Ph. D. Thesis

Complex Stable Manifold Theorems and Generalized Complex Hénon Mappings

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Chapter 1

Introduction

The recent areas of interest in differentiable dynamical systems include the following two subjects:

(1) **non-uniformly hyperbolic dynamical systems; and**

(2) **complex dynamical systems.**

In this thesis, we will mainly concentrate on the **non-uniformly hyperbolic complex dynamics** related to the above two subjects. We will study the Pesin stable manifolds of holomorphic diffeomorphisms on complex manifolds and the generalized complex Hénon mappings of $\mathbb{C}^2$ from the dynamical point of view. We begin to give a brief review of the Pesin stable manifold theory in Section 1.1, 1-dimensional complex dynamical systems in Section 1.2, 2-dimensional complex dynamical systems in Section 1.3, and then give an outline of this thesis in Section 1.4.

For the convenience, we first recall some basic definitions and notations which will be used in the context.

The basic goal of the theory of dynamical systems is to understand the eventual or asymptotic behavior of an iterate process. The set of periodic points and the nonwandering point set are very important sets to such an iterate process. Let X be a topological space, and $f : X \to X$ a continuous mapping. The *forward orbit of* x_0 *under* f is defined as the set $\text{Orb}^+(x_0, f) := \{x \in X : x = f^n(x_0), n = 0, 1, 2, \cdots\}$, where $f^0 :=$ identity mapping, $f^1 := f$ and f^n is the n-th iterate of f. If f is a diffeomorphism, we define the *backward orbit of* x_0 *under* f as the forward orbit of x_0 under f^{-1} and denote it by $\text{Orb}^-(x_0, f)$. A *fixed point* of f is a point x such that $f(x) = x$; a *periodic point* is a fixed point for an iterate of f; that is, there exists a strictly positive integer n such that $f^n(x) = x$. If x is a periodic point, the positive integer $m := \min\{n : f^n(x) = x\}$ is called the period of x. We denote by $P(f)$ the set of periodic points of f. The *period set* of f is the set of periods of all periodic points of f. A point x of X is *wandering* if it has a neighborhood U such that $f^k(U) \cap U = \emptyset$ for all positive integer k. A point x is *nonwandering* if the above does not hold. We denote by $\Omega(f)$ the set of nonwandering points of f.

Now we recall some definitions in differentiable dynamical systems. Let $X := M$ be a n-dimensional smooth Riemannian manifold, and $f : M \to M$ a diffeomorphism. The subset $\Lambda \subset M$ is said to be *f-invariant* if $f^{-1}(\Lambda) = \Lambda$. The tangent mapping Df from the tangent bundle TM into itself is defined pointwisely as $Df_x : T_xM \to T_{f(x)}M$, and the mapping Df_x is an isomorphism of the respective linear spaces. A Riemannian metric on the manifold M defines an inner product (and hence a norm) on each tangent space T_xM. An f-invariant compact subset $\Lambda \subset M$ is said to be *hyperbolic* if for each point $x \in \Lambda$ there exists a pair of linear subspaces E_x^s and E_x^u of the tangent space T_xM, such that

(H1) $T_xM = E^s_x \oplus E^u_x$, $\dim E^s_x := s(x)$, $\dim E^u_x := u(x)$, $s(x) + u(x) = n$, with $s(x)$ and $u(x)$ depending continuously on $x \in \Lambda$;

(H2) $Df_x(E^s_x) = E^s_{f(x)}$ and $Df_x(E^u_x) = E^u_{f(x)}$;

(H3) there exist two constants $c > 0$ and $0 < \lambda < 1$ such that

$$\|Df^n_x(v_s)\| \leq c\lambda^n \|v_s\| \quad \text{and} \quad \|Df^{-n}_x(v_u)\| \leq c\lambda^n \|v_u\|$$

for $v_s \in E^s_x, v_u \in E^u_x$, and $n > 0$.

Note that the definition of hyperbolicity does not depend on the choice of Riemannian metric of M.

If the whole manifold M is a hyperbolic set of f, then f is called an *Anosov diffeomorphism.* If the nonwandering set $\Omega(f)$ is a hyperbolic set of f, and the set of periodic points $P(f)$ is a dense subset of $\Omega(f)$, then we say that f satisfies *Axiom A.*

If x_0 is a fixed point of f, the set

$$W_s(x_0, f) := \{x \in M : \lim_{n \to +\infty} dist(f^n(x), x_0) = 0\}$$

is called *stable manifold of* x_0, and the set

$$W_u(x_0, f) := \{x \in M : \lim_{n \to -\infty} dist(f^n(x), x_0) = 0\}$$

is called *unstable manifold of* x_0. Similarly, we can define stable manifold and unstable manifold for periodic point and for compact f-invariant set.

Let x_0 and y_0 be two fixed points of f, a point x is said to be a *heteroclinic point of* $W_s(x_0, f)$ *and* $W_u(y_0, f)$ if $x \in W_s(x_0, f) \cap W_u(y_0, f)$; a point x is said to be a *homoclinic point of* x_0 if $x \in W_s(x_0, f) \cap W_u(x_0, f) - \{x_0\}$. We say that two submanifolds M_1 and M_2 of M intersect *transversally* at the point $x \in M$ if $T_xM_1 \oplus T_xM_2 = T_xM$.

1.1 Pesin stable manifold theory

The earlier results on differentiable dynamical systems had been mostly geometric and restricted to *hyperbolic* (Anosov, 1967) or *Axiom-A* systems (Smale, 1967). The hyperbolic dynamical system theory stands today as a very solid and rather well understood region of the large world of Dynamical Systems. The complement of the hyperbolic systems in the world of dynamical systems, which was called *the dark realm of dynamics* by Palis [Pa2], was much "bigger" than the dynamists thought in the sixties. Palis gave an interesting pictorial view of how hyperbolicity stands via its complement in the World of Dynamical Systems in the sixties and nineties:

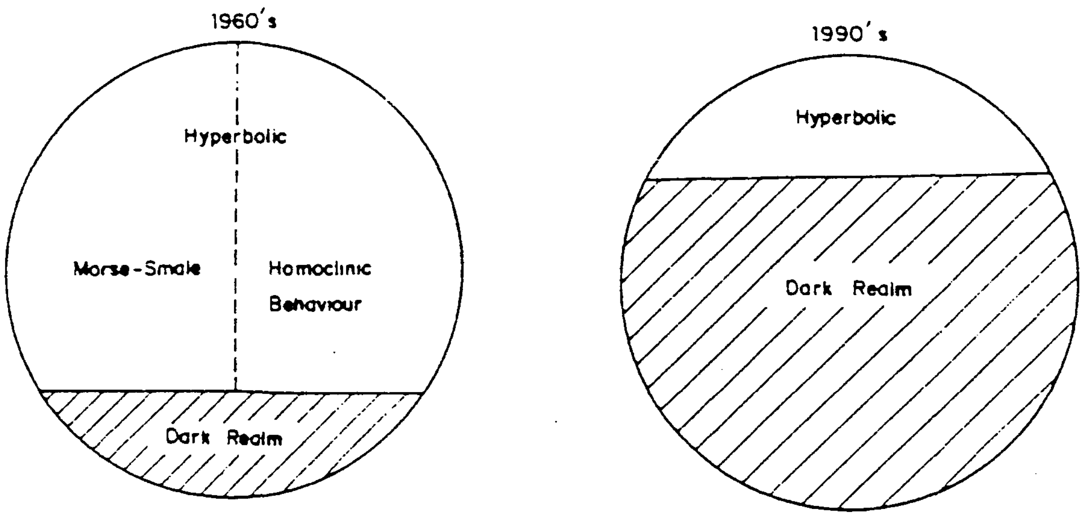

Figure 1.1

Pesin's theory (Pesin, 1976-77) extended some of these geometric results in the sixties to more interesting and complicated non-uniformly hyperbolic differentiable dynamical systems (which contain uniformly hyperbolic ones as special cases), but working now *almost everywhere* with respect to some ergodic measure.

Let M be a smooth compact Riemannian manifold, f a C^2-diffeomorphism, and μ an f-invariant probability measure. **Pesin stable manifold theorem** says that for μ-a.e. $x \in M$, the stable set

$$W^s(x,f) = \{y \in M : \limsup_{k\to\infty} \frac{1}{k} \log d(f^k(y), f^k(x)) < 0\}$$

is in fact **an immersed Euclidean space.**

Pesin theory is one of the most remarkable theories in the history of dynamics. It has much to do with Lypunov exponent, Hausdorff dimension, entropy, ergodicity, mixing, Smale horseshoe and chaotic phenomenon. It is a very *interesting* and *difficult* subject, because it is heavily dependent on differential geometry, functional analysis and ergodic theory.

The original contribution of Pesin ([P1], 1976 and [P2], 1977) has been extended by many mathematicians, notably Katok ([Ka1], 1980), Katok and Strelcyn ([KS], 1986), Ledrappier and Young ([LY], 1984), Mañé ([Mé], 1983), Ruelle ([Ru1],1979 and [Ru2], 1982) and Pugh and Shub ([PS], 1989). A clear treatment of the Pesin stable manifold theory for diffeomorphisms on compact smooth Riemannian manifolds is given by Fathi-Herman-Yoccoz in [FHY].

1.2 1-dimensional complex dynamical systems

Recently there has been an increasing interest in complex dynamical systems. The study of 1-dimensional complex dynamical systems, which originated in the 1920's with the work of two French mathematicians Fatou [Fa] and Julia [Ju], remained undeveloped for almost fifty years. The important turning point is the observation of Mandelbrot ([Ma] 1982) of the well known set which bears his name as well as the very beautiful computer graphics images (cf. [PR] and [Ba]) that typically accompany these dynamical systems. Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational mapping on the Riemann sphere $\overline{\mathbb{C}}$. A decisive step made by Fatou and Julia is the decomposition of the sphere into two invariant subsets: an open set on which the family of iteration $\{f^n\}$ is a normal family in the sense of Montel, and a perfect set coinciding with the closure of the set of repelling periodic points of f. The first set is called *Fatou set of* f and denoted by $F(f)$, the second one is called *Julia set of* f and denoted by $J(f)$.

The Julia set of f has many nice properties. We list some of them as follows(see [De] for the proofs):

J1. $J(f) \neq \emptyset$;

J2. $J(f) = \{z \in \overline{\mathbb{C}} : \{f^n\}$ *is not normal at* $z\}$;

J3. $J(f) = J(f^m)$ *for all integers* m;

J4. $f|_{J(f)}$ *is topological mixing*;

J5. $J(f)$ *is a perfect set*;

J6. *Every repelling periodic point of* f *admits homoclinic points. Moreover, homoclinic points are dense in* $J(f)$;

J7. f *is chaotic (in the sense of Devaney) on* $J(f)$.

Sullivan [Su1,2] has completed the description of the dynamics on the Fatou set for rational mappings by using Teichmuler theory and the theory of Fuchsian and Kleinian groups. Let the degree of the rational mapping f is at least 2, Sullivan proved that every component Ω of the Fatou set $F(f)$ is eventually periodic, i.e., there exist two positive integers m and n, such that $f^{m+n}(\Omega) = f^m(\Omega)$. In other words, there is no wandering domains for the Fatou set.

The simplest and interesting 1-dimensional complex dynamical systems should be the quadratic polynomials

$$p_c(z) = z^2 + c.$$

For this family of polynomials, the Mandelbrot set is defined as the subset of C

$$M := \{c \in \mathbb{C} : J(p_c) \text{ is connected } \}.$$

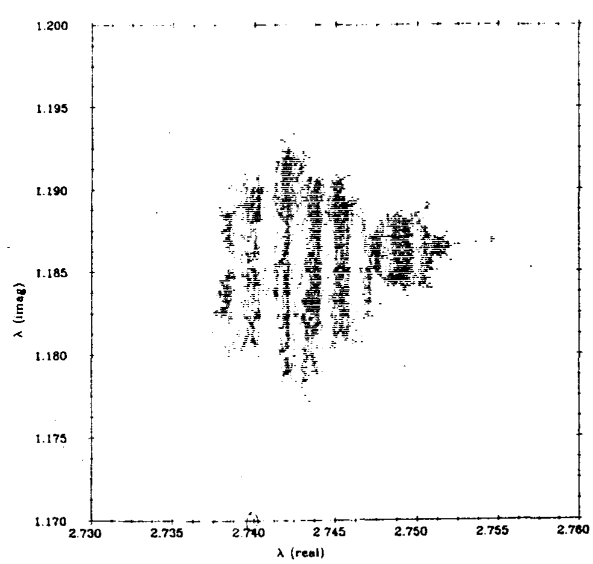

Figure 1.2. First detailed picture of the Mandelbrot set for $z \to z^2 - c$(March 1980)

The initial computer pictures of the Mandelbrot set M seemed to indicate that M had more than one "main body" and that it might be disconnected (see Figure 1.2). In fact, Douady and Hubbard [DH] proved that the Mandelbrot set is connected (see Figure 1.3). It is still unknown, however, whether M is also **locally connected.** Yoccoz gave some partial answer in [Yz].

Usually the Mandelbrot sets have self-similar structure and have non-integral Hausdorff dimension, in other words, they are *fractals* (see Figure 1.3).

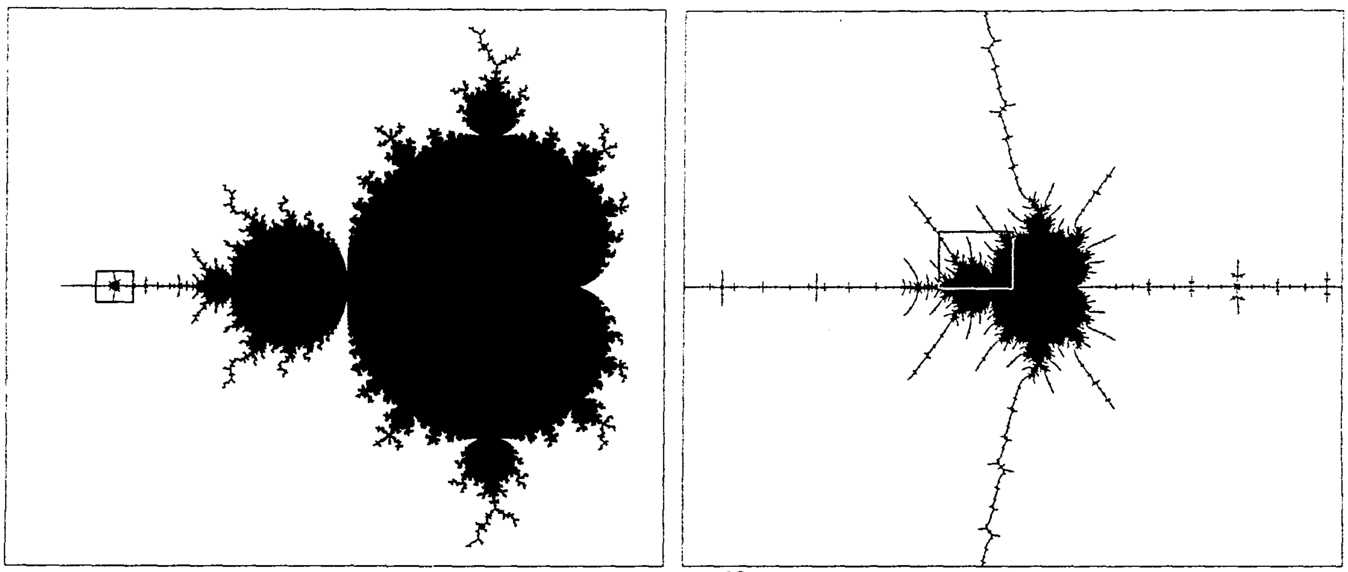

Figure 1.3. The picture of the Mandelbrot set for $z \to z^2 + c$

There are several good surveys on one-dimensional complex dynamics, see for examples [Br], [Bl], [Cn], [Ly], and [Mi].

1.3 2-dimensional complex dynamical systems

In 1969 and 1976, Hénon ([Hé1] and [Hé2]) introduced the mappings

$$H(x,y) = (y, y^2 + c - ax) \quad \text{with} \quad a \neq 0$$

of R^2 and he demonstrated numerically for certain values of the parameters H appeared the existence of *strange attractors. an attractor* is defined as a compact invariant set Γ having a dense orbit and whose stable set $W^s(\Gamma) := \{x \in \mathrm{R}^2 : \lim_{n\to\infty} dist(H^n(x), \Gamma) = 0\}$ has non-empty interior. We call an attractor *strange* if it has a dense orbit with positive Lypunov exponent. This numerical result of Hénon has finally established rigorously by Benedicks and Carleson ([BC], see also [MV]) in 1991.

If we want to study the complex 2-dimensional dynamical systems, the formula $H(x,y) = (y, y^2 + c - ax)$ can also be used to define a diffeomorphism of $\mathbb{C}^2$ where $a, c \in \mathbb{C}$. It is easy to see that the polynomial automorphisms are the simplest nontrivial model for studying the 2-dimensional complex dynamical systems. In 1986, Hubbard and Oberste-Vorth ([Hu] and [HO]) began to study the complex Hénon mappings of $\mathbb{C}^2$ and introduced some important invariant sets which bear some analogues to the Fatou and Julia sets of the one-dimensional case. They defined that $K^+ := K^+(H)$ is the closed set consisting of all points whose forward orbit under H remains bounded, and $K^- := K^-(H)$ is the closed set consisting of all points whose backward orbit under H remains bounded, and the intersection is denoted by $K(H) := K^+(H) \cap K^-(H)$. Naturally, they also defined that $J^+(H) := \partial K^+(H)$, $J^-(H) := \partial K^-(H)$ and $J(H) := J^+(H) \cap J^-(H)$. They showed in [H] and [HO] that the topology of these sets are very complicated.

