of a cosmological constant and some yet unspecified non-baryonic dark matter component). The fact that WMAP data on cosmological parameters are reaching impressive precision [8, 9, 10, 11] is giving a strong support to this point of view. On the other hand, the proposal of alternative gravitational theories, which can be considered serious competitors of general relativity, has proved to be a rather challenging objective.

Despite these successes, there are some difficulties, both observational and theoretical, suggesting that general relativity is in the best case only an effective field theory, providing a good but incomplete description of gravitational phenomena, thus requiring a suitable extension.

First of all, it must be mentioned the well-known cosmological constant problem [12, 13], *i.e.* the difficulty of justifying the (small) value of the observed (positive) cosmological constant within our present theoretical frameworks. The naive expectation based on straightforward dimensional analysis would be to have either a zero cosmological constant, or a Planckian one. It turns out that the observed cosmological constant (inferred by the acceleration parameter), is very small (and positive) but not zero. Moreover, its value corresponds to a very small energy scale which has no particular role in our present theories of particle physics. (However, if one is interested in the numerology, the geometric mean of the present Hubble scale and the Planck scale corresponds roughly to the energy scale of neutrino masses).

A second aspect which deserves a discussion regards the fact that general relativity does not provide equally good descriptions for all the gravitational systems we know. Indeed, while on the scales of Earth, solar system and cosmological observations the theory is able to predict correctly the observed phenomenology, there are some difficulties in understanding some features of galactic dynamics. In particular, the comparison of the observed rotation curves, *i.e.* the angular velocity of stars in spiral galaxies as a function of the radial distance, with the ones predicted using the baryonic content alone, shows a rather puzzling disagreement. The current paradigm to account for this is the so called Cold Dark Matter (CDM) one. In this framework, the matter content of the universe is modified: besides the baryonic component, *i.e.* the particles and fields described by the Standard Model (SM) of particle physics, some additional particles are postulated. These additional fields are typically obtained from extensions of the SM[1].

Despite the rather compelling theoretical motivations, the CDM paradigm is still not able to completely grasp the physics of galaxies (rotation curves, satellites, cusps, etc.). On the other hand, some modified theories of gravity, Milgrom's MOND [16] and its relativistic implementations (Bekenstein's TeVeS [17]) seem to be able to give better explanations to the observations [18]. These models represent a dramatic departure from general relativity on galactic scales (and in particular on the outer part of the galactic disk) and their viability in describing the other gravitational systems is not yet clear. For instance, they are not completely satisfactory in describing the dynamics of clusters of galaxies (see [19]) and gravitational lensing [20, 21].

Besides these issues, there are well known theoretical reasons that point out that general relativity is an incomplete theory. Perhaps the most stringent one is the presence of singularities within the theory. The singularity theorems by Hawking and Penrose [1, 3] state that, under rather general conditions, solutions of Einstein's equation will be singular. This means that the theory is losing its predictivity in describing the evolution of the universe, when singularities in the future are present, or that it is impossible to retrodict what was the initial state when a past singularity is present. In both cases, the presence of singularities is clearly signaling that general relativity is not a complete theory, and that it requires a suitable completion already at the classical level.

Gravitation and quantum theory

The obvious candidate to overcome this conceptual difficulty is the inclusion of quantum mechanics in the picture. We already know that quantum mechanics, and in particular quantum field theory (QFT), on which the Standard Model of particle physics is built, is able to describe with great accuracy the microscopic phenomena involving elementary particles. However, the understanding of the merging of gravity and quantum theory into a unique scheme is a major conceptual problem.

The troubles start already at the level of semiclassical gravity [22], *i.e.* when gravity is described by a classical metric, while matter fields are described by quantum theory. Perhaps the most striking theoretical discovery in this area is that black holes do emit thermal radiation [23, 24], and hence that they evaporate. This phenomenon is just the gravitational counterpart of a well known phenomenon in quantum field theory, *i.e.* particle creation in an external field (see [25] for a discussion). However, it was soon realized that Hawking radiation can be a threat for one of the

[1]It is interesting to notice that despite its huge success in accounting for the properties of elementary particles (see for instance [14] for an overview of the experimental data), the common attitude of particle physicists is that the Standard Model is just an effective field theory describing the low energy limit of some more fundamental theory. A lot of efforts has therefore gone in the direction of providing viable extensions of the SM by including larger symmetry groups (Grand Unified Theories, GUT) or by introducing supersymmetry (SUSY), like, for instance, in the Minimal Supersymmetric Standard Model (MSSM) [15].

pillars of quantum mechanics: the unitarity of the operator describing the evolution in time of the state of the system.

To be explicit, consider the standard picture of formation of a black hole from the collapse of a star. In the asymptotic past, apart from the small distorsion due to the (essentially classical) matter of the star, spacetime is well approximated by a portion of Minkowski spacetime, and the quantum fields can be safely considered to be in their vacuum state.

However, when the collapse takes place and the black hole is formed, after a transient, Hawking radiation is produced and the black hole starts evaporating. The issue of the final stage of evaporation is still not completely understood. There are some proposals like remnants and baby universes, which are logical possibilities that cannot be discarded *a priori*, even though they seem to be disfavored (see [26, 27, 28, 29]).

In the scenario when the evaporation of black hole is complete, which is for some aspects a more pleasant alternative, there is a very serious menace towards quantum mechanics, and in particular unitarity. In the specific situation of complete evaporation, all that is left over of the black hole is a region of almost flat spacetime filled with almost thermal radiation. The key point is that the final state, if it is truly a thermal state, is described by a mixed density matrix $\hat{\rho}_{th}$, while, in the asymptotic past, the initial state (essentially the Boulware vacuum) is described in terms of a pure density matrix.

In the case of black holes, then, a process taking a pure density matrix into a mixed density matrix seems to be realized: this is in clear contradiction with the existence of an underlying unitary evolution as in the standard quantum theory. Another way to state what happens is that it is impossible to map the asymptotic states, initial and final, one into the other by means of a unitary $S-$matrix operator [30, 31].

This major conceptual issue, called the information loss problem [32, 33], still does not have a definite answer. To give up unitarity by replacing the $S-$matrix by a superscattering operator, like the $\$-$matrix discussed in [31], does not seem to be a viable possibility [34]. On the contrary, there are encouraging evidences that a careful quantum gravitational treatment (implementing unitarity) should fix the problem [35, 36, 37, 38, 39].

The program of quantizing gravity, however, is still far from a satisfactory conclusion. It is by now pretty clear that the formulation of a theory of quantum spacetime requires some fundamental conceptual revolution. The two major candidates to accomplish the target of "quantizing gravity" are Loop Quantum Gravity (LQG, see for a review [40, 41]) and Superstring/M-Theory (see for instance [42, 43, 44]). Despite their important contributions to the formulation of a quantum theory of gravity (in the case of string theory suggesting also the possibility of having a unification of all the fields/interactions within a single structure), they are still not in the shape of a complete and satisfactory theory of quantum gravity.

In recent years, a new approach to these old problems has been gaining momentum and many authors have been advancing the idea that gravity could all in all be an intrinsically classical/large scale phenomenon similar to a condensed matter state made of many atoms [45]. In this sense gravity would not be a fundamental interaction but rather a large scale/number effect, a phenomenon emerging from a quite different dynamics of some elementary quantum objects. To support this perspective, many examples can be brought up, starting from the causal set proposal [46], passing to group field theory [47], quantum graphity models [48] and other approaches (see *e.g.* [49]) which will be mentioned in the rest of this thesis.

All these models and many others share a common scheme: they consider a fundamental theory which is not gravitational and examine, using different techniques often borrowed from condensed matter physics, how space, time and their dynamics could emerge in a suitable regime. In this sense a leading inspirational role also been played by another stream of research which goes under the name of "analogue models of gravity" [50]. These are condensed matter systems which have provided toy models showing how at least the concepts of a pseudo-Riemannian metric and Lorentz invariance of matter equations of motion can be emergent. For example, non-relativistic systems which admit some hydrodynamic description can be shown to have perturbations (phonons) whose propagation is described, at low energies, by hyperbolic wave equations on an effective Lorentzian geometry. While these models have not provided so far also an analogy for the dynamical equations of gravity, they do have provided a new stream of ideas about many other pressing problems in gravitation theory (see for example recent works on the origin of the cosmological constant in emergent gravity [51]).

Gravity as a collective phenomenon?

This idea is certainly fascinating, and it is not at all a wild speculation. As in the case of singularities, it is general relativity itself that is pointing out this possibility. As it is well known, black holes (more specifically their event horizon) behave as thermodynamical systems. In fact, it has been proven that the laws of black hole mechanics, *i.e.* the rules, dictated by general relativity, governing their evolution, can be put in correspondence with the laws of thermodynamics [52]. Similarly, it has been pointed out that some aspects of gravitational collapse closely resembles features of critical phenomena [53, 54].

This analogy with thermodynamics is not the only clue we have. The logic behind the thermodynamic interpretation of black hole mechanics has been pushed to the rather extreme consequence that the Einstein's equation can be interpreted as an equation of state [55, 56]. This fact strongly suggests that general relativity itself might be just a thermodynamical limit of some more fundamental theory, that the metric is no more fundamental than the pressure or any other state function for a thermodynamical system and that it is just a way to summarize the global behavior of a large number of degrees of freedom which are completely different in nature.

The work done during the PhD, which will be discussed in this thesis, can be put in this kind of "third way" approach to gravity, nowadays denoted as the "emergent gravity" program. The general point of view can be summarized in the idea that spacetime, and hence its geometry encoded in a pseudo-Riemannian manifold $(\mathscr{M}, g)$, is not a fundamental structure. Rather it is a kind of collective phenomenon, similar to the appearance of phonons in a Bose–Einstein condensate, which is generated by the underlying dynamics of the really fundamental constituents of spacetime.

There are already a number of encouraging progresses in this directions, which we have already mentioned and that will be discussed in the rest of the thesis, that are giving support to the whole picture of emergent gravity. Instead of describing them here, it is perhaps more important to connect this particular view of gravity as a collective phenomenon with other approaches to quantum gravity which are somehow "more orthodox".

One of the most intriguing approach to quantum gravity is (super-)string theory. Despite the difficulties which are still preventing us from formulating a phenomenologically viable low energy theory, *i.e.* a careful derivation of the standard model in full details from first principles, the theory itself represents a valuable attempt to give a unified quantum description of all the fields that we observe, from gravity to elementary particles. The technical tool which is allowing this is the string.

A more "conservative" approach, at least under a certain point of view, is represented by loop quantum gravity. In LQG, one is quantizing a theory which is almost[2] Einstein's theory of gravity, but makes use of different variables (essentially, instead of quantizing the metric, one quantizes the holonomy). In this picture, the microscopic structure of spacetime at the Planck scale is not a continuum geometry, bur rather a polymeric structure [40].

Both these approaches and their consequences are still not completely understood, at least as far as the macroscopic nature of spacetime is concerned. In both these approaches, the microscopic constituents are given (strings, loops) as well as their dynamics, at least in some approximation. However, this is not the end of the story. One has to combine them into macroscopic configurations, and then one has to see how the microscopic dynamics induces the laws describing the behavior of macroscopic spacetime (see [57] for additional comments).

There is an analogy, in theoretical physics, which is enlightening on this point. In condensed matter systems the laws governing the behavior of macroscopic objects are just obtained from the Schrödinger equations for the atoms. However, the description of macroscopic bodies is far from being a trivial manipulation of the Schrödinger equation. It might happen that qualitatively new phenomena appear at the macroscopic level which cannot be predicted from the behavior of a single microscopic constituent. To be more clear, let us consider the phenomenon of superconductivity [58, 59]. It involves the behavior of a large number of particles, it involves the mechanism of Cooper-pairing (the formation of bound states of pairs of electrons with opposite spins), and it possesses many features which cannot be understood if not in terms of collective phenomena. There is a qualitative difference in the behavior of the microscopic constituents, the atoms and the electrons, and the behavior of the macroscopic system, the superconductor.

There is no reason why this should not be the case for the gravitational interaction. Schematically, assume that the fundamental building block is a Planck-size tetrahedron, and that you are given the rules for gluing the tetrahedra in all the possible ways. This could be the case, for instance, of causal dynamical triangulations [60, 61, 62]. You will still have the problem of constructing a macroscopic manifold out of the microscopic building blocks and then of understanding how the rules of gluing the tetrahedra are translated into Einstein's equations. This procedure is by no means trivial and one has to take into account the possible complications due to the emergence of collective phenomena.

Again, this statement is fascinating, and again, it is not a wild speculation. There are some cases in which this seems to happen. For instance, in more sophisticated formulations of LQG, like Group Field Theory (GFT) [47], which is a kind of covariant implementation of the LQG program, it

[2]The theory introduces an additional constant, the Barbero–Immirzi parameter, which is related to a topological term which is not relevant at the classical level.

has been shown that in some suitable regime the behavior of the theory resembles those of matter fields living on an effective spacetime [63]. In another very well known setting, the conjectured AdS/CFT correspondence [64] between weakly coupled string theory on a five-dimensional Anti-de Sitter (AdS) spacetime and conformal field theories in a four dimensional Minkowski spacetime, one might look at the duality in the direction CFT to AdS by saying that the strongly coupled limit of the CFT shows the emergence of gravitational-like phenomena (in an appropriate sense).

Summarizing the ideas just presented, we can say something about the general picture of emergent gravity. The general goal of the program is to understand how the qualitative idea that gravity is a sort of collective phenomenon can be realized in concrete models, and, most ambitiously, to give a precise derivation of general relativity from a more fundamental theory. The objective would be to understand what features are necessary for a theory to be able to produce general relativity in a suitable limit. This could be accomplished not only by means of the construction of working toy models but also by formulating no-go theorems (when possible) which will useful to select good ideas from hopeless models. It is important to keep in mind that this program is not in opposition to other approaches to quantum gravity like strings or loops. It is somehow more general, including them as particular cases, as the previous discussion should have clarified.

Plan of the thesis

This thesis is not a review of all the approaches to emergent gravity. Rather, it is the exposition of (part of) the work done during the PhD. As such, it will be largely influenced by the particular perspective of analogue models [50], and it will be devoted to the discussion of the specific questions raised by them. Nevertheless, there are deep relationships with other approaches, due to the great generality of the questions one is trying to answer to, *i.e.* whether gravity could be an emergent phenomenon.

There are two main issues raised by analogue models:

- what is the most general geometrical structure that can emerge?
- What are the features of its dynamics?

Accordingly, the presentation is split into two parts, which however should be seen just as different faces of the same problem.

The first part, from chapters 2 to chapter 5, is devoted the discussion aspects of emergent spacetimes, *i.e.* to the exploration of the effective geometry (and its symmetries) that naturally arises in analogue models. In chapter 2, the reader can find some introductory material concerning the Bose–Einstein condensates (BEC) analogue models, and some remarks about the generalization to multi-components systems. The key lesson is that what is called Finsler geometry, and not Riemannian geometry, is the geometrical framework describing emergent spacetimes.

In Chapter 3, the status of Lorentz symmetry (which is tightly related with the concepts of pseudo-Riemannian geometry) is discussed. A number of different approaches to Lorentz symmetry violation/deformation will be discussed. Moreover, some scenarios involving relativity beyond the setting of special relativity are considered, as well as the necessity of going beyond the tools of pseudo-Riemannian geometry of Minkowski spacetime (with some hints pointing again on Finsler geometry).

In Chapter 4 some basic material on Finsler geometry is presented. It is intended to give a rather sketchy but self-consistent overview of the concepts of this generalization of Riemannian geometry, its peculiarities, as well as the precise conditions under which a Finsler structure reduces to a Riemannian one.

In Chapter 5, finally, it will be shown how Finsler geometry can be used both within the context of analogue models and in general emergent spacetimes, as well as in the discussion of Lorentz violating theories. It will be shown that the relation between ideas like modified dispersion relations and rainbow geometry (energy dependent metrics) can be rigorously interpreted in geometrical terms by means of Finsler structures. The difficulties of a Finsler scenario will be discussed too, in particular concerning the existence of a single spacetime geometry within a Finslerian scheme and the fate of Lorentz invariance at low energy.

The second part of the thesis, from Chapter 6 to 8, concerns the investigation of some dynamical aspects of emergent gravity. Chapter 6 contains the discussion of some important points concerning the emergence of gauge invariance, the induced gravity program, the Weinberg–Witten theorem and some technical points that will be used especially in chapter 8.

In chapter 7 we will discuss the case of gravitational dynamics of BEC systems, showing that despite the fact that they are definitely not good analogues for gravitational theory, they do provide some useful insights into certain aspects of gravitational theories.

Leaving the non-relativistic setup of the BEC, in Chapter 8, it will be discussed a toy model defined in four dimensional Euclidean space which makes possible to emerge a fully diffeomorphism invariant and, in an appropriate sense, background independent theory of scalar gravity (most notably Norström's theory) with Lorentzian signature, suggesting that time might be an emergent concept as well.

Chapter 9 is instead devoted to the concluding remarks and to the summary of the concepts presented in this work, as well as to an overview of future perspectives.

The original contributions

To help the reader, it might be useful to highlight very concisely the original contributions to the discussion of some of the problems which will be mentioned in the course of the thesis.

As it will be discussed in chapter 3, the issue of Lorentz invariance and the viability of Planck-scale modified dispersion relations have taken the attention of the community for several years. Nevertheless, the geometrical interpretation of non-quadratic dispersion relation was not understood. As it will be discussed in chapter 5, extending the analysis of [234], there is a direct relation between modified dispersion relations and (four dimensional, pseudo-)Finsler geometry. The relevance of this relationship is not only taxonomical: it shows how questions about the fate of Lorentz invariance can have a very precise geometrical formulation even without introducing concepts like non-commutative geometry. Furthermore, it shows also how a violation of Lorentz invariance does not necessarily imply the splitting of spacetime into space and time. Instead, within certain limits, it can be understood of a unique anisotropic geometrical structure of the four dimensional spacetime itself.

The relationship between the Finsler structures and the mass of the particles is complicated: particles with different masses will see different structures. It is therefore important to confront this phenomenon with the familiar mechanism for generation of masses in the standard model, i.e. the Higgs phenomenon. This has been done in [241]. The main result is that the Higgs mechanism does introduce some additional difficulties in a Finsler scenario. These will be discussed in chapter 5. The main point is that in a Finsler scenario there seems to be a tension between the spacetime symmetries (the geometry of spacetime) and the internal symmetries (the geometry of the gauge fields). This poses serious questions about the compatibility of Finsler structures with spontaneously broken gauge theories.

In the second part of the thesis we will embark into the analysis of emergent gravity systems. Here the exploration is at very early stages and there are a lot of different aspects to be understood yet. Following [283], in chapter 7, we will use Bose–Einstein condensates to build up a model of emergent gravity which we will use to extract some lessons for gravity. To date it is the only analogue model addressing the issue of the dynamics of gravity in an explicit way. Several lessons can be learned, especially about the difference between the notions of locality given at the macroscopic and microscopic layers of the theory, as well as about the possible origin for a small cosmological constant.

In [295] (here discussed in chapter 8) we have addressed two points: the issue of emergence of time and diffeomorphism invariance. Usually, time and diffeomorphism invariance are concepts which are introduced at the very beginning in all the models for gravity which have been proposed so far. We have shown that they are not necessarily needed at a fundamental level. Rather, they can be emergent properties of the effective degrees of freedom. This discussion clearly shows how, in the emergent gravity program, each of the several assumptions at the basis of gravitational theories and of spacetime geometry in a more general sense must be considered carefully.

Part I

Kinematics

Chapter 2

Preliminaries

2.1 Acoustic spacetimes

The fact that gravity is a relatively weak force makes very difficult to observe directly (in a laboratory) the behavior of matter in strong gravitational fields. There are some examples of phenomena involving strong gravity in astrophysics, like neutron stars, neutron star binaries, black holes. However these are situations in which we are not able to control all the details of the system. Despite the huge amount of work done by theorists, phenomena like Hawking radiation and cosmological particle production cannot be observed in actual gravitational systems (with the exception of the cosmic microwave background radiation (CMB) which is a rather indirect test of cosmological particle creation). Therefore, it is impossible to test the theoretical ideas behind these phenomena, at least if we use the ordinary gravitational field, *i.e.* spacetime geometry.

A crucial point must be kept in mind: these effects involve just the fact that quantum fields live on curved spacetimes, the latter not being necessarily solutions of some sort of Einstein equations. Therefore, one could imagine to realize *effective curved spacetimes* in systems which could be realized in laboratory, thus opening the possibility of detecting and testing directly some features of QFT in curved spacetimes.

In fact, since the work of Unruh [65], it has been shown that it is possible to simulate analogue of curved spacetimes, even containing analogues of black holes, using condensed matter systems, like, for instance, perfect fluids. The general idea behind this fact is that, when a fluid is flowing with a given velocity field, sound waves are dragged by the fluid flow. Hence, if a supersonic flow is realized in some region, sound waves cannot go upstream. This is in analogy with what happens with trapped regions in general relativity. Using this very simple idea it can be easily realized that some sort of sonic analogue of black holes spacetimes can be realized. This general expectation can be formalized in a rigorous mathematical theorem [66], which shows how in some situations the properties of sound propagation can be exactly described by a curved pseudo-Riemannian metric.

Theorem Consider an inviscid, barotropic and irrotational fluid. Sound waves, *i.e.* linearized perturbations around a given fluid configuration, are solutions of a Klein Gordon equation for a massless scalar field living on a curved metric,

$$g_{\mu\nu} = \frac{\rho}{c} \left(\begin{array}{ccc} -(c^2 - v^2) & \vdots & -\vec{v} \\ \cdots & \cdot & \cdots \\ -\vec{v} & \vdots & \mathbb{I}_3 \end{array} \right) \tag{2.1}$$

This metric tensor is called the *acoustic metric*. It depends algebraically on the properties of the fluid flow: the local density ρ, the velocity field $\vec{v}$, and the speed of sound c. Therefore, the dynamics for this metric is not described by some sort of Einstein equations written in terms of some curvature tensors. Rather, the acoustic metric is a solution of the fluid equations, *i.e.* the continuity equation and Euler equation.

During the last years this result has been extended to other condensed matter systems, like non-linear electrodynamic systems, gravity waves, etc.. In all these systems, under certain circumstances, an effective Lorentzian spacetime emerges, even though the fundamental system is described by equations written in a Galilean spacetime. A rather complete survey on the topic of analogue models can be found in [50]. For a detailed discussion of ^{3}He models, see [67]. In what follows only a small class of analogue models will be considered. Nevertheless, they will be enough to describe the main ideas of analogue models and emergent spacetimes.

2.1.1 Horizons, black holes

The aforementioned theorem allows us to use the same language we use in the case of spacetimes in general relativity to the emerging geometrical structures in fluids (see [68] for an extended analysis). However, the fact that these analogue models are defined through systems which are described by Newtonian mechanics implies that they have some specific features.

The manifold structure behind these analogues will be typically Galilean spacetime $\mathcal{M}_G = \mathbb{R}^3 \times \mathbb{R}$ in which the fluid flow takes places. There can be some exceptions, like flows in thin pipes in which spacetime is effectively two-dimensional, as well as the introduction of topological defects. The manifold and the acoustic metric g specify the acoustic spacetime $(\mathcal{M}_G, g)$.

Given that the Galilean spacetime does have a privileged direction, given by the Galilean time, some global properties of the geometry will apply to the acoustic spacetime. In particular, some causality properties, like for instance stable causality, are inherited from the underlying physical spacetime. Therefore these analogue models are not able to simulate spacetimes with pathological causal structures (*e.g.* time machines).

It is worth mentioning also that metrics of the form (2.1) are very special: it is immediately realized that spatial sections ($t = const$) are conformally flat, the conformal factor being related to the density ρ. This puts restrictions on which kind of geometries can be realized with this class of analogues. For instance, the Kerr solution cannot be simulated since it does not admit flat spatial slices. (However, see [69] for the possibility of simulating an equatorial slice of Kerr in an acoustic analogue.) Despite the fact that this class of metrics is not the most general four dimensional metric that one can construct, it is rich enough to contain interesting examples, *e.g.* black hole spacetimes.

A *sonic point* is defined to be a point in which the speed of the fluid is equal to the speed of sound. Sonic points play an important role. Indeed, considering the case of steady flows, the vector $\partial/\partial t$ is a Killing vector of the acoustic metric (2.1). On a sonic point, this Killing vector, associated to time translations, has vanishing norm, *i.e.* it is a null vector, while in supersonic regions it becomes spacelike (of course with respect to the acoustic metric). Therefore, using the language of general relativity, supersonic regions are ergoregions, while their boundaries, made of sonic points, are ergospheres. It is worth noticing that while horizons usually separate a region of supersonic flow from a region of subsonic flow, there can be some sort of *extremal* situations in which the horizon separates two subsonic (or two supersonic) regions [70].

Acoustic spacetimes naturally acquire the definition of trapped surfaces as in general relativity. They can be defined in a very clear way from the point of view of the fluid flow. Let us consider a closed two-surface. If the fluid flow is directed inward everywhere on the surface, and if the fluid flow is supersonic, then no sound wave can escape from the surface: it will be dragged inside. In this situation, the surface is said to be outer-trapped. If, conversely, the supersonic fluid flow is directed outward, the surface is said to be inner-trapped (or anti-trapped). Of course, a more rigorous definition of trapped surfaces based only on the local properties of geometry should be given in terms of the expansion of bundles of null geodesics emanating from the surface itself [1, 3]. However, the definition given in terms of the properties of the fluid flow is perfectly equivalent, providing at the same time a more immediate physical intuition.

Having defined trapped surfaces, it is immediate to define acoustic trapped regions as those containing outer-trapped surfaces. The boundary of a trapped region is called the acoustic apparent horizon. The (future) event horizon is defined as in general relativity as the region of the acoustic manifold which is not in the causal past of the asymptotic infinity, or, equivalently, the region from which sound waves cannot escape. An analogous definition can be given for the past event horizon.

This very concise discussion shows how acoustic spacetimes can mimic important structures that we find in general relativity, at least as far as only kinematical features are considered. A more detailed discussion on the causal structure of acoustic spacetimes can be found in [70].

2.2 BEC generalization

Analogue models based on perfect fluids have the avantage of being rather easy to construct, at least in principle. However, in practice, real fluids are never perfect, and hence instead of the rather simple continuity and Euler equations for perfect fluid one should use the full Navier–Stokes equations. These will be relevant especially where fluid flow will begin to develop turbulence. A second reason why the perfect fluid analogues have a limited interest is that they are essentially classical. The quantum mechanical effects of motion of molecules are completely hidden by the overwhelmingly large thermal fluctuations. Therefore, they are not fit to simulate quantum field theories in curved spacetimes.

An alternative to perfect fluids is represented by Bose–Einstein condensates. A BEC is a particular phase of a system of identical bosons in which a single energy level has a macroscopic occupation number. They are realized by using ultra-cold atoms (see [71] and [72] for recent reviews). Under these extreme conditions, quantum phenomena become a key ingredient in determining the macroscopic properties of the system. The analysis of these systems, therefore, opens a new area of investigation for quantum fields in curved spacetime [73, 74], as we will discuss later.

In this section we will give a rather concise and self-contained description of the formalism required for the description of a BEC. In particular, a discussion of second quantization which will be used later in the thesis to present a generalization of the standard scenario. For complete expositions see [71, 72, 75].

2.2.1 Second quantization and the mean field approximation

The formalism of second quantization allows us to work in full generality with many body systems without having to deal explicitly with a wavefunction. This allows us to treat in a very easy way even those situations in which the number of particles is not constant. It is based on the annihilation-creation algebra introduced in the discussion of the spectrum of the harmonic oscillator in quantum mechanics. Given that we will be interested in bosonic systems, this discussion is sufficient for our purposes. A formalism for fermions can be developed along the same lines [75].

As said, the basic ingredient is the algebra of creation and annihilation operators:

$$[\hat{b}, \hat{b}^\dagger] = 1. \tag{2.2}$$

The number operator $\hat{N}$ is defined to be:

$$\hat{N} = \hat{b}^\dagger \hat{b}. \tag{2.3}$$

The commutation relations are:

$$[\hat{b}, \hat{N}] = \hat{b}, \qquad [\hat{b}^\dagger, \hat{N}] = -\hat{b}^\dagger. \tag{2.4}$$

The number operator is Hermitian, and its eigenstates satisfy

$$\hat{N}|n\rangle = n|n\rangle, \qquad n \in \mathbb{R}. \tag{2.5}$$

Due to the algebra of the operators, it is easy to see that:

$$\hat{b}|n\rangle = \alpha|n-1\rangle, \tag{2.6}$$

where the coefficient α is fixed to be:

$$|\alpha|^2 = ||\hat{b}|n\rangle||^2 = n. \tag{2.7}$$

In order for the space to be a Hilbert space, the norm of any state must be positive definite. Consequently, the eigenvalues of the number operators are indeed all the natural numbers. The state $|0\rangle$ is the vacuum state, and it is annihilated by the annihilation operator b:

$$\hat{b}|0\rangle = \mathbf{0}. \tag{2.8}$$

If one considers an orthonormal basis of single particle states labelled by some index k, described by wavefunctions $u_k(\mathrm{x})$, obeying some orthonormality conditions

$$\int_V d^3x\, u_k^*(\mathrm{x}) u_h(x) = \delta_{kh}, \qquad \sum_k u_k^*(\mathrm{x}) u_k(\mathrm{y}) = \delta^3(\mathrm{x}-\mathrm{y}), \tag{2.9}$$

one can construct the multi-particle states associating to each single particle state a copy of the above mentioned algebra, $\hat{b}_k, \hat{b}_k^\dagger$, together with the assumption that different copies of the algebra are mutually commuting, to form what is called the (bosonic) Fock space. It contains all the states of the form:

$$|n_1, ..., n_k\rangle, \tag{2.10}$$

which corresponds to configurations with n_1 particles in the (single particle) state 1, n_2 particles in the (single particle) state 2, etc.. The algebra of the creation and annihilation operators guarantees that the states are automatically symmetrized, as required by Bose–Einstein statistics.

The field operators are defined as:

$$\hat{\Psi}(\mathrm{x}) = \sum_k u_k(\mathrm{x}) b_k, \qquad \hat{\Psi}^\dagger(\mathrm{x}) = \sum_k u_k^*(\mathrm{x}) b_k^\dagger, \tag{2.11}$$

which are obeying the commutation relations:

$$[\hat{\Psi}(\mathrm{x}), \hat{\Psi}(\mathrm{y})^\dagger] = \delta^3(\mathrm{x} - \mathrm{y}), \tag{2.12}$$

with the other commutators being zero. The interpretation of these operators is clear: the operator $\hat{\Psi}^\dagger(\mathrm{x})$ creates a particle at the point x, while the operator $\hat{\Psi}(\mathrm{x})$ destroys a particle at the point x. This formalism is particularly useful in discussing the properties of many particle systems. All the observables quantities, like energy, angular momentum, interactions between particles, charges, etc. are translated in terms of second quantized operators defined through the field operators $\hat{\Psi}, \hat{\Psi}^\dagger$.

The formalism just described can be used to study a BEC of many atoms in a box of volume V, in the dilute gas approximation. In this limit it is possible to describe the atoms via a second-quantized field operator

$$\hat{\Psi} = \frac{1}{\sqrt{V}} \sum_k \hat{a}_k e^{ik\cdot \mathrm{x}}, \tag{2.13}$$

whose evolution is encoded in the Hamiltonian $\hat{H}_0$

$$\hat{H}_0 = \int \hat{\Psi}^\dagger(\mathrm{x}) \left(-\frac{\hbar}{2m}\nabla^2 - \mu + \frac{\kappa}{2}|\hat{\Psi}|^2 \right) \hat{\Psi}(\mathrm{x}) \mathrm{d}^3\mathrm{x}, \tag{2.14}$$

which generates the operator equation

$$i\hbar \frac{\partial}{\partial t}\hat{\Psi} = [\hat{H}_0, \hat{\Psi}] = -\frac{\hbar^2}{2m}\nabla^2\hat{\Psi} - \mu\hat{\Psi} + \kappa|\hat{\Psi}|^2\hat{\Psi}. \tag{2.15}$$

It is a non linear Schrödinger-like equation for the field operator $\hat{\Psi}$.

The operator $\hat{\Psi}$ has dimension $L^{-3/2}$: the quantum average of its modulus square on a given state represents the number density of atoms. The mass m is the mass of the atoms. The energy μ is the chemical potential and has dimension ML^2T^{-2}. The constant κ, of dimension ML^5T^{-2}, represents the strength of the two-bodies interaction between atoms. We neglect higher order contributions to the second quantized Hamiltonian.

It is important to clarify a point. Despite the fact that (2.14) does possess a global $U(1)$ invariance

$$\hat{\Psi}(\mathrm{x}) \to e^{-ia}\hat{\Psi}(\mathrm{x}), \qquad a \in \mathbb{R}, \tag{2.16}$$

this symmetry is not required by consistency of the theory, as it happens for instance with unitarity[1]. The $U(1)$ symmetry is usually enforced because one is often interested in cases in which the total number of particles $\hat{N}_{tot}$ is a conserved quantity:

$$[\hat{N}_{tot}, \hat{H}_0] = 0. \tag{2.17}$$

The total number operator will be the Noether charge of this $U(1)$ global symmetry. This is the most common situation: typically the bosons considered in BEC experiments are atoms, whose number is certainly conserved in the condensation process. Nevertheless, there are cases, discussed later in this thesis (chapter 7), in which there is no reason to require such a conservation law, either because the system is open, or because the bosons which are condensing are collective modes, not necessarily having some conserved global charge.

The description of the condensation mechanism within the second quantization formalism is usually treated by means of a mean field approximation, *i.e.* it is assumed that the ground state of the system $|\Omega\rangle$ is such that the field operator $\hat{\Psi}$ develops a non-zero vacuum expectation value,

$$\langle\Omega|\hat{\Psi}(x)|\Omega\rangle = \psi(x), \tag{2.18}$$

where $\psi(x)$, called condensate wavefunction, is a classical complex field playing the role of the order parameter of the phase transition associated to the condensation. This vacuum expectation value motivates the splitting of the field operator $\hat{\Psi}$

$$\hat{\Psi}(x) \approx \psi(x)\mathbb{I} + \hat{\chi}(x), \tag{2.19}$$

where ψ is associated to the condensate, while $\hat{\chi}$ is associated to the non-condensed fraction. As a first approximation, then, the properties of the condensate can be described by means of this representation.

It turns out that the mean field approximation is not the most rigorous method to treat a BEC system. In particular, the particle-number-conserving approach developed in [76, 77] has been proven to give more accurate predictions for the physical properties of the condensate in settings in which the number of atoms, N, is fixed. This dimensionless quantity, then, is used to expand systematically the equation for the evolution of the operator $\hat{\Psi}$, in powers of $N^{1/2}$ (see also the box at the end of section 2.3). While this method was shown to provide a more accurate description of BEC systems, it has also shown that the mean field approximation gives already very good predictions for quasi-static configurations (for a discussion and references, see [76, 77]).

In all the situations in which we are interested in, we consider an idealized case where the confining potential is almost constant, both in space and in time. Moreover, we neglect boundary effects due to the finite size of the trap used in practice to confine the condensate, and we assume that all the other experimental parameters are weakly time-dependent. In this case, then, the mean field approximation is well motivated.

[1] One could think, motivated by our derivation of the field operators that this $U(1)$ is related to the $U(1)$ symmetry of the wave function in quantum mechanics. This is not the case. Unitarity is enforced by the Hamiltonian being a Hermitian operator.

2.2.2 Gross–Pitaevski and its fluid interpretation

In this framework, the lowest order approximation for the description of the dynamics of the condensate is obtained by replacing (2.18) into the field equation (2.15), neglecting completely the fluctuation part. The resulting equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi - \mu\psi + \kappa|\psi|^2\psi, \tag{2.20}$$

is called the Gross–Pitaevski equation. It describes the properties of the field ψ, the condensate wavefunction, and it can be used to predict a number of properties of the condensate.

By a suitable redefinition it can be put into a fluid form. Indeed, using the so-called Madelung representation

$$\psi(t, \mathrm{x}) = n_c^{1/2}(t, \mathrm{x})e^{-i\theta(t,\mathrm{x})/\hbar}, \tag{2.21}$$

it takes the form of a system of two coupled differential equations for the fields n_c, θ:

$$\frac{\partial n_c}{\partial t} - \nabla n_c \cdot \frac{\nabla\theta}{m} = 0, \tag{2.22}$$

$$\frac{\partial\theta}{\partial t} = -\frac{\hbar^2}{2m}\frac{\nabla^2 n_c^{1/2}}{n_c^{1/2}} + \frac{1}{2m}(\nabla\theta)^2 - \mu + \kappa n_c. \tag{2.23}$$

The first equation has the form of a continuity equation. It is just the statement of the conservation of the Noether current associated with the $U(1)$ invariance of the system we discussed above. The fact that the quantity $\nabla\theta/m$ has the dimensions of a velocity makes possible to define the velocity of the condensate as:

$$\vec{v} = -\frac{\nabla\theta}{m}. \tag{2.24}$$

The velocity field is obtained as a gradient of a potential. Hence, the flow of a BEC is automatically irrotational. Of course, this is not forbidding the existence of singular vortex lines.

The second equation requires some discussion, in particular the first term in the right hand side. The term

$$V_{quantum} \equiv -\frac{\hbar^2}{2m}\frac{\nabla^2 n_c^{1/2}}{n_c^{1/2}}, \tag{2.25}$$

is called the quantum potential. It is proportional to $\hbar^2$, to the Laplacian of the number density of the condensate, and it is suppressed by the mass of the atoms. Therefore, as long as the condensate is not too inhomogeneous, it can be safely neglected. This regime is called the *hydrodynamic regime*, since equation (2.23) reduces to the Bernoulli equation for a perfect fluid. Taking its gradient, one obtains again the Euler equation.

This proves an important fact. In the hydrodynamic regime, the equations for the condensate are identical to those for a irrotational, barotropic perfect fluid. Hence, the theorem proved for the sound waves does apply also in this case: the excitations over the condensate will be propagating over an effective acoustic metric, having the same form discussed in the previous section.

The crucial difference making the BEC so appealing is that now the excitations moving on this effective metric are essentially described by a quantum field. Therefore, a BEC analogue model can simulate quantum theory of a scalar field over a curved effective spacetime.

2.3 Phonons

In the next section we will give a brief overview of the possibilities that this result offers. Before that, it is important to give a careful analysis of the physical properties of the excitations, because they will be relevant in chapter 7. To this purpose, we will consider the case of homogeneous condensate, for which n_c, θ are constants. It is easy to see that, while the global $U(1)$ allows us to set $\theta = 0$, the equations of motion give $n_c = \mu/\kappa$.

To evaluate the spectrum of the excitations, one has to find the equation for the field $\hat{\chi}$. This is obtained from the equation (2.15), inserting the mean field ansatz, and neglecting the nonlinear terms which are defining the interactions among the excitations. The desired equation reads:

$$i\hbar\frac{\partial\hat{\chi}}{\partial t} = -\frac{-\hbar^2}{2m}\nabla^2\hat{\chi} + \mu\hat{\chi} + \mu\hat{\chi}^\dagger. \tag{2.26}$$

This equation tells that the time evolution mixes the creation and annihilation operators. Hence $\hat{\chi}$ *is not* the field representing the propagating particles. To find this latter, it is convenient to rewrite the previous equation in momentum space:

$$i\hbar\frac{\partial\hat{a}_k}{\partial t} = \left(\frac{\hbar^2k^2}{2m} + \mu\right)\hat{a}_k + \mu\hat{a}^\dagger_{-k}. \tag{2.27}$$

The propagating modes will be described by operators $\hat{b}_\mathrm{k}$ for which

$$i\hbar\frac{\partial\hat{b}_\mathrm{k}}{\partial t} = \omega(\mathrm{k})\hat{b}_\mathrm{k}. \tag{2.28}$$

These operators can be found from the operators $\hat{a}$ by means of a so-called Bogoliubov transformation.

2.3.1 Bogoliubov transformations

Let us go back to the case of a single copy of the creation-annihilation algebra:

$$[\hat{a}, \hat{a}^\dagger] = 1.$$

Taking a linear combination, it is possible to define the operator:

$$\hat{b} = \alpha\hat{a} + \beta\hat{a}^\dagger.$$

The requirement that the commutation relation $[\hat{b}, \hat{b}^\dagger] = 1$ holds does imply that

$$|\alpha|^2 - |\beta|^2 = 1. \tag{2.29}$$

Such a transformation is called a Bogoliubov transformation, and α, β are the Bogoliubov coefficients. It is a linear mapping between two copies of the creation-annihilation algebra. Without loss of generality the coefficients α, β can be chosen to be real, and to be parametrized by a single real variable r as:

$$\alpha(r) = \cosh(r), \qquad \beta(r) = \sinh(r), \tag{2.30}$$

whence the inverse transformation:

$$\hat{a} = \cosh(r)\hat{b} - \sinh(r)\hat{b}^\dagger. \tag{2.31}$$

An important property of the Bogoliubov transformations is that the Fock vacua will be inequivalent:

$$\hat{a}|0_a\rangle = 0, \qquad \hat{b}|0_a\rangle \neq 0. \tag{2.32}$$

and, in particular:

$$\langle 0_a|\hat{N}_b|0_a\rangle = |\beta|^2 = \sinh^2(r). \tag{2.33}$$

This construction can be generalized to the interesting case of larger number of copies of the creation and annihilation algebras. For more details see [75].

2.3.2 Bogoliubov dispersion relation

The technique of Bogoliubov transformations is enabling us to find the eigenmodes of the Hamiltonian for the excitations. In our case, we need to use a Bogoliubov transformation of the form:

$$\hat{b}_{\mathrm{k}} = \alpha(k)\hat{a}_{\mathrm{k}} + \beta(k)\hat{a}^\dagger_{-\mathrm{k}}. \tag{2.34}$$

Notice that, even though there are only two coefficients for each k, these Bogoliubov transformations are mixing the algebras for the mode k and −k. The fact that there are only two coefficients is due to conservation of momentum. The operator $\hat{b}_{\mathrm{k}}$ destroys an excitation of momentum k: this can be realized only through annihilation of a mode with momentum k or through creation of a mode with momentum −k. Finally, rotational symmetry of the condensate allow us to say that the Bogoliubov coefficients will depend only to the norm of the momentum, not on its direction.

Using these facts, it is straightforward to get:

$$\frac{\alpha\beta}{\alpha^2+\beta^2} = \frac{\mu}{2(\epsilon(k)+\mu)}, \qquad \epsilon(k) = \frac{\hbar^2k^2}{2m} + \mu, \tag{2.35}$$

$$\omega^2(k) = \left(\frac{\hbar^2k^2}{2m}\right)^2 + \frac{\mu\hbar^2}{m}k^2. \tag{2.36}$$

These equations are determining the Bogoliubov coefficients which are defining the propagating modes, quasi-particles which are commonly called phonons, as well as their spectrum. The first equation has a nice interpretation. For small values of k, $\alpha\beta \approx 1$, which means that the propagating modes are collective excitations which are made by atoms and holes, while, for $k \gg k_*$, $\alpha\beta \to 0$, which means that the phonons are basicly atoms, at short distances.

The dispersion relation for the phonons is telling that they have a gapless spectrum,

$$\lim_{k\to 0}\omega(k) = 0, \tag{2.37}$$

and that at low energy the dispersion relation is linear:

$$\omega(k) \approx c_s\hbar k, \tag{2.38}$$

with the speed of sound defined by

$$c_s^2 = \frac{\mu}{m} = \frac{n_c\kappa}{m}. \tag{2.39}$$

The fact that the quasi-particles are gapless is a direct consequence of an important fact. The condensation mechanism, with the selection of a preferred value for the condensate wavefunction, represents a breaking of the global $U(1)$ symmetry associated with number conservation. The phonons are nothing else than the Goldstone bosons associated to this symmetry breaking, and hence they must be gapless (for a discussion of Goldstone theorem and its relevance in condensed matter systems see, for instance, [78]). The linearity of the spectrum at small momenta means that we can see the phonons as relativistic massless particles, with the speed of sound playing the role of the speed of light. In some sense we see that in the infrared, long-range regime there is an approximate Lorentzian symmetry emerging out of a Galilean system. Of course this matches with the observation we have already made with the analysis of the Gross–Pitaevski equation.

This result generalizes the situation of sound waves in perfect fluids. This emergent Lorentz invariance of the spectrum is only an approximate symmetry, not an exact one. The dispersion relation has an additional term, whose interpretation is very easy from the physical point of view: in the far ultraviolet, in the short distance regime, the phonons have the dispersion relation of the nonrelativistic atoms. This is the counterpart in the energy spectrum of what is happening at the level of field operators with the Bogoliubov coefficients: in the same kinematical regime, the phonic field operators are indistinguishable from the atomic operators.

2.3.3 The healing length

The scale which is ruling the crossover is the healing length scale. Indeed, in the Bogoliubov dispersion relation, the quartic term becomes comparable to the quadratic term for

$$k_*^2 = \frac{4\mu m}{\hbar^2} = \frac{1}{L^2}. \tag{2.40}$$

The healing length plays several roles in the physics of the BEC. As we have seen, it determines the properties of the dispersion relation: it gives the energy scale at which the low energy Lorentz invariance of the spectrum is broken. It represents the energy scale at which the phononic regime connects to the atomic regime. This is also suggested by the behavior of the Bogoliubov coefficients, as we have seen: above the energy scale set by the healing length, the quasi-particle operators become closer and closer to the atomic operators.

The healing length does represent also the typical scale of the dynamics of the condensate [72, 75]. To see this, it is instructive to consider the case in which the condensate is confined in the half space defined by $x > 0$. Assuming that the system is invariant under translations along the transverse directions y, z, and that it is static, we can parametrize the condensate wave-function as:

$$\psi(x) = n_c^{1/2} f(x), \qquad n_c = \mu/\kappa. \tag{2.41}$$

The boundary conditions are $f(0) = 0$ and $f(x \to \infty) = 1$. In this case, the equation for the condensate becomes:

$$L^2 \frac{d^2 f}{dx} - f(x) + f^3(x) = 0, \tag{2.42}$$

with L being the length scale defined in (2.40). The solution of this equation with the given boundary conditions is:

$$f(x) = \tanh\left(\frac{x}{\sqrt{2}L}\right), \tag{2.43}$$

as a direct calculation can show. This solution shows that the healing length L is the length scale needed to the condensate to pass from the "cut" at the boundary $\psi(0) = 0$ to the bulk value $\psi(x \to \infty) = n_c^{1/2}$.

2.3.4 The depletion factor

The fact that the ground state is defined to be the vacuum of the Fock space of quasi-particles rather than particles has a very important consequence. We have said that the condensation can be understood as a macroscopic occupation number of a single state. However, there is always a fraction of atoms which is not in the condensed phase. This can be seen by inspection of the quantity

$$\mathfrak{n}_{\mathrm{k}} = \langle \Omega | \hat{a}_{\mathrm{k}}^{\dagger} \hat{a}_{\mathrm{k}} | \Omega \rangle, \tag{2.44}$$

which is the (expectation value of the) number of atoms in the state of momentum k, when the ground state is $|\Omega\rangle$, the vacuum of the quasi-particles. By means of the Bogoliubov transformation between the particle and quasi-particle basis it is immediate to realize that:

$$\mathfrak{n}_{\mathrm{k}} = |\beta(\mathrm{k})|^2. \tag{2.45}$$

The *depletion factor*, *i.e.* the ratio between the total number of particles not in the condensate and the total number of particles, is obtained from these expectation values by summing on the states:

$$d = \frac{\Delta N}{N} = \frac{V}{(2\pi)^3 N} \int d^3k |\beta(\mathrm{k})|^2. \tag{2.46}$$

Without entering into the details of the calculation (see [75]) it turns out that

$$d = \frac{8}{3} \left(\frac{N a^3}{\pi V} \right)^{1/2}, \tag{2.47}$$

where the scattering length a is related to the coupling constant κ by:

$$\kappa = \frac{4\pi\hbar^2}{m} a. \tag{2.48}$$

It is interesting to note that the stronger is the interaction (encoded into the scattering length) the larger is the depletion factor. Therefore, its smallness, at fixed density, is guaranteed by the weakness of the interaction.

In particular, if we introduce the length $l = (V/N)^{1/3}$ which represents the size of the typical cell occupied by just one atom, the smallness of the depletion factor is due to the smallness of the ratio a/l, *i.e.* to the fact that the interaction of the atoms is very short range with respect to the typical distance between atoms. Therefore, the smallness of the depletion factor is related also to the diluteness of the system of bosons.

2.3.5 The Bogoliubov–de Gennes formalism

In the derivation of the Gross–Pitaevski equation (2.20) the contribution of the particles out of the condensate phase has been completely neglected. Consequently, the Gross–Pitaevski is an accurate description of the properties of the condensate if and only if the depletion factor is very small.

An improvement to the Gross–Pitaevski formalism is obtained by introducing systematically these corrections. The equation obtained in this way, which is an improved mean field description based on (2.18), is called the Bogoliubov–de Gennes equation.

It is useful to briefly sketch the derivation. Starting from (2.15), and inserting (2.18) in it, one obtains:

$$\frac{i\hbar}{\partial t}(\psi + \hat{\chi}) = \left(-\frac{\hbar^2\nabla^2}{2m} - \mu\right)(\psi + \hat{\chi}) + \kappa(\psi^* + \hat{\chi}^\dagger)(\psi + \hat{\chi})(\psi + \hat{\chi}). \tag{2.49}$$

Taking the vacuum expectation value of this expression, we obtain the modified equation for the condensate

$$i\hbar\frac{\partial\psi}{\partial t} = \left(-\frac{\hbar^2\nabla^2}{2m} - \mu\right)\psi + \kappa|\psi|^2\psi + 2\kappa\mathfrak{n}\psi + \kappa\mathfrak{m}\psi^*, \tag{2.50}$$

where we have introduced the anomalous density and mass

$$\mathfrak{n} = \langle\hat{\chi}^\dagger\hat{\chi}\rangle, \qquad \mathfrak{m} = \langle\hat{\chi}^2\rangle, \tag{2.51}$$

and where we have neglected the corrections due to $\langle|\hat{\chi}|^2\hat{\chi}\rangle$. This equation is known as the Bogoliubov–de Gennes equation. It is an improvement with respect to the Gross–Pitaevski since it is able to take into account the fact that the depletion factor is small but non vanishing.

The equation for the field $\hat{\chi}$ can be obtained subtracting this equation from the equation (2.49):

$$i\hbar\frac{\partial\hat{\chi}}{\partial t} = \left(-\frac{\hbar^2\nabla^2}{2m} - \mu\right)\hat{\chi} + 2\kappa n_c\hat{\chi} + \kappa\psi^2\hat{\chi}^\dagger + 2\kappa(|\hat{\chi}|^2 - \mathfrak{n})\psi + \kappa(\hat{\chi}^2 - \mathfrak{m})\psi^* + O(\hat{\chi}^3). \tag{2.52}$$

This formalism will be exploited in chapter 7 to show how it is possible to simulate some sort of gravitational dynamics in Bose–Einstein condensates.

$1/N$ **Expansion in BECs** It is useful to give a description of Bose–Einstein condensation in dilute gases by means of an asymptotic expansion of the field operator $\hat{\Psi}$. The dimensionless parameter used to define such an expansion is the total number of atoms N. Bose–Einstein condensation corresponds to the fact that the occupation number of a state is of order N, while the other levels are almost empty. This can be included into the mean field approximation by making the number N to explicitly appear in the decomposition of the field operator

$$\hat{\Psi} = \psi + \sum_{l=2}^{\infty} N^{-n/2}\psi_{(n)} + \sum_{n=1}^{\infty} N^{-n/2}\hat{\chi}_{(n)}.$$

The term $\psi_{(1)}$ is absent since it amounts to a redefinition of the condensate field ψ. This decomposition can be inserted into the field equation (2.15). In the limit in which $N \to \infty$, (2.15) formally reduces to the Gross–Pitaevski equation for the mean field (2.20), while the corrections due to the excitations decouples. A more careful analysis shows that in this limit, the field equation for the many body system effectively generates a hyerarchy of equations, one for each order of the expansion. The equation corresponding to $N^{-1/2}$ is the equation of motion for the first order excitations, which we have used to define the phonons. Higher order equations contain all the details of the dynamics of the quasi-particles and their back-reaction on the condensate.
Of course, this description is only approximate, and gives good results as long as N is large, *i.e.* when the non-condensate fraction is very small.

2.4 BEC-based analogue models

The discussion of the physical properties of the BECs clearly shows their potential relevance in simulating quantum phenomena in curved (acoustic) spacetimes. Indeed, the low energy effective field theory of the phonons is a quantum theory of a scalar field on the acoustic spacetime defined by the condensate. Despite being a very simple case, it is enough to reproduce, at least in principle, interesting phenomena of particle creation in curved spacetime. The two main effects which have been considered in the past are Hawking radiation from an acoustic black hole and cosmological particle creation in an expanding acoustic Friedman–Robertson–Walker metric.

