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Free-discontinuity problems: calibration and approximation of solutions.

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Thesis submitted for the degree of *Doctor Philosophiae*

Academic Year 2000 – 2001

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Contents

Introduction

It is well known that mathematical models of many problems in Fracture Mechanics, Image Segmentation, Static Theory of Liquid Crystals and other applied sciences, involve the minimization of integral functionals defined in spaces of discontinuous functions and consisting of a bulk term and of a surface energy part concentrated along the (free) discontinuity zone (whence the name of "free-discontinuity problems"). The appropriate space where a weak formulation leading to existence results can be done, has been proved to be SBV, the space of special functions of bounded variation introduced by De Giorgi and Ambrosio ([27]); in this setting such functionals take the form

$$\int_\Omega g(x, \nabla u)\, dx + \int_{S_u} \varphi(x, [u], \nu_u)\, d\mathcal{H}^{N-1}, \tag{1}$$

where we denoted by ∇u, S_u, $[u]$ and ν_u the approximate gradient, the jump set, the jump of u and the approximate normal vector to S_u, respectively (we refer to Chapter 2 for the precise definitions). A prototypical example is given by the Mumford-Shah functional

$$F_{\beta,g}(u) = \int_{\Omega\setminus S_u} |\nabla u|^2\, dx + \mathcal{H}^{N-1}(S_u) + \beta \int_{\Omega\setminus S_u} (u-g)^2\, dx, \tag{2}$$

where $\beta > 0$ and $g \in L^\infty(\Omega)$. It was introduced in [44] in the framework of a variational approach to Image Segmentation.

The existence theory for free discontinuity problems has been developed by Ambrosio in [6]; however, his results, which are based on compactness arguments, do not provide any information about the behaviour of the solutions. Actually, one of the most relevant mathematical features of the Mumford-Shah functional and of functionals like (1), is a deep lack of convexity, which causes non-uniqueness of the solutions and makes the exhibition of explicit minimizers a very difficult task, even in terms of numerical approximations. To overcome these difficulties, in the last years a lot of work has been addressed to performing variational approximations (in the sense of De Giorgi's Γ-convergence) via smooth functionals defined in Sobolev spaces, for which the numerical treatment is easier. In this way one can also achieve the goal of defining a parabolic evolution model as limit of the gradient flows of the approximating functionals.

The aim of this thesis is twofold. In the first part we attack the problem of finding explicit solutions, confining our treatment, for simplicity, to the Mumford-Shah functional and to the homogeneous version

$$F_0(u) := \int_\Omega |\nabla u|^2\, dx + \mathcal{H}^{N-1}(S_u), \tag{3}$$

which occurs in the theory of interior regularity for minimizers of $F_{\beta,g}$; in the second part we deal with the variational approximation of general free-discontinuity problems.

Let us start with the description of Part I. First of all, we recall that every minimizer must satisfy suitable equilibrium conditions, which can be obtained by considering different types of variations: according to the classical terminology, we will call them Euler-Lagrange equations. For example, it is immediate to see that if u minimizes $F_{\beta,g}$, then it solves $\Delta u = \beta(u-g)$ in the complement of S_u, with homogeneous Neumann conditions along S_u. Concerning F_0, if S_u is regular, the following equilibrium conditions are satisfied (see [45]):

i) u is harmonic on $\Omega \setminus S_u$;

ii) the normal derivative of u vanishes on both sides of S_u;

iii) the mean curvature of S_u is equal to the difference of the squares of the tangential gradients of u on both sides of S_u.

Here and in the sequel, we will say that u is an extremal for F_0 (or an F_0-extremal) if it satisfies conditions **i)**, **ii)** and **iii)** above. Due to the lack of convexity, the extremality conditions do not imply minimality, not even local minimality, as elementary examples show. The theory of calibration recently developed by Alberti, Bouchitté & Dal Maso in [2], provides us with a sufficient condition for optimality. We will use this method to exhibit a wide class of non-trivial minimizers and to prove that, in many situations, Euler-Lagrange equations imply the minimality in small domains (as it happens for several classical problems of the Calculus of Variations). Before entering the discussion of our results, we want to describe the basic idea of the calibration method, focusing our attention, for simplicity, on the homogeneous functional F_0.

Given u in $SBV(\Omega)$, let u^+ and u^- denote its limits on the two side of the discontinuity set S_u, so that $u^+ > u^-$, and let ν_u be the normal unit vector to S_u pointing towards u^+; the *complete graph* of u, denoted by Γ_u, is the boundary of the subgraph of u, oriented by the inward normal ν_{Γ_u}. In other words, it consists of the union of the usual graph and of all "vertical" segments joining $(x, u^-(x))$ and $(x, u^+(x))$, with x varying in S_u (see Figure 1 below). Let $u : \Omega(\subset \mathbb{R}^N) \to \mathbb{R}$ be an

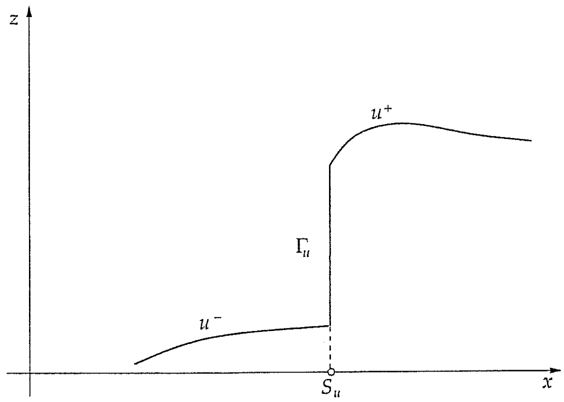

Figure 1: The complete graph of u.

extremal for F_0; a vectorfield $\phi = (\phi^x, \phi^z) : \Omega \times \mathbb{R} \to \mathbb{R}^N \times \mathbb{R}$ is a *calibration* for u in Ω with respect to F_0, if it satisfies the following conditions:

(a) $\text{div}\phi = 0$ in the sense of distributions;

(b) $|\phi^x(x,z)|^2 \leq 4\phi^z(x,z)$ for $x \in \Omega$, $z \in \mathbb{R}$;

(c) $\phi^x(x,u(x)) = 2\nabla u(x)$ and $\phi^z(x,u(x)) = |\nabla u(x)|^2$ for $x \in \Omega \setminus S_u$;

(d) $\displaystyle\int_{u^-(x)}^{u^+(x)} \phi^x(x,z)\,dz = \nu_u(x)$ for $x \in S_u$;

(e) $\left|\displaystyle\int_{t_1}^{t_2} \phi^x(x,z)\,dz\right| \leq 1$ for $x \in \Omega$, t_1, $t_2 \in \mathbb{R}$.

The principle of calibrations says that the existence of such a ϕ implies that u is a *Dirichlet-minimizer* of F_0 in Ω, which means that u minimizes F_0 among the functions in $SBV(\Omega)$ with the same values as u on $\partial\Omega$. Assuming that ϕ is regular enough, the proof of this fact is very simple and short. First of all, given $v \in SBV(\Omega)$, consider the flux of ϕ through Γ_v, which is given by

$$\int_{\Gamma_v} \phi \cdot \nu_{\Gamma_v}\, d\mathcal{H}^N = \int_\Omega [\phi^x(x,v) \cdot \nabla v(x) - \phi^z(x,v)]\, dx + \int_{S_v} \left[\int_{v^-}^{v^+} \phi^x(x,z)\,dz\right] \cdot \nu_v(x)\, d\mathcal{H}^{N-1}(x) \; ; \tag{4}$$

from (4), taking into account **(e)** and the inequality

$$\phi^x(x,v) \cdot \nabla v(x) - \phi^z(x,v) \leq |\nabla v|^2,$$

which is true for every $x \in \Omega \setminus S_v$ thanks to **(b)**, we have

$$\int_{\Gamma_v} \phi \cdot \nu_{\Gamma_v}\, d\mathcal{H}^N \leq F_0(v) \qquad \forall v \in SBV(\Omega). \tag{5}$$

Note also that by **(c)**, **(d)** and (4), it turns out

$$F_0(u) = \int_{\Gamma_u} \phi \cdot \nu_{\Gamma_u}\, d\mathcal{H}^N. \tag{6}$$

We are now in a position to conclude: indeed, for every function v which agrees with u on the boundary of Ω, by (5) and (6), we have

$$F_0(u) = \int_{\Gamma_u} \phi \cdot \nu_{\Gamma_u}\, d\mathcal{H}^N = \int_{\Gamma_v} \phi \cdot \nu_{\Gamma_v}\, d\mathcal{H}^N \leq F_0(v),$$

where the second equality follows from the divergence theorem, since ϕ is divergence-free and Γ_u and Γ_v have the same boundary.

Let us point out that in [2] the theory of calibration method has been developed for general free-discontinuity problems. Concerning the non-homogeneous Mumford-Shah functional $F_{\beta,g}$, a calibration with respect to this functional will be a vectorfield satisfying all the conditions above except **(b)** and **(c)**, which have to be replaced by

(b)' $|\phi^x(x,z)|^2 \leq 4\left[\phi^z(x,z) + \beta(z - u(x))^2\right]$ for $x \in \Omega$, $z \in \mathbb{R}$;

(c)' $\phi^x(x,u(x)) = 2\nabla u(x)$ and $\phi^z(x,u(x)) = |\nabla u(x)|^2 - \beta(g(x) - u(x))^2$ for $x \in \Omega \setminus S_u$.