Friedland and Milnor [FM] have classified the group G consisting of all complex polynomial automorphisms on $\mathbb{C}^2$ up to conjugation into two classes. The first

class contains the affine mappings and the "elementary" mappings which have the form $h(x,y) = (x + p(y), y)$. **The second class** consists of the finite compositions of "generalized Hénon " mappings. Each "generalized Hénon " mapping g has the form

$$g(x,y) = (y, p(y) + ax),$$

where $p(y)$ is a monic polynomial of degree at least 2 and a is a non-zero complex number. Let we fix f to be a finite composition of "generalized Hénon" mappings. The dynamical properties of the mappings in the first class are very simple, for example, the topological entropy is zero and the period set is finite [FM]. The mappings in the second class are very interesting. These diffeomorphisms provide examples of simple forms with very complicated dynamics and have been studied intensively in recent years. Bedford and Smillie obtained a series results for f in the second class. They proved in [BS2] that *$J^+(f)$ is the closure of the stable manifold $W^s(p,f)$ and the $J^-(f)$ is the closure of the unstable manifold $W^u(p,f)$ for any saddle periodic point p.* This was conjectured by J. Hubbard. If f is hyperbolic on $J(f)$, Bedford and Smillie proved in [BS1] that *f is topological mixing on $J(f)$, and that $J^+(f), J^-(f), K^+(f)$ and $K^-(f)$ are connected* which was conjectured by Friedland and Milnor in [FM]. In [BS3], they proved that *f is ergodic with respect to the equilibrium measure μ* (see Section 5.1 for the definition of μ) and *μ is a measure with maximal entropy.* Smillie proved in [Si] that *the topological entropy of f is* $\log d$ where d is the degree of f. Fornaess and Sibony considered a class of special complex Hénon mappings

$$g(z,w) = (z^2 + c + aw, az)$$

when $z^2 + c$ has attractive cycle of order k and $|a| << 1$. They proved in [FS] that *when $k > 1$ the boundary of the basin of attraction is not a topological manifold and is of Lebesgue measure zero.* They also studied the topological structure of $K^{\pm}(g), J^{\pm}(g), K(g)$ and $J(g)$. Although we have mentioned some substantial contributions above, the dynamics of f is still not well understood and significant problems are still being made. One can find some interesting open problems in the problem lists [Be] and [CFGK].

1.4 The outline of this thesis

In this thesis, we will combine the techniques of Pesin theory and the techniques of powerful complex analysis theory to study the complex Pesin stable manifolds and the dynamics of the complex polynomial automorphisms of $\mathbb{C}^2$. In other words , we will study the *intersection part* of non-uniformly hyperbolic dynamical systems and complex dynamical systems. Our results presented here mainly come from the papers [Wu 1,2,3] and one joint paper with Verjovsky [VW]. In the following we give an outline of this thesis.

Pesin stable manifold theory, in general, works only for **compact** smooth Riemannian manifolds and smooth diffeomorphisms. At a regular point in the sense of Pesin, the stable manifold passing through this point is an immersed smooth submanifold. In Chapter 2, we will work on holomorphic diffeomorphisms f on **compact complex** manifolds M. Of cause, we expect to get some stronger results under such stronger assumptions. By using *Cauchy-Riemann condition* of the holomorphic diffeomorphisms f and the *complex structure* of the complex manifolds M, we apply Pesin stable manifold theorem (note: we may consider the complex manifold and the holomorphic diffeomorphism as a smooth Riemannian manifold and a smooth diffeomorphism respectively. In this sense, we can apply Pesin theory) to prove that the Pesin stable manifolds at the regular points in the sense of

Pesin are actually **immersed complex manifolds**[Wu 1]. For proving above results, we first prove the complex Oseledec multiplicative ergodic theorem by using the Cauchy-Riemann condition of f, and then prove that every tangent space of the Pesin stable manifold is a complex linear space.

If we say that the "complex world" and the "real world" are in the different side of a river, then the Cauchy-Riemann condition and the complex structure play the role of a "bridge" which successfully links up the two "worlds".

In Chapter 3, we obtain the same result as in Chapter 2 for typical non-compact complex manifold $\mathbb{C}^n$ [Wu 2]. Since the number of compact complex manifolds are very limited and some important complex manifolds are not compact(e.g. $\mathbb{C}^n$), it will be very interesting and useful to have a version of complex manifold theorem for holomorphic diffeomorphisms on non-compact complex manifolds. We have not tried to present such a complex manifold theorem in the greatest generality. For the purpose of our applications in Chapter 5, we only consider the holomorphic diffeomorphisms of $\mathbb{C}^n$ and give a version of complex stable manifold theorem for this case. Of course, we also need some kind of compactness condition. We mainly study the holomorphic diffeomorphism f of $\mathbb{C}^n$ with *an f-invariant compact subset* $K \subset \mathbb{C}^n$. In this case, we make use of the non-linear ergodic theorem obtained by Ruelle [Ru1] (instead of using Pesin stable manifold theorem in Chapter 2) and the Cauchy-Riemann condition to prove the complex multiplicative ergodic theorem and the complex stable manifold theorem. More precisely, for any f-invariant Borel probability measure μ *supported on* K, there is a Borel set $\Gamma \subset K$ such that $\mu(\Gamma) = 1$ and for any $p \in \Gamma$, the stable set

$$W^s(p, f) = \{z \in \mathbb{C}^n : \limsup_{k\to\infty} \frac{1}{k} \log |f^k(z) - f^k(p)| < 0\} \tag{1.1}$$

is actually an **immersed complex manifold**. This result will be used and extended in Chapter 5 for generalized complex Hénon mappings of $\mathbb{C}^2$.

In Chapter 4, we study the stable manifolds of automorphisms F of $\mathbb{C}^n$ at hyperbolic fixed points [Wu3] without using the Pesin theory. If $F(0) = 0$, we denote $\lambda_1, \cdots, \lambda_n$ the all eigenvalues of the tangent map $DF(0)$. If $\lambda_1, \cdots, \lambda_n$ satisfy the condition $(*)$ (see Definition 4.2.1), we show that the stable manifold at point 0

$$W_s(0, F) = \{z \in \mathbb{C}^n : \lim_{l \to \infty} F^l(z) = 0\},$$

is an injectively immersed complex submanifold biholomorphically equivalent to $\mathbb{C}^{n-k}$, where $F^l = F \circ F^{l-1}, F^1 = F$ and $n - k$ is the cardinal number of the eigenvalues of $DF(0)$ whose absolute values are less than 1.

This result extends the well known classical result that *the basin of attraction of a sink is biholomorphic equivalent to* $\mathbb{C}^n$(cf. [RR]).

In Chapter 5, we study the dynamical properties of complex polynomial automorphisms of $\mathbb{C}^2$. In [Wu 1,2] [VW], we obtained some results related to heteroclinic points and homoclinic points of saddle periodic points, introduced a new notation *S-Julia set* as an analogue of the Julia set, and extended some beautiful results in [BS 1,2,3] and [FS] under the *hyperbolic condition* to more general *non-uniformly hyperbolic cases.*

We fix f to be a composition of finite generalized complex Hénon mappings.

Following [H], define

$$K^{\pm} = \{p \in \mathbb{C}^2 : \{f^{\pm n}(p) : n = 0, 1, 2, ...\} \quad \text{is bounded}\},$$

and $J^{\pm} = \partial K^{\pm}, K = K^+ \cap K^-$ and $J = J^+ \cap J^-$.

Friedland and Milnor [FM] proved that K is an f-invariant compact subset of $\mathbb{C}^2$. Bedford and Smillie [BS1,2,3] introduced an f-invariant Borel measure μ on K and they proved that f is ergodic with respect to μ. Using these properties and applying Complex Stable Manifold Theorem obtained in Chapter 3, we prove (Theorem 5.2.1) that for μ-almost every $p \in K$, the stable and unstable manifolds $W^s(p, f)$, $W^u(p, f)$ are immersed complex manifolds and *biholomorphically equivalent to* $\mathbb{C}$. Moreover, we also show (Theorem 5.2.2) that for μ-almost every $p \in K$,

$$W^s(p, f) \subset J^+ \text{ and } W^u(p, f) \subset J^-.$$

If p is a saddle periodic point (see 5.2 for definition), then we have an interesting fact: the stable manifold $W_s(p, f)$ (resp. the unstable manifold $W_u(p, f)$) at p defined in [BS1,2,3] coincides with $W^s(p, f)$ (resp. $W^u(p, f)$) (Theorem 5.2.4). Then the above statement and Theorem 1 of [BS2] imply that the closure of $W^s(p, f)$ (resp. $W^u(p, f)$) is exactly J^+ (resp. J^-).

Remark that some authors called $W_s(p, f)$ (resp. $W_u(p, f)$) the *stable manifold* (resp. *unstable manifold*) of f at p, and called $W^s(p, f)$ (resp. $W^u(p, f)$) the *strong stable manifold* (resp. *strong unstable manifold*) of f at p. In this thesis, we only use the different notations to distinguish them without giving different names.

In Sections 5.3, we prove that for any two saddle points p and q of f, $W^s(p, f)$ and $W^u(q, f)$ have non-empty transversal intersections, and $W^s(q, f)$ and $W^u(p, f)$ have non-empty transversal intersections. In other words, $W^s(p, f)$ and $W^u(q, f)$ have transversal *heteroclinic points* and $W^s(q, f)$ and $W^u(p, f)$ have transversal *heteroclinic points.* An easy consequence of this result is that *for any saddle periodic point p of f, f admits transversal homoclinic points.* This theorem is, in some sense, an extension of a remarkable result of Katok [Ka2] from the real 2-dimensional case to the complex 2-dimensional case. Thus by using complex variable method we overcome the difficulties of studying a dynamical system in real dimension 4.

In Section 5.4, we try to find a good analogue of the Julia set for the complex generalized Hénon mapping in $\mathbb{C}^2$. The notation of Hubbard and Oberste-Vorth (cf. [BS1]) suggests $J = J^+ \cap J^-$ as 2-dimensional analogue of the Julia set. Friedland and Milnor [FM] proved that K is an f-invariant compact subset of $\mathbb{C}^2$. Bedford and Smillie [BS1,2,3] introduced an f-invariant Borel measure μ on K (see section 5.1 for the definition of μ) and proved that f is ergodic with respect to μ. They suggest, in [BS3], that the set $J^* = support(\mu)$ may better carry this analogy because it contains a dense subset of the saddle periodic points ([BS3, Theorem 3.4]) and J^* is a perfect set ([BS3, Remark following Theorem 2.5]). In Section 5.4, we introduce another analogue, called S-Julia set, which seems to be a better candidate for 2-dimensional Julia set. *The closure of all periodic saddle points of f, denoted by $SJ(f)$ or simply by SJ, is called the S-Julia set of f* in Chapter 5. As mentioned in Corollary 5.4.2, $J^* \subseteq SJ \subseteq J$. We prove that SJ has following properties which are parallel to the ones of the Julia set J in one complex variable case:

SJ1. $SJ(f) \neq \emptyset$ (Corollary 5.4.2);

SJ2. $SJ(f) \subset \{z \in \mathbb{C}^2 : \{f^n\}$ *is not normal at* $z\}$ (Theorem 5.4.9);

SJ3. $SJ(f) = SJ(f^m)$ *for all integers* m (Corollary 5.4.2);

SJ4. $f|_{SJ(f)}$ *is topological mixing* (Theorem 5.4.3);

SJ5. $SJ(f)$ *is a perfect set* (Remark 5.4.5);

SJ6. *Every saddle point of f admits transversal homoclinic points. Moreover, the transversal homoclinic points are dense in $SJ(f)$* (Theorems 5.4.4 and 5.4.11);

SJ7. *f is chaotic (in the sense of Devaney) on $SJ(f)$* (Theorem 5.4.7).

Some results in Chapter 5 were proved independently by Bedford, Lyubich & Smillie in recent Stony Brook preprint no.8(1992).

Chapter 2

Complex Pesin stable manifold theorem I: the compact case

2.1 Introduction

Pesin [P1,2] and Ruelle [R] established a remarkable stable manifold theorem for $C^{1+\epsilon}$ diffeomorphisms f of compact Riemannian manifolds M as following: given an f-invariant probability Borel measure μ, then for μ-a.e. p, the stable set at p

$$W^s(p,f) = \{q \in M : \limsup_{m\to\infty} \frac{1}{m} \log d(f^m(q), f^m(p)) < 0\}$$

is an immersed Euclidean space.

The unstable set at p, denoted by $W^u(p,f)$, is defined analogously by using f^{-1} instead of f. So we will mainly concentrate on the stable sets.

In this chapter, we study the dynamical properties of holomorphic diffeomorphisms f on n-dimensional compact complex manifolds M. Of course, M and f can be regarded as the Riemannian manifold and the diffeomorphism respectively if we forget the complex structure of M. We will denote $M^C := M$ if we consider M as an n-dimensional complex manifold, $M^R := M$ if we consider M as a 2n-dimensional real manifold. If we consider M as a real manifold and f as a real analytic diffeomorphism, we can apply Pesin's stable manifold theorem and obtain a family of stable manifolds. The goal of this chapter is to prove that such stable manifolds are actually immersed *complex* manifolds.

In Section 2.2, we first induce the Riemannian metric on M^R from the Hermitian metric on M^C by using the complex structure on M^C. Then Oseledec's multiplicative ergodic theorem works. Since our mapping f is holomorphic, by the Cauchy-Riemann condition of f together with the condition of regularity of the point in M^R, we have Lemma 2.2.5 and Corollary 2.2.6. As a consequence of these results and Oseledec multiplicative ergodic theorem, we obtain the complex multiplicative ergodic theorem on complex manifolds (Theorem 2.2.8).

In Section 2.3, we first recall Pesin's local stable manifold theorem in Theorem 2.3.1 and a nice property of local stable manifold in Theorem 2.3.2. This property says that *every point of the local stable manifold passing through a regular point is forward regular.* By this property, we can easily check that the conditions of Lemma 2.2.5 and Corollary 2.2.6 are satisfied. This implies that *the tangent space at every point of the local stable manifold is a complex linear space, thus the local stable manifold is a complex submanifold of M^C.* The globalization of the local complex stable manifold can be done in the standard way, see, for example, the proof of global stable manifold theorem in [FHY].

2.2 Complex multiplicative ergodic theorem I

2.2.1 Notations and some calculations

Let M be a compact complex manifold of complex dimension n, and $f : M \to M$ a holomorphic diffeomorphism. For any point $p \in M$, let $(U_p; z_1, ..., z_n)$ be a complex local coordinate system defined on a neighborhood U_p of the point p, and let x_k and y_k be the real and imaginary parts of z_k, respectively. If M is considered as a 2n-dimensional analytic real manifold, then $(U_p; x_1, ..., x_n, y_1, ..., y_n)$ is a real analytic local coordinate system of M. For the sake of clearness, we denote $M^C = M$ if we consider M as a complex manifold, $M^R = M$ if we consider M as a real analytic manifold.

Define

$$(\frac{\partial}{\partial z_k})_p = \frac{1}{2}[(\frac{\partial}{\partial x_k})_p - i(\frac{\partial}{\partial y_k})_p], \quad (\frac{\partial}{\partial \overline{z_k}})_p = \frac{1}{2}[(\frac{\partial}{\partial x_k})_p + i(\frac{\partial}{\partial y_k})_p]$$

then clearly $\{(\frac{\partial}{\partial z_k})_p : 1 \leq k \leq n\}$ is a basis of the tangent space T_pM^C and $\{(\frac{\partial}{\partial x_k})_p, (\frac{\partial}{\partial y_k})_p : 1 \leq k \leq n\}$ is a basis of the tangent space T_pM^R at point p.

For any given complex manifold M^C, we can introduce a positive definite Hermitian metric (see [Ok]) on M^C. We fix now a positive definite Hermitian structure H_C on M^C, i.e., for a complex local coordinate system $(U_p; z_1, ..., z_n)$, the mapping $H_C : \ T_pM^C \times T_pM^C \to C$ satisfies

(i) $\forall \lambda_1, \lambda_2 \in C, \xi_1, \xi_2, \eta \in T_pM^C$, $H_C(\lambda_1\xi_1 + \lambda_2\xi_2, \eta) = \lambda_1 H_C(\xi_1, \eta) + \lambda_2 H_C(\xi_2, \eta)$;

(ii) $\forall \xi, \eta \in T_pM^C$, $\overline{H_C(\xi, \eta)} = H_C(\eta, \xi)$;

(iii) $\forall \xi \in T_pM^C$ and $\xi \neq 0$, $H_C(\xi, \xi) > 0$.

The Hermitian metric is naturally given as:

$$\|\xi\|_H = \sqrt{H_C(\xi,\xi)}.$$

For applying the multiplicative ergodic theorem in the real case, we will induce the Riemannian metric on M^R from the Hermitian structure H_C by using the complex structure on M^C.

In the theory of complex manifold, the typical complex structure J_p ($p \in M^C$) on T_pM^R is defined as a real linear endomorphism $J_p : T_pM^R \longrightarrow T_pM^R$ satisfying

$$J_p(\frac{\partial}{\partial x_k}) = \frac{\partial}{\partial y_k}, \quad J_p(\frac{\partial}{\partial y_k}) = -\frac{\partial}{\partial x_k} \quad (1 \le k \le n). \tag{2.1}$$

Thus we can rewrite the basis of T_pM^R as $\{(\frac{\partial}{\partial x_k})_p, J_p(\frac{\partial}{\partial x_k})_p : 1 \le k \le n\}$.

Define a map $\psi_p : \ T_pM^C \longrightarrow T_pM^R$ by

$$\psi_p[\sum_{k=1}^{n}(a_k + ib_k)\frac{\partial}{\partial z_k}] = \sum_{k=1}^{n}(a_k\frac{\partial}{\partial x_k} + b_kJ_p\frac{\partial}{\partial x_k})$$

where $a_k, b_k \in R$. Obviously ψ_p is an isomorphism.