The most important part in designing an analogue model is to control the physical properties of the condensate itself in such a way to reproduce the desired acoustic metric. In the case of the perfect fluid we have discussed above, the procedure is pretty straightforward: by controlling the velocity profile one can obtain the various properties of the acoustic metric. In the case of the condensate, an alternative way has been envisaged to achieve this goal. In the case of the BEC the scattering length, related to the interatomic interactions, can be modified by acting directly on the condensate, by means of the so-called Feshbach resonance [72]. By using this technique, one can make the speed of sound a position dependent function. Therefore, instead of keeping fixed the speed of sound and changing the velocity profile, in order to have subsonic and supersonic regions it is enough to keep the flow's velocity fixed and to change the speed of sound.

The possibility of simulating black hole spacetimes and hence to have some sort of grasp on the Hawking effect has been put forward since the works [73] (see also [74]). Similarly, it has been proposed that BEC could be used to realize analogue of expanding Friedman–Robertson–Walker (FRW) spacetimes, and hence to have the possibility of check in a laboratory phenomena which are typical of inflationary scenarios, like particle creation [79, 80]. In this case, it becomes manifest how important is the possibility of controlling the scattering length in realizing a curved acoustic metric.

The method of Feshbach resonance is so effective that one could realize physical situations of rather exotic phenomena which do not have an analogue in the standard approach to quantum field theories, namely, signature changes events [81]. By tuning the value of the scattering length, and in particular making it negative, one can immediately realize that $c_s^2 < 0$, which means that the acoustic metric becomes a Riemannian one (signature $(++++)$).

BEC-based analogue models, therefore, do represent a very nice opportunity to discuss many different phenomena which are of interest for theoretical physics. There are two fundamental lessons to be learned from the study of these systems. First of all, testing phenomena of quantum field theories in curved spacetimes which have been only predicted and never measured directly, given the difficulty of detecting them in strong gravitational fields generated by the astrophysical sources, is an important achievement *per se*, making theoretical prediction subject to experimental verification.

An additional reason is related to the so called trans-Planckian problem. In the derivation of Hawking radiation and the spectrum of inflationary perturbations, one is implicitly assuming that the effective field theory description (QFT in curved spacetime) is holding up to arbitrarily high energies/arbitrarily small scales (for a discussion and references see, for instance [82]). This is certainly an untenable assumption. Almost certainly, at sufficiently small scales, the EFT picture will break down (see for instance the discussion about the breaking of locality [83, 84, 85]), higher dimensional operators will become more and more relevant up to the point that the fundamental theory of spacetime (quantum gravity/strings/...) might be required to perform the calculations.

The trans-Planckian problem, then, opens the door to this (mostly unknown) physical effects on low energy predictions. In this respect, it is important to understand how much the predictions of Hawking's spectrum and particle creation in an inflationary model are affected by modifications to the UV behavior of the theory. In the case of BEC, we are in complete control of the trans-Planckian physics of the system, *i.e.* the properties of the model below the healing length.

As it has been shown, phonons do not have an exact relativistic dispersion relation: the dispersion relation receives corrections which are more relevant the higher is the energy, with the healing length scale playing the role of the separation scale between the relativistic branch and the trans-phononic branch of the spectrum. The robustness of Hawking radiation and particle creation against high energy modified dispersion relations has been discussed in several works [86, 87, 88, 89, 90, 91, 92, 93, 94]. Similarly, while a signature change event would lead to an infinite particle production, the presence of high energy modifications of the spectrum regularizes the divergencies, predicting a finite result [81].

The potentialities of these analogue systems are further highlighted by some works where the issue of the backreation of the emitted radiation on spacetime (*i.e.* the precise way in which Hawking radiation leads dynamically to the evaporation of the black hole) is explicitly considered [95, 96].

While experimental data are still missing, rather important progresses have been done recently on both the experimental side and on the numerical side. In fact, very recently it has been discussed the realization of a BEC analogue of a black hole [97]. Even though this is certainly encouraging, still this is not enough for the purposes of detecting directly Hawking quanta. On the other hand, the production of Hawking radiation from an acoustic black hole in a BEC has been detected in numerical simulations [98]. Finally, some numerical estimates on the effects of cosmological particle creation have been done in [99].

It is clear that the underlying physics, for these analogue models, is non-relativistic: some Lorentz-violating effects are gradually turned on as the energy is increased. Even though this is a possibility, this might not be the case for "real" spacetime, and it might be that other kind of UV completions of our low energy effective field theories would lead to unexpectedly large effects on the low energy predictions of Hawking spectrum and cosmological particle production. To decide this matter, of course, a consistent theory of quantum gravity is needed.

2.5 Multi-component systems: normal modes analysis

The analysis done so far has involved the discussion of systems where there is just a single field describing the excitations. This may suffice to investigate phenomena like particle creation, as we have discussed previously. However, it is important to understand how generic is the emergence of a single pseudo-Riemannian metric when several components are present. In addition to the obvious interest in understanding the conditions necessary to have the emergence of a Lorentzian structure, it turns out that this analysis gives indications about what happens when this is not the case. It is instructive to discuss two cases, where the physical features are clear. These will be paradigmatic for the cases in which more fields are involved.

Accidental symmetries It is important to clarify some terminology: according to the common definition, an accidental symmetry is an exact symmetry which is an outcome of the requirements of Lorentz invariance, gauge invariance and renormalizability of a given Lagrangian, once the field content and the gauge group are specified (see [100], section 12.5 for more details). For instance, in the Standard Model, baryon and lepton numbers are accidental symmetries. Here we refer to the Lorentz invariance emerging in some kinematical regime of analogue models as an accidental symmetry even if it is only an approximate one. Despite this, it is interesting to note that it is indeed an accidental approximate symmetry according to the above definition. It is not imposed a priori as an invariance of the Lagrangian. Rather, it is a consequence of other specified symmetries via the Goldstone's theorem.

2.5.1 Birefringent crystals

The first case to be discussed is the propagation of light in a crystal. On scales larger than the size of the cell, a crystal is a homogeneous medium: all its points are equivalent, having the same physical properties. However, not all the directions are equivalent: in general a crystal is anisotropic, the anisotropy being related to the preferred directions selected by the fundamental cell.

Therefore, the propagation of signals, like sound waves or light, is affected by the symmetry properties. In the case of the propagation of light (for the propagation of sound waves see for instance [101]), the Maxwell equations in vacuum are replaced by effective equations where (see [102] for a complete treatment) the overall electromagnetic properties of the material are included in the fields $\mathbf{D}, \mathbf{H}$, related to the electric and magnetic fields $\mathbf{E}, \mathbf{B}$ by means of constitutive relations,

$$\mathbf{D} = \left.\frac{\partial F}{\partial \mathbf{E}}\right|_{B=const}; \qquad \mathbf{H} = \left.\frac{\partial F}{\partial \mathbf{B}}\right|_{E=const}, \tag{2.53}$$

where F is the free energy of the system. While in general these relations can be very complicated, in the case of weak fields the constitutive relations can be approximated by linear relations:

$$D_i = \varepsilon_0 \varepsilon_{ij}(\omega) E_j, \qquad H_i = \mathfrak{m}_{ij}(\omega) B_j. \tag{2.54}$$

The tensors $\epsilon_{ij}, \mathfrak{m}_{ij}$ are given by the components of the matrix of second derivatives of the free energy as a function of the EM fields. The matrix $\mathfrak{m}_{ij}(\omega)$ is simply the inverse of the magnetic permittivity

$$\mathfrak{m}_{ij}(\omega) = \frac{1}{\mu_0} (\mu^{-1})_{ij}(\omega). \tag{2.55}$$

The conditions (2.54) are very general: we can simplify the analysis by assuming that we are dealing with a medium for which the magnetic permittivity tensor is just the identity tensor[2]. Note also that one can include *a priori* the possibility that the tensors $\varepsilon, \mathfrak{m}$ are frequency dependent, *i.e.* dispersive, as it is the case in practice. From now on it is assumed that the medium is not dispersive.

[2]Actually, this is the most common situation. Nevertheless, there are materials which do have interesting magnetic properties. See for instance [103].

Under these assumptions it is possible to perform an orthogonal transformation of coordinates so that ε is diagonal:

$$\varepsilon = \mathrm{diag}(\epsilon_1, \epsilon_2, \epsilon_3), \tag{2.56}$$

where the three principal values are not necessarily identical. Once the rotation is done, the analysis of Maxwell equations follows pretty straightforwardly. Considering the plane wave ansatz

$$\mathbf{E}(t,x) = \mathbf{e}\, e^{-i(\omega t - \mathbf{k}\cdot\mathbf{x})}, \qquad \mathbf{H}(t,x) = \mathbf{h}\, e^{-i(\omega t - \mathbf{k}\cdot\mathbf{x})}, \tag{2.57}$$

and defining the vector refractive index vector $\mathbf{n}$ as:

$$\mathbf{n} = \frac{c}{\omega}\mathbf{k}, \tag{2.58}$$

Maxwell's equations can be rewritten as:

$$\mathbf{h} = \mathbf{n}\times\mathbf{e}, \qquad \mathbf{d} = -\mathbf{n}\times\mathbf{h}. \tag{2.59}$$

Substituting the first in the second, one gets

$$\mathbf{d} = n^2\mathbf{e} - (\mathbf{n}\cdot\mathbf{e})\mathbf{n}, \tag{2.60}$$

which must be compared with the constitutive relation giving $\mathbf{d}$ in terms of $\mathbf{e}$. This leads to the following equation:

$$(n^2\delta_{ik} - n_i n_j - \varepsilon_{ij})e_j = 0. \tag{2.61}$$

In order for this equation to have a nonzero solution for e_j, the determinant of the linear operator acting on the polarization of the electric field must be zero. So that we obtain what is called the Fresnel equation:

$$\det(n^2\delta_{ik} - n_i n_j - \varepsilon_{ij}) = 0. \tag{2.62}$$

This equation is an equation for the refractive index vector $\mathbf{n}$. The shape of the space of solutions of this equation depends on the shape of the tensor ε, which encodes the EM properties of the medium we are considering. For instance, if $\varepsilon_{ij} = \varepsilon\delta_{ij}$ we have an isotropic medium, and we can see that $\mathbf{n}$ lives on the sphere of radius $\varepsilon^{1/2}$. The geometric interpretation is obviously given in term of a single Riemannian metric, leading to a single pseudo-Riemannian structure, with the speed of light in vacuum replaced by the speed of light in the crystal.

In the case of uniaxial crystals, we have that $\varepsilon_{ij} = \mathrm{diag}(\varepsilon_1, \varepsilon_1, \varepsilon_2)$. This leads to a Fresnel equation which is factorizable:

$$(n_x^2 + n_y^2 + n_z^2 - \varepsilon_1)(\varepsilon_1 n_x^2 + \varepsilon_1 n_y^2 + \varepsilon_2 n_z^2 - \varepsilon_1\varepsilon_2) = 0. \tag{2.63}$$

Therefore, this equation gives rise to two bilinear forms, with which one can define two pseudo-Riemannian metrics, thus leading to a bi-metric theory. The two photon's polarizations are travelling at different speeds. This phenomenon is called bi-refringence.

In the case of biaxial crystals, we have three principal axis with three distinct eigenvalues, the factorization of the Fresnel determinant into two quadratic terms is no longer possible: we have to solve an algebraic equation of the fourth degree:

$$-\varepsilon_3 n_z^2(n_x^2 + n_y^2 + n_z^2) - \varepsilon_2(\varepsilon_2(\varepsilon_3 - n_x^2 - n_y^2) +$$

$$-\varepsilon_3(n_x^2+n_z^2)+n_x^2(n_x^2+n_y^2+n_z^2))+$$
$$-\varepsilon_2(-\varepsilon_3(n_y^2+n_z^2)+n_y^2(n_x^2+n_y^2+n_z^2))=0. \tag{2.64}$$

In the three dimensional vector space where the vector **n** lives, the surface defined by this equation is a complicated self-intersecting quartic surface. The features of this surface are the origin of the interesting optical properties of this class of crystals [104].

Notice the relation between the symmetry group of the crystal and the corresponding geometrical structure: the more anisotropic is the crystal, the more we break Lorentz invariance in the analogue model. In the most general case, Lorentz invariance is not an approximate symmetry in these analogue systems, even at low energies.

2.5.2 Two components BEC

Another interesting class of analogue systems with a rather rich phenomenology is given by the two components BEC systems [105, 106]. In these systems, one has two species of bosons, rather than only one as in the BEC discussed previously.

These two components can be thought to be two different hyperfine levels of the same atom. The two components are interacting, typically by laser coupling, which is inducing transitions between the two components. The analysis of the spectrum of the quasi-particles is considerably more difficult. Nevertheless, in the hydrodynamic regime (when the quantum potential is neglected), the spectrum can be obtained analytically. For the detailed calculation, which is based on a suitable generalization of the Madelung representation, see [105].

The final outcome of the analysis is pretty clear. The Hamiltonian describing the 2BEC contains a number of parameters: the masses of the atoms, the chemical potentials, the strengths of the atomic interaction, and the coupling describing the induced transitions between the components. According to the values of these constants, the phononic branch of the spectrum shows different behaviors. In particular, one can distinguish three geometrical phases: for generic values of the parameters, the analogue geometry is not Riemannian, but rather Finslerian[3]. If some tuning is made, it is possible that, instead of this Finslerian structure, a bi-metric (pseudo-Riemannian) will describe the propagation of the phonons (which are of two kinds, in these 2BEC systems). Finally, if more tuning is performed by the experimenter on the various constants describing the properties of the system, the two families of phonons perceive the same Lorentzian geometry at low energy. In this specific corner of the parameter space Lorentz invariance as an approximate symmetry is recovered at low energies.

Of course, this clean description holds only in the hydrodynamic limit, when the quantum potentials are neglected. As in the case of single BEC, the higher the energy, the large will be the contribution of the corrections to the dispersion relations (see (2.36)). Again, as we shall prove later, dispersion can be seen as the manifestation of Finsler geometry. Therefore, in 2BECs there are two ways in which we can see Finsler geometry emerge: first, by a generic choice of the physical properties of the system, and secondly by the appearance of dispersion.

[3]Given that the concepts of Finsler geometry will not be defined until chapter 4, the reader can use an operative definition of Finsler geometry (which will be discussed much later in this thesis) in terms of the dispersion relation. If the dispersion relation is not quadratic ($g^{\mu\nu}k_\mu k_\nu = m^2$ for some $g^{\mu\nu}$), the corresponding geometrical structure is Finslerian rather than Riemannian.

2.6 From analogue models to Finsler geometry

The analysis of analogue models has shown that Lorentz invariance is an accidental approximate symmetry in a rather large class of condensed matter systems. This feature, however, is not generic. If we take a general system with many components, extending the analysis of the previous section, the normal modes analysis [107] shows that each mode will have, in general, different speed of propagation. In the typical case, having n modes, the dispersion relation can be expressed as:

$$\omega^{2n} = Q_{i_1 \ldots i_{2n}} k^{i_1} \ldots k^{i_{2n}}. \tag{2.65}$$

In general this equation defines a (possibly self-intersecting) algebraic hypersurface of degree n in momentum space. Given the homogeneity property $k \to \lambda k, \omega \to \lambda \omega$, this hypersurface is nothing else than the generalization of a cone.

The microscopic details of the system are encoded in the tensor Q. According to this, there are different emergent geometrical structures. Tuning the microscopic parameters, it might happen that the tensor Q factorizes into a product of metric tensors:

$$Q_{i_1 \ldots i_{2n}} = g^{(1)}_{i_1 i_2} \ldots g^{(n)}_{i_{n-1} i_n}, \tag{2.66}$$

In this case, the emergent spacetime structure is pretty clear: there is an effective multimetric structure, where each mode is propagating on a different light cone determined by the corresponding metric structure. By additional tuning, it might be possible to reduce this multi-metric structure to a single Lorentzian structure, when $g^{(1)} = g^{(2)} = \ldots = g^{(n)}$. We can say, then, that the emergence of an effective Lorentzian spacetime is not generic: some conditions have to be satisfied in order for a single metric to be recovered. This is often translated in terms of symmetryes of the underlying model. For instance, anisotropies in crystals are directly related to the appearance of birefringence.

Of course, even in the most symmetric case, the emergent Lorentz invariance is only an approximate symmetry of the spectrum. For instance, in condensed matter systems the underlying spacetime symmetry is Galilean invariance, which has rather different properties than Lorentz invariance. This fact is often contained into corrections to the lowest order/low energy equations for the perturbations. In systems like BEC, the quantum potential gets more important the larger is the energy of the phonon, modifying the linear dispersion relation in the way described by the Bogoliubov dispersion relation (2.36). Dispersion is in general a feature that we should expect in analogue spacetimes. Of course, dispersion is not compatible with a notion of pseudo-Riemannian geometry.

However, the discussion made so far should have clarified that in analogue models the most general situation is the one in which the tensor Q is not factorized. In this case, while the dispersion relation is still homogeneous (hence there is no dispersion), it is generally anisotropic: modes moving in different directions will propagate with different speeds. Again, this feature cannot be described by Lorentzian geometry.

It is interesting that, despite that dispersion and anisotropic propagation do not have a direct interpretation in terms of Lorentzian geometry, they do have a geometrical interpretation in terms of a suitable metric generalization of Riemannian geometry, namely Finsler geometry. Finsler geometry will be extensively discussed in chapters four and five. We can anticipate the main result: the most general metric geometrical structure emerging from analogue models is Finsler geometry.

It should be said that analogue models are not just an exercise in mathematical physics. In fact, analogue models can provide relevant insights and useful ideas about those situations where spacetime is emergent from a complicated underlying quantum gravity dynamics. In the next chapter it will be shown that the ideas we have already discussed in the area of analogue models have a counterpart in many theoretical frameworks designed to describe the semiclassical limit of quantum spacetime near the Planck scale, as well as their phenomenological consequences.

Chapter 3

Special Relativity beyond the Planck scale

3.1 Is Lorentz invariance a fundamental symmetry?

The modern description of spacetime is based on the concepts of pseudo-Riemannian geometry: spacetime locally looks like a piece of Minkowski spacetime, and the gravitational field, which is dynamical, is encoded in the metric tensor and its curvature. General relativity explains with great accuracy how this happens at a macroscopic level, from millimeter to cosmological scales (with the interesting exception of some aspects of galactic dynamics).

In typical experiments in a laboratory we can neglect curvature, and treat spacetime as an effective fixed Minkowski spacetime over which the matter fields are evolving, including the gravitational field, if needed, in the Newtonian limit. In general this approximation holds whenever the radius of curvature is much larger than the typical size of the experiment. For instance, this is the typical situation for high energy physics experiments, where gravitational effects are completely negligible with respect to the other phenomena involved. In that case a non-dynamical flat Minkowski spacetime is the background spacetime over which the SM of particle physics is defined.

Lorentz invariance is deeply rooted in this picture. Being more explicit, the isometry group of Minkowski spacetime is the Poincaré group, containing translations as well as the Lorentz group. Poincaré invariance plays a crucial role in particle physics, being the spacetime symmetry enforced onto the Lagrangian, while at the same time providing the classification of fields in terms of its irreducible representations.

Given that the role of translations is somehow of secondary interest, in what follows we will mainly be interested in the invariance under the Lorentz group, which is the part of the group deserving more careful analysis. For analogous reasons, the discrete symmetries of time reversal and space inversion (T,P) and charge conjugation (C), and their combination (particularly CPT) will be considered in less detail, even though they play an important role in the general topic of the extension of the standard model.

It is worth stressing that the origin of Minkowski spacetime is tied to the implementation of the relativity principle, *i.e.* the equivalence of all inertial frames, and of the requirement that the speed of light is an invariant quantity under a change of frame. We will re-examine this statement later, showing that the role of the speed of light is less important than what one would imagine. For the moment, the simple derivation of Minkowski spacetime from these two postulates will suffice.

As a side remark, it should be mentioned that, even though in Minkowski spacetime geometry is nondynamical, general relativity can be seen as the gauge theory of the Lorentz group. In an appropriate sense, the Lorentz symmetry is promoted from global symmetry to gauge symmetry (see, for instance, the Einstein–Cartan–Sciama–Kibble formalism [108, 109, 110, 111]), following rather closely the procedure followed by "ordinary" gauge theories. Therefore, renouncing to Lorentz invariance has deep implications also at the dynamical level, when gravity is introduced as another force, and this should be appropriately taken into account (see, for instance, [112, 113]).

Contrary to the group of rotations, which is compact, $SO(3,1)$ is a non-compact group. This has consequences at different levels. Among them, the fact that even though we have tested Lorentz invariance for relatively small values of the boost parameters we cannot conclude anything about the invariance under arbitrarily large boosts.

That something new might happen for large boost parameters might be guessed by thinking about Lorentz contraction. This effect corresponds to the transformation of large spatial distances in a reference frame into ultra short spatial distances in another reference frame, sufficiently boosted with respect to the first. This picture clearly does not take into account a simple argument coming from naive analysis of covariant quantum gravity. At small distances, quantum fluctuations of the metric are expected to be so large that the very definition of a classical metric tensor seems to be impossible [114, 115].

The standard point of view is that at scales comparable with the Planck length, quantum gravitational effects will be so relevant that the standard picture of flat Minkowski spacetime will break down. This fact is rather remarkable: sticking to the picture of pseudo-Riemannian geometry, one should expect that the smaller is the spacetime volume considered, the better flat geometry should be an approximation to the actual curved geometry, and all the gravitational effects should disappear. However, this is an essentially classical argument and discards completely any form of quantum fluctuations: it is precisely because of these fluctuations that spacetime at smallest scales is expected to be very different from a standard manifold.

Clearly this expectation is in contrast with Lorentz invariance as we know it. In special relativity, the *linearity* of Lorentz transformations implies that it is impossible within the formalism to introduce a privileged length scale: if we take two vectors $v, \lambda v$ with $\lambda \in \mathbb{R}$, the transformed vectors will be $\Lambda v, \lambda \Lambda v$, $\forall \lambda$, while the 4-length, which is a relativistic invariant, is homogeneous in λ. Consequently, there is nothing distinguishing large distances from small distances. This also shows that any modification of spacetime geometry based on extensions of Riemannian geometry, where the metric tensor still plays a role in specifying physical distances, will suffer from this problem about introducing scales distinguishing short distance physics from large distance physics.

Given that spacetime geometry is pseudo-Riemannian on large scales, since we do believe that at length scales shorter than the Planck one some fundamentally new behavior should emerge, taking Lorentz invariance too seriously up to arbitrarily high energy scales (or, equivalently, large boost parameters) would be contradictory, given that Poincaré invariance in its standard presentation is defined through isometries of Minkowski spacetime. What should emerge is that if Minkowski spacetime is replaced by some other structure, at short distances, then the Poincaré group should be replaced by a corresponding group of invariance.

Therefore, besides the obvious interest in the testing Lorentz symmetry as a symmetry of nature *per se*, the detection of Lorentz-violating effects in high energy phenomena could be relevant for the investigation of the microscopic structure of spacetime. Indeed, there are quite a number of quantum gravity scenarios in which Poincaré invariance is broken: string field theory VEVs [116], some brane-world scenarios [117], spacetime foam models [118], semiclassical spin-network calculations in Loop Quantum Gravity [119, 120], non-commutative geometry studies [121, 122], as well as, more recently, group field theory motivated scenarios [123] and Hořava's proposal of an anisotropic Lifshitz point theory of quantum gravity [124].

Of course all these ideas are just hints: they cannot be considered as proofs that Lorentz invariance is not a true fundamental symmetry of nature. Nevertheless they are certainly suggestions that Lorentz violation could be a theoretical possibility for most of the scenarios we are presently considering to deal with quantum gravity, even though when abandoning Lorentz invariance a number of difficulties must be overcome. As we have mentioned briefly, this would have important implications for both the implementations of a relativity principle and for the dynamics of spacetime as well. We will come back to the consequences for the relativity principle later in this chapter.

In the past a lot of work has been done in this sense, both from the theoretical side in the direction of Lorentz symmetry deformation (deformed/doubly special relativity, DSR) [125, 126, 127], as well as in the direction of building particle physics models with Lorentz Invariance Violation (LIV) effects included [116, 128, 129, 130, 131]. These theoretical developments have been accompanied by the analysis of the constraints coming from experiments and astrophysical observations [132, 133, 134, 135], with increasing accuracy as long as new tests are proposed.

For the present discussion, it is not essential to follow closely a specific theoretical model which could give a solid ground on these speculations. Rather, only a generic feature is explicitly considered: it will be assumed that there is a modified structure of spacetime, hopefully geometrical in nature, from which one could extract an effective field theory from which to draw predictions for low energy phenomenology through a suitably formulated effective field theory. The general point of view, then, is that spacetime, even in vacuum, due to its complicated microscopic dynamics, acquires some sort of nontrivial optical properties. As such, spacetime cannot be expected to display Lorentz invariance.

3.2 Tests of Lorentz invariance

Having discussed the theoretical ideas motivating Lorentz violating effects, it is worth giving a brief overview of the concrete tests of Lorentz invariance. A thorough review can be found in [134]. See also [136] for more recent discussion.

Before entering into some details, a remark is in order. The concept of test theory is playing a crucial role in this kind of analysis. In order to test a particular symmetry of a given theory one has to introduce a suitable extension of it including all the terms which are invariant under the remaining symmetries (and eventually satisfying other requirements, like renormalizability). The parameters of this extension will control the deviations from the original theory. Measurements of physical observables will be important in putting constraints on these parameters and therefore enabling a quantitative statement about the validity of the particular symmetry at hand.

3.2.1 Robertson–Mansouri–Sexl formalism

The most immediate framework parametrizing kinematical deviations from Lorentz invariance is the so called Robertson–Mansouri–Sexl (RMS) framework [137, 138]. The existence of a preferred frame, where the propagation of light is isotropic, is assumed. The transformations to other frames are of the form:

$$\begin{aligned} t' &= (t - \vec{\epsilon} \cdot \vec{\mathrm{x}})/a, , \\ \vec{\mathrm{x}}' &= d^{-1}\vec{\mathrm{x}} - (d^{-1} - b^{-1})(\vec{v} \cdot \vec{\mathrm{x}})\vec{v}/(v^2) + \vec{v}t/a, \end{aligned} \tag{3.1}$$

where a, b, d are functions of the relative velocity $\vec{v}$ between the two frames and $\vec{\epsilon}$ is an arbitrary vector encoding the particular synchronization procedure used. In the case of special relativity, with Einstein's clock synchronization, $a = b^{-1} = (1 - v^2/c^2)^{1/2}$ and $d = 1$.

In this formalism nothing is said about the dynamics of the fundamental constituents of which rods and clocks are made. These coordinate transformations are referring only to the relations between the readings of assigned sets of rods and clocks. Changing sets, nothing guarantees that the new coefficients a, b, d will agree with the ones of the old set. To deal properly with this difficulty one should be able to incorporate this framework into a dynamical one, like the standard model extension we will consider later.

In common situations where the RMS frameworks is applied, speeds are small and hence, instead of the general parametrization given above, one can use a simplified one

$$\begin{aligned} a &= 1 - (\alpha - 1/2)v^2 + o(v^2), \\ b &= 1 + (\beta + 1/2)v^2 + o(v^2), \\ d &= 1 + \delta\, v^2 + o(v^2), \end{aligned} \tag{3.2}$$

and constraints on α, β, δ are provided by the various experiments. This parametrization is particularly effective for the class of experiments which have been considered since the early days of special relativity (*e.g.* Michaelson–Morley, Ives–Stilwell, Kennedy–Thorndike, see [139, 140, 141, 142] for the most recent constraints).

3.2.2 Lorentz Invariance violation in effective field theories

The formalism briefly outlined in the previous section is modeled after the low velocity tests which are based on interferometric techniques. Clearly, today we can count on a whole new class of experiments: from high energy phenomena at particle colliders to the observation of ultra high energy cosmic rays. It turns out that it is possible to use the phenomena involved in them in such a way to test Lorentz invariance.

The first step for such a program is to formulate a suitable test theory. Since the standard model of particle physics is formulated as a quantum field theory in Minkowski spacetime, and since we already know that this effective field theory description gives a good account for the physical properties of elementary particles and their interaction, the most immediate test theory would be an effective field theory extending the SM by the inclusion of Lorentz violating terms. Of the three basic assumptions on which the Lagrangian of the SM is based, only gauge invariance and power counting renormalizability should be kept, while Lorentz invariance is of course dropped.

This program has been endeavored by Colladay and Kostelecky [129] about ten years ago. For the sake of clarity, only the main points will be discussed.

The idea in the background of the formulation of the theory is that Lorentz invariance, while present in the microscopic theory, is somehow spontaneously broken. For instance, it might happen that some tensor fields acquire some nonvanishing vacuum expectation value (as it seems to happen in string field theory [116]) which results in a Lorentz violating vacuum. This assumption might seem of little use, but it happens to have a role in giving the effective field theory a solid formal ground. In fact, if Lorentz invariance is spontaneously broken, one can work with an effective field theory obeying the standard axioms, namely microcausality, positivity of energy, energy-momentum conservation, so that the standard quantization techniques can be used.

Besides this technical point, the extension is formulated by asking that all the terms included are gauge invariant (under the full $SU(3) \times SU(2) \times U(1)$) and power counting renormalizable. This means that all the operators introduced must have mass dimension of four or less. With the matter content fixed to be given by the standard leptons and quarks (spinors of mass dimension 3/2), a Higgs doublet (scalar, mass dimension 1) and the required gauge bosons (gauge vectors, mass dimension 1), the extension is uniquely fixed. The complete list of all the operators to be included is not particularly relevant for the present discussion. The interested reader can find the complete Lagrangian, as well as a more detailed analysis, in [129].

An important fact must be kept in mind. Given the constraint of renormalizability, the coefficients of the additional operators are dimensionless or have positive mass dimension. For instance, one has operators like

$$k_\mu \bar{\Psi} \gamma^\mu \Psi, \tag{3.3}$$

where k_μ is a (constant) vector with dimensions of a mass[1].

This fact implies that all the dimension three and four operators are not explicitly suppressed by a large mass, for instance the Planck mass associated to quantum gravity, and hence their smallness with respect to the Lorentz invariant operators is not a priori guaranteed. It is implicitly assumed that the microphysics of Lorentz symmetry breaking would be able to provide all these coefficients with a suppression factor given by some powers of the dimensionless ratio $r = M_{EW}/M_{Pl}$ between the electroweak scale and the Planck scale. This fact cannot be decided within the model itself: only a full fledged theory of the mechanism of Lorentz violation can be able to explain this kind of "naturalness" problem.

This extension of the Standard model has been studied thoroughly, and the various coefficients have been constrained with various methods. A rather complete and up-to date list of the constraints can be found in [143].

[1]Unless otherwise specified, we will use units in which $\hbar = c = 1$. However, sometimes we will introduce explicitly the speed of light c, to make some points more transparent.

3.2.3 Dispersion relations

The possibility of using high energy particles, especially of astrophysical origin, has opened new possibilities in testing Lorentz invariance. The most immediate object that can be tested is the dispersion relation, *i.e.* the relation between the energy and the momentum of a particle. The relativistic dispersion relation

$$E^2 = m^2c^4 + p^2c^2, \tag{3.4}$$

is the only one compatible with Lorentz invariance.

A manifestation of Lorentz violation could be therefore the modification of the dispersion relation to a more general form. It is generally assumed that there is a preferred frame, taken to be the one of the cosmic microwave background, in which the dispersion relation has the form:

$$E^2 = m^2c^4 + \sum_{n\geq 1} \frac{c^n}{E_P^{n-2}} F_{(n)}^{i_1...i_n} p_{i_1}...p_{i_n}, \tag{3.5}$$

where the Planck energy E_P has been introduced to make the tensors $F_{(n)}$ dimensionless. The velocity c has been kept in the formalism as a mere constant. It does not represent necessarily the speed of propagation of some signal, or some limit speed as in special relativity.

Moreover, let us stress that all the coefficients appearing in this dispersion relation can depend on the particular kind of particle considered. The most simple example of such a situation is the case in which the dispersion relation has the form:

$$E^2 = m_a^2c^4 + \eta_a p^2c^2, \tag{3.6}$$

where η_a is a dimensionless coefficient depending on the species, labeled by the letter a. In this case, the limit speed would be given by $\eta_a^{1/2}c$, and not c, and hence it would be different for species with different η_a.

Another manifestation of a deformed dispersion relation is the phenomenon of dispersion: wavepackets with different energy might move with different speeds. This is what happens, for instance, to visible light moving in a medium: different colors travel with different speeds. In general, the origin of dispersion is the fact that the group velocity is in general a function of the energy:

$$v_g = \frac{\partial E}{\partial p}(E). \tag{3.7}$$

Consequently, wavepackets having different energies, emitted at the same time from a given source, will arrive at different times at the detector, the difference in the time of flight being related to the difference in the group velocities (and of course to the distance travelled). For instance, if the dispersion relation of photons were,

$$E^2 = p^2c^2 + \frac{\eta_4}{M_P^2c^4}p^4, \tag{3.8}$$

an easy estimate shows that, for energies much smaller than the Planck scale, the group velocity

$$v_g(p) \approx \left(1 + \frac{7}{2}\eta_4 \frac{p^2}{M_P^2c^2}\right)c. \tag{3.9}$$

An observation of a difference in the time of arrival of two wavepackets, with different energies, which have been emitted at the same time, directly leads to difference in the speed of propagation.

Even though a modified dispersion relation is not a complete specification of a dynamical framework, *i.e.* a definition of an extension of a field theory like the standard model, it still allows to discuss their effects on the kinematics of physical processes (see for instance [132]). As it happens for scattering processes among classical particles, in evaluating the amplitude for a given process in quantum field theory, one has to insert a Dirac delta function to enforce the conservation of energy and momentum. In this Dirac delta one has to insert the momenta of the ingoing and outgoing particles, which are assumed on shell. Some processes might be forbidden, then, just by kinematical arguments.

As an example, in standard QED, the vacuum Čerenkov effect, $e^- \to e^- + \gamma$, is forbidden by kinematical arguments. However, the introduction of a modified dispersion relation might change the situation. It might happen that an energy threshold E_* for the process appears: for electrons of energies larger than E_* the process does take place, while for energies lower than E_* the process is kinematically forbidden. The physical consequence of this process would be that a beam of electrons leaving a region where they have been accelerated to an energy $E > E_*$ will lose energy due to this mechanism. The longer the beam is travelling, the larger will be the amount of energy carried away by Čerenkov photons, and ultimately the electrons will be reduced to have energy E_*. Of course, the knowledge of the dispersion relation alone allows to establish whether the reaction can take place or not examining only the kinematics. The computation of the rate of the reaction, *i.e.* the efficiency with which the energy is subtracted from the beam, requires the formulation of a field theory.

3.2.4 Modified dispersion relations and the effective field theory approach

To fully understand the relevance of dispersion relations it is not enough to limit the analysis to these rather general kinematical effects. First of all, as it has been already mentioned, the fact that a reaction is allowed does not suffice to specify its rate. In order to do so, a transition amplitude must be evaluated, and hence a full dynamical framework must be given. Furthermore, there are phenomena which cannot be understood just in terms of the dispersion relation (*e.g.* effects involving polarization of particles with spin), and, in general, it is impossible to recover from the dispersion relation alone other dynamical features (new interactions) which can be handled only within a field-theoretic framework.

A full treatment in terms of an effective field theory must be given. In other words, one has to introduce the operators corresponding to the modified dispersion relation into a formalism like the one of the SME of Colladay and Kostelecky. This has been done initially by Myers and Pospelov [144], who considered the case of cubic modification to the dispersion relation.

The dispersion relation is encoded in the kinetic term, the part of the Lagrangian quadratic in the fields. A modification of the dispersion relation implies a modification of this part of the Lagrangian. To do it consistently, the Lagrangian must be supplemented with all the operators up to dimension five satisfying the following requirements

1. they are quadratic in the fields;
2. they contain up to one more derivative with respect to the standard Lorentz invariant La-

grangian;

3. they are gauge invariant;
4. they are either Lorentz invariant or all the possible terms constructed with a preferred timelike vector field n^μ;
5. they are not reducible to lower dimensional operators when equations of motion are used;
6. they are not reducible to a total derivative.

These requirements are sufficient to specify the general structure of the Lagrangian for the fields involved in the Standard Model, *i.e.* scalars, spinors and gauge vectors.

In general one obtains that not only the dispersion relation is deformed in different ways for different particles, but also that different polarizations of the same field do propagate differently. For instance, parametrizing such dispersion relations as

$$E^2 = m^2c^4 + p^2c^2 + \frac{\eta_3}{E_P}p^3c^3,$$

one could expect left-handed electrons having a value of $\eta_3 = \eta_L$ different from the one, η_R, ot the right handed electrons. This is the phenomenon of birefringence. Birefringence opens the doors for a different class of constraints, based on the observation of the behavior of polarization of beams of particles.

Despite the logic of this extension seems pretty clear, one easily realizes that nothing is preventing large modifications of the part of the Lagrangian involving only dimension three and four operators due to the presence of higher dimensional operators. We already know that if there are such terms which are not Lorentz invariant they must be extremely tiny. However, in an effective field theory scenario like the one considered here, even though these dangerous terms are absent at the tree level, they can be generated through radiative corrections [144, 145]. Because of these quantum corrections, higher dimensional operators can generate relevant corrections to three and four dimensional operators which are already severely constrained, thus posing a sort of naturalness problem. In general, one would like to have within the effective field theory a custodial symmetry, *i.e.* a symmetry protecting the lower dimensional operators from large quantum corrections. It has been suggested that supersymmetry could play this role [146, 147]. In fact, in supersymmetric scenarios, some dangerous Lorentz violating operators of dimension three and four are explicitly forbidden, therefore improving this naturalness problem.

A construction of an effective field theory with Lorentz violation shows clearly two things: first of all, once Lorentz invariance is broken, and æther-like structures (as the vector field n^μ mentioned above) are introduced, a lot of phenomenology is produced. This rather large amount of effects can be used in principle to constrain the parameters related to violations of Lorentz invariance. The price to pay is that an additional fine-tuning problem has to be solved: lower dimensional operators, the ones for which the constraints are more tight since they are essentially the part of the Lagrangian relevant for low energy phenomena, must be protected against large quantum corrections. This is of course a challenge that any viable candidate for an extension of the Standard Model must successfully address in order to be taken seriously.

3.2.5 Quantum gravity phenomenology

As it has been said, these theoretical investigations have been motivated by the question about the status of Lorentz invariance and in general about the high energy structure of spacetime. These are complemented by a plethora of experimental tests and observations, which are collectively denoted as Quantum Gravity Phenomenology (QGP) [148], see also [149, 150] for recent discussions and references.

Lorentz invariance violation (LIV) is only one particular aspect of this rather broad area of research. Indeed, as a consequence of the particular quantum gravity model considered, there can be several phenomena which can be used to detect them. For instance, quantum decoherence [151, 152], effects on cosmological perturbations [153], cosmological variations of the coupling constants [154, 155], black holes at TeV scale [156, 157] and violation of discrete symmetries [158, 159].

In particular, the investigation of LIV is deeply related to the CPT invariance of the Lagrangian. In relativistic quantum field theory, the so-called CPT theorem ensures that CPT invariance holds provided that Lorentz invariance holds [100]. The deep connection between the discrete CPT transformation and spacetime symmetries is further strengthened by the so-called anti-CPT theorem [160]: in a quantum field theory which is unitary and local, the breaking of CPT implies the breaking of Lorentz invariance[2]. For a discussion of the status of CPT and its experimental tests, see [151, 152].

The fact that it is possible to put constraints on LIV effects, even the Planck suppressed ones, should not come as a surprise. Indeed, even though the typical size of the anomalous effects can be estimated to be controlled by the ratio $r = E_{EW}/E_{Pl}$ at most, one could imagine some experimental situations in which they are actually amplified, through the appearance of some large number N which makes them detectable.

This is not at all a new situation in physics. For instance, some models of grand unification do predict the proton decay process, which is strictly forbidden in the standard model (essentially by baryon number conservation). Typically, the rate of the process is suppressed by some powers of the ratio m_p/M_{GUT}, where m_p is the mass of the proton and M_{GUT} is the energy scale associated to the grand unification model. The typical order of magnitude of the latter is $10^{13}GeV$. Despite this huge suppression, it is possible to constrain the processes leading to proton decay by comparing the predictions with the present lower bound to the proton's lifetime ($\geq 10^{31}$ years). This is a particular example in which we are able to put constraints on ultra-high energy physics even with very low energy experiments.

For what concerns LIV due to Planck scale phenomena, the same kind of reasoning does apply. With suitably designed experiments/observations one can amplify tiny effects in such a way to bring them in the sensitivity range of the apparatus used, therefore allowing an experimental detection. In this sense, astrophysical phenomena are the most interesting ones, both for the large energies that can be reached and for the large distances which must be covered by the signals to reach the observers on the Earth. There has been a rather large effort, in the last years, in the direction of testing Lorentz invariance by means of astrophysical phenomena. Again, for reviews see, for instance, [134, 135], while for a discussion of more recent results see [136].

[2] Notice, however, that there can be Lorentz violating terms which are invariant under CPT.

3.3 Special Relativity revisited: axiomatic approach

We have shown how Lorentz invariance violation can be constrained. The analysis presented was not related, in principle, to an extension of relativity theory to more general frameworks. It was the analysis of mere Lorentz symmetry breaking. It is important then to turn the attention to the possibility of having different implementations of the relativity principle, which are not necessarily related to the Lorentz group. Indeed, Lorentz violation as it is described by the Standard Model extension is deeply influenced by the idea that there is a preferred reference frame, pinpointed by the æther field n^μ. This sounds like a step back with respect to the relativity principle, which ensures the absence of such a privileged frame.

It should be mentioned that this preferred frame is not necessarily a non-dynamical, background structure superimposed on spacetime. It can be dynamically generated in a Lorentz-covariant theory from non-zero vacuum expectation values of some vector or tensor fields which are transforming in a nontrivial way. An early example was given by Kostelecky and Samuel in an analysis of string field theory [116]. Another model in which a preferred frame is introduced dynamically is the so called Einstein-æther theory [161], in which the familiar Einstein–Hilbert action for the gravitational field is supplemented with the action of a vector field, the æther field, which is constrained to have unit norm. In this way, the idea of a preferred frame is included without making reference to a specific background quantity, not specified by an equation of motion, related to the old idea of absolute space and in deep contrast with the present concept of spacetime.

It is interesting to reconsider the familiar derivation of special relativity and of Minkowski spacetime. As we have already said, the common presentation makes use of two postulates:

- the relativity principle: the equivalence between all inertial reference frames;
- the constancy of the speed of light: the speed of propagation of luminous signals is the same in all reference frames.

These two postulates lead directly to Minkowski spacetime and to the Poincaré group as the relativity group relating all the inertial observers.

A less known fact is that the postulate about the constancy of the speed of light is somehow redundant. Indeed, von Ignatowski [162, 163] already realized that the second postulate is of secondary relevance: actually, it can be removed without changing much the outcome. It is useful to present in a concise form the main points of the argument, while a complete discussion can be found in [164].

To begin with, one needs to carefully list all the relevant axioms concerning spacetime and its properties, as well as to give a short discussion of their physical motivations.

First of all, each inertial observer is describing space and time by assigning to each event four numbers, three specifying the spactial position x, y, z, while the fourth, t, is related to the chronological ordering between the events. Therefore, each observer is representing spacetime as $\mathbb{R} \times \mathbb{R}^3$. Notice that a precise notion of space has been introduced. Space has the structure of the three dimensional Euclidean space. With this structure one directly relates the three numbers x, y, z to physical distances. Another observer will represent spacetime with a different copy of

$\mathbb{R} \times \mathbb{R}^3$, and will use the coordinates t', x', y', z'.

The next assumption we are going to use is the fact that the transformations relating the two reference frames are linear [165, 166]. This assumption is deeply related to the fact that we want spacetime to be homogeneous, *i.e.* that there are no privileged points in spacetime. In particular, consider two inertial reference frames, and let x^μ, y^μ be the coordinates used by two observers to label spacetime events, accordingly to the readings of their rods and clocks. The most general relation between the two coordinates system is:

$$y^\mu = f^\mu(x). \tag{3.10}$$

The relationship between infinitesimal displacements, or, equivalently, the relationships between velocities/momenta of particles in the two frames are given by the differential of this map, *i.e.*

$$dy^\mu = \frac{\partial f^\mu}{\partial x^\nu} dx^\nu. \tag{3.11}$$

Therefore, in order for this relationship to be independent from the position in spacetime of the two reference frames it is needed that

$$f^\mu(x) = \Lambda^\mu_\nu x^\nu + a^\mu, \tag{3.12}$$

where all the coefficients appearing in this map are depending on the state of relative motion between the frames (the relative velocity).

Alternatively one could restrict the coordinate transformations (3.10) by the requirement that inertial frames are mapped into inertial frames, or, in other words, that they live invariant:

$$\frac{d^2\vec{x}}{dt^2} = 0. \tag{3.13}$$

The most general transformation compatible with this condition is [167]:

$$f^\mu(x) = \frac{\Lambda^\mu_\nu x^\nu + a^\mu}{b_\rho x^\rho + d}. \tag{3.14}$$

If it is required that the transformation is everywhere regular, the linearity of the transformation easily follows from $b_\rho = 0$.

It is also required that these relativity transformations must form a group: taking three inertial frames K, K', K'', the map between K, K'' should be evaluated by composition of the mappings between K, K' and K', K''. If this happens, the composition is a closed operation in the set of the relativity transformations. Besides associativity (which is guaranteed by linearity), the relativity transformations with their composition form a group if they include the identity transformation (trivial) and if for every group element there is an inverse, *i.e.* for any transformation from K to K' there is a transformation from K' to K such that the net transformation, when composing them, is just the identity. Invertibility is guaranteed by the invertibility of the matrix Λ.

A crucial physical requirement must be satisfied. In general, one would like that the relation of cause-effect is independent from the reference frame considered to describe a particular phenomenon. Since the concept of causality in terms of light cones cannot be used, one has to introduce a slightly different condition. This is called the pre-causality condition: if, in one reference frame, one considers two events such that $t_1 > t_2$ at the same spatial position, then, in any other reference frame this chronological order is respected, namely $t'_1 > t'_2$. Notice that this condition is weaker than the causality relation of special relativity. There, the proposition holds between any timelike separated events. In terms of our transformation matrices, this implies that $\Lambda^0_0 > 0$.

It is interesting that the outcome of this set of axioms is a one parameter family of group of transformations, with the parameter having the dimensions of a speed. Of course, for finite values of this parameter one obtains Lorentz transformations, while the limit case of infinite speed one obtains the Galilei group. It is remarkable that the existence of this parameter, as well as its invariance under change of reference frame are consequences of other assumptions rather than a fundamental postulate.

Therefore, in special relativity, the postulate about the constancy of the speed of light is not necessary. In this axiomatic approach, the relation between the invariant speed and the speed of propagation of light can be seen as a result of experimental observation, rather than a theoretical assumption.

This discussion shows that in the construction of special relativity quite a number of assumptions have to be used, at different levels. Going beyond special relativity, therefore, requires the relaxation of one or more of these postulates. In the following, the consequences of the relaxation of the isotropy postulate will be examined.

3.4 Anisotropic relativity and Finsler geometry

It is instructive to discuss a case in which the axiomatic derivation presented in the previous section does not necessarily lead to special (or Galilean) relativity. The example has been known since the 30's, and it involves a two dimensional spacetime [168, 169].

In two dimensions, space is one-dimensional. Hence, with respect to the list of assumptions discussed in the previous section, there is only one slight difference concerning isotropy. In two or more space dimensions, isotropy is related to rotational symmetry. In one dimension, there is no notion of rotation. However, one can still speak about space inversion $x \to -x$, which is a discrete symmetry. It is interesting to consider the case in which it is not required the equivalence between the two possible orientations of the line representing space.

As we have said, given two inertial frames, the mapping between them is linear:

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}, \tag{3.15}$$

where $\alpha, \beta, \gamma, \delta$ are functions of the velocity v at which the primed frame is moving with respect to the unprimed frame. Arranging the coordinates so that the origins of the two reference frames coincide at $t = t' = 0$, the trajectory of $O'(x' = 0)$ obeys:

$$\begin{pmatrix} t'(\tau) \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} t(\tau) \\ vt(\tau) \end{pmatrix}, \tag{3.16}$$

where τ is an intrinsic parameter on the world line of O'. Consequently we get

$$\gamma = -v\delta. \tag{3.17}$$

The point $O(x=0)$ is moving with respect to O' at a velocity given by v^*. We are not assuming here that $v^* = -v$, we just ask that $v^* = v^*(v)$, and that this relation is invertible. Notice that $0^* = 0$. From:

$$\begin{pmatrix} t'(\tau) \\ v^* t'(\tau) \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} t(\tau) \\ 0 \end{pmatrix}, \tag{3.18}$$

we get:

$$\gamma = \alpha v^*. \tag{3.19}$$

Using these relations we easily see that the transformation matrix has the form

$$\alpha(v) \begin{pmatrix} 1 & \tilde{\beta} \\ v^* & -v^*/v \end{pmatrix}. \tag{3.20}$$

The precausality condition gives a constraint on the sign of the function $\alpha(v)$. Indeed, when acting on the vector $(\Delta t, 0)$ corresponding to the separation between two events which in one reference frame do have the same spatial location, the transformed vector:

$$\begin{pmatrix} \Delta t' \\ \Delta x' \end{pmatrix} = \alpha(v) \begin{pmatrix} 1 & \tilde{\beta} \\ v^* & -v^*/v \end{pmatrix} \begin{pmatrix} \Delta t \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha(v)\Delta t \\ \alpha(v) v^* \Delta t \end{pmatrix}, \tag{3.21}$$

must show the same chronological order between the events, *i.e.* $\Delta t' > 0$ if $\Delta t > 0$, which is equivalent to say $\alpha(v) > 0$.

Straightforward manipulations lead to the inverse matrix:

$$\frac{1}{\alpha(v)(1+\tilde{\beta} v)} \begin{pmatrix} 1 & \tilde{\beta} \\ v & -v/v^* \end{pmatrix}. \tag{3.22}$$

For consistency, this must be the transformation matrix associated to v^*:

$$\frac{1}{\alpha(v)(1+\tilde{\beta} v)} \begin{pmatrix} 1 & \tilde{\beta}(v) v/v^* \\ v & -v/v^* \end{pmatrix} = \alpha(v^*) \begin{pmatrix} 1 & \tilde{\beta}(v^*) \\ (v^*)^* & -(v^*)^*/v^* \end{pmatrix}. \tag{3.23}$$

This matrix equation is giving:

$$v^{**} = v. \tag{3.24}$$

Further properties can be found looking at the composition of transformations. The composition of two relativity transformations must be such that the set of the latter forms a group, with the composition defining the group multiplication. In general, nothing is asked about the commutativity of the relativity group. In this case, the group turns out to be abelian. Indeed, if the group a Lie group, abelianity is a consequence of the fact that the group has just one parameter (the velocity). A physical motivation could be that, composing two boosts, one ends up with the same final frame whatever is the order of the composition (of course if one has only one spatial dimension as in the present case). With this requirement, by inspection of the product of the matrices, one finds the condition

$$\alpha(v_1)\alpha(v_2)(1 + v_2^* \tilde{\beta}(v_1)) = \alpha(v_1)\alpha(v_2)(1 + v_1^* \tilde{\beta}(v_2)), \tag{3.25}$$

from which it is obtained simply that:

$$v_2^* \tilde{\beta}(v_1) = v_1^* \tilde{\beta}(v_2). \tag{3.26}$$

Assuming that the velocities are non vanishing:

$$\frac{\tilde{\beta}(v_1)}{v_1^*} = \frac{\tilde{\beta}(v_2)}{v_2^*}. \tag{3.27}$$

Since this holds for any pair of velocities, one can conclude that:

$$\tilde{\beta} = \kappa v^*, \tag{3.28}$$

with κ an integration constant. This allows us to say that the matrix element in the first row, second column of the composed matrix must be exactly κ times the second row, first column matrix element. After trivial algebra one finds the condition:

$$v_1^*(1 + \frac{v_2^*}{v_2}) = v_2^*(1 + \frac{v_1^*}{v_1}). \tag{3.29}$$

There are two possibilities: A) $v^* = -v$ and B) $v^* \neq -v$.

- A) The transformation matrix has the form

$$\alpha(v) \begin{pmatrix} 1 & -\kappa v \\ -v & 1 \end{pmatrix}. \tag{3.30}$$

Looking back at the inverse matrix one gets:

$$\alpha(-v) = \frac{1}{\alpha(v)(1 - \kappa v^2)}. \tag{3.31}$$

If we apply a spatial reflection, $x, x' \to -x, -x'$, the observer O' will be seen moving with velocity $-v$ with respect to O. Consequently:

$$\alpha(-v) \begin{pmatrix} 1 & \kappa v \\ v & 1 \end{pmatrix} = \mathbb{P}\alpha(v) \begin{pmatrix} 1 & -\kappa v \\ -v & 1 \end{pmatrix} \mathbb{P}, \tag{3.32}$$

where

$$\mathbb{P} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.33}$$

is the matrix representing spatial reflections. From this, one easily concludes that,

$$\alpha(v) = \alpha(-v) = (1 - \kappa v^2)^{-1/2}. \tag{3.34}$$

If $\kappa = 0$ one obtains the familiar Galileian transformations in two dimensions. These transformations leave invariant the line element dx^2 defining distances between points in space. If otherwise $\kappa \neq 0$ one can easily recognize that the transformation is a Lorentz boost, and that the quantity $c = \kappa^{-1/2}$ represent an invariant velocity.