In [2], the authors provide easy and short proofs of some natural minimality results; in all their applications they deal with minimizers presenting either a vanishing gradient or an empty discontinuity set and such a simple structure allows quite simple constructions. In this thesis we will face the more complicated problem of performing calibrations also for candidate functions with both non-vanishing gradient and nonempty discontinuity set. We will consider only functions with regular discontinuity set; however, some of our results can be extended also to the case when the singular set S_u presents a triple junction (see [37]), while it is completely open the problem of calibrating minimizers with a cracktip.

As a matter of fact, we do not know of any general recipe to find calibrations but each time we have to restart and look for a particular construction working in the particular case under consideration. However, in all the constructions we are going to perform, some common features are present that we try now to describe. First of all they are made by blocks: we decompose the open set U (where we want to define the calibration ϕ) in a finite family of Lipschitz subsets $(A_i)_{i=1,\dots,k}$ and we take ϕ coinciding in A_i with a suitable divergence-free vectorfield ϕ_i; of course, in order to guarantee that ϕ is globally divergence-free in the sense of distributions, the ϕ_i's must satisfy a suitable transmission condition along the interfaces. Our constructions present more or less the structure shown in the following figure:

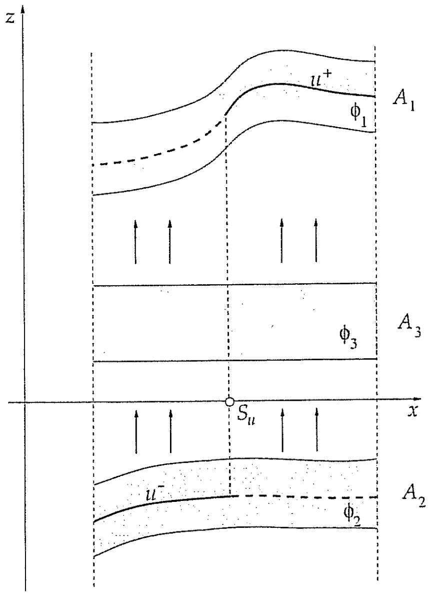

Figure 2: The structure of the calibration.

there are two thin slabs (A_1 and A_2) around the two "branches" of the regular part of graph u and a block A_3 in the "middle"; the other regions are simply transition regions where ϕ is taken purely vertical. Each block plays a different role depending on his location. The slabs A_1 and A_2 are the core of the calibration and the definition of ϕ_1 and ϕ_2 the more delicate point, since it is here that we have to exploit all the information contained in the equilibrium conditions satisfied by u; the block A_3 acts as a corrector: it annihilates the tangential component introduced in A_1 and A_2 and injects the missing normal component in order to fulfill condition **(d)**. The proofs of our minimality results usually proceed as follows: at the beginning we define the candidate calibration ϕ in such a way that conditions **(a)**, **(b)**, **(c)** and **(d)** are automatically satisfied; we let the definition depend on some parameters and in the rest of the (very long and technical) proof we show that a careful choice of them allows to achieve also condition (e). The scheme of that part of the proof is more or less the following. Exploiting the geometrical structure of the vectorfield, we show that condition (e) is proved if we verify that, for every $x_0 \in S_u \cap U$, the function

$$I_{x_0}(t) := \left| \int_{u^-}^{u^+} \phi^x(x_0 + t\nu_u(x_0), z)\, dz \right| \quad \text{(we can think } u^+ \text{ and } u^- \text{ suitably extended beyond } S_u\text{)}$$

has a strict maximum for $t = 0$; then, the Euler-Lagrange equations imply that $I'_{x_0}(0) = 0$ while tuning the parameters leads to $I''_{x_0}(0) < 0$, which gives the desired result.

Let us now describe in deeper details the content of the first part of the thesis.

In Chapter 2 (which contains the results of [26]), we prove that if u is an extremal for the homogeneous Mumford-Shah functional F_0 on a two-dimensional domain, and if S_u is a straight segment connecting two boundary points, then, for every $(\overline{x}_1, \overline{x}_2) \in S_u$ (under the additional technical assumptions that $\nabla u(\overline{x}_1, \overline{x}_2) \neq 0$ and $\partial^2_{\tau\tau} u(\overline{x}_1, \overline{x}_2) \neq 0$, where ∂_τ denotes the tangential derivative), there exists a neighbourhood U of $(\overline{x}_1, \overline{x}_2)$ such that u minimizes F_0 in U with respect to its own boundary values on ∂U (i.e. it is a Dirichlet-minimizer). In Section 2.2 we treat the special case of

$$u(x_1, x_2) := \begin{cases} x_1 & \text{if } x_2 > 0, \\ -x_1 & \text{if } x_2 < 0 \end{cases} \tag{7}$$

and we give the first example of calibration for a discontinuous function which is not locally constant. Although it is the simplest one, such a case involves most of the technical difficulties of the general one. From the point of view of calibrations, the interaction between the (non-vanishing) gradient and the (nonempty) discontinuity set is reflected in the fact that we have to guarantee simultaneously conditions (c) and (d), which push in opposite directions. Indeed, condition (c) forces ϕ^x to be tangential to S_u on the graph of u while (d) says that ϕ^x must be on the average orthogonal to S_u for $x \in S_u$ and t between $u^-(x)$ and $u^+(x)$; so we have, in some way, to "rotate" ϕ^x and this must be done carefully, without compromising the other conditions. Fashioned upon this simple geometric idea of "rotating" ϕ^x, our candidate calibration ϕ, near the graph of u (i.e. in the blocks A_1 and A_2 of Figure 2), takes the form

$$(A(x_1, x_2, z)2e_1, 1),$$

where $A(x_1, x_2, z)$ is a suitable orthogonal matrix which, in view of condition (c), must coincide with the identity for $z = u(x_1, x_2)$ (see Figure 3 below). In treating general extremal functions w with a rectilinear discontinuity set, we perform the change of variable $(x_1, x_2) \mapsto (w(x_1, x_2), v(x_1, x_2))$, where v is the harmonic conjugate of w; under this map, which is supposed to be conformal in a neighbourhood of the point $(\overline{x}_1, \overline{x}_2) \in S_u$ (whence the technical assumption $\nabla w(\overline{x}_1, \overline{x}_2) \neq 0$), w is tranformed into the function u of (7) so that we can "recycle" the construction above. Concerning the definition of ϕ in A_1 and A_2, this procedure turns out to be equivalent to taking

$$\phi := \left(A(w, v, z)2\nabla w, |\nabla w|^2\right),$$

where A is the matrix used for the function in (7).

In Chapter 3 (which contains the results of [39]) we deal with general F_0-extremals defined in two-dimensional domains, whose discontinuity set can be now any analytic curve joining two boundary points. The use of a new technique enables us to prove that the Dirichlet-minimality holds not only in a neighbourhood of each point of S_u, but, actually, in a uniform neighbourhood of the whole S_u (see

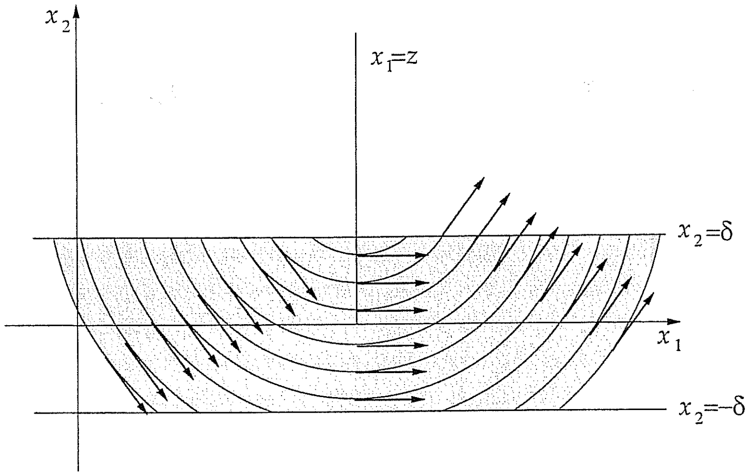

Figure 3: Section of the set A_1 at z=constant.

Theorem 3.1.1). Let us underline that the analyticity assumption for S_u does not seem too restrictive: it has been proved that the regular part of the discontinuity set of a minimizer is of class C^∞ and it is a conjecture that it is analytic (see [9]).

The main novelty concerns the definition of ϕ in the blocks A_1 and A_2. We begin with the following remark: if we fibrate a neighbourhood of the graph of u by the graphs of a family of harmonic functions $(v_t)_{t\in\mathbb{R}}$ and we let ϕ be equal to

$$(2\nabla v_t, |\nabla v_t|^2)$$

on the graph of v_t, then ϕ is divergence free (see Lemma 3.2.1). We perform such a construction using the family

$$v_t = u + tv,$$

where v is a suitable harmonic function with gradient orthogonal to S_u, which does the work of injecting some of the normal component needed to fulfill condition **(d)**. Note that this technique resembles the classical method of the Weierstrass fields, where the proof of the minimality is achieved by considering the gradient field of a family of extremals, foliating a neighbourhood of the graph of u.