The Hermitian structure H on the real vector space T_pM^R can be defined as a map $H : T_pM^R \times T_pM^R \longrightarrow C$ such that: for any $\xi, \eta \in T_pM^R, H(\xi,\eta) = H_C(\psi_p^{-1}(\xi), \psi_p^{-1}(\eta))$.

Define $F : T_pM^R \times T_pM^R \longrightarrow R$ by $F(\xi,\eta) = \frac{1}{2}(H(\xi,\eta) + \overline{H(\xi,\eta)})$. Because $\xi \neq 0$ implies $\psi_p^{-1}(\xi) \neq 0$, then for $\xi \neq 0$,

$$F(\xi,\xi) = \frac{1}{2}(H(\xi,\xi) + \overline{H(\xi,\xi)}) = H(\xi,\xi) = H_C(\psi_p^{-1}(\xi), \psi_p^{-1}(\xi)) > 0. \tag{2.2}$$

Thus F is a C^∞ Riemannian structure on M^R, and we can define a Riemannian norm:

$$\|\xi\| = \sqrt{F(\xi,\xi)}, \quad \text{for } \ \xi \in T_pM^R.$$

Clearly, for any $\xi \in T_pM^R$, we have

$$\|\psi_p^{-1}(\xi)\|_H = \|\xi\|. \tag{2.3}$$

This implies that for every two points $p, q \in M$,

$$d_H(p,q) = d(p,q)$$

where $d_H(\cdot,\cdot)$ is the induced distance on M^C by $\|\cdot\|_H$, $d(\cdot,\cdot)$ is the induced distance on M^R by $\|\cdot\|$.

For any given $p \in M$, let $(U_p; z_1, ..., z_n)$ be a local coordinate system on U_p, and $(U_{f(p)}; w_1, ..., w_n)$ a local coordinate system on $U_{f(p)}$. If we decompose z_k and w_k into their real and imaginary parts, i.e., $z_k = x_k + iy_k$ and $w_k = u_k + iv_k$ $(1 \le k \le n)$, we can express f as

$$w_k(z) = f_k(z_1, ..., z_n) = u_k(x_1, ..., x_n, y_1, ..., y_n) + iv_k(x_1, ..., x_n, y_1, ..., y_n)$$

for $1 \le k \le n$.

Throughout this section we denote

$$\tilde{f}(q) = (u_1(q), ..., u_n(q), v_1(q), ..., v_n(q)), \quad for \quad q = (x_1, ..., x_n, y_1, ..., y_n) \in U_p.$$

We first consider the real tangent map at point $p \in M$:

$$D_p\tilde{f} : T_pM^R \longrightarrow T_pM^R$$

which is a real linear map and for any $1 \le \alpha \le n$,

$$(D_p\tilde{f})\frac{\partial}{\partial x_\alpha} = \sum_{k=1}^{n}(\frac{\partial u_k}{\partial x_\alpha}\frac{\partial}{\partial u_k} + \frac{\partial v_k}{\partial x_\alpha}\frac{\partial}{\partial v_k}),$$

$$(D_p\tilde{f})\frac{\partial}{\partial y_\alpha} = \sum_{k=1}^{n}(\frac{\partial u_k}{\partial y_\alpha}\frac{\partial}{\partial u_k} + \frac{\partial v_k}{\partial y_\alpha}\frac{\partial}{\partial v_k}).$$

Similarly, the complex tangent map at point p

$$D_pf : T_pM^C \longrightarrow T_pM^C$$

is defined as a complex linear map such that for any $1 \leq \alpha \leq n$,

$$(D_p f)\frac{\partial}{\partial z_\alpha} = \sum_{k=1}^{n} \frac{\partial f_k}{\partial z_\alpha}\frac{\partial}{\partial w_k}.$$

Since f is holomorphic, u_k and v_k satisfy the *Cauchy-Riemann condition*:

$$\frac{\partial u_k}{\partial x_\alpha} = \frac{\partial v_k}{\partial y_\alpha}, \quad \frac{\partial u_k}{\partial y_\alpha} = -\frac{\partial v_k}{\partial x_\alpha} \qquad (1 \leq \alpha, k \leq n). \tag{2.4}$$

By (2.4), it is easy to check that:

$$\psi_p^{-1}(D_p\tilde{f})\frac{\partial}{\partial x_\alpha} = (D_p f)\frac{\partial}{\partial z_\alpha}, \quad \psi_p^{-1}(D_p\tilde{f})\frac{\partial}{\partial y_\alpha} = i(D_p f)\frac{\partial}{\partial z_\alpha} \;\; (1 \leq \alpha \leq n).$$

Generally, for any $v = \sum_{k=1}^{n}(a_k + ib_k)\frac{\partial}{\partial z_k} \in T_pM^C$,

$$\begin{aligned}(D_p f)v &= (D_p f)\sum_{k=1}^{n}(a_k + ib_k)\frac{\partial}{\partial z_k} \\ &= \psi_p^{-1}(D_p\tilde{f})[\sum_{k=1}^{n}(a_k\frac{\partial}{\partial x_k} + b_k\frac{\partial}{\partial y_k}] \\ &= \psi_p^{-1}(D_p\tilde{f})\psi_p(v).\end{aligned} \tag{2.5}$$

This gives the following lemma:

Lemma 2.2.1. *For every* $p \in M, m \in \mathbf{N}$ *and* $v \in T_pM^C$,

$$\|(D_p f^m)v\|_H = \|(D_p\tilde{f}^m)\psi_p(v)\|.$$

In particular,

$$\|(D_p f^m)\frac{\partial}{\partial z_\alpha}\|_H = \|(D_p\tilde{f}^m)\frac{\partial}{\partial x_\alpha}\| = \|(D_p\tilde{f}^m)\frac{\partial}{\partial y_\alpha}\| \;\; (1 \leq \alpha \leq n).$$

Proof: From (2.3) and (2.5), we have $\|(D_p f)v\|_H = \|(D_p\tilde{f})\psi_p(v)\|$. If we replace f by f^m, then $\|(D_p f^m)v\|_H = \|(D_p\tilde{f}^m)\psi_p(v)\|$. Particularly,

$$\|(D_p f^m)\frac{\partial}{\partial z_\alpha}\|_H = \|(D_p\tilde{f}^m)\psi_p(\frac{\partial}{\partial z_\alpha})\| = \|(D_p\tilde{f}^m)\frac{\partial}{\partial x_\alpha}\|;$$

$$\|(D_pf^m)(i\frac{\partial}{\partial z_\alpha})\|_H = \|(D_p\tilde{f}^m)\psi_p(i\frac{\partial}{\partial z_\alpha})\| = \|(D_p\tilde{f}^m)\frac{\partial}{\partial y_\alpha}\|.$$

Since

$$\|(D_pf^m)\frac{\partial}{\partial z_\alpha}\|_H = \|(D_pf^m)(i\frac{\partial}{\partial z_\alpha})\|_H,$$

then we have

$$\|(D_pf^m)\frac{\partial}{\partial z_\alpha}\|_H = \|(D_p\tilde{f}^m)\frac{\partial}{\partial x_\alpha}\| = \|(D_p\tilde{f}^m)\frac{\partial}{\partial y_\alpha}\| \ \ (1 \leq \alpha \leq n).$$

We finish the proof of this lemma. □

2.2.2 Lyapunov exponents and regular points

We recall some definitions and notations from [P1]:

Definition 2.2.2. For any $p \in M^R, v \in T_pM^R$, the number

$$\chi^+(p,v) = \limsup_{m\to+\infty} \frac{1}{m} \log \|(D_p\tilde{f}^m)v\| \tag{2.6}$$

is called the *upper Lyapunov exponent* of the tangent vector v at point p.

It can be shown (see [Os]) that χ^+ is measurable and satisfies the following properties of the *characteristic exponents*: for any point $p \in M^R, v_1, v_2, v \in T_pM^R$,

1. $-\infty < \chi^+(p,v) < +\infty$, for $v \neq 0$, and $\chi^+(p,0) = -\infty$;
2. $\chi^+(p,av) = \chi^+(p,v)$ for any $a \in \mathrm{R} - \{0\}$;
3. $\chi^+(p,v_1+v_2) \leq \max\{\chi^+(p,v_1), \chi^+(p,v_2)\}$.

By using the above properties 1-3 of χ^+, it is easy to prove that the function χ^+ defined on the tangent bundle TM^R takes on at most $2n$ values other than $-\infty$ on each tangent vector space T_pM^R and generates a filtration

$$L_1(p) \subset L_2(p) \subset ... \subset L_{s(p)}(p) = T_pM^R \tag{2.7}$$

of every such space. Namely, there are real numbers

$$\lambda_1(p) < \lambda_2(p) < ... < \lambda_{s(p)}(p) \tag{2.8}$$

such that $\chi^+(p,v) = \lambda_j(p)$ for $v \in L_j(p) \setminus L_{j-1}(p)$. The numbers $\lambda_j(p)$ are called the *j-th upper Lyapunov exponent of $\tilde{f}$ at point* p and the number $k_j(p) = dimL_j(p) - dimL_{j-1}(p)$ is called *the multiplicity of the j-th exponent.*

Remark 2.2.3. (see [P2] §3)

(i) $\lambda_j(p)$ and $k_j(p)$ don't depend on the Riemannian metric;

(ii) $s(p), k_j(p)$, and $L_j(p)$ $(1 \le j \le s(p))$ depend measurably on p;

(iii) The upper Lyapunov exponent χ^+ is $\tilde{f}$-invariant, more precisely, for any $p \in M^R$,

$$\lambda_j(\tilde{f}(p)) = \lambda_j(p), k_j(\tilde{f}(p)) = k_j(p), (D_p\tilde{f})L_j(p) = L_j(\tilde{f}(p)).$$

In general, the limit of $\frac{1}{m}\log\|(D_p\tilde{f}^m)v\|$ may not exist. Even if the limit existed for all $v \in T_pM^R$, the asymptotic behavior of $D_p\tilde{f}^m$ may have a pathology as $m \to \infty$. Such a pathology is prevented by the condition of regularity ([P1,2]) which in particular guarantee the existence of the limit as $m \to \infty$ for any non-zero tangent vector $v \in T_pM^R$. The Oseledec multiplicative theorem ([Os]) implies that for any Borel probability $\tilde{f}$-invariant measure the set of regular points has full measure.

For completeness, we review the definitions of forward regularity, backward regularity and regularity. A point $p \in M^R$ is said to be *forward regular* if there exist real numbers $\lambda_1(p) < \cdots < \lambda_{s(p)}(p)$ and a decomposition of the tangent space at p into $T_pM^R = \mathrm{E}_1(p) \oplus \cdots \oplus \mathrm{E}_{s(p)}(p)$ such that for every non-zero tangent vector $v \in \mathrm{E}_j(p)$,

$$\lim_{m \to +\infty} \frac{1}{m} \log \|(D_p \tilde{f}^m)v\| = \lambda_j(p), \tag{2.9}$$

and

$$\lim_{m \to +\infty} \frac{1}{m} \log |Jac(D_p \tilde{f}^m)| = \sum_{j=1}^{s(p)} \lambda_j(p) \dim \mathrm{E}_j(p). \tag{2.10}$$

A point p is said to be *backward regular* if (2.9) and (2.10) hold for $m \to -\infty$. A point p is said to be *regular* if it is both forward regular and backward regular.

We will use the following property of forward (resp. backward) regular point which can be found in [P1]:

Proposition 2.2.4. *([P1]) Let p be a forward (resp. backward) regular point, then for each integer $m \in \mathbf{Z}$, the point $\tilde{f}^m(p)$ is forward (resp. backward) regular.*

2.2.3 Complex multiplicative ergodic theorem I

Definition 2.2.2 and Lemma 2.2.1 give the following lemma:

Lemma 2.2.5. *For any forward regular point $p \in M^R$, there are subspaces $W_j(p)$ $(j = 1, \ldots, s(p))$ of the tangent space T_pM^R such that*

(i) $L_j(p) = \oplus_{k=1}^{j} W_k(p)$ for $j = 1, \ldots, s(p)$;

(ii)
$$\lim_{m\to+\infty}\frac{1}{m}\log\|(D_p\tilde{f}^m)v\| = \lambda_j(p) \quad \text{uniformly in} \quad v\in W_j(p);$$
(iii) *for any* $1\le\alpha\le n$,
$$\lim_{m\to+\infty}\frac{1}{m}\log\|(D_p\tilde{f}^m)\frac{\partial}{\partial x_\alpha}\| = \lim_{m\to+\infty}\frac{1}{m}\log\|(D_p\tilde{f}^m)\frac{\partial}{\partial y_\alpha}\|.$$

Proof: (i) and (ii) follow from the formula (2.7) and (2.8). (iii) follows from Lemma 2.2.1. □

An easy consequence of this lemma is the following:

Corollary 2.2.6. *For any forward regular point* $p\in M^R$, *the dimension of* $W_j(p)$ *is even* ($j=1,...,s(p)$). *Moreover,* $\frac{\partial}{\partial x_\alpha}\in W_j(p)$ *if and only if* $\frac{\partial}{\partial y_\alpha}\in W_j(p)$.

Proof: By Lemma 2.2.5(iii), we have that
$$\frac{\partial}{\partial x_\alpha}\in W_j(p) \;\text{ if and only if }\; \frac{\partial}{\partial y_\alpha}\in W_j(p).$$
Therefore the real dimension of $W_j(p)$ is even. □

Since
$$T_pM^R = W_1(p)\oplus ...\oplus W_{s(p)}(p),$$
$$\begin{aligned}T_pM^C &= \psi_p^{-1}(T_pM^R) = \psi_p^{-1}(W_1(p)\oplus ...\oplus W_{s(p)}(p))\\ &= [\psi_p^{-1}(W_1(p))]\oplus ...\oplus[\psi_p^{-1}(W_{s(p)}(p))]\end{aligned}$$

Define
$$W_j^C(p) = \psi_p^{-1}(W_j(p)), \quad (\text{for } 1\le j\le s(p)),$$

then

$$T_pM^C = W_1^C(p) \oplus \ldots \oplus W_{s(p)}^C(p). \tag{2.11}$$

By Lemma 2.2.1, Lemma 2.2.5 and Corollary 2.2.6, we can rewrite Lemma 2.2.5 in the complex version:

Lemma 2.2.7. *For any forward regular point $p \in M^C$, there are subspaces $W_j^C(p)$ of the tangent space T_pM^C, such that*

(i)

$$T_pM^C = \oplus_{j=1}^{s(p)} W_j^C(p);$$

(ii)

$$\lim_{m\to+\infty} \frac{1}{m} \log \|(D_pf^m)v\|_H = \lambda_j(p) \quad \text{uniformly in} \quad v \in W_j^C(p).$$

Applying Oseledec multiplicative ergodic theorem (cf.[FHY] or [Os]), We have the following

Theorem 2.2.8. (complex multiplicative ergodic theorem I) *There exists a Borel Set B in the complex manifold M^C which has the following properties:*

(i) Every point of B is regular;

(ii) B is invariant under f and has measure 1 for every f-invariant probability Borel measure on M^C;

(iii) For every $p \in B$, there exists a splitting of the tangent space $T_pM^C = \oplus_{j=1}^{s(p)} W_j^C(p)$ and real numbers $\lambda_1(p) < \lambda_2(p) < \ldots < \lambda_{s(p)}$ such that, for any positive definite Hermitian metric $\|\cdot\|_H$ on M^C:

(a) $W_j^C(p), \lambda_j(p)$ and $s(p)$ are Borel measurable functions of p, moreover

$W_j^C(f(p)) = T_pf(W_j^C(p))$ *and* $\lambda_j(p), s(p)$ *are invariant under* f*;*

(b) $\forall v \in W_j^C(p)$, $v \neq 0$,

$$\lim_{m\to\pm\infty} \frac{1}{m} \log \|(D_pf^m)v\|_H = \lambda_j(p).$$

Proof: The only thing needed to be proved here is (iii), but this is an easy consequence of Lemma 2.2.7 and the regularity of p. □

2.3 Complex stable manifold theorem I

In this section, we will keep all the notations given in 2.2.

Let $[\lambda, \mu], \lambda < \mu < 0$, be a compact interval in the real line R. Denote

$$B_{\lambda,\mu} = \{p \in B : \lambda_j(p) \notin [\lambda, \mu], j = 1, ..., s(p)\}.$$

where B is the regular points set (see Theorem 2.2.8). Remark that $B_{\lambda,\mu}$ is invariant under f. For $p \in B_{\lambda,\mu}$, we define

$$E^s(p) = \oplus_{\lambda_j(p)<\lambda} W_j(p) \text{ and } E^u(p) = \oplus_{\mu<\lambda_j(p)} W_j(p).$$

We recall Pesin's local stable manifold theorem:

Theorem 2.3.1. ([P1][FHY][Ru1]) *Given $\epsilon > 0$ and $\rho \in]e^\lambda, e^\mu[$, then there exist:*
A) two Borel functions $\delta_\epsilon : B_{\lambda,\mu} \to]0, \infty[$ and $\gamma_\epsilon : B_{\lambda,\mu} \to [1, \infty[$;
B) $\forall p \in B_{\lambda,\mu}$, a Lipschitz map:

$$\phi_p : B^s(p, \delta_\epsilon(p)) = \{v \in E^s(p) | \|v\| \leq \delta_\epsilon(p)\} \longrightarrow E^u(p);$$

such that:
(i) $\widetilde{W}^s_{loc}(p) = exp_p[graph\phi_p]$ is a submanifold in M^R of class C^∞, where exp_p is the exponential map from T_pM to M;
(ii) $p \in \widetilde{W}^s_{loc}(p)$;
(iii) $T_p\widetilde{W}^s_{loc}(p) = E^s(p)$;
(iv) $\forall y, z \in \widetilde{W}^s_{loc}(p), \forall m \geq 0, d(\tilde{f}^m(y), \tilde{f}^m(z)) \leq \gamma_\epsilon(p)\rho^m d(y, z)$.