This group of transformations leaves the Minkowski line element $ds^2 = -\kappa^{-1}dt^2 + dx^2$ invariant. Therefore, in this case, one can conclude that the structure of spacetime is the one of special relativity.

- The second possibility is that $v^* \neq -v$. If this holds, one easily sees that there is a second integration constant, ξ such that:

$$\frac{v^*}{1 + \frac{v^*}{v}} = \frac{1}{\xi}. \tag{3.35}$$

This condition implies that the relation between the velocity v and its reciprocal is given by:

$$v^* = \frac{-v}{1 - \xi v}. \tag{3.36}$$

In this case it is clear how the operation $\mathbb{P}$ does not connect $\alpha(-v)$ and $\alpha(v^*)$ as in the previous case. In this case we have an anisotropy, related to the parameter ξ.

The evaluation of the function α can be done in a different way. Assuming that the group is a Lie group, any finite transformation can be obtained by exponentiating the generator,

$$L(v(\eta)) = \exp(\eta T), \tag{3.37}$$

where η is the Lie group parameter and $v(\eta)$ is the relationship existing between the velocity and the parameter η is called the rapidity. In standard Lorentz boosts, the rapidity is related to the velocity by $(1 - v^2/c^2)^{-1/2} v/c = \sinh(\eta)$. On the other hand, the generator can be evaluated by taking the derivative

$$T \equiv \left. \frac{dL(v(\eta))}{d\eta} \right|_{\eta=0}. \tag{3.38}$$

Using the chain rule:

$$T = \left. \frac{dL(v(\eta))}{d\eta} \right|_{\eta=0} = \left. \left(\frac{dL(v(\eta))}{dv} \frac{dv}{d\eta} \right) \right|_{\eta=0}. \tag{3.39}$$

Hence, taking the parametrization in which

$$\left. \frac{dv}{d\eta} \right|_{\eta=0} = 1, \tag{3.40}$$

one can use directly (3.20) to determine the shape of the generator of the modified boosts. Using:

$$\left. \frac{dv^*}{dv} \right|_{v=0} = -1, \qquad \left. \frac{d}{dv} \left(\frac{v^*}{v} \right) \right|_{v=0} = -\xi, \tag{3.41}$$

and introducing $a = d\alpha/dv$ evaluated at $v = 0$, the generator takes the form:

$$T = a\mathbb{I}_2 - \begin{pmatrix} 0 & \kappa \\ 1 & \xi \end{pmatrix}. \tag{3.42}$$

Exponentiating this generator one gets the desired transformation. In the case of special relativity, $\xi = 0$, and invariance under the spatial reflection $\mathbb{P}$ forces $a = 0$. In the anisotropic case, this is not true anymore. Of the three parameters a, κ, ξ it can be shown that only two are relevant. In fact, by a suitable redefinition of the coordinates t, x which amounts to a different choice of the synchronization procedure,

$$\begin{pmatrix} t \\ x \end{pmatrix} \to G^{-1} \begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} 1 & \xi/2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}, \tag{3.43}$$

the generator becomes:

$$T' = a'\mathbb{I}_2 - \begin{pmatrix} 0 & \kappa' \\ 1 & 0 \end{pmatrix}, \tag{3.44}$$

where the primed quantities can be evaluated from the unprimed ones using the transformation G, and the fact that $T' = G^{-1}TG$. In these coordinates the nature of the transformation is very clear. Besides the standard boost (Lorentzian or Galilean depending on the value of κ'), there is an exponential factor. After standard manipulations (see [169]) The finite transformation reads:

$$L(v(\eta)) = e^{a'\eta} \begin{pmatrix} \cosh(\sqrt{\kappa'}\eta) & -\kappa' \sinh(\sqrt{\kappa'}\eta) \\ -\sinh(\sqrt{\kappa'}\eta) & \cosh(\sqrt{\kappa'}\eta) \end{pmatrix}, \tag{3.45}$$

which leads immediately to:

$$v(\eta) = \kappa' \tanh(\sqrt{\kappa'}\eta). \tag{3.46}$$

The relation between the velocity of one frame with respect to the other and the parameter of the boost is an odd function, so that $\eta \to -\eta$ implies that $v \to -v$.

This set of transformations no longer leaves a quadratic line element invariant. In particular, the Lorentz contraction of lengths is controlled not only by the familiar relativistic gamma factor: there is an overall rescaling depending on the exponential factor $\exp(a'\eta)$ which is not an even function of η. This is due to the fact that boost with speed v is not just a boost with speed $-v$ composed with a reflection of the spatial axis. As a consequence, in this model there is a kinematical anisotropy between the two possible orientations of the spatial direction.

For values of $a' \neq 0$, Minkowski spacetime (and in particular its geometrical structure) is definitely not the spacetime structure emerging from the relativity principle. Nevertheless, it is easy to see that there is an object, which is a generalization of a quadratic line element, which is invariant under these transformations:

$$ds^2 = (\kappa^{-1}dt^2 - dx^2)^{1-2b}(n_\mu dx^\mu)^b, \tag{3.47}$$

where b is a calculable dimensionless parameter depending on a' (in particular $(b = -a')$) and n_μ is a constant null vector field, such that $n_\mu n^\mu = 0$.

For a more detailed discussion of the implications of this result, see [169]. Most notably, when $b = 0$ the standard two dimensional Minkowski spacetime is recovered. Therefore, in two dimensions, the relativity principle allows a wider range of geometrical possibilities.

$\kappa = 0, b = 0$	Galilean relativity
$\kappa = 0, b \neq 0$	anisotropic Galilean relativity
$\kappa \neq 0, b = 0$	special relativity
$\kappa b \neq 0$	anisotropic relativity

3.4.1 Towards a generalization in $3 + 1$ dimensions

The $1 + 1$ dimensional case showed how kinematical anisotropy could lead to interesting alternative possibilities in the implementation of the relativity principle without modifying too much the language needed to describe spacetime. What has been obtained is that the notion of invariant line element needs to be extended from the quadratic case to a more general case, namely Finsler norms, which is the topic of the next chapters.

To make Finsler geometry to be a compelling alternative to pseudo-Riemannian geometry, this two dimensional toy model needs to be extended to higher dimensions. Notably, this has been done several times. Anisotropic models of relativity theory have been periodically rediscovered by different communities in different epochs [170, 171, 172, 173, 174].

There is a crucial feature, distinguishing these generalizations from the the two dimensional version. In fact, while in two dimension the theory is maximally symmetric, with the number of generators being three (two translations and one boost) as for the Poincaré group in two dimensions, in $3 + 1$ it is impossible to have an anisotropic realization which is maximally symmetric. As we shall see, the number of generators is typically eight (four translations and four boosts-rotations).

This anisotropic extension of special relativity is now known as Very Special Relativity (in the following just VSR). This particular relativistic mode has been proposed as a theory in which relativistic invariance is reduced due to the presence of a preferred null vector field. In VSR, the Finsler line element is[3]

$$ds^2 = (\eta_{\mu\nu} dx^\mu dx^\nu)^{1-b} (n_\mu dx^\mu)^{2b}, \tag{3.48}$$

where n_μ is a constant null vector field, and b a real parameter. In units where $c = 1$ (or equivalently $\kappa' = 1$), the corresponding modified dispersion relation is:

$$(\eta^{\mu\nu} p_\mu p_\nu)^{1-b} (n^\mu p_\mu)^{2b} = m^2, \tag{3.49}$$

where we are raising and lowering indices through the Minkowski metric $\eta_{\mu\nu}$.

For massless particles, the dispersion relation is just the special relativistic one. Notice that, despite its Finsler nature, the line element (3.48) is just obtained from the Minkowski one with a disformal transformation (see the box in the next chapter). In particular, the causality relations are the same as in special relativity, with the speed of light still representing a limit speed.

The symmetry group which leaves invariant this Finsler line element is a subgroup of the Weyl group which leaves invariant the direction of the four vector n^μ, besides leaving invariant the Minkowski metric, up to rescalings. This group has eight generators: four translations, a combination of the boost along n and the identity $N_n - \xi\mathbb{I}$, the rotation around the spatial part of n J_n, and two combinations of the boosts and rotations in the transverse directions[4].

To conclude with VSR, it is interesting to mention that there some constraints are already available. The value of b can be constrained by means of low energy experiments (see [174]) and it turns out that $b \leq 10^{-26}$.

Other anisotropic extensions have been proposed (see [172]), but VSR represents the minimal modification of special relativity, since its symmetry group is the largest possible (not containing the Poincaré group).

3.4.2 Finsler geometry as a test theory for Minkowski spacetime

One of the pillars of general relativity is that spacetime is locally Minkowskian, *i.e.* spacetime is a pseudo-Riemannian manifold, whose geodesics represent the possible trajectories of test particles. This is called the metric postulate. One could see special relativity and local Lorentz invariance as consequences of this fact. Of course, this assumption is as questionable as any other postulate of general relativity. What is needed to test the metric postulate is some sort of test theory, where geometry is suitably generalized in such a way to contain pseudo-Riemannian geometry as a special case, when a certain set of parameters characterizing the test theory itself are set to have particular values. In the case of VSR, Minkowski spacetime is recovered when $b = 0$. In all the other cases, the geometrical structure of spacetime differs from Minkowski spacetime.

In other extensions [175], the metric is supplemented by an independent connection which could be used to define the motion of free test particles in terms of its autoparallel curves. Again, the structure of the nonmetricity and the torsion tensors parametrize the deviation from general

[3]For the details for the formulation of a field theory in this case, we refer to [174].

[4]See for instance [174] for an accurate discussion of the details.

relativity, and in particular can be used to test the metric postulate. This kind of perspective towards Finsler geometry has been taken in a number of works, in the past. See [176, 177, 178, 179].

This comparison immediately requires a specification. In the case of VSR, the parameter b cannot be predicted by the relativity principle alone, exactly as the invariant speed, which will become the limit speed for signals, the speed of light, cannot be predicted by the derivation *á la* von Ignatowsky. Within a picture in which spacetime is dynamical, the parameter b can be thought as a global parameter summarizing the microscopic properties of spacetime, in the same way as the electric and magnetic properties of a medium can be summarized in two tensors. The dynamical origin of these properties are beyond the reach of the effective theory. Accordingly, generalizations of VSR will not be able to predict the precise values of all the parameters characterizing the deviations from Minkowski spacetime, unless a more fundamental theory, from which they can be derived, is proposed.

Finally, it is interesting to observe that Finsler geometry seems to establish a common language for both emergent spacetimes (analogue models) and Lorentz violating scenarios, at least as far as geometry is concerned. This connection can be used to make clear what are the structural similarities between the two approaches. This point of view will be further expanded in Chapter 5.

3.5 Introducing the Planck scale

The previous section dealt with the cases in which the spacetime symmetries are realized linearly. Lorentz transformations, and in particular the boosts, act in such a way to introduce an invariant speed.

It is a rather general expectation that when gravity is correctly described as a quantum phenomenon, the Planck length, defined from the Newton's gravitational constant, should play a role as a sort of threshold: when probing spacetimes on microscopic scales beyond the Planck length quantum gravitational phenomena should become more and more important. Hence, a description of spacetime in terms of flat Minkowski spacetime should become inaccurate. See for instance the discussion about the emergence of a minimum length in [180, 181].

This picture clashes with special relativity: Minkowski spacetime is scale invariant. This is manifest in the fact that boosts are not leaving any length invariant. From the mere intuition, this picture of spacetime becomes more and more inaccurate, the higher is the energy of the phenomena considered. As Minkowski spacetime locally approximates any curved spacetime, in a sort of decoupling limit when the source for the gravitational field becomes so small that spacetime can be considered practically flat, one should be able to define a suitable extension of Minkowski spacetime which can encode the net effect of quantum fluctuations of the metric tensor at the smallest scales.

This effective spacetime should be endowed by an important property: it should not be scale-invariant. While at large distances the properties of Minkowski spacetimes must be recovered, at distances shorter than a given characteristic scale, which can be thought to be the Planck scale, the description of spacetime should be encoding in an effective way the averaging over the quantum fluctuations of the metric.

In this sense, one could imagine to extend the derivation of special relativity in such a way to add a length scale into the game. This length scale need not necessarily to be an additional invariant quantity: it could be as well a threshold scale for some particular phenomena, like large deviations from linearity of the transformations between inertial frames.

These ideas have been put forward in different contexts. For instance, in [182] it has been proposed to introduce some nonlinear relativity transformations. However, the attempt seems to fail since these transformations are just standard Lorentz transformations in non-cartesian coordinates.

More recently, there a new perspective about the effective description of quantum spacetime has been proposed. This goes under the name of Doubly (or Deformed) Special Relativity (DSR) [125, 126, 127]. See also [183, 184] for up-to-date discussions. Essentially, in this framework, the algebra of the Poincaré group is suitably deformed in order to accommodate an energy scale, playing the role of a distinguished scale.

For the sake of clarity, we hereby report the form of the DSR1 algebra, *i.e.* the deformations of the Poincaré algebra commutations relations characterizing this particular version of doubly special relativity. The modified commutators are:

$$[K^i, K^j] = -i\epsilon^{ijk}\left(J^k \cosh\frac{P^0}{\kappa} - \frac{P^k}{4\kappa^2}P^m J_m\right), \tag{3.50}$$

$$[K^i, P^j] = i\delta^{ij}\kappa \sinh\frac{P^0}{\kappa}, \tag{3.51}$$

where κ is a constant with the dimensions of an energy (typically, the Planck scale), P^0, P^j are the familiar generators of time and space translations, K^i are the boost generators and the J^i are the rotation generators. The dispersion relation, defined as the (first) Casimir of the resulting, deformed, algebra, is:

$$\left(2\kappa \sinh\frac{P^0}{2\kappa}\right)^2 - P^2 = m^2, \tag{3.52}$$

which reduces to the standard relativistic dispersion relation (and, correspondingly, the algebra reduces to the standard Poincaré algebra) in the limit $\kappa \to \infty$.

The entire DSR program could be looked at from a different point of view: it has been proposed [185] that some sort of deformed special relativity should emerge when large deviations from what it is called an ideal measurement should be expected. In the case of the gravitational interaction, the naive expectation is that this should happen for particles with sufficiently high energy. Indeed, since gravity is coupled to the stress energy tensor, the higher is the energy content of the system the larger will be the gravitational interactions. It is better to specify this statement with a more explicit discussion.

Let e^a_μ be an assignment of a tetrad, which maps vector quantities into the frame quantities in the usual way:

$$V^a = e^a_\mu V^\mu. \tag{3.53}$$

The tetrad field can be seen as the gravitational field itself. Indeed, the relation between the metric and the tetrad is:

$$g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu, \tag{3.54}$$

and this relationship allows a reformulation of general relativity in terms of the tetrad field alone. Consider now a particle, with a given momentum π^μ. This is the intrinsic momentum, the one that a perfect measurement should be able to pinpoint. The momentum measured by the observer associated to the tetrad at hand is expressed by:

$$p^a = e^a_\mu \pi^\mu, \tag{3.55}$$

according to the standard rules of general relativity. However, from the discussion about the dynamics of spacetime at the smallest scales, one should not be surprised that the mapping between the "true momentum" π^μ and the measured one p^a should be affected by gravitational effects. In other words, one should expect that:

$$p^a = e^a_\mu(\pi)\pi^\mu, \tag{3.56}$$

where $e^a_\mu(\pi)$ is denoting the fact that the tetrad actually becomes a function of the momentum of the particle which is used to probe spacetime. There are two important consequences of this ansatz. First of all, symmetries which are realized linearly in π^μ are no more linear in p^a. Hence, if one has Lorentz invariance implemented linearly in π^μ (exact quantum gravity), the resulting symmetry on p^a will not be linear any more.

$$(p')^a = e^a_\mu(\pi')(\pi)'^\mu = e^a_\mu(\Lambda\pi)\Lambda^\mu_\nu e^\nu_b(\pi)e^b_\rho(\pi)\pi^\rho = \Lambda^a_b(p)p^b. \tag{3.57}$$

The second consequence is that if the relation between the tetrad and the metric tensor is unchanged, the metric tensor itself becomes some sort of momentum dependent object:

$$g_{\mu\nu}(\pi) = \eta_{ab}e^a_\mu(\pi)e^b_\nu(\pi). \tag{3.58}$$

This object has been dubbed as *rainbow metric*. It has been shown to arise in rather different contexts: for instance, averaging over quantum fluctuations [186, 187], from renormalization group arguments in quantum gravity [188, 189], as well as indications from quantum reference frames [190].

In fact, another way to derive DSR is to formulate of an effective field theory for matter fields where the gravitational field has been integrated away. While it has been proved that in 2+1 dimensions the gravitational field can be integrated away producing an effective DSR description for point particles [191], in 3+1 there are only indications that this is the case [192].

In general, DSR is based on the idea that the relativity group is acting non-linearly in momentum space. While this is easy to understand, the entire DSR idea faces some severe problems. The first question that needs to be solved is the fact that the deformation of the Poincaré algebra needs to be specified, otherwise it would be arbitrary. Moreover, one has to understand if two different realizations are equivalent or not, or, in other words, if it is possible to trivialize the action of the deformed group into a standard special relativity scenario with a suitable reparametrization [193].

The DSR program must also face saturation problems: for instance, if one asks that the Planck energy is an invariant energy, one should be worried about the fact that this scale should not be an upper bound for the energy of systems of particles, which would lead to the unpleasant feature of forbidding macroscopic objects.

The most important difficulty regards the realization of DSR in real space. Indeed, while in momentum space the realization is pretty clear, the translation in coordinate space requires a more careful discussion. Some implementations were considering an action of the relativity group which is mixing coordinate and momenta [194].

More recently, it has been realized that the most natural scenarios to implement the DSR program (if there is such a possibility) is the one in which ordinary commutative spacetime is replaced by non-commutative one. Most notably, the fact that spacetime should be described effectively by a noncommutative geometry was put forward by Doplicher and collaborators already in [195]. In that case it was argued that the gravitational interaction should be responsible of the emergence of an effective noncommutativity of the canonical type.

In a canonical commutativity scenario, coordinates are promoted to operators such that:

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}\mathbb{I}, \tag{3.59}$$

where θ is a tensor which encodes the effects of the complicated microstructure of spacetime. Another typical form of noncommutativity often considered is the Lie-algebraic structure of $\kappa-$Minkowski spacetime [196]

$$[x^i, x^j] = 0 \qquad [t, x^i] = \frac{i}{\kappa}x^i, \tag{3.60}$$

where again, a deformed version of the Poincaré group, namely $\kappa-$Poincaré, is playing the role of the relativity group.

The most general parametrization of noncommutative spacetimes including the most studied cases is the following:

$$[x^\mu, x^\nu] = -\frac{i}{\kappa^2}\theta^{\mu\nu}\mathbb{I} - \frac{i}{\kappa}C^{\mu\nu}_{\alpha}x^\alpha - iZ^{\mu\nu}_{\alpha\beta}J^{\alpha\beta}, \tag{3.61}$$

where $\theta^{\mu\nu}, C^{\mu\nu}_{\alpha}, Z^{\mu\nu}_{\alpha\beta}$ are tensors and $J^{\alpha\beta}$ are the familiar generators of the boosts and rotations (*i.e.* the generators of the Lorentz group). According to the values of the tensors we have

- canonical noncommutativity, if $Z^{\mu\nu}_{\alpha\beta} = 0, C^{\mu\nu}_{\alpha} = 0$;
- Lie-type noncommutativity (*e.g.* $\kappa-$Minkowski), for $\theta^{\mu\nu} = 0, Z^{\mu\nu}_{\alpha\beta} = 0$;
- Snyder-type noncommutativity [197], for $\theta^{\mu\nu} = 0, C^{\mu\nu}_{\alpha} = 0$.

The role of noncommutative geometry in providing an effective description of spacetime beyond special relativity, when quantum gravitational effects start to become visible, has been considered by several communities. Besides the works we have already mentioned, this possibility has been suggested also from string theory arguments [198]. The emergence of an effective non-commutative spacetime in quantum gravity has also been proposed within the group field theory approach [123, 199].

This noncommutative scenario is not without major difficulties: without suitable cancelations (typically due to other symmetries), noncommutative quantum field theories, when analyzed from a perturbative point of view, do show an unpleasant feature, called UV/IR mixing [200]. While in common quantum field theories there are decoupling theorems (see for instance [201]) guaranteeing that high energy modes are not affecting low energy degrees of freedom, in these situations there are cases in which what would be thought to be a source of modifications in the UV behavior of the theory, for instance the tensor $\theta \approx 1/M_P^2$, does seem to affect the low energy/long range limit of the theory.

Besides these issues, in general the symmetry groups associated to these noncommutative spacetimes are not standard Lie groups, but rather quantum groups. The implications of this point on the formulation of quantum field theories are parts of a still ongoing research program. Some preliminary results are pointing towards some radical departures from the notions we are familiar with in the realm of quantum field theory. For instance, it has been suggested that, instead of standard bosonic and fermionic statistics, some “twist” might be present [202].

3.6 In summary

It is better to recollect the various ideas discussed so far. It seems fair to say that

- it is a plausible idea that spacetime does not have the structure of Minkowski spacetime all the way up to and past the Planck scale,
- therefore the symmetry group of spacetime might be different from standard Poincaré group, at least at short scales.

In all this, it is implicit that the notion of geometry encoded into a pseudo-Riemannian metric $g_{\mu\nu}(x)$ breaks down. The simplest generalization we can use to approximate this is to use some sort of rainbow geometry over a commutative spacetime, $g_{\mu\nu}(x,\pi)$, once we have properly defined it.

In fact, it has been shown that an appropriate relaxation of the axioms of the derivation *á la* von Ignatowski of special relativity points towards metric structures which are more general than pseudo-Riemannian geometry. In principle, one could expect either a deformation of the metric structure (and hence some sort of dispersion relation), or a deformation of the algebra of the symmetry generators, like in some DSR proposals.

A more radical solution could be given by the introduction of noncommutativity in spacetime, albeit this approach is still rather far away from the formulation of a satisfactory and phenomenologically viable theory with which predict new phenomena/put constraints.

The ultimate goal of all these approaches should be the formulation of a test theory like the one proposed by Colladay and Kostelecky and further extended to include higher dimensional operators. Of course, the fact that this particular test theory is an effective field theory should not be seen as a necessary condition. One can imagine test theories which are not effective field theories. For instance, the D-particle model [203] cannot be described by an effective field theory.

However, the overall goal is clear. It is important to formulate models which can encode in an effective way some characteristic quantum gravity features, in the particular flavor predicted by the favorite quantum gravity model one is advocating, and therefore predict the particular ways in which spacetime properties will differ from those discussed in special relativity. The search for possible extensions is important not only as a mathematical exercise *per se*, but also for concrete reasons. While quantum gravity effects are expected to be Planck suppressed, there are physical observations that we can make which are able to amplify tiny effects and make them detectable. Quantum gravity phenomenology does open a window on the phenomena at the Planck scale, making them not only subjects of theoretical speculations, but also of experimental analysis, giving us the opportunity to test some ideas on the fundamental structure of spacetime.

Of course, there are many ways in which models can be built, and it is still not clear which of the options currently on the market do represent viable possibilities. Only a careful examinations of all their features and consequences, and comparison with experimental data we are accumulating about ultra high energy physics could clarify this point.

As a closing remark in this very short overview on Lorentz invariance and the DSR perspective an instructive example should be mentioned. The concepts discussed so far sound more or less reasonable: the idea that spacetime at the Planck scale should differ from the spacetime described in relativity is old and nowadays well rooted in the quantum gravity community.

Nevertheless a sanity check is always needed, as the following example shall clarify. It has been already proposed in the past that the relativity theory should be extended in such a way to incorporate a privileged energy scale [204]. This energy scale was just the proton mass, which plays the role of the Planck scale for phenomena involving the strong interaction. If one insists on this, one has to abandon Minkowski spacetime in favor of some new geometrical structure (in the mentioned paper the author refers to Finsler geometry, quite interestingly). Of course, today we know that this is not the case. Therefore, this example shows how misleading could turn out to be the entire DSR approach. Obviously, this is not a matter that can be decided *a priori*. Only a development of a full fledged test theory, together with a careful analysis of its experimental consequences can tell us something and eventually falsify this scenario.

Chapter 4

Finsler geometry: basics

In this chapter we are going to review some basic facts about Finsler geometry. The material presented here is very introductory. We will be concerned with the concepts related mainly to flat Finsler spaces, without treating the notions of connection and curvature. For detailed discussions on Finsler geometry we refer to [205, 206] (see also [207] for an alternative presentation). We will consider the geometry of spaces, not of spacetimes. The extension of the definitions to the case of indefinite Finsler spaces will be discussed in the next chapter. If not stated otherwise, the dimensionality of the manifold is arbitrary.

4.1 Finsler spaces: definition

Definition *Let V be a (real) vector space and let $F : V \to \mathbb{R}$ be a continuous function. The function F is called a norm if:*

(a) $F(v) > 0$, $\forall v \in V$. Moreover $F(v) = 0 \Leftrightarrow v = 0$;

(b) $F(\lambda v) = |\lambda| F(v)$, $\forall \lambda \in \mathbb{R}, \forall v \in V$;

(c) the triangular inequality holds, $F(u) + F(v) \leq F(u + v), \forall u, v \in V$.

The second condition can be slightly modified, restricting the condition of homogeneity to $\lambda > 0$. While this certainly leads to a structure which do have different properties than a standard norm, with an abuse of language this kind of function is sometimes called a norm. In this specific case the condition of homogeneity of the norm is called *positive homogeneity.* In the rest of the thesis we will not use positively homogeneous norms unless otherwise specified. Notice also that there is no restriction upon the dimension of the vector space.

Besides the norm, another important structure can be introduced. It is called the Finsler metric tensor:

$$g_{ij}(v) \equiv \frac{1}{2} \frac{\partial^2 F^2}{\partial v^i \partial v^j}, \tag{4.1}$$

4.1.1 Basic theorems

Given a positive definite bilinear form (*i.e.* a scalar product) on V, represented by the matrix b_{ij}, a norm can be defined:

$$F(v) = (b_{ij} v^i v^j)^{1/2}. \tag{4.2}$$

In this case, the Finsler metric tensor defined in (4.1) is given by b_{ij} and the norm is Riemannian. However, not all the norms are defined through scalar products, as we shall see later.

The homogeneity property of the norm plays often an important role. In particular, Euler's theorem and its consequences provide some useful properties that can be exploited in the analysis. The discussion of these issues can be found in appendix A. Euler's theorem can be cast in the following form.

Euler's Theorem *Let f be a real-valued function on $\mathbb{R}^D$, differentiable away from the zero vector. The two following statements are equivalent.*

A. f is positively homogeneous of degree s,

$$f(\lambda v) = \lambda^s f(v), \ \forall \lambda > 0.$$

B. The function f satisfies the following partial differential equation

$$v^i \frac{\partial f}{\partial v^i} = s f(v).$$

Here we shall mention a rather important consequence of the homogeneity property of the norm that deserves some comments. The matrix defined in (4.1) is homogeneous of degree zero:

$$g_{ij}(\lambda v) = g_{ij}(v), \tag{4.3}$$

as one can easily check. This fact has an immediate implication. Since the matrix g is homogeneous of degree zero, in general it is not continuous at $v = 0$: if we take two non-collinear vectors u, w ($v \neq \alpha w$, for some α), such that $g_{ij}(v) \neq g_{ij}(w)$, then taking the limit of $g_{ij}(\lambda v), g_{ij}(\lambda w)$ for $\lambda \to 0$ we obtain two different results, for the homogeneity property of the matrix g. Hence, the limit of $g_{ij}(v), v \to 0$ does not exist and hence the tensor g_{ij} is not continuous in $v = 0$. There is of course a case in which the continuity holds on the entire space V, and it is the case in which $g_{ij}(v) = b_{ij}$, *i.e.* a norm induced by a scalar product.

It follows from the homogeneity of the norm that the norm itself can be always written as:

$$F(v) = (g_{ij}(v) v^i v^j)^{1/2}. \tag{4.4}$$

This follows from homogeneity and Euler's theorem. Indeed

$$g_{ij}(v) v^i v^j = \frac{1}{2} \frac{\partial^2 F^2}{\partial v^i \partial v^j} v^i v^j = \frac{1}{2} \frac{\partial F^2}{\partial v^i} v^i = F^2,$$

where we have used Euler's theorem twice.

It can be shown that the validity of the triangular inequality is equivalent to the fact that the matrix (4.1) is positive definite. This statement is a part of a more general theorem on Finsler norms (see, for instance [206], section 1.2B)

Theorem *Let F be a positively homogeneous, nonnegative real-valued function on $\mathbb{R}^D$ such that it is $\mathcal{C}^\infty$ on $\mathbb{R}^D \setminus 0$, and the matrix defined by (4.1) is positive definite. Then the function is positive on $\mathbb{R}^D \setminus 0$ ($F(v) > 0$, $\forall v \neq 0$), it obeys the triangular inequality as well as the fundamental inequality*

$$w^i \frac{\partial F}{\partial v^i}(v) \leq F(w), \tag{4.5}$$

the equality holding if and only if $w = \alpha v$, for some $\alpha > 0$.

The proof is omitted. It can be found for instance in [206], section 1.2B. This theorem is crucial for two reasons. The first one is that it gives a direct computational criterion to decide whether a function is a norm, reducing the task of checking the triangle inequality to a check of the positive definiteness of a matrix. The second important consequence comes from a closer inspection of the inequality (4.5). Let us elaborate a bit on this.

From Euler's theorem we know that

$$v^i \partial_{v^i} F(v) - F(v) = 0,$$

hence (4.5) can be rewritten as

$$F(v) + \partial_{v^i} F(v)(w^i - v^i) \leq F(w)$$

This has a nice graphical interpretation. It is telling that at any point v, the graph of the function F is convex. In particular the homogeneity property implies that the graph is a convex cone emanating from the origin of $\mathbb{R}^{D+1}$.

Moreover, given that the norm is positive for every vector different from zero, (4.5) can be rewritten as

$$g_{ij}(v) w^i v^j \leq F(w)F(y), \tag{4.6}$$

which is nothing else than the generalization of the Cauchy–Schwarz inequality usually discussed for scalar products.

The fundamental inequality has some deep implications in issues on global Finsler geometry, which are irrelevant for the present discussion. It has however an interesting application in the definition of angle. In Euclidean geometry the metric tensor defines the angle $\theta(w, u)$ between the two vectors w, u through:

$$\cos(\theta(w,u)) = \frac{b_{ij} w^i u^j}{(b_{mn} u^m u^n)^{1/2} (b_{hk} w^h w^k)^{1/2}}. \tag{4.7}$$

This is a direct consequence of the Cauchy–Schwarz inequality: the right hand side is always bounded in modulus. Notice that the angle is symmetric in the two vector arguments, $\theta(w, u) = \theta(u, w)$. For a Finsler norm, we can define the notion of angle between vectors in the same way:

$$\cos\left(\theta(w,u)\right) = \frac{g_{ij}(w) w^i u^j}{F(w)F(u)}. \tag{4.8}$$

The crucial difference with the angle defined by bilinear forms (to which this definition reduces when the norm comes from a Euclidean scalar product) is that it is not symmetric in the vector arguments any more. In general we have that $\theta(w, u) \neq \theta(u, w)$. The implications of this will be discussed later in this chapter, when we will speak about orthogonality.

Disformal transformations Looking at equation (4.4) one could imagine to be able to define a Finsler norm giving the matrix $g_{ij}(v)$ from the beginning. However this is not as simple as it seems. Indeed, a general symmetric matrix $g_{ij}(v)$, even homogeneous of degree zero, is not necessarily a second derivative of a scalar function of the vector argument. Comparing (4.4) with (4.1) one can get a necessary condition for $g_{ij}(v)$:

$$\frac{\partial g_{jm}(v)}{\partial v^i} v^m + \frac{\partial g_{im}(v)}{\partial v^j} v^m = -\frac{1}{2} \frac{\partial^2 g_{mn}(v)}{\partial v^i \partial v^j} v^m v^n.$$

To see this, one could imagine to give a generalization to Finsler norms of conformal transformations of the metric, namely $g_{ij}(v) = \Omega^2(v) b_{ij}$. It is easily found that, in order for g_{ij} to be the matrix of the second derivatives of the Finsler norm obtained from it, $\Omega(v)^2$ is constrained to satisfy

$$\frac{\partial \Omega^2}{\partial v^i \partial v^j} = -2 \left(\frac{b_{im} v^m}{b_{hk} v^h v^k} \frac{\partial \Omega^2}{\partial v^j} + \frac{b_{jm} v^m}{b_{hk} v^h v^k} \frac{\partial \Omega^2}{\partial v^i} \right).$$

Therefore, the generalization of conformal transformations to Finsler norms is properly defined only in terms of the norm itself, $\tilde{F}(v) = \Omega(v) F(v)$, with $\Omega > 0$ a positively homogeneous function of degree zero. The corresponding metrics will not be conformally related. Instead, the two metrics will be related by a kind of disformal transformation,

$$\tilde{g}_{ij}(v) = \Omega^2(v) g_{ij}(v) + \left(\frac{\partial \Omega^2}{\partial v^i} g_{jk}(v) v^k + \frac{\partial \Omega^2}{\partial v^j} g_{ik}(v) v^k \right) + \frac{1}{2} \frac{\partial \Omega^2}{\partial v^i \partial v^j} F^2(v),$$

which is similar to the disformal transformations often encountered in generalized (bi-metric) theories of gravity, where the physical and the gravitational metrics are related by

$$\tilde{g}_{\mu\nu} = \Omega^2(x) g_{\mu\nu} + V_\mu(x) V_\nu(x), \tag{4.9}$$

where $\Omega(x)$ and V_x are parametrizing the deformation of the metric. See for instance [209] for a discussion of the possible role of these transformations in gravity.

4.1.2 The Cartan tensor

The matrix $g_{ij}(v)$ is not the only interesting structure derived from a norm. Another relevant structure is the third derivative of the square of the norm (if it is differentiable three times), often called the Cartan tensor:

$$C_{ijk}(v) \equiv \frac{1}{2} \frac{\partial^3 F^2}{\partial v^i \partial v^j \partial v^k}. \tag{4.10}$$

It is a completely symmetric tensor (if the norm is at least $\mathcal{C}^3$) homogeneous of degree (-1) in the argument. As a consequence it has an important property:

$$C_{ijk}(v) v^i = C_{ijk}(v) v^j = C_{ijk}(v) v^k = 0, \tag{4.11}$$

that can be found by applying the Euler's theorem on homogeneous functions (again, see the appendix A).

The Cartan tensor is what distinguishes Riemannian geometry from generic Finslerian geometry. Since the Cartan tensor is a tensor, if it is different from zero in one coordinate chart, it will be different from zero in any coordinate chart. It is easy to show that if a norm is induced by a scalar product, then the Cartan tensor is zero. Therefore, by means of a coordinate transformation we cannot turn a Finsler norm into a Riemannian one.

There is a theorem [208] which provides a characterization of Riemannian spaces within the domain of Finsler spaces by making use of the properties of the Cartan tensor.

Theorem (Deicke) *Let F be a norm on $\mathbb{R}^D$. The following three statements are equivalent.*

A. F is Euclidean, $F^2(v) = b_{ij}v^iv^j$ for some positive definite b_{ij}.

B. $C_{ijk} = 0$ for all i, j, k.

C. $C_k = g^{ij}C_{ijk} = 0$ for all k.

The proof of this theorem is omitted.

Banach spaces, Hilbert spaces and metric spaces Given a (real) vector space V, it acquires the structure of a Banach space once a norm is defined over it. If, on the other hand, instead of a norm one defines a scalar product, *i.e.* a positive definite bilinear form, one speaks about a Hilbert space. Clearly, a Hilbert space is a Banach space, while the converse is not true. In Finsler geometry, the tangent space to each point of the manifold is a Banach space. When it is a Hilbert space, the geometry is Riemannian.
To completely fix the terminology, we define a metric space a set X over which a distance function $d : X \times X \to \mathbb{R}$ is defined, with:

- $d(x, y) \geq 0$, $\forall\, x, y \in X$, and $d(x, y) = 0 \Leftrightarrow x = y$;
- $d(x, y) = d(y, x)$, $\forall\, x, y \in X$;
- $d(x, y) + d(y, z) \geq d(x, z)$, $\forall\, x, y, z \in X$

In the case of Finsler geometry (and hence in Riemannian geometry too), the distance between two points is defined to be the length of the shortest geodesic connecting them. Therefore, a Finsler manifold is naturally endowed with a metric structure.

4.1.3 Finsler spaces

Having discussed the properties of a norm, we can pass to the definition of a Finsler space.

Definition A Finsler space is a pair $(\mathscr{M}, F)$, where $\mathscr{M}$ is a manifold of dimension D and $F_{-}(-)$ is a pointwise defined norm, *i.e.* $F_m(-) : T_m\mathscr{M} \to \mathbb{R}$, $\forall m \in \mathscr{M}$.

We can conclude therefore that Riemannian spaces are special cases of Finsler spaces, where the norm is defined by a scalar product. It is important to make this statement more explicit, making a comparison with other generalization of Riemannian geometry.

Through a norm one can define a notion of distance. Indeed, one can define the notion of length of an arc of curve in the obvious way:

$$\ell(C) = \int_{\tau_0}^{\tau_1} F(x, \dot{x})d\tau. \tag{4.12}$$

Due to the homogeneity property of the norm this definition is reparametrization invariant. Take two points $p, q \in \mathscr{M}$, and consider all the curves connecting these two points. The minimum[1] of the lengths of all these curves defines a distance between the two points.

We can hence say that *Finsler geometry is a generalization of Riemannian geometry*, but it is still based on the existence of a distance function. It generalizes the way in which distances are defined. Moreover, the linearity properties, often used in Riemannian geometry due to the fact that the metric tensor is bilinear, cannot be used in Finsler geometry, where the metric tensor is not defining a bilinear form, in general. Hence, Finsler geometry is fundamentally different from metric-affine theories [210], where the geometry of a manifold is described in terms of a metric tensor and an independent affine connection. The analysis of the geodesic equation will make this statement rigorous.

Let us make a side remark. As we have shown, the metric tensor and the Cartan tensor are not continuous at $v = 0$. Therefore, in defining the various structures needed in Finsler geometry it is customary to remove the zero section of the tangent bundle. The resulting bundle is called the *slit tangent bundle*,

$$\breve{T}\mathscr{M} \equiv T\mathscr{M} \setminus \mathbf{0}. \tag{4.13}$$

This bundle is particularly useful since typically Finslerian quantities are regular on it, since the continuity problems typically arise on the zero section. Therefore, most of the definitions are more transparent when given in terms of the slit tangent bundle.

4.1.4 The indicatrix

Before giving some examples, it is useful to introduce the notion of indicatrix. The indicatrix I_m is defined as $I_m = \{v \in T_m\mathscr{M} : F_m(v) = 1\}$. In Riemannian geometry the indicatrix is an ellipsoid (solution of a quadratic equation). It is interesting to note that due to the homogeneity property of the norm, the latter can be specified just in terms of the indicatrix (see for instance [211], chapter 15). Given a closed convex hypersurface J in $T_m\mathscr{M}$ with center in $v = 0$ (*i.e.* symmetric under reflections through the origin of the tangent space), and such that every ray emanating from $v = 0$ intersects J exactly once, we can define a norm $F^{(J)}$ in the following way. For any vector $w \in T_m\mathscr{M}$ there is a vector $\lambda w = u \in J$, for our hypothesis that J intersects each ray exactly once. Hence, one can define the norm of the vector w as

$$F^{(J)}(w) = \frac{1}{\lambda}. \tag{4.14}$$

In order for the norm to be defined in this way to obey the triangular inequality, the indicatrix must be convex. If it is desired, one can relax the property of symmetry around $v = 0$. This corresponds to the definition of positively homogeneous Finsler norms.

This construction allows us to have a clear visual picture of the relationships between Riemannian and Finslerian geometries. While Riemannian metrics correspond to all the ellipsoids containing the origin, Finsler norms correspond to all the closed and convex hypersurfaces, containing the origin and intersecting all the rays emanating from it just once.

[1]Actually, to be rigorous, one should take the Infimum, *i.e.* the maximum lower bound to the length of a curve connecting the two points.

Therefore we can say that while Riemannian geometry boils down to Euclidean geometry in a sufficiently small neighborhood, Finsler geometry can be distinguished from Euclidean geometry even in a single point. We will make this qualitative statement more rigorous in the rest of the chapter. What we can say already here is that Finsler geometry is a *locally anisotropic* extension of Riemannian geometry.

4.2 Examples

We have already seen that Riemannian spaces are particular cases of Finsler spaces where the norm is defined through a positive definite quadratic form. It is useful at this point to make a short list of other simple examples of Finsler structures. Other examples can be found in the book by Asanov [207].

4.2.1 Randers spaces

Let us assume to have a Riemannian space, with metric b_{ij}, and a vector field A_i. The function:

$$F(v) = (b_{ij}v^i v^j)^{1/2} + A_i v^i, \tag{4.15}$$

defines a positively homogeneous norm (pseudo-norm). The metric tensor is given as usual by:

$$g_{ij} = b_{ij} + A_i A_j + \frac{1}{2}\frac{b_{ik}v^k A_j + b_{jk}v^k A_i}{(b_{mn}v^m v^n)^{1/2}} + \frac{1}{2}\frac{b_{ij}A_k v^k}{(b_{mn}v^m v^n)^{1/2}} - \frac{1}{2}\frac{b_{ih}v^h b_{jl}v^l A_k v^k}{(b_{mn}v^m v^n)^{3/2}}. \tag{4.16}$$

There are some conditions for this norm to represent a Finsler norm. The first one is positivity: $F(v) > 0$, $\forall v \neq 0$. The fact that b_{ij} satisfies the Cauchy–Schwarz inequality implies that:

$$|A_i v^i| \leq (A_m A_n b^{mn})^{1/2}(v^h v^k b_{hk})^{1/2}. \tag{4.17}$$

The positivity condition therefore leads to

$$(b_{ij}v^i v^j)^{1/2} > -A_i v^i \geq -\mathrm{sgn}(A_i v^i)(A_m A_n b^{mn})^{1/2}(v^h v^k b_{hk})^{1/2}. \tag{4.18}$$

Hence, if $A_i v^i$ is positive the inequality is trivially verified. If the sign of $A_i v^i$ is -1, the inequality holds if and only if $A_i A_j b^{ij} < 1$.

The second condition to be checked is the positive definiteness of the matrix (4.16). We refer to [206] for the detailed discussion of this point.

The Finsler spaces of this class are called Randers spaces. They were discussed by Randers [212] with the purpose of introducing some extra structures besides the pseudo-Riemannian metric attached to spacetime in order to implement in a geometrical way the notion of "arrow of time". Of course, in order to do this, one should be able to rigorously extend the definitions of Finsler geometry to what would be the generalization of Lorentzian signature for Finsler spaces. The discussion of this issue is postponed to the next chapter, where applications of Finsler geometry to physics will be discussed in detail.

There is another obvious instance in which Randers spaces do appear in physics (with the abovementioned caveat about the Lorentzian signature): indeed the Lagrangian for a charged point particle moving in a given spacetime and external electromagnetic field takes exactly the same form of (4.15). However, given that particles with different charges will see different Finsler structures, for the analysis of the system it is simpler to speak about the motion of particles in a Lorentzian manifold under the influence of an external electromagnetic field rather than motion of particles in a multi-Finsler structure[2].

4.2.2 Berwald–Moor

Despite the Randers structures are pretty familiar to physicists, albeit under other names, they were not the first Finslerian structures to be introduced, hystorically. Riemann itself introduced the quartic line element, back in his dissertation *"Über die Hypothesen, welche der Geometrie zu Grunde liegen"* in 1854 [213]:

$$ds^4 = Q_{ijhk} dx^i dx^j dx^h dx^k. \tag{4.19}$$

However, he later focused on the simpler case of quadratic line elements, what we know today as Riemannian geometry. It was Finsler that later started developing the theory of more general metric spaces.

Nowadays these spaces are called Berwald–Moor spaces. They are defined through totally symmetric rank four tensors Q_{ijhk}. The norm is:

$$F(v) = (Q_{ijhk} v^i v^j v^h v^k)^{1/4}. \tag{4.20}$$

The corresponding Finsler metric can be easily derived:

$$g_{ij}(v) = 3 \frac{Q_{ijhk} v^h v^k}{(Q_{mnhk} v^m v^n v^h v^k)^{1/2}} - 2 \frac{Q_{ilhk} v^l v^h v^k Q_{jlhk} v^l v^h v^k}{(Q_{mnhk} v^i v^j v^h v^k)^{3/2}}. \tag{4.21}$$

In this case, positivity is guaranteed from the very definition, provided that the tensor Q_{ijhk} is nondegenerate (*i.e.* there is no vector $\bar{v}$ such that $Q_{ijhk} \bar{v}^i \bar{v}^j \bar{v}^h \bar{v}^k = 0$). On the other hand, the positive definiteness must be carefully checked.

As a particular case one could consider some sort of bi-metric Finsler structures, where the rank four tensor is the symmetrized product of two Riemannian metrics

$$Q_{ijhk} = b_{(ij} c_{hk)}. \tag{4.22}$$

As in the case of Randers spaces, these structures do appear in physics, as we have seen in the introductory chapter, for instance in the study of the propagation of light rays or sound waves in crystals.

4.2.3 (α, β) spaces

The class of spaces discussed by Randers can be further generalized. These spaces are characterized by two structures, a bilinear form a_{ij} and a vector field b_i. These two objects define two homogenous

[2] A multi-Finsler structure is the Finslerian counterpart of a multi-metric structure. Instead of having a manifold with a given number of metrics, we have a manifold with a given number of norms.

functions on the tangent space:

$$\alpha(v) = \sqrt{a_{ij}v^i v^j} \qquad \beta(v) = b_i v^i. \tag{4.23}$$

An (α, β) space is a special Finsler space where the norm is specified as

$$F(v) = \alpha(v)\Phi\left(\frac{\beta(v)}{\alpha(v)}\right), \tag{4.24}$$

where Φ, apart from the constraints due to the definition of a norm, is arbitrary.

A particular example, which has been already encountered in the previous chapter and that will be discussed in the following, is

$$f(v) = (\alpha(v))^{1-b}(\beta)^b. \tag{4.25}$$

As we have mentioned in the previous chapter, this is the class of Finsler norms introduced in very special relativity.

4.3 The Legendre transform

Up to now we have provided the definitions of objects defined in the tangent bundle, involving positions and velocities. However, for physical applications we have to refer to momenta, rather than to velocities. Since momenta are defined as tangent forms, and not as tangent vectors, we have to extend the analysis to the co-tangent bundle. The bridge between the tangent and the co-tangent bundle is provided by the Legendre transform. Here we show how to formulate this extension. For a detailed discussion we refer to [205] and [214].

In Riemannian geometry the inverse metric is defining straightforwardly a scalar product between forms. In order to do the same in Finsler geometry one has to take care of the dependence of the metric on the tangent vector. In Riemannian geometry we have that

$$\omega_\mu(v) = g_{\mu\nu}(x)v^\nu. \tag{4.26}$$

Since the metric tensor is defined through a non degenerate matrix, we can invert this relation to express a vector in terms of its dual form

$$v^\mu(\omega) = g^{\mu\nu}(x)\omega_\nu. \tag{4.27}$$

This shows how we can define a scalar product on forms, once we have defined a scalar product on vectors:

$$\langle\omega_1, \omega_2\rangle_{forms} \equiv \langle v(\omega_1), v(\omega_2)\rangle_{vectors}. \tag{4.28}$$

The strategy to generalize the duality between vectors and forms defined in Riemannian geometry to the case of Finsler spaces, is to perform a Legendre transform of a suitable power of the Finsler norm. We define the form dual to a given vector as:

$$\omega_\mu = g_{\mu\nu}(x, v)v^\nu. \tag{4.29}$$

This relation can be written as:

$$\omega(v)_\mu = \frac{1}{2}\frac{\partial F^2(x, v)}{\partial v^\mu}. \tag{4.30}$$

Clearly, if g is a non degenerate Finsler metric, then the map just defined between forms and vectors is invertible. Therefore, as promised, we define a norm on forms through the following Legendre transform

$$\frac{1}{2}G^2(x,\omega) = v^\mu(\omega)\omega_\mu - \frac{1}{2}F^2(x, v(\omega)), \tag{4.31}$$

or, equivalently,

$$G(x,\omega) = F(x, v(\omega)). \tag{4.32}$$

The tensor obtained from this norm plays the same role of the inverse metric tensor in Riemannian geometry, and it is simply given by

$$h^{\mu\nu}(x,\omega) = \frac{1}{2}\frac{\partial^2 G^2(x,\omega)}{\partial\omega_\mu \partial\omega_\nu}. \tag{4.33}$$

We can connect this tensor to the inverse metric $g^{\mu\nu}(x,v)$ just using the definition of G.

$$h^{\mu\nu}(x,\omega) = g^{\mu\nu}(x, v(\omega)). \tag{4.34}$$

As a side remark, one should note that the homogeneity properties of the structures defined in the tangent space percolate on the structures defined on the cotangent bundle, as it is easily realized from the homogeneity properties of the Legendre transform. In fact, if the vector v is rescaled to λv, the corresponding form ω is rescaled to $\lambda\omega$, thus implying that the function $G(\omega)$ is homogeneous of degree one in ω_μ.

This procedure is just the generalization to a Finsler space of the procedure given in Riemannian geometry, which is just a special case of Finsler spaces as we have already mentioned. This discussion shows that Finsler geometry can be presented either in the tangent space formulation, or in the co-tangent space formulation [214]. It is a matter of convenience which of them one wants to work with. Given that Finsler geometry can be seen as the geometry of the reparametrization invariant actions, the two choices correspond to the presentation of the action in the Lagrangian (tangent space) or in the Hamiltonian (co-tangent space) formalism. We will discuss this point later, when we will connect modified dispersion relations to Finsler spaces, showing that the situation is more rich than one could naively expect from just the analysis of the homogeneity properties of the norm.

Kähler manifolds It is interesting to see that an expression like (4.1) to define the metric tensor has some similarities with the definition of the metric in a Kähler manifold. A Kähler manifold is a complex manifold endowed with a complex structure J and a Hermitian metric g, such that the Kähler form $\Omega(X,Y) = g(JX,Y)$ is closed ($d\Omega = 0$) (For more details see for instance [215]). In this case the metric g is called the Kähler metric. It can be shown that any Kähler metric can be written as

$$g_{\mu\bar{\nu}} = \frac{\partial}{\partial z^\mu}\frac{\partial}{\partial \bar{z}^\nu}\mathcal{K}(z,\bar{z}).$$

The function $\mathcal{K}$ is called the Kähler potential. In Finsler geometry the norm is playing a role similar to the Kähler potential, generating the metric through second derivatives, even though the two structures are fundamentally different and should not be confused.

4.4 Geodesics, connections, a sketch

As we have seen in the first section, Finsler spaces are metric spaces, in the sense that it is possible to define a distance function $d : \mathscr{M} \times \mathscr{M} \to \mathbb{R}$ wich assigns to each pair of points the length of the shortest curve connecting it. As in Riemannian spaces we call these curves geodesics. To find their equation one can use their definition, *i.e.* the fact that they do minimize the length functional:

$$\ell = \int_{\tau_0}^{\tau_1} F(x, \dot{x}) d\tau.$$

The Euler-Lagrange equations obtained from the variation of this action, when we fix the parametrization $F(x, \dot{x}) = 1$, read:

$$\ddot{x}^k + G^k_{ij}(x, \dot{x})\dot{x}^i \dot{x}^j = 0, \tag{4.35}$$

where

$$G^k_{ij}(x, \dot{x}) = \frac{1}{2} g^{ih}(x, \dot{x}) \left(-\frac{\partial g_{jk}(x, \dot{x})}{\partial x^h} + \frac{\partial g_{jh}(x, \dot{x})}{\partial x^k} + \frac{\partial g_{kh}(x, \dot{x})}{\partial x^j} \right), \tag{4.36}$$

are generalizing the Christoffel's coefficients, which are obtained where the Finsler metric is just a Riemannian one.

This equation shows clearly what is the qualitatively new feature of Finsler geometry with respect to Riemannian geometry. Indeed, the geodesic equation can be seen as a particular instance of the equation of parallel transport. In Riemannian geometry geodesics can be defined as curves of shortest length as well as straightest curves (autoparallel curves). These definitions are equivalent. In the non-Riemannian geometry of metric-affine spaces, these two class of curves do not necessarily coincide, depending on the relationship between the metric and the connection. However, in both cases, the equation determining the transport of a vector along a curve has the same structure:

$$\frac{dV^i}{d\tau} + \Gamma^i_{jk}(x)\dot{x}^j V^k = 0, \tag{4.37}$$

where the Γs are the connection coefficients in the given coordinate system used. In these settings they do depend on the point of the manifold where the curve passes through, but not on its direction. In Finsler spaces, the connection coefficients $G^i_{jk}(x, \dot{x})$ *do depend on the direction of the curve.* This means that to describe geometry as the properties of geometrical figures and their transformations it is not enough to specify their position but also their orientation, *i.e.* the pair $(x, \dot{x}) \in T\mathscr{M}$. This is somehow obvious since Finsler spaces represents metric spaces where the metric tensor is anisotropic: it depends on the direction in which it is evaluated.