We are also interested in a different type of minimality: in Theorem 3.1.1 we compare u with competitors of the form $u + w$, where the perturbation w can be very large, but vanishes outside a fixed small neighbourhood of S_u; we wonder if a minimality property is preserved also when the perturbation has L^∞-norm very small outside a (small) neighbourhood of S_u, but support possibly coinciding with $\overline{\Omega}$.

Accordingly, we say that $u \in SBV(\Omega)$ is a local graph-minimizer in Ω if there exists a neighbourhood U of the complete graph of u such that $F_0(u) \leq F_0(v)$, for every $v \in SBV(\Omega)$ with the same values as u on $\partial\Omega$ and with complete graph contained in U (see the figure below).

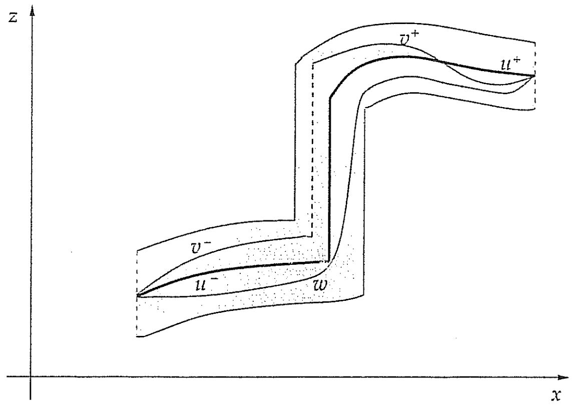

Figure 4: v and w are admissible competitors for the graph-minimality.

In [2] it is proved that any harmonic function defined in Ω is a local graph-minimizer whatever Ω is. If the function presents some discontinuities, it turns out that the graph-minimality may fail when Ω is too large, even in the case of rectilinear discontinuities, as the counterexample given in Section 3.3 shows.

Therefore, we have to add some restrictions on the domain Ω. To this aim, we introduce a suitable notion of capacity which seems useful to describe the right interplay between S_u and Ω. Given an open set A (with Lipschitz boundary) and a portion Γ of ∂A (with nonempty relative interior in ∂A), we define

$$K(\Gamma, A) := \inf \left\{ \int_A |\nabla v|^2 \, dx : \; v \in H^1(A), \; \int_\Gamma v^2 d\mathcal{H}^1 = 1, \text{ and } v = 0 \text{ on } \partial A \setminus \Gamma \right\}.$$

Note that if $A_1 \subset A_2$, and $\Gamma_1 \subset \Gamma_2$, then $K(\Gamma_1, A_1) \geq K(\Gamma_2, A_2)$, which suggests that if $K(\Gamma, A)$ is very large, then A is thin in some sense. In Subsection 4.3.1 (see Theorem 3.1.4) we prove that given an analytic curve Γ, there exists a positive constant $C(\Gamma)$, depending explicitly on the length and on the curvature of Γ, such that if Ω is Γ-admissible (that is $\Gamma \cap \overline{\Omega}$ connects two points of $\partial\Omega$ and $\Omega \setminus \Gamma$ has two connected components Ω_1 and Ω_2), and if $u \in SBV(\Omega)$ is an extremal satisfying $S_u = \Gamma \cap \Omega$ and

$$\min_{i=1,2} K(\Gamma \cap \Omega, \Omega_i) > C(\Gamma)(\|\partial_\tau u^+\|^2_{C^1(\Gamma\cap\Omega)} + \|\partial_\tau u^-\|^2_{C^1(\Gamma\cap\Omega)}),$$

then u is a local graph-minimizer in Ω.

Remark that the condition above imposes a restriction on the size of Ω depending on the behaviour of u along S_u: if u has large or very oscillating tangential derivatives, we have to take Ω quite small to satisfy it. In the special case of a locally constant function u, such a condition is always fulfilled and therefore u is a local graph-minimizer whatever Ω is, in agreement with a result proved in [2]. In the final part of the chapter we investigate some properties of $K(\Gamma, \Omega)$ and we present an application of Theorem 3.1.4.

In Chapter 4 (containing the results of [40]) we are interested in the minimizers of the non-homogeneous Mumford-Shah functional $F_{\beta,g}$ (see (2)). We recall that in the context of Image Segmentation, for $N = 2$, g is the input grey level function, the function u which minimizes $F_{\beta,g}$ represents a piecewise smooth approximation of g (the processed image) and S_u the set of relevant contours. It is intuitive that taking β very large forces the solution u to be close to g. This assertion is made precise by the following simple computation: assuming for simplicity that g belongs to $SBV(\Omega)$ and satisfies

$$F_{\beta,g}(g) = \int_{\Omega\setminus S_g} |\nabla g|^2\, dx + \mathcal{H}^{n-1}(S_g) = C < +\infty, \tag{8}$$

and letting u_β be a minimum point of $F_{\beta,g}$, we have

$$\int_\Omega (u_\beta - g)^2\, dx \leq \frac{F_{\beta,g}(u_\beta)}{\beta} \leq \frac{F_{\beta,g}(g)}{\beta} = \frac{C}{\beta},$$

that is $u_\beta \to g$ in $L^2(\Omega)$ as $\beta \to +\infty$. This suggests that, in agreement with our expectations, if β is large, u_β is an "accurate" reconstruction of the original image g. Actually, T.J.Richardson in [48] has proved also the convergence of the discontinuity sets in dimension two: more precisely, he has shown that if g satisfies (8) and is of class $C^{0,1}$ outside any neighbourhood of the singular set S_g, then

$$S_{u_\beta} \to S_g \text{ in the Hausdorff metric} \qquad \text{and} \qquad \mathcal{H}^1(S_{u_\beta}) \to \mathcal{H}^1(S_g),$$

as $\beta \to +\infty$. In the main theorem of the chapter (see Theorem 4.3.2), using the calibration method, we are able to prove that, under suitable assumptions on the regularity of Ω, g, and S_g, the following stronger result holds true.

Suppose that Γ is a closed hypersurface of class $C^{2,\alpha}$ contained in the N-dimensional domain Ω (satisfying in turn suitable regularity assumptions), and let g belong to $W^{1,\infty}(\Omega\setminus\Gamma)$, with $S_g = \Gamma$ and $\inf_{x\in\Gamma}(g^+(x) - g^-(x)) > 0$ (where g^+ and g^- denote the upper and the lower traces of g on Γ). Then, there exists $\beta_0 > 0$, depending only on Γ, on the $W^{1,\infty}$-norm of g, and on the size of the jump of g along Γ, such that, for $\beta \geq \beta_0$, $F_{\beta,g}$ admits a unique absolute minimizer u_β which satisfies $S_{u_\beta} = \Gamma$ and

$$\begin{cases} \Delta u_\beta = \beta(u_\beta - g) & \text{in } \Omega\setminus\Gamma \\ \partial_\nu u_\beta = 0 & \text{on } \partial(\Omega\setminus\Gamma); \end{cases} \tag{9}$$

in other words $F_{\beta,g}$ not only approximates but, in fact, reconstructs the regular contours exactly, when the fidelity parameter β is sufficiently large. Note that, differently from the previous chapters, we are concerned now with global minimizers. We point out that the case of g equal to the characteristic function of a regular set and the case of g regular in the whole Ω have been already treated in [2] and require a simpler construction. The starting point for the definition of the calibration is similar to the one of Chapter 3: if we fibrate a neighbourhood of the graph of u_β by the graphs of a family of functions $(v_t)_{t\in\mathbb{R}}$ all satisfying (9) and we let ϕ be equal to

$$(2\nabla v_t, |\nabla v_t|^2 - \beta(v_t - g)^2)$$

on the graph of v_t, then ϕ is divergence-free. But here there is a new difficulty due to the fact that we are looking for a vectorfield defined in the whole Ω: actually, we have to suitably modify the construction above in order to make it working "globally". Another technical difficulty originates from the need of estimating how quickly the gradient of u_β changes. Indeed suppose that **(d)** holds true; then, if near Γ the gradient suddenly becomes orthogonal to Γ or abruptly increases its modulus, it could happen that condition **(f)** is violated; this risk can be bypassed by carefully estimating the L^∞-norm of the Hessian matrix $\nabla^2 u_\beta$ with respect to β: this is what we do in Section 4.2 where, using some tools of sectorial operators theory and interpolation theory, we prove that for any positive γ sufficiently small, there exists a constant K, independent of β, such that the solution u_β of (9) satisfies

$$\|\nabla^2 u_\beta\|_\infty \leq K\beta^{\frac{1}{2}+\gamma}\|g\|_{W^{1,\infty}}.$$

In order to perform such an estimate we need to assume that Γ is of class $C^{2,\alpha}$, for some $\alpha > 0$ (α can be arbitrarily small), however, at least in dimension two, this regularity assumption is close to optimal, since, by the Bonnet Regularity Theorem (see [11]), in a neighbourhood of any regular point, the discontinuity set is of class $C^{1,1}$, for every $g \in L^\infty(\Omega)$.