The following theorem is very useful in our case.

Theorem 2.3.2. ([P1, Prop. 2.3.1] or [KS, p.40]) *Let $p \in B$ and $q \in \widetilde{W}^s_{loc}(p)$, then q is forward regular, and*

$$s(q) = s(p), \lambda_j(q) = \lambda_j(p), k_j(q) = k_j(p), j = 1, ..., s(p).$$

Lemma 2.3.3 *Let $p \in B$ and $q \in \widetilde{W}^s_{loc}(p)$. Then*
(i) $\psi_q^{-1} T_q \widetilde{W}^s_{loc}(p)$ is a complex vector space;
(ii) $\widetilde{W}^s_{loc}(p)$ is actually a complex submanifold.

Proof: From Theorem 2.3.2, we know that q is a forward regular point. Corollary 2.2.6 tells us that $\frac{\partial}{\partial x_\alpha} \in T_q \widetilde{W}^s_{loc}(p)$ if and only if $\frac{\partial}{\partial y_\alpha} \in T_q \widetilde{W}^s_{loc}(p)$. This implies that $\psi^{-1} T_q \widetilde{W}^s_{loc}(p)$ is a complex subspace of $T_q w_C$. By proposition 2.1 in [Ok, p.371], we know that $\widetilde{W}^s_{loc}(p)$ is actually an immersed complex manifold. □

In the following of this section , we simply denote the complex submanifold $\widetilde{W}^s_{loc}(p)$ by $W^s_{loc}(p)$. Clearly, $T_p W^s_{loc}(p) = (E^C)^s(p) = \oplus_{\lambda_j(p)<\lambda} W_j^C(p)$.

Noting that $W^s_{loc}(p)$ is the complex submanifold of M^C, we can translate the global stable manifold theorem [FHY,theorem 17] into the following complex version:

Theorem 2.3.4. *(**complex stable manifold theorem I**) Let $p \in B$ (see Theorem 2.2.8), then the set*

$$W^s(p, f) = \{q \in M^C : \limsup_{m\to\infty} \frac{1}{m} \log d_H(f^m(p), f^m(q)) < 0\}$$

is an immersed complex manifold of dimension $dim(E^C)^s(p)$ and is also the image of a C^∞ injective immersion of a real Euclidean space of dimension $2dim(E^C)^s(p)$, *such that:*

(i) $dimW^s(p,f)$ $=dim(E^C)^s(p)$, *where* $(E^C)^s(p) = \oplus_{\lambda_j(p)<0} W_j^C(p)$;

(ii) $T_pW^s(p,f) = (E^C)^s(p)$;

(iii)

$$W^s(p,f) = \cup_{m\geq 0} f^{-m}(W^s_{loc}(f^m(p))),$$

where $\log \rho \in]\lambda_{j_0}(p), 0[$ *and* $\lambda_{j_0}(p) = \max\{\lambda_j(p) < 0\}$;

(iv)

$$W^s(p,f) = \{q \in M^C : \limsup_{m\to\infty} \frac{1}{m} \log d_H(f^m(p), f^m(q)) \leq \lambda_{j_0}(p)\}.$$

Proof: Using Lemma 2.3.3, we can get the complex version of Theorem 2.3.1. Then the proof of statements (i),(ii),(iii),(iv) and that $W^s(p,f)$ is the injective immersion of a real Euclidean space of dimension $2dim(E^C)^s(p)$ is same as the one of Theorem 17 in [FHY]. What we need proving now is that $W^s(p,f)$ is an immersed complex manifold. By Theorem 2.3.2 and Proposition 2.2.4, it is easy to see that every point q of $W^s(p,f)$ is forward regular. Then by a same argument as in the proof of Lemma 2.3.3, we can prove that $W^s(p,f)$ is actually an immersed complex manifold. □

Corollary 2.3.5. *Let* $p \in B$ *and* $q \in W^s(p,f)$, *then* q *is forward regular, and*

$$s(q) = s(p), \lambda_j(q) = \lambda_j(p), k_j(q) = k_j(p) \text{ for } j = 1, ..., s(p).$$

Proof: This corollary is an easy consequence of Proposition 2.2.4, Theorem 2.3.2 and Theorem 2.3.4(iii). □

Chapter 3

Complex Pesin stable manifold Theorem II: the $\mathbb{C}^n$ case

3.1 Introduction

In this chapter, we study the dynamical properties of the holomorphic diffeomorphisms f on $\mathbb{C}^n$ with an f-invariant compact subset K of $\mathbb{C}^n$. Of course, $\mathbb{C}^n$ and f may be regarded as $\mathbb{R}^{2n}$ and the real analytic diffeomorphism $\tilde{f}$ on $\mathbb{R}^{2n}$ respectively, if we forget the complex structure of $\mathbb{C}^n$ as we did in Chapter 2. In Section 3.2, we first fix some notations, then recall the Oseledec multiplicative ergodic theorem and the nonlinear ergodic theorem from [Ru1]. In Section 3.3, we make use of the ergodic theorems of Section 3.2 and Cauchy-Riemann condition to prove the complex multiplicative ergodic theorem(Theorem 3.3.3), the local complex stable manifold theorem(Theorem 3.3.4), and the global complex stable manifold theorem(Theorem 3.3.5).

3.2 Ergodic theorem for measurable maps

Let $\mathbb{C}^n$ be the n-dimensional complex Euclidean space, and R^n the n-dimensional real Euclidean space. For the point of $\mathbb{C}^n$ we shall use the notation $z = (z_1, ..., z_n)$, where $z_j = x_j + iy_j \in \mathbb{C}$ and x_j, y_j are real numbers. The absolute value of a complex number z_j will be denoted by $|z_j|$, and for $z \in \mathbb{C}^n$, we define

$$|z|_C = \sqrt{|z_1|^2 + ... + |z_n|^2}.$$

Define a real linear invertible map $\psi : \mathbb{C}^n \to \mathrm{R}^{2n}$ by

$$\psi(z_1, ..., z_n) = (x_1, ..., x_n, y_1, ..., y_n),$$

where $z_j = x_j + iy_j$ for $1 \le j \le n$. For any point $(x, y) = (x_1, ..., x_n, y_1, ..., y_n) \in \mathrm{R}^{2n}$, we define the norm of (x, y) by

$$|(x, y)|_R = \sqrt{|x_1|^2 + ... + |x_n|^2 + |y_1|^2 + ... + |y_n|^2}.$$

Obviously, we have

$$|z|_C = |\psi(z)|_R \qquad \text{for all} \quad z \in \mathbb{C}^n. \tag{3.1}$$

We recall a general version of the multiplicative ergodic theorem from [Ru1]:

Theorem 3.2.1: *Let $(M, \mathcal{B}, \rho)$ be a Borel probability space, $\tau : M \to M$ a Borel measurable map with Borel measurable inverse preserving ρ, and $T : M \to GL(\mathrm{R}^m)$ a Borel measurable map to the invertible real $m \times m$ matrices, such that*

$$\log^+ \|T(\cdot)\|, \log^+ \|T^{-1}(\cdot)\| \in L^1(M, \rho),$$

where $\log^+ a = \max\{0, \log a\}$.

Write:

$$T_x^k = T(\tau^{k-1}x)\cdot \cdots \cdot T(\tau x)\cdot T(x) \tag{3.2}$$

$$T_x^{-k} = T^{-1}(\tau^{-k}x)\cdot \cdots \cdot T^{-1}(\tau^{-2}x)\cdot T^{-1}(\tau^{-1}x). \tag{3.3}$$

Then there is $\Gamma \in \mathcal{B}$ *such that* $\tau\Gamma = \Gamma, \rho(\Gamma) = 1$, *and for each* $x \in \Gamma$, *there is a direct sum decomposition of* R^m *into linear subspaces*

$$\mathrm{R}^m = \mathrm{E}^{(1)}(x) \oplus \cdots \oplus \mathrm{E}^{(s(x))}(x)$$

and real numbers $\lambda^{(1)}(x) < \cdots < \lambda^{(s(x))}(x)$, *where* $\lambda^{(1)}(x)$ *is never* $-\infty$, *such that*

(a) $s(x), \lambda^{(j)}(x)$, *and* $m^{(j)}(x) = \dim \mathrm{E}^{(j)}(x)$ *are* τ*-invariant and measurable;*

(b)

$$\lim_{k\to\pm\infty} \frac{1}{k}\log|T_x^k u|_R = \lambda^{(j)}(x) \quad \text{if} \quad 0 \neq u \in \mathrm{E}^{(j)}(x).$$

(c) $T(x)\mathrm{E}^{(j)}(x) = \mathrm{E}^{(j)}(\tau x)$ *for* $1 \le j \le s(x)$*;*

(d) If $u \in \mathrm{R}^m$, *then the limit*

$$\lim_{k\to\pm\infty} \frac{1}{k}\log|T_x^k u|_R = \chi(x,u)$$

exists and is finite. If $\lambda \in \mathrm{R}$, *the linear subspace*

$$V_x^\lambda = \{u \in \mathrm{R}^m : \chi(x,u) \le \lambda\}$$

is a measurable function of $x \in \Gamma$.

Proof: See [Ru1]. □

If $S \in GL(\mathbb{C}^n)$, we define $T = \psi \circ S \circ \psi^{-1}$. T_x^k and S_x^k for $k \in Z$ are defined as in (3.2)-(3.3). Then for any $u \in \mathrm{R}^{2n}$,

$$|T_x^k u|_R = |\psi \circ S_x^k \circ \psi^{-1}(u)|_R = |S_x^k \circ \psi^{-1}(u)|_C.$$

Therefore we have the following remark:

Remark 3.2.2 (1) If $S \in GL(\mathbb{C}^n)$, then Theorem 3.2.1 is true provided that we replace T by S, T_x^k by S_x^k, R^m by $\mathbb{C}^n$, $\mathrm{E}^{(j)}(x)$ by $\mathrm{E}_C^{(j)}(x)$, and $|\cdot|_R$ by $|\cdot|_C$, where $\mathrm{E}_C^{(j)}(x) = \psi^{-1}(\mathrm{E}^{(j)}(x))$ for $j = 1, ..., s(x)$.
(2) Every $\mathrm{E}_C^{(j)}(x)$ is a complex linear subspace of $\mathbb{C}^n$. The reason is that

$$|S_x^k(z)|_C = |S_x^k(iz)|_C \quad \text{for} \quad z \in \mathbb{C}^n.$$

Definition 3.2.3: The set Γ in Theorem 3.2.1 is called *the regular set of* τ, $\lambda^{(j)}(x)$ $(1 \leq j \leq s(x))$ are called the *Lypunov exponents of* τ *at* x, and $m^{(j)}(x)$ is called the *multiplicity of* $\lambda^{(j)}(x)$.

In what follows we will denote by $B(\alpha)$ the open ball of radius α centered at 0, by $\overline{B}(\alpha)$ its closure, and by $H(\overline{B}(1), 0; \mathbb{C}^n, 0)$ the space of diffeomorphisms holomorphic in $B(1)$ and continuous on $\overline{B}(1)$.

We recall a nonlinear ergodic theorem from [Ru1] and we will use the notations introduced in Remark 3.2.2:

Theorem 3.2.4: *Let* $(M, \mathcal{B}, \rho)$ *be a probability space and* $\tau : M \to M$ *a measurable map preserving* ρ. *Let* $x \to F_x$ *map* M *to* $H(\overline{B}(1), 0; \mathbb{C}^n, 0)$. *We write*

$$F_x^k = F_{\tau^{k-1}x} \circ \cdots \circ F_{\tau x} \circ F_x$$

and denote by $S(x)$ *the derivative of* F_x *at* 0. *We assume that* $x \to S(x), \|F_x\|_1$ *are measurable and that*

$$\int_M \log^+ \|F_x\|_1 \rho(dx) < +\infty. \tag{3.4}$$

Let $[a,b], a < b$, be a compact interval in R. *Denote by $\Gamma_{a,b}$ the subset of Γ which consists of the points x such that $\lambda^{(j)}(x) \notin [a,b]$ for $1 \le j \le s(x)$. Clearly, $\Gamma_{a,b}$ is invariant under τ.*

There are then measurable functions $\beta > \alpha > 0, \gamma > 1$ on $\Gamma_{a,b}$ with the following properties:

(a) If $x \in \Gamma_{a,b}, \lambda \in]a,b[$, the set

$$\nu_x^\lambda = \{u \in B(\alpha(x)) : \|F_x^k u\| \le \beta(x) e^{k\lambda} \text{ for all } k \ge 0\}$$

is a complex submanifold of $B(\alpha(x))$, tangent at 0 to the complex linear subspace $V_x^\lambda = \{u \in \mathbb{C}^n : \chi(x,u) \le \lambda\}$;

(b) If $u, v \in \nu_x^\lambda$, then

$$\|F_x^k u - F_x^k v\| \le \gamma(x) \|u - v\| e^{k\lambda}.$$

If ρ is ergodic, there exists $\gamma' \ge \gamma$ measurable on $\Gamma_{a,b}$ with the property:

(b') If $u, v \in \nu_x^\lambda$, then

$$\|F_x^k u - F_x^k v\| \le \gamma'(x) \|u - v\| e^{ka}.$$

Proof: See [Ru1]. □

Although in reference [Ru1] it was not mentioned that $E^{(j)}(x)$ as actually complex subspaces, this in fact is true by Remark 3.2.2. It is important that the proof of above theorem in [Ru1] needs this fact. This is why we gave the Remark 3.2.2 before Theorem 3.2.4.

3.3 Ergodic theorem and stable manifold theorem for holomorphic diffeomorphisms of $\mathbb{C}^n$

3.3.1 Notations

Let $f : \mathbb{C}^n \to \mathbb{C}^n$ be a holomorphic diffeomorphism described by a n-tuple $f(z) = (f_1(z), ..., f_n(z))$. If we decompose z_j and f_j into their real and imaginary parts, i.e., $z_j = x_j + iy_j$ and $f_j = u_j + iv_j$ $(1 \leq j \leq n)$, we can express f as

$$f_j(z_1, ..., z_n) = u_j(x_1, ..., x_n, y_1, ..., y_n) + iv_j(x_1, ..., x_n, y_1, ..., y_n)$$

for $1 \leq j \leq n$.

Throughout this section we denote

$$\tilde{f}(q) = \psi \circ f \circ \psi^{-1}(q) = (u_1(q), ..., u_n(q), v_1(q), ..., v_n(q)),$$

for $q = (x_1, ..., x_n, y_1, ..., y_n) \in \mathrm{R}^{2n}$.

Let $p \in \mathbb{C}^n$, we denote the Jacobian matrix of f at point p by

$$J_p(f) = \begin{pmatrix} \frac{\partial f_1}{\partial z_1}(p) & \cdots & \frac{\partial f_1}{\partial z_n}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial z_1}(p) & \cdots & \frac{\partial f_n}{\partial z_n}(p) \end{pmatrix}.$$

Similarly, we denote the Jacobian matrix of $\tilde{f}$ at point $q \in \mathrm{R}^{2n}$ by

$$J_q\tilde{f} = \begin{pmatrix} \frac{\partial u_1}{\partial x_1}(q) & \cdots & \frac{\partial u_1}{\partial x_n}(q) & \frac{\partial u_1}{\partial y_1}(q) & \cdots & \frac{\partial u_1}{\partial y_n}(q) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial u_n}{\partial x_1}(q) & \cdots & \frac{\partial u_n}{\partial x_n}(q) & \frac{\partial u_n}{\partial y_1}(q) & \cdots & \frac{\partial u_n}{\partial y_n}(q) \\ \frac{\partial v_1}{\partial x_1}(q) & \cdots & \frac{\partial v_1}{\partial x_n}(q) & \frac{\partial v_1}{\partial y_1}(q) & \cdots & \frac{\partial v_1}{\partial y_n}(q) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial v_n}{\partial x_1}(q) & \cdots & \frac{\partial v_n}{\partial x_n}(q) & \frac{\partial v_n}{\partial y_1}(q) & \cdots & \frac{\partial v_n}{\partial y_n}(q) \end{pmatrix}.$$

Then the complex tangent map of f at point $p \in \mathbb{C}^n$,

$$D_p f : \mathbb{C}^n \to \mathbb{C}^n,$$

is defined by

$$(D_p f)z = J_p z^T,$$

where A^T stands for the transposed matrix of matrix A.