As a consequence, in Finsler geometry, in order to give the definition of a connection (and hence of curvature tensors) one has necessarily to work in the tangent bundle $T\mathscr{M}$, and treat it as the fundamental structure. The discussion of connections, curvatures and related topics is outside the scope of this discussion. For a complete presentation see [205, 206].

4.5 Isometries

In any physical theory the crucial role is played by the symmetries of the system which one is going to describe. This implies that we have to provide a definition of symmetry which is suitable to define, for instance, Noether charges.

When speaking about a Riemannian structure over a manifold, we define an isometry to be a diffeomorphism $f : \mathscr{M} \to \mathscr{M}$ which leaves the metric structure invariant:

$$f_* g = g. \tag{4.38}$$

For the case of Finsler geometry this definition still applies. As in Riemannian geometry, when considering a vector field over the manifold as the generator of a one parameter family of diffeomorphisms, we can define the Lie derivative of the metric tensor in the familiar way, since the Lie derivative is not a structure related to any metric or connection defined over the manifold. The only difference with the standard treatment is that we have to take into account the fact that the metric tensor depends on a tangent vector too, when computing the Lie derivative. Therefore, given a vector field Y on $\mathscr{M}$, we obtain that the Lie derivative of the metric tensor (see [205]) evaluated on the field Y along the direction of the vector field ξ, in components, is given by:

$$\pounds_\xi g_{\mu\nu}(x, Y(x)) = \xi^\rho \partial_\rho g_{\mu\nu}(x, Y) + \partial_\mu \xi^\rho g_{\rho\nu}(x, Y) + \partial_\nu \xi^\rho g_{\rho\mu}(x, Y) + \\ + 2C_{\sigma\mu\nu}(x, Y)\xi^\rho \partial_\rho Y^\sigma. \tag{4.39}$$

The Killing equation for the Finsler metric tensor is obtained asking that the Lie derivative of the metric tensor vanishes. As it is easily seen, the Killing equation is a linear (system of) PDE for the Killing vector field ξ. Since the concept of Lie derivative is independent from the metric structure, here we do not collect the various derivatives of the metric tensor appearing in this equation into a combination of connection coefficients, as it is done, for instance, in [205]. Accordingly, the formal properties of the Lie derivative are unchanged. In particular, the property:

$$\pounds_{[X,Y]} T = \pounds_X \pounds_Y T - \pounds_Y \pounds_X T, \tag{4.40}$$

where T is a "velocity-dependent" tensor, is still valid, as an explicit calculation can prove. This property is crucial, since it tells us that the Killing vectors of a Finsler metric define a Lie algebra, with the product given by the Lie bracket between vector fields.

Number of Killing vectors

A crucial point is the number of Killing vectors. In Riemannian geometry, we have that a Killing vector field satisfies the two equations:

$$\xi_{i;j} - \xi_{j;i} = 0; \qquad \xi_{i;jk} = -R^h_{jki}\xi_h. \tag{4.41}$$

The first equation is just the Killing equation, while the second is a condition which follows from the Killing equation and the Bianchi identities for the curvature tensor. This equation tells us that a Killing vector field is uniquely determined once we specify its value and its first derivatives at a single point: all the coefficients of the Taylor expansion are uniquely determined through these initial data and the curvature of the manifold. As a consequence, a Killing vector field is determined by $N(N + 1)$ parameters, where N is the number of dimensions of the manifold. The Killing equation tells us that these parameter must satisfy $N(N + 1)/2$ linear equations. Therefore there are at most $N(N + 1)/2$ Killing vector field on a Riemannian manifold.

To see in which cases this number is maximal, one needs to consider some integrability conditions for the Killing equation. In particular, using (4.41) and the Bianchi identities, we can prove that there is another relation which must be satisfied by a Killing vector field [216]:

$$\xi_m(R^m_{kij;l} - R^m_{lij;k}) + \xi_{m;l}R^m_{kij} - \xi_{m;k}R^m_{lij} + \xi_{i;m}R^m_{jkl} + \xi_{m;j}R^m_{ikl} = 0. \tag{4.42}$$

These conditions reduce the number of independent parameters. The only case in which this equation reduces to a tautology is the one in which the Riemannian space is maximally symmetric [216], *i.e.* its curvature tensor has the form $R_{ijkl} = R(g_{ik}g_{jl} - g_{il}g_{jk})/N(N-1)$, with R constant.

The discussion of symmetries in Finsler spaces should proceed in the same way. First of all, we do not want to consider the case in which there is "curvature", since we already know that even in Riemannian geometry curvature reduces the symmetry of the manifold, in general. We therefore consider the content of the Killing equation for flat Finsler spaces. We mimick the procedure followed in Riemannian geometry, finding that the second derivative of the Killing vector field must satisfy:

$$g_{ik}\partial_j\partial_m\xi^k + C_{imk}\partial_j\partial_h\xi^k y^h + C_{ijk}\partial_m\partial_h\xi^k y^h - C_{jmk}\partial_i\partial_h\xi^k y^h = 0. \tag{4.43}$$

In order to argue that the only solution to these equations is $\partial^2\xi = 0$ we need to discuss what happens to geodesics under the Killing flow. Since the metric is left invariant, then the length of curves is left invariant too, and therefore the geodesics are sent into geodesics. In particular, this implies that, under the (now infinitesimal) Killing flow:

$$x^\mu \to x'^\mu = x^\mu + \epsilon\xi^\mu, \tag{4.44}$$

the geodesic equation is left invariant. In the particular situation in which we are working (*i.e.* flat Finsler spaces), this implies that it must be:

$$\frac{d^2x^\mu}{d\tau^2} = 0 \leftrightarrow \frac{d^2x'^\mu}{d\tau^2} = 0. \tag{4.45}$$

Trivial algebra leads to the desired result:

$$\partial_\mu\partial_\nu\xi^\rho = 0. \tag{4.46}$$

As in Riemannian geometry, then, the Killing vector field can be determined from the vector at a point and its first derivatives, or, equivalently, a Killing vector field is determined at most by $N(N+1)$ real parameters.

To determine how many of them are free one has to count the equations which have to be solved. If we consider the Killing equations in Finsler spaces, we have to consider an $N-$parameter family of $N(N+1)/2$ equations, where the parameters are the component of the tangent vector on which we evaluate our Finsler metric tensor. We can look at these equations in another way. We can fix the tangent vector to be a special one, $y \neq 0$, and we ask that the Killing vector field leaves invariant not only the metric on that point, but also all its derivatives. This is equivalent to the replacement of the metric with its Taylor expansion around a given point.

To see how this is working, we can derive the Killing equation w.r.t. the tangent vector to obtain a new, linearly independent, set of linear equations for the Killing vector and its derivatives:

$$C_{ikm}\partial_j\xi^k + C_{jkm}\partial_i\xi^k + 2C_{ijk}\partial_m\xi^k + 2C_{ijkm}\partial_h\xi^k y^h = 0. \tag{4.47}$$

This equation reduces the number of free parameters at our disposal, unless $C_{ijk} = 0$, which is precisely the case in which the Finsler metric reduces to a Riemannian one. In particular, this last equation is precisely the condition that the Cartan tensor is left invariant under the Lie transport:

$$\pounds_\xi C_{ijk} = 0. \tag{4.48}$$

Similarly, we can see that the Killing equation implies that:

$$\pounds_\xi C^{(n)}_{i_1 \ldots i_n} = 0, \tag{4.49}$$

where we have introduced the notation:

$$C^{(n)}_{i_1 \ldots i_n} = \frac{\partial^n F^2}{\partial y^{i_1} \ldots \partial y^{i_n}}, \tag{4.50}$$

for a generalization of the Cartan tensor to higher ranks. This means that there is a whole family of conditions which must be satisfied by the Killing vectors. Some of them will be redundant, while others will reduce the number of free parameters, *i.e.* the number of Killing vectors. In general, the minimum number of equations is obtained when $C_{ijk} = 0$, which corresponds to the case of Riemannian geometry (in the flat case). In practice we cannot present an argument to tell that this is the only reasonable situation: there could be situations in which some of the higher order Killing equations we are considering are linearly dependent among themselves, reducing then the number of equations to be solved. What we can say is that since the number of equations we have to solve greater than the $N(N+1)/2$ given by the standard Killing equations for the metric, we have that a Finsler space can admit at most $N(N+1)/2$ Killing vectors. The next sections will be devote to the analysis of the conditions which are guaranteeing that a Finsler norm is actually determined by a Riemannian metric, and in particular the proof of the statement that Riemannian spaces are the most isotropic Finsler spaces.

4.6 Back to Riemannian geometry

The description of Finsler geometry given in this chapter naturally leads to the question: if we assume that a manifold is endowed with some sort of Finsler structure, what are the conditions which are forcing it to be Riemannian?

In this section we will briefly discuss two of these conditions. They are related with the very familiar notion of orthonormal frames, *i.e.* a basis of the tangent space made by normalized vectors mutually orthogonal. It is important to understand how this concept can be used in Finsler geometry and what are the consequences of requiring the existence of them together with some requirements about their mutual relations.

4.6.1 Orthogonality

The core of Finsler geometry is represented by the definition of a norm over a vector space. As we have already mentioned, a norm is not necessarily given by a scalar product. Therefore, in Finsler geometry there is no obvious definition of angle. In fact one can define some notion of angle, as we have briefly discussed in the section 4.1, but this is by no means unique. (See also [205] for more details.)

A more simple notion that is in any case useful is that of orthogonality of vectors. In Riemannian geometry two vectors u, v are orthogonal ($u \perp v$) if

$$b_{ij}u^i v^j = 0.$$

In Finsler geometry two vectors are said to be orthogonal ($u \perp_F v$) if:

$$g_{ij}(u)u^i v^j = \frac{1}{2}\frac{\partial F^2}{\partial u^i}(u)v^i = 0. \tag{4.51}$$

Of course, this notion of orthogonality is equivalent to the requirement that the angle between two vectors is $\pi/2$, see (4.8). It is clear that the orthogonality relation $\perp_F$ is not symmetric. In general $u \perp_F v$ does not imply $v \perp_F u$. Notice that, if we have positively homogeneous norms, $u \perp_F v$ does not even imply $-u \perp_F v$.

It is striking that there is some sort of rigidity in the notion of orthogonality. In particular, it has been proved that in three dimensions the orthogonality relation is symmetric if and only if the norm is obtained from a quadratic form. The theorem does not hold in two dimensions, for which there are counterexamples that can be presented. There can be two dimensional Finsler spaces for which the orthogonality relation is symmetric and yet they are not Riemannian.

Even though, to the best of our knowledge, the theorem is not proved for four dimensions, even in the Riemannian signature, for physical applications three spatial dimensions are enough (neglecting the possibility of having additional spatial dimensions, which are a logical possibility, even though in this thesis it is always assumed that spacetime is four dimensional).

Indeed, an observer should be able to label to each point in spacetime with four numbers. These will correspond to readings of clocks and rods. In particular, the observer can define space through the equation $t = const.$ We assume that this coordinate system is such that the Finsler norm does not depend on coordinates. On each of this constant time slices, the four dimensional norm $F_4(\dot{t}, \dot{x}, \dot{y}, \dot{z})$ will induce a three dimensional one $F_3(\dot{x}, \dot{y}, \dot{z})$ in a canonical way. It is this norm which is establishing the orthogonality relations. In particular, if we ask that the statement $u \perp_F v \Leftrightarrow v \perp_F u$ holds for any pair (u, v) of spatial vectors, this norm must be a Riemannian one, *i.e.* $F_3(u) = \sqrt{b_{ij}u^i u^j}$. Notice that this argument is not sufficient for arguing that F_4 must be Riemannian as well (more precisely pseudo-Riemannian). However, it represents a strong restriction. The symmetry of the orthogonality relations for spatial vectors is forbidding any spatial anisotropy and hence is forcing the Finsler structure of spatial slices to be Riemannian.

4.6.2 A theorem by Helmholtz

There is another way in which we can characterize Riemannian spaces among Finslerian ones, and it involves the number of Killing vectors allowed. There is a theorem due to Helmholtz stating that if the number of Killing vectors is maximal (*i.e.* $D(D+1)/2$ for a D dimensional space) then the space is a maximally symmetric Riemannian space. In other words, this theorem is the precise mathematical statement that a Euclidean space is the most symmetric flat Finsler space, and that any other Finsler space must be locally anisotropic. Here we will give a sketch of a proof (see also [217] and [211]).

Let us consider a Finsler space $(\mathbb{R}^D, F)$, where F is a norm, continuous on the entire tangent space $V = T_p\mathbb{R}^D$, $\forall p \in \mathbb{R}^D$ and differentiable in $V \setminus 0$. Let us moreover consider the homogeneous case $\partial_x F = 0$. Under a diffeomorphism, and in particular under an isometry, the tangent vectors transform linearly:

$$\tilde{\xi}^i = A^i_j(p,\omega)\xi^j, \tag{4.52}$$

where ω are the parameters of the transformation. Let us assume that the isometry group is a Lie group. This implies that the matrix A can be seen as the exponential of a matrix $T(p)$, the generator, with parameter ω

$$A = \exp(\omega T(p)). \tag{4.53}$$

It is straightforward to see that

$$F(A\xi) = F(\xi) \Leftrightarrow \frac{\partial F}{\partial \xi^i} T^i_j \xi^j = 0. \tag{4.54}$$

This is a useful characterization of the isometries in terms of their generators.

It is convenient to generalize the notion of indicatrix, defining the hypersurfaces:

$$I_a = \{\xi \in V | F(\xi) = a\}. \tag{4.55}$$

By definition $F(I_a) = a$, and the indicatrix corresponds to I_1.

The hypersurfaces I_a are compact. Indeed, given the coordinate system in V we are using, define an auxiliary norm:

$$||\xi|| = (\delta_{ij}\xi^i\xi^j)^{1/2}, \tag{4.56}$$

where δ_{ij} is the Kronecker one, without loss of generality. Let us consider the unit sphere in this norm.

$$S_1 = \{\xi \in V| \, ||\xi|| = 1\}. \tag{4.57}$$

On S_1, the function F has a maximum and a minimum, respectively M, m since it is continuous. On the sphere with radius a/m, $S_{a/m}$, F reaches the minimum value a (by positive homogeneity of F). Hence $S_{a/m}$ fully includes I_a. If it were not the case, a would not be a minimum of F on $S_{a/m}$. Alternatively, one could use $S_{2a/m}$, showing that there cannot be any intersection. In particular, for every vector on I_a we have:

$$||\xi|| \leq \frac{a}{m}. \tag{4.58}$$

Therefore the hypersurfaces I_a are indeed bounded. Closedness follows from the fact that the indicatrix has one and only one intersection with every ray emanating from the origin of the vector space.

The other ingredient we need for our argument is that the only allowed symmetries are the rotations. Indeed, invariance under A implies:

$$\frac{\partial F}{\partial \xi^i} T^i_j \xi^j = 0. \tag{4.59}$$

Let us look at the characteristic curves, defined as solutions of the differential equation:

$$\dot{\xi}^i(t) = T^i_j \xi^j(t), \tag{4.60}$$

in the tangent space. On these curves F is constant, as it is easily shown. If we pass to complex vectors, $\mathbb{R}^D \to \mathbb{C}^D$, a general solution of these equations is:

$$\xi^i(t) = \sum_{n=1}^{D} w^i_n e^{\mu_n t}, \tag{4.61}$$

where the ws are (complex) integration constants, and μ_n are the solutions of the secular equation:

$$\det(T - \mu \mathbb{I}_D) = 0. \tag{4.62}$$

This equation is a polynomial equation of degree D in μ. In general, the D solutions of this equation are complex. Furthermore, if $\mu = a + ib$ is a solution, $\mu^* = a - ib$ is a solution too. This allows us to use trigonometric functions in place of complex exponentials. Therefore we can rewrite the most general solution to the equation for the characteristic curves as:

$$\xi^i(t) = \sum_{n=1}^{D} (u_n^i e^{a_n t} \sin(b_n t) + w_n^i e^{a_n t} \cos b_n t). \tag{4.63}$$

We see that the norm of this vector under grows without bounds as t increases, unless the a_n are all zero.

To show this with an example, fix a given n, and $u_n \cdot w_n = 0, ||u_n||^2 = ||w_n||^2 = 1$ with the scalar product defined by δ_{ij}. The modulus of this vector has the property that

$$||\xi^i(t)|| = e^{a_n t}, \tag{4.64}$$

so it grows without bound if $t \to \infty$, if $a_n > 0$ (of course, if $a_n < 0$ the growth is for negative values of the parameter). If it grows without bound it will eventually cross the sphere which defines the boundary of the domain where the indicatrix must be confined. As a consequence, then, all the eigenvalues must be pure imaginary numbers. On the other hand this implies that the generators of isometries must take the form:

$$\mathrm{diag}(ib, -ib, 0, ..., 0), \tag{4.65}$$

which is the diagonal form of an antisymmetric matrix, *i.e.* the familiar generator of a rotation.

We can now go to the most important part we are interested in, showing that the maximal symmetry group of a Finsler norm is $SO(D)$. If the symmetry is maximal, the Finsler space is of Riemannian type.

To prove this it is enough to realize that in the most symmetric situation, one has that all the generators of rotations are included. These will generate $SO(D)$. The second part of the statement follows by observing that the orbit of this $SO(D)$ is an ellipsoid (we are including the possibility that the axis are not cartesian). The equation of an ellipsoid is:

$$b_{ij}\xi^i\xi^j = a^2. \tag{4.66}$$

for b positive definite bilinear form. This can be seen easily from the fact that the orbit of a vector under each generator is an ellipse, which obeys a quadratic equation. As a consequence, we have proved that on each I_a,

$$F(\xi)|_{I_a} = (b_{ij}\xi^i\xi^j)^{1/2} = a^2. \tag{4.67}$$

Homogeneity of the norm let us extend this equality on the whole vector space, *i.e.* $F(\xi) = (b_{ij}v^i v^j)^2$. Therefore if the (connected part of the) isometry group of the norm is $SO(D)$ the norm itself is Euclidean.

This clearly shows that the most symmetric Finslerian spaces are Riemannian spaces. Of course, the inclusion of curvature must be taken into account. However the result does not change substantially ([211] Ch. 15): the most symmetric Finsler spaces are maximally symmetric Riemannian spaces.

Chapter 5

Finsler geometry and emergent spacetimes

The previous discussion has shown that Finsler geometry could be a promising candidate for a generalized spacetime geometry beyond Riemannian geometry. This idea is not new at all: see, for instance, [176, 177, 178, 179].

In order to apply the concept of Finsler geometry to spacetime it is necessary, however, to extend the definitions to the case of a Lorentzian signature. This is an operation that must be done with care even in the case of Riemannian geometry, see [218]. In the case of Finsler geometry this task proves to be even harder, as it will be clear from the discussion in the second section of the chapter.

The central topic of this chapter is the discussion of the relation between modified dispersion relations and Finsler geometry. Indeed, it turns out that, despite the mentioned difficulties, Finsler geometry is able to put in relation different areas of theoretical physics in which Lorentz invariance has been questioned, either in emergent spacetime scenarios, like in analogue models, or in some quantum spacetime scenarios. This is possible since, as we will show, modified dispersion relations, generically present in all these frameworks, are nothing else than the manifestation of some Finsler structure for spacetime. Of course, there are exceptions to this rule: for instance, non-commutative spacetimes cannot be reduced to Finsler spaces.

Before discussing these important points, it is useful to show how Finsler geometry, in the Riemannian signature, does appear even in a familiar situation: the optical geometry of stationary spacetimes [219, 220].

5.1 Optical Geometry

In general relativity, as well as in all the metric theories of gravity, photons are moving along null geodesics[1]. The study of the structure of null geodesics can be used to describe the some properties of geometry, at least in some particular cases.

Consider a static spacetime $(\mathscr{M}, g)$, where g is a Lorentzian metric with a timelike, hypersurface-

[1] This is not completely true. Quantum corrections can introduce nonminimal coupling terms like $R_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}$: these terms can change the properties of propagation of photons. See [221].

orthogonal Killing vector field. Let Σ be the three-dimensional hypersurface associated to it. It is possible to find an adapted coordinate system in which the metric takes the form:

$$ds^2 = -f(x)dt^2 + h_{ij}(x)dx^i dx^j, \tag{5.1}$$

where t is a coordinate for the transverse direction, h is the three dimensional metric induced on Σ, and the x^i are coordinates on the hypersurface.

Null geodesics in $\mathscr{M}$ are defined by:

$$-f(x)\left(\frac{dt}{d\lambda}\right)^2 + h_{ij}(x)\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda} = 0, \tag{5.2}$$

or, equivalently

$$\dot{t} = \pm\left(\tilde{h}_{ij}(x)\dot{x}^i\dot{x}^j\right)^{1/2}, \qquad \tilde{h}_{ij}(x) = \frac{1}{f(x)}g_{ij}(x), \tag{5.3}$$

where λ is an affine parameter on the null geodesic and the dot denotes the differentiation with respect to it. Therefore, instead of working with the full four dimensional geometry of $\mathscr{M}$, one can use an induced metric structure on Σ. Indeed, one can introduce a distance function on the hypersurface by assigning to each pair of points the optical distance Δt needed for a null ray to connect them. This procedure generates a new Riemannian space $(\Sigma, \tilde{h})$. This is called the optical space. It turns out that null geodesics of the metric g in $\mathscr{M}$ are projected onto geodesics of the optical metric $\tilde{h}$ on Σ. For more details see [219] and references therein.

In the case of stationary spacetimes, the situation changes slightly. In particular, the optical geometry turns out to be described by a Finsler structure. For this class of spacetimes, there is an adapted coordinate system such that the metric reads:

$$ds^2 = -f(x)dt^2 + v_i(x)(dtdx^i + dx^i dt) + h_{ij}(x)dx^i dx^j. \tag{5.4}$$

Repeating the calculation done for the static case, one easily gets:

$$\Delta t = \int \frac{dt}{d\lambda}d\lambda = \int d\lambda \frac{v_i\dot{x}^i \pm \sqrt{(v_i\dot{x}^i)^2 + f(x)h_{ij}(x)\dot{x}^i\dot{x}^j}}{f(x)}, \tag{5.5}$$

where the sign $\pm$ is introduced again to distinguish past from future directed geodesics. With an obvious redefinitions of the various quantities:

$$A_i = \frac{v_i}{f}, \qquad \tilde{h}_{ij} = \frac{h_{ij}}{f} + A_i A_j, \tag{5.6}$$

the expression for the optical distance Δt becomes

$$\Delta t = \int d\lambda \left(\pm\sqrt{\tilde{h}_{ij}\dot{x}^i\dot{x}^j} + A_i\dot{x}^i\right), \tag{5.7}$$

which is the integral of a Finsler norm of the Randers type. A comment is in order at this point. Once we fix the null geodesics to be future directed, the induced optical distance is not a symmetric function (and hence should not be called a distance). Given two points, P_1, P_2 on Σ,

$$\Delta t(P_1, P_2) \neq \Delta t(P_2, P_1), \tag{5.8}$$

the difference given by the presence of the vector field A_i, which induces an anisotropy between past and future directed geodesics.

5.1.1 Example: the Kerr black hole

The most simple example of stationary (but not static) spacetime is given by the Kerr solution. In Kerr coordinates, the metric is given by

$$ds^2 = -f(r,\theta)(du + a\sin^2\theta d\phi)^2 + 2(du + a\sin^2\theta d\phi)(dr + a\sin^2\theta d\phi)+$$
$$+ (r^2 + a^2\cos^2\theta)(d\theta^2 + \sin^2\theta d\phi^2), \tag{5.9}$$

where:

$$f(r,\theta) = 1 - \frac{2mr}{r^2 + a^2\cos^2\theta}. \tag{5.10}$$

The induced line element on $du = 0$ hypersurfaces is easily seen to be:

$$ds_u^2 = -f(r,\theta)a^2\sin^4\theta d\phi^2 + 2a\sin^2\theta d\phi(dr + a\sin^2\theta d\phi) + (r^2 + a^2\cos^2\theta)(d\theta^2 + \sin^2\theta d\phi^2). \tag{5.11}$$

This defines the quantity:

$$B(\dot{x}) = -f(r,\theta)a^2\sin^4\theta\dot{\phi}^2 + 2a\sin^2\theta\dot{\phi}(\dot{r} + a\sin^2\theta\dot{\phi}) + (r^2 + a^2\cos^2\theta)(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) \tag{5.12}$$

which is a bilinear form for 3-vectors tangent to the surface $u = const$.

The four dimensional line element, evaluated on the vector $v = (\dot{u}, \dot{r}, \dot{\theta}, \dot{\phi})$ gives:

$$-f\dot{u}^2 + 2A(\dot{x})\dot{u} + B(\dot{x}) = 0, \tag{5.13}$$

where

$$A(\dot{x}) = -2af(r,\theta)\sin^2\theta\dot{\phi} + 2(\dot{r} + a\sin^2\theta\dot{\phi}). \tag{5.14}$$

When solving for $\dot{u}$ for lightlike curves:

$$\dot{u} = \frac{A(\dot{x}) \pm \sqrt{A^2 + fB}}{f}. \tag{5.15}$$

Notice that this norm is nontrivial even though B is a degenerate form. For instance, in the case of Minkowski spacetime, corresponding to the case $m = 0, a = 0$, this expression reduces to:

$$\dot{u} = \dot{r} \pm \sqrt{\dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)} \tag{5.16}$$

Inside the square root there is not only the induced metric on surfaces $du = 0$, which is the degenerate bilinear form $r^2(d\theta^2 + \sin^2\theta d\phi^2)$. Rather, the optical distance turns out to be the line element of flat Euclidean space.

5.2 Lorentzian Signature: pseudo-Finsler geometry?

All the definitions and the properties of Finsler geometry presented in chapter 4 have been given in the Euclidean signature. The rigorous extension of the various concepts to what would be the pseudo-Finsler case is still not completely clear. In this section we shall consider this issue in details, trying to point out the main problems encountered in the achievement of this goal.

One of the first attempts done in this direction is due to Beem and collaborators [222, 223, 224, 225]. The core of the procedure is to relax the assumption of positive definiteness of the metric and replace it with the requirement that it is a non-degenerate matrix, with Lorentzian signature.

However, it might be more useful to follow the physical intuition. Unfortunately, there are not many works dealing with the foundations and physical interpretation of pseudo-Finslerian spaces (see also [226, 107, 227]).

As it is clear from our definition, when discussing the issue of Lorentzian signature, the first point concerns the reality conditions to be applied on the norm. Even when discussing Minkowski spacetime, which is the Lorentzian version of Euclidean space, one has to face the fact that the norm, defined as the square root of the scalar product of a tangent vector with itself, is a complex number. In particular, if we use the signature $(-+++)$, we see that timelike vectors will have norm of the form ib, where i is the imaginary unit and b is a (positive) real number, while for null vectors and spacelike vectors the norm gives a non-negative real number. This simple observation on a very well understood case suggests that it is better to generalize the definition of a norm we gave in the axioms of Finsler geometry to

$$F : T_pM \to \mathbb{C}, \tag{5.17}$$

keeping the homogeneity property unchanged, $F(\lambda v) = |\lambda| F(v), \lambda \in \mathbb{R}_+$. If we follow this path, it is clear that the distance function $d(p, q)$ between pairs of points p, q of the manifold is no more a real function. This matches the fact that in a Lorentzian spacetimes points are not only close or far away one from the other, but are also related by causality relations: one needs a way to distinguish far away in the past/future from just far away in "space".

This reality condition of the distance function in pseudo-Riemannian spacetimes is bypassed using the Synge's world function [228]:

$$\Omega(p, q) = \frac{1}{2(\tau_1 - \tau_0)} \int_{\tau_0}^{\tau_1} g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta d\tau, \tag{5.18}$$

where the integral is defined on a geodesic connecting the two points, and where τ is an affine parameter on it. Of course, this definition is not reparametrization invariant, and the prefactor $(\tau_1 - \tau_0)$ takes care of this point, making this definition invariant under affine reparametrizations $\tau \to a\tau + b$, with $a, b \in \mathbb{R}$. If we were considering, for instance, Finslerian spacetimes which are generalization of the Berwald–Moor metrics to the Lorentzian case, like the case of light in crystals, we could try to avoid the use of complex numbers just using some sort of generalized world function:

$$\Omega_d(p, q) = \frac{1}{d!(\tau_1 - \tau_0)^{d-1}} \int_{\tau_0}^{\tau_1} Q_{\mu_1 \dots \mu_d} \dot{x}^{\mu_1} \cdots \dot{x}^{\mu_d} d\tau. \tag{5.19}$$

Despite the fact that this works for some classes of metrics (see, for instance, the discussion in [229], Appendix C), some interesting cases would be left outside. The trick of replacing F with F^d (with d chosen suitably in order to give rise to real quantities) does not work in the case of Randers space(times), where the norm is given by (4.15).

It is clear, at this point, that there are no general rules which allow us to work with real numbers only. We therefore ask that the norm is in general a complex function of degree of homogeneity one.

The second ingredient we need in Finsler geometry is of technical nature. We need that the norm is at least twice differentiable in the vector arguments, and that the matrix of the second derivatives is positive definite. To generalize this statement to pseudo-Finsler geometry we need to speak about the structure of the matrix which is emerging from the norm. If we keep the definition (4.1), we see that, being the norm a complex function, the metric has in general complex entries. Clearly, any object which is defined through the norm should be looked at as a complex function of vectors, leading to some great difficulties. For instance, one has to be sure that at least the connection coefficients entering in the geodesic equation are real. If the line element is a complex-number-valued function of the vectors, one cannot prove that the connection coefficients are real. However, if they are complex, since the geodesic equation is an equation for a curve in a manifold which is locally diffeomorphic to $\mathbb{R}^N$, one can conclude that the variational problem is ill posed, and that the considered system does not make sense at all.

This short discussion shows that probably to consider the most general complex norms, and to try to define out of them some notion of pseudo-Finsler structure applying blindly the definition (4.1) is not the right way to proceed. At this stage, a case-by-case analysis should be more helpful. As we have already said, a restriction could be based on an analogy with Lorentzian geometry [222, 223, 224, 225]: we ask that the Finsler metric $g_{\alpha\beta}(v)$, as defined with the derivatives of the norm, at fixed v, is a Lorentzian metric with signature $(-+++)$. This requirement intuitively leads us to the definition of Lorentzian version of Finsler geometry as the Finsler structure which locally possesses light cones, and consequently timelike and spacelike vectors.

However, considering the very simple physical examples we have proposed, it seems quite natural to allow multiple light cones structures, *i.e.* situations in which there is a family of (a finite number of) nested (topological) cones in tangent space on which the norm gives zero. This is the case of bi-refringence (or multi-refringence in the case of several polarizations). We cannot give here a general treatment of the different pathologies which can be encountered (see, for instance, the discussion in [229, 230, 231]).

In order to deal with the complex nature of the norm, we could even insist in giving some extended definition of a metric tensor. For instance, we could define it as:

$$g_{\mu\nu}(x,v) = \frac{1}{2}\frac{\partial^2 |F(v)|^2}{\partial v^\mu \partial v^\nu}, \tag{5.20}$$

using the modulus squared of the norm, which is certainly a real number. Such a definition is certainly useful when we want to discuss the motion of test particles. However, one has still to prove that such a definition is indeed physically motivated, and that it gives rise to the correct metric tensor. For instance, in the case of a pseudo-Riemannian norm, this definition does not give rise to the pseudo-Riemannian metric from which the norm is defined. A direct inspection shows a singular behavior on the light cone due to the change of sign of the norm.

In general, the correct formulation of a pseudo-Finslerian structure which can include all the interesting cases we have seen is still an open problem. It seems that the major difficulty is to keep all the physical quantities (for instance, Noether charges) regular and possibly real. There are encouraging works in this direction, but a sufficiently general solution is still missing. In what follows, we will bypass this issue, in the sense that we will try to discuss the various models in such a way to make clear what is the geometrical interpretation of otherwise clear physical concepts, without being too worried about the more formal aspects of the mathematical correspondence. This latter point requires further investigation.

As it will be discussed in the section about higher order derivative equations, we can handle these situations, despite we lack a notion of metric with signature $(-+++)$ even in a Finslerian sense. Therefore, we can propose a broader definition of pseudo-Finsler structure, inspired by the causal structure of a pseudo-Riemannian manifold. Let us be more explicit on this last point.

We say that two points are causally connected in a given Finsler spacetime if there is a (not necessarily smooth) null curve connecting them. This definition applies in any situation, even in the multi-refringent case, since automatically selects the fastest light signal we have at our disposal. We might say that we have a pseudo-Finsler structure when the metric structure encodes a causality relation, *i.e.* a partial ordering of points which establishes whether they can be connected or not by a signal.

This definition of causality, however, makes explicit use of the matter content of the universe. This is not different from what happens in GR: there, matter fields have kinetic terms which are universally built with the metric tensor in order to satisfy the equivalence principle. There is therefore only one possible choice for the definition of a causal structure. However, when local Lorentz invariance is broken, for instance with some form of bi-metricity, the definition of the causal structure given in terms of the light cones of a single pseudo-Riemannian metric becomes useless, and we have operatively to pass to the definition given in terms of connectivity of points through particles and fields.

This points clarify what is going to replace the concept of light cone, once we do not have a metric structure to define it. The metric structure itself, as we will show later, will be then defined through the dispersion relation.

5.3 The geometric optics limit

In the previous chapter we have implicitly assumed that test particles are moving along geodesics of some Finsler structure. However, it is important to make a connection with the case in which the particle at hand is a very peaked wavepacket of a given field, treated in the geometric optics approximation.

Given that it will be crucial for the rest of the chapter, we will briefly review the geometrical optic limit of wave propagation. For convenience, let us consider a complex scalar field obeying the Klein–Gordon equation

$$\Box\phi = 0, \tag{5.21}$$

in flat Minkowski spacetime. A convenient way of describing solutions to this equation is the so called eikonal parametrization:

$$\phi(x) = A(x)e^{-iS(x)/\epsilon}, \tag{5.22}$$

where $A(x)$ is the amplitude of the field, $S(x)$ is a real function called the eikonal, and ϵ is a real parameter. It is introduced because this parametrization is particularly convenient for situations in which the phase of the wave is rapidly varying. This will correspond to the limit $\epsilon \to 0$. Using

(5.22) into the field equation leads to:

$$\Box A - A\frac{\partial_\mu S \partial^\mu S}{\epsilon^2} = 0, \tag{5.23}$$

$$A\Box S + 2\partial_\mu A \partial^\mu S = 0. \tag{5.24}$$

In the limit $\epsilon \to 0$ the first equation essentially gives:

$$\partial_\mu S \partial^\mu S = 0. \tag{5.25}$$

This is called the eikonal equation, and it is a useful tool to investigate the approximate behavior of wave propagation at short distances. It is easy to see that it is nothing else than the Hamilton–Jacobi equation for a particle with Hamiltonian:

$$\mathcal{H}(p) = p_\mu p^\mu, \qquad p_\mu = \partial_\mu S. \tag{5.26}$$

In other words, in the eikonal approximation the propagation of waves can be described by rays, *i.e.* point particles travelling along geodesics of a given geometry.

To make more explicit and convincing this correspondence, we need to introduce another concept: the characteristic curves. These are allowing us to pass from Hamilton–Jacobi to Hamilton equations for point particles, thus making clear how the geometric optics limit arises. The eikonal equation, which we slightly generalize to the form $H(x, \partial S)$ to include a possible explicit dependence on the position (for instance, the presence of a curved metric $g^{\mu\nu}(x)$) is a partial differential equation of the first order. It is usually supplemented by some initial condition, *i.e.* the value of the function S is specified on a given initial surface. For instance, we can consider the case of a $t = 0$ slice in Minkowski spacetime, and give the initial condition

$$S(t = 0, \mathrm{x}) = f(\mathrm{x}). \tag{5.27}$$

The method of characteristic curves allows us to generate the solution S with the specified initial condition.

Let us consider the curves described by the canonical system:

$$\dot{x}^\mu = \frac{\partial H}{\partial p_\mu} = \eta^{\mu\nu} p_\nu, \qquad \dot{p}_\mu = -\frac{\partial H}{\partial x^\mu} = 0. \tag{5.28}$$

Along these curves, the function H is a constant:

$$\frac{dH(x,p)}{d\tau} = \frac{\partial H}{\partial x^\mu}\dot{x}^\mu + \frac{\partial H}{\partial p_\mu}\dot{p}_\mu = \frac{\partial H}{\partial x^\mu}\frac{\partial H}{\partial p_\mu} - \frac{\partial H}{\partial p_\mu}\frac{\partial H}{\partial x^\mu} = 0. \tag{5.29}$$

Let us now consider the problem at hand. The initial condition for the PDE results into an initial condition for the characteristic curves:

$$x^0(\tau = 0) = 0, \qquad x^i(\tau = 0) = x^i_0, \qquad p_i = \partial_i S = \partial_i f(x), \tag{5.30}$$

while we have that the initial condition for p_0 is given implicitly by

$$H(x(\tau = 0), \partial_\mu S(\tau = 0)) = 0. \tag{5.31}$$

In order to be able to obtain $p_0(0)$ from this equation, it must happen that:

$$\frac{\partial H}{\partial p_0} \neq 0. \tag{5.32}$$

If this holds, then the initial condition for p_0 can be given. In our particular case, it is easy to see that:

$$p_0 = \pm (p_i p^i)^{1/2}. \tag{5.33}$$

The ambiguity in the sign is due to the two possible choices of the orientations of the parametrization. As we have said, along the characteristic curves H is constant. This means that if $H = 0$ on the initial surface, it will be zero along the volume generated by the flow of the initial surface along the characteristic curves. Therefore the solution of the equation satisfying the given initial condition can be generated by these curves.

This is the main idea behind the method of characteristic curves for the study of the properties of the solutions to partial differential equation. Of course, many important issues have been ignored here, like for instance the problem of caustics (intersection of characteristic curves), and the reader is referred to [232] for a detailed discussion of the technical points. For our purposes, it is sufficient to understand that there is a rigorous way to connect a partial differential equation like a wave equation to a set of curves. In physical terms, this mapping is the duality between wavefronts (PDE side) and rays (particle side).

It is very important to see what is the meaning of the function S resulting from this procedure. First of all, given that the PDE does not contain S, but only its first derivative, it is clear that only the variation of S makes sense. Along a characteristic curve, by definition, we have that

$$\frac{dS}{d\tau} = \frac{\partial S}{\partial x^\mu} \dot{x}^\mu = p_\mu \dot{x}^\mu. \tag{5.34}$$

Hence, on a segment of characteristic curve, the difference between the values of the eikonal function at the extrema of the segment is nothing else than the on-shell value of the action of a fictitious point particle moving on it,

$$\Delta S = \int_{\tau_0}^{\tau_1} d\tau \, p_\mu \dot{x}^\mu \,. \tag{5.35}$$

What we have just proved is that we can trade the eikonal equation

$$H(x, \partial S) = 0, \tag{5.36}$$

for the system of equations obtained from the variation of the action:

$$I = \int d\tau \left(p_\mu \dot{x}^\mu - \lambda H(x, p) \right), \tag{5.37}$$

where λ is a Lagrange multiplier introduce to enforce the constraint $H = 0$. Notice that $\Delta S = I$, on shell.

As a special case, one might notice that in the case of a massless Klein–Gordon field, waves in the geometric optics limit do propagate on null geodesics of the metric tensor: in fact, this is the meaning of the equations for the rays obtained from the variation of the functional I just discussed.

Even from this very sketchy presentation the main point should be clear: even though the description of the motion of point particles is apparently of limited interest, it does represent the geometric optics (*i.e.* the short wavelength) limit of a field theory. Therefore, the study of the behavior of point particles is not less important than the study of the behavior of field theories.

5.4 Higher order derivative field theory

As we have shown in the chapter 2, there is a natural relation between propagation of waves in media and Finsler geometry. This is related to the shape of the dispersion relation. In this section we give the proof that to a given PDE, when we look at the equivalent of geometric optics approximation, we can associate a Finsler metric. The result is that the rays of PDEs can be described by the geodesics of a related Finsler structure. What we are going to do is to generalize the procedure considered in the previous section, using the appropriate framework and language, to the case of several fields and higher order derivatives.

A fact is unchanged: the geometrical interpretation of wave propagation comes when one considers the eikonal approximation, which provides the dual description to wavefronts, *i.e.* the rays. We shall see then that if we have a higher order differential linear equation, once some sort of hyperbolicity condition is satisfied, the rays are described by trajectories of particles, whose dynamics is determined by the PDE itself, exactly in the same way in which it is happening for second order wave equations.

The logic is the same: given a higher order partial differential equation, we are going to make the eikonal approximation to obtain the so-called Fresnel equation. We shall then recall the characteristic curve method to solve the Fresnel equation. This will show how to solve this equation is equivalent to consider the propagation of a point particle. The link between this class of partial differential equations and Finsler spacetimes is established in the last part, where we shall perform a Legendre transform from the Hamiltonian formalism to the Lagrangian formalism which will allow us to determine the Finsler norm associated to a given dispersion relation, *i.e.* to a given PDE. Finally, we are going to discuss the inverse path: how to associate to a Finsler norm a PDE.

The material presented here is quite standard, and can be found in several places. The material concerning PDEs can be found for instance in [232, 233] . A related discussion on the in the context of emergent gravity scenarios can be found in [107].

5.4.1 Eikonal approximation

Let us consider without loss of generality a generic multiplet of scalar (complex) fields, $\phi^A(x)$, with $A = 1, ..., n$ on $\mathcal{M}$, which is not necessarily endowed with a pseudo-Riemannian metric, at this stage. We assume that the equations of motion are a set of linear wave equations[2]:

$$\mathscr{D}_{AB}\phi^B = 0 \Leftrightarrow \left(\sum_{n\geq 1}^{N} (i)^n (\ell)^{n-2} T_{AB}^{\mu_1 \dots \mu_n}(x)\partial_{\mu_1}...\partial_{\mu_n} - M_{AB} \right) \phi^B = 0, \tag{5.38}$$

where we introduce the parameter ℓ with the dimension of a length in order to have only dimensionless tensors T_{AB}, and where M_{AB} is the mass matrix (which can be position dependent, in principle). Notice that if we ask that the matrices appearing in the equations are constants, then we can solve the equation exactly, since we can use the plane wave expansion (*i.e.* the familiar normal modes analysis):

$$\phi_A(x) = \chi_A \exp(-ik_\mu x^\mu), \tag{5.39}$$

[2] If we are dealing with highly non-linear theories, this condition can be satisfied by performing the linear analysis around a given classical solution.

the differential equation becoming just the on-shell condition for the wave-vector k_μ.

A comment is needed. Here we are considering the case in which the PDE are arising from a variational principle. In this way, one can see that the coefficients $\eta_{(n)}$ are real, and the matrices T_{AB}, M_{AB} are symmetric (according to our choice to make explicit the appearance of the imaginary unit i in (5.38)). For latter convenience, it will be useful to rewrite the mass matrix as[3] $M_{AB} = \alpha_{AB}/\ell^2$, where α_{AB} is now dimensionless. While this choice is natural from the point of view of dimensional analysis, we have to consider the possibility that the mass matrix contains entries of the typical size m^2, where m is a mass scale that can be different from $1/\ell$. This means that the definition of α could hide some of the features we are trying to uncover.

To find approximate solutions of (5.38) when the coefficients of the PDE are position dependent we can use the eikonal approximation. As we have seen, this amounts to look for solutions of the form:

$$\phi_A(x) = \chi_A(x) \exp(-iS(x)/L), \tag{5.40}$$

where we introduced the functions χ_A which is a multiplet representing the "polarization" of the field multiplet, while S is the eikonal function, giving the phase of the wave. The length L is introduced in order to develop an approximation method which is going to give us an alternative equation for the fields, simpler to solve. The rough idea, which we are going to develop with care, is that the length L represents the length over which the phase S/L has a change of the order of the unity. Equivalently, we are testing the theory at the scale L. Of course, if L is much smaller than the scale over which the parameters entering in the differential equation are varying significantly, so that with a good approximation they can be considered as constants, it is even easier to describe the behavior of the solutions.

First of all, let us consider separately each term appearing in the sum in 5.38 when applied to the eikonal ansatz:

$$(i)^n \ell^{n-2} T^{\mu_1 \dots \mu_n}_{(n)AB}(x) \partial_{\mu_1 \dots \mu_n} (\chi_B \exp(-iS/L)). \tag{5.41}$$

This expression gives rises to several terms which we can classify according to the power of L entering in the expansion:

$$\ell^{n-2} T^{\mu_1 \dots \mu_n}_{(n)AB}(x) \sum_{k=0}^{n} (i)^{n-k} \frac{1}{L^k} W^B_{(n,k)\mu_1 \dots \mu_n} \exp(-iS/L), \tag{5.42}$$

where we have used the rather implicit notation $W^B_{(n,k)\mu_1 \dots \mu_n}$ to denote the results of the various differentiations and rearrangements. It is important to notice that these functions W are $L-$independent. If we factorize the $1/L^n$ we obtain:

$$\frac{\ell^{n-2}}{L^n} T^{\mu_1 \dots \mu_n}_{(n)AB}(x) \sum_{k=0}^{n} (i)^k L^k W^B_{(n,k)\mu_1 \dots \mu_n} \exp(-iS/L). \tag{5.43}$$

The core of the eikonal approximation consists in observing that in the limit in which the length L becomes very small (with respect to the length over which the various functions appearing in the equation have a significant variation[4]), of this term there is only the $k = 0$ contribution surviving

[3] We are using units in which $\hbar = 1 = c$.

[4] Notice that the eikonal approximation, then, works if and only if the polarization χ is a slowly varying function of the position.

the limit, *i.e.* the terms

$$\frac{\ell^{n-2}}{L^n} T^{\mu_1 \ldots \mu_n}_{(n)AB}(x) \chi^B \partial_{\mu_1} S \ldots \partial_{\mu_n} S \exp(-iS/L). \tag{5.44}$$

The other are just corrections which must be taken into account if one is seeking for the exact solution. If we introduce the dimensionless quantity $\theta = \ell/L$, we see that in the eikonal approximation Eq. 5.38 can be rewritten as:

$$O_{AB}(\partial S)\chi^B = \left[\sum_{n\geq 1}^{N} \theta^{n-2} T^{\mu_1 \ldots \mu_n}_{(n)AB}(x) \partial_{\mu_1} S \ldots \partial_{\mu_n} S + \frac{1}{\theta^2} \alpha_{AB} \right] \chi^B(x) = 0, \tag{5.45}$$

which is known as the eikonal equation. The parameter θ is entering in this equation with different powers. Therefore, according to the regime we are probing, some terms will be more relevant than others. In order for the eikonal equation to have a non trivial solution, *i.e.* a non-zero polarization χ, it must happen that the matrix O_{AB} is singular, corresponding to the condition of having vanishing determinant:

$$\det O_{AB}(\partial S) \equiv \mathcal{F}(\partial S) = 0. \tag{5.46}$$

This equation, a first order PDE in S, is known as the Fresnel equation. Its solution gives the function S, which is proportional to the phase of the wave, *i.e.* it gives the shape of the wavefronts.

5.4.2 PDE and characteristic curves

The eikonal approximation transforms a higher order differential equation into the Fresnel equation, *i.e.* a first order non-linear PDE. The characteristic curves method will allow to find solutions to the Fresnel equation, and, most important, to understand its geometrical content.

In the $D+1$ vector space defined by (τ, x), embed the Cauchy surface (again, for all the details see [232]) $x^\mu(\xi^i)$, $\mu = 1, ..., D, i = 1, ..., D-1$, ξ coordinates on the surface. The initial data for $S(x)$ can be used to define the co-dimension two surface $(S_0(x(\xi)), x^\mu(\xi^i))$. The solution to the Cauchy problem is obtained just defining the curves in the (s, x) space which corresponds to

$$\mathcal{F}(x(\tau), (\partial S)(x(\tau))) = const, \tag{5.47}$$

which are leaving from the initial data surface.

Let us introduce the notation $\pi_\mu = \partial_\mu S$. To construct the characteristic curves in (τ, x), we will consider curves in the (τ, x^μ, π_ν) space, used as an auxiliary space, and then projecting down the result on the (τ, x) space.

The Fresnel equation, then, can be written as:

$$\mathcal{F}(x, \pi) = 0, \tag{5.48}$$

$$\pi_\mu = \partial_\mu S(x). \tag{5.49}$$

Considering the derivative of F with respect to τ leads to

$$\dot{\mathcal{F}}(x, \pi) = 0 \Leftrightarrow \frac{\partial \mathcal{F}}{\partial x^\mu} \dot{x}^\mu + \frac{\partial \mathcal{F}}{\partial \pi_\mu} \dot{\pi}_\mu = 0. \tag{5.50}$$

This equation is satisfied if we have

$$\dot{x}^\mu = \frac{\partial \mathcal{F}}{\partial \pi_\mu}, \qquad \dot{\pi}_\mu = -\frac{\partial \mathcal{F}}{\partial x^\mu}. \tag{5.51}$$

These are nothing else than the Hamilton equations. It is clear that, along these curves, $\dot{\mathcal{F}} = 0$, and therefore the initial data can be propagated, generating the corresponding solution to the Cauchy problem. Using the definition of π, we can see that:

$$\frac{dS(x(\tau))}{d\tau} = \pi_\mu \dot{x}^\mu. \tag{5.52}$$

Therefore,

$$S(x) - S(x_0) = \int \pi_\mu \dot{x}^\mu d\tau, \tag{5.53}$$

where the integral is evaluated along the curve connecting the two points. This implies that S is the classical action for point particles described by the Hamiltonian associated to $\mathcal{F}(x,p)$. This Hamiltonian is obtained from the Fresnel equation precisely using the relation $\pi_\mu = \partial_\mu S$. Moreover, since the Hamiltonian is forced to be zero from the equation of motion, we see that the classical action to be used is *reparametrization invariant*[5]

$$S_{cl} = \int \left(\pi_\mu dx^\mu - \lambda \mathcal{F} d\tau\right). \tag{5.54}$$

The solutions to the Hamilton's equations are the rays, the basic tools of geometric optics, which represent the duals of the wavefronts of wave optics. Note that $\dot{x}$ describes the tangent vector to the ray's trajectory, while π_μ is not related directly to it, since it describes the wavefront: π is precisely the gradient of the function determining the wavefront (if we were speaking in terms of Riemannian geometry, it would be the normal to the surface).

This discussion shows how is it possible to formulate a generalization of the treatment of the ray-optics approximation to the case of higher order PDE. The logic is that one trades the complicate solution of a PDE for a Hamilton–Jacobi equation for a corresponding classical point particle.

5.5 From modified dispersion relations to geometry

We have shown how solving the Fresnel equation is equivalent to look at the equations of motion for a classical point particle, whose dynamics is given by the PDE in the way we have briefly discussed, and in particular is described by a suitable variational principle.

The next step is to recover a geometrical interpretation for this motion, as it has been done for the case of Klein Gordon equation. Having the action in the Hamiltonian formalism, we need to perform a Legendre transformation, which will express the action in the tangent bundle, giving a more direct relation with geometry. This has been discussed in details in [234], where we showed how a modified dispersion relation can be interpreted in terms of a (pseudo-)Finsler structure.