In Subsection 4.3.2 we extend, for $N = 2$, Theorem 4.3.2 to the case when Ω has piecewise smooth boundary (say a curvilinear polygon) and Γ touches the boundary.

As an application of our results, we give in Section 4.4 a proof of the following fact: if u_0 is regular enough outside a smooth singular set S_{u_0}, then the gradient flow $u(x,t)$ of u_0 for the homogeneous functional F_0 keeps, at least for small times, the singular set of $u(\cdot,t)$ equal to S_{u_0}, while u evolves in $\Omega \setminus S_{u_0}$ according to the heat equation with Neumann boundary conditions on $\partial(\Omega \setminus S_{u_0})$. In dimension one, this result was proved by Gobbino (see [29]), with a slightly different definition of gradient flow.

Let us switch now to the second part of the thesis which concerns, as we said before, the variational approximation of free-discontinuity problems. When the volume part of the energy is given by $\int_\Omega |\nabla u|^2\,dx$, heuristic considerations suggest to use, as approximating functionals, energies of the form

$$\frac{1}{\varepsilon}\int_\Omega f(\sqrt{\varepsilon}|\nabla u|)\,dx,$$

where $f : [0,+\infty) \to [0,+\infty)$ is quadratic near the origin and with finite limit at infinity. However, an easy convexity argument shows that energies of this kind Γ-converge to the zero functional. Various methods have been developed to bypass this convexity constraint, most of them exploiting the De Giorgi's suggestion of replacing the functionals above with suitable non-local versions (see [16], [30], [23]). The approach we consider in Chapter 5 (which contains the results of [41]) is based on singular perturbations and consists in adding a "small" term depending on higher derivatives: the idea is to impose a bound on the oscillations of minimizing sequences by penalizing abrupt changes of the gradient. So we are led to consider energies of the form

$$\frac{1}{\varepsilon}\int_\Omega f(\sqrt{\varepsilon}|\nabla u|)\,dx + r(\varepsilon)\int_\Omega \|\nabla^2 u\|^2\,dx, \tag{10}$$

where $r(\varepsilon)$ is a function which vanishes as $\varepsilon \to 0^+$.

The first progress in this direction was made by Alicandro, Braides & Gelli in [4]: they showed that the one dimensional functionals

$$\frac{1}{\varepsilon}\int_0^1 f(\sqrt{\varepsilon}|u'|)\,dx + \varepsilon^3\int_0^1 |u''|^2\,dx,$$

with $f(t) = \alpha t^2 \wedge \beta$, Γ-converge with respect to the L^1-norm to the functional

$$\alpha\int_0^1 |u'|^2\,dx + c(\beta)\sum_{S_u}\sqrt{u^+ - u^-},$$

where $c(\beta)$ is a suitable constant depending on β; later Alicandro & Gelli treated the N-dimensional case (see [5]). We aim to extend the results above to general functionals of the form (10), where f is still quadratic near the origin, but possibly unbounded. In fact we face the problem in a more general framework, by investigating the asymptotic behaviour of

$$F_\varepsilon(u) := \int_\Omega f_\varepsilon(|\nabla u|)\,dx + (r(\varepsilon))^3 \int_\Omega \|\nabla^2 u\|^2\,dx, \tag{11}$$

where (f_ε) is any family of positive non-decreasing functions with a convex or convex-concave shape (i.e. there exists $x_\varepsilon > 0$ such that f_ε is convex in $[0, x_\varepsilon]$ and concave in $[x_\varepsilon, +\infty)$); let us remark that such a structure assumption is quite natural for this kind of problems (see, for example, [17], [18], [31]). In the main theorem of the chapter (Theorem 5.1.2) we prove that the Γ-limits of (11) are related to the pointwise limits of $f_\varepsilon(t)$ and of $r(\varepsilon)f_\varepsilon(t/r(\varepsilon))$: if for an infinitesimal subsequence (ε_n) we have

a) $f_{\varepsilon_n} \to g$ pointwise,

b) $r(\varepsilon_n)f_{\varepsilon_n}(\cdot/r(\varepsilon_n)) \to b$ pointwise,

then (F_{ε_n}) Γ-converges to a functional F defined on $BV(\Omega)$ and taking the form

$$F(u) = \int_\Omega f(|\nabla u|)\,dx + \int_{S_u} \varphi(u^+ - u^-) + C|D^c u|, \tag{12}$$

where C (possibly equal to $+\infty$, meaning that F is finite only on SBV), f, and φ can be characterized in terms of g and b.

The "regularizing" effect due to the presence of the second derivatives in the approximating functionals, determines a restriction on the regularity and on the growth of the jump-function φ, which turns out to satisfy the growth condition

$$C_1(\sqrt{z} - 1) \le \varphi(z) \le C_2(z+1) \qquad \forall z \ge 0,$$

for suitable C_1, $C_2 > 0$, whenever $\lim_{t\to 0^+} b(t)/t \neq 0$; moreover, we always have $\varphi(0) = 0$. In particular, the Mumford-Shah functional is not reachable by our procedure. However, since for any positive, convex, and superlinear function g and for any positive and concave function b with $\lim_{t\to 0^+} b(t)/t = +\infty$, it is possible to construct a family (f_ε) and a rescaling function $r(\varepsilon)$ such that conditions a) and b) above are fulfilled, we see that a wide class of free-discontinuity functionals with φ satisfying (5.1.57) can be approximated. Letting b vary among the possible choices, we may conjecture to recover most of the admissible asymptotic behaviours as the following fact seems to suggest: all the functions of the form $\varphi(t) = ct^\gamma$, with $c > 0$ and γ varying in $[1/2, 1]$ are reachable, and for every $\gamma \in [1/2, 1)$ a function φ can be generated such that

$$\lim_{z\to+\infty} \frac{\varphi(z)}{z^\gamma} = +\infty \qquad \text{and} \qquad \lim_{z\to+\infty} \frac{\varphi(z)}{z^{\gamma+\varepsilon}} = 0 \qquad \forall \varepsilon > 0.$$

As announced, in Section 5.2 we apply our theorem to prove that if f is quadratic near the origin, sublinear, and concave at infinity, there exists a rescaling function $r(\varepsilon)$ (explicitly given in terms of f) such that the family (10) Γ-converges, up to passing to a subsequence, to a free-discontinuity functional like (12). All the possible Γ-limits of that family are classified. The rescaling $r(\varepsilon)$ is unique up to asymptotic equivalence, in the sense that when we use functions with a different behaviour near the origin, we obtain in the limit either $F \equiv 0$ or the functional $\alpha \int |\nabla u|^2\,dx$ defined only on $H^1(\Omega)$.

In a recent paper ([13]) Bouchitté, Dubs & Seppecher considered the one-dimensional functionals

$$F_\varepsilon(u) := \int_I \frac{|u'|^2}{1+(\varepsilon|u'|)^p}\,dx + \varepsilon^{\frac{3p}{p-1}\vee 4}\int_I |u''|^2\,dx$$

defined in $W^{2,2}(I)$ and proved that they Γ-converge to the functional F (defined in $SBV(I)$) given by

$$F(u) := \int_I |u'|^2\,dx + k_p \sum_{x\in S_u}(u^+-u^-)^{\frac{4-p}{2+p}\vee 0}.$$

When $p \le 2$ their result is a particular case of ours (but it is proved by the use of different techniques); on the contrary, the case $p > 2$ is not included in our treatment since the potential $f(t)$ becomes decreasing and degenerates at infinity; note that the use of a degenerate potential allows the approximation of the Mumford-Shah functional (the case $p > 4$).

Let us also point out that our theorem applies to the study of the singular perturbations of the rescaled Perona-Malik energy

$$\frac{1}{\varepsilon}\int_\Omega \log(1+\varepsilon|\nabla u|^2)\,dx :$$

we will show that the right rescaling function is given by $r(\varepsilon) = \frac{\varepsilon}{\log\frac{1}{\varepsilon}}$ and that the family

$$\frac{1}{\varepsilon}\int_\Omega \log(1+\varepsilon|\nabla u|^2)\,dx + \left(\frac{\varepsilon}{\log\frac{1}{\varepsilon}}\right)^3\int_\Omega \|\nabla^2 u\|^2\,dx$$

Γ-converges to

$$\int_\Omega |\nabla u|^2\,dx + c\int_{S_u}\sqrt{u^+-u^-}\,d\mathcal{H}^{N-1},$$

with $c > 0$ explicitly computable (see Example 5.2.8). The Perona-Malik functional was introduced in the context of Image Processing. Let us briefly recall the problem: if g is the input grey level function representing the original image, the simplest way to smooth and denoise it is to apply a gaussian convolution kernel; this procedure turns out to be equivalent to letting g evolve according to the heat equation, i.e. to taking as processed image the solution $u(x,t)$ of the heat diffusion equation

$$\frac{\partial}{\partial t}u = \Delta u \qquad u(x,0) = g(x), \tag{13}$$

computed at time t ("t" can be seen as a scale parameter: the greater it is, the smaller is the scale at which the smoothing occurs).