The real tangent map of $\tilde{f}$ at point $q \in \mathrm{R}^{2n}$,

$$D_q \tilde{f} : \mathrm{R}^{2n} \to \mathrm{R}^{2n},$$

is defined by

$$(D_q \tilde{f})(x,y) = (J_q \tilde{f})(x,y)^T.$$

As usual, we introduce the first-order linear partial differential operator

$$\frac{\partial}{\partial z_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}), \qquad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}).$$

Since f is holomorphic, u_k and v_k satisfy the Cauchy-Riemann condition:

$$\frac{\partial u_k}{\partial x_j} = \frac{\partial v_k}{\partial y_j}, \ \frac{\partial u_k}{\partial y_j} = -\frac{\partial v_k}{\partial x_j} \qquad (1 \le j,k \le n). \tag{3.5}$$

By Cauchy-Riemann condition, it is easy to check that for any given point $p \in \mathbb{C}^n, q = \psi(p) \in \mathrm{R}^{2n}$,

$$\psi((D_p f)z) = (D_q \tilde{f})(\psi(z)) \quad \text{for all} \quad z \in \mathbb{C}^n.$$

This implies that

$$|(D_p f)z|_C = |(D_q \tilde{f})(\psi(z))|_R \quad \text{for all} \quad z \in \mathbb{C}^n. \tag{3.6}$$

or

$$|(D_p f)\psi^{-1}(x,y)|_C = |(D_q \tilde{f})(x,y)|_R \quad \text{for all} \quad x,y \in \mathrm{R}^n. \tag{3.7}$$

This gives that

$$|(D_pf)(iz)|_C = |(D_pf)z|_C = |(D_q\tilde{f})(\psi(z))|_R \quad \text{for all} \quad z \in \mathbb{C}^n. \tag{3.8}$$

Since f^k is also a holomorphic diffeomorphism,

$$|(D_pf^k)(iz)|_C = |(D_pf^k)z|_C = |(D_q\widetilde{f^k})(\psi(z))|_R \quad \text{for all} \quad z \in \mathbb{C}^n.$$

But $\widetilde{(f^k)} = \psi \circ f^k \circ \psi^{-1} = (\psi \circ f \circ \psi^{-1})^k = (\tilde{f})^k$, we have the following lemma:

Lemma 3.3.1 *Given any point $p \in \mathbb{C}^n$, then for all $k \in \mathbb{N}$ and all $z \in \mathbb{C}^n$,*

$$|(D_pf^k)(iz)|_C = |(D_pf^k)z|_C = |(D_q(\tilde{f})^k)(\psi(z))|_R \quad \text{for all} \quad z \in \mathbb{C}^n. \tag{3.9}$$

Let $K \subset \mathbb{C}^n$ be an f-invariant compact subset. For any f-invariant Borel probability measure μ_K on K, we define

$$\mu(A) = \mu_K(A \cap K)$$

for any Borel set A in $\mathbb{C}^n$. Clearly μ is an f-invariant Borel probability measure on $\mathbb{C}^n$.

If we define $\Sigma = \psi(K), \rho(B) = \mu(\psi(B))$ for any Borel set B in $\mathbb{R}^{2n}$, then $\Sigma \subset \mathbb{R}^{2n}$ is an $\tilde{f}$-invariant compact subset, and ρ is an $\tilde{f}$-invariant Borel probability measure on $\mathbb{R}^{2n}$ with $\rho(\Sigma) = 1$.

3.3.2 Complex multiplicative ergodic theorem II

Applying Theorem 3.2.1 with $M = \mathbb{R}^{2n}, \tau = \tilde{f}, T(q) = J_q\tilde{f}$, we have the following version of the multiplicative ergodic theorem for $\tilde{f}$. Note that in this case, the conditions of Theorem 3.2.1 are all automatically satisfied.

Theorem 3.3.2: *Then there is a Borel set* Γ *in* R^{2n} *such that* $\tilde{f}\Gamma = \Gamma, \rho(\Gamma) = 1$, *and for each* $q \in \Gamma$, *there is a direct sum decomposition of* R^{2n} *into linear subspaces*

$$\mathrm{R}^{2n} = \mathrm{E}^{(1)}(q) \oplus \cdots \oplus \mathrm{E}^{(s(q))}(q)$$

and real numbers $\lambda^{(1)}(q) < \cdots < \lambda^{(s(q))}(q)$, *where* $\lambda^{(1)}(q)$ *is never* $-\infty$, *such that*

(a) $s(q), \lambda^{(j)}(q)$, *and* $m^{(j)}(q) = \dim \mathrm{E}^{(j)}(q)$ *are* $\tilde{f}$*-invariant and Borel measurable;*

(b)

$$\lim_{k\to\pm\infty} \frac{1}{k} \log |D_q \tilde{f}^k u|_R = \lambda^{(j)}(q) \quad \text{if} \quad 0 \neq u \in \mathrm{E}^{(j)}(q).$$

(c) $(D_q\tilde{f})\mathrm{E}^{(j)}(q) = \mathrm{E}^{(j)}(\tilde{f}(q))$ *for* $1 \leq j \leq m$.

For any $p \in \mathbb{C}^n, q = \psi(p)$, define $t(p) = s(q)$ and $\mathrm{E}_C^{(j)}(p) = \psi^{-1}\mathrm{E}^{(j)}(q)$ for $1 \leq j \leq s(q)$. Then we have

$$\mathbb{C}^n = \psi^{-1}(\mathrm{R}^{2n}) = \mathrm{E}_C^{(1)}(p) \oplus \cdots \oplus \mathrm{E}_C^{(t(p))}(p).$$

By (3.9), for any $p \in \mathbb{C}^n, q = \psi(p)$, if $0 \neq \psi(u) \in \mathrm{E}^{(j)}(q)$, i.e. $0 \neq u \in \mathrm{E}_C^{(j)}(p)$, we have

$$\lim_{k\to\pm\infty} \frac{1}{k} \log |(D_p f^k)(iu)|_C = \lim_{k\to\pm\infty} \frac{1}{k} \log |(D_p f^k)(u)|_C = \lambda^{(j)}(q), \tag{3.10}$$

This gives that

$$z \in \mathrm{E}_C^{(j)}(p) \ \text{ iff } \ iz \in \mathrm{E}_C^{(j)}(p).$$

or, equivalent

$$\mathrm{E}_C^{(j)}(p) \text{ is a complex linear subspace of } \mathbb{C}^n. \tag{3.11}$$

Hence we obtain the following complex multiplicative ergodic theorem:

Theorem 3.3.3 (complex multiplicative ergodic theorem II): *If we denote $\Lambda = \psi^{-1}(\Gamma)$, then $f(\Lambda) = \Lambda, \mu(\Lambda) = 1$, and for each $p \in \Lambda$, there is a direct sum decomposition of $\mathbb{C}^n$ into complex linear subspaces*

$$\mathbb{C}^n = \mathrm{E}_C^{(1)}(p) \oplus \cdots \oplus \mathrm{E}_C^{(t(p))}(p),$$

such that

(a) $t(p), \lambda^{(j)}(p)$, *and* $m^{(j)}(p) = \dim \mathrm{E}_C^{(j)}(p)$ *are* f*-invariant and Borel measurable;*

(b)

$$\lim_{k \to \pm\infty} \frac{1}{k} \log |D_p f^k u|_C = \lambda^{(j)}(p) \quad \text{if} \quad 0 \neq u \in \mathrm{E}_C^{(j)}(p).$$

(c) $(D_p f)\mathrm{E}_C^{(j)}(p) = \mathrm{E}^{(j)}(f(p))$ *for* $1 \leq j \leq m$.

As usual, we call set Λ in Theorem 3.3.3 *the regular set of* f, the points in set Λ *the regular points of* f.

3.3.3 Complex stable manifold theorem II

If we assume in Theorem 3.2.4 that $M = \mathbb{C}^n, \tau = f, F_p = \psi_{f(p)}^{-1} \circ f \circ \psi_p$, where the map $\psi_p : \mathbb{C}^n \to \mathbb{C}^n$ is defined by $\psi_p(z) = z + p$, then the conditions of Theorem 3.2.4 are all satisfied and hence we have the following theorem for f:

Theorem 3.3.4: *Let f, K, μ and Λ as before, let $[a, b], a < b$, be a compact interval in* R, *and let $\Lambda_{a,b}$ be the subset of Λ which consists of the points p such that $\lambda^{(j)}(p) \notin [a, b]$ for $j = 1, \cdots, s(p)$. Then there exist Borel measurable functions $\beta > \alpha > 0, \gamma >$*

1 *on* $\Lambda_{a,b}$ *with the following properties:*

(a) If $p \in \Lambda_{a,b}, \lambda \in]a,b[$*, the set*

$$W^{\lambda}_{loc}(p,f) = \{z \in B(p,\alpha(p)) : |f^k(z) - f^k(p)| \leq \beta(p)e^{k\lambda} \text{ for all } k \geq 0\}$$

is a complex submanifold of $B(p,\alpha(p))$*, tangent at* 0 *to* $V^{\lambda}_p = \{u \in \mathbb{C}^n : \chi(p,u) \leq \lambda\}$*;*

(b) If $z, z' \in W^{\lambda}_{loc}(p,f)$*, then*

$$|f^k(z) - f^k(z')| \leq \gamma(p)|z - z'|e^{k\lambda}.$$

If μ *is ergodic, there exists* $\gamma' \geq \gamma$ *Borel on* $\Lambda_{a,b}$ *with the property:*

(b') If $z, z' \in W^{\lambda}_{loc}(p,f)$*, then*

$$|f^k(z) - f^k(z')| \leq \gamma'(p)|z - z'|e^{ka}.$$

Proof: Assume that in Theorem 3.2.4 that $M = \mathbb{C}^n, F_p = \psi^{-1}_{f(p)} \circ f \circ \psi_p$ where $\psi_p : \mathbb{C}^n \to \mathbb{C}^n$ is defined by $\psi_p(z) = z + p$, then

$$f_p(0) = \psi^{-1}_{f(p)} \circ f \circ \psi_p(0) = f(p) - f(p) = 0.$$

Clearly $f_p \in H(\overline{B}(1), 0; \mathbb{C}^n, 0)$, and

$$S(p) = DF_p(0) = D(\psi^{-1}_{f(p)} \circ f \circ \psi_p(z))|_{z=0} = D(f(z+p) - f(p))|_{z=0} = f'(p).$$

So $S(p)$ is holomorphic (of cause measurable) and hence $\|F_p\|_1$ is continuous (of cause measurable) and bounded on the compact subset Λ. Thus

$$\int_{\mathbb{C}^n} \log^+ \|f_p\|_1 \mu(dp) = \int_{\Lambda} \log^+ \|f_p\|_1 \mu(dp) < +\infty.$$

We have now verified that all the conditions in Theorem 3.2.4 are satisfied. Note that

$$\begin{aligned}
F_p^k(z) &= F_{f^{k-1}(0)} \circ \cdots \circ F_{f(p)} \circ F_p(z) \\
&= (\psi^{-1}_{f^k(p)} \circ f \circ \psi_{f^{k-1}(p)}) \circ \cdots \circ (\psi^{-1}_{f^2(p)} \circ f \circ \psi_{f(p)}) \circ (\psi^{-1}_{f(p)} \circ f \circ \psi_p)(z) \\
&= \psi^{-1}_{f^k(p)} \circ f^k \circ \psi_p(z) \\
&= \psi^{-1}_{f^k(p)}(f^k(z+p)) \\
&= f^k(z+p) - f^k(p).
\end{aligned}$$

Then we apply Theorem 3.2.4, it is easy to check that this theorem is true. □

Note that $W^s_{loc}(p,f)$ is a complex submanifold of $\mathbb{C}^n$, we can easily globalize the complex local stable manifold $W^s_{loc}(p,f)$ to the global stable manifold $W^s(p,f)$ for $f : \mathbb{C}^n \to \mathbb{C}^n$ in a usual way (cf. [FHY,theorem 17]). In fact, we could globalize the local stable manifold by analytic continuation. The following theorem is a complex version of global stable manifold theorem:

Theorem 3.3.5(complex stable manifold theorem II) *Let $p \in \Lambda_{a,b}$ (see Theorem 3.3.4), then the set*

$$W^s(p,f) = \{z \in \mathbb{C}^n : \limsup_{k\to\infty} \frac{1}{k} \log |f^k(z) - f^k(p)| < 0\}$$

is an immersed complex manifold of dimension $\dim V_p^s$ and is also the image of a C^∞ injective immersion of a real Euclidean space of dimension $2\dim V_p^s$, such that:
(i) $\dim W^s(p,f) = \dim V_p^s$, *where* $V_p^s = \oplus_{\lambda_j(p)<0} \mathrm{E}_C^{(j)}(p)$;
(ii)The tangent space of $W^s(p,f)$ at p $T_pW^s(p,f) = V_p^s$;
(iii) $W^s(p,f) = \cup_{k\geq 0} f^{-k}(W^s_{loc}(f^n(p),f))$, *where λ related to $W^s_{loc}(f^n(p),f)$ (see Theorem 3.3.4) is in belongs to the open interval $]\lambda^{(j_0)},0[$, here* $\lambda^{(j_0)}(p) = \max\{\lambda^{(j)}(p) <$

0};

(iv) $W^s(p,f) = \{z \in \mathbb{C}^n : \limsup_{k\to\infty} \frac{1}{k} \log |f^k(z) - f^k(p)| \leq \lambda_{j_0}(p)\}$.

Proof: The proof of the statements (i),(ii),(iii),(iv) and the statement that $W^s(p,f)$ is the injective immersion of an real Euclidean space of dimension $2\dim V_p^s$ is same as the one of Theorem 17 in [FHY]. What we need to prove now is that $W^s(p,f)$ is an immersed complex manifold. But this is a easy consequence of Theorem 2.4, (iii) and the statement that $W^s(p,f)$ is the injective immersion of an real Euclidean space. □

Chapter 4

Stable manifolds of Aut($\mathbb{C}^n$) at hyperbolic fixed points

4.1 Introduction

In this chapter, we study the stable manifolds of automorphisms F of $\mathbb{C}^n$ at hyperbolic fixed points. If $F(0) = 0$, we denote $\lambda_1, \cdots, \lambda_n$ the all eigenvalues of the tangent map $DF(0)$. If $\lambda_1, \cdots, \lambda_n$ satisfy the condition $(*)$ (see Definition 4.2.1 in section 4.2), we show that the stable manifold at point 0

$$W_s(0, F) = \{z \in \mathbb{C}^n : \lim_{l \to \infty} F^l(z) = 0\},$$

is an injectively immersed complex submanifold biholomorphically equivalent to $\mathbb{C}^{n-k}$, where $F^l = F \circ F^{l-1}, F^1 = F$ and $n - k$ is the cardinal number of the eigenvalues of $DF(0)$ whose absolute values are less than 1.

Our result can be considered as an extension of the following classical result related to the Fatou-Bieberbach domain.

Theorem 4.1.1(cf. [RR] for the proof) *Suppose that F is an automorphism of $\mathbb{C}^n$ and $F(0) = 0$. If all the eigenvalues $\lambda_1, \cdots, \lambda_n$ of $DF(0)$ satisfy $|\lambda_j| < 1$, then the stable manifold $W_s(0, F)$ is biholomorphically equivalent to $\mathbb{C}^n$.*

Note that in Theorem 4.1.1, the stable manifold is complex n-dimensional and hence is actually a submanifold of $\mathbb{C}^n$. If the hyperbolic fixed point is not a sink, the stable manifold in general is only an **immersed** submanifold of $\mathbb{C}^n$ according to the stable manifold theory.

4.2 Stable manifold theorem of automorphisms of $\mathbb{C}^n$

4.2.1 The statement of the main theorem

A map F of $\mathbb{C}^n$ is said to be an automorphism of $\mathbb{C}^n$ if it is a biholomorphic map from $\mathbb{C}^n$ onto $\mathbb{C}^n$. The set of all automorphisms of $\mathbb{C}^n$ forms a group under composition, denoted by $Aut(\mathbb{C}^n)$. If $F \in Aut(\mathbb{C}^n)$, we write $DF(z)$ for the complex derivative or tangent map of F at z and write $\lambda_1(z), \cdots, \lambda_n(z)$ for all eigenvalues of $DF(z)$.

Definition 4.2.1 If $F \in Aut(\mathbb{C}^n)$ and $F(0) = 0$, let $\lambda_1, \cdots, \lambda_n$ be the eigenvalues of $DF(0)$. We will denote by l the vector $(l_1, \cdots, l_n)$ with nonnegative integer components, λ^l will denote the product $\lambda_1^{l_1}\cdots\lambda_n^{l_n}$, and $|l|$ will denote the $l_1+\cdots+l_n$. We say that $\lambda_1,\cdots,\lambda_n$ satisfy the *condition* $(*)$ if there exist constants $C>0$ and $\nu>0$ such that

$$|\lambda_s-\lambda^l|\geq C|l|^{-\nu}$$

for any $1\leq s\leq n$, and for any l with $|l|\geq 2$.

For an automorphism F of $\mathbb{C}^n$, if $F(p)=p$ and all the eigenvalues $\lambda_1(p),\cdots,\lambda_n(p)$ of $DF(p)$ satisfy $|\lambda_i(p)|<1$, Rosay and Rudin[RR] proved that the stable manifold of F at point p

$$W_s(p)=\{z\in\mathbb{C}^n:\lim_{l\to\infty}F^l(z)=p\}$$

is biholomorphically equivalent to $\mathbb{C}^n$, that is, there exists a biholomorphic map from $W_s(p)$ onto $\mathbb{C}^n$.

In the next section, we will make use of Siegel linearization theorem to prove the following theorem .

Theorem 4.2.2 *Suppose that $F\in Aut(\mathbb{C}^n)$, the point 0 is a hyperbolic fixed point of F, and that all eigenvalues $\lambda_1,\cdots,\lambda_n$ of $DF(0)$ satisfy condition $(*)$, then the stable manifold*

$$W_s(0,F)=\{z\in\mathbb{C}^n:\lim_{l\to\infty}F^l(z)=0\}$$

is an injectively immersed complex submanifold and is biholomorphically equivalent to $\mathbb{C}^{n-k}$, where $F^l=F\circ F^{l-1},F^1=F$ and $n-k$ is the cardinal number of the eigenvalues λ_i with $|\lambda_i|<1$.

Some similar results were obtained by Bedford and Smillie[BS1] and Wu[Wu2] when F is the so-called "generalized Hénon" mapping of $\mathbb{C}^2$. But the techniques are completely different from here.