The starting point is the action

$$S = \int \left(dx^\mu p_\mu - \lambda(\mathcal{C}(p,\ell,m) - m^2)d\tau\right), \qquad \mathcal{C}(p,\ell,m) = h^{\mu\nu}(p,\ell,m)p_\mu p_\nu, \tag{5.55}$$

[5] In fact, depending on the specific shape of the modified mass shell, the action can be solely invariant under positive, orientation-preserving reparametrizations.

where λ is the Lagrange multiplier implementing the Hamiltonian constraint $\mathcal{C}$, the mass-shell condition. The rest mass, *i.e.* the energy at zero momentum, has been singled out explicitly. For notational convenience[6], we shall use sometimes $M = \ell^{-1}$. From the action, we identify the standard symplectic form:

$$\{x^\mu, p_\nu\} = \delta^\mu_\nu, \qquad \{x^\mu, x^\nu\} = 0 = \{p_\mu, p_\nu\}. \tag{5.56}$$

Hamilton equations provide the relation between momentum and speed, up to the factor λ encoding the freedom in the choice of the parameter τ. They are

$$\dot{x}^\mu = \lambda\{x^\mu, \mathcal{C}\}, \tag{5.57}$$

$$\dot{p}^\mu = \lambda\{p^\mu, \mathcal{C}\} = 0. \tag{5.58}$$

The last equation is due to the fact that we have assumed that the matrix $h^{\mu\nu}$ does not depend on the coordinates. In order to deduce the Lagrangian, we have to invert the relation (5.57), which is in general a highly non-linear equation: this is not always possible, and, correspondingly, the Lagrangian picture of the theory cannot be given. Despite this possible obstruction, in practice we can try to solve the problem of finding the Lagrangian restricting the phase space to a suitable domain. This approach is motivated by the fact that typically the modified dispersion relation is given as a power series in ℓ, and implicitly truncated to a given order. This is possible because we are interested in the low-momentum regime. After performing the inversion, we obtain

$$p_\mu = f(\dot{x}, \lambda, M, m), \tag{5.59}$$

which we can plug back into the action (5.55)

$$S \to \int \dot{x}^\mu p_\mu(\dot{x}, \lambda, M, m) d\tau, \tag{5.60}$$

We can then vary over λ to obtain an equation allowing us to eliminate the Lagrange multiplier: we express it in terms of $\dot{x}$ and insert its value back in the action (5.60),

$$S \to \int \dot{x}^\mu p_\mu(\dot{x}, \lambda(\dot{x}, M, m), M, m) d\tau = \int F(\dot{x}, M, m) d\tau = \int \mathcal{L}\, d\tau. \tag{5.61}$$

Since the action is initially reparametrization invariant, the final action that we have obtained is still reparametrization invariant, which means that the function F, identified as the Lagrangian $\mathcal{L}$ is by construction homogenous of degree one, that is a general norm on the tangent bundle. Just as in the well known case of the relativistic particle, even in the case of modified dispersion relations the action can be defined by using a line element as the Lagrangian. The Euler–Lagrange equations provide then the geodesic equations. Thus, we can conclude that *a free particle with a modified mass-shell moves along a geodesic of a Finsler spacetime.*

The Legendre transformation described above is a very general method, it is useful to look at specific examples to understand the features and difficulties one can meet. The usual relativistic particle is the simplest (best known) example to illustrate the procedure. We have the action given by

[6] We are always working in units $\hbar = c = 1$.

$$S = \int dx^\mu p_\mu - \lambda(\mathcal{C} - m^2)d\tau, \qquad \mathcal{C} = \eta^{\mu\nu} p_\mu p_\nu. \tag{5.62}$$

The Hamilton equations read then

$$\dot{x}^\mu = \lambda\{x^\mu, \mathcal{C}\} = 2\lambda p^\mu, \Rightarrow p^\mu = \frac{1}{2\lambda}\dot{x}^\mu, \tag{5.63}$$

$$\dot{p}^\mu = \lambda\{p^\mu, \mathcal{C}\} = 0. \tag{5.64}$$

Expressing p in terms of the speed $\dot{x}$ is straightforward, so that we obtain the new action

$$S = \int \left(\frac{1}{2\lambda}\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu - \lambda(\frac{1}{4\lambda^2}\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu - m^2)\right) d\tau, \qquad \mathcal{C} = \eta^{\mu\nu} p_\mu p_\nu. \tag{5.65}$$

Variation in terms of λ leads to

$$\lambda = \frac{\sqrt{\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}}{2m}.$$

(Note also that λ is indeed a homogenous function of degree one of the velocities). Plugging this last equation in (5.65), we obtain the standard action:

$$S = m \int \sqrt{\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} d\tau \tag{5.66}$$

A similar analysis can be done in the case of the modified form of the mass-shell condition. The Legendre transformation is however more involved and one needs to proceed by perturbations.

For instance, consider the case of an action of the form:

$$S = \int dx^\mu p_\mu - \lambda(\mathcal{C} - m^2)d\tau, \qquad \mathcal{C} = \eta^{\mu\nu} p_\mu p_\nu + \alpha\ell^2|\vec{p}|^4. \tag{5.67}$$

The Hamilton equations read then

$$\dot{t} = \lambda\{t, \mathcal{C}\} = 2\lambda p_0, \qquad \dot{x} = 2\lambda(p + 2\alpha\ell^2 p^3), \tag{5.68}$$

$$\dot{p}^\mu = \lambda\{p^\mu, \mathcal{C}\} = 0. \tag{5.69}$$

We invert the first equation considering neglecting terms $O((\alpha\ell^2p^2)^2)$. This is possible if we restrict the phase space to momenta which are $\ell^2p^2 \ll 1$.

When using this approximation, the Finsler norm can be evaluated as we have described. The action reads:

$$S = m \int F d\tau, \tag{5.70}$$

$$F \approx \sqrt{\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} + \alpha m^2 \frac{\dot{\mathrm{x}}^4}{(\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu)^{3/2}}. \tag{5.71}$$

It is important to notice that the norm is mass dependent. This could have been expected from the beginning just from dimensional analysis. Given that the dispersion relation involves a mass scale M, and given that the norm must be dimensionless, the mass of the particle must be used to compensate M and give rise to dimensionless terms. Notice also that the massless case requires some extra care.

5.5.1 From Finsler to field theory

The discussion we made about the Fresnel equation, as we have already pointed out, is just the generalization of the procedure usually considered for second order PDE. The general line of reasoning is to map a PDE into an ODE system, this ODE system having an interpretation in terms of point particles. In particular, we can see that the Hamiltonian for the point particle and the differential operator for the wave equation are related by $p_\mu \leftrightarrow i\partial_\mu$. In particular, we have seen that $i\partial_\mu \to p_\mu$ explicitly via the geometric optics approximation. This correspondence leads us from a differential operator to a Finsler norm. To close the circle, we have to understand how we can associate to a Finsler norm a differential operator playing the role of its Laplacian (or Dalambertian if we are in Lorentzian signature).

In Finsler geometry there is no universal recipe to do this operation. There are already some proposals about the definition of a Laplace operator associated to a Finsler space [235, 236], but they do not seem to fit our purposes. In light of the previous discussion we feel that the more physical approach is probably the inverse of the PDE-characteristic curve approach. When we are given a Finsler norm, we build the action, Legendre transform it (taking care of the constraint due to reparametrization invariance), and get the Hamiltonian. Then, we build the Laplacian with the "correspondence principle":

$$p_\mu \to i\partial_\mu, \qquad H(p) = 0 \to H(i\partial)\phi = 0. \tag{5.72}$$

Of course, this prescription is effective whenever we have a single scalar field. However, if we have a multiplet of fields, we have to deal with the issue of bi-refringence, *i.e.* the possibility that each polarization propagates with a different mass shell condition (which is encoded in the fact that the Fresnel equation admits a multi-valued solution). To solve this issue we would like to recall what is the procedure adopted in the case of spinor and vector fields in Minkowski spacetime. These fields are defined through the representation theory of the Poincaré group. The Lagrangian is not built just using the d'Alembert operator on the fields bilinear, as it is manifest in the Dirac Lagrangian which makes use of the gamma matrices. Analogously, the Lagrangian for the vector field is not built with $A^\alpha \Box A_\alpha$, but with $F^{\alpha\beta}F_{\alpha\beta}$. The analysis of the equation of motion through the Fresnel equation we have defined above gives the correct identification of the mass shell condition corresponding to particles of given mass in Minkowski spacetime.

In the case of general Finsler backgrounds, the situation is basicly the same. However, a supplementary discussion is needed in the case of bi-refringence. In these situations, the Fresnel equation defines a Finsler structure related to a multi-sheeted light cone structure. To reconstruct the wave equation in the case of spinor and vector fields, one should repeat the group-theoretical analysis done in Special Relativity and build consistently their Lagrangian. A consistent construction should lead to a Fresnel equation producing the Finsler structure we have started with.

Let us consider in detail the case of a birefringent vector field, whose Lagrangian is given by:

$$L = -\frac{1}{4}G^{\alpha\beta\gamma\beta}F_{\alpha\beta}F_{\gamma\delta}. \tag{5.73}$$

In the case of Lorentz invariant case, the tensor G is built with the Minkowski metric. In the general case, the equation of motion for the field V_α is

$$G^{\alpha\beta\gamma\delta}\partial_\beta F_{\gamma\delta} = 0. \tag{5.74}$$

If we go for the plane wave approximation $V_\alpha = v_\mu e^{-ik_\alpha x^\alpha}$, we find the equation

$$G^{\alpha\beta\gamma\delta}k_\beta k_\gamma v_\delta = 0. \tag{5.75}$$

This last condition gives the Fresnel equation:

$$\det(G^{\alpha\beta\gamma\delta}k_\beta k_\gamma) = 0. \tag{5.76}$$

This equation can be rewritten as:

$$Q^{\alpha_1\ldots\alpha_8}k_{\alpha_1}\ldots k_{\alpha_8} = 0, \tag{5.77}$$

where Q is a completely symmetric tensor of type $(0, 8)$, if we work in four dimensions, completely determined by the tensor G. The tensor Q is the object with which we define the norm, while the tensor G is the object which relates each polarization to the corresponding sheet of the light cone. In the case in which we start from the Finsler metric, *i.e.* the tensor Q, one can find the tensor G just using the existing relation with the tensor Q provided by the Fresnel equation.

5.6 A condition for monometricity

As we have seen, given a modified dispersion relation, changing the mass corresponds, in general, to a change of the Finsler structure. Here we want to derive a condition on the mass shell ensuring that this does not happen.

The key point is that, in the Lagrangian formalism, after having introduced the so-called einbein variable λ, the action is given by:

$$I = \int d\tau \left(\frac{F^2(\dot{x})}{4\lambda} + \lambda m^2\right). \tag{5.78}$$

In this action, the mass compares just in the second term, by hypothesis. If we introduce the "reduced" Lagrangian:

$$\bar{L} = L - \lambda m^2 = \frac{F^2(\dot{x})}{4\lambda}, \tag{5.79}$$

we observe that the mass does not appear and that this is a homogeneous function of degree (-1) of the variable λ. Therefore, by Euler's theorem, it must obey:

$$\lambda\frac{\partial\bar{L}}{\partial\lambda} + \bar{L} = 0. \tag{5.80}$$

In terms of the canonical variables,

$$\bar{L} = p_\mu\dot{x}^\mu - \lambda\mathcal{M}(p), \tag{5.81}$$

where the momenta can be expressed in terms of the velocities by inverting the relation:

$$\dot{x}^\mu = \lambda\frac{\partial\mathcal{M}}{\partial p}. \tag{5.82}$$

We can generically write:

$$p_\mu = f_\mu\left(\frac{\dot{x}}{\lambda}\right), \tag{5.83}$$

which is the general functional dependence of the momenta on the velocities in these systems endowed with reparametrization invariance.

Equation (5.80) can be written as:

$$\lambda \frac{\partial p_\mu}{\partial \lambda} \dot{x}^\mu - \lambda \mathcal{M}(p) - \lambda^2 \frac{\partial \mathcal{M}}{\partial p_\mu} \frac{\partial p_\mu}{\partial \lambda} + p_\mu \dot{x}^\mu - \lambda \mathcal{M}(p) = 0. \tag{5.84}$$

Using the relation between the momenta and the velocities we get:

$$\lambda \left(\frac{\partial \mathcal{M}}{\partial p_\mu} p_\mu - 2\mathcal{M}(p) \right) = 0. \tag{5.85}$$

The solutions of this equation are either $\lambda = 0$ or

$$\frac{\partial \mathcal{M}}{\partial p_\mu} p_\mu - 2\mathcal{M}(p) = 0. \tag{5.86}$$

By Euler's theorem this is equivalent to say that the mass shell function is a homogeneous function of degree two of the momenta. If this happens, changing the mass in the dispersion relation does not imply a change into the Finsler structure in the Lagrangian formalism.

As a consequence of the fact that the dispersion relation is a homogeneous function, there is no new constant with energy dimensions in it. All the additional coefficients can be only dimensionless or with speed dimensions. Therefore, the only situations in which it is meaningful to speak about Finsler geometry as the geometry of spacetime are the ones where only velocity scales are meaningful (see for instance the case of very special relativity). It seems that no DSR program where a mass scale (*e.g.* the Planck mass) plays some key role can be realized within Finsler geometry. The price to pay would be to abandon the idea of having a unique geometrical framework to describe the motion of all particles and fields.

5.7 The Higgs mechanism for mass generation

The important fact on which this discussion is based is the conclusion that a MDR can be seen as the manifestation of Finsler structure of spacetime, which therefore represents the geometrical theory corresponding to higher order derivatives field theories. We have already seen that the Finsler norm associated to a given mass shell function depends on the mass of the particle considered. In the Standard Model of particle physics masses as generated through a Higgs mechanism [237, 238, 239, 240]. It is therefore interesting to see what happens at the level of field theory when modified dispersion relations are introduced into the Lagrangian. This particular issue has beed carefully examined in [241].

Constraints on LIV in the Higgs sector have already been considered in [242, 243], at the level of the lowest dimensional operators of the standard model extension with LIV. It is therefore natural to ask what happens when higher dimensional operators are included, in particular to understand the way in which the Finsler structure can affect the low energy phenomenology of a spontaneously broken gauge theory through a Higgs mechanism, including the effects of higher dimensional operators in a systematic way.

Obviously this specific issue is of a key relevance for the development of Finsler extensions of the Standard Model (SM). In particular, it is interesting to focus on the dynamics of the gauge bosons alone, neglecting the fermionic content of the theory. Two cases are going to be considered. First, the class of modified dispersion relations of the polynomial form. Second, we will examine the case of a field theory defined in the framework of very special relativity.

To simplify further the treatment of the first case, it is enough to consider the situation proposed by renormalization group (RG) arguments [189], in which we have that all the massless particles are described by the standard (pseudo-)Riemannian geometry Lagrangian, while the massive ones have a MDR. If it is assumed that the Higgs has a MDR, a direct calculation shows that there is a non-trivial percolation of the LIV terms for the Higgs into lower dimensional operators for the gauge bosons, which cannot be predicted from an analysis like the one proposed in the LIV standard model extension of [129], and whose suppression is ruled by the hierarchy between the UV energy scale responsible for the Lorentz violating operators and the scale of symmetry breaking.

In the case of very special relativity (VSR) it will turn out that there are some rather crucial difficulties in realizing a Higgs mechanism in such Finsler scenarios, thus disfavouring this class of anisotropic relativistic models as a viable description of spacetime.

5.7.1 Polynomial MDR

In the effective field theory approach [144], it is customary to introduce the MDR according to the classification of the operators in terms of their canonical dimensions. This fits perfectly with MDR which admit a polynomial expansion in terms of the momenta. This is possible, by dimensional arguments, if a dimensionfull quantity, an energy scale, is introduced in the theory. Typically, this high energy scale related to LIV is identified with the Planck mass M_P, which is the scale at which new gravitational physics, and hence geometry, as we have discussed in chapter 3. In what follows, instead of making such an assumption, a generic UV cutoff $\Lambda_{\rm UV}$ is used, without specifying its origin.

As it has already been noted, the main difficulty in this EFT of Lorentz violation is represented by the large number of operators which must be added to the SM Lagrangian, at least in absence of some guiding principle which can be used to restrict the possible additional terms to a specific class.

In [189] it has been shown that it is reasonable to expect that, due to RG effects, the geometry felt by fields can be energy dependent. In particular, it was shown that, as a consequence, the various fields get a MDR according to their mass: while massive fields have a modified mass shell, the massless ones do not, their mass shell being always the light cone relative to the (low energy) Minkowski metric.

These results can be used to introduce a specific class of models, with simple arguments based on symmetry principles. At high energy, the masslessness of gauge fields is protected by gauge invariance, which is unbroken in the high energy phase of the theory, and then, by the RG argument, their dispersion relation is unchanged. On the contrary, there is no symmetry protecting the Higgs Lagrangian from acquiring additional terms producing a MDR, since gauge invariance is not limiting enough the shape of the potential term, and in particular allows a mass term.

It is interesting, therefore, to discuss what happens in this scenario, where the gauge fields have the standard kinetic terms, while the Higgs field's Lagrangian has a modification according to the particular MDR associated to it.

Abelian Higgs model

To start with, consider the case of an abelian Higgs model. The Lagrangian for the Lorentz invariant case is given by[7]:

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \eta^{\mu\nu}D^{\dagger}_{\mu}\phi^{\dagger}D_{\nu}\phi - V(\phi), \tag{5.87}$$

where $D_{\mu} = \partial_{\mu} + igA_{\mu}$ is the covariant derivative containing the gauge field A_{μ}, $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$, g is the gauge coupling and $V(\phi)$ is the potential, which has been taken to have the form $-\mu^2|\phi|^2 + \lambda|\phi|^4$ just to discuss a specific model, without losing generality.

Add to the scalar field Lagrangian a term which modifies the kinetic term, for example a p^4-like term

$$\frac{\eta}{{\Lambda_{\rm UV}}^2}M^{\mu\nu\rho\sigma}(D_{\mu}D_{\nu}\phi)^{\dagger}D_{\rho}D_{\sigma}\phi, \tag{5.88}$$

where $\eta \sim O(1)$, $\Lambda_{\rm UV}$ is the high energy scale corresponding to the physics generating this term and M is a tensor whose form is not specified here. We have to consider what happens in the case when ϕ gets a vacuum expectation value (*vev*). In particular, writing $\phi = ((v/\sqrt{2}) + \varphi)e^{i\theta}$, where $v = \mu/\lambda^{1/2}$ is the *vev*, we get the mass term for the gauge field as usual, as well as an additional contribution coming from the MDR, which is easily obtained:

$$D_{\mu}D_{\nu}\phi \rightarrow (\partial_{\mu} + igA_{\mu})(\partial_{\nu} + igA_{\nu})(v/\sqrt{2} + \varphi). \tag{5.89}$$

This term generates new interactions between the (now massive) gauge boson A_{μ} and the field φ, which modify the ones already present in the Lorentz invariant Lagrangian. Moreover, there is a whole new part to be included in the action for the gauge boson alone, modifying its propagator. In particular, there is the term:

$$\frac{\eta}{2}\left(\frac{gv}{\Lambda_{\rm UV}}\right)^2 M^{\mu\nu\rho\sigma}(\partial_{\mu}A_{\nu} - igA_{\mu}A_{\nu})(\partial_{\rho}A_{\sigma} + igA_{\rho}A_{\sigma}). \tag{5.90}$$

This term goes directly into the renormalizable part of the action related to the gauge boson. Notice that this amounts to a new quartic self-interaction governed by the tensor $M^{\mu\nu\rho\sigma}$, which includes the effect of Lorentz violation, and, most important, the modification of the kinetic term at the level of dimension four operators. We recognize the combination $gv = M_A$ is nothing but the mass of the gauge field.

For different MDR for the Higgs field, the discussion is similar. For example, for a p^{2n} modification[8], we get a p^{2n-2} modification of the kinetic term for the massive gauge boson, inherited from the scalar field, as well as new self-interactions.

These terms are highly constrained from astrophysical observations for particles like photons and electrons (*i.e.* the QED sector), but not for the bosons $W^{\pm}, Z^0$, for which the analysis of Lorentz invariance have not been considered yet, at least with the same accuracy. As it stands, however, this discussion is not completely satisfactory for a phenomenological analysis, since we are

[7]Here high energy physics conventions are used: the Minkowski metric is given by diag$(+,-,-,-)$.

[8]The discussion of p^{2n+1} modifications, is exactly the same, even though it is clear that for that class one should consider necessarily the fate of the invariance of the Lagrangian under discrete symmetries, C, P, T. For the simplicity of the discussion, here we consider only even powers of the momenta.

still discussing the abelian case, while we should discuss the most general case of nonabelian gauge theories. What we can conclude, at this stage, is that LIV in the form of a MDR is propagating in the Lagrangian of a spontaneously broken gauge theory in a non trivial way even at the tree level, without taking into account quantum corrections, whose role could be even more important [145].

However, despite being potentially relevant, besides being suppressed by suitable powers of the cutoff $\Lambda_{\rm UV}$ required by dimensional analysis, all the new terms are multiplied by the dimensionless ratio $r = M_A^2/{\Lambda_{\rm UV}}^2$, between the square of the mass of the gauge boson and the square of the UV cutoff. If the MDR were related to the Planck length and $v \approx$ TeV the electro-weak (EW) scale, we could conclude that $r \approx 10^{-32}$, thus enhancing the Planck suppression through the large hierarchy between the EW scale and the Planck scale.

The bottom line of this discussion is that the modification of the dispersion relation induced on the gauge boson is more suppressed than expected, and thus we can detect it only in extremely accurate precision tests of our models[9].

The Non-Abelian case

To fully understand the implications of a MDR in a realistic model for particle physics, we have to discuss the case of non-abelian gauge fields. Let us consider the case of a non-abelian gauge group, like for instance an $SU(N)$ gauge theory, with generators of the Lie algebra given by the matrices T_A, and gauge fields G^A_μ. Let us consider a Lorentz invariant Higgs model:

$$L = -\frac{1}{4}\mathrm{Tr}(G_{\mu\nu}G^{\mu\nu}) + |D_\mu\Phi|^2 - V(\Phi). \tag{5.91}$$

As before, let us add a MDR term, p^4-like, and discuss what happens if the Higgs multiplet gets a *vev*. The term to be added can be written as:

$$\frac{\eta}{{\Lambda_{\rm UV}}^2} M^{\mu\nu\rho\sigma}(D_\mu D_\nu \Phi)^\dagger D_\rho D_\sigma \Phi. \tag{5.92}$$

If the potential V allows a non-vanishing *vev* of the multiplet Φ, so to break spontaneously the gauge symmetry, we have the gauge bosons mass terms generated by the standard kinetic term of the Higgs and, as before, new contributions coming from the additional term encoding the MDR. Now:

$$D_\rho D_\sigma \Phi \to (\partial_\rho + igT_A W^A_\sigma)(\partial_\sigma + igT_A W^A_\rho)(\langle\Phi\rangle + \varphi). \tag{5.93}$$

Neglecting the terms describing the interaction of the gauge bosons with the field φ, we have a new part for the action of the gauge bosons alone:

$$M^{\mu\nu\rho\sigma}[(-iT_A\partial_\mu W^A_\nu - gT_AT_BW^A_\mu W^B_\nu)V]^\dagger[(+iT_C\partial_\rho W^C_\sigma - gT_CT_DW^C_\rho W^D_\sigma V)], \tag{5.94}$$

[9]Therefore, to accomplish this task one should move beyond the tree level analysis and consider the first quantum corrections to the SM Lagrangian.

These new terms include new self-couplings to be added to those given by the original Lagrangian, plus a dimension four operator which has to be added to the standard kinetic term and which modifies the dispersion relation of the gauge bosons at the quadratic level, as in the abelian case.

The power counting argument on the strength of the modification is left unchanged. The structure is otherwise the same as in the abelian case. In principle, we could expect three features:

- (additional) mixing between different gauge bosons induced by the MDR;
- modification of the dispersion relation at the level of p^{2n-2} instead of p^{2n};
- additional three and four gauge bosons interactions.

All these points must be taken into account as potential sources for new physics, beyond the predictions of SM.

Without loss of generality, let us consider the special case of a spontaneously broken $SU(2)\times U(1)$ gauge theory, with a Higgs doublet, which is relevant for the SM dynamics. To avoid confusion on conventions, we write down step by step the Lagrangian, in order to make the comparison with the standard case easier. The part of the Lagrangian involving only the Higgs doublet and the gauge fields is given by:

$$L_0 = -\frac{1}{4}\mathrm{Tr}(\mathbf{F}_{\mu\nu}\cdot\mathbf{F}^{\mu\nu}) - \frac{1}{4}G_{\mu\nu}G^{\mu\nu} + (D_\mu\Phi)^\dagger D^\mu\Phi + \mu^2\Phi^\dagger\Phi - \frac{\lambda}{4}(\Phi^\dagger\Phi), \tag{5.95}$$

where the SU(2) gauge fields, W^i_μ, and the U(1) gauge vector B_μ have field strengths given respectively by:

$$\mathbf{F}_{\mu\nu} = \partial_\mu\mathbf{W}_\nu - \partial_\nu\mathbf{W}_\mu - g\mathbf{W}^\mu\times\mathbf{W}^\nu, \qquad G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu, \tag{5.96}$$

where we have used the compact notation $\mathbf{W}^\mu = \tau_i W^i_\mu$, with τ the Pauli matrices, and where g, g' are the two dimensionless coupling constants. The covariant derivative to be applied on the Higgs doublet is given, as usual, by:

$$D_\mu = \partial_\mu + i\frac{g}{2}\mathbf{W}_\mu + i\frac{g'}{2}B_\mu. \tag{5.97}$$

Working in the unitary gauge, we parametrize the *vev* of the doublet in the form:

$$\langle\Phi\rangle = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix}, \tag{5.98}$$

whence the standard massive gauge bosons Lagrangian can be easily computed. Let us now suppose that the Higgs doublet, due to fuzziness of spacetime at small scales, shows a modified dispersion relation of the p^4 kind and therefore let us add to (5.95) the term

$$L_{p^4} = -\frac{\eta}{{\Lambda_{\mathrm{UV}}}^2}M^{\mu\nu\rho\sigma}\left(D_\mu D_\nu\Phi^\dagger\right)D_\rho D_\sigma\Phi. \tag{5.99}$$

Clearly, the addition of this term has no effect on the value of the *vev*, but it does have an effect on the shape of the kinetic terms for the gauge bosons, as well as new interactions. Let us neglect this last issue, and let us focus on the contributions to the kinetic terms. The additional terms are easily identified since they come from the terms

$$\partial_\mu(D_\nu\langle\Phi\rangle). \tag{5.100}$$

After some algebra, we conclude that, in addition to the standard kinetic terms and to the dynamically generated mass terms, the free part of the Lagrangian for the gauge bosons contains the following contribution:

$$-\eta g^2\frac{v^2}{{\Lambda_{\rm UV}}^2}M^{\mu\nu\rho\sigma}\left[(\partial_\mu W_{1\nu}+i\partial_\mu W_{2\nu})(\partial_\rho W_{1\sigma}-i\partial_\rho W_{2\sigma})+\left(\frac{g'}{g}\partial_\mu B_\nu-\partial_\mu W_{3\nu}\right)\left(\frac{g'}{g}\partial_\rho B_\sigma-\partial_\rho W_{3\sigma}\right)\right]. \tag{5.101}$$

As it is easily seen, the mixing terms between the fields B and W_3 can be removed in the standard way, if we introduce the combinations Z^μ, A^μ

$$Z^\mu = \cos\theta_W W_3^\mu - \sin\theta_W B_\mu, \tag{5.102}$$

$$A^\mu = \cos\theta_W B^\mu + \sin\theta_W W_3^\mu, \tag{5.103}$$

with

$$\cos\theta_W = \frac{g}{(g^2+g'^2)^{1/2}}, \quad \sin\theta_W = \frac{g'}{(g^2+g'^2)^{1/2}}, \tag{5.104}$$

and therefore the additional term[10] coming from (5.99) after symmetry breaking becomes

$$-\eta g^2\frac{v^2}{{\Lambda_{\rm UV}}^2}M^{\mu\nu\rho\sigma}\partial_\mu W_{1\nu}\partial_\rho W_{1\sigma}-\eta g^2\frac{v^2}{{\Lambda_{\rm UV}}^2}M^{\mu\nu\rho\sigma}\partial_\mu W_{2\nu}\partial_\rho W_{2\sigma}+$$
$$-\eta(g^2+g'^2)\frac{v^2}{{\Lambda_{\rm UV}}^2}M^{\mu\nu\rho\sigma}\partial_\mu Z_\nu\partial_\rho Z_\sigma. \tag{5.105}$$

The particle spectrum is easily deduced. The masses are the same of the Lorentz-invariant case: there are two massive gauge bosons, $W_{1,2}$, which have mass given by $M_W = gv/2$, a Z boson with $M_Z = M_W/\cos\theta_W$ and a massless gauge field which represents the electromagnetic field, associated to the residual $U(1)$ gauge invariance of the model. However, while this residual gauge invariance protects the A_μ from dangerous terms containing the (Lorentz violating) tensor $M^{\mu\nu\rho\sigma}$, the other gauge bosons have a modified dispersion relation at the level of dimension four operators given by

$$-4\eta\frac{M_W^2}{{\Lambda_{\rm UV}}^2}M^{\mu\nu\rho\sigma}\partial_\mu W_{\pm\nu}\partial_\rho W_{\pm\sigma}, \tag{5.106}$$

for the bosons $W^\pm$, while for the Z^0 the modification is given by:

$$-4\eta\frac{M_Z^2}{{\Lambda_{\rm UV}}^2}M^{\mu\nu\rho\sigma}\partial_\mu Z_\nu\partial_\rho Z_\sigma. \tag{5.107}$$

Notice that this modification is mass dependent: the $W^\pm$ bosons will receive a modification which is (slightly) smaller of the one for the Z^0.

Higher order derivative operators contribute in a similar way. Let us consider, for instance, a term like:

$$\frac{\eta_{(n)}}{{\Lambda_{\rm UV}}^{2n-2}}M^{\mu_1\cdots\mu_{2n}}(D_{\mu_1}...D_{\mu_n}\Phi)^\dagger D_{\mu_{n+1}}...D_{\mu_{2n}}\Phi. \tag{5.108}$$

[10] Here we assume that the matrix M is real.

When the Higgs gets a *vev*, the modification of the kinetic term for the gauge bosons is easily seen to be

$$\frac{\eta_{(n)}}{{\Lambda_{\rm UV}}^{2n-2}} M^{\mu_1 \dots \mu_{2n}} (\partial_{\mu_1} \dots \partial_{\mu_{n-1}} D_{\mu_n} \langle \Phi \rangle)^\dagger \partial_{\mu_{n+1}} \dots \partial_{\mu_{2n-1}} D_{\mu_{2n}} \langle \Phi \rangle. \tag{5.109}$$

In particular, the matrix structure is the same as the p^4 modification, and therefore we can conclude that the photon Lagrangian will not get modifications, while the massive gauge bosons will receive p^{2n-2} modifications to their propagators, which will be suppressed by the ratio $(M^2_{\rm boson}/{\Lambda_{\rm UV}}^2)$, besides the standard suppression given by powers of the UV cutoff. In general, the MDR will not change the diagonalization procedure necessary to extract the mass eigenstates representing the physical propagating modes: they will be given by the same combinations as in the Lorentz invariant case. What is different is just the shape of the dispersion relation/kinetic term. In the specific case we have considered, the $SU(2) \times U(1)$, there was no modification at all of the Weinberg's angle. This is totally general, being related to the fact that the kinetic term is a field bilinear: adding derivatives we do not touch the matrix structure, hence we do not introduce extra sources of mixing, as we naively expected.

The final outcome of this discussion is pretty easy to understand: the MDR of the Higgs propagates in the Lagrangian of the massive gauge bosons in such a way to produce a MDR which is not of the type which we are inserting at the beginning. In particular, the corrections in the form of p^{2n} operators for the Higgs become effectively p^{2n-2} terms for the massive gauge bosons. Nevertheless, as we have shown, these modifications to standard model are further suppressed by $M^2_{\rm boson}/{\Lambda_{\rm UV}}^2$, which means that if the SSB scale and the LIV scale are too far away these terms are negligible, at least at the classical level. This large suppression can make these new terms still compatible with present constraints on dimension four operators: larger modifications would be already ruled out.

The modification to the MDR of the massive gauge bosons Lagrangian is polarization dependent. This can be understood easily since gauge symmetry is broken, and since the would-be Goldstone bosons coming from the Higgs multiplet are included in the gauge fields corresponding to the broken generators, becoming their longitudinal component, have a different dispersion relation with respect to the transverse polarizations. This ultimately results in a polarization dependent dispersion relation. Correspondingly, the residual gauge invariance $U(1)_{em}$ protects the photon from acquiring Lorentz violating terms, at least at the tree level.

Therefore, in order to present an extension of the SM taking into account a sort of energy-dependence geometrical structure of spacetime, when a Higgs mechanism is invoked to have SSB, the analysis of the Lagrangian must be done with care, since new terms appear which cannot be expected naively from just the basic principles one is using, like gauge invariance. Moreover, the extra suppression given by the dimensionless ratio $(M_{\rm boson}/\Lambda_{\rm UV})^2$ cannot be obtained from dimensional analysis alone.

5.7.2 Very Special Relativity

The discussion of the Higgs model in the case of polynomial dispersion relations has highlighted that a MDR for the Higgs field does not produce a MDR of the same kind for the gauge boson. This means that to really believe that a MDR is the manifestation of a modified geometrical structure which wants to be universal as (pseudo-)Riemannian geometry is for SM, then it should at least be compatible with a Higgs mechanism. If not, in order to save the fundamental role of geometry, we must find an alternative scheme of SSB which is compatible with the particular geometrical structure.

In the previous section we have considered a situation in which the geometrical structure is particle dependent. It is interesting, therefore, to consider now a specific model of a "universal", particle independent, modified geometrical structure and to consider on this background the simplest version of a spontaneously broken gauge theory.

Very special relativity has been proposed as a theory in which relativistic invariance is reduced due to the presence of a preferred null vector field. In VSR, the Finsler line element is[11]

$$ds^2 = (\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu)^{1-b}(n_\mu\dot{x}^\mu)^{2b}, \tag{5.110}$$

where n_μ is a constant null vector field, and b a real parameter. As we have already said, the corresponding modified dispersion relation is:

$$(\eta^{\mu\nu}p_\mu p_\nu)^{1-b}(n^\mu p_\mu)^{2b} = m^2, \tag{5.111}$$

where we are raising and lowering indices through the Minkowski metric $\eta_{\mu\nu}$.

For massless particles, the dispersion relation is just the special relativistic one. Notice that, despite its Finsler nature, the line element (5.110) is just obtained from the Minkowski one with a (Finsler-like) disformal transformation. In particular, the causality relations are the same as in special relativity.

Let us suppose that we have an abelian Higgs model on this Finsler spacetime. The Lagrangian will be:

$$L = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \phi^\dagger(\eta^{\mu\nu}D_\mu D_\nu)\phi + m^2\phi^\dagger\left(\frac{in^\mu D_\mu}{m}\right)^B\phi - \frac{\lambda}{2}|\phi|^4, \tag{5.112}$$

where the kinetic term of the scalar field is determined by the MDR through the replacement $p_\mu \to i\partial_\mu$ and then asking that gauge invariance holds, replacing partial derivatives with gauge covariant derivatives. Here we have introduced the notation $B = 2b/(1+b)$ to make expressions simpler to manipulate. Notice that this Lagrangian is the only one compatible with gauge invariance and the VSR relativity group [174].

Clearly this Lagrangian has a non-polynomial form which could make difficult an explicit treatment of the interactions, even at the perturbative level. By dimensional analysis, since there is no energy parameter entering in the dispersion relation which could be used as an expansion parameter to produce a series of differential operators of increasing order, the truncation this kinetic term at a given order of differentiation is impossible.

The outcome of the SSB is the appearance of massive gauge bosons. This implies that we have to obtain a term which reads:

$$m^2\int d^4k\,\eta^{\mu\nu}\tilde{A}_\mu(-k)\left(\frac{n^\rho k_\rho}{m}\right)^B\tilde{A}_\nu(k), \tag{5.113}$$

[11] For the details for the formulation of a field theory in this case, we refer to [174].

in momentum space. This term has the correct transformation properties with respect to the spacetime symmetry group to represent the mass term appropriate for the given dispersion relation.

To see what happens in this Finsler setting, let us consider in detail the case of the free scalar field with the only potential which is allowed by the relativistic symmetry group of the model. The field equation, neglecting the possible coupling with the gauge field, is:

$$\Box\phi - \mu^2 \left(i\frac{n^\rho \partial_\rho}{\mu} \right)^B \phi + \lambda |\phi|^2 \phi = 0, \tag{5.114}$$

Notice that this model does not have a smooth limit when $b \to 0$, since the function x^a is not analytic in $x = 0$, if a is not a positive integer. In the case $b = 0$, which corresponds to the special relativistic case, this equation admits the constant solutions:

$$\phi = v e^{i\theta}, \tag{5.115}$$

with $v = \mu/(2\lambda)^{1/2}, \theta \in [0, 2\pi)$. However, if $b \neq 0$, it is easy to see that the only constant solution has $v = 0$. This is consistent with the fact that one of the boosts is mixed with a dilatation, which does not leave the field ϕ invariant. Therefore, the only constant solution which is compatible with the transformation properties of the field under the relativity group is the identically vanishing solution. A *vev* for ϕ would break the VSR group to a smaller group, in the specific case, the subgroup of $SO(3,1)$ which leaves invariant the vector n^μ. Despite being a logical possibility, the link with the Finsler line element (5.110) would be weakened, since it is true that it is left invariant by this smaller group, but the Minkowski line element would be left invariant as well.

We can conclude then that a Higgs mechanism with a Higgs field taking a constant *vev* is incompatible with the VSR scenario, whose spacetime symmetry group forbids in the Lagrangian any operator which would be able to produce such a constant expectation value. It is worth noting that even a scenario in which a fermion-antifermion condensate is formed, $\langle \bar{\psi}\psi \rangle$, is problematic for the same reason. Under the relativity group of VSR, the wave function of the fermion undergoes dilatations, again making the condensation mechanism incompatible with the relativity group. This means that the problems we are encountering are quite independent from the specific model adopted.

The only other alternative to this scenario is the case of the condensation of some other operator, which has the dimensions of a squared mass, and which is allowed by the symmetries of the system, in particular a scalar which can define uniquely the mass of the gauge boson in every reference frame connected by a relativity transformation. It is easily seen that an operator which could do the job is given by:

$$\mathcal{O} = \phi^\dagger \left(\frac{i n^\alpha \partial_\alpha}{\mu} \right)^{-B} \phi. \tag{5.116}$$

In particular, it is easy to see that the equation of motion (5.114) has the following plane wave solutions:

$$\begin{aligned} \phi &= v(k) e^{-ik_\alpha x^\alpha}, \\ v^2(k) &= \frac{1}{\lambda} \left[\eta^{\alpha\beta} k_\alpha k_\beta - \mu^2 \left(\frac{n^\alpha k_\alpha}{\mu} \right)^B \right]. \end{aligned} \tag{5.117}$$

Notice that this $v(k)$ has the correct transformation properties under the relativity group to represent the amplitude of a scalar field. In the special case in which $k^2 = 0$, after elementary algebra one sees that

$$\mathcal{O} = -\frac{\mu^2}{\lambda}, \tag{5.118}$$

which is left invariant by all the reference frame transformations considered. This kind of operators can be therefore used to build interaction terms with the gauge fields to produce, in certain regimes of the theory, a mass term for the gauge bosons. However, to do so, we have to deeply modify the Lagrangian for the would be gauge field in a way which includes couplings which cannot be obtained just with the minimal coupling prescription $\partial_\alpha \to D_\alpha$.

A potential difficulty is that in order for the Lagrangian to be gauge invariant, the field A_α must enter either through the field strength $F_{\alpha\beta}$, or through the covariant derivative D_α. In particular, in order to get a mass term for a linear equation of motion of the massive gauge boson, we need an operator containing two covariant derivatives, at most. To saturate the vector indices, we need a metric tensor. Finally, for the Lagrangian to be a $U(1)$ scalar we need the combination $\phi^\dagger\phi$. The term just described is nothing but:

$$(iD_\alpha\phi)^\dagger(iD_\beta\phi)\eta^{\alpha\beta}, \tag{5.119}$$

which is already in the Lagrangian and cannot produce the desired mass term. Any other kind of operator can only involve fractional derivatives, and therefore cannot give rise to bilinear expressions in the field A_α, but to highly non-polynomial operators like, for example,

$$\eta^{\alpha\beta}(iD_\alpha\phi)^\dagger\left(\frac{in^\gamma D_\gamma}{\mu}\right)^{-B}(iD_\beta\phi), \tag{5.120}$$

whose physical content, in terms of Feynman diagrams, is not clear at all.

A crucial difficulty, which makes this approach useless, is the fact that a background solution with a generic $k_\mu \neq n_\mu$ automatically breaks the VSR group, since the only vector which is left covariant by the symmetry group is n_μ itself. It is interesting to note that, if we consider this latter possibility, since n_μ is a null vector, we obtain that the corresponding amplitude $v(n)$ given by formula (5.117) actually vanishes, therefore making impossible for our program to have a successful conclusion of generating massive gauge bosons via gauge symmetry rearrangement.

5.8 Summary

The discussion about modified dispersion relations has clarified one point. It is possible to connect them to a modified geometrical structure, Finsler geometry (with the caveats we have mentioned about the definition of pseudo-Finsler geometry). This is indeed an important piece of information. It is ensuring that, even though the dispersion relation is not quadratic, a particle is moving in some form of geometry along its geodesics. This is important especially for giving a rigorous formulation of what has been called "rainbow" geometry, *i.e.* a geometry which is "momentum-dependent". We have shown that a rainbow geometry is nothing else than Finsler geometry in momentum space.

In general, scenarios with broken Lorentz invariance, associated to modified dispersion relations, might still have some sort of geometrical interpretation. This fact could be an extremely useful ingredient in model building, especially in providing coherent frameworks for anisotropic extensions of the standard model. As it has been discussed in chapter 3, when dropping the assumptions of Lorentz invariance, one has to add to the Lagrangian of the standard model all the operators compatible with renormalizability and gauge invariance, as well as the presence of preferred structures like a timelike aether field n^{μ} and generalizations to tensors of higher rank. The inclusion of dispersion relations requires the inclusion of higher dimensional operators in a systematic way.

While this procedure is certainly correct within an EFT approach, the inclusion of all the possible terms leads to a proliferation of free parameters, which must be constrained by experiments and observations. Moreover, one has to take into account quantum corrections to the tree level Lagrangian. In general, it has been shown that higher dimensional Lorentz violating operators are percolating down to operators of dimensions three and four [145], which are the most constrained ones, posing an additional naturalness problem.

This problem of course requires a solution, if on is interested in formulating a phenomenologically viable model with Lorentz symmetry violation. As we have already mentioned, it has been suggested that a custodial symmetry is needed. Of course there is a possible alternative: exactly in the same way in which the pseudo-Riemannian geometry of Minkowski spacetime is reducing the number of possible operators that can be included in a Lagrangian, it might happen that the same happens when a pseudo-Finsler structure is present.

It is important to remember that Finsler geometry has already been considered in the past in this kind of context. Let us cite only the works which have a direct connection with this discussion, where Finsler geometry has been proposed as an alternative geometrical structure for special relativity in 1+1 dimensions [168, 169], as well as anisotropic extensions to 3+1 dimensions [171, 172, 245, 173, 174], as a possible way to avoid the GZK cutoff [244], as a test theory for the metric postulate in general relativity [176, 177, 178, 179], and as the most general spacetime geometry which can be realized in an emergent geometry scenario [107, 246, 50, 106].

The analysis of the geometrical interpretation of a modified dispersion relation was encouraging, in this direction. However, the problem of the mass-dependence of the Finsler norms shows that the attempt is hopeless. Indeed, a change in the mass amounts to a change of the corresponding Finsler structure. The same happens when a Higgs mechanism is invoked to generate mass terms. In fact, it has been shown that a Higgs mechanism is an additional source for a naturalness problem, at least potentially, allowing a direct generation of low dimensional operators for the massive gauge bosons out of higher dimensional operators in the Higgs sector.

We have described the interplay of the Higgs mechanism with MDR. We have shown that a MDR for the Higgs field corresponds to new physics in the gauge sector of the theory, which could be tested, in principle, by precision tests in the massive gauge bosons sector. We have seen that, however, in the most simple cases of polynomial dispersion relations the new physics is additionally suppressed by the hierarchy between the SSB scale and the high energy scale which is associated to the dispersion relation. On one side, this makes them quite difficult to detect, but on the other side, this means that they could be still compatible with present bounds on Lorentz symmetry violation. In particular this is relevant for dimension three and four operators, for which the ratio r is crucial to make them small enough to be compatible with observations.

In this respect, and we stress again this point, the effect of SSB is to translate an order one LIV effect in the Higgs sector into a largely suppressed LIV effect in the massive gauge bosons sector, this suppression being naturally small since it is just the square of the ratio between the EW scale and the Planck scale. Even though radiative corrections could modify this analysis, we can say that the Higgs mechanism can naturally realize a scenario in which LIV is very suppressed[12].

Besides the large suppression, it is worth to note that, for instance, the dimension four operators obtained from dimension six operators after symmetry breaking, are not gauge invariant[13]: therefore, they cannot be naively predicted from the standard arguments [129, 144], for which one uses operators which are gauge invariant. Of course, being not gauge invariant, these operators can appear only for the gauge bosons corresponding to the broken symmetries, leaving untouched the photon's and the gluons' Lagrangian. For these fields, the source for Lorentz violating terms (if there is any LIV for these fields) must be independent: for instance, we can consider models in which also the massless fields do have a MDR.

For what concerns phenomenological investigations, these effects could be negligible for the EW theory: the EW scale and the high energy scale associated with the LIV, the Planck scale for instance, are widely separated, and therefore the massless ration $(M_{EW}/\Lambda_{\text{UV}})^2$ is extremely tiny. In a GUT scenario, where the difference between the GUT scale and the high energy scale Λ_{UV} can be significantly smaller, the effects could be more evident, at least in principle. However, an experimental detection requires the analysis of the physics of the massive bosons coming from the breaking of the GUT symmetry, which is not accessible for our present technology.

Despite producing a potentially interesting phenomenology, the class of polynomial MDR has the important conceptual drawback of being described by a family of Finsler geometries, parametrized by the mass of the particles one is considering [234], losing the uniqueness of the geometrical background: it is impossible to find a unique Finsler metric describing these new kinetic terms. Consequently, it is difficult to believe that these situations can come from a coherent underlying geometrical theory which should describe a "semiclassical" structure of spacetime.

Therefore, it is interesting to see what happens in a fully consistent Finsler setting, in which all the particles see a single Finsler metric. In the specific case of VSR, we have shown that the SSB mechanism *à la* Higgs cannot be realized in the usual way, without breaking (very special) relativistic invariance. This is interesting because it is a spontaneous symmetry breaking of a spacetime symmetry through a scalar *vev*. Moreover, the transformation law for the scalar field under certain changes of reference frame involves a dilatation factor, which is a sort of global $U(1)$ transformation with a complex parameter. Touching the gauge symmetry necessarily touches the relativistic symmetry: they are deeply entangled, in this specific model.

Even without doing explicit calculations to check the radiative corrections, it is interesting to note that a VSR field theory, containing only spinors interacting through gauge fields, at the massless level, has an enhanced degree of symmetry: it is a conformal theory, invariant under the whole Weyl group, since the vector n^μ never appears in the Lagrangian[14]. Nevertheless, we already

[12]Notice that we have just turned the problem of the smallness of the coefficients to the hierarchy problem: $M_{EW}/M_P \approx 10^{-16}$, which requires an independent explanation.

[13]Of course, this is just an artifact of the spontaneous breaking of the symmetry. Gauge invariance is always a symmetry of the action, while it is just realized in a non-linear way.

[14]In [247, 174] the authors considered nonlinear self-interactions like $(\bar{\psi}\gamma^\alpha n_\alpha \psi)^b(\bar{\psi}\psi)^{1-b}$, but this is a non-polynomial term whose physical meaning is not clear. Certainly, it does not describe the mass term of free spinors, since it leads to an equation of motion which is nonpolynomial.

know that scale invariance is broken by quantum corrections via trace/conformal anomalies. In the case of very special relativity, this would amount to the breaking of the relativistic symmetry to a subgroup of the Lorentz group, as we have already discussed. This would spoil the Finsler line element (5.110) of its privileged role as the only line element left invariant by the relativity transformations. For instance, since the symmetry would be reduced to a subgroup of the standard Lorentz group, the standard Minkowski line element is left invariant too, and one could introduce the preferred vector n^μ in other ways, which are not directly linked to a Finsler norm[15].

In general, the discussion of radiative corrections can be crucial, in LIV scenarios [145], since very suppressed higher dimensional operators can percolate on dimension three and four operators, when taking into account higher loops corrections. Here the situation is the same. Apart the SM vertices, which can be consistently renormalized, the new gauge bosons interactions can be particularly dangerous, since loops can amplify them. For example, in the simple scenario we have discussed in section 5.7 where a p^4 modification was considered, the resulting four bosons interaction will have a dimensionless coupling given by $M_{EW}^2/{\Lambda_{\rm UV}}^2$. The relevance of this coupling changes dramatically when we include this vertex in the calculation of the self energy of a gauge boson: by simple arguments we see that the contribution becomes of the order of the EW scale:

$$\frac{M_{EW}^2}{{\Lambda_{\rm UV}}^2}\int^{\Lambda_{\rm UV}}\frac{d^4k}{k^2-M_{EW}^2}\simeq\frac{M_{EW}^2}{{\Lambda_{\rm UV}}^2}{\Lambda_{\rm UV}}^2=M_{EW}^2, \tag{5.121}$$

without any further suppression. In order to protect lowest order operators from these dangerous radiative corrections, we need some form of custodial symmetry which compensates this kind of contribution with another one, with the opposite sign. This can be implemented, for example, providing a SUSY extension of the theory [146, 147]: a fermionic loop with the same amplitude but opposite sign would cancel this dangerous “order one” radiative correction.

At this point a comment on fermions must be made. If we suppose that the mass generation mechanism for them is given through Yukawa couplings with the Higgs, we see that there is no (tree-level) LIV/MDR induced by the Higgs. Of course, it is conceivable that fermionic fields acquire directly a MDR due to QG effects, without necessarily passing through the Higgs, and of course, loop corrections will produce as well modifications to the propagators.

To conclude, it is clear that the Higgs mechanism fits particularly well in Lorentz invariant theories, while it is difficult to reconcile it with different geometrical structures while preserving their fundamental role. In particular, it is well designed to generate masses for gauge bosons in Lorentz-invariant gauge theories, while destroys the geometrical interpretation in Finsler backgrounds like the one of VSR. The key point is that, while in the SM all the kinetic terms are formed using field bilinears and at most two (gauge covariant) derivatives, MDR require more complicated expressions which are not trivial to manipulate when we introduce the decomposition of the Higgs field into the *vev* and fluctuations. Conversely, if we want to preserve a unique geometrical background, we need another mechanism for SSB which avoids this difficulty.

Looking at the problem from a different and more ambitious perspective we could say that an accurate study of the properties of the electroweak symmetry breaking could shed light onto spacetime structure beyond GR, even though only in a very indirect way.

[15] Notice, however, that strictly speaking Riemannian geometry is a special case of Finsler geometry.

It has been also shown that the unique case in which the connection between MDR and Finsler structures is independent from the mass of the particles, the dispersion relation is specified by homogeneous functions. In this case, there is no additional energy scale implied in the game. Instead there is an additional dimensionless parameter. Therefore, to connect these scenarios with some theory of spacetime at the Planck scale is hard: spacetime still looks the same at different scales.

The program seems to fail also for the intrinsic difficulties encountered in associating a pseudo-Finsler structure to different sorts of Lorentz violating models. For instance, it has been showed that in the case of birefringent optics, the introduction of a single regular Finsler structure requires the resolution of some regularity problems (see [248, 230, 231]).

In addition, as we have shown, with the remarkable exception of two dimensional spaces, the most symmetric Finsler spaces are Riemannian. Despite the theorem was proved in the Euclidean signature, it seems reasonable that the result holds in the Lorentzian. This means that, in order to have genuinely Finslerian structures, some anisotropies must be introduced. These anisotropies are encoded in preferred tensors which are absent in special relativity. Besides the compatibility with observations, they pose the conceptual question about their dynamical origin.

Despite these negative sides, the discussion has highlighted an important fact: there are key ideas connecting the research areas of emergent gravity, analogue models and high energy models of Lorentz violation. They share the same kind of structures, as well as the same kind of problems, like the existence of a unique underlying geometrical structure which is keeping under control radiative corrections and in general naturalness problems. In this sense, the discussion of Finsler geometry has shown how these areas can be unified under the general flag of spacetime geometry beyond pseudo-Riemannian spacetimes.

Part II

Dynamics

Chapter 6

Emergent Gravitational Dynamics

6.1 Introduction

So far the attention was focused on the issue of emergent spacetimes, *i.e.* the way in which the evolution of fields can be described by means of an effective metric (for instance the acoustic metric of the BEC). As it has already been stressed, despite interesting, this feature is purely kinematical: nothing is said about the dynamics of the metric itself.

The natural question is, then, whether it is possible that the gravitational interaction, *i.e.* the metric *and* its dynamics, is an emergent phenomenon. As it has been mentioned in the introductory chapter, gravitation manifests some peculiar features which are typical of thermodynamical systems: black hole thermodynamics, the interpretation of Einstein equations as an equation of state and the behavior of critical collapse resembling a typical phase-transition behavior. Of course these facts are not a proof that the gravitational interaction is not a fundamental force and that it is a collective phenomenon, some sort of thermodynamical limit of a more fundamental theory. Nevertheless, it is certainly an interesting possibility.

The second part of this thesis is devoted to the presentation of some work done in the direction of obtaining an emergent gravitational theory out of a non-gravitational one. Before entering into the core of the discussion, it is useful to give a brief discussion about the general issues underlying the emergent gravity framework.