The main drawback of this approach is that it produces an inconditional smoothing which cannot distinguish between objects and contours, since also edges begin soon to diffuse! To overcome these difficulties Perona and Malik proposed in [47] a model of selective smoothing where the contours are preserved as much as possible: it consists in replacing (13) by the nonlinear equation

$$\frac{\partial}{\partial t}u = \operatorname{div}\left(\frac{\nabla u}{1+|\nabla u|^2}\right) \qquad u(x,0) = g(x), \tag{14}$$

which is the gradient flow of the (Perona-Malik) functional $\int_\Omega \log(1+|\nabla u|^2)\,dx$. The underlying idea is the following: where $|\nabla u|$ is large, in particular, near the edges, the diffusion is low and the contour is "kept", while far from the edges, where the gradient is small, u diffuses as in the heat equation. Note that the simultaneous smoothing and edge detection effects of the equation strongly depend on the particular structure of the function $\log(1+t^2)$: the quadratic behaviour near the origin is responsible of the denoising process while the concave and sublinear behaviour at infinity is responsible of the edge detection. Our Γ-convergence result says that there is an alternative procedure, based on minimizing the (rescaled) energy instead of considering its gradient flow, which exploits the structure of $\log(1+t^2)$ generating again a smoothing and edge detection effect (in the final chapter of the thesis we will explain a further way to produce such an effect starting from rescaled Perona-Malik energies).

Actually, the same considerations apply to all functions f satisfying our structure assumptions and we can think the functionals $\int f(|\nabla u|)\,dx$ as "generalized Perona-Malik energies" giving rise to "generalized Perona-Malik equations" of the form

$$\frac{\partial}{\partial t}u = \mathrm{div}\,(g(|\nabla u|)\nabla u) \qquad u(x,0)=g(x),$$

with g bounded and decreasing to 0 when $|\nabla u|$ is large.

We want to mention, as a further application of our main result, the study of the asymptotic behaviour of the family

$$\frac{1}{\varepsilon}\int_\Omega f(\varepsilon|\nabla u|)\,dx + \varepsilon^3\int_\Omega \|\nabla^2 u\|^2\,dx,$$

where f is non-decreasing, differentiable at the origin, with non-zero derivative, and concave at infinity: the Γ-limit turns out to be a functional defined in $BV(\Omega)$ and taking the form

$$f'(0)\int_\Omega |\nabla u|\,dx + \int_{S_u} \varphi(u^+ - u^-)\,d\mathcal{H}^{N-1} + f'(0)|D^c u|, \tag{15}$$

with φ explicitly characterized in terms of f. Again, as f varies among all the admissible potentials, a wide class of jump-functions (satisfying (5.1.57)) can be generated (see Theorem 5.2.12 and Example 5.2.13).

Some final remarks are in order. All the convergence results we mentioned above are completely proved in the one-dimensional case; in N dimensions one can prove the following. Let (F_n) be a sequence of one-dimensional functionals converging to F and denote by (F_n^N) and F^N their respective N-dimensional versions; then we show that $\Gamma\text{-}\lim_n F_n^N(u) = F^N(u)$ if u satisfies

$$\exists\, u_k \to u \quad \text{s.t} \quad \mathcal{H}^{N-1}(S_{u_k}) < +\infty \quad \text{and} \quad F^N(u_k) \to F^N(u).$$

The class of such functions coincides with the whole space if F^N is finite in BV so that in this case the Γ-convergence is completely proved; we believe that the same occurs when F^N is defined in SBV but, at the moment, such a technical result is not available, and, in fact, the representation of the Γ-limit is performed for functions with discontinuity set of finite $\mathcal{H}^{N-1}$-measure. Let us finally remark that these difficulties arise in the proof of the Γ-lim sup inequality; on the other hand, the Γ-lim inf inequality is completely proved as well as the equicoerciveness of the approximating functionals which guarantees the convergence of minimizers.

In Chapter 6 (which contains the results of [42]) we deal again with the Perona-Malik functional, but following a different approach, more suitable to numerical applications in Image Segmentation. Considering that the structure of a digital image is simply a lattice of picture elements (the so-called *pixels*) it is natural to use techniques based on finite differences. In this context, considering the Gobbino's paper [30] , Chambolle proposed in [20] a functional of the form

$$F_\varepsilon(u) = \varepsilon^2 \sum_{x\in\Omega\cap\varepsilon\mathbb{Z}^2} \sum_{\substack{\xi\in\mathbb{Z}^2\\ x+\varepsilon\xi\in\Omega}} \frac{1}{\varepsilon} f\left(\frac{|u(x+\varepsilon\xi)-u(x)|^2}{\varepsilon|\xi|^2}\right)\rho(\xi)\,,$$

where Ω is a two-dimensional domain, the function $f : [0, +\infty) \to [0, +\infty)$ is non-decreasing, continuous and satisfies

$$f'(0) = 1 \qquad \text{and} \qquad f(+\infty) = 1,$$

while the convolution term $\rho : \mathbb{Z}^2 \to [0, +\infty)$ is even and satisfies

$$\rho(0) = 0\,, \qquad \sum_{\xi \in \mathbb{Z}^2} \rho(\xi) < +\infty\,, \qquad \rho(\xi) > 0 \quad \text{if } |\xi| = 1\,, \text{ and} \qquad \rho(\xi) = \rho(\xi^\perp)\,. \tag{16}$$

Chambolle proved that the Γ-limit is the anisotropic Mumford-Shah functional given by

$$c_\rho \int_\Omega |\nabla u|^2 \, dx + \int_{S_u} \Phi(\nu) \, d\mathcal{H}^1, \tag{17}$$

where

$$c_\rho := \frac{1}{2} \sum_{\xi \in \mathbb{Z}^2} \rho(\xi) \quad \text{and} \qquad \Phi(\nu) := \sum_{\xi \in \mathbb{Z}^2} \rho(\xi) |\nu \cdot \hat{\xi}|$$

($\hat{\xi}$ stands for $\frac{\xi}{|\xi|}$). In our main theorem we prove that (17) is the Γ-limit also of the following discrete Perona-Malik functionals

$$F_\varepsilon(u) = \varepsilon^2 \sum_{x \in \Omega \cap \varepsilon \mathbb{Z}^2} \sum_{\substack{\xi \in \mathbb{Z}^2 \\ x + \varepsilon \xi \in \Omega}} \frac{1}{a_\varepsilon |\xi|} \log \left(1 + a_\varepsilon |\xi| \frac{|u(x + \varepsilon \xi) - u(x)|^2}{\varepsilon^2 |\xi|^2} \right) \rho(\xi)\,,$$

where $a_\varepsilon = \varepsilon \log \frac{1}{\varepsilon}$ and ρ satisfies (16). Numerical experiments are in progress and the results will appear in the final version of [42].

The results contained in Chapters 2 have been obtained in collaboration with G. Dal Maso and M.G. Mora and are published in [26], while the results of Chapter 3, obtained in collaboration with M.G. Mora, are published in [39]; the content of Chapters 4 and 5 corresponds to the papers [40] and [41] respectively and finally, the results stated in Chapter 6 will appear soon in a joint paper with M. Negri (see[42]).

Chapter 1

Preliminary results

In this chapter we fix the main notation and collect some preliminary results that we shall need in the sequel.

1.1 BV functions

1.1.1 Definitions and general properties

In this subsection we fix notations and we briefly recall basic definitions and properties from the theory of BV functions: for a general treatment we refer to [9]. The Lebesgue measure and the $(N-1)$-dimensional Hausdorff measure of a set $B \subset \mathbb{R}^N$ are denoted by $\mathcal{L}^N(B)$ and $\mathcal{H}^{N-1}(B)$ respectively. We will often write $|B|$ instead of $\mathcal{L}^N(B)$. Given a measure μ we denote its total variation by $|\mu|$; moreover $\mu\lfloor B$ denotes the restriction of the measure μ to the set B given by $(\mu\lfloor B)(A) = \mu(B \cap A)$.

Let $\Omega \subset \mathbb{R}^N$ be an open set, let $u : \Omega \to \mathbb{R}$ be a measurable function, and let $x \in \Omega$. We denote by $u^+(x)$ and $u^-(x)$, respectively, the upper and lower limit of u at x, defined by

$$\begin{aligned} u^+(x) &:= \inf\left\{t \in \mathbb{R} : \lim_{\rho\to 0+} \frac{|\{y \in \Omega : |x-y| < \rho,\ u(y) > t\}|}{\rho^N} = 0\right\}, \\ u^-(x) &:= \sup\left\{t \in \mathbb{R} : \lim_{\rho\to 0+} \frac{|\{y \in \Omega : |x-y| < \rho,\ u(y) < t\}|}{\rho^N} = 0\right\}. \end{aligned}$$

If $u^+(x) = u^-(x) \in \mathbb{R}$, then the common value of $u^+(x)$ and $u^-(x)$ is called the *approximate limit* of u at the point x, and is denoted by $\text{ap-}\lim_{y\to x} u(y)$.

We say that u is a *function of bounded variation* in Ω, and we write $u \in BV(\Omega)$, if $u \in L^1(\Omega)$ and its distributional derivative is a vector-valued measure Du with finite total variation $|Du|(\Omega)$. Given $u \in BV(\Omega)$, we denote by by J_u the set where $u^+ > u^-$ and by S_u the *essential discontinuity set* of u made up of those points x which are not Lebesgue points. It turns out that $J_u \subseteq S_u$ and $\mathcal{H}^{N-1}(S_u \setminus J_u) = 0$. For every $x \notin S_u$ we denote by $\tilde{u}(x)$ the approximate limit of u at x.