4.2.2 The proof of Theorem 4.2.2

For proving the theorem, we begin to study the lower triangular holomorphic maps (see definition below). Suppose that the holomorphic map $G = (g_1, \cdots, g_n) : \mathbb{C}^n \to \mathbb{C}^n$ has the form

$$\begin{cases} g_1(z) &= \lambda_1 z_1, \\ g_2(z) &= \lambda_2 z_2 + h_2(z_1), \\ \cdots, & \\ g_k(z) &= \lambda_k z_k + h_k(z_1, \cdots, z_{k-1}), \\ g_{k+1}(z) &= \lambda_{k+1} z_{k+1} + h_{k+1}(z_1, \cdots, z_k), \\ \cdots, & \\ g_n(z) &= \lambda_n z_n + h_n(z_1, \cdots, z_{n-1}). \end{cases} \tag{4.1}$$

where $\lambda_i \in \mathbb{C}$, h_i is a holomorphic function of $(z_1, \cdots, z_{i-1})$, $h_i(0) = 0$ for $i = 1, \cdots, n$. We call such maps *lower triangular.*

It is clear that the matrix representation of $DG(0)$ is lower triangular, that $DG(0)$ is invertible if and only if $\lambda_i \neq 0$ for all i, and that $G \in Aut(\mathbb{C}^n)$ if and only if $\lambda_i \neq 0$ for all i.

Lemma 4.2.3 *Let $G = (g_1, \cdots, g_n)$ be a lower triangular automorphism of $\mathbb{C}^n$ and have the form of (4.1). If $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_k| > 1 > |\lambda_{k+1}| \geq \cdots \geq |\lambda_n| > 0$ for some positive integer k with $1 < k < n$, we set*

$$\mathbb{C}^n = E_1 \oplus E_2$$

where

$$E_1 = \{z = (z_1, \cdots, z_n) \in \mathbb{C}^n : z_{k+1} = \cdots = z_n = 0\},$$

$$E_2 = \{z = (z_1, \cdots, z_n) \in \mathbb{C}^n : z_1 = \cdots = z_k = 0\}.$$

Then $G^l(z) = (g_1^{(l)}(z), \cdots, g_n^{(l)}(z)) \to 0$ *as* l *tends to infinity, uniformly on compact subsets of* E_2*, and*

$$\bigcup_{l=1}^{\infty} G^{-l}(V) = E_2$$

for every neighborhood V *of 0 in* E_2*, i.e.,* $W_s(0, G) = E_2$.

Proof: Let $E \subset E_2$ be compact, then for each $z \in E$, $g_j^{(l)}(z) = 0$ for $j = 1, \cdots, k$ and all positive integers l. In this case, we can rewrite (4.1) as following

$$\begin{cases} g_j(z) &= 0 \quad j = 1, 2, \cdots, k, \\ g_{k+1}(z) &= \lambda_{k+1} z_{k+1}, \\ g_{k+2}(z) &= \lambda_{k+2} z_{k+2} + \tilde{h}_{k+2}(z_{k+1}), \\ \cdots, & \\ g_n(z) &= \lambda_n z_n + \tilde{h}_n(z_{k+1}, \cdots, z_{n-1}). \end{cases} \tag{4.2}$$

where $\tilde{h}_j(z_{k+1}, \cdots, z_{j-1}) = h_j(0, \cdots, 0, z_{k+1}, \cdots, z_{j-1})$ for $(k+2 \le j \le n)$.

We define $\|\cdot\|_E$ to be the sup-norm over E. Since $g_{k+1}^{(l)}(z) = \lambda_{k+1}^l z_{k+1}$ and $|\lambda_{k+1}| < 1$, it follows that $\|g_{k+1}^{(l)}\|_E \to 0$ as $l \to \infty$. Assume now that $k+1 < i \le n$ and that

$$\lim_{l \to \infty} \|g_j^{(l)}\|_E = 0 \quad \text{for} \quad k+1 \le j < i.$$

Since $\tilde{h}_i(0) = 0$, it implies that

$$\lim_{l \to \infty} \|\tilde{h}_i(g_{k+1}^{(l)}, \cdots, g_{i-1}^{(l)})\|_E = 0.$$

By (4.2), we know that

$$g_i^{(l+1)} = \lambda_i g_i^{(l)} + \tilde{h}_i(g_{k+1}^{(l)}, \cdots, g_{i-1}^{(l)}).$$

Therefore, $\forall \epsilon > 0$, there exists a positive integer L, such that when $l \ge L$,

$$|g_i^{(l+1)}| \le |\lambda_i||g_i^{(l)}| + \epsilon \quad \text{on} \quad \text{E} .$$

This gives that

$$\limsup_{l\to\infty} \|g_i^{(l)}\|_E \le \frac{\epsilon}{1-|\lambda_i|}$$

for all $\epsilon > 0$.

By induction on i, we have

$$\|g_j^{(l)}\|_E \to 0 \quad as \quad l \to \infty \quad for \quad j = k+1, \cdots, n.$$

This gives $W_s(0,G) \supset E_2$. In the following, we will prove that $W_s(0,G) \subset E_2$.

If $z = (z_1, \cdots, z_n) \in W_s(0,G)$, then

$$|g_j^{(l)}(z)| \to 0 \quad as \quad l \to \infty \quad for \quad j = 1, \cdots, n.$$

Since $g_1^{(l)}(z) = \lambda_1^l z_1$ and $|\lambda_1| > 1$, it follows that $z_1 = 0$ and $g_1^{(l)}(z) = 0$ for all positive integers l.

Therefore $g_2^{(l)}(z) = \lambda_1^l z_2$. A similar argument implies that $z_2 = 0$ and $g_2^{(l)}(z) = 0$ for all positive integers l because $|\lambda_2| > 1$.

By induction, we prove that

$$z_j = 0 \quad \text{and} \quad g_j^{(l)}(z) = 0$$

for $j = 1, \cdots, k$ and all positive integers l. Thus $z \in E_2$. We have now proved that $W_s(0,G) = E_2$. This statement is obviously equivalent to the following one: *given any neighborhood V of 0 in E_2,*

$$\bigcup_{l=1}^{\infty} G^{-l}(V) = E_2.$$

This completes the proof. □

Let's recall Siegel linearization theorem (cf.[Ar] or [Ym]).

Theorem 4.2.4 (Siegel's Theorem) *Assume that $F \in Aut(\mathbb{C}^n)$ and the point 0 is a hyperbolic fixed point of F. If the eigenvalues $\lambda_1, \cdots, \lambda_n$ of $DF(0)$ satisfy the condition $(*)$, then in some neighborhood of 0, F is biholomorphically equivalent to the linear map $DF(0)z$. More precisely, there exists a biholomorphic map H_0 defined on a small neighborhood of an open ball $B_r = \{z \in \mathbb{C}^n : |z| < r\}$ for some $r > 0$ such that: (i) $H_0(0) = 0$; (ii) $DH_0(0) = I$; and (iii)*

$$H_0 \circ F(z) = DF(0)H_0(z) \quad for \quad z \in B_r. \tag{4.3}$$

Remark 4.2.5: As stated in [Ar], it seems as if one needs the condition $(*)$ in a neighborhood of the origin, however the proof of the Siegel's theorem in [Ar] depends only on the condition $(*)$ at the origin. A clear statement of this theorem can be found in [Ym].

We can choose the coordinates such that the matrix representation of $DF(0)$ is lower triangular and has the form

$$A := \begin{pmatrix} \lambda_1 & & & & \\ a_{21} & \lambda_2 & & & \\ \cdot & \cdot & \cdot & & \\ \cdot & \cdot & & \cdot & \\ \cdot & \cdot & & & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot\ \cdot & \lambda_n \end{pmatrix}$$

with $|\lambda_1| \geq \cdots \geq |\lambda_k| > 1 > |\lambda_{k+1}| \geq \cdots \geq |\lambda_n| > 0$.

If we define $G : \mathbb{C}^n \to \mathbb{C}^n$ by $G(z) = Az$. Then clearly $G \in Aut(\mathbb{C}^n)$.

It is easy to see that (4.3) is equivalent to the following

$$H_0(z) = G^{-1} \circ H_0 \circ F(z) \quad for \quad z \in B_r. \tag{4.4}$$

By the Stable Manifold Theorem at hyperbolic fixed point (cf. [PM] or [Wu2]) and Siegel's Theorem, it is easy to prove that there exists a $r_0 > 0$ small enough with $r_0 < r$, such that (i) V_{r_0} is a complex submanifold of $\mathbb{C}^n$, where V_{r_0} is the connected component of $W_s(0, F) \cap B_{r_0}$ containing the origin; (ii) $F(V_{r_0}) \subset V_{r_0}$; and (iii) $W_s(0, F) = \bigcup_{l \geq 0} F^{-l}(V_{r_0})$ is an injectively immersed complex submanifold. We fix such a positive constant r_0. If we set $U_0 = V_{r_0}$ and $U_l = F^{-l}(U_0)$ for $l \geq 1$, then $U_l \subset U_{l+1}$ for all nonnegative integers l.

Now we begin to extend H_0 from U_0 to the stable manifold $W_s(0, F)$.

By (4), we have

$$H_0(z) = G^{-1} \circ H_0 \circ F(z) \quad for \quad z \in U_0.$$

Note that $F, G^{-1} \in Aut(\mathbb{C}^n)$ and H_0 is holomorphic on U_0, we define

$$H_1(z) = G^{-1} \circ H_0 \circ F(z) \quad for \quad z \in U_1,$$

then H_1 is holomorphic on U_1 and $H_1(z) = H_0(z)$ for $z \in U_0$.

Similarly, we define

$$H_2(z) = G^{-1} \circ H_1 \circ F(z) \quad for \quad z \in U_2,$$

then H_2 is holomorphic on U_2 and $H_2(z) = H_1(z)$ for $z \in U_1$.

By induction, we can define

$$H_{l+1}(z) = G^{-1} \circ H_l \circ F(z) \quad for \quad z \in U_{l+1},$$

such that H_{l+1} is holomorphic on U_{l+1} and $H_{l+1}(z) = H_l(z)$ for $z \in U_l$.

If we define $H : W_s(0, F) \to \mathbb{C}^n$ by

$$H(z) = H_l(z) \quad for \quad z \in U_{l-1},$$

then H is well-defined and satisfies

(i) H is holomorphic on $W_s(0,F)$;

(ii)

$$H(z) = G^{-1} \circ H \circ F(z), \quad for \quad z \in W_s(0,F),$$

or, equivalently,

$$H \circ F(z) = G \circ H(z) \quad for \quad z \in W_s(0,F). \tag{4.5}$$

Proof of theorem 4.2.2: We are going to prove that H is our required biholomorphic map from $W_s(0,F)$ as an immersed complex submanifold onto E_2 (see Lemma 4.2.3 for the definition of E_2).

If $z_1, z_2 \in W_s(0,F)$ and $H(z_1) = H(z_2)$, then (4.5) implies $H(F(z_1)) = H(F(z_2))$. Inductively, we have $H(F^l(z_1)) = H(F^l(z_2))$ for all nonnegative integers l. When l is sufficiently large, both $F^l(z_1)$ and $F^l(z_2)$ are close to 0, but H is one-to-one in a small neighborhood of 0. Thus $F^l(z_1) = F^l(z_2)$. It implies that $z_1 = z_2$. Therefore H is injective.

By (4.5), we have

$$H^{-1} \circ G \circ H(z) = F(z) \quad for \quad z \in W_s(0,F).$$

Thus

$$H^{-1} \circ G^l \circ H(z) = F^l(z) \quad for \quad z \in W_s(0,F).$$

Then

$$\begin{aligned} \lim_{l\to\infty} F^l(z) = 0 \quad iff \quad & \lim_{l\to\infty} G^l(H(z)) = 0 \\ iff \quad & H(z) \in E_2. \ \text{(by Lemma 4.2.3)} \end{aligned}$$

So this implies that $H(W_s(0,F)) = E_2$. We have now proved that H is a biholomorphic map from $W_s(0,F)$ onto E_2. □

Chapter 5

Generalized complex Hénon mappings

5.1 Introduction and notations

In this chapter, we will study the dynamical properties of complex polynomial automorphisms of $\mathbb{C}^2$.

Let f be a polynomial automorphism of $\mathbb{C}^2$ which consists of the finite compositions of "generalized Hénon" mappings (see [FM] for details), i.e.,

$$f = f_1 \circ \ldots \circ f_m,$$

where f_j has the form

$$f_j(x, y) = (y, p_j(y) - a_j x)$$

for a monic polynomial $p_j(x)$ of degree at least 2 and a non-zero complex number a_j. For such an f, we define the *degree of* f by $d(f) = \prod_{j=1}^{m} d(f_j)$. The dynamical properties of these polynomial automorphisms are nontrivial because the topological entropy of f is positive, in fact Smillie [Si] proved that $h(f) = \log d$, where $d \geq 2$ is the degree of f.

Following [Hu] and [BS1], we define

$$K^{\pm} := \{p \in \mathbb{C}^2 : \{f^{\pm k}(p) : n = 0, 1, 2, ...\} \quad \text{is bounded}\},$$

and $J^{\pm} := \partial K^{\pm}, K := K^+ \cap K^-$ and $J := J^+ \cap J^-$. These interesting sets have been studied by several authors (e.g., [FM], [BS1,2,3], [FS]).

Friedland and Milnor proved, Lemma 3.5 in [FM], that there exists a closed disk $D_\kappa = \{z \in \mathbb{C} : |z| \leq \kappa\}$ (where κ is a positive constant) such that the nonwandering set $\Omega(f) \subset D_\kappa \times D_\kappa$, and moreover

$$\Omega(f) \subset K(f) \subset D_\kappa \times D_\kappa.$$

Clearly, $\Omega(f)$ and $K(f)$ are f-invariant compact subsets of $\mathbb{C}^2$. So we can study the dynamics of f on K.

Now we recall the definition of the equilibrium measure μ of K from [BS1,3]. In [BS1], Bedford and Smillie introduced the functions

$$G^{\pm}(p,q) = \lim_{k \to \infty} \frac{1}{d^k} \log^+ |(f^{\pm k}(p,q)|.$$

The functions give the rate of escape of the orbit of (p,q) to infinity in forward and backward time. It was shown in [BS1] that $G^{\pm}$ is continuous on $\mathbb{C}^2$ and is a pluriharmonic Green function of $K^{\pm}$. The currents $\mu^{\pm}$ were defined in [BS3] as

$$\mu^{\pm} = \frac{1}{2\pi} dd^c G^{\pm}$$

where

$$dd^c = 2i \sum_{1 \le j,k \le 2} \frac{\partial^2}{\partial z_j \partial \overline{z}_k} dz_j \wedge d\overline{z}_k.$$

The equilibrium measure μ of K then was defined as $\mu = \mu^+ \wedge \mu^-$. It was proved in [BS3] that (i) *μ is an f-invariant Borel measure on K* and (ii) *f is ergodic with respect to μ.*

It should be pointed out that the definition of stable/unstable manifolds given in [BS1,2,3] and our definition of stable/unstable manifold are different. Following [BS1,2,3], we set

$$W_s(p,f) = \{q \in \mathbb{C}^2 : \lim_{k \to \infty} d(f^k(p), f^k(q)) = 0\}$$

$$W_u(p,f) = \{q \in \mathbb{C}^2 : \lim_{k \to -\infty} d(f^k(p), f^k(q)) = 0\}.$$

Bedford and Smillie proved that if f is hyperbolic on J, then $W_s(p,f)$ and $W_u(p,f)$ ($\forall p \in J$) are immersed complex manifolds biholomorphically equivalent to $\mathbb{C}$. In Section 5.2 we use the properties of μ and apply Theorem 3.3.5 to prove several more general results. In Theorem 5.2.1 , we prove that for μ-a.e. $p \in K$, the stable manifold $W^s(p,f)$ is an injectively immersed holomorphic copy of $\mathbb{C}$. In Theorem 5.2.2, we show that for μ-a.e. $p \in K$, $W^s(p,f) \subset J^+$ and $W^u(p,f) \subset J^-$. Moreover, if p is a saddle point, then we have (Theorem 5.2.4) that $W^s(p,f) = W_s(p,f)$ and $W^u(p,f) = W_u(p,f)$.

In Section 5.3, we study the heteroclinic points and homoclinic points of saddle points. We prove that in Theorem 5.3.2 that for any two saddle points p and q, $W^s(p,f)$ and $W^u(q,f)$ have transversal heteroclinic points and $W^u(p,f)$ and $W^s(q,f)$ have transversal heteroclinic points. This implies that for any saddle point p of f, f admits transversal homoclinic points. There are some interesting consequences which are listed in Corollary 5.3.3.

In Section 5.4, we introduce a new notation $SJ(f)$ as an analogue of the Julia set. $SJ(f)$ is defined as the closure of the set of saddle points of f, and we call it *S-Julia set of f*. We prove that $SJ(f)$ bears most analogies with the Julia set of one variable complex polynomial. For example, it is proved that $SJ(f)$ is an invariant perfect set, $f|_{SJ(f)}$ is topological mixing and f is chaotic on $SJ(f)$.

5.2 Stable manifolds

Bedford and Smillie [BS1, Theorem 5.4] proved that if f is **hyperbolic** on J, then for any point $p \in J$, The stable manifold and unstable manifold at point p are immersed holomorphic copies of $\mathbb{C}$. In the following Theorem 5.2.1, we will drop the condition of hyperbolicity and prove that for μ-a.e. $p \in K$, the same conclusion holds. Here the measure μ is the equilibrium measure on K which was defined in Section 5.1. In other words, Theorem 5.2.1 contains the Bedford and Smillie's result as a special case.