First of all, the idea that the fundamental interactions can be, after all, not so fundamental, is not a new idea in theoretical physics. In a rather old work, Bjorken [249, 250] has showed how it might be possible that a $U(1)$ gauge interaction like electromagnetism can be generated in a purely fermionic theory with a four fermions interaction. It is interesting to give a sketch of the idea. The system is described by the following generating functional:

$$Z[J_\mu] = \int \mathcal{D}\psi\mathcal{D}\bar{\psi} \exp(-\frac{i}{\hbar}(S[\psi,\bar{\psi}] - \bar{\psi}\gamma^\mu\psi J_\mu)), \tag{6.1}$$

where the action is given by:

$$S[\psi,\bar{\psi}] = \int d^4x\bar{\psi}\left(i\gamma^\mu\partial_\mu - m + \frac{G}{2}(\bar{\psi}\gamma^\nu\psi)^2\right)\psi, \tag{6.2}$$

with G being the coupling constant associated to the four fermions vertex. Of course, this model is non-renormalizable by power counting, and hence it must be seen as an effective theory valid at sufficiently small energies (with respect to the energy scale set by G). The properties of this model are better understood by introducing a Gaussian integration over an auxiliary vector field A_μ in the partition function

$$Z[J_\mu] = N \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}A \exp\left(-\frac{i}{\hbar} S[\psi, \bar{\psi}] - \frac{1}{2G}(A_\mu - G\bar{\psi}\gamma_\mu\psi)^2 - \bar{\psi}\gamma^\mu\psi J_\mu) \right), \tag{6.3}$$

where N is a normalization factor. The term introduced cancels precisely the quartic term in the original action for the fermions, giving us the possibility of integrating away the fermions by the standard techniques of evaluation of functional determinants. The integration leads to

$$Z[J_\mu] = N' \int \mathcal{D}A \exp\left(-i \int d^4x \left(\frac{A^2}{2G} - V(A - J) \right) \right), \tag{6.4}$$

where V is the effective potential, and it is given, formally, by:

$$i \int d^4x V(A) = \log \det(i\gamma^\mu(\partial_\mu + A_\mu) - m) - \log \det(i\gamma^\mu \partial_\mu - m). \tag{6.5}$$

By means of this integration over closed fermionic loops, the field A_μ inherits its own dynamics. The lowest order effective action generated in this way gives:

$$S_{eff} = \int d^4x \left(\frac{A^2}{2G} + \frac{1}{4e^2} F_{\mu\nu}F^{\mu\nu} + \frac{\lambda}{4}(A^2)^2 \right), \tag{6.6}$$

where e is a coupling constant given by

$$\frac{1}{e^2} = \frac{1}{12\pi^2} \log \frac{\Lambda^2}{m^2}, \tag{6.7}$$

with Λ an UV cutoff regulating the integrals.

Clearly, this theory is not gauge invariant, since there are terms containing A^2 in the effective action. Nevertheless, the potential can have a Mexican-hat shape, being minimized by non-vanishing values of A_μ. In this case, the theory would be show a spontaneous Lorentz symmetry breaking. The massless Nambu–Goldstone modes associated to this symmetry breaking might be interpreted as photons, having the same kind of coupling to the fermions, *i.e.* minimal coupling.

This sketch shows how it is possible to have composite vector mediators arising from four-fermions interactions (see also [251, 252, 253]), and gives some support to the idea that gravity might be an emergent phenomenon as well. Clearly, many difficulties must be addressed and, to date, there is no complete working model of emergent gravity. Nevertheless, it is a topic deserving further research.

6.2 Induced gravity versus emergent gravity

This very simple example put forward a question: is it possible to generalize the mechanism to gravity? Before considering this very general point, it is interesting to mention the mechanism of induced gravity. In the previous example, a central role is played by quantum corrections to the tree level action. It is instructive to see how these can be relevant within a context in which gravity is introduced.

In a very famous paper [254], Sakharov suggested the idea that the dynamics of the gravitational field coud be seen as a manifestation of the "elasticity of spacetime" with respect to the matter fields living over it. Technically, the story goes as follows [255]. Assume to have a certain set of matter fields, denoted collectively by ϕ. Assume that they propagate over a metric $g_{\mu\nu}$ which is, at least at a first glance, just an external field, without any sort of dynamics. The only equations of motion are, at the tree level,

$$(\Box_g - m^2)\phi = 0, \tag{6.8}$$

for the matter fields[1]. However, quantum corrections significantly change the picture.

By simple manipulations of the functional integrals, it can be seen that loop corrections due to the dynamics of the matter fields can generate an effective action for the gravitational field. At one loop:

$$S_{1-loop}[g] \approx \log\det\left(\Box_g - m^2\right). \tag{6.9}$$

The standard Schwinger–DeWitt technique [22, 256] shows that this term involves a number of divergent contributions, which need appropriate counterterms to be renormalized. In particular, among these terms there are what we call the cosmological constant term and the Einstein–Hilbert action.

This picture has been refined by Adler [257, 258], essentially by assuming that the gravitational dynamics is due to heavy fields. In that scheme it is even possible to compute the value of the Newton's gravitational constant. In the same spirit, but with a slightly different twist, there are proposals of some models of gravity (and gauge interactions) entirely arising from fermionic loops [259, 260, 261, 262]. These attempts are certainly very important in the discussion of the nature of gravitational interaction, and must be taken into account

For instance, one can apply the same logic to the case of BEC [263]: there, the loop corrections due to the quantum phononic field might generate cosmological constant and Einstein–Hilbert terms. Nevertheless, there is a key difference: while in Sakharov's proposal the dynamics of the gravitational field is produced entirely by means of loop corrections, in a BEC there is already a dynamics for the condensate inducing a dynamics for the acoustic metric, *i.e.* the Gross–Pitaevski equation. In these systems, then, the loop contributions are just subdominant corrections.

Induced gravity is certainly an option that should be considered. Nevertheless, it is perhaps not appropriate to think of induced gravity as an issue of emergent gravity. Indeed, there is a difference between the present discussion and the one presented in the previous section. Reduced to the bone, the mechanism described by Bjorken is essentially the formation of a composite field, by some sort of fermionic condensation:

$$A_\mu \propto \langle \bar{\psi}\gamma_\mu\psi \rangle. \tag{6.10}$$

Subsequently, the dynamics of the field A_μ is induced by the quantum corrections.

The case of induced gravity is fundamentally different in the first step: the pseudo-Riemannian structure of spacetime is postulated *a priori*. It is assumed that the matter fields are living on some sort of spacetime. What is not assumed is the fact that the metric tensor does obey some equation

[1] Of course, for fermions one should consider the corresponding Dirac equation and consistently take into account the small differences with respect to the bosonic case. Small modifications are present for particles with different spins. Nevertheless, the general result is unchanged.

of motion. Sakharov's induced gravity, then, is an issue in which only the gravitational dynamics is induced by quantum corrections, and not the entire geometrical picture. Stated in another way, in Sakharov's picture, the graviton is not a composite particle.

In this sense, then, it is important to distinguish this scheme from a more radical emergent picture, where all the concepts needed to speak about spacetime geometry are emerging from a pre-geometric phase.

6.3 The Weinberg–Witten theorem: a no-go theorem?

The idea of having the graviton as a composite particle/derived concept is certainly a fascinating idea. However, there are limitations to what it is possible to do. In particular, there is a theorem, due to Weinberg and Witten [264], which is often presented as a crucial (fatal, in fact) obstruction for a successful emergent gravity program.

The theorem states precise limits for the existence of consistent theories with massless particles. It has two parts, and it says that (quoting from [264]):

1. A theory that allows the construction of a Lorentz-covariantly conserved four-vector current J^μ cannot contain massless particles of spin $j > 1/2$ with nonvanishing values of the conserved charge $\int J^0 d^3x$.
2. A theory that allows for the construction of a conserved Lorentz covariant energy-momentum tensor $\theta^{\mu\nu}$ for which $\int \theta^{0\nu} d^3x$ is the energy-momentum four-vector cannot contain massless particles of spin $j > 1$.

For a careful discussion of the proof of the theorem, and for references, see [265]. For additional comments, see [266, 267]

Crucial ingredients for the proof of this theorem are Lorentz invariance and the nonvanishing of the charges obtained from Lorentz covariant vectors and tensors. Interestingly, the gauge bosons like the gluons and the graviton are not forbidden since the current for the gluons is not Lorentz-covariant conserved, and the graviton does not possess a covariant stress-energy tensor (but rather a pseudo-tensor).

This theorem, then, poses rather strong constraints on the possible theories that can be built in Minkowski spacetime. Of course, gravity is not just the theory of a spin-2 particle in Minkowski spacetime. Nevertheless, it surely makes sense to consider the linearized theory in sufficiently small neighborhoods. In this limit, then, the theorem does apply.

With this caveat in mind, we can say that in an emergent gravity program this theorem must be taken appropriately into account and appropriately evaded. There are (at least) two "obvious" way out:

- allow for Lorentz symmetry breaking, or
- make the spacetime manifold to emerge as well.

The first option is rather straighforward, and it is essentially what could be pursued within scenarios like the one considered in analogue models, in which a preferred time function is specified.

However, this is also a (conceptually high) price to pay: a step back from Minkowski spacetime to the notions of absolute space and time. Moreover, and most importantly, there is the issue of recovering a low energy approximate Lorentz invariance: as we have already seen, this is not at all an easy task without the requirement that the theory possesses some additional symmetries which are able to realize this feature.

The second option is probably the most viable, conceptually appealing, but most demanding in terms of new concepts to be introduced. If no reference is made to a background Minkowski spacetime, but rather it is assumed that the graviton emerges in the same limit in which the manifold emerges, then there is no obvious conflict with the WW theorem. Simply, what is called the gauge symmetry in terms of fields living of spacetime is the manifestation of an underlying symmetry acting on the fundamental degrees of freedom in the limit when they are reorganized in terms of a spacetime manifold and fields (gauge fields and gravitons in particular).

There are already two examples of this possibility, namely matrix models and quantum graphity models. In both cases, the very notion of spacetime manifold is immaterial for the foundations of the theory. The manifold and the metric are derived concepts, obtained in precise dynamical regimes of the theory. The interested reader can find additional comments and references in [48, 268].

The bottom line of this very concise overview of the WW theorem is clear: to obtain a realistic model of emergent gravity one must ask for very special mechanisms to be at work in the model. Without these, the theory would not be able to give a meaningful limit.

6.4 Emerging what? Some examples

In an emergent gravity approach, there are several features we would like to reproduce. Of course, the main achievement one is looking for is the emergence of a viable dynamics for the gravitational field, compatible with the present constraints. Notice that this does not mean necessarily Einstein's theory, but in general any modification of it whose predictions are within the experimental ranges. Furthermore, one could be interested in asking more refined questions. For instance, what is the origin of diffeomorphism invariance, whether this requirement is really compelling as it seems and what are the specific properties that must be included into the fundamental theory such that the emergent theory does possess this features.

Diffeomorphism invariance represent a gauge invariance for the gravitational field, and its presence has a lot of consequences at different levels. To make the reader fully appreciate this point, it will suffice to stress that one of them is to circumvent the Weinberg–Witten theorem. Therefore, to understand how one can make diffeomorphism invariance to emerge is a key step for the program of making the graviton to pop-out of a non-gravitational model.

In this closing section, some comments about diffeomorphism invariance and time (as the signature of the metric) are made, with the hope to clarify some points which will be used in the following.

6.4.1 Diffeomorphism invariance

One of the cornerstones of general relativity is the invariance of the theory under diffeomorphisms, that is to say, the physical indistinguishability between a solution to Einstein's equations $(\mathscr{M}, g)$ and its pull-back $(\mathscr{M}, \phi_* g)$ under a diffeomorphism $\phi : \mathscr{M} \to \mathscr{M}$.

This indistinguishability needs some further specifications (see [269] for a careful treatment of this subject). An aspect of diffeomorphism invariance is the possibility of rewriting the equations of motion in a general covariant form, *i.e.* making them invariant in form under and arbitrary change of coordinate system. While this is certainly an important feature of general relativity, this is not at all its peculiarity. In fact, any equation of motion can be written into a general covariant form. For instance, the Klein-Gordon equation for a massless scalar field on Minkowski spacetime,

$$\Box_\eta \phi = 0, \tag{6.11}$$

is formally equivalent to the set of equations

$$\Box_g \phi = 0, \qquad R^\mu_{\nu\rho\sigma}(g) = 0, \tag{6.12}$$

where $R^\mu_{\nu\rho\sigma}$ is the Riemann tensor associated to the metric $g_{\mu\nu}$. With the trick of inserting an auxiliary field $g_{\mu\nu}$ and by imposing a constraint on it, the vanishing of the Riemann tensor, the Lorentz-invariant equation has been turned into a generally invariant system of equations [270].

Nevertheless, the two formulations are only formally equivalent. Their physical content is in fact very different, as we shall argue.

An interesting feature of general invariance is that it leads to the possibility of generating new solutions of the field equations. For instance, consider Einstein's equations in vacuum. Let x^μ be a local coordinate chart, and let $g_{\mu\nu}(x)$ a solution of the equations, specified in that coordinate chart. Let $f^\rho(x)$ and invertible map. Then, as one can immediately verify, the metric

$$\tilde{g}_{\mu\nu}(x) = \frac{\partial f^\rho}{\partial x^\mu}(x)\frac{\partial f^\sigma}{\partial x^\nu}(x) g_{\rho\sigma}(x), \tag{6.13}$$

is another solution of the equations. Of course, this mapping f^ρ can be seen as the coordinate chart action of a diffeomorphism, and therefore the result is not a surprise.

This property, immediately poses a question about the interpretation of the coordinates. In fact, assume to solve the Einstein equations with given initial conditions on a Cauchy surface Σ. Let $(D^+(\Sigma), g)$ be a solution[2]. Let us now play the game of constructing a new solution by suitably chosing a function f^ρ. The function is chosen to be the identity $f^\rho(x) = x^\rho$ on the whole $D^+(\Sigma)$ except from a small region $\mathscr{H}$ in the interior (such that $\mathscr{H}$ and Σ do not intersect), where the function differs from the identity (without losing the invertibility). This defines a new solution $\tilde{g}_{\mu\nu}$ with the same initial conditions, which however differs in the neighborhood $\mathscr{H}$ from the solution g. One can be convinced of this by examining, for instance, the values of the Kretschmann invariants $K = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ of the two solutions at the same point $x_0^\mu \in \mathscr{H}$. By designing the function f in the way we described, it is clear that $K_g(x_0) \neq K_{\tilde{g}}(x_0)$.

This poses an immediate question about the loss of predictability of the theory, at least if one follows the perspective that the coordinates do have some physical interpretation, for instance as

[2]Here $D^+(\Sigma)$ is the future domain of dependence, see [1].

readings of rods and clocks. This argument, known as the "hole argument", shows that such a position is untenable. In order to extract the physical content from a diffeomorphism invariant theory one has to introduce explicitly "rods and clocks", which are able to map the mathematical description into the physical one.

In this sense, it is clear that the operation of rewriting the Klein–Gordon equation into a diffeo-invariant form is only formal. The empirical content of the two formulations is rather different. In fact, while in the first (Lorentz-invariant) form the coordinates do have a *direct* meaning of readings of rods and clocks, which therefore enables us to speak immediately about the properties of the field ϕ at the event x, the second formulation cannot give this possibility. In order to be equivalent to the first formulation, more fields are needed: these fields will turn abstract points of a manifold, without any physical meaning, into spacetime events.

This brief discussion should help the reader in distinguishing some formal aspects of diffeomorphism invariance from more physical aspects. For further readings we refer to the extant literature (see, for instance, [269] and references therein).

6.4.2 What about time?

It is sometimes stated that in general relativity time is somehow an illusory concept [271, 272, 273, 274]. For instance, a careful examination of the canonical formalism, the ADM formalism [275], shows that time evolution closely resembles a gauge transformation, with the Hamiltonian constraint playing the role of the gauge transformation generator. While this is certainly true, this particular point of view hides the fact that there is a privileged class of foliations for the spacetime manifold.

In general relativity time is encoded in a very peculiar way within the theory. In fact, there is no preferred notion of time, given that every timelike curve corresponds to a clock of a particular observer living on that world-line and that there is no way to prefer one observer with respect to the others. Therefore time is implemented by means of the signature of the metric, and this will be taken as our definition.

Of course, this is a very preliminary definition, which is unable to grasp a fundamental property of time that we experience in everyday life: its arrow. For a critical assessment of this point, see [276]. This point is supported by simple examination of some solutions of Einstein's equations, like for instance Gödel universe and in general spacetimes with closed timelike curves [1, 277].

To take the signature as the definition of time, is particularly compelling when coming back to the Hamiltonian version of general relativity. Indeed, while the above remark about the interpretation of time evolution in terms of gauge transformations gets a part of the issue, it does not take into account another important fact, *i.e.* the precise form of the Hamiltonian constraint, which does take into account explicitly the signature of the four dimensional metric.

In Einstein's gravity (and all the generalizations considered so far, with some exceptions), the signature of the metric is a background quantity, *i.e.* it is not specified by the equations of motion, but rather it is given *a priori*.

In all the analogue models proposed so far, with the notable exception of the mentioned signature change events in BECs [81], the Lorentzian signature of the metric is induced by the presence of an underlying time direction. Somehow, time seems to be a feature which cannot be easily realized in a timeless system. Even in the case of BECs, the signature is turned to Euclidean from Lorentzian, and not the inverse.

Nevertheless it might be interesting to understand how fundamental is the notion of time (defined as the signature of the metric). In [278], it has been shown that general relativity can be seen as the Higgs phase of a $GL(4)$ gauge theory, where the order parameter controlling the (spontaneous) breaking of the symmetry from $GL(4)$ to $SO(3,1)$ is a multiplet of scalars κ_{ab}, symmetric in the two $GL(4)$ indices, which is taking a non-zero vacuum expectation value, and in particular $\kappa_{ab} = \eta_{ab}$, the Minkowski metric in a Lorentz frame. While this is certainly promising, there are still many points to be clarified. Nevertheless, it certainly suggests that the signature of the metric might not be necessarily a fixed background, but might result from some dynamical selection process. Together with the motivations from analogue models, this suggests a very natural question: can we realize a toy model in which time emerges from a timeless system?

Chapter 7

Emergent gravitational dynamics: the case of BEC

7.1 Introduction

As we have already discussed in chapter 2, the work done in the area of analogue models for gravity so far has been mainly focused on the analysis of kinematical features of quantum field theories in curved spacetimes. In particular, encouraging steps have been done in the direction of studying effects like Hawking radiation and cosmological particle creation in the context of Bose–Einstein condensates, their robustness against high energy modifications of the dispersion relation, and possible experimental detection.

Despite the obvious relevance of these investigations for the general understanding of quantum field theories in curved spacetimes, they are considering aspects that rely only on kinematical features of the theory. Whether the acoustic metric is a solution of Einstein equations (or modifications) or not is irrelevant for the phenomena under consideration.

The extension of the study of emergent spacetimes, *i.e.* the way in which an effective pseudo-Riemannian structure is generated, to the study of emergent gravity, *i.e.* the way in which gravitational dynamics is induced from the equations of motion of the underlying degrees of freedom, represents a key issue. In all the analogue models mentioned in this thesis, it is pretty clear from the onset that the acoustic metric is generally found as a solution of fluid equations which are certainly not equivalent to Einstein equations. The main reason is that they are non-relativistic theories defined over Galilean spacetime. Hence, their relevance for the understanding of gravitational dynamics is limited, at least at first sight.

In the previous chapter we have seen that this is indeed a general problem of "emergent gravity" scenarios. While an effective Lorentzian metric is generically possible to be realized, it is very difficult to get for it a gravitational theory which obeys Einstein equations (*i.e.* to have an emergent spin-2 graviton), without starting from the very beginning with a theory which is already very close to general relativity. In particular, to be able to simulate Einstein gravity with a condensed matter system requires that the rigid scaffolding provided by Galilean spacetime must disappear, at least in some limit. This seems hopeless.

Nevertheless, in these analogue models there is an emergent geometry, over which the excitations of sufficiently small energy are propagating. Moreover, given the very definition of the analogue model as a dynamical system obeying some equation of motion, this Lorentzian metric does have some form of dynamics. In an appropriate sense, then, in analogue models there can be a sort of gravitational dynamics, provided that we understand how gravity is defined. Obviously, the dynamical equations will be very different from Einstein equations. For instance, in the case of acoustic metric in fluids, the components of the metric are determined by solving the continuity and Euler equations for the fluid flow. However, despite this *a priori* obstruction, it might be the case that some useful general lessons can be extracted anyway.

The first possibility is that instead of obtaining exactly general relativity, these gravitational theories will be of different sort. Indeed there are several gravitational theories different from Einstein's, differing in various aspects from it. For a review of some possible metric theories see for instance [279]. Of course many alternative theories have been ruled out (for instance the one proposed by Nordström, which will be discussed in the next chapter). In general there are strong constraints (see [4]) that are reducing the space of theories of concrete physical interest.

The fact that the emergent gravitational theory is not general relativity does not mean that the investigation of the dynamics of the geometry of analogue models is always useless. Indeed general relativity is based on several structures and assumptions (metric, signature, equivalence principle, Einstein's equations...). Different theories, based on different sets of assumptions, even though not realistic, might shed a light on some particular issues related to some specific aspect, otherwise hidden by mere technical difficulties. In this sense, analogue models could play the role of toy models for certain aspects of gravitational dynamics.

There are not many works in the area of emergent gravitational dynamics. There are some works in the area of string-net condensation (see [280, 281, 282]) addressing the issue. However, these works focus just on the emergence of a spin-two graviton at the linearized level. While this is certainly important, still there are other directions worth exploring. In particular, the coupling with matter fields and the nature of vacuum energy remain somehow obscure in these models.

It turns out that a BEC model can be used as a toy model to learn rather intriguing lessons, which are the subject of this chapter. We will follow closely the material presented in [283]. We have seen that BEC systems do have the structure to naturally describe quantum field theories in curved spacetimes. The quantum fields are represented by the quasi-particles, the collective excitations over the condensate, while the acoustic metric is generated by the condensate itself. It is interesting to investigate whether we can rewrite the equations of motion for the acoustic metric in the form of some known gravitational equation. Of course, given that the fundamental system is described by a nonrelativistic equation, we can at most obtain some modification of Poisson equation for some form of Newtonian gravitational potential.

We have seen in chapter 2 that the equations governing the dynamics of BECs are the Bogoliubov–de Gennes equations. From their general form we can expect to get some sort of semiclassical nonrelativistic quantum gravity. In the familiar picture (Fig. 7.1) it corresponds to a rather unexplored corner (see however [284, 285, 286]). While this is certainly a regime where little can be extrapolated for the dynamics of the graviton, still one can extract general ideas which are valid independently from the detailed form of the equations of motion. The two main lessons arising from

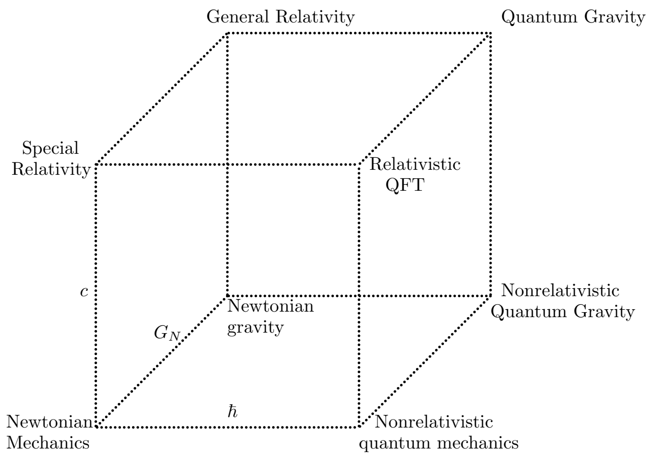

Figure 7.1: According to the inclusion of specific combinations of the three fundamental constants $\hbar, c, G_N$, the various theories can be assigned to one of the vertices of a cube.

this models concern locality, as one expect from the fact that the phonons are collective degrees of freedom, and, most surprisingly, about the cosmological constant.

7.2 BECs dynamics

The program of obtaining some sort of Poisson equation out of the Bogoliubov–de Gennes theory cannot be carried on in a standard BEC. Indeed, phonons are massless excitations. Since we want phonons to enter the Poisson equation as a source term, we have to circumvent the impossibility of treating massless particles in the framework of Newtonian mechanics. This is easily done by making phonons massive. Concretely, this is done by introducing a new term in the Hamiltonian which will softly break the usual $U(1)$ symmetry associated to number conservation and therefore will allow the quasi-particles to acquire a mass. Essentially, the quasi-particles will be pseudo-Goldstone bosons [78, 100]: their spectrum, instead of being gapless, is gapped. One can expect, then, that instead of massless quasi-particles, the collective modes above the condensate will be massive.

In order to do so, the standard Hamiltonian $\hat{H}_0$ described in chapter 2 needs to be slightly modified, by introducing a term which is (softly) breaking the $U(1)$ symmetry in (2.15).

$$\hat{H}_0 \to \hat{H} = \hat{H}_0 + \hat{H}_\lambda, \qquad \hat{H}_\lambda = -\frac{\lambda}{2}\int \mathrm{d}^3\mathrm{x}\left(\hat{\Psi}(\mathrm{x})\hat{\Psi}(\mathrm{x}) + \hat{\Psi}^\dagger(\mathrm{x})\hat{\Psi}^\dagger(\mathrm{x})\right). \tag{7.1}$$

The parameter λ has the same dimension as μ (see Chapter 2). With this new Hamiltonian, the non-linear equation (2.15) becomes

$$i\hbar\frac{\partial}{\partial t}\hat{\Psi} = [\hat{H}, \hat{\Psi}] = -\frac{\hbar^2}{2m}\nabla^2\hat{\Psi} - \mu\hat{\Psi} + \kappa|\hat{\Psi}|^2\hat{\Psi} - \lambda\hat{\Psi}^\dagger. \tag{7.2}$$

The addition of this term implies that the whole dynamics of the system must be reconsidered. The analysis presented in the subsection (7.2.3) will show how $\hat{H}_\lambda$ generates a mass for the quasi-particle. Even though $\hat{H}_\lambda$ both creates and destroys pairs of atoms, it is not difficult to check that

$\hat{H}_\lambda$ is not commuting with the number operator $\hat{N}$,

$$[\hat{H}_\lambda, \hat{N}] = -\lambda \int \mathrm{d}^3\mathrm{x} \left(\hat{\Psi}(\mathrm{x})\hat{\Psi}(\mathrm{x}) - \hat{\Psi}^\dagger(\mathrm{x})\hat{\Psi}^\dagger(\mathrm{x}) \right) \tag{7.3}$$

while unitarity is preserved. In fact, when applied on a state with a definite number of atoms n we have:

$$|n\rangle \to |n-2\rangle + |n+2\rangle, \tag{7.4}$$

which means that an eigenstate of the number operator evolves into a superposition of states with different occupation numbers. However, the expectation value of the number of operator on its eigenstates is still constant

$$i\hbar \frac{\partial}{\partial t} \langle n|\hat{N}|n\rangle = \langle n|[\hat{N}, \hat{H}]|n\rangle = \langle n|[\hat{N}, \hat{H}_\lambda]|n\rangle \propto \langle n|n-2\rangle - \langle n|n+2\rangle = 0. \tag{7.5}$$

Given this crucial difference with the standard description of BECs, a formalism like the particle-number-conserving one [76, 77] cannot be immediately used for these new models. However, given the important improvements in the description of inhomogeneous condensates provided by this formalism it will be interesting to extend the particle-number-conserving method in a suitable way, in order to be able to control to which accuracy we can trust the standard mean field approximation we are using. In fact, we are assuming that the addition of the new term in the Hamiltonian will not destroy the stability properties of the mean field theory, and in particular the fact that the mean field theory, for nearly homogeneous condensates and trapping potentials, in weakly time-dependent regimes, does offer a good description of the condensate dynamics.

This point of view (and its implications for the condensation mechanism) deserves some further specifications. While the condensation process can be easily understood as a macroscopic occupation number of an energy level, there are several approaches to describe it mathematically. The mean field approach is particularly convenient: we say that the system of N bosons has condensed whenever the field $\hat{\Psi}$ develops a non-zero vacuum expectation value (vev)

$$\langle\Omega|\hat{\Psi}|\Omega\rangle = \psi, \tag{7.6}$$

where ψ is the condensate wave-function. If this mean field is non-vanishing, we have that the two point correlation function

$$G(\mathrm{x}, \mathrm{y}) = \langle\hat{\Psi}^\dagger(\mathrm{x})\hat{\Psi}(\mathrm{y})\rangle \approx \psi^*(\mathrm{x})\psi(\mathrm{y}), \tag{7.7}$$

tends to a non-zero constant when x, y are infinitely separated, *i.e.* the system develops long range correlations [72].

The mean field method is based on the assumption that the ground state of the system is not the vacuum state of the Fock space, $|0\rangle$, but rather it is similar to a coherent state [287]. For a single mode Fock space, a coherent state is defined to be:

$$|z\rangle = e^{-|z|^2/2} e^{-z\hat{b}^\dagger}|0\rangle, \tag{7.8}$$

and it is easy to see that it is an eigenstate of the annihilation operator:

$$\hat{b}|z\rangle = z|z\rangle. \tag{7.9}$$

In the case of BEC, the fact that the state $u_0(\mathrm{x})$ is macroscopically occupied (*i.e.* there are N_0 bosons in the state 0, with $N_0/N \approx 1$) can be formalized by taking:

$$|\Omega\rangle \approx e^{-N_0/2} e^{-i(N_0)^{1/2}\hat{a}_0^\dagger}|0\rangle. \tag{7.10}$$

On this states, the field operator $\hat{\Psi}$ behaves like a c-number:

$$\hat{\Psi}(\mathrm{x})|\Omega\rangle \approx \sqrt{N_0}u_0(\mathrm{x})|\Omega\rangle, \tag{7.11}$$

where the approximation is due to the fact that interactions are introducing some corrections. This property of the ground state motivates the splitting of the field operators into the classical part, which deals with the condensed phase, and small residual fluctuations, describing the states which are close to the ground state:

$$\hat{\Psi} = \psi\mathbb{I} + \hat{\chi}. \tag{7.12}$$

This is the basic idea of the mean field approximation: the field operator $\hat{\Psi}$ is approximated by a classical field, which is describing the condensate, while the fluctuations, assumed to be small, are still encoded in a field operator.

The fact that the solutions to the Gross–Pitaevski equation leads to a non-vanishing mean field, in light of the discussion about the realization of a regime of long range correlations (see (7.7)), ensures that a condensation has taken place. In this sense, the addition of the new term into the Hamiltonian should not forbid the condensation, provided that λ, μ, κ are such that the condensate wavefunction can be different from zero.

From a different point of view, one can imagine to keep λ very small with respect to all the other energy scales present in the theory, making the new term a tiny perturbation of the system. Of course, nonperturbative effects can spoil this picture. It is perhaps important to stress that some non-perturbative effects in BEC have been already discussed in chapter 2. For instance, the depletion factor, evaluated in (2.47), is not analytic in the parameter κ, around $\kappa = 0$. Developments in condensed matter theory are necessary to decide this subtle point.

7.2.1 $U(1)$ breaking: discussion

The role of the $U(1)$ symmetry in standard BEC has been carefully discussed. It is related to the fact that the number of atoms is conserved. The breaking of this symmetry therefore is connected with the failure of this charge to be conserved. There are at least two possible scenarios to implement this.

A rather natural option is to have an open system. Concretely, one could imagine to have a condensate which is able to exchange particles with some sort of reservoir, in such a way to preserve, on average, their number. Several settings in this sense could be conceived, *e.g.* with coupling with suitably tuned lasers.

A second important instance where the number of constituents is not necessarily a conserved charge is the one in which the constituents themselves are some sort of collective degrees of freedom. A system which has some similarities with the one discussed in this chapter is represented by the excitations in the so quantum Heisenberg ferromagnet (see, for instance [288]) defined with a spin system. In this case, the fundamental operators give rise to effective degrees of freedom, called magnons, whose Hamiltonian, in general, is not $U(1)$ invariant. For more details see [289], and references therein.

7.2.2 The condensate wave-function

We consider the dynamics generated by (7.2), from which we want to extract the equation of motion for the condensate ψ. The evolution of the mean field ψ is easily determined in terms of the eigenstates $|E\rangle$ of the Hamiltonian $\hat{H}$:

$$i\hbar\frac{\partial}{\partial t}\psi = i\hbar\frac{\partial}{\partial t}(\langle E|\hat{\Psi}|E\rangle) = \langle E|i\hbar\frac{\partial}{\partial t}\hat{\Psi}|E\rangle = -\frac{\hbar^2}{2m}\nabla^2\psi - \mu\psi - \lambda\psi^* + \kappa|\psi|^2\psi + 2\kappa\mathfrak{n}_E\psi + \kappa\mathfrak{m}_E\psi^*, \tag{7.13}$$

where $\mathfrak{m}_E = \langle E|\hat{\chi}^2|E\rangle, \mathfrak{n}_E = \langle E|\hat{\chi}^\dagger\hat{\chi}|E\rangle$ encode the effect of the non-condensate atoms. This is the generalization of the Bogoliubov-de Gennes (BdG) equation for the condensate wave-function to the case $\lambda \neq 0$.

In the standard case, if we have N particles in the condensate, the number density of the non-condensate fraction is of order $1/N$ with respect to the number density of the condensate. In particular, the terms $\mathfrak{m}, \mathfrak{n}$ are of order $1/N$. This can be safely exported to our case, with the slight modification of the meaning of the number N, which does represent only the average number of particles in the condensate (see (7.5)).

At zeroth order in the $1/N$ expansion, we have the generalization of the Gross-Pitaevski (GP) equation:

$$i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\nabla^2\psi - \mu\psi - \lambda\psi^* + \kappa|\psi|^2\psi. \tag{7.14}$$

The time independent homogeneous solution to the GP equation is

$$n_c = |\psi|^2 = \frac{\mu + \lambda}{\kappa}, \tag{7.15}$$

where we have fixed the phase of the condensate to be zero. In 7.2.5 we will show that this is not an arbitrary choice, but rather a consequence of the situation we want to describe once we impose a stability condition for the quasi-particles.

As in the standard case, we define the healing length ξ as the length scale at which the kinetic term is of the same order of magnitude of the self-interaction term in the Hamiltonian:

$$\frac{\hbar^2}{2m\xi^2} = \kappa n_c \Leftrightarrow \xi^2 = \frac{\hbar^2}{2m\kappa n_c}. \tag{7.16}$$

Again, this length represents the spatial scale needed for the condensate to pass from the value $n_c = 0$ at the boundary of the region where it is confined to the bulk value n_c, as one can easily show with a calculation similar to the one presented in subsection 2.3.3. Also, in this case, the healing length represents the scale of the dynamical processes involving the deformation of the condensate wavefunction. This will have a crucial impact on the emergent gravitational dynamics in these systems.

7.2.3 Quasi-particles

The equation of motion for the particles out of the condensate is obtained by subtracting the equation for the condensate (7.14) from the equation for $\hat{\Psi}$ given in (7.2). We are interested in the propagating modes, so we neglect the self-interactions. We obtain:

$$i\hbar\frac{\partial}{\partial t}\hat{\chi} = -\frac{\hbar^2}{2m}\nabla^2\hat{\chi} + (2\kappa|\psi|^2 - \mu)\hat{\chi} + (\kappa\psi^2 - \lambda)\hat{\chi}^\dagger. \tag{7.17}$$

Let us consider the case of homogeneous condensate with density n_c given above. In this situation we have:

$$i\hbar\frac{\partial}{\partial t}\hat{\chi} = -\frac{\hbar^2}{2m}\nabla^2\hat{\chi} + (\mu + 2\lambda)\hat{\chi} + \mu\hat{\chi}^\dagger. \tag{7.18}$$

If we decompose the field $\hat{\chi}$ in its plane wave components, we can rewrite this equation as

$$i\hbar\frac{\partial}{\partial t}\hat{a}_{\mathrm{k}} = \frac{\hbar^2 k^2}{2m}\hat{a}_{\mathrm{k}} + (\mu + 2\lambda)\hat{a}_{\mathrm{k}} + \mu\hat{a}^\dagger_{-\mathrm{k}}. \tag{7.19}$$

The mixing between $\hat{a}$ and $\hat{a}^\dagger$ due to the evolution in time becomes then apparent. We therefore pass to the quasi-particle operators $\hat{\phi}(\mathrm{x})$

$$\hat{\phi}(\mathrm{x}) = \frac{1}{\sqrt{V}}\sum_{\mathrm{k}}\hat{b}_{\mathrm{k}}e^{i\mathrm{k}\cdot\mathrm{x}}, \tag{7.20}$$

which are related to the particle operators through the Bogoliubov transformation

$$\hat{a}_{\mathrm{k}} = \alpha(\mathrm{k})\hat{b}_{\mathrm{k}} + \beta(\mathrm{k})\hat{b}_{-\mathrm{k}}, \quad \text{with } |\alpha|^2(k) - |\beta|^2(k) = 1. \tag{7.21}$$

The coefficients α, β are only functions of $k = |\vec{\mathrm{k}}|$, since the condensate is homogeneous and isotropic. The equation of evolution for the quasi-particles is then given by

$$i\hbar\frac{\partial}{\partial t}\hat{b}_{\mathrm{k}} = \mathcal{E}(k)\hat{b}_{\mathrm{k}}, \tag{7.22}$$

with the energy

$$\mathcal{E}(k) = \left(\frac{\hbar^4 k^4}{4m^2} + 4\lambda(\mu + \lambda) + \frac{\mu + 2\lambda}{m}\hbar^2 k^2\right)^{1/2}. \tag{7.23}$$

The Bogoliubov coefficients are given by:

$$\alpha^2(k) = \frac{A(k) + \mathcal{E}(k)}{2\mathcal{E}(k)}, \qquad \beta^2(k) = \frac{1}{2\mathcal{E}(k)}\frac{\mu^2}{A(k) + \mathcal{E}(k)}, \tag{7.24}$$

where we have introduced the quantity

$$A(k) = \frac{\hbar^2 k^2}{2m} + \mu + 2\lambda. \tag{7.25}$$

The high energy limit of these coefficients is:

$$\lim_{k\to\infty}\alpha^2(k) = 1, \qquad \lim_{k\to\infty}\beta^2(k) = 0, \tag{7.26}$$

which means that at large wave-number (and hence large momentum), the quasi-particle operators coincide with the particle operators. This matches with the behavior of the energy, which becomes just the energy of a non-relativistic particle of mass m, just like a free atom. The dispersion relation (7.23) suggests the introduction of the following quantities:

$$c_s^2 = \frac{\mu + 2\lambda}{m}, \qquad \mathcal{M}^2 = 4\frac{\lambda(\mu + \lambda)}{(\mu + 2\lambda)^2}m^2. \tag{7.27}$$

Here c_s plays the role of the speed of sound, while $\mathcal{M}$ plays the role of a rest mass for the quasi-particle. Since $\mathcal{M}$ is proportional to λ, we clearly see that it is the term $\hat{H}_\lambda$ that generates the mass of the quasi-particle. When $\lambda \to 0$, that is when $\hat{H} \to \hat{H}_0$, the quasi-particle becomes massless, *i.e.* a phonon, and the speed of sound reduces to the usual one in BEC. Perturbation theory, therefore, should be a viable strategy to compute the various physical properties of these systems.

Notice that, in order to have a non-negative mass square term, and to avoid a tachyonic instability, we have to require $\lambda \geq 0$. In standard BEC, one usually assumes that the chemical potential μ is positive: indeed if it were negative, there could not be any condensation. In our case, we can relax this requirement and obtain that $\mu > -\lambda$ as a condition. In the following we consider $\mu > 0$, in order to be able to consider the case in which the correction we are inserting is very small, without affecting dramatically the condensation. Indeed, it is easy to see that a condensation can take place even in a system with this soft $U(1)$ breaking by checking the behavior of the two points correlation function $G(\mathrm{x}, \mathrm{y}) = \langle \hat{\Psi}^{\dagger}(\mathrm{x})\hat{\Psi}(\mathrm{y})\rangle$. It is immediate to realize that, in the case of homogeneous condensate, this correlation function describes long range correlations, since the mean field ψ is non-vanishing (cf. equation (7.15)).

$\mathcal{M}$ is proportional to m, the mass of the atoms. Defining the ratio $\zeta = \lambda/\mu$, we introduce the function $F(\zeta)$

$$\mathcal{M}^2 = F(\zeta)m^2 = 4\frac{\zeta(1+\zeta)}{(1+2\zeta)^2}m^2. \tag{7.28}$$

Under our assumptions, we have that $\zeta \geq 0$. It is then straightforward to check that on this domain $F(\zeta)$ is a monotonic (increasing) function and that

$$F(0) = 0, \qquad \lim_{\zeta \to +\infty} F(\zeta) = 1. \tag{7.29}$$

We conclude therefore that the mass of the quasi-particles $\mathcal{M}$ is always bounded by the mass of the atoms, $\mathcal{M} \in [0, m)$.

It is also interesting to notice that using the variable ζ, the speed of sound is:

$$c_s^2 = \frac{1+2\zeta}{1+\zeta}\frac{\kappa n_c}{m}. \tag{7.30}$$

For ζ small, we then have $c_s^2 \approx \kappa n_c/m$, which is the standard result, while, for $\zeta \to \infty$, $c_s^2 \to 2\kappa n_c/m$.

7.2.4 The various regimes for the MDR

Before moving on to the gravitational dynamics, let us discuss briefly the content of the dispersion relation (7.23) for the quasi-particles, rewritten using c_s and $\mathcal{M}$.

$$\mathcal{E}(p) = \left(\frac{p^4}{4m^2} + c_s^2 p^2 + \mathcal{M}^2 c_s^4\right)^{1/2}, \tag{7.31}$$

where we are using the obvious notation $p = \hbar k$ to simplify the shape of the expressions. Let us define the characteristic momenta p_A, p_B and p_C such that

$$\frac{p_A^4}{4m^2} = c_s^2 p_A^2, \qquad \frac{p_B^4}{4m^2} = \mathcal{M}^2 c_s^4, \qquad c_s^2 p_C^2 = \mathcal{M}^2 c_s^4, \tag{7.32}$$

so that they are explicitly

$$p_A^2 = 4m^2 c_s^2, \qquad p_B^2 = 2m\mathcal{M}c_s^2, \qquad p_C^2 = \mathcal{M}^2 c_s^2. \tag{7.33}$$

They are related by

$$p_C^2 = 2F(\zeta)p_B^2 = 4F^2(\zeta)p_A^2. \tag{7.34}$$

If $\zeta \ll 1$, which will be the regime we shall consider, we have also that

$$p_C \ll p_B \ll p_A. \tag{7.35}$$

Taking into account (7.35), the characteristic momenta define different regimes:

- If $p \gg p_A$, the term p^4 dominates, the dispersion relation (7.31) is well approximated by $\mathcal{E} \sim p^2/2m$, we are in the trans-phononic regime.
- If on the contrary we have $p_C \ll p \ll p_A$, we can safely neglect the term of order p^4, we are then in the relativistic regime since the dispersion relation (7.31) is well approximated by $\mathcal{E} \sim (p^2c_s^2 + \mathcal{M}^2c_s^4)^{\frac{1}{2}}$. The quasi-particle is then relativistic, when the speed of sound c_s is playing the role of the speed of light.
- If we are in the regime where $p \ll p_C$, this means that the quasi-particle has a speed much smaller than c_s, so that this is the Galilean limit of the relativistic regime. We are then dealing with a Galilean quasi-particle. The rest mass $\mathcal{M}c_s^2$ provides the usual constant shift of the Galilean energy $\mathcal{E} \sim \mathcal{M}c_s^2 + p^2/2\mathcal{M}$.

7.2.5 The fluid description

We have already seen that the standard Gross–Pitaevski (GP) equation describing a BEC admits an interesting fluid interpretation, through the Madelung representation. We are considering now the GP equation given in (7.14)

$$i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\nabla^2\psi - \mu\psi - \lambda\psi^* + \kappa|\psi|^2\psi, \tag{7.36}$$

and we want to use the Madelung representation for the complex field ψ:

$$\psi = \sqrt{n_c}e^{-i\theta/\hbar}. \tag{7.37}$$

When replacing this into the GP equation, dividing by the phase and splitting the resulting expression into the real and imaginary parts we obtain two equations:

$$\dot{n}_c + \vec{\nabla}\cdot(n_c\vec{v}) = -\frac{\lambda}{\hbar}n_c\sin\left(\frac{2\theta}{\hbar}\right), \tag{7.38}$$

$$\dot{\theta} = V_{\text{quantum}} + \frac{m}{2}v^2 - \mu - \lambda\cos\left(\frac{2\theta}{\hbar}\right) - \kappa n_c, \tag{7.39}$$

where we have introduced the velocity field $\vec{v} = -\vec{\nabla}\theta/\hbar$, and

$$V_{\text{quantum}} = -\frac{1}{\sqrt{n_c}}\frac{\hbar^2}{2m}\nabla^2\sqrt{n_c}, \tag{7.40}$$

is the familiar quantum potential term. These two equations, in the case $\lambda = 0$, reduce to the usual form of the continuity equation and the Euler equation for a perfect fluid, once we neglect the quantum potential term. On the other hand, when $\lambda \neq 0$ the $U(1)$ invariance is broken, and the number operator is no more conserved by the Hamiltonian evolution.

It is interesting to see what happens when we consider the case of homogeneous condensates, $\partial_\mu n_c = \partial_\mu v^i = 0$. From the first equation we get:

$$\sin\left(\frac{2\theta}{\hbar}\right) = 0 \Leftrightarrow \theta = \frac{l\pi}{2}\hbar, \quad l \in \mathbb{Z}. \tag{7.41}$$

This result implies that not only $\vec{v}$ is constant, but that actually vanishes. Inserting this result in the second equation we obtain:

$$n_c = \frac{\mu + \cos(l\pi)\lambda}{\kappa}. \tag{7.42}$$

The analysis of the quasi-particle dynamics in the case of homogeneous condensates has shown that the case $\lambda < 0$ corresponds to a negative mass square term, *i.e.* tachyonic behavior: the energy of a quasi-particle would get an imaginary part leading to exponential growing and damping of modes. Since a choice of the phase of the condensate such as $\cos(l\pi) = -1$ would be completely equivalent to a change of sign of λ, thus leading to instabilities, without repeating the analysis done to obtain the energy spectrum of the quasi-particles, we see that $\cos(l\pi) = 1$ is required for the stability of the condensate.

7.3 Locality and the UV scale of BEC

Before going on with the description of the gravitational theory in a BEC, it is better to make some comments on the first aspect of the physics of BECs which could be relevant in our understanding of macroscopic physics. This comment holds also for the standard BEC. However at this point it is more clear what are its implications.

We have seen that in a BEC there are two levels. There are the atoms, represented by the field operators $\hat{\Psi}(\mathrm{x})$, and there are the quasi-particles, which are the propagating modes (eigenstates of the Hamiltonian). In other field theories, for instance in scenarios in which some symmetry is spontaneously broken, the relation between the "fundamental" fields ϕ^A and the propagating modes ω^A is just a linear and local relation:

$$\omega^A(x) = M^A_B \phi^B(x). \tag{7.43}$$

The matrix M^A_B is the mixing matrix. In BECs the situation is conceptually different. Instead of a relation like (7.43), we have a Bogoliubov transformation:

$$\omega^A(\mathrm{k}) = M^A_B(\mathrm{k})\phi^B(\mathrm{k}), \tag{7.44}$$

where we are working in momentum space. This relation is always linear. However, it is nonlocal:

$$\omega^A(\mathrm{x}) = \int d^3y K^A_B(\mathrm{x}, \mathrm{y})\phi^B(\mathrm{y}), \tag{7.45}$$

where the kernel K is determined by the Bogoliubov coefficients:

$$K^A_B(\mathrm{x}, \mathrm{y}) = \int d^3k \, M^A_B(\mathrm{k}) e^{-i\mathrm{k}\cdot(\mathrm{x}-\mathrm{y})}. \tag{7.46}$$

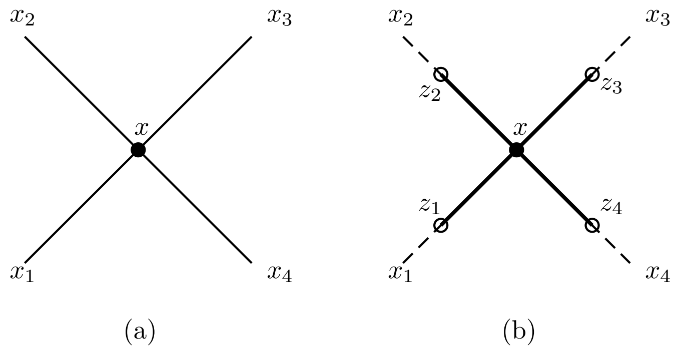

Figure 7.2: (a) Feynman diagram of a four-particles vertex. (b) Lowest order Feynman diagram for the interaction of four quasi-particles (dashed lines), induced by the interactions of particles (solid line). The circles denote the conversion of particles into quasi-particles at intermediate points. They represent the action of the Bogoliubov transformations.

This is the mathematical statement of the fact that quasi-particles are collective degrees of freedom. In particular, there is a mismatch between the notion of locality of the quasi-particle with respect to the notion of locality of the atoms. In this sense we say that there are two notions of locality, one microscopic and one macroscopic, and that they are distinct. A more precise explanation is needed. Separately, the two classes of operators ϕ, ω do obey canonical equal time commutation relations:

$$[\phi^A(\mathrm{x}), (\phi^B)^\dagger(\mathrm{y})] = \delta^{AB}\delta^3(\mathrm{x}-\mathrm{y}), \qquad [\omega^A(\mathrm{x}), (\omega^B)^\dagger(\mathrm{y})] = \delta^{AB}\delta^3(\mathrm{x}-\mathrm{y}), \tag{7.47}$$

which are a direct consequence of the fact that Bogoliubov transformations are preserving the algebra of the creation-annihilation operators. Therefore, as long as we use only one of the two families, there is no way in which a deviation from standar local quantum field theory can be manifest. However, the mixed commutators are nontrivial. It is straightforward to see that:

$$[\phi^A(\mathrm{x}), (\omega^B)^\dagger(\mathrm{y})] = (K^{AB})^*(\mathrm{x}, \mathrm{y}), \tag{7.48}$$

Of course, this fact becomes crucial when the effective Lagrangian describing the physics of the quasi-particles involves terms mixing particle and quasi-particle operators.

In the case of the BEC it is pretty clear how the underlying dynamics induces in the action for the quasi-particles an interaction term of the form

$$L_{\mathrm{int}} = -\frac{\kappa}{4}\left((\phi^A)^\dagger\phi_A\right)^2 \tag{7.49}$$

which explicitly involves the particle fields, rather than the quasi-particles. Hence, when computing the effects of the interaction terms, *e.g.* scattering processes between the quasi-particles, the nonlocality encoded in the kernel K will necessarily enter the physical quantities (see figure 7.2).

This feature has pretty interesting implications and links with the research done in quantum gravity. The (effective) quantum field theories describing high energy physics are based on the assumption on locality, which is entering the physical observables at different levels. To give just an example, any violation of locality would result into a violation of bounds on cross sections which are essentially kinematical in nature, depending just on the fact that the underlying theory is local.

A detection of a violation of locality could be an indication that the (effective) quantum field theory we are using at low energies, namely the standard model of particle physics, is an effective field theory of quasi-particles over a condensate. Since we know that it is impossible to obtain the entire condensate dynamics out of a local quantum field theory based on quasi-particles (without knowing the Bogoliubov transformations), we could imagine that if the same picture holds for the real worlds, and in particular for gravitons, the difficulties in quantizing gravity are due to the fact that we are insisting in using the quasi-particle basis instead of using the particle one.

It is worth mentioning the fact that the status of locality has been questioned in quantum gravity. In relativistic quantum field theory, locality of the theory is encoded in the fact that, considering for instance the case of a real scalar field,

$$[\phi(x), \phi(y)] = 0, \tag{7.50}$$

if the points x, y are space-like separated.

When we pass to the realm of quantum gravity, the metric tensor is promoted to a quantum description. Whatever this description will turn out to be, quantum fluctuations of the metric tensor, especially on very small scales, are expected to be very large, leading to the concept of spacetime foam [114, 115]. Therefore, at very small scales, the notions of time-like, space-like and light-like separation of two points are basicly useless, if gravity is quantized. This has the obvious consequence that at sufficiently small scales any effective field theory based on locality and microcausality should break down.

The violation of locality in the low energy effective field theory has been explicitly checked within the formalism of string field theory, especially in the particularly relevant case of black hole spacetimes [290]. Violation of locality in black hole physics is crucial since it is a loophole in the argument by Hawking for information loss in black hole evaporation. In this respect spacetimes with horizons are special since they can magnify microscopic details of spacetime structure, or, in other words, bring the details of ultraviolet physics down to infrared scales. For a discussion on the role of locality in black hole physics see, for instance, [85, 83, 84, 291].

Coming back to the case of BECs it is interesting to note that the nonlocality is controlled by a very specific energy scale. We have seen that the Bogoliubov coefficients depend on the momentum of the quasi-particles in such a way that quasi-particles operators are becoming particle operators for very high momenta. The scale of the crossover is given by the healing length scale, which represent then the typical scale associated with nonlocality.

7.4 Gravitational dynamics

So far we have described the physics of the system in the case of homogeneous condensate ψ, and its implications for locality of physics of phononic observers. The next step is the analysis of the inhomogeneous condensate, and hence the promised emergence of a gravitational dynamics. To simplify further the analysis it is better to consider the case of condensates which are nearly, but not exactly, homogeneous: this will correspond to the case of weak gravitational field. This limitation is consistent with the formalism we are using. Indeed the mean field method certainly is not a good approximation in regions where there are large variations in density (see for instance vortex cores).