The *complete graph* of a function $u \in BV(\Omega)$ is the set

$$\Gamma_u := \{(x,z) \in \Omega \times \mathbb{R} : u^-(x) \le z \le u^+(x)\}.$$

If $u \in BV(\Omega)$, then it can be proved that S_u is countably $(\mathcal{H}^{N-1}, N-1)$ rectifiable, i.e.

$$S_u = N \cup \bigcup_{i\in\mathbb{N}} K_i,$$

where $\mathcal{H}^{N-1}(N)=0$, and each K_i is a compact set contained in a C^1 hypersurface; as a consequence we have that for $\mathcal{H}^{N-1}$-a.e. $x \in S_u$ it is possible to define an approximate tangent plane $T_x(S_u)$ and therefore an approximate normal unit vector $\nu_u(x)$ which can be chosen in such a way that

$$\lim_{\rho \to 0^+} \int_{B_\rho^{\nu_u(x)}(x)} |u(y)-u^+(x)|\, dy = 0,$$

where $B_\rho^{\nu_u(x)}(x) := \{y \in B_\rho(x) : (y-x)\cdot \nu_u(x) > 0\}$ (here and in the sequel, given x and y in $\mathbb{R}^N$, we denote the scalar product of x and y by $x \cdot y$). For every $u \in BV(\Omega)$, by the Radon-Nykodim Theorem we can write $Du = D^a u + D^s u$, where $D^a u$ is absolutely continuous and $D^s u$ is singular with respect to the Lebesgue measure. We denote the density of $D^a u$ with respect to the Lebesgue measure by ∇u. Moreover, we denote the restriction of $D^s u$ to S_u by $D^j u$, and the restriction of $D^s u$ to $\Omega \backslash S_u$ by $D^c u$. It turns out that $D^j u = (u^+ - u^-)\nu_u \mathcal{H}^{N-1} \lfloor S_u$ so that in particular

$$|D^s u| = |D^c u| + (u^+ - u^-)\mathcal{H}^{N-1} \lfloor S_u.$$

We will say that a set E is of *finite perimeter* in Ω if χ_E (i.e. the characteristic function of E) is of bounded variation in Ω. We define $\partial^* E \cap \Omega := S_{\chi_E} \cap \Omega$ the *reduced boundary* of E in Ω. Let us recall now the Fleming-Rishel *coarea formula*. Let u be a Lipschitz function and let v belong to $BV(\Omega)$. Then for almost every $t \in \mathbb{R}$ we have that $\{x \in \Omega : u > t\}$ is a set of finite perimeter in Ω and

$$\int_\Omega v|\nabla u|\, dx = \int_{-\infty}^{+\infty} \left(\int_{\partial^*\{u>t\}\cap\Omega} \tilde{v}\, d\mathcal{H}^{N-1} \right) dt \tag{1.1.1}$$

We say that u is a *special function of bounded variation*, and we write $u \in SBV(\Omega)$, if $u \in BV(\Omega)$ and $D^c u = 0$. For each $p \geq 1$ the space of all functions $u \in SBV(\Omega)$ such that

$$\nabla u \in L^p(\Omega) \qquad \text{and} \qquad \mathcal{H}^{N-1}(S_u) < +\infty$$

is denoted by $SBV^p(\Omega)$. We consider also the larger space $GBV(\Omega)$, which is composed by all measurable functions $u : \Omega \to \mathbb{R}$ whose truncations $u_k = (u \wedge k) \vee (-k)$ belong to $BV(\Omega)$ for every $k > 0$; finally we set

$$GSBV := \{u \in GBV(\Omega) : |D^c u_k| = 0 \ \forall k > 0\} = \{u \in L^1(\Omega) : u_k \in SBV(\Omega) \ \forall k > 0\},$$

and

$$GSBV^p(\Omega) := \{u \in L^1(\Omega) : u_k \in SBV^p(\Omega) \ \forall k > 0\}.$$

Every $u \in GBV(\Omega) \cap L^1_{loc}(\Omega)$ has a countably $(\mathcal{H}^{N-1}, N-1)$ rectifiable discontinuity set S_u.

We conclude this subsection by recalling a "slicing" result due to Ambrosio (see [7]) and a L^1-precompactness criterion by slicing proved in [3]. We introduce first some notation. Let $\xi \in S^{N-1}$ and let $\Pi_\xi := \{y \in \mathbb{R}^N : y \cdot \xi = 0\}$ be the linear hyperplane orthogonal to ξ. Given $E \subset \mathbb{R}^N$ we denote by $E_\xi \subseteq \Pi_\xi$ the orthogonal projection of E on Π_ξ and for $y \in \Pi_\xi$ we set $E_\xi^y := \{t \in \mathbb{R} : y + t\xi \in E\}$. Finally for $u : E \to \mathbb{R}$ we define $u_\xi^y : E_\xi^y \to \mathbb{R}$ by $u_\xi^y(t) := u(y+t\xi)$.

Theorem 1.1.1 *a) Let $u \in BV(\Omega)$. Then, for all $\xi \in S^{N-1}$ the function u_ξ^y belongs to $BV(\Omega_\xi^y)$ for $\mathcal{H}^{N-1}$-a.e. $y \in \Pi_\xi$. For such y we have*

$$\begin{aligned} (u_\xi^y)'(t) &= \nabla(y+t\xi)\cdot\xi \qquad \textit{for a.e. } t \in \Omega_\xi^y, \\ S_{u_\xi^y} &= (S_u)_\xi^y, \\ u_\xi^y(t\pm) &= u^\pm(y+t\xi) \qquad \textit{or} \qquad u_\xi^y(t\pm) = u^\mp(y+t\xi), \end{aligned}$$

according to the case $\nu_u \cdot \xi > 0$ *or* $\nu_u \cdot \xi\rangle < 0$ *(the case* $\nu_u \cdot \xi\rangle = 0$ *being negligible). Moreover we have*

$$\int_{\Pi_\xi} |D^c u_\xi^y|(A_\xi^y)\, d\mathcal{H}^{N-1}(y) = |D^c u \cdot \xi|(A),$$

for all open subset $A \subseteq \Omega$, *and for all Borel functions* g

$$\int_{\Pi_\xi} \sum_{t \in S_{u_\xi^y}} g(t)\, d\mathcal{H}^{N-1}(y) = \int_{S_u} g(x)|\nu_u \cdot \xi|\, d\mathcal{H}^{N-1}.$$

b) Conversely, if $u \in L^1(\Omega)$ *and for all* $\xi \in \{e_1, \dots, e_N\}$ *and for a.e.* $y \in \Pi_\xi$ $u_\xi^y \in BV(\Omega_\xi^y)$ $(SBV(\Omega_\xi^y))$ *and*

$$\int_{\Pi_\xi} |Du_\xi^y|\, d\mathcal{H}^{N-1}(y) < +\infty,$$

then $u \in BV(\Omega)$ $(SBV(\Omega))$.

Given a family $\mathcal{F}$ of functions, for every $\xi \in S^{N-1}$ and $y \in \Pi_\xi$ we set $\mathcal{F}_\xi^y := \{u_\xi^y : u \in \mathcal{F}\}$; moreover we say that a family $\mathcal{F}'$ is δ-close to $\mathcal{F}$ if $\mathcal{F}'$ is contained in a δ-neighbourhood of $\mathcal{F}$.

Lemma 1.1.2 *Let* $\mathcal{F}$ *be a family of equiintegrable functions belonging to* $L^1(A)$ *and assume that there exists a basis of unit vectors* $\{\xi_1, \dots, \xi_N\}$ *with the property that for every* $i = 1, ..., N$, *for every* $\delta > 0$, *there exists a family* $\mathcal{F}_\delta$ δ*-close to* $\mathcal{F}$ *such that* $(\mathcal{F}_\delta)_{\xi_i}^y$ *is precompact in* $L^1(A_{\xi_i}^y)$ *for* $\mathcal{H}^{N-1}$*-a.e* $y \in A_{\xi_i}$. *Then* $\mathcal{F}$ *is precompact in* $L^1(A)$.

1.1.2 Semicontinuity and relaxation in BV and SBV

Let $f : \mathbb{R} \to [0, +\infty]$ be convex. Then we define the *recession function* f^∞ of f by

$$f^\infty(z) = \lim_{t \to +\infty} \frac{f(tz)}{t}.$$

Let $\theta : \mathbb{R} \to [0, +\infty]$ be lower semicontinuous and such that there exists $\lim_{t \to 0^+} \theta(t)/t$. Then we can define the *recession function* θ^0 of θ by

$$\theta^0(z) = \lim_{t \to 0^+} \frac{\theta(tz)}{t}.$$

The functions f^∞ and θ^0 turn out to be 1-homogeneous. For every g, $h : \mathbb{R} \to [0, +\infty]$, we define the *inf-convolution of* g *and* h as the function $g \triangle h$ given by

$$(g \triangle h)(z) = \inf\{g(x) + h(z - x) : x \in \mathbb{R}\}.$$

Finally we recall that given a function $F : X \to \mathbb{R} \cup +\infty$, where X is a topological space, we denote by $\overline{F}$ the *relaxed functional* of F, i.e. the greatest lower semicontinuous (with respect to the X-topology) functional which is less than F.