Theorem 5.2.1 *For μ almost every $p \in K$, the stable and unstable manifolds $W^s(p, f)$, $W^u(p, f)$ are immersed complex manifolds and biholomorphically equivalent to $\mathbb{C}$.*

In order to prove above theorem, we need some well known results in ergodic theory. Birkhoff Ergodic Theorem is the first major result in ergodic theory. It has several versions. For our purpose, we recall one version of it for a measure-preserving map of a σ-finite measure space. A σ-finite measure on a measurable space $(X, \mathcal{B})$ is a transformation $m : \mathcal{B} \to \mathbf{R}^+ \cup \{\infty\}$ such that

(i) $m(\emptyset) = 0$;

(ii) $m(\cup_{k=1}^{\infty} B_k) = \sum_{k=1}^{\infty} m(B_k)$ whenever $\{B_k\}$ is a sequence of pairwise disjoint members of $\mathcal{B}$; and

(iii) there exists a countable collection $\{A_k\}_{k=1}^{\infty}$ of elements of $\mathcal{B}$ with $m(A_k) < \infty$ for all k and $\cup_{k=1}^{\infty} A_k = X$.

The probability measure provides an example of a σ-finite measure.

Theorem 5.A (Birkhoff ergodic theorem)(cf. [Wa, Theorem 1.14 & Remark])

(1) *Suppose that* $(X, \mathcal{B}, m)$ *is* σ*-finite,* $T : (X, \mathcal{B}, m) \to (X, \mathcal{B}, m)$ *is measure-preserving and* $f \in L^1(m)$*, then*

$$\frac{1}{m}\sum_{j=0}^{k-1} f(T^j x)$$

converges a.e. to a function $f^* \in L^1(m)$*. Also* $f^* \circ T = f^*$ *a.e. and if* $m(X) < \infty$*, then* $\int f^* dm = \int f dm$*.*

(2) *If* T *is ergodic, then* f^* *is constant a.e. and so if* $m(X) < \infty$*,*

$$f^* = \frac{1}{m(X)} \int f dm \qquad \text{a.e.}$$

If $(X, \mathcal{B}, m)$ *is a probability space and* T *is ergodic we have that for any* $f \in L^1(m)$*,*

$$\lim_{m\to\infty} \frac{1}{m}\sum_{j=0}^{m-1} f(T^j x) = \int f dm \qquad \text{a.e.}$$

Theorem 5.B(cf. [KS, Part III, Proposition 2.2]) *Let* $(M, \mathcal{M}, \mu)$ *be a measure space of finite measure,* $f : M \to M$ *a measurable measure preserving mapping, and* G *a positive finite measurable function defined on* M *such that*

$$\log^- \frac{G \circ f}{G} \in L^1(M, \mu), \quad \textit{where} \quad \log^- a := \min\{\log a, 0\}.$$

Then

$$\lim_{n\to\infty} \frac{1}{n} \log G(f^n x) = 0 \quad \mu \quad \textit{almost everywhere},$$

and

$$\int_M \log \frac{G \circ f}{G} d\mu = 0.$$

Proof of Theorem 5.2.1: Let $\lambda_1 \leq \lambda_2$ be the Lyapunov exponents of μ. Since the entropy $h_\mu(f) > 0$ (see [FM] or [Si]) and $h_\mu(f)$ is less than or equal to the sum of positive Lyapunov exponents, then $\lambda_2 > 0$. Since $h_\mu(f^{-1}) = h_\mu(f) > 0$, then $-\lambda_1 > 0$, i.e., $\lambda_1 < 0$. Thus

$$\lim_{k\to\infty} \frac{1}{k} \log \|D_p f^k(v)\| = \lambda_1 < 0 \tag{5.1}$$

holds for all $0 \neq v \in T_pW^s(p,f)$ and for μ-a.e. $p \in K$.

By Theorem 3.3.5, we know that for μ-a.e. $p \in K$, $W^s(p,f)$ is an immersed complex manifold which is diffeomorphic to R^2. This implies that for μ-a.e. $p \in K$, $W^s(p,f)$ is biholomorphically equivalent to either $\mathbb{C}$ or complex unit disk D.

Set $S = \{p \in K : W^s(p,f)$ is biholomorphically equivalent to $D\}$. Clearly $f^{-1}(S) = S$. Since f is ergodic with respect to μ, $\mu(S) = 0$ or $\mu(S) = 1$.

Assume that $\mu(S) = 1$, i.e., $W^s(p,f)$ is biholomorphically equivalent to D for μ-a.e. $p \in K$. In the following, we want to derive a contradiction.

For any $p \in S$, $W^s(p,f)$ is biholomorphically equivalent to D, so we can define the Poincare-Bergman metric, denoted by $\|\cdot\|_P$, on $T_pW^s(p,f)$.

By Theorem 3.3.3, we can assume that $RS \subset S$ is the set of regular points in S with $\mu(RS) = 1$.

For $p \in RS$ and every non-zero tangent vector $v \in T_pW^s(p,f)$, define

$$F(p) = \log \frac{\|D_pf(v)\|}{\|v\|} \text{ and } G(p) = \log \frac{\|v\|}{\|v\|_P}.$$

Note that (i) F and G are independent of v because $W^s(p,f)$ is one-dimensional and (ii) the Poincare-Bergman metric coincides with the Kobayashi pseudodistance in our case. So by the distance-decreasing property of Kobayashi pseudodistance for holomorphic mappings, we have

$$\|D_pf(v)\|_P = \|v\|_P$$

for any non-zero tangent vector $v \in T_pW^s(p,f)$. Thus a simple calculation gives

$$F(p) = G(f(p)) - G(p). \tag{5.2}$$

Note that any probability Borel measure is σ-finite and μ is f-invariant, so Birkhoff ergodic theorem (see Theorem 5.A(1)) is true when applied to the function $F(p)$. Then the limit $\frac{1}{k}\sum_{j=0}^{k-1} F(f^j(p))$ exists for μ almost every $p \in K$ as k tends to $+\infty$. Moreover, Since f is ergodic, then by Theorem 5.A(2) and (5.2), we have

$$\lim_{k\to\infty} \frac{1}{k}\sum_{j=0}^{k-1} F(f^j(p)) = \int F d\mu = \int (G\circ f - G)d\mu, \;\; \mu \;\; a.e. \;\; p \in K. \tag{5.3}$$

It is easy to check that G and f satisfy the condition in Theorem 5.B, then we have

$$\int (G\circ f - G)d\mu = 0. \tag{5.4}$$

(5.3) and (5.4) imply that

$$\lim_{k\to\infty} \frac{1}{k}\sum_{j=0}^{k-1} F(f^j(p)) = \int (G\circ f - G)d\mu = 0, \;\; \mu \;\; a.e. \;\; p \in K. \tag{5.5}$$

But

$$\begin{aligned}
\lim_{k\to\infty} \frac{1}{k}\sum_{j=0}^{k-1} F(f^j(p)) &= \lim_{k\to\infty} \frac{1}{k}\log\frac{\|D_pf^k(v)\|}{\|v\|} \\
&= \lim_{k\to\infty} \frac{1}{k}[\log\|D_pf^k(v)\| - \log\|v\|] \\
&= \lim_{k\to\infty} \frac{1}{k}\log\|D_pf^k(v)\| \\
&= \lambda_1 < 0, \;\; \mu \;\; a.e. \;\; p \in K.
\end{aligned}$$

Hence this contradicts (5.5). This shows that for μ-a.e. $p \in K$, the stable manifold $W^s(p,f)$ is equivalent to $\mathbb{C}$. By a similar argument, $W^u(p,f)$ is biholomorphically equivalent to $\mathbb{C}$ for μ-a.e. $p \in K$. □

Bedford and Smillie in [BS2] proved that for every saddle periodic point p, the stable/unstable manifold at p was contained in J^+/J^-. Here we have a similar but more general result.

Theorem 5.2.2 *For μ almost every $p \in K$,*

$$W^s(p,f) \subset J^+ \text{ and } W^u(p,f) \subset J^-.$$

Proof: Let $p \in K$ be a regular point. Thus $\{f^k(p)\}_{k=0}^{+\infty} \subset K$ is obviously bounded. This implies that $W^s(p,f) \subset K^+$. Given $q \in W^s(p,f)$, we assume that $q \notin J^+$, i.e., $q \in W^s(p,f) \setminus J^+ \subset K^+ \setminus J^+ = \mathrm{int}K^+$. By Lemma 3.4 in [BS1], $\{f^k\}$ is a normal family on $\mathrm{int}K^+$. It follows that the sequence of the norms of Jacobian matrices $\{\|J_q f^k\|\}_{k=0}^{+\infty}$ is bounded. Let $\lambda_1 < 0 < \lambda_2$ be the Lyapunov exponents defined in the proof of Theorem 5.2.1. Since $q \in W^s(p,f)$, there are positive integer k_0 large enough, constant $C > 0$, and λ with $\lambda_1 \leq \lambda < 0$ such that

$$|f^k(q) - f^k(p)| \leq Ce^{\lambda k} \quad \text{for} \quad k \geq k_0.$$

Hence there exists a constant $C' > 0$, such that

$$\|J_{f^k(q)}f - J_{f^k(p)}f\| \leq C'e^{\lambda k} \quad \text{for} \quad k \geq k_0. \tag{5.6}$$

By (5.6), we can apply the perturbation theorem (cf. Theorem 4.1 in [Ru1], note that this theorem is proved in [Ru1] for real case, but it is easy to get the complex version, the arguments are same as what we did in Theorem 3.3.4) to the Jacobian matrix of f and obtain

$$\lim_{k\to+\infty} \frac{1}{k} \log \|(J_q f^k)\| = \lambda_2 > 0.$$

Since $\lambda_2 > 0$, the sequence $\{\|(J_q f^k)\|\}_{k=0}^{\infty}$ is unbounded, a contradiction. □

We say $p \in \mathbb{C}^2$ is a *saddle periodic point* if (i) p is a periodic point with period m; and (ii) one of the eigenvalues is strictly large that 1 in absolute value, and the other is strictly less than 1 in absolute value. We recall a theorem in [BS2]:

Theorem 5.2.3([BS2, Theorem 1]) *Let p be a saddle periodic point of f. Then*

$$J^+ = \overline{W_s(p,f)} \quad and \quad J^- = \overline{W_u(p,f)}.$$

If p is a saddle periodic point, we will show that $W_s(p,f)$ (resp. $W_u(p,f)$) coincide with $W^s(p,f)$ (resp. $W^u(p,f)$).

We need the following property of stable/unstable manifold at a saddle periodic point.

Theorem 5.2.4 *Let p be a saddle periodic point of f, Then*

$$W^s(p,f), W^u(p,f), W_s(p,f) \ \ and \ \ W_u(p,f)$$

are biholomorphically equivalent to $\mathbb{C}$.

Proof: In Theorem 5.4 of [BS1], Bedford and Smillie proved that $W_s(p,f)$ and $W_u(p,f)$ ($\forall p \in J$) are biholomorphically equivalent to $\mathbb{C}$ if J is a hyperbolic set of f. If we replace J by the orbit of p, then f is hyperbolic on the orbit of p. Then Bedford and Smillie's argument applies also to our case.This gives that $W_s(p,f)$ and $W_u(p,f)$ are biholomorphically equivalent to $\mathbb{C}$. Note that every saddle periodic point is regular and by Theorem 5.2.1, $W^s(p,f)$ and $W^u(p,f)$ are immersed complex manifolds and biholomorphically equivalent to $\mathbb{C}$. So, the proof is finished. □

As a consequence of Theorem 5.2.3 and Theorem 5.2.4, we conclude the following interesting result:

Corollary 5.2.5 *Let p be a saddle periodic point of f. Then*

$$W^s(p,f) = W_s(p,f) \quad and \quad W^u(p,f) = W_u(p,f).$$

Hence

$$J^+ = \overline{W^s(p,f)} \quad and \quad J^- = \overline{W^u(p,f)}.$$

Proof: We prove this theorem only for the case $W^s(p,f)$, the proof for the case $W^u(p,f)$ is similar. From Theorem 5.2.4 we know that $W_s(p,f)$ and $W^s(p,f)$ are biholomorphically equivalent to $\mathbb{C}$. So there exist two biholomorphisms

$$h_1 : W^s(p,f) \to \mathbb{C} \quad and \quad h_2 : W_s(p,f) \to \mathbb{C}.$$

Clearly $W^s(p,f) \subset W_s(p,f)$. Assume that $W^s(p,f) \neq W_s(p,f)$, then $h_2(W^s(p,f)) \neq \mathbb{C}$ and is simply connected. The simply connectedness of $h_2(W^s(p,f))$ is due to the fact that $W^s(p,f)$ is an immersed complex manifold and biholomorphically equivalent to complex plane $\mathbb{C}$. But

$$h_1 \circ h_2^{-1} : h_2(W^s(p,f)) \to \mathbb{C}$$

is still a biholomorphism. This contradicts the Riemannian Mapping Theorem (cf. [BG]). We finish the proof. □

Remark 5.2.6: From [BS3] we know that there are infinitely many saddle periodic points. For any two different saddle periodic points p and q, we have

$$W^s(p,f) \subset J^+ - W^s(q,f) = \overline{W^s(q,f)} - W^s(q,f). \tag{5.7}$$

But both $W^s(p,f)$ and $W^s(q,f)$ are biholomorphically equivalent to complex plane $\mathbb{C}$. The formula (5.7) tells us that the stable manifolds at saddle points are very intricate from the geometrical point of view.

5.3 Transversal heteroclinic and homoclinic points

We first recall the definition of saddle points:

Definition 5.3.1. We say $p \in \mathbb{C}^2$ is a *saddle point* of f, if (i) p is a periodic point with period m; and (ii) one of the eigenvalues of $Df^m(p)$ is strictly large than 1 in absolute value, and the other is strictly less than 1 in absolute value.

By Theorem 5.2.1, the stable/unstable manifold at saddle point p is actually an injectively immersed 1-dimensional complex submanifold in $\mathbb{C}^2$.

The following theorem is the main theorem of this section:

Theorem 5.3.2. *Let p, q be two different saddle points of f, then there exist x, y such that $W^s(p, f)$ intersects $W^u(q, f)$ at x transversally, and $W^u(p, f)$ intersects $W^s(q, f)$ at y transversally. Consequently, both p and q admit transversal homoclinic points.*

Before proving this theorem, we list some easy consequences whose proofs are standard (cf. [De]).

Corollary 5.3.3. *(i) Every saddle point of f admits a transversal homoclinic point. (ii) There exists an f-invariant hyperbolic set Γ such that $f|_\Gamma$ is topologically conjugate to a topological Markov chain and the topological entropy of $f|_\Gamma$ is positive. (iii) There exists a positive integer N such that the period set of all the periodic points of f^N equals to the set of all positive integers.*

Note that Corollary 5.3.3(iii) gives a partial answer of a conjecture of Friedland and Milnor[FM, p.97] that *there must exist periodic points of all sufficiently large periods.*

In order to prove Theorem 5.3.2, we collect some results from [BS1,Theorem 5.4], [BS2, Theorem 1] and Theorem 5.2.1 and Corollary 5.2.5 (see [Wu2, Theorem 3.4 and Corollary 3.5]) and restate them as the following theorem.

Theorem 5.3.4. *Let p be a saddle point of f, then*
(i) $W^s(p,f)$ and $W^u(p,f)$ are injectively immersed complex manifolds biholomorphically equivalent to $\mathbb{C}$.
(ii) $\overline{W^s(p,f)} = J^+$ and $\overline{W^u(p,f)} = J^-$.

By Theorem 5.3.4(i), for any saddle point p, we may assume that

$$\phi : \mathbb{C} \to W^s(p,f)$$

is a biholomorphic diffeomorphism from $\mathbb{C}$ onto $W^s(p,f)$ (here $W^s(p,f)$ is considered as an injectively immersed complex manifold). If q is another saddle point of f, then by Theorem 5.3.4(ii), we have

$$\overline{W^s(p,f)} = J^+ = \overline{W^s(q,f)}.$$

Then there exist $t_j \in \mathbb{C}$ with $\lim_{j\to\infty} |t_j| = \infty$, such that

$$\lim_{j\to\infty} \phi(t_j) = q.$$

Using these notations, we will prove the following lemma:

Lemma 5.3.5. *For any real positive number a, there exist a sequence of positive numbers $\{r_j\}$ such that*

$$|\phi(t) - \phi(t_j)| \leq a \quad \text{for all} \quad t \in B(t_j, r_j),$$

and there exists $T_j \in \partial B(t_j, r_j)$ with

$$|\phi(T_j) - \phi(t_j)| = a,$$

where $B(t_j, r_j)$ is the ball centered at t_j with radius r_j and $\partial B(t_j, r_j)$ stands for the boundary of the ball $B(t_j, r_j)$.

Proof: For a fixed j, we consider the real valued function $m_j : [0, \infty) \to [0, \infty)$ defined by

$$m_j(r) = \max_{t \in \overline{B}(t_j, r)} |\phi(t) - \phi(t_j)|.$$

Claim: (i) $m(\cdot)$ is continuous and strictly increasing;

(ii) $\max_{t \in \overline{B}(t_j, r)} |\phi(t) - \phi(t_j)| = \max_{t \in \partial B(t_j, r)} |\phi(t) - \phi(t_j)|$;

(iii) $\lim_{r \to \infty} m_j(r) = \infty$.

Proof of claim: (i) is obvious from the definition of $m(\cdot)$.

(ii)Since $\phi(t) - \phi(t_j) := (\phi_1(t) - \phi_1(t_j), \phi_2(t) - \phi_2(t_j)) : \overline{B}(t_j, r) \to \mathbb{C}^2$ is holomorphic on $\overline{B}(t_j, r)$, the Maximum Modulus Theorem(cf. [Co]) implies that

$$\begin{aligned} \max_{t \in \overline{B}(t_j, r)} |\phi(t) - \phi(t_j)| &= \max_{t \in \overline{B}(t_j, r)} \max\{|\phi_1(t) - \phi_1(t_j)|, |\phi_2(t) - \phi_2(t_j)|\} \\ &= \max_{t \in \partial B(t_j, r)} \max\{|\phi_1(t) - \phi_1(t_j)|, |\phi_2(t) - \phi_2(t_j)|\} \\ &= \max_{t \in \partial B(t_j, r)} |\phi(t) - \phi(t_j)|. \end{aligned}$$

This proves (ii).