In an asymptotically flat spacetime, in order to identify the Newtonian gravitational potential it is necessary to evaluate the non-relativistic limit of the geodesic equation in a weak gravitational field [2]. In the asymptotic region there is a coordinate system such that the metric can be written as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. The tensor $h_{\mu\nu}$ encodes the deviation from exact flatness, *i.e.* the gravitational field. In this regime, it is easy to show that the Newtonian gravitational field is identified with the component h_{00}:

$$\Phi_N(\mathrm{x}) = -\frac{1}{2} h_{00}(\mathrm{x}). \tag{7.51}$$

In the context of standard BEC (*i.e.* dealing with the non-linear equation (2.15)), the quasi-particles travel in an emergent metric ds^2 determined in terms of the homogenous condensate ψ.

$$ds^2 = \frac{n_c}{m c_s}\left[-\left(c_s^2 - v^2\right) dt^2 - 2 v_i dt d\mathrm{x}^i + \delta_{ij} d\mathrm{x}^i d\mathrm{x}^j\right], \tag{7.52}$$

where m is the mass of the atoms and c_s and $\vec{v}$ depend on the properties of the condensate $\psi = \sqrt{n_c} e^{i\theta}$, through

$$c_s = \frac{\kappa n_c}{m}, \qquad \vec{v} = \frac{1}{m} \vec{\nabla}\theta.$$

Considering that the condensate is homogenous, the density and velocity profiles become constant, *i.e.* respectively $n_c = n_\infty$, $\vec{v} = \vec{v}_\infty$. With the coordinate transformation,

$$dT = dt, \qquad dX^i = d\mathrm{x}^i - v^i_\infty dt, \tag{7.53}$$

the line element (7.52) is rewritten as:

$$ds^2_\infty = -c^2_\infty dT^2 + d\mathbf{X}^2. \tag{7.54}$$

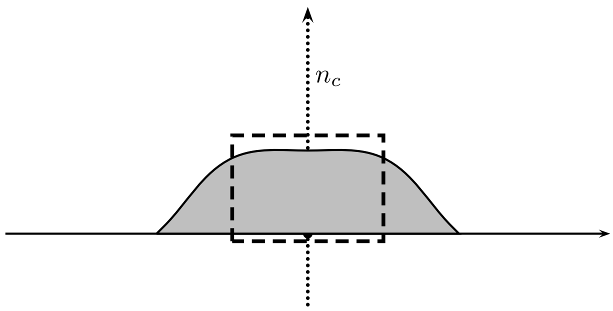

Figure 7.3: Schematic configuration of BEC wavefunction, confined within a given region. The box highlights the region of interest.

The condition of asymptotic flatness for the acoustic spacetime can be translated into the condition of asymptotic homogeneity for the condensate. We require then that only in a small region of space, in the bulk, the condensate deviates from perfect homogeneity (see the region enclosed by the box in figure 7.3)

We consider therefore some small deviation from the asymptotic values of the velocity and of the density:

$$n_c = n_\infty(1 + 2u(\mathrm{x})), \qquad \vec{v} = \vec{v}_\infty + \vec{w}(\mathrm{x}), \qquad \text{with } u \ll 1, \quad w \ll v. \tag{7.55}$$

This implies in particular a rescaling of the speed of sound.

$$c_s^2 = \frac{\kappa n_c}{m} = c_\infty^2(1 + 2u(\mathrm{x})).$$

The acoustic line element (7.52) becomes then

$$ds^2 = \frac{n_c}{mc_s}\frac{mc_\infty}{n_\infty}\left(-(c_s^2 - v^2)dt^2 - 2v_i d\mathrm{x}^i dt + \delta_{ij} d\mathrm{x}^i d\mathrm{x}^j\right), \tag{7.56}$$

where we have introduced a constant prefactor mc_∞/n_∞ in order to have the conformal factor asymptotically normalized to one. Using (7.55), together with the coordinate change (7.53), the acoustic line element has the form:

$$ds^2 = ds_\infty^2 - 3u(X)c_\infty^2 dT^2 - 2w_i(X)dTdX^i + u(X)\delta_{ij}dX^i dX^j, \tag{7.57}$$

at first order in u, w_i. Consequently, we see that

$$h_{00}(X) = -3c_\infty^2 u(X), \tag{7.58}$$

so that the gravitational field is encoded in the *number density perturbation* of the condensate wave-function ψ,

$$\Phi_N(X) = \frac{3}{2}c_\infty^2 u(X), \tag{7.59}$$

while it is independent from velocity perturbations, which therefore can be discarded.

This result allows a simplification in the choice of the physical situation: it is enough to discuss the case in which the condensate wavefunction has a constant phase, while its modulus slightly deviates from perfect homogeneity. It is convenient to introduce the parametrization:

$$\psi = \left(\frac{\mu + \lambda}{\kappa}\right)^{1/2}(1 + u(\mathrm{x})), \tag{7.60}$$

where $u(\mathrm{x})$ is a dimensionless function and it is assumed to be very small. In practice, we will assume that it is associated with a localized inhomogeneity of the condensate. At infinity (but still inside the region highlighted in Fig. 7.3) we ask that $u \to 0$. Notice that the wavefunction (7.60) is real, due to the fact that we are really interested in number density fluctuations, and not on fluctuations in the velocity profile. This simplification reduces the number of independent functions without making the system trivial, as it will be shown.

7.4.1 The gravitational potential for the quasi-particles

Having discussed the setup in which we are working, we can reconsider the quasi-particle dynamics in this new system. It is not necessary to recover some sort of acoustic metric. While it is certainly interesting, given that the quasi-particle are massive and that we are interested in the nonrelativistic limit, the notion of acoustic metric is of little interest, in this particular case: it has been used just as a guide to isolate a candidate for the gravitational potential.

The first step, then, is to see if there is a term in the equation of motion for quasi-particles (7.17), which can be identified as an external potential term. This will allow us to check the conjecture that it will be given by the number density perturbation $u(\mathrm{x})$, as well as the precise coefficient relating it to the familiar Newtonian potential (having dimensions of the square of a velocity). The next step will be to take this potential and plug it into the Bogoliubov–de Gennes equation (7.13), which is describing its dynamics.

To identify the Newtonian potential, the diagonalization of the Hamiltonian in (7.17) for the field $\hat{\chi}$ must be done again, including now the fluctuations of the condensate wavefunction. In this case, the diagonalization procedure is more involved: we have to deal with the non-commuting operators ∇^2 and u. We can not perform it in an exact way. However, we are interested in the Galilean regime for the quasi-particle spectrum, when $p_C \gg p$. It is then a reasonable approximation to neglect all the terms involving the commutators $[\hat{p}^2/2m, u(\mathrm{x})]$, which are largely suppressed (with respect to the other terms appearing in the equations) by the mass of the atoms and from the smallness of $u(\mathrm{x})$.

With these simplifying assumptions, the Hamiltonian for the quasi-particles in the non-relativistic limit is

$$\hat{H}_{quasip.} \approx \mathcal{M}c_s^2 - \frac{\hbar^2\nabla^2}{2\mathcal{M}} + 2\frac{(\mu+\lambda)(\mu+4\lambda)}{\mathcal{M}c_s^2}u(\mathrm{x}), \tag{7.61}$$

where the mass of the quasi-particle $\mathcal{M}$ and for the speed of sound c_s are given in (7.27). We first recognize the constant shift $\mathcal{M}c_s^2$ of the energy due to the rest mass in the Galilean regime. This term is not affecting the discussion in any way and can be subtracted without physical consequences. The term proportional to $u(\mathrm{x})$ can be clearly interpreted as an external potential. If we want to identify it with the gravitational potential Φ_{grav}, we need to have

$$2\frac{(\mu+\lambda)(\mu+4\lambda)}{\mathcal{M}c_s^2}u(\mathrm{x}) = \mathcal{M}\Phi_{\mathrm{grav}} \Leftrightarrow \Phi_{\mathrm{grav}}(\mathrm{x}) = \frac{(\mu+4\lambda)(\mu+2\lambda)}{2\lambda m}u(\mathrm{x}), \tag{7.62}$$

where $\mathcal{M}$ is the mass of the quasi-particles. Note that this identification is formal, and relies on the way in which the gravitational potential enters the Schroedinger equation for a non-relativistic quantum particle. We should always work with u: our definition of Φ_{grav} is dictated from the analogy we want to make with Newtonian gravity. For instance, we see that this definition becomes singular when we deal with massless quasi-particles, *i.e.* when $\lambda \to 0$. This must be expected: when λ vanishes the quasi-particles become massless phonons, for which the coupling to a Newtonian gravitational potential cannot be defined in terms of their mass density.

7.4.2 The Modified Poisson equation

Now that we have identified a candidate for the Newton potential Φ_{grav} from the quasi-particles dynamics, we need to check that it satisfies some sort of Poisson equation. Since the gravitational potential is deduced from ψ – as small deviations from perfect homogeneity (c.f. (7.60)) – the Poisson equation should be deduced from the BdG equation (7.13). With the natural assumption that the potential is reacting instantaneously to the change of distribution of matter, we can neglect the time derivative and (7.13) becomes

$$\left(\frac{\hbar^2}{2m}\nabla^2 - 2(\mu+\lambda)\right)u(\mathrm{x}) = 2\kappa\left(\mathfrak{n}(\mathrm{x}) + \frac{1}{2}\mathfrak{m}(\mathrm{x})\right). \tag{7.63}$$

We have seen in section 7.2.2 that the terms $\mathfrak{m}(\mathrm{x})$ and $\mathfrak{n}(\mathrm{x})$ are functions of the atoms $\hat{\chi}$ outside the condensate and therefore of the quasi-particle $\hat{\phi}$, through the Bogoliubov transformation (7.21). Therefore they can be interpreted as the source in the (modified) Poisson equation. We examine now different types of source: either localized particles or plane-waves.

Localized sources

The most natural source to consider for the Poisson equation is a single quasi-particle $\hat{\phi}$ at a given position x_0. However, point-like distributions give rise to divergencies. We consider therefore a quasi-particle which is localized around the point x_0, with a non-zero spread to regularize these divergencies. We consider a quasi-particle in a state of the form:

$$|\zeta_{\mathrm{x}_0}\rangle = \int \mathrm{d}^3\mathrm{x}\zeta_{\mathrm{x}_0}(\mathrm{x})\hat{\phi}^\dagger(\mathrm{x})|\Omega\rangle, \qquad \text{with} \qquad \int \mathrm{d}^3\mathrm{x}|\zeta_{\mathrm{x}_0}(\mathrm{x})|^2 = 1 \Leftrightarrow \langle\zeta_{\mathrm{x}_0}|\zeta_{\mathrm{x}_0}\rangle = 1. \tag{7.64}$$

ζ_{x_0} encodes the spreading of the particle around x_0 since

$$\langle\zeta_{\mathrm{x}_0}|\hat{\phi}^\dagger(\mathrm{x})\hat{\phi}(\mathrm{x})|\zeta_{\mathrm{x}_0}\rangle = |\zeta_{\mathrm{x}_0}(\mathrm{x})|^2. \tag{7.65}$$

We can now determine the value for the anomalous mass $\mathfrak{m}$ and anomalous density $\mathfrak{n}$ when the quasi-particle is in the state $|\zeta_{\mathrm{x}_0}\rangle$. An explicit calculation, given in appendix B, gives

$$\mathfrak{n}(\mathrm{x}) = \left|\int \mathrm{d}^3\mathrm{z} f(\mathrm{x}-\mathrm{z})\zeta_{\mathrm{x}_0}(\mathrm{z})\right|^2 + \left|\int \mathrm{d}^3\mathrm{z} g(\mathrm{x}-\mathrm{z})\zeta_{\mathrm{x}_0}(\mathrm{z})\right|^2 + \frac{1}{V}\sum_{\mathrm{k}}\beta^2(\mathrm{k}), \tag{7.66}$$

$$\mathfrak{m}(\mathrm{x}) = 2\left(\int \mathrm{d}^3\mathrm{z}_1 g(\mathrm{x}-\mathrm{z}_1)\zeta^*_{\mathrm{x}_0}(\mathrm{z}_1)\right)\left(\int \mathrm{d}^3\mathrm{z}_1 f(\mathrm{x}-\mathrm{z}_2)\zeta_{\mathrm{x}_0}(\mathrm{z}_2)\right) + \frac{1}{V}\sum_{\mathrm{k}}\alpha(\mathrm{k})\beta(\mathrm{k}), \tag{7.67}$$

where we have introduced the functions f, g depending on the Bogoliubov coefficients α and β

$$f(\mathrm{x}) = \frac{1}{V}\sum_{\mathrm{k}}\alpha(\mathrm{k})e^{i\mathrm{k}\cdot\mathrm{x}}, \qquad g(\mathrm{x}) = \frac{1}{V}\sum_{k}\beta(\mathrm{k})e^{-i\mathrm{k}\cdot\mathrm{x}}. \tag{7.68}$$

The quantities $\mathfrak{n}_\Omega$ and $\mathfrak{m}_\Omega$ with

$$\mathfrak{n}_\Omega = \frac{1}{V}\sum_{\mathrm{k}}\beta^2(\mathrm{k}), \qquad \mathfrak{m}_\Omega = \frac{1}{V}\sum_{\mathrm{k}}\alpha(\mathrm{k})\beta(\mathrm{k}), \tag{7.69}$$

are vacuum contributions independent from the presence of actual quasi-particles. They are related to the inequivalence of the particle and quasi-particle vacua, and it can be easily seen that:

$$\mathfrak{n}_\Omega = \langle\Omega|\hat{\chi}^\dagger(\mathrm{x})\hat{\chi}(\mathrm{x})|\Omega\rangle, \qquad \mathfrak{m}_\Omega = \langle\Omega|\hat{\chi}(\mathrm{x})\hat{\chi}(\mathrm{x})|\Omega\rangle. \tag{7.70}$$

The functions f, g encode the fact that quasi-particles are collective degrees of freedom and therefore intrinsically some non-local objects. This non-locality is due to the Bogoliubov transformation (7.21). Quasi-particles and atoms (*i.e.* local particles) coincide only if we have $\alpha(\mathrm{k}) = 1, \beta(\mathrm{k}) = 0$, and therefore $f(\mathrm{x}) = \delta^3(\mathrm{x})$, while $g(\mathrm{x}) = 0$. Since this is not the case, the anomalous mass and the anomalous density will show an intrinsic non-locality. The spreading characterized by $|\zeta_{\mathrm{x}_0}\rangle$ encodes some extra non-local effect, introduced by hand for regularization purposes. Therefore this feature is not as fundamental as the non-locality introduced by the Bogoliubov transformation.

The equation (7.63) becomes then:

$$\left(\frac{\hbar^2}{2m}\nabla^2 - 2(\mu+\lambda)\right)u(\mathrm{x}) = 2\kappa\left(\tilde{\mathfrak{n}}(\mathrm{x}) + \frac{1}{2}\tilde{\mathfrak{m}}(\mathrm{x})\right) + 2\kappa\left(\mathfrak{n}_\Omega + \frac{1}{2}\mathfrak{m}_\Omega\right), \tag{7.71}$$

where we have introduced the quantities

$$\tilde{\mathfrak{n}}(\mathrm{x}) = \mathfrak{n}(\mathrm{x}) - \mathfrak{n}_\Omega, \qquad \tilde{\mathfrak{m}}(\mathrm{x}) = \mathfrak{m}(\mathrm{x}) - \mathfrak{m}_\Omega, \tag{7.72}$$

which represent the contribution of actual quasi-particles to the anomalous density and anomalous mass, respectively. By dimensional analysis, the terms $\mathfrak{n}, \mathfrak{m}$ have the dimensions of number densities. Since in Newtonian gravity the source for the gravitational field is a mass density, we introduce the mass density distribution:

$$\rho_{\text{matter}}(\mathrm{x}) = \mathcal{M}\left(\tilde{\mathfrak{n}}(\mathrm{x}) + \frac{1}{2}\tilde{\mathfrak{m}}(\mathrm{x})\right). \tag{7.73}$$

With this definition, we can rewrite (7.71) as an equation for the field Φ_{grav}:

$$\left(\nabla^2 - \frac{1}{L^2}\right)\Phi_{\text{grav}} = 4\pi G_N^{loc}\rho_{\text{matter}} + \Lambda, \tag{7.74}$$

where we have defined

$$G_N^{loc} \equiv \frac{\kappa(\mu+4\lambda)(\mu+2\lambda)^2}{4\pi\hbar^2 m\lambda^{3/2}(\mu+\lambda)^{1/2}}, \qquad \Lambda \equiv \frac{2\kappa(\mu+4\lambda)(\mu+2\lambda)}{\hbar^2\lambda}(\mathfrak{n}_\Omega + \frac{1}{2}\mathfrak{m}_\Omega), \tag{7.75}$$

$$L^2 \equiv \frac{\hbar^2}{4m(\mu+\lambda)}. \tag{7.76}$$

This particular choice of notation is motivated by the comparison of (7.74) with the Newtonian limit of Einstein equations with a cosmological constant.

For this reason, we can identify these three quantities as the analogous of the Newton constant, the analogous of the cosmological constant and a length scale which represents the range of the interaction, as we are going to discuss below.

To get a better grasp of the physics of the modified Poisson equation (7.74), we can look at its solution for a given distribution of quasi-particle ρ_{matter}.

As it is well known, a solution for the equation

$$\left(\nabla^2 - \frac{1}{L^2}\right)\Phi(\mathrm{x}) = 4\pi G_N^{loc}\mathcal{M}\delta^3(\mathrm{x}-\mathrm{z}), \tag{7.77}$$

is given by the Yukawa potential

$$\Phi_{\mathrm{Y}}(\mathrm{x};\mathrm{z}) = \frac{G_N^{loc}\mathcal{M}e^{-|\mathrm{x}-\mathrm{z}|/L}}{|\mathrm{x}-\mathrm{z}|}. \tag{7.78}$$

On the other hand, a solution for the equation

$$\left(\nabla^2 - \frac{1}{L^2}\right)\Phi(\mathrm{x}) = \Lambda, \tag{7.79}$$

is just given by the constant solution

$$\Phi_\Lambda = -L^2\Lambda. \tag{7.80}$$

Notice the peculiarity of this solution. It does not give rise to a gravitational acceleration since the gradient is trivially zero. Therefore the only effect of this term is to shift the overall density of the condensate.

Einstein equations with cosmological constant: the Newtonian limit For a discussion see [292]. Here we report just a sketch. If we restore the standard units, Einstein equations read:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G_N}{c^3} T_{\mu\nu}$$

In general the solutions will be asymptotically de Sitter, not asymptotically flat spacetimes. However, in a sufficiently small neighborhood (typically much smaller than the de Sitter radius $\Lambda^{-1/2}$), it is still meaningful to investigate the deviations from flat spacetime, $g_{\mu\nu} \approx \eta_{\mu\nu} + \epsilon h_{\mu\nu}$, where we have introduced $\epsilon \ll 1$ (at least formally) to remember that this is an expansion for a small deviation. Of course the deviation from flatness is due to both the cosmological constant and the stress energy tensors. Therefore, the Einstein equation should be rewritten as:

$$G_{\mu\nu} + \epsilon\Lambda g_{\mu\nu} = \frac{8\pi G_N}{c^3} \epsilon T_{\mu\nu}$$

In the weak field expansion (and after gauge fixing, see for instance [3] for the details), then, the linearized equations read:

$$-\epsilon\Box\bar{h}_{\mu\nu} + \epsilon\Lambda\eta_{\mu\nu} + \underline{\underline{\epsilon^2\Lambda h_{\mu\nu}}} = \frac{8\pi G_N}{c^3} \epsilon T_{\mu\nu}$$

The underlined term is of order ϵ^2, and hence must be discarded, because we are truncating the series at the lowest order. As it has been already mentioned, the interesting equation for the Newtonian limit is the 00 component. If we introduce the notation $\rho_\Lambda = c^2\Lambda/8\pi G_N$, the relevant equation becomes:

$$-\Box\bar{h}_{00} = \frac{8\pi G_N}{c^2}(\rho + \rho_\Lambda),$$

where everything is expressed in terms of mass densities (M/L^3). Here we have just used the weak field hypothesis. Now we have to use the $c^2 \to \infty$ one to get the nonrelativistic limit. The result is the well known:

$$\nabla^2\Phi_N = 4\pi G_N(\rho + \rho_\Lambda)$$

Therefore, the cosmological constant enters the Poisson equation as a constant source term. It is not a mass term for the graviton. To confirm this with a heuristic argument one could check that, when the metric of de Sitter spacetime is written is spherical coordinates, the $g_{00} = -(1 - \Lambda r^2/3)$ does satisfy this equation with $\rho = 0$.

The linearity of equation (7.74) allows us to use these results to write down a solution for a generic distribution of matter (*i.e.* quasi-particles) as

$$\Phi_{\rm grav}(\rm x) = \int \rho_{\rm matter}(\rm z)\Phi_{\rm Y}(\rm x;z)d^3z + \Phi_\Lambda. \tag{7.81}$$

Solutions of (7.74) are therefore constructed from the Yukawa potential smeared out due to the non-locality of the quasi-particle (with an extra global shift due to the cosmological constant). The Yukawa potential is typically encoding some short range interaction, characterized by the scale L which is simply related to the healing length (7.16),

$$L^2 = \frac{\xi^2}{2}. \tag{7.82}$$

Although this a very short range for the gravitational interaction, this outcome should not come as a surprise. In fact, the healing length (c.f. (7.16)) characterizes the typical length over which a condensate can adjust to density gradients. Since density inhomogeneities encode the gravitational interaction, one should expect them to be damped over a distance of the order the healing length.

In the context of relativistic field theory, the short interaction scale for gravity would be translated in a massive graviton, with mass given by

$$M_{grav}^2 = \frac{\hbar^2}{L^2 c_s^2} = 4\frac{\mu+\lambda}{\mu+2\lambda} m^2.$$

We can then compare the masses of the quasi-particles $\mathcal{M}$, graviton M_{grav} and atoms m,

$$0 \leq \mathcal{M} < m < \sqrt{2}m < M_{grav} \leq 2m, \tag{7.83}$$

which shows the hierarchy of the energy scales present in this system. We notice that the graviton is then always more massive than the quasi-particles, and that this interaction is of very short range, since the ξ is much shorter than the acoustic Compton length[1] of the quasi-particles. In particular, we cannot tune the parameters of the system in such a way to make M_{grav} arbitrarily small, in order to be closer to reality.

Other sources: plane wave states

While a quasi-particle localized in a given point in space is certainly the most natural source for gravity from the Newtonian perspective, it is interesting also to see what happens when instead we consider quasi-particles with a definite momentum $\mathrm{p} = \hbar\mathrm{k}$. Let us focus first on the special case of a 1-particle state with momentum p, that is $|\mathrm{p}\rangle = \hat{b}^\dagger_{\mathrm{k}}|\Omega\rangle$. The anomalous mass and the anomalous densities become then

$$\mathfrak{n}(\mathrm{x}) = \langle\Omega|\hat{b}^\dagger_{\mathrm{k}}\hat{\chi}^\dagger(\mathrm{x})\hat{\chi}(\mathrm{x})\hat{b}_{\mathrm{k}}|\Omega\rangle, \qquad \mathfrak{m}(\mathrm{x}) = \langle\Omega|\hat{b}^\dagger_{\mathrm{k}}\hat{\chi}(\mathrm{x})\hat{\chi}(\mathrm{x})\hat{b}_{\mathrm{k}}|\Omega\rangle. \tag{7.84}$$

To express them in terms of quasi-particles, we need to perform the Bogoliubov transformations (7.21). As we recalled in the previous section, we can not specify these transformations exactly, due to the presence of the potential u. However, the corrections to the Bogoliubov coefficients $\alpha(k), \beta(k)$, evaluated in the case $u = 0$ in (7.24), provide some modifications to the expressions of $\mathfrak{n}$ and $\mathfrak{m}$ which are relevant only beyond the linear order in u, which means beyond the approximation we are using. We can then safely neglect these corrections. Using the Bogoliubov transformation, we obtain explicitly

$$\mathfrak{n}_k(x) = \frac{\alpha^2(k)+\beta^2(k)}{V} + \mathfrak{n}_\Omega, \qquad \mathfrak{m}_k(x) = 2\frac{\alpha(k)\beta(k)}{V} + \mathfrak{m}_\Omega. \tag{7.85}$$

where we recognize the contribution of the vacuum $\mathfrak{n}_\Omega$, $\mathfrak{m}_\Omega$.

The generalization to the case of states containing a definite number of quasi-particles with a given momentum follows in the same way. For these states denoted as $|n(k_1),...,n(k_n)\rangle$, one obtains:

$$\mathfrak{n}_k(x) = \sum_i n(k_i)\frac{\alpha^2(k_i)+\beta^2(k_i)}{V} + \mathfrak{n}_\Omega, \qquad \mathfrak{m}_k(x) = 2\sum_i n(k_i)\frac{\alpha(k_i)\beta(k_i)}{V} + \mathfrak{m}_\Omega. \tag{7.86}$$

[1]We are using c_{sound} instead of c_{light} to define all these scales. We have to use the natural units for a hypothetical phononic observer.

In all these expressions, besides the Bogoliubov coefficients, we recognize the terms $n(k_i)/V$, which are the number densities of quasi-particles in a given eigenstate of momentum. These number densities, however, are not giving immediately the source term to be inserted into (7.74), since they are weighted by the Bogoliubov coefficients. This is the representation, in momentum space, of the non-locality we have discussed in position space.

We rewrite (7.74) with a source term made by a single particle with a given momentum $\mathrm{p} = \hbar\mathrm{k}$.

$$\left(\nabla^2 - \frac{1}{L^2}\right)\Phi_{\mathrm{grav}} = 4\pi G_N(k)\rho_{\mathrm{matter}} + \Lambda, \tag{7.87}$$

where we have $\rho_{\mathrm{matter}} = \mathcal{M}/V$ since we have just one particle of mass $\mathcal{M}$, while Λ, L are defined as in (7.75)-(7.76). We encode the effect of the Bogoliubov coefficients, and hence of non-locality, in the "running" Newton constant

$$G_N(k) = \left(\alpha^2(k) + \beta^2(k) + \alpha(k)\beta(k)\right) G_N^{loc}. \tag{7.88}$$

The discussion of the solution to this equation is even simpler than the localized state, given that the source term is just constant. Consequently, $\Phi_{\mathrm{grav}} = const$ is a solution as in the case with purely vacuum contribution. For what concerns the physical effects of this kind of gravitational field, we have to plug this constant solution into (7.61): this amounts just to a shift of the energy, which is, however, momentum dependent, leading to observable relative energy shifts if different momenta are considered.

7.5 Summary

In an analogue gravity model based on a BEC system, the degrees of freedom are separated into the atoms that condense and the ones which do not. Quasi-particles are then collective degrees of freedom constructed from the non-condensed atoms. The dynamics of the quasi-particles can be described, in a given regime, in terms of the propagation of particles over an effective curved spacetime metric, which is a function of the density n_c and the velocity profile $\vec{v}$ of the condensate. In this sense, it is natural to expect that gravitational degrees of freedom are encoded in the condensate. Dynamics of the latter is encoded in the BdG equation (7.13), which is essentially Galilean. Hence, we can not expect to recover the Einstein equations in this context [263]. Nevertheless, one can still try to interpret (7.13) as some sort of Poisson equation for some type of Newtonian gravity.

However, quasi-particles are massless in usual BEC systems and hence they cannot be considered as sources for the gravitational field in the Poisson equation. We introduced therefore a new term $\hat{H}_\lambda$ in the dynamics of the BEC which softly breaks the $U(1)$ symmetry and consequently, as we showed in section 7.2.3, generates a mass gap for the quasi-particles. We showed explicitly that the presence of this small symmetry breaking term does not prevent a condensation from happening and still allows a mean field description (which is sufficiently accurate for our purposes). Then, following the usual general relativistic argument, we have argued, in section (7.4), that the Newtonian potential Φ_N has to be related to small inhomogeneities in the condensate density (while perturbations in the velocity profile do not contribute at first order as gravitational degrees of freedom). This conjecture, based on the analysis of a standard BEC system, was then confirmed by a specific analysis of the modified BEC dynamics for an almost homogenous condensate.

The endpoint of this investigation can be then summarized in the following two equations

$$\vec{F} = \mathcal{M}\vec{a} = -\mathcal{M}\vec{\nabla}\Phi_{\text{grav}}, \tag{7.89}$$

$$\left(\nabla^2 - \frac{1}{L^2}\right)\Phi_{\text{grav}} = 4\pi G_N \rho + \Lambda, \tag{7.90}$$

where $\mathcal{M}$ is the mass of the quasi-particle acquired via the soft $U(1)$ symmetry breaking induced by (7.1), L is proportional to the healing length, Λ plays the role of the cosmological constant and G_N is an effective coupling constant that depends on the condensate microphysics and the form of the matter source.

For what regards the latter we have considered two cases: a localized quasi-particle state and a set of plane waves. In the first case the analogue Newton constant is indeed momentum and position independent and the solution of the modified Poisson equation (7.90), has the form of a smeared Yukawa potential. The smearing is due to the fact that quasi-particles are intrinsically non-local objects, being collective degrees of freedom. When considering plane-waves as sources, we have instead that, due to the momentum dependence of the Bogoliubov transformation, G_N is running with the momentum and the solution for the gravitational potential is a constant (albeit a different one for different momenta). One should however be careful: while it is common in quantum field theory (QFT) to encounter the notion of running coupling constants, the origin of the running here is rather peculiar. Indeed, in QFT the running is due to quantum corrections to the tree level/classical action, here the running is due to the inequivalence between the ground state of the Fock spaces of atoms and quasi-particles. Paraphrasing what has been done in the context of emergent geometry, where the notion of "rainbow geometry" has been introduced, we could speak about "rainbow dynamics".

We have also obtained naturally a cosmological constant in the model: vacuum gravitates, even though in a very peculiar way. It is induced by the terms $\langle\Omega|\hat{\chi}^\dagger\hat{\chi}|\Omega\rangle$, $\langle\Omega|\hat{\chi}\hat{\chi}|\Omega\rangle$, where Ω is the state with no quasi-particles. It is entirely due to the (unavoidable) inequivalence between the quasi-particle vacuum and the particle vacuum and cannot be put to zero just tuning the parameters. It represents an interesting alternative to known mechanisms to generate a cosmological constant (see also [51] for similar ideas about the nature of the vacuum energy in condensed matter systems).

Let us compare this point with the standard cosmological constant problem. If one thinks to the cosmological constant is generated by the zero point energy associated to each mode, in the case of BECs a naive expectation would be that the cosmological constant term would be set at the characteristic UV scale represented by the inverse of the healing length. This would lead to a very large vacuum energy. However, we have seen that the cosmological constant, in this specific case, is linked to the depletion factor (see Eq.(2.47)), and hence naturally suppressed by a factor $1/N$ in the expansion in the inverse of the number of atoms. It is the condensation mechanism itself guaranteeing a naturally small cosmological constant. Of course, the worse is the condensation, the larger is the number of atoms out of the condensate and hence the bigger will be the cosmological constant. This fact has a rather peculiar interpretation.

To have an emergent Lorentzian structure, it is essential that there is a mean field which represents the emergent spacetime. This was encoded in the splitting of the field operators $\hat{\Psi} \approx \psi + \hat{\chi}$. However, for a mean field approach to make sense, fluctuations around the mean field should be relatively small:

$$\frac{\langle \hat{\chi}^2 \rangle}{|\psi|^2} \ll 1. \tag{7.91}$$

Therefore, the smallness of the analogue of the cosmological constant term in the BEC is deeply intertwined with the very definition of mean field, *i.e.* how good is the picture of quasi-particles moving in a classical Lorentzian spacetime. Of course this poses the question of whether this mechanism do apply also for spacetime in which we actually live.

In conclusion, BEC as an analogue model for gravity presents many differences with a realistic gravity theory as we expected. We do not get general relativity in a condensate. However

- there is an emergent Lorentzian metric describing the propagation of the phonons;
- there is a mismatch between microlocality and macrolocality;
- there is an emergent Newtonian gravitational theory, which is very short range;
- in this theory vacuum gravitates;
- the source term for the gravitational field inherits the nonlocality properties of the phonons;
- the cosmological constant is naturally small provided that the depletion factor is small.

This short list should be encouraging. Indeed, despite the limited applicability of the results, the suggestions for realistic theories of gravity, in particular for quantum gravity and the role of locality, and, perhaps more interestingly, for the cosmological constant problem are definitely worth of further investigation.

It is clear that the same observation we have made for the case of emergent Lorentz invariance here does apply for the emergent gravitational dynamics. Here a single field has been introduced. What is really interesting is to see what happens if several different species are present. In that case, besides the issue of having a short range rather than a long range interaction, also the coupling to the gravitational field must be carefully discussed. Indeed, in order to have some sort of equivalence principle, all the fields must be coupled to the gravitational field in the same way.

The natural setup to discuss these issues is the 2-BEC model [105]: in fact in this case one could treat a multi-particle system whose richness could allow a closer mimicking of Newtonian gravity with a long range potential. However, the fact that emergent gravity has to be Newtonian in a BEC-based analogue model seems to be unavoidable since the gravitational potential depends on the condensate, which is typically described by non-relativistic equations. A possible way to avoid this issue is either to consider relativistic BEC [293, 294] (however in this case we would still expect to get only some type of scalar gravity), or to change completely paradigm and identify gravity not in the condensate but among the perturbations around the condensate (see for example [295]). In the next chapter, we will try to address these issues.

Chapter 8

Towards emergent diffeo invariance

8.1 Introduction

In the previous chapter it has been showed how it is possible to understand a number of relevant lessons about gravitation from the study of a relatively simple analogue model based on a BEC system. Nevertheless, the obstructions to realize a perfect dynamical analogue of general relativity out of a condensed matter system are still there. Therefore, despite their potential relevance in understanding some features of gravity, the conclusions following the discussion of the particular toy model must be taken with care.

In chapter 6 two main issues have been considered: the way in which time appears in general relativity and the invariance of the theory under diffeomorphisms. In discussing the general framework of analogue models, it has been shown that it is rather easy to emerge a notion of Lorentzian metric, under rather general conditions, even when one starts from Galilean structures. What it is impossible to emerge is the notion of time: the time of the laboratory is percolating down to the effective level, giving rise directly to the Lorentzian nature of the acoustic geometry.

In general, time is always assumed as an underlying structure in emergent gravity: typically, one considers some sort of dynamical system in which, in a given regime, a metric structure appears. Consequently, the notion of time of the emergent system is inherited from the underlying notion of time of the fundamental theory.

It is worth mentioning that there are interesting situations in which time disappears: Bose–Einstein condensate analogue models of signature change events have indeed been considered in the literature together with the associated particle production (see *e.g.* [81]). In these works the fundamental Lagrangian is non-relativistic as usual, while the emergent metric for the perturbations (*i.e.* the acoustic metric considered in chapter 2) can have Lorentzian or Riemannian signature depending on the experimental possibility of changing at will the sign of the atomic interaction. In this case time is disappearing, in an certain sense. Of course, as it is customary in these systems, higher energy corrections are ensuring that the time of the laboratory is not completely lost.

The natural question to ask, then, is whether the opposite can happen, *i.e.* if time, *defined* as the Lorentzian signature of the metric, can be an emergent concept in an otherwise time-less system.

Secondly, it is important to understand whether diffeomorphism invariance can be an emergent feature as well, *i.e.* if it is possible to realize systems in which a background scaffolding like for instance Galilean spacetime disappears at the effective level, giving rise to an effective theory which shares this important property with GR.

Expanding the analysis of [295], we will discuss one example in which both of these features are realized in a system defined in a background made by a four dimensional flat Euclidean *space*.

Two points must be stressed. As in the analysis of analogue spacetimes the emergent Lorentz symmetry is an approximate accidental symmetry of the low energy effective theory of the propagating modes, here, the emerging Lorentzian structure will suffer of the same limitation: high energy corrections will show clearly that the underlying theory is not Lorentzian. Analogously, the presence of a background structure, spoiling diffeomorphism invariance, will be encoded into higher order corrections to the equations of motion of the emergent degrees of freedom which will make the underlying scaffolding visible.

There is a crucial difference with other approaches to emergent gravity: there is a background structure here, namely a flat Euclidean space. There is a sort of preferred scaffolding provided by this structure, and the fundamental symmetry group will be the Euclidean group $ISO(4)$, made by four dimensional rotations $SO(4)$ and four dimensional translations. The key point is that these structures will be hidden at the effective level, as it will be proved.

The origin of this Euclidean space is unspecified: it is somehow assumed that the fundamental theory gives birth to it, as well as to objects like fields living on it. The fact that the nature of this space is not specified should not deviate the attention from the central issue, which is the fact that even though at some level there is such a structure, yet it is possible that the effective dynamics is not seeing it at all.

This opens a brand new class of possibilities for emergent gravity scenarios: even models based on flat space alone could give rise to interesting dynamics and should not be discarded *a priori*.

8.1.1 Euclidean/Lorentzian vs Elliptic/Hyperbolic

Before entering into the bulk of the discussion, it is worth to make a step back and examine the interplay between the signature of the metric and the notion of time in linear second order partial differential equations PDE. These equations have typically the form:

$$a^{\mu\nu}\partial_\mu\partial_\nu f + b^\mu\partial_\mu f + cf = 0. \tag{8.1}$$

According to the signature of the matrix a, one can classify the PDE into different categories. For instance, considering a four dimensional manifold to be the domain of definition of the equation, if the signature of a is $(++++)$, then the equation is said to be elliptic. If the signature is $(-+++)$ the equation is said to be hyperbolic.

The qualitative difference between these two cases is the existence of characteristic curves, in the second case, which are propagating the data specified on a given Cauchy surface in a peculiar way, essentially along what one would call the light cones.

This has also the practical consequence of requiring a careful specification of the boundary conditions. To give a familiar example, when considering Green's functions in field theory, while in the case of elliptic PDEs there is only a way to specify boundary conditions at infinity, in the case of hyperbolic equations there are several possibilities. Indeed, one can define advanced, retarded and Feynman's Green's functions.

Therefore, in the case of linear second order partial differential equations the link between the signature of the metric and the character of the equation is immediate. This is the reason why we associate to Lorentzian metrics a very special role, representing particular geometrical structures encoding directly a notion of time.

In the case of more general classes of PDE which are nonlinear, but still linear in the second order derivatives, the same distinction between elliptic and hyperbolic equations does apply. What is relevant is the signature of the tensor a which now depends on the function f and its first derivatives. It is clear that in these cases, the link between the signature of a metric tensor and the character of the PDE itself is weaker: in general, the tensor a will not be given directly by the metric tensor. This will be the technical point we will exploit in the following to make time to appear in a Euclidean system.

8.2 Emergence of time

8.2.1 The model

As explained in the Introduction, it is assumed that the fundamental (unknown) theory gives rise in some unspecified dynamical limit to simple structures such as $\mathbb{R}^4$ equipped with the Euclidean metric $\delta^{\mu\nu}$, and a set of scalar fields $\Psi_i(x_\mu)$, $i = 1, ..., N$ ($x_\mu \in \mathbb{R}^4$) with their Euclidean Lagrangian $\mathcal{L}$. Since this fundamental theory is unknown, one can choose such Lagrangian to be of the simple shape

$$\mathcal{L} = F(X_1, ..., X_N). \tag{8.2}$$

with $X_i = \delta^{\mu\nu}\partial_\mu\Psi_i\partial_\nu\Psi_i$. By construction this Lagrangian is invariant under the Euclidean group $ISO(4)$, and under the transformations $\Psi_i \to \Psi_i + c_i$, with constant c_i.

One might imagine that this class of Lagrangian is rather exotic. In fact, it is not the case. Recently, a lot of attention has been caught by the family of cosmological models containing scalar fields possessing a non-trivial kinetic term. These models go under the name of K-essence [296]. Another interesting example is represented by nonlinear electrostatics. The electric field is the gradient of the electro-static potential $\vec{E} = \vec{\nabla}\phi$. In non-linear electrostatics the Lagrangian of the field is given by some function of the modulus square of the electric field, $F(E^2) = F((\nabla\phi)^2)$.

The equations of motion are easily derived from the variation of the action built with this Lagrangian. For a given field Ψ_i

$$\partial_\mu\left(\frac{\partial F}{\partial X_i}\partial^\mu\Psi_i\right) = 0 = \Sigma_j\left(\frac{\partial^2 F}{\partial X_i\partial X_j}\partial_\mu X_j\right)\partial^\mu\Psi_i + \frac{\partial F}{\partial X_i}\partial_\mu\partial^\mu\Psi_i. \tag{8.3}$$

The starting point is to consider a specific solution of the above equations of motion, ψ_i. This solution will play the role of the background fluid flow in the analogue models discussed in chapter 2. In general the solution of this class of equations is a rather difficult task, because of the nonlinearities. However, it is interesting to notice that affine functions $\psi_i = \alpha^i_\mu x^\mu + b_i$ are solutions of the equations of motion. For the present purposes it is sufficient to consider only these background solutions. As a consequence, all the quantities evaluated on them will be constants, since everything is defined only in terms of the gradients of the fields.

The next step is to consider the perturbations φ_i around these affine solutions. For $\Psi_i = \psi_i + \varphi_i$, the kinetic term X_i becomes then

$$X_i \to \overline{X}_i + \delta X_i, \quad \text{with} \quad \overline{X}_i = \delta^{\mu\nu}\partial_\mu\psi_i\partial_\nu\psi_i \quad \text{and} \quad \delta X_i = 2\partial_\mu\psi_i\partial^\mu\varphi_i + \partial_\mu\varphi_i\partial^\mu\varphi_i. \tag{8.4}$$

This splitting has an important feature. It does not break the symmetry of the original Lagrangian: the transformation $\Psi_i \to \Psi_i + c_i$ is translated into $\varphi_i \to \varphi_i + c_i$.

A straightforward analysis shows that there are choices of F such that the Lagrangian for the perturbations φ_i is invariant under the Poincaré group $ISO(3,1)$, at least in an approximate sense. To see this, and obtain conditions of F and ψ_i for this to happen, one has to compute the Lagrangian for the perturbations φ_i. This is easily done expanding (8.2) using (8.4).

$$\begin{aligned} F(X_1, .., X_N) \to \quad & F(\overline{X}_1, .., \overline{X}_N) + \sum_j \left.\frac{\partial F}{\partial X_j}\right|_{\overline{X}} \delta X_j \\ & + \frac{1}{2}\sum_{jk} \left.\frac{\partial^2 F}{\partial X_j \partial X_k}\right|_{\overline{X}} \delta X_j \delta X_k + \frac{1}{6}\sum_{jkl} \left.\frac{\partial^3 F}{\partial X_j \partial X_k \partial X_l}\right|_{\overline{X}} \delta X_j \delta X_k \delta X_l + .. \end{aligned} \tag{8.5}$$

The first term $F(\overline{X}_1, .., \overline{X}_N)$ is the Lagrangian for the classical solution ψ_i. The second term, the one linear in δX_j, contains a term linear in $\partial_\mu\varphi_i$, which is zero since the configuration ψ_i is a solution of the equations of motion. We can also identify the quadratic contribution for $\partial_\mu\varphi_k\partial_\nu\varphi_k$:

$$\text{for } k \neq l, \quad \partial_\mu\varphi_k\partial_\nu\varphi_l \left(2 \left.\frac{\partial^2 F}{\partial X_k \partial X_l}\right|_{\overline{X}} \partial^\mu\psi_k\partial^\nu\psi_l \right), \tag{8.6}$$

$$\text{for } k = l, \quad \partial_\mu\varphi_k\partial_\nu\varphi_k \left(\left.\frac{\partial F}{\partial X_k}\right|_{\overline{X}} \delta^{\mu\nu} + \frac{1}{2} \left.\frac{\partial^2 F}{(\partial X_k)^2}\right|_{\overline{X}} \partial^\mu\psi_k\partial^\nu\psi_k \right). \tag{8.7}$$

The contribution (8.6) is a source of mixing between different fields. Of course, this is not a conceptual difficulty: with a suitable transformation in field space one can undo this mixing and put the system in a diagonal form (in field space). In this way, all the fields will be decoupled, at least at the lowest order in pertubation theory. To simplify the analysis, without loss of generality, it is possible to demand that these mixing terms vanish. This of course restricts the possible shapes of F. For example, by chosing $\left.\frac{\partial^2 F}{\partial X_k \partial X_l}\right|_{\overline{X}} = 0$, if $k \neq l$. For instance, one could use from the beginning a function like

$$F(X_1, .., X_N) = f_1(X_1) + ... + f_N(X_N). \tag{8.8}$$

In this case the field equations for each field are decoupled, and mixing is prevented at every stage. Indeed one could have worked from the very beginning with a single field, and showing therefore the emergence of time for a single field. However, in order to proceed further into the construction of a model of gravity and matter, several fields are needed.

It is possible to identify in (8.7) the effective or emergent metrics for each field φ_k, (taking into account (8.8))

$$g_k^{\mu\nu} \equiv \left.\frac{df_k}{dX_k}\right|_{\overline{X}_k} \delta^{\mu\nu} + \frac{1}{2} \left.\frac{d^2 f_k}{(dX_k)^2}\right|_{\overline{X}_k} \partial^\mu\psi_k\partial^\nu\psi_k. \tag{8.9}$$

In fact, these are the inverse metrics from which the actual metrics can be derived once invertibily conditions are imposed.

Two comments are in order, here. First of all, all the higher order terms responsible for the interactions among the fields have been discarded. This has been done for two (related) reasons. One is that, in perturbation theory, the linear term in the equations of motion is defining the propagating modes, *i.e.* the notion of particles, and only this part is necessary to give a definition of a metric tensor. The second reason is that the higher order interaction terms are given by derivative couplings, due to the symmetry $\Psi_i \to \Psi_i + c_i$ of the original Lagrangian, which is an exact symmetry also for the fields φ_i. As a consequence, as long as one is dealing with low energy/long range phenomena, these can be safely discarded.

The second comment involves a sort of monometricity constraint that must be imposed at this stage. Since a priori $f_i \neq f_j$ and $\psi_i \neq \psi_j$ if $i \neq j$, one is dealing with a multi-metric structure: each field sees its own metric. However, a mono-metric structure can be enforced by constraining the solution ψ_k and the derivatives of f_k at $\overline{X}_k$ to be independent of k

$$f_k = f, \quad \psi_k = \psi, \quad \forall k. \tag{8.10}$$

At this point it has been shown that the perturbations around a solution of the field equations on a Riemannian manifold can propagate, for suitably chosen Lagrangians, on an effective geometry which is not the fundamental one, $\delta_{\mu\nu}$, but rather a rank 2 tensor constructed from it and partial derivatives of the chosen background solution. Note that, in order for this to be possible, it is crucial to have a Lagrangian with non-canonical kinetic terms as, it can be clearly evinced from the second contribution to the metrics in equation (8.9).

According to the properties of the solutions considered, this effective inverse metric can be of three different types. It can be Riemannian, degenerate or pseudo-Riemannian. It is this latter possibility which is particularly interesting for the emergent gravity scenario. In fact, one can even ask that the metric (8.9) is the Minkowski metric $\eta_{\mu\nu}$. This will put some constraints on the derivative of f, evaluated at $\overline{X} = \partial^\mu\psi\partial_\mu\psi$.

In order to do so, a particular solution of the equations of motion, $\bar{\psi}$, has to be specified. As it has been seen, it can be chose to be an affine function of the coordinates, $\bar{\psi} = \alpha^\mu x_\mu + \beta$. The underlying $SO(4)$ symmetry can be conveniently exploited to simplify the algebra: one can always make a rotation such that

$$\bar{\psi} = \alpha x_0 + \beta. \tag{8.11}$$

The choice of the coordinate x_0 is completely arbitrary, what only matters is that there is one specific coordinate which is pinpointed by the gradient of the background solution. The last step is to ask for the metric to have the signature $(-, +, +, +)$. This puts some constraints on the value of the derivatives of f

$$\begin{aligned} &\left.\frac{df}{dX}\right|_{\overline{X}} + \frac{1}{2}\left.\frac{d^2 f}{(dX)^2}\right|_{\overline{X}} \partial^0\bar{\psi}\partial^0\bar{\psi} < 0, \\ &\left.\frac{df}{dX}\right|_{\overline{X}} + \frac{1}{2}\left.\frac{d^2 f}{(dX)^2}\right|_{\overline{X}} \partial^a\bar{\psi}\partial^a\bar{\psi} > 0, \quad a = 1, 2, 3 \end{aligned} \tag{8.12}$$

which, using (8.11), imply

$$\left.\frac{df}{dX}\right|_{\overline{X}} + \frac{\alpha^2}{2}\left.\frac{d^2 f}{(dX)^2}\right|_{\overline{X}} < 0, \qquad \left.\frac{df}{dX}\right|_{\overline{X}} > 0. \tag{8.13}$$

Note that, due to the choice of a solution of the form (8.11), the conditions (8.12) are not only implying a pseudo-Riemannian signature but also the constancy of the metric components, which hence can be easily rescaled so to take the familiar Minkowskian form diag$(-1, +1, +1, +1)$.

Of course, there are many possible choices of $f(X)$ and α which can fulfill the above requirements. For example, one could pick up the simple case

$$f(X) = -X^2 + X, \qquad \frac{1}{3} < \alpha^2 < \frac{1}{2}. \tag{8.14}$$

However, in what follows the particular form of $f(X)$ is not really important, as well as the value of α. What it is assumed is that equations (8.13) are satisfied.

To summarize, since $g_k^{\mu\nu} \equiv \eta^{\mu\nu}$, $\forall k$, the (free) perturbations φ_i are propagating on an effective Minkowski spacetime, even though the fundamental theory is Euclidean (c.f. (8.2)).

8.2.2 The Lorentzian signature

Having described the fundamental ideas, it is possible to pass to the detailed discussion of the emergence of time in this system.

First of all, the model considered does not posses any fundamental speed scale (indeed, the metric has been kept dimensionless at every stage). This fact should be expected given that the fundamental theory is defined over Euclidean space. At this level, there is no coordinate with time dimension and therefore one cannot define a constant with speed dimension. The invariant speed c appearing in the Minkowski metric of special relativity, which will relate the length x_0 to an actual time parameter t, could be determined by first introducing a definition of clock, which defines a coordinate with time dimension (as it would be natural to do given the hyperbolic form of the equations of motion for the perturbations) and then by defining c as the signal speed associated to light cones in the effective spacetime. Noticeably, a similar situation has been encountered in the axiomatic derivation of special relativity [162] where, as it has beed discussed in chapter 3, given a list of axioms, one derives the existence of a universal speed, observer independent, which is identified as the speed of light only *a posteriori*, when the actual experiment on the propagation of signals is done.

Second, and more important, a comment is needed about our choice of the background solution around which the perturbations and their dynamics are defined. It is obvious that within our model this choice is arbitrary. It simply shows that there are some background solutions $\bar{\psi}$ for which a pseudo-Riemannian metric can emerge. Obviously, different background solutions could lead to alternative metrics, *e.g.* one could also obtain the Euclidean metric $\delta_{\mu\nu}$ (for example if ψ is constant). While it is conceivable that in a more complicate model there could be some mechanism for selecting the specific background solution that leads to an emergent Lorentzian signature, it is not obvious at all that such a feature should be built in the emergent theory. In fact, one generally minimizes an energy functional to select the ground state of the theory. However, when a Lorentzian regime emerges from a Euclidean setup, as in this model, there is no initial notion of time and hence no energy functional to minimize. It is therefore unclear how a ground state could be selected from within the emergent system.

On the other hand, it is also conceivable that the actual background solution in which the initial system of fields (8.2) emerges from the fundamental (pre-geometric) theory, can be depending on the conditions for which the "condensation" of the fundamental objects takes place. In this sense, the right ground state or background solution would be selected from minimizing some functional defined at the level of the atoms of space-time. To use an analogy, the same fundamental constituents, *e.g.* carbon atoms, can form very different materials, diamond or graphite, depending on the external conditions during the process of formation. Similarly, in a Bose–Einstein condensation the characteristics of the background solution (the classical wave function of the condensate), such as density and phase, are determined by physical elements (like the shape of the EM trap or the number and kind of atoms involved) which pre-exist the formation of the condensate. Summarizing, the action principle alone cannot be of help into the selection of the "true vacuum". Some other ingredient, coming from the microstructure, is needed.

The outcome of this first part is pretty clear: the properties of the fundamental Lagrangian and of the background solutions have been identified so that the perturbations φ_i have a kinetic term determined by a single Minkowski metric,

$$\mathcal{L}_{\text{eff}}(\varphi_1, ...\varphi_N) = \sum_i \eta^{\mu\nu} \partial_\mu \varphi_i \partial_\nu \varphi_i. \tag{8.15}$$

In this sense, this is a toy-model for the emergence of the Poincaré symmetry. This construction can be seen as a generalization of the typical situation in analogue models of gravity where one has Poincaré symmetries emerging from fundamental Galilean symmetries. However, let us stress that in this case no preferred system of reference is present in the underling field theory given that the fundamental Lagrangian is endowed with a full Euclidean group $ISO(4)$. Moreover, the emergence of a pseudo-Riemannian metric, in this model, is free of the usual problems encountered in the context of continuous signature change (*e.g.* degenerate metrics) given that the former arises as a feature of the dynamics of perturbations around some solution of the equations of motion. Accordingly, one can see that the invariance under Lorentz transformations is only an approximate property of the field equations (as usual for emergent systems), valid up to some order in perturbation theory. In particular, if one analyzes the third order contribution in (8.5) one gets, in the mono-metric case ($F(X_1, ..., X_N) = f(X_1) + ... + f(X_N)$, and $\psi_k = \psi$, $\forall k$)

$$\partial_\alpha \varphi_k \partial_\beta \varphi_k \partial_\gamma \varphi_k \left(\left.\frac{d^2 f}{(dX_k)^2}\right|_{\overline{X}} \partial^\alpha \psi \delta^{\beta\gamma} + \frac{1}{6} \left.\frac{d^3 f}{(dX_k)^3}\right|_{\overline{X}} (\partial^\alpha \psi \partial^\beta \psi \partial^\gamma \psi) \right). \tag{8.16}$$

This contribution is clearly not Lorentz invariant if the solution ψ pinpoints a specific direction, as for example when the Minkowski metric is emergent. As a matter of fact our theory will show æther like effects beyond second order. The crossover scale, *i.e.* the lenght scale at which this term becomes important depends on the specific form of the function F as well as the value of the parameter α.