The following relaxation result is proved in [12].

Theorem 1.1.3 *(Relaxation in BV) Let $f : [0,+\infty) \to [0,+\infty)$ be a non-decreasing convex function and let $\varphi : [0,+\infty) \to [0,+\infty)$ be a concave function. Let $F : BV(\Omega) \to [0,+\infty]$ be defined by*

$$F(u) := \begin{cases} \displaystyle\int_\Omega f(|\nabla u|)\,dx + \int_{S_u} \varphi(u^+ - u^-)\,d\mathcal{H}^{N-1} & \text{if } u \in SBV^2(\Omega) \cap L^\infty(\Omega), \\ +\infty & \text{otherwise.} \end{cases} \tag{1.1.2}$$

Then the relaxed functional of F with respect to the L^1-metric is given on BV by

$$\overline{F}(u) := \int_\Omega f_1(|\nabla u|)\,dx + \int_{S_u} \varphi_1(u^+ - u^-)\,d\mathcal{H}^{N-1} + \left(f^\infty(1) \wedge \varphi^0(1)\right)|D^c u|,$$

where $f_1 := f \triangle \varphi^0$ and $\varphi_1 := \varphi \triangle f^\infty$.

It is possible to prove that $f\triangle\varphi^0 = [f \wedge (\varphi^0 + f(0))]^{**}$, where h^{**} denotes the convexification of h, i.e. the greatest convex and lower semicontinuous function which is smaller than h and, analogously, $\varphi\triangle f^\infty = \mathrm{sub}\,[\varphi \wedge (f^\infty + \varphi(0))]$ where $\mathrm{sub}\,h$ denotes the subadditive envelope, i.e the greatest lower semicontinuous and subadditive function which is smaller than h. Given two Borel functions $\varphi :]0,+\infty[\to [0,+\infty)$ and $f : [0,+\infty] \to [0,+\infty)$, we consider the functional F defined by

$$F(u) = \begin{cases} \displaystyle\int_\Omega f(|\nabla u|)\,dx + \int_{S_u} \varphi(u^+ - u^-)\,d\mathcal{H}^{N-1} & \text{if } u \in GSBV(\Omega), \\ +\infty & \text{otherwise.} \end{cases} \tag{1.1.3}$$

In [7] the following semicontinuity result is proved.

Theorem 1.1.4 *(Ambrosio's Semicontinuity Theorem) Let $\Omega \subset \mathbb{R}^N$ be an open bounded set. Let $f : [0,+\infty) \to [0,+\infty)$ be a non-decreasing convex function such that $f^\infty(1) = +\infty$ and let $\varphi :]0,+\infty] \to [0,+\infty)$ be a non-decreasing subadditive function such that $b^0(1) = \infty$. Then the functional F defined in (1.1.3) is lower semicontinuous with respect to the L^1 convergence.* □

1.1.3 A density result in SBV

In analogy with the strong density results of smooth functions in $W^{1,p}(\Omega)$, functions in $SBV^p(\Omega)$ can be approximated in a "strong sense" by functions which have a "regular" jump set and are smooth outside. This can be formally expressed as follows.

Let Ω be an open bounded subset in $\mathbb{R}^N$ with Lipschitz boundary and denote by $\mathcal{W}(\Omega)$ the space of all function $w \in SBV(\Omega)$ enjoying the following properties:

i) $\mathcal{H}^{N-1}(\overline{S}_w \setminus S_w) = 0$;

ii) $\overline{S}_w$ is the intersection of Ω with the union of a finite number of pairwise disjoint $(N-1)$-simplexes;

iii) $w \in W^{k,\infty}(\Omega \setminus \overline{S}_w)$ for every $k \in \mathbb{N}$.

Cortesani and Toader have proved in [24] the following density result.

Theorem 1.1.5 *Let $u \in SBV^p(\Omega) \cap L^\infty(\Omega)$. Then there exists a sequence $(w_j)_j$ in $\mathcal{W}(\Omega)$ such that $w_j \to u$ strongly in $L^1(\Omega)$, $\nabla w_j \to \nabla u$ strongly in $L^p(\Omega,\mathbb{R}^N)$, $\lim_j \|w_j\|_\infty = \|u\|_\infty$ and*

$$\limsup_{j\to\infty} \int_{S_{w_j}} \phi(w_j^+, w_j^-, \nu_{w_j})\,d\mathcal{H}^{N-1} \le \int_{S_u} \phi(u^+, u^-, \nu_u)\,d\mathcal{H}^{N-1},$$

for every upper semicontinuous function $\phi : \mathbb{R}\times\mathbb{R}\times S^{N-1} \to [0,+\infty)$ *such that* $\phi(a,b,\nu) = \phi(b,a,-\nu)$, *for every* a, $b \in \mathbb{R}$ *and for every* $\nu \in S^{N-1}$.

Remark 1.1.6 Under the additional assumption that $1 < p \leq 2$ the structure of the jump set of the functions w_j given by Theorem 1.1.5 can be further improved by using a capacitary argument. In particular for $N = 2$ and and $p = 2$, we can suppose that $\overline{S_{w_j}}$ is made up of a finite family of pairwise disjoint segments compactly contained in Ω.

1.2 Euler-Lagrange equations for the Mumford-Shah functional and regularity of the solutions

Let us consider the (non-homogeneous) Mumford-Shah functional

$$F_{\beta,g}(u) := \int_\Omega |\nabla u|^2\,dx + \mathcal{H}^{N-1}(S_u) + \beta\int_\Omega |u-g|^2\,dx\,, \qquad u \in SBV(\Omega); \tag{1.2.1}$$

when $\beta = 0$ we will write

$$F_0(u) := \int_\Omega |\nabla u|^2\,dx + \alpha\mathcal{H}^{N-1}(S_u), \tag{1.2.2}$$

and we will call it the *homogeneous Mumford-Shah functional.*

Definition 1.2.1 *We say that* $u \in SBV(\Omega)$ *is a* Dirichlet minimizer *of* $F_{\beta,g}$ *if* $F_{\beta,g}(u) < +\infty$ *and* $F_{\beta,g}(u) \leq F_{\beta,g}(v)$ *for any* $v \in SBV(\Omega)$ *with the same trace as* u *on* $\partial\Omega$. *Analogously we define Dirichlet minimizers for* F_0. *We say that* $u \in SBV(\Omega)$ *is an* absolute minimizer of $F_{\beta,g}$ if $F_{\beta,g}(u) < +\infty$ and $F_{\beta,g}(u) \leq F_{\beta,g}(v)$ for any $v \in SBV(\Omega)$.

Definition 1.2.2 *The* deviation from minimality $\mathrm{Dev}(u,A)$ *of a function* $u \in SBV(\Omega)$ *(satisfying* $F_0(u) < +\infty$*) in* A *open subset of* Ω *is defined as the smallest* $\lambda \in [0,+\infty]$ *such that*

$$F_0(u) \leq F_0(v) + \lambda,$$

for any $v \in SBV(\Omega)$ *such that* $\mathrm{supp}(u-v) \subset\subset A$. *Moreover, given* $\lambda \in [0,+\infty)$ *we will say that* u *is a* λ-quasi-minimizer of F_0 if, for all balls $B_\rho(x) \subseteq \Omega$, we have $\mathrm{Dev}(u, B_\rho(x)) \leq \lambda\rho^N$. The class of all λ-quasi-minimizers will be denoted by $\mathcal{M}_\lambda(\Omega)$. Finally we will say that u is a *quasi-minimizer* of F_0 if there exists $\lambda \in [0,+\infty)$ such that $u \in \mathcal{M}_\lambda(\Omega)$.

Note that $\mathrm{Dev}(u,\Omega) = 0$ means that u is a Dirichlet minimizer of F_0; moreover any Dirichlet minimizer of $F_{\beta,g}$ is a λ-quasi-minimizer of F_0 with $\lambda = 4\beta\omega_N\|g\|_\infty^2$ (ω_N denotes the measure of the unit ball).

Theorem 1.2.3 *If* $u \in SBV(\Omega)$ *is a quasi-minimizer of* F_0, *there exists an* $\mathcal{H}^{N-1}$*-negligible set* $\Sigma \subset \overline{S}_u \cap \Omega$ *relatively closed in* Ω *such that* $\Omega \cap \overline{S}_u \setminus \Sigma$ *is a hypersurface of class* $C^{1,1/4}$.