Using the Maximum Modulus Theorem again, we then have that the function $m(\cdot)$ is in fact strictly increasing, consequently r_j is unique for fixed j.

(iii) Theorem 5.3.4(i) and Liouville's theorem (cf. [Co]) imply that

$$\lim_{r \to \infty} m(r) = \infty,$$

otherwise ϕ would be a constant map which is impossible.

The claim guarantees that this lemma is true. □

Since $\lim_{j \to \infty} \phi(t_j) = q$, we have

Lemma 5.3.6 *For any ϵ with $0 < \epsilon < \frac{a}{2}$, there exists a positive integer N_0 such that*

$$|\phi(t) - q| \leq a + \epsilon, \quad \text{for all } t \in B(t_j, r_j) \text{ with } j \geq N_0,$$

and

$$\lim_{j \to \infty} |\phi(T_j) - q| = a.$$

Proof of theorem 5.3.2: We begin by making a renormalization of the sequence of disks $\{B(t_j, r_j)\}$. From now on, we fix the positive constant a. Denote $B := B(0,1) \subset \mathbb{C}$ and $B_j := B(t_j, r_j) \subset \mathbb{C}$. Define $\psi_j : B \to B_j$ by

$$\psi_j(t) = r_j e^{i\alpha_j} t + t_j$$

where α_j is the argument of the vector $T_j - t_j$. Then $T_j = t_j + r_j e^{i\alpha_j}$. Clearly $\psi_j(1) = T_j$ and ψ_j can be considered as a biholomorphic map from $\overline{B}$ onto $\overline{B}_j$. The notation $\overline{B}$ stands for the closure of the unit ball B. The notation $\mathcal{H}(\overline{B}, \mathbb{C}^2)$ will stand for the set of holomorphic mappings from a neighborhood of $\overline{B}$ into $\mathbb{C}^2$ with the topology of uniform convergence.

Denote $g_j := \phi \circ \psi_j$, then by Lemma 5.3.6 $\{g_j\}$ is a bounded sequence of biholomorphic maps on $\overline{B}$. Using the coordinates of $W^s(p)$ given by ϕ and by Montel's theorem (cf. [GR]), there exist a subsequence $\{j_k\}$, such that

$$g_{j_k} \to g \in \mathcal{H}(\overline{B}, \mathbb{C}^2) \text{ and } g'_{j_k} \to g' \in \mathcal{H}(\overline{B}, \mathbb{C}^2),$$

where the convergence means the uniform convergence.

By Hurwitz's theorem (cf. [Co]), g is a nonsingular holomorphic map or g is identically equal to a constant. But the later case is precluded by the condition

$$|g(1) - g(0)| = |g(1) - q| = a > 0.$$

This implies that $g(B)$ is a simply connected complex submanifold containing the point q.

Let $E^s(q)$ and $E^u(q)$ be the complex tangent space of $W^s(q)$ and $W^u(q)$ at point q respectively. Since q is a saddle point , $E^s(q)$ must intersect $E^u(q)$ transversally at q.

(i) If $g'(0) \in E^u(q)$, then $g(B)$ intersects $W^s(q)$ transversally at q. This implies that $g_{j_k}(B)$ intersects $W^s(q)$ transversally, i.e., $\phi(B_{j_k})$ intersects $W^s(q)$ transversally provided k is large enough. This shows that $W^s(p) \cap W^s(q) \neq \emptyset$. But $p \neq q$, this is a contradiction.

(ii) If $g'(0) \in E^s(q)$. The same argument as in case (i) implies that $W^u(q)$ intersects $W^s(p)$ transversally. In fact, $\phi(B_{j_k})$ intersects $W^u(q)$ transversally, provided k is large enough.

(iii) If $g'(0) = a_s e_s + a_u e_u, a_s, a_u \in \mathbb{C}^*$ and $0 \neq e_s \in E^s(q), 0 \neq e_u \in E^u(q)$, then the same arguments as in case (i) imply that $W^s(p) \cap W^s(q) \neq \emptyset$ (see Figure 1). This is impossible by the definition of stable manifold.

Therefore only case (ii) is possible.

This finishes the proof of theorem 5.3.2. □

5.4 S-Julia set

In this section, we will give a definition of S-Julia set for f which is similar to the definition of the Julia set of one variable complex polynomials. we will prove that the S-Julia set of f bears most analogies with the Julia set. The S-Julia set seems to be a better candidate for the analogue of the Julia set in the case of two complex variable polynomials.

Definition 5.4.1 Let f be a finite composition of "generalized" Hénon mappings. *The S-Julia set* of f, denoted by $SJ(f)$ or simply by SJ, is the closure of the set of all saddle points of f.

Corollary 5.4.2. *(i)* $SJ(f^n) = SJ(f)$, *for all integers* n, *and* $f^{-1}(SJ(f)) = SJ(f)$.
(ii) $J^* \subseteq SJ \subseteq J$.

Proof: (i) is obvious by Definition 5.4.1.
(ii) By Theorem 5.3.2, for any saddle point p we have

$$p \in W^s(p,f) \cap W^u(p,f) \subseteq \overline{W^s(p,f)} \cap \overline{W^u(p,f)} = J^+ \cap J^- = J.$$

Since J is a closed set, this implies that J contains the closure of the set of all saddle points, i.e., $J \supseteq SJ$.

By [BS3, Theorem 3.4], J^* contains a dense subset of the saddle points, then $J^* \subseteq SJ$ by Definition 5.4.1. □

Before stating our main results of this section, we recall some well-known results which we will use in the following proofs.

λ-lemma. *Let f be a C^1 diffeomorphism with p as a hyperbolic periodic point, and let D^u be a u-disk in $W^u(p,f)$. Let D be u-disk meeting $W^s(p,f)$ transversely at some point x. Then $\cup_{n\geq 0} f^n(D)$ contains u- disks arbitrarily C^1 close to D^u.*

Smale Homoclinic Theorem. *Let f be a C^1 diffeomorphism with a hyperbolic periodic point p having a transversal homoclinic point x. Then there is an integer $n > 0$ such that f^n has a closed invariant set Λ containing x and p so that $f^n|_\Lambda$ is topologically equivalent to the 2-symbol shift automorphism. Moreover, Λ is a hyperbolic set for f^n and the homoclinic point x is in the closure of the hyperbolic periodic points of f.*

λ-lemma can be found in [Pal], [PM] or [Ne] and Smale Homoclinic Theorem can be found in [Sa] or [Ne].

Theorem 5.4.3. *f is topological mixing on $SJ(f)$, i.e., for any two non-empty open sets $U, V \subset SJ(f)$, there exists a positive integer N such that $f^n(U) \cap V \neq \emptyset$ for any positive integer $n \geq N$.*

Before proving this theorem, we first present following key lemma.

Lemma 5.4.4. *Let p, q be two different saddle points of f. Then for any neighborhoods U of p, V of q, there exists a positive integer N such that*

$$f^n(U \cap SJ(f)) \cap (V \cap SJ(f)) \neq \emptyset \quad \text{for all} \quad n \geq N. \tag{5.8}$$

Proof: By Theorem 5.3.2, there exist two points x, y such that $W^s(p,f)$ intersects $W^u(q,f)$ at point x transversally, and $W^u(p,f)$ intersects $W^s(q,f)$ at point y transversally. Then the λ-lemma implies that the point y is the limit of a sequence transversal homoclinic points $\{y_n\}$ of point q, see Figure 5.1.

According to the Smale Homoclinic Theorem, for each y_n, there exists an integer $n_0 > 0$ such that f^{n_0} has a closed hyperbolic f^{n_0}-invariant set Γ_n containing q and y_n so that $f^{n_0}|_{\Gamma_n}$ is topologically equivalent to the shift automorphism (Σ_2, σ). Consequently, y_n is in the closure of the hyperbolic periodic points of f in Γ_n.

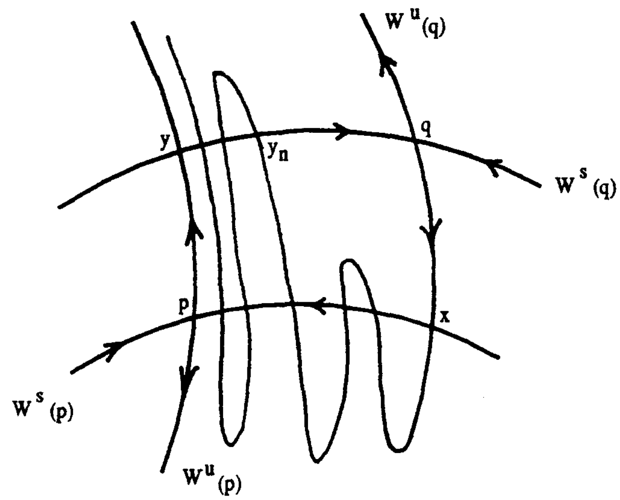

Figure 5.1

We begin to prove that y_n is actually in the closure of saddle points in Γ_n. Since y_n is a transversal homoclinic point of q and Γ_n is f^{n_0}-invariant, $f^{n_0 k}(y_n) \to q$ as $k \to \infty$ and $f^{n_0 l}(y_n) \in \Gamma_n$ for all integer l. The definition of hyperbolic set tells us that the splitting of the tangent space, $T_z\mathbb{C}^2 = E^s(z) \oplus E^u(z)$, depends continuously upon $z \in \Gamma_n$, in particular, the dimension of the stable tangent subspace at z, denoted by $s(z) := dim E^s(z)$, depends continuously upon $z \in \Gamma_n$ (cf. [Ka1]). Of course also, $u(z) := dim E^u(z)$ is continuous on $z \in \Gamma_n$. This means that

$$s(f^{n_0 k}(y_n)) = s(q) = 1 \quad \text{and} \quad u(f^{n_0 k}(y_n)) = u(q) = 1 \tag{5.9}$$

provided k is large enough. But $s(f(z)) = s(z)$ and $u(f(z)) = u(z)$, so (5.9) implies that

$$s(y_n) = u(y_n) = 1. \tag{5.10}$$

Since y_n is an accumulation point of a sequence of hyperbolic periodic points, say $\{p_{n,l}\}_{l=1}^{\infty}$, in Γ_n, the same argument as above shows that

$$s(p_{n,l}) = s(y_n) = 1 \quad \text{and} \quad u(p_{n,l}) = u(y_n) = 1 \tag{5.11}$$

provided l is large enough. Hence y_n is an accumulation point of a sequence of saddle points, i.e.,

$$y_n \in SJ(f). \tag{5.12}$$

Consequently, the point $y = \lim_{n\to\infty} y_n$ is also an accumulation point of a sequence of saddle points of f, i.e.,

$$y \in SJ(f). \tag{5.13}$$

For any neighborhoods U of p, V of q, it is clear that there exists an integer $n_1 > 0$ such that

$$f^n(y) \in V \;\text{ and }\; f^{-n}(y) \in U \;\text{ for all }\; n \geq n_1. \tag{5.14}$$

Then for all integer $k \geq 0$, we have

$$f^{n_1+k}(y) \in f^{2n_1+k}(U) \cap V. \tag{5.15}$$

The facts that $y \in SJ(f)$ and $f^{-1}(SJ(f)) = SJ(f)$ together with (5.15) imply that

$$f^n(U \cap SJ(f)) \cap (V \cap SJ(f)) \neq \emptyset \;\text{ for all }\; n \geq 2n_1. \tag{5.16}$$

This finishes the proof of this lemma. □

Remark 5.4.5. In fact, the proof of Lemma 5.4.4 also shows that any saddle point is accumulated by a sequence of pairwise different saddle points. Hence $SJ(f)$ has a very nice property which is similar to one of the Julia set in the case of one complex variable: *$SJ(f)$ is a perfect set.*

Proof of Theorem 5.4.3 Given any two non-empty open set $A, B \subset SJ(f)$. Since both A and B contain infinitely many saddle points, we may choose small non-empty open sets $A_1 \subset A$, $B_1 \subset B$ such that $A_1 \cap B_1 = \emptyset$ and there exist a saddle point $p \in A_1$, another saddle point $q \in B_1$. By Lemma 5.4.4, there exists a positive integer N such that

$$f^n(A_1) \cap B_1 \neq \emptyset \quad \text{for all} \quad n \geq N.$$

This implies that

$$f^n(A) \cap B \neq \emptyset \quad \text{for all} \quad n \geq N.$$

Hence f is topological mixing on $SJ(f)$. □

There are a series of interesting consequences of Theorem 5.3.2 and Theorem 5.4.3. We list some of them below. First, we recall the definition of chaos in the sense of Devaney (cf. [De]).

Definition 5.4.6. Let X be a metric space, a mapping $F : X \to X$ is said to be *chaotic in the sense of Devaney* on X if

(a) F has sensitive dependence on initial conditions, i.e., there exists $\delta > 0$ such that, for any $x \in X$ and any neighborhood U of x, there exists $y \in U$ and an integer $n \geq 0$ such that $|F^n(x) - F^n(y)| > \delta$;

(b) F is topologically transitive, i.e., for any two non-empty open sets $U, V \subset X$ there exists some positive integer m such that $F^m(U) \cap V \neq \emptyset$;

(c) Periodic points of F are dense in X.

Theorem 5.4.7. *f is chaotic on $SJ(f)$ in the sense of Devaney.*

Proof: As topological mixing implies topological transitive, the conditions (b) and (c) of chaos for the map $f|_{SJ}$ are satisfied by Theorem 5.4.3 and Definition 5.4.1. What we need to prove now is that $f|_{SJ}$ has sensitive dependence on initial conditions.

Since SJ is compact and infinite, we may assume that

$$0 < R = \max_{p,q \in SJ} |p - q| < \infty.$$

We take $\delta = \frac{R}{10}$ in the Definition 5.4.6. For any given $p \in SJ$ and any given neighborhood U of p, we divide the proof into the following two cases.

(i) If p is a saddle point, then there exists another saddle point q such that $|p-q| > \frac{R}{3}$. By Theorem 5.3.2, there exists a transversal heteroclinic point y of $W^u(p, f)$ and $W^s(q, f)$. Then

$$\lim_{n\to\infty} f^n(y) = q \quad \text{and} \quad \lim_{n\to\infty} f^{-n}(y) = q. \tag{5.17}$$

This implies that there is a positive integer N such that $f^{-N}(y) \in U$. Let the period of p is m, then

$$\lim_{l\to\infty} |f^{ml}(p) - f^{ml}(f^{-N}(y))| = \lim_{l\to\infty} |p - f^{ml-N}(y)| = |p - q| > \frac{R}{3}. \tag{5.18}$$

Obviously, there exists a positive integer L large enough such that

$$|f^{mL}(p) - f^{mL}(f^{-N}(y))| > \frac{R}{10}.$$

(ii): If p is not a saddle point, then there exist infinitely many saddle points in U. Suppose that

$$|f^k(p) - f^k(z)| \le \delta \quad \text{for any} \quad z \in U. \tag{5.19}$$

We will derive a contradiction below.

For any $y, y' \in U$ and for any positive integer k, we have

$$|f^k(y) - f^k(p)| \geq |f^k(y) - f^k(y')| - |f^k(y') - f^k(p)| \geq |f^k(y) - f^k(y')| - \delta.$$

Hence by (14), we have

$$|f^k(y) - f^k(y')| \leq 2\delta = \frac{R}{5} \quad \text{for any} \quad y, y' \in U \quad \text{and all integer} \quad k. \tag{5.20}$$

If we choose y a saddle point, then a similar argument as in the first case and (5.18) imply a contradiction to (5.20). □

Let us recall the definition of a normal family.

Definition 5.4.8: Let $\{F_n\}$ be a family of holomorphic mappings defined on an open set $U \subset \mathbb{C}^m$ into $\mathbb{C}^k$. The family is called *a normal family* if every sequence of the family $\{F_n\}$ has a subsequence with either

(i) converges uniformly on every compact subset of U, or

(ii) converges uniformly to ∞ on U.

The family $\{F_n\}$ is called *not normal at point* $z_0 \in U$ if the family fails to be a normal family in every neighborhood of z_0.

Theorem 5.4.9. *The family of iterates $\{f^n\}$ is not normal at any point $z_0 \in SJ(f)$.*

Proof: Without loss of generality, we may assume that z_0 is a fixed saddle point because any point in SJ is accumulated by a sequence of saddle points. Assume that $\{f^n\}$ is a normal family on a neighborhood U of z_0. Since $f^n(z_0) = z_0$ for any integer n, it follows that $f^n(z)$ does not converge to ∞ on U. Thus there is a subsequence $\{f^{n_j}\}$ which converges uniformly to a holomorphic mapping F on U. Hence $|Df^{n_j}(z_0)| \to |DF(z_0)|$ as $j \to \infty$. Assume that $Df(z_0)$ has the form

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad \text{with} \quad |\lambda_1| > 1 \quad \text{and} \quad |\lambda_2| < 1.$$

Then $Df^{n_j}(z_0)$ has the form

$$\begin{pmatrix} \lambda_1^{n_j} & 0 \\ 0 & \lambda_2^{n_j} \end{pmatrix}.$$

Clearly $|Df^{n_j}(z_0)| \to \infty$. This contradiction establishes the result. □

Theorem 5.4.10. *The topological entropy of $f|_{SJ(f)}$ is $\log d$, where d is the degree of f.*

Proof: By [BS3, Corollary 4.5], $h_{top}(f|_{J^*}) = h_{top}(f|_J) = \log d$. This equality together with Corollary 5.4.2(ii) imply the conclusion of this theorem. □

Theorem 5.4.11. *Both the homoclinic points and the heteroclinic points of f are dense in $SJ(f)$.*

Proof: This is a direct consequence of the proof of Theorem 5.3.2. □

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