8.2.3 Conformal symmetry

It is important to discuss the spacetime symmetries of the Lagrangian for the perturbations:

$$L = \partial_\mu \varphi_i \partial^\mu \varphi_i. \tag{8.17}$$

This Lagrangian leads to the EOM:

$$\Box_\eta \varphi_i = \eta^{\mu\nu} \partial_\nu \partial_\nu \varphi_i = 0. \tag{8.18}$$

It has already been recognized that this equation is invariant under Poincaré group. However, at the lowest order in perturbation theory, the particular choice of the action, and in particular the existence of the symmetry under field translations $\varphi_i \to \varphi_i + c_i$ implies that there are no terms containing explicitly the fields φ_i themselves. For instance, a mass term is explicitly forbidden by this symmetry. Therefore, this symmetry in field space is enhancing spacetime symmetry to the whole conformal group, obtained supplementing the Poincaré group with the conformal transformations:

$$x^\mu \to \lambda x^\mu, \qquad x^\mu \to \frac{x^\mu + b^\mu x^2}{1 + 2b^\alpha x_\alpha + x^2}. \tag{8.19}$$

This fact is of paramount importance for the discussion of the emergent gravitational theory.

8.3 Towards Nordström gravity

So far, results which are familiar in the area of analogue gravity have been just generalized and extended, with the (important) difference that time has emerged from a time-less system. However, as said, a typical drawback of analogue gravity models is related to the fact that they show only the emergence of a background Lorentzian geometry while they are unable to reproduce a geometrodynamics of any sort. In what follows, it will be showed that this model overcomes this drawback and indeed is able to describe the emergence of a theory for scalar gravity. This theory will come out to be the only known other theory of gravitation, apart from GR, which satisfies the strong equivalence principle [297], *i.e.* Nordström gravity.

Nordström theory of gravity [298] was one the first attempts to realize a relativistic theory of gravity. The rough idea is to replace the Poisson equation for the (scalar) gravitational field with a Klein-Gordon equation:

$$\nabla^2 \phi = 4\pi G_N \rho \to \Box_\eta \phi = 4\pi G_N T, \tag{8.20}$$

where T is the trace of the stress energy tensor of the matter fields. In doing so, one has in mind that the theory is defined over a background non-dynamical Minkowski spacetime.

The theory was given a diffeo-invariant formulation by Einstein and Fokker [299]. The single equation of motion for the gravitational field was replaced by geometric equations

$$R(g) = 24\pi G_N T_g \qquad C_{\mu\nu\rho\sigma}(g) = 0, \tag{8.21}$$

in which the gravitational field is encoded into a conformally flat metric tensor (here $C_{\mu\nu\rho\sigma}$ is the Weyl tensor). This theory, despite being diffeomorphism invariant, is background dependent, since a single Minkowski spacetime is pinpointed in any solution [300]. Notice that, given its diffeomorphism invariance, this theory does suffer from the hole-problem. Such a theory has been immediately ruled out: it cannot be related to the description of the physical gravitational field we experience, because it does predict that no gravitational bending of any conformally coupled matter field should be observed. The observation of the bending of light cannot be explained in Nordström theory, given that the Maxwell action is conformally invariant. Despite this, it is still interesting to study such a theory since there are some structural analogies with general relativity, like diffeomorphism invariance. For additional comments and alternative perspectives on this particular scalar theory of gravity, the reader can see [301, 302, 303].

After having discussed the general features of Nordström gravity, it is possible to describe how one can recover this relativistic scalar gravity theory with a careful manipulation of a Lagrangian of the type (8.2), when the ground state is such that the perturbations are living (at the lowest order in perturbation theory) in a Minkowski spacetime. So, let us start from the truncated Lagrangian for the perturbations (8.15) that has been obtained in the previous section. This Lagrangian can simply be rewritten in terms of the (real) multiplet $\varphi = (\varphi_1, ..., \varphi_N)$ as

$$\mathcal{L}_{\text{eff}}(\varphi) = \eta^{\mu\nu}(\partial_\mu \varphi)^T(\partial_\nu \varphi). \tag{8.22}$$

Besides the conformal symmetry discussed in section 8.2.3, this system has a global $O(N)$ symmetry which has emerged as well from the initial Lagrangian (8.2). This global symmetry is not an accidental one: it is the byproduct of the mono-metricity constraint. It is hence quite natural to rewrite the multiplet φ by introducing an amplitude characterized by a scalar field $\Phi(x)$ and a multiplet $\phi(x)$ with N components such that

$$\begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_N \end{pmatrix} = \Phi \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_N \end{pmatrix}, \qquad \text{with } |\phi|^2 \equiv \sum_i \phi_i^2 = \ell^2. \tag{8.23}$$

Here ℓ is an arbitrary length parameter introduced to keep the dimension right. In particular, Φ is dimensionless and ϕ has the dimension of a length. Φ is the field invariant under $O(N)$ transformations, whereas ϕ does transform under $O(N)$. As it will become clear later, this field redefinition will provide the means to identify gravity and matter degrees of freedom. This is just a generalization of the Madelung representation encountered in the context of BECs. Of course, this parametrization breaks the manifest conformal invariance of the equations for the perturbations, as the presence of the dimensionful quantity ℓ clearly shows.

The Lagrangian for the perturbations (8.22) reads now as

$$\mathcal{L}_{\text{eff}}(\varphi_1, ...\varphi_N) \to \mathcal{L}_{\text{eff}}(\Phi, \phi_1, ...\phi_N) = \ell^2 \eta^{\mu\nu}\partial_\mu \Phi \partial_\nu \Phi + \sum_i \Phi^2 \eta^{\mu\nu}\partial_\mu \phi_i \partial_\nu \phi_i + \lambda(|\phi|^2 - \ell^2), \tag{8.24}$$

where λ is a Lagrange multiplier introduced to enforce the constraint on the norm. The part of the Lagrangian containing the fields ϕ_i is nothing else than the Lagragian for a non-linear sigma model given in terms of the fields ϕ_i. The associated equations of motion are

$$\eta^{\mu\nu}(\ell^2 \partial_\mu \partial_\nu \Phi - \Phi \sum_i \partial_\mu \phi_i \partial_\nu \phi_i) = 0, \tag{8.25}$$

$$\eta^{\mu\nu}(2\partial_\mu \Phi \partial_\nu \phi_i + \Phi^2 \partial_\mu \partial_\nu \phi_i + \frac{1}{\ell^2}\partial_\mu \phi_j \partial_\nu \phi_k \delta^{jk} \phi_i) = 0, \tag{8.26}$$

$$|\phi|^2 - \ell^2 = 0. \tag{8.27}$$

In order to obtain these equations one has to eliminate the Lagrange multiplier λ. This is the reason why λ has disappeared.

It is a rather striking fact that it is possible rewrite these equations as the equations of motion of Nordström gravity coupled to some given matter fields. Introducing the (conformally flat) metric

$$g_{\mu\nu}(x) = \Phi^2(x)\eta_{\mu\nu}, \tag{8.28}$$

the equations of motion (8.26) can be simply rewritten as

$$(\sqrt{-g})^{-1}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi_i) + \frac{1}{\ell^2}g^{\mu\nu}\partial_\mu\phi_j\partial_\nu\phi_k\delta^{jk}\phi_i = \Box_g\phi_i + \frac{1}{\ell^2}g^{\mu\nu}\partial_\mu\phi_j\partial_\nu\phi_k\delta^{jk}\phi_i = 0, \tag{8.29}$$

where $\Box_g$ is the Dalambertian associated to the metric g and used that $\sqrt{-g} = \Phi^4$ and $g^{\mu\nu} = \Phi^{-2}\eta^{\mu\nu}$. (Incidentally, this trick works only in four dimensions.) To be consistent, to make the change of variables $\Phi \to g_{\mu\nu}$ well defined, one should be complete the system of equations written in terms of $g_{\mu\nu}$ with the constraint ensuring that $g_{\mu\nu}$ is conformally flat, that is

$$C_{\alpha\beta\gamma\delta}(g) = 0, \tag{8.30}$$

where $C_{\alpha\beta\gamma\delta}$ is the Weyl tensor.

Eq. (8.29) shows that (8.28) does reveal an important feature: it is showing that the emergent theory does admit a gravitational interpretation. Moreover *the gravitational degree of freedom should be encoded in the scalar field* Φ, whereas *matter should be encoded in the* ϕ_i. We are therefore aiming at a scalar theory of gravity with actions:

$$S_{\text{eff}} = \int dx^4\sqrt{-\eta}\,\mathcal{L}_{\text{eff}} = S_{\text{grav}} + S_{\text{matter}}, \tag{8.31}$$

$$S_{\text{grav}} = \ell^2\int dx^4\sqrt{-\eta}\,\eta^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi, \tag{8.32}$$

$$S_{\text{matter}} = \int dx^4\sqrt{-\eta}\left(\sum_i\Phi^2\eta^{\mu\nu}\partial_\mu\phi_i\partial_\nu\phi_i + \lambda(|\phi|^2 - \ell^2)\right), \tag{8.33}$$

where the volume element $\sqrt{-\eta} = 1$ has been explicitly introduced to make clear that these actions are given in flat spacetime.

It is easy to see that the very same actions can be recast in the form of actions in a curved spacetime endowed with the metric (8.28). In particular for the matter action in (8.33) one has

$$\begin{aligned} S_{\text{matter}} &= \int dx^4\left(\sum_i\Phi^2\eta^{\mu\nu}\partial_\mu\phi_i\partial_\nu\phi_i + \lambda(|\phi|^2 - \ell^2)\right) = \\ &= \int\sqrt{-g}dx^4\left(\sum_i g^{\mu\nu}\partial_\mu\phi_i\partial_\nu\phi_i + \lambda'(|\phi|^2 - \ell^2)\right), \end{aligned} \tag{8.34}$$

where the Lagrange multiplier has been suitably rescaled to λ'. This allows to construct the stress-energy tensor $T_{\mu\nu}$ for the non-linear sigma model, and its trace $\mathbf{T}$ with respect to the metric g:

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}} = \sum_i\left(\partial_\mu\phi_i\partial_\nu\phi_i - \frac{1}{2}g_{\mu\nu}(g^{\alpha\beta}\partial_\alpha\phi_i\partial_\beta\phi_i)\right),$$

$$\mathbf{T} = g^{\mu\nu}T_{\mu\nu} = -\Phi^{-2}\sum_i\eta^{\mu\nu}\partial_\mu\phi_i\partial_\nu\phi_i.$$

These expressions are valid only on shell, since the term in λ' vanishes there. Off shell this would not be true, of course.

Finally, the above result, together with the recognition that the Ricci scalar $\mathbf{R}$, associated to the metric $g_{\mu\nu}$, can be written as $\mathbf{R} = -6\Box_\eta\Phi/\Phi^3$, allows us to rewrite Eq. (8.25) as the Einstein–Fokker equation

$$\Box_\eta\Phi = \frac{1}{\ell^2}\eta^{\mu\nu}\Phi\sum_i \partial_\mu\phi_i\partial_\nu\phi_i \quad \Leftrightarrow \quad \mathbf{R} \ = \frac{6}{\ell^2}\mathbf{T}. \tag{8.35}$$

Consequently, one can gather together the equations of motion (8.29), (8.30), (8.35), obtained by introducing the metric (8.28), to obtain the system of equations

$$\mathbf{R} \ = \frac{6}{\ell^2}\mathbf{T}, \qquad C_{\alpha\beta\gamma\delta} = 0. \tag{8.36}$$

$$\Box_g\phi_i + \frac{1}{\ell^2}g^{\mu\nu}\partial_\mu\phi_j\partial_\nu\phi_k\delta^{jk}\phi_i = 0, \qquad |\phi|^2 - \ell^2 = 0. \tag{8.37}$$

These equations of motion are those for Nordström gravity

$$\mathbf{R} = 24\pi G_\mathrm{N}\,\mathbf{T}, \qquad C_{\alpha\beta\gamma\delta} = 0, \tag{8.38}$$

coupled to a non-linear sigma model. Indeed, the rewriting of (8.25)-(8.27) into the form (8.36)-(8.37), is a special case of the procedure suggested by Einstein and Fokker so to cast Nordström gravity in a geometrical form [299].

We see from the above equation that the Newton constant G_N in our model has to be proportional to ℓ^{-2}. However, in identifying the exact relation between the two quantities, some care has to be given to the fact that the stress-energy tensors appearing respectively in equation (8.36) and equation (8.38) do not share the same dimensions. This is due to the fact that the fields ϕ_i have the dimension of a length rather than the usual one of an energy. This implies that in order to really compare the expressions one has to suitably rescale our fields with a dimensional factor, Ξ, which in the end would combine with ℓ so to produce an energy, $\dim[\ell\,\Xi] =$ energy. In particular, is easy to check that one has to assume $4\pi\ell^2\Xi^2 \equiv E^2_{\mathrm{Planck}}$ in order to recover the standard value of G_N (assuming c as the observed speed of signals and $\hbar$ as the quantum of action). As a final remark, it must be stressed that the scale ℓ is completely arbitrary within the emergent system and in principle should be derived from the physics of the "atoms of spacetime" whose large N limit gives rise to (8.2). This is obvious once it is recalled that this system considered is invariant under conformal transformations, at least at the lowest order.

Accidentally, the above discussion also shows that, once the fields are suitably rescaled so to have the right dimensions, the constraint appearing in Eq.(8.37) is fixing the norm of the multiplet to be equal to the square of the Planck energy. This implies that the interaction terms in the aforementioned equation are indeed Planck-suppressed and hence negligible at low energy. This should not be a surprise, given that in the end $\ell\Xi$ is the only energy scale present in our model. It is conceivable that more complicate frameworks, possibly endowed with many dimensional constants, will introduce a hierarchy of energy scales and hence break the degeneracy between the scale of gravity and the scale of matter interactions.

8.4 The emergent model

Having discussed all the technical details, it is possible now to glue together the different parts into a single picture.

Initially, we have considered some scalar fields living in a Euclidean space, and showed that there exists a class of Lagrangians such that the perturbations around some classical solutions $\bar{\psi}$ propagate in a Minkowski spacetime. In this case $\bar{\psi}$ is essentially picking up a preferred direction, so that we have a spontaneous symmetry breaking of the Euclidean symmetry. The apparent change of signature is free of the problems usually met in signature change frameworks since the way in which this happens is structurally different. Indeed, while in common scenarios signature changes happen in the same layer, namely on the same manifold in different places, here the change of signature is instead of a different nature, being a mismatch of signature between to distinct layers: the fundamental one of the underlying Euclidean space, and the effective one of the excitations.

The point is that *Lorentzian signature can emerge from a fundamental Euclidean theory* and this process can in principle be reconstructed by observers living in the emergent system. In fact, while from the perturbations point of view, *a priori*, it is difficult to see the fundamental Euclidean nature of the world, this could be guessed from the fact that some Lorentz symmetry breaking would appear at high energy (in our case in the form of a non-dynamical æther field, related to the background solution $\bar{\psi}$).

Taking the equations of motion for these excitations above this particular class of Euclidean "condensate", using a natural field redefinition (which is adapted to the symmetry required to have a monometric theory), we have shown that it is possible to identify from the perturbations φ_i, a scalar field Φ encoding gravitational degrees of freedom and a set of scalar fields ϕ_i (a non-linear sigma model) encoding matter. In this sense, *gravity and matter are both emergent at the same level.* This feature is rather unusual within the standard framework of analogue models of gravity where one usually identifies the analogue of the gravitational degrees of freedom with the "background" fields, *i.e.* the condensate or the solution ψ of the equations of motion. This is what happens in the case of BEC, for instance.

If we would follow this line of thought in looking for a theory of gravitational dynamics, we would be led to the conclusion that, in order to have an emergent theory which is diffeomorphism invariant, the fundamental field theory (8.2) must be endowed with diffeomorphisms invariance from the very beginning. This would imply that one would have to obtain gravity from a theory which is already diffeomorphisms invariant and hence most probably with a form very close to some known theory of gravitation. For these reasons, we do expect that if an emergent picture is indeed appropriate for gravitation, then it should be of the sort presented here, with *both matter and gravity emerging at the same level.* Of course, an alternative scenario is that a full fledged theory of gravity emerges together with the notion of manifold in a single step from the eventual semiclassical/large number limit of the fundamental objects. This happens for instance in matrix models and quantum graphity models, as we have already mentioned.

It is interesting to note that this simultaneous emergence of matter and gravity is associated not only to an emergent local Lorentz invariance for the perturbations dynamics but it is associated also to the possibility of writing the equations in a diffeomorphism invariant way. One could say that diffeomorphism invariance has emerged as well. In fact, we saw how the equations of motion (8.25) and (8.26) could be rewritten in a completely equivalent way using a conformally flat metric (8.28). Most noticeably, they can be rewritten in an evidently diffeomorphisms invariant form, from the point of view of "matter fields observers". Following the standard hole argument, this also implies that the coordinates x_μ, used to parameterized our theory, do not have any physical meaning from the point of view of the ϕ_i "matter observers". They are merely parameters. Here is a crucial difference with respect to the standard observation that any equation can be put in a diffeomorphism invariant form.

This discussion should make clear that this kind of diffeomorphism invariance is not a kind of fake invariance (see the box about the Stuckelberg fields), but it does represent a true property of the system. Stated in another way, we can say that the systems of equations derived from (8.2) are just a gauge-fixed form of the equations of Nordström gravity coupled to a nonlinear sigma model. This gauge fixing, however, is not enough to conclude that the theory has the same empirical content of a theory of some scalar massless fields in Minkowski spacetime, as the discussion in chapter 6 about the difference between coordinates as parameters and coordinates as readings should have clarified.

Stueckelberg mechanism There is an interesting counterpart in field theory of this rewriting of field equations in an apparent diffeo-invariant way. If one considers the case of a massive vector field,

$$L = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - m^2 A^\mu A_\mu,$$

one easily sees that this Lagrangian is not invariant under the field transformation $A_\mu \to A_\mu + \partial_\mu f$. However, if one introduces two auxiliary fields, a vector and a scalar B_μ, θ, and replaces the starting Lagrangian with:

$$-\frac{1}{4}B^{\mu\nu}B_{\mu\nu} - m^2(B^\mu - \partial^\mu\theta)(B_\mu - \partial_\mu\theta),$$

with $B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$, one easily realizes that there is an underlying $U(1)$ gauge invariance $B_\mu \to B_\mu + \partial_\mu f, \theta \to \theta - f$. However, upon examination of the equations of motion, one easily realizes that the physical content of the two theories is the same. They are the same theory written in different variables, and hence, they do have the same symmetries. The apparent $U(1)$ gauge symmetry is fictitious.

In agreement with the fact that diffeomorphisms invariance is emergent in our system, it can be noted that the cubic contribution (8.16) ends up breaking it at the same level it breaks Lorentz invariance. Moreover, the derivation obviously holds for small perturbations φ_i, and hence small Φ, implying that in this framework one would predict strong deviations from the weak field limit of the theory whenever the gravitational field becomes very large.

Coming back to the emergence of diffeomorphism invariance, we note that Nordström gravity is also a nice framework for discussing the subtle distinction between background independence and diffeomorphisms invariance [300]. We call background some geometrical degrees of freedom that are not dynamical. For example, in general relativity the topology of the manifold and its dimension, or the signature of the metric, can be considered as background quantities. We can therefore have some specific background structures while still having diffeomorphisms invariance. Nordström gravity is encoded in conformally flat metrics. If one considers fields which are conformally coupled to the metric (such as the electromagnetic field), these fields only see the metric $\eta_{\mu\nu}$ which is of course not dynamical. The Minkowski metric can be see then as a background structure: this is what one may call a "prior geometry" (*e.g.* see [2]). One may hence say that diffeomorphism invariance is somewhat of a weaker form in Nordström gravity with respect the one present in general relativity.

In particular, while the essence of diffeomorphism invariance in GR is encoded in the associated Hamiltonian constraints, these are not defined in the present formulation of Nordström gravity. Furthermore, in the most general implementations of Norström theory, quantities can be built which manifestly include the background structure $\eta_{\mu\nu}$ and hence are not diffeomorphism invariant. However, within our model, the prior geometry cannot be detected. Indeed, in order to detect the Minkowski background, one should be able to propose a method to pinpoint the conformal factor Φ^2 in the relation $g_{\mu\nu} = \Phi^2\eta_{\mu\nu}$. However, a careful analysis shows that this is actually impossible. Let us elaborate on this point. If we perform a conformal transformation, $x^\mu \to \bar{x}^\mu(x)$, the equations of motions associated to (8.22) are transforming like

$$\Box_\eta\varphi_i = 0 \to \Box_{\bar{\eta}}\varphi_i = 0, \tag{8.39}$$

where η and $\bar{\eta}$ are two different Minkowski metrics related by some conformal factor $\lambda(x)$. Therefore, η and $\bar{\eta}$ are indistinguishable, due to conformal invariance the equations of motion for φ_i. Hence, what appears to be a background structure, namely $\eta_{\mu\nu}$, is ambiguously defined, and the coordinates x^μ in which the equations of motion for the fields φ_i are written have no operational meaning, they are mere labels. Furthermore, this ambiguity in the definition of what would be called a background structure implies an ambiguity on the definition of the conformal factor relating the physical metric to the would-be background structure. In this sense, within this very specific implementation of the model which has conformal invariance, there is no Minkowski geometry as a background. There is a background structure, which is the conformal structure of Minkowski spacetime. This is a mild limitation of our simple toy model as a diffeomorphism invariant, background independent system.

Of course, the above discussion holds only at the lowest order in the fields φ_i. As previously discussed, higher orders in perturbation theory will produce terms like (8.16) producing a breaking of the conformal symmetry and hence the appearance of the background structures, *i.e.* the Euclidean space and the $\partial_\mu\bar{\psi}$ which have selected the timelike direction.

Finally, Nordström gravity is only a scalar gravity theory, which has been falsified by experiments (*e.g.* the theory does not predict the bending of light). In order to obtain a more physical theory, in particular general relativity, one should surely look for more complicated emergent Lagrangians than (8.2). Of course, one would in this case aim to obtain the emergence of a theory characterized by spin-2 gravitons (while in Nordström theory the graviton is just a scalar). This would open a door to a possible conflict with the Weinberg–Witten theorem [264] discussed in chapter 6. However, there are many ways in which such a theorem can be evaded and in particular one may guess that analogue models inspired mechanisms like the one discussed here will generically lead to Lagrangian which show Lorentz and diffeomorphism invariance only as approximate symmetries for the lowest order in the perturbative expansion.

It is unclear which sort of generalization may still lead to some viable gravitational theory from the perturbations dynamics. For example, the simple addition of a potential will in general prevent the selection of a preferred direction, except in regions where the potential is almost flat. Moreover, it would also spoil the metric interpretation of the theory. For example, the terms $|\varphi|^n$ for $n \geq 1$ and $\neq 4$ cannot be rewritten as an interaction between the matter field fields ϕ living on the conformal metric $\Phi^2\eta_{\mu\nu}$, when using the change of variables (8.23) (although it is interesting to note that a $|\varphi|^4$ term would give Nordström gravity with a cosmological constant). However, this "rigidity" of the model is most probably due to its simplicity: considering a more complex emergent field theory with fields such as spinors or tensors could possibly allow to have a preferred direction pinpointed while giving rise to more physical Lagrangians for the perturbations.

Another natural question concerns the extendibility of this class of models to fields with different spins. For instance, to extend this toy model in such a way to make it more physically motivated, one should be able to include fermions and gauge bosons. In this sense it would be interesting to see how gauge invariance can emerge in a scenario like the one described above. These represents interesting directions for future developments.

8.5 The role of symmetries

In concluding this chapter, it is interesting to discuss in details the features that allowed the construction of such a toy model. In particular, it is important to stress the role of symmetries, in order to make clear the way in which they enter at the various levels. As in the case of selecting Riemannian geometry out of Finsler geometry, here there are some symmetries which are absolutely essential: it is only due to their presence that we do have an emergent gravitational system possessing a geometrical nature.

We have seen that in order to produce a working model, a number of properties must be assumed. First of all, there is an underlying $ISO(4)$ symmetry which allows us to use particularly simple affine solutions. This $ISO(4)$, when spontaneously broken, can lead to an approximate Poincaré invariance. Moreover, the masslessness of the resulting modes is promoting this Poincaré invariance to a full conformal invariance, which is approximate as well. This conformal invariance is the key symmetry which hides the background structure, forbidding a low energy observer to detect a background metric structure (there is only a background conformal structure).

Conformal invariance seems to be deeply intertwined with the possibility of writing down the resulting equations of motion in the form of a system of diffeomorphism invariant equations, as we have seen. However, in order for the Lagrangian (8.22) to be conformal invariant, there must be an overall $O(N)$ symmetry between the fields. This symmetry is just the other side of the coin of the mechanism leading to the monometricity. If two fields move in different metrics, clearly this $O(N)$ is broken and the entire model fails to provide a geometric picture, let alone a diffeo-invariant one.

In general, one should expect that in any situation in which the metric is an emergent structure, there should be a mechanism taking care of the fact that different matter fields should propagate over the same geometry. In this picture, where a manifold is given from the beginning, the role of internal and spacetime symmetries is crucial. The behavior we have described is not general at all. Of course, one could conclude that this kind of models is somehow contrived and unnatural. However, there is also a positive side: given that symmetries (both of the equations of motion and of the ground state) play a crucial role in the emergence mechanism, the fact that our universe seems to be ruled, at large scales, by general relativity, suggests that not all the pre-geometric scenarios are viable, and that there are rather strong constraints on what are the possible mechanism of emergence.

Chapter 9

Conclusions

So far, we have discussed a number of different topics which might seem quite disconnected, at a superficial glance. However, there is a unique underlying theme, which might be summarized into the question:

What if gravity and spacetime are emergent concepts?

This question is not new at all, and in the introduction we have already discussed some issues related its importance. In concluding the thesis, it is perhaps useful to make a schematic summary of the key ideas that have been touched so far.

General relativity, as any other theory, is based on certain fundamental objects and on their symmetries. It is certainly a rather compelling and elegant theory, and it does have a rather large observational and experimental basis. Nevertheless, we know that it is not the whole story, since it is not complete (singularities, quantum theory...).

Assuming that general relativity is a kind of thermodynamical theory is a bold step. For sure, there are motivations for this. There are some evidences from condensed matter systems, which are displaying phenomena like the emergence of acoustic metrics together with some primitive form of gravitational dynamics. There are other evidences from different approaches to quantum gravity (*e.g.* matrix models). There is the general argument we already mentioned in the introduction regarding the fact that whatever are the fundamental building blocks of (quantum) spacetime, the recovery of a classical spacetime is essentially a condensed-matter-like problem of assembling several microscopic degrees of freedom into macroscopic structures and that, as it happens in the case of condensed matter systems, this procedure might lead to qualitatively new phenomena at macroscopic scales.

To be concrete, the analogy with Bose–Einstein condensates is illuminating. The fundamental theory is given in terms of N bosons, living in a given Galilean spacetime (or a portion of it). In general, there is no geometrical interpretation of this system, apart from the geometrical aspects inherited by the spacetime in which the bosons are living. Nevertheless, in certain dynamical regimes a phase transition takes place, the system condenses, and a mean field description replaces the many body one. It is precisely this mechanism that generates an effective geometry which is rather different than the original one: a Lorentzian spacetime appears, replacing the original

Galilean spacetime, even though only within a certain approximation. This phenomenon is possible only because of the cooperation of a large number of bosons in a specific macroscopic configuration.

However, after the observations made in the course of the previous chapters, it should be clear that there are conditions which should be satisfied in order for a model to produce an emergent system which has the same properties we observe for spacetime and matter fields living on it.

The discussion about the kinematical aspects have shown an important fact about Lorentz invariance. If spacetime is not fundamental, neither its symmetries can be, at least in their realization. In the analogue models we have described in this thesis, in general, the particles do have modified dispersion relations, which are Lorentz-violating. It has been shown that this is related to the fact that the geometrical structure felt by these particles is not a Lorentzian geometry, but one of its generalizations, namely Finsler geometry.

The fact that in a system which is not Lorentz-invariant the emergent geometrical structure is non-Riemannian, and Finslerian in particular, should not be a surprise. It has been shown that Finsler geometry is essentially the most general metric theory based on line elements. Therefore, in absence of additional requirements, there is no reason why only Riemannian geometries should be selected. We have made this statement rather quantitative in showing that it is an argument about symmetries which is selecting Riemannian structures within Finslerian ones: in fact, we have seen that Riemannian spaces are the most symmetric among all the Finsler spaces.

Of course, the selection of the Riemannian solutions will be a consequence of the dynamical equations of the theory. A comparison might help in making this statement more clear. In recent years, it has been proposed to generalize the theory of gravity in terms of metric affine theories (see, for instance [175]), in which the metric $g_{\mu\nu}$ is supplemented by an affine connection $\Gamma^{\mu}_{\nu\rho}$. The equations of motion will specify:

- the dynamics of matter fields;
- the dynamics of the metric tensor;
- the relationships between the connection and the metric tensor.

This last conditions will be essentially a compatibility condition. In the case of the Palatini approach to general relativity, it is just the condition that the connection is compatible with the metric (and that, if it is symmetric, it can be only the Levi-Civita one).

If there is a generalization of general relativity in terms of Finsler geometry[1], among all the equations, there shall be one involving the Cartan tensor $C_{\mu\nu\rho}(v)$. In particular, there should be a generalization such that, in a certain regime, for certain kinds of couplings between matter and geometry, one of the equations becomes

$$C_{\mu\nu\rho}(v) = 0, \tag{9.1}$$

thus selecting Riemannian geometry.

Alternatively, if one imagines that the fundamental equations are given in terms of non-geometrical variables, like in the case of condensed matter systems, there should be particular conditions en-

[1]For a discussion of this point, see for instance [207]. For an alternative approach, see [304].

suring that there is a unique emergent metric, and that it is Riemannian[2]. This in general is a restriction upon the set of solutions, and it is still not clear how to impose it in a generic way. Nevertheless, in order for an emergent gravity model to be successful, there must be a specific mechanism controlling this particular feature, *i.e.* the emergence of monometricity.

In all the examples studied, from analogue models to high energy extensions of the standard model, it has been shown that without Lorentz invariance and without any kind of custodial symmetry, it is unlikely to protect low energy physics from large Lorentz violating effects. This point represents an obstruction in formulating a phenomenologically viable theory encoding Lorentz violation. On the other side, it narrows the set of possible models, by the requirement of some sort of higher degree of symmetry instead of Lorentz invariance, and thus it is telling us something deep about Nature, at least potentially.

The various aspects of the relationship between Riemannian and Finslerian geometry should have clarified how crucial is the role of symmetries in selecting the geometrical structure appearing in a model of emergent spacetime. The second part of the thesis reinforces this point of view: we have shown how symmetries are crucial also in determining the dynamical aspects of emergent gravity models.

From the work presented here we can draw some other conclusions about the dynamics of gravity. It has been argued that diffeomorphism invariance is a property that can appear in a system which does possess a rigid scaffolding, at least in an appropriate sense. Of course, this is unlikely, but we have discussed in details a toy model in which this happens. This phenomenon is essentially due to the presence of symmetries which are hiding the scaffolding and leaving us with coordinates which do not have the interpretation of readings of clocks and rods.

It is interesting also to reconsider the possibility of having from the onset a notion of time. It often happens, in emergent models, that time is present from the very beginning as a form of absolute structure, an assumption in deep contrast with the interpretation of time given in general relativity. With an argument which is similar to the interpretation of coordinates as readings versus parameters, it has been argued that despite the presence of this absolute structure, at the emergent level it is only relevant what the “phononic” observers will perceive as time. For them, the only meaningful notion of time should be the one given in relational terms.

Of course, the investigations about these more formal aspects of emergent gravity are very fascinating, but they are somehow a bit far away from more phenomenological aspects which might be more interesting in light of a falsification procedure.

In this sense, the discussion about the physics of Bose–Einstein condensate suggests many ideas to be used for phenomenology. First of all, the most apparent feature of the gravitational dynamics of BEC is the modification of the Poisson equation. There are essentially three levels at which the modifications enter.

To begin with, there is a mass term for the graviton. In covariant theories[3], the mass of the graviton is set to zero by a gauge symmetry (consequence of diffeo-invariance) which is explicitly forbidding any mass term in the Pauli–Fierz Lagrangian. Nowadays we do have some pretty strong

[2]For a related discussion of this point, within the context of the so called premetric electrodynamics [305], see [306].

[3]And in four dimensions. In three dimensions the story changes due to topological reasons.

limits on the mass of the graviton [307] as well as some good theoretical reasons [308, 309, 312, 310, 311, 313, 314] which are disfavouring (covariant) theories with massive gravitons. Therefore, a theory in which the range of the interaction is basicly the Planck scale like the one we have in BEC is certainly unphysical. Nevertheless, if it turned out that the graviton were actually massive (with a very small mass, of course) the analogy with a BEC could give a suggestion for the nature of this mass. Apart from the (spontaneous) breaking of the gauge invariance (diffeomorphisms) forbidding the mass term for the graviton, one could associate the mass scale to the fundamental dynamical scale at which the fundamental theory acts, exactly as the healing length is the typical dynamical scale of the condensate, or as the mass of the Z^0 is associated to the dynamical scale of electroweak theory, etc..

It should also be mentioned the fact that a modification of gravity, in the form of a suitably modified Poisson equation, could be a welcome feature to address the problems that we presently face at the galactic scales. Despite the rather compelling aspects of cold dark matter scenarios, it is clear that in order to explain the observations of the various properties of galaxies some new ingredient should be included. Milgrom's MOND is certainly one interesting attempt, but as we said in the introduction it cannot be seen as a final answer. There are some attempts to formulate covariant theories, based on the idea that a modification of gravity at the galactic scale is introduced within a cold dark matter scenario (see, for instance, [315]), but the problem is still open. Perhaps some insights coming from modified gravitational theories in analogue systems could give us some further hints toward a better understanding of this puzzle.

The second level of novelty which is appearing in BEC systems is represented by the nonlocality effects. All the nonlocal effects in the theory of phonons are due to a unique reason: phonons are collective degrees of freedom. Therefore, all the local effects of the fundamental theory of particles are turned into non-local phenomena. The breaking of locality is certainly a most intriguing fact. Locality is deeply rooted in our present understanding of physical phenomena, and, consequently, is assumed in all our theories. Nevertheless, as any other feature of physical theories, what we can say is that so far locality has passed all the experimental tests. It might happen that a deeper understanding of space and time at the quantum level might involve a radical departure from this notion (and there are already indications that this might be the case [290, 83, 84, 85]).

The departure from locality does have some physical implications. For instance, in the case of phonons in BECs, we have seen that interaction vertex are replaced by "blobs": the interaction is smeared over the typical scale of nonlocality (the healing length, for BECs). This is certainly a feature which can be detected in particle physics phenomena, perhaps at very high energy (comparable to the Planck scale), and might be related to the microstructure of spacetime. Of course, these are speculations, but nevertheless one could figure out some experiments which could detect peculiar signals due to the breakdown of locality. For instance, by a careful analysis of the properties of the $S-$matrix, one realizes that the axioms of local quantum field theory imply some specific bounds on the analytical properties of the scattering amplitudes [316]. Therefore, the search for a violation of these bounds would represent a concrete way to study some possible signatures of microscopic features of spacetime.

The third point at which the Poisson equation gets modified is the presence of a vacuum contribution to the gravitational field, *i.e.* a cosmological constant. Vacuum is expected to gravitate.

Nevertheless, we do have a theoretical problem in explaining the way in which this happens, and in particular in predicting the actual value of the cosmological constant [12, 13]. Of course, one could assume a pragmatic attitude saying that, after all, it is "just" another coupling constant appearing into the action describing Nature, so its value might not be more interesting than the value of the fine structure constant, or the Newton's constant. This is certainly an acceptable point of view, but it somehow clashes with another point of view: unification.

We do have some reasons to conjecture that electroweak and strong interactions might be the residual of a larger family of gauge interactions in which they are unified. It would be fascinating if this unification picture would include all the interactions, by involving gravitation as well. With all the possible caveats, string theory points in that direction. In a unification scheme, all the coupling constant should be calculable from the properties of the fundamental theory and the symmetry breaking patterns.

Lacking such a picture, one could rely only on naturalness arguments. These point toward two possibilities: either vanishing cosmological constant (in a SUSY framework) or a Planckian one. Both these options cannot explain the observed value.

An alternative, statistical origin for the cosmological constant has been advocated within the causal set approach [317]: in that framework, the cosmological constant is seen as a kind of statistical residual fluctuation around the zero value, its smalless being explained by the large number of Planck-sized volumes existing in a macroscopic universe.

The discussion about analogue models has shed some light on this issue from a different point of view. The key idea is that the cosmological constant is a byproduct of the large number dynamics of the model, and that it is the smaller the larger is the system. Essentially, the large number between the energy scale of the cosmological constant and the Planck energy can be seen, in this perspective, like a sort of Avogadro number, counting how many "atoms" are building up the observed universe.

In the case of the BEC, there is a further intriguing aspect. In fact, the smallness of the cosmological constant term is linked to a diluteness condition, rather than to the number of atoms. In fact, the term from which the cosmological constant term is originated is proportional to the so-called depletion factor (see (2.47)), which measures the ratio between the fraction of atoms which are not condensed and the condensed phase. It depends on two different physical conditions: diluteness of the Bose gas and the value of the scattering length. It can be seen also as the r.m.s. of the fluctuations around the condensate. This latter aspect has a nice interpretation: the smaller is the cosmological constant, the better is the mean field description, *i.e.* the more accurate is the notion of acoustic spacetime. It would be interesting to understand whether the same applies to spacetime in which we live: is the cosmological constant small for the same reason why we perceive a classical spacetime (*i.e.* why do we have to deal with a mean field description rather than full quantum gravity)?

This thesis should have convinced the reader of two basic points. Despite the fact that analogue models and emergent gravity models are not reproducing exactly all the features of real gravitational phenomena, they are a really nice laboratory for studying in concrete systems some otherwise inaccessible phenomena. Furthermore they can give some (non-conventional) inspirations to look at the problems the community is facing when dealing with gravity.

In this sense, it is worth to stress again the fact that a condensed matter point of view in quantum gravity might be useful should not come as a surprise. In building spacetime from the Planck scale to the cosmological scale one could expect that collective phenomena take place. In this respect, the troubles with quantum gravity could be interpreted as the results of the attempt of deducing the properties of the condensate by using only local quantum field theory of phonons. This is impossible, as we have seen. Nonetheless, the situation is not completely hopeless. As in the case of the physics of phonons (nonlocality, cosmological constant), there can be some residual features which could be of help in understanding what are the new ingredients to be taken into account, and guide us towards a solution of the quantum gravity problem.

In this sense, then, this line of research represents a most valuable resource in suggesting new paths for the investigation of the fundamental nature of gravity and spacetime.

Appendix A

Euler's theorem and its consequences

The homogeneity of the norm and of its derivatives with respect to the vector argument is a fundamental ingredient for the development of the theory of Finsler spaces. Therefore we give here a brief review of the Euler's theorem on homogeneous functions, specializing the results to the case of Finsler metrics.

Let us begin with the theorem: *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function away from the origin of $\mathbb{R}^n$. Then the two following statements are equivalent:*

a. f is positively homogeneous of degree n.

$$f(\lambda v) = \lambda^n f(v), \qquad \forall \lambda > 0. \tag{A.1}$$

b. The radial derivative of f is proportional to f itself:

$$v^i \frac{\partial f}{\partial v^i} = n f(v). \tag{A.2}$$

While we omit the proof of this theorem (see, for instance, [206]), we want to consider some consequences.

If a function f is homogeneous of degree n, then we can prove that the function $\partial f / \partial v^i$ is homogeneous of degree $n-1$. The proof is obtained just taking the derivative of (A.2) with respect to v^j:

$$\frac{\partial}{\partial v^j}\left(v^i \frac{\partial f}{\partial v^i}\right) = n \frac{\partial f}{\partial v^j}$$

After trivial manipulations, we obtain:

$$v^i \frac{\partial}{\partial v^i}\left(\frac{\partial f}{\partial v^j}\right) = (n-1)\frac{\partial f}{\partial v^j}, \tag{A.3}$$

whence, using the Euler's theorem, we have that $\partial f / \partial v^i$ is a positively homogeneous function of degree $n-1$.

Let us now consider the specific case of Finsler norms. Since F is a positively homogeneous function of degree 1, we have that the function $\partial F / \partial y^i$ is positively homogeneous of degree zero.

The Euler's theorem says that:

$$v^i \frac{\partial}{\partial v^i}\left(\frac{\partial F}{\partial v^j}\right) = 0, \tag{A.4}$$

which says that the matrix $\partial^2 F/\partial v^i \partial v^j$ is degenerate, having v^i as an eigenvector with eigenvalue zero. This is the reason why, when using F as a Lagrangian, the Legendre transform to obtain the Hamiltonian is singular.

From the definition we have given, we see that the Finsler metric is a positively homogeneous function of degree zero. Indeed, the square of the norm is an homogeneous function of degree two. Hence, following Euler's theorem,

$$\frac{\partial F^2}{\partial v^i} v^i = 2F^2, \tag{A.5}$$

which implies that $\partial F^2/\partial v^i$ is homogeneous of degree one. Applying again the Euler's theorem we obtain that:

$$2g_{ij}(v)v^i = \frac{\partial^2 F^2}{\partial v^i \partial v^j} v^i = \frac{\partial F^2}{\partial v^j}, \tag{A.6}$$

whence it follows that $g_{ij}(v)$ is homogeneous of degree zero. This means that the Finsler metric depends on the direction of the vector only, and not on its norm.

The homogeneity property of the metric results into another important identity. The Cartan tensor, being the derivative of the metric tensor with respect to the vector, is a positively homogeneous function of degree -1. This follows from the application of the Euler's theorem on the metric, in analogy with what has been done for the previous cases. Since the metric tensor is homogeneous of degree zero, and given that the Cartan tensor is the derivative of the metric, the Euler's theorem applied to the metric tensor results into the following important identity:

$$C_{ijk}v^i = C_{ijk}v^j = C_{ijk}v^k = 0. \tag{A.7}$$

Appendix B

Source term

In this appendix we provide the details of the calculation of the source term for the Poisson equation corresponding to a localized source (c.f. 7.4.2). We have to evaluate the expressions:

$$\mathfrak{n}(\mathrm{x}) = \langle \zeta_{x_0} | \hat{\chi}^\dagger(\mathrm{x}) \hat{\chi}(\mathrm{x}) | \zeta_{x_0} \rangle, \qquad \mathfrak{m}(\mathrm{x}) = \langle \zeta_{x_0} | \hat{\chi}(\mathrm{x}) \hat{\chi}(\mathrm{x}) | \zeta_{x_0} \rangle, \tag{B.1}$$

where

$$|\zeta_{x_0}\rangle = \int \mathrm{d}^3\mathrm{z}\, \zeta_{x_0}(\mathrm{x}) \hat{\phi}^\dagger(\mathrm{z}) |\Omega\rangle. \tag{B.2}$$

Let us describe it for $\mathfrak{n}$, since $\mathfrak{m}$ can be evaluated following the same steps. First, one has to write explicitly $\mathfrak{n}$ in terms of the field operators:

$$\mathfrak{n} = \int \mathrm{d}^3\mathrm{z}_1 \mathrm{d}^3\mathrm{z}_2\, \zeta^*_{x_0}(\mathrm{z}_1) \zeta_{x_0}(\mathrm{z}_2) \langle \Omega | \hat{\phi}(\mathrm{z}_1) \hat{\chi}^\dagger(\mathrm{x}) \hat{\chi}(\mathrm{x}) \hat{\phi}^\dagger(\mathrm{z}_2) | \Omega \rangle. \tag{B.3}$$

Let us evaluate then the expectation value inside the integral. To do this, it is necessary to replace the expansion of the field operators in plane waves, and then to use the Bogoliubov transformation:

$$\begin{aligned} \langle \Omega | \hat{\phi}(\mathrm{z}_1) \hat{\chi}^\dagger(\mathrm{x}) \hat{\chi}(\mathrm{x}) \hat{\phi}^\dagger(\mathrm{z}_2) | \Omega \rangle = \frac{1}{V^2} \sum_{\mathrm{k},\mathrm{k}',\mathrm{h},\mathrm{h}'} e^{i\mathrm{h}\cdot\mathrm{z}_1} e^{-i\mathrm{k}\cdot\mathrm{x}} e^{i\mathrm{k}'\cdot\mathrm{x}} e^{-i\mathrm{h}'\cdot\mathrm{z}_2} \times \\ \times \langle \Omega | \hat{b}_\mathrm{h} (\alpha(\mathrm{k}) \hat{b}^\dagger_\mathrm{k} + \beta(\mathrm{k}) \hat{b}_{-\mathrm{k}}) (\alpha(\mathrm{k}') \hat{b}_{\mathrm{k}'} + \beta(\mathrm{k}') \hat{b}^\dagger_{-\mathrm{k}'}) \hat{b}^\dagger_{\mathrm{h}'} | \Omega \rangle. \end{aligned} \tag{B.4}$$

It is easy to see that, in this last expression, there are only two non-vanishing terms

$$\begin{aligned} \langle v \rangle = \langle \Omega | \hat{b}_\mathrm{h} (\alpha(\mathrm{k}) \hat{b}^\dagger_\mathrm{k} + \beta(\mathrm{k}) \hat{b}_{-\mathrm{k}}) (\alpha(\mathrm{k}') \hat{b}_{\mathrm{k}'} + \beta(\mathrm{k}') \hat{b}^\dagger_{-\mathrm{k}'}) \hat{b}^\dagger_{\mathrm{h}'} | \Omega \rangle = \alpha(\mathrm{k}) \alpha(\mathrm{k}') \langle \Omega | \hat{b}_\mathrm{h} \hat{b}^\dagger_\mathrm{k} \hat{b}_{\mathrm{k}'} \hat{b}^\dagger_\mathrm{h} | \Omega \rangle + \\ + \beta(\mathrm{k}) \beta(\mathrm{k}') \langle \Omega | \hat{b}_\mathrm{h} \hat{b}_{-\mathrm{k}} \hat{b}^\dagger_{-\mathrm{k}'} \hat{b}^\dagger_\mathrm{h} | \Omega \rangle. \end{aligned} \tag{B.5}$$

Using the algebra of the operators $\hat{b}, \hat{b}^\dagger$, it is easy to see that the expression reduces to:

$$\langle v \rangle = \alpha(\mathrm{k}) \alpha(\mathrm{k}') \delta_{\mathrm{h},\mathrm{k}} \delta_{\mathrm{h}',\mathrm{k}'} + \beta(\mathrm{k}) \beta(\mathrm{k}') \left(\delta_{\mathrm{k},\mathrm{k}'} \delta_{\mathrm{h},\mathrm{h}'} + \delta_{\mathrm{h},-\mathrm{k}'} \delta_{\mathrm{k},-\mathrm{h}'} \right). \tag{B.6}$$

Consequently,

$$\mathfrak{n}(\mathrm{x}) = A(\mathrm{x}) + B(\mathrm{x}) + C(\mathrm{x}), \tag{B.7}$$

where

$$A(\mathrm{x}) = \frac{1}{V^2} \int \mathrm{d}^3\mathrm{z}_1 \mathrm{d}^3\mathrm{z}_2 \sum_{\mathrm{k},\mathrm{k}',\mathrm{h},\mathrm{h}'} e^{i\mathrm{h}\cdot\mathrm{z}_1} e^{-i\mathrm{k}\cdot\mathrm{x}} e^{i\mathrm{k}'\cdot\mathrm{x}} e^{-i\mathrm{h}'\cdot\mathrm{z}_2} \zeta^*_{x_0}(\mathrm{z}_1) \zeta_{x_0}(\mathrm{z}_2) \alpha(\mathrm{k}) \alpha(\mathrm{k}') \delta_{\mathrm{h},\mathrm{k}} \delta_{\mathrm{h}',\mathrm{k}'}\,, \tag{B.8}$$

$$B(\mathrm{x}) = \frac{1}{V^2} \int \mathrm{d}^3\mathrm{z}_1 \mathrm{d}^3\mathrm{z}_2 \sum_{\mathrm{k},\mathrm{k}',\mathrm{h},\mathrm{h}'} e^{i\mathrm{h}\cdot\mathrm{z}_1} e^{-i\mathrm{k}\cdot\mathrm{x}} e^{i\mathrm{k}'\cdot\mathrm{x}} e^{-i\mathrm{h}'\cdot\mathrm{z}_2} \zeta^*_{x_0}(\mathrm{z}_1)\zeta_{x_0}(\mathrm{z}_2)\beta(\mathrm{k})\beta(\mathrm{k}')\delta_{\mathrm{k},\mathrm{k}'}\delta_{\mathrm{h},\mathrm{h}'} \ , \tag{B.9}$$

$$C(\mathrm{x}) = \frac{1}{V^2} \int \mathrm{d}^3\mathrm{z}_1 \mathrm{d}^3\mathrm{z}_2 \sum_{\mathrm{k},\mathrm{k}',\mathrm{h},\mathrm{h}'} e^{i\mathrm{h}\cdot\mathrm{z}_1} e^{-i\mathrm{k}\cdot\mathrm{x}} e^{i\mathrm{k}'\cdot\mathrm{x}} e^{-i\mathrm{h}'\cdot\mathrm{z}_2} \zeta^*_{x_0}(\mathrm{z}_1)\zeta_{x_0}(\mathrm{z}_2)\beta(\mathrm{k})\beta(\mathrm{k}')\delta_{\mathrm{h},-\mathrm{k}'}\delta_{\mathrm{k},-\mathrm{h}'} \ . \tag{B.10}$$

To manipulate these expression, it is useful to recall the representation of the Dirac delta in a Fourier series:

$$\delta^3(\mathrm{x}_1 - \mathrm{x}_2) = \frac{1}{V}\sum_{\mathrm{k}} e^{-i\mathrm{k}(\mathrm{x}_1 - \mathrm{x}_2)}, \tag{B.11}$$

and that the distribution ζ_{x_0} is normalized,

$$\int \mathrm{d}^3\mathrm{x} |\zeta_{x_0}(\mathrm{x})|^2 = 1. \tag{B.12}$$

After straightforward passages we obtain

$$A(\mathrm{x}) = \left| \int \mathrm{d}^3\mathrm{z} f(\mathrm{x}-\mathrm{z})\zeta_{x_0}(\mathrm{z}) \right|^2, \tag{B.13}$$

$$B(\mathrm{x}) = \frac{1}{V}\sum_{\mathrm{k}} \beta^2(\mathrm{k}), \tag{B.14}$$

$$C(\mathrm{x}) = \left| \int \mathrm{d}^3\mathrm{z} g(\mathrm{x}-\mathrm{z})\zeta_{x_0}(\mathrm{z}) \right|^2, \tag{B.15}$$

and, finally:

$$\mathfrak{n}(\mathrm{x}) = \left| \int \mathrm{d}^3\mathrm{z} f(\mathrm{x}-\mathrm{z})\zeta_{x_0}(\mathrm{z}) \right|^2 + \left| \int \mathrm{d}^3\mathrm{z} g(\mathrm{x}-\mathrm{z})\zeta_{x_0}(\mathrm{z}) \right|^2 + \frac{1}{V}\sum_{\mathrm{k}} \beta^2(\mathrm{k}) \ . \tag{B.16}$$

where we have introduced the functions:

$$f(\mathrm{x}) = \frac{1}{V}\sum_{\mathrm{k}} \alpha(\mathrm{k}) e^{i\mathrm{k}\cdot\mathrm{x}}, \qquad g(\mathrm{x}) = \frac{1}{V}\sum_{\mathrm{k}} \beta(\mathrm{k}) e^{-i\mathrm{k}\cdot\mathrm{x}}. \tag{B.17}$$

Notice that, as a consequence of $\alpha(\mathrm{k}) = \alpha(-\mathrm{k})$, $\beta(\mathrm{k}) = \beta(-\mathrm{k})$ and of the fact that these coefficients can be chosen to be real, the functions f, g are real functions.

Applying the same procedure to the term $\mathfrak{m}(\mathrm{x})$ we obtain

$$\mathfrak{m}(\mathrm{x}) = 2\left(\int \mathrm{d}^3\mathrm{z}_1 g(\mathrm{x}-\mathrm{z}_1)\zeta^*_{x_0}(\mathrm{z}_1) \right) \left(\int \mathrm{d}^3\mathrm{z}_1 f(\mathrm{x}-\mathrm{z}_2)\zeta_{x_0}(\mathrm{z}_2) \right) + \frac{1}{V}\sum_{\mathrm{k}} \alpha(\mathrm{k})\beta(\mathrm{k}). \tag{B.18}$$

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