For a proof of the theorem see [9] (Theorem 8.1). In the following we focus our attention on the necessary optimality conditions near the regular points of S_u. So let u be a Dirichlet minimizer of $F_{\beta,g}$ and let $A \subset \Omega$ be an open subset such that $S_u \cap A$ is a graph i.e.

$$\overline{S}_u \cap A = \{(z,\psi(z)) : z \in D\},$$

for some open subset $D \subset \mathbb{R}^{N-1}$ and $\psi : D \to \mathbb{R}$. Denote $A^+ := \{(z,t) \in A : t > \psi(z)\}$ and $A^- := \{(z,t) \in A : t < \psi(z)\}$. By comparing u with $v = u + \varepsilon\varphi$, where φ is a smooth function vanishing in a neighbourhood of $\partial A^\pm \setminus \overline{S}_u$, from the minimality of u one obtain

$$\int_{A^\pm} (\nabla u \cdot \nabla\varphi + \beta(u-g)\varphi)\, dx = 0;$$

the equation above says that u is a weak solution of the following problem

$$\begin{cases} \Delta u = \beta(u-g) & \text{in } A^\pm, \\ \partial_\nu u = 0 & \text{on } \partial A^\pm \cap \overline{S}_u. \end{cases} \tag{1.2.3}$$

In particular, taking $\beta = 0$, we obtain that every minimizer of F_0 is harmonic in A^+ and A^-. Combining Theorem 1.2.3 with the regularity results for the solutions of problem (1.2.3), we can state the following theorem.

Theorem 1.2.4 *If u is a Dirichlet minimizer of $F_{\beta,g}$ and $\overline{S}_u \cap A$ is the graph of a function ψ of class $C^{1,\gamma}$ ($\gamma < 1$), then u has a $C^{1,\sigma}$ extension on each side of $\overline{S}_u \cap A$ for some $\sigma \leq \gamma$ (we can take $\sigma = \gamma$ if $N = 2$).*

The Euler-Lagrange equation (1.2.3) has been obtained considering only variations of u and keeping S_u fixed. By considering also variations of S_u we expect to obtain a transmission condition of u along S_u accounting for the interplay between the bulk and the surface part. Before writing such a condition, let us recall the notion of mean curvature.

Definition 1.2.5 *Let M be a C^2 $(N-1)$-dimensional manifold, A an open set, $\{e_1, ..., e_N\}$ the canonical basis of $\mathbb{R}^N$, and $\nu : M \cap A \to S^{N-1}$ a C^1 unit normal vector field. Then the* mean curvature vector $\mathbf{H}_M$ *is defined by*

$$\mathbf{H}_M(x) := \text{div}^M \nu(x)\nu(x) \qquad \forall x \in A \cap M,$$

where

$$\text{div}^M \nu(x) = \sum_{k=1}^{N-1} \nabla^M \nu_k(x) \cdot e_k \tag{1.2.4}$$

(∇^M denotes the tangential gradient, i.e. the projection of the gradient on the tangent plane T_xM). The quantity $H_M := \text{div}^M \nu(x)$ is called scalar mean curvature associated with ν.

Note that, differently from H_M, the mean curvature vector does not depend on the orientation of ν. The following formula is straightforward consequence of the definition (1.2.4):

$$\text{div}^M \nu(x) = \text{div}\tilde{\nu}(x) - \partial_{\nu(x)}\tilde{\nu}(x) \cdot \nu(x), \tag{1.2.5}$$

where $\tilde{\nu}$ is any C^1 extension of ν.

If M is the graph of a C^2 function $\psi : D \subset R^{N-1} \to \mathbb{R}$ and ν is the upper normal given by

$$\nu(x) = \frac{(-\nabla\psi, 1)}{\sqrt{1 + |\nabla\psi|^2}},$$

then one can check that

$$H_M = -\mathrm{div}\left(\frac{\nabla\psi}{\sqrt{1+|\nabla\psi|^2}}\right). \tag{1.2.6}$$

Also the following formula for $N = 2$ will be useful in the sequel: letting $\gamma \subset \mathbb{R}^2$ be a smooth curve, ν a (smooth) unit normal vector field to γ, and $\xi \mapsto (x(\xi), y(\xi)))$ a parameterization of γ by arc-length, denoting by $\mathrm{curv}\,\gamma$ the scalar (mean) curvature of γ with respect to ν, we have

$$\mathrm{curv}\,\gamma(\xi) = -(\ddot{x}(\xi), \ddot{y}(\xi)) \cdot \nu(\xi); \tag{1.2.7}$$

since the two vectors in (1.2.7) are parallel, it follows that

$$[\mathrm{curv}\,\gamma(\xi)]^2 = (\ddot{x}(\xi))^2 + (\ddot{y}(\xi))^2. \tag{1.2.8}$$

We recall the following generalized divergence theorem.

Theorem 1.2.6 *Let $M \subset \Omega$ be a C^2 $(N-1)$-dimensional manifold with no boundary in Ω. Then*

$$\int_M \mathrm{div}^M \eta \, d\mathcal{H}^{N-1} = \int_M \eta \cdot \mathbf{H}_M \, d\mathcal{H}^{N-1} \qquad \forall \eta \in [C_c^1(\Omega)]^N. \tag{1.2.9}$$

If $\phi : A \setminus \overline{S}_u \to \mathbb{R}$ is a function having continuous extensions on each side of $\overline{S}_u \cap A$, we denote by ϕ^+ (respectively by ϕ^-) the upper (lower) trace of ϕ on $\overline{S}_u \cap A$ from A^+ (from A^-) and we set

$$[\phi]^{\pm} = \phi^+ - \phi^-.$$

Note that thanks to Theorem 1.2.4, if u is a minimizer and $\overline{S}_u \cap A$ is the graph of $C^{1,\gamma}$ function, then the traces of ∇u on $\overline{S}_u \cap A$ exist in a classical sense. We state now the announced transmission condition satisfied by u on S_u.

Theorem 1.2.7 *Let u be a Dirichlet minimizer for $F_{\beta,g}$, suppose that g is of class C^1 and that $\overline{S}_u \cap A$ is the graph of a $C^{1,\gamma}$ function. Then, for any $\eta \in [C_c^1(\Omega)]^N$, we have*

$$\int_{S_u \cap A} \left[|\nabla u|^2 + \beta(u-g)\right]^{\pm} \eta \cdot \nu \, d\mathcal{H}^{N-1} = \int_{S_u \cap A} \mathrm{div}^{S_u} \eta \, d\mathcal{H}^{N-1}, \tag{1.2.10}$$

where ν is the upper normal to $\overline{S}_u \cap A$.

Recalling (1.2.9), the theorem above says that if u is a Dirichlet minimizer, near the regular points of S_u we have that

$$H_{S_u} = \left[|\nabla u|^2 + \beta(u-g)\right]^{\pm} \qquad \text{on } S_u \cap A \tag{1.2.11}$$

in a weak sense. In fact it is possible to prove that the function ψ such that $\mathrm{graph}\,\psi = \overline{S}_u \cap A$ is a weak solution of

$$-\mathrm{div}\left(\frac{\nabla\psi}{\sqrt{1+|\nabla\psi|^2}}\right) = \left[|\nabla u|^2 + \beta(u-g)\right]^{\pm}; \tag{1.2.12}$$

in particular, if u is a Dirichlet minimizer of F_0 ($\beta = 0$), we have

$$-\text{div}\left(\frac{\nabla\psi}{\sqrt{1+|\nabla\psi|^2}}\right) = \left[|\nabla u|^2\right]^{\pm}. \tag{1.2.13}$$

Using (1.2.12), it is possible to prove that, as soon as we know that $\overline{S}_u \cap A$ is of class $C^{1,\gamma}$, we may obtain as much regularity as we want by assuming enough regularity on g.

Theorem 1.2.8 *Let u be a Dirichlet minimizer of $F_{\beta,g}$ and let $\overline{S}_u \cap A$ be the graph of a $C^{1,\gamma}$ function ψ. If $g \in C^{k,\delta}(A)$ for some $k \geq 1$, $\delta \leq 1$, there exists σ depending only on N, δ, and ψ such that ψ is of class $C^{k+2,\sigma}$ and u admits a $C^{k+2,\sigma}$ extension on each side of $\overline{S}_u \cap A$. In particular for every Dirichlet minimizer of F_0 we have that S_u is of class C^∞ near its regular points.*

The following conjecture is still an open problem.
Conjecture (De Giorgi). *If u is a Dirichlet minimizer of F_0 then S_u is analytic near its regular points.* We conclude the section with the following conjecture, stated for $N = 2$ by Mumford & Shah in [45].
Conjecture (Mumford-Shah) *If u is a minimizer of $F_{\beta,g}$ then $\overline{S}_u$ is locally in Ω the union of finitely many $C^{1,1}$ embedded arcs.*

In [45] it has been proved that if the conjecture is true then only two kind of singularities can occur inside Ω: either a curve ends at some point, the so-called cracktip, or three curves meet forming equal angles of $2\pi/3$, the so-called triple junction.

1.3 The calibration method

In this section we state the theorem on calibrations in the form that we will need (which is a particular case of the more general statement of [2]); for the sake of completeness we present also some simple applications taken from [2].

First we introduce a more general notion of minimality that will be useful in the sequel. Let U be an open subset of $\Omega \times \mathbb{R}$ with Lipschitz boundary whose closure can be written as

$$\overline{U} := \{(x,t) \in \overline{\Omega} \times \mathbb{R} : \tau_1(x) \leq t \leq \tau_2(x)\}, \tag{1.3.1}$$

where the functions τ_1, $\tau_2 : \overline{\Omega} \to [-\infty, +\infty]$ satisfy $\tau_1 < \tau_2$.

Definition 1.3.1 *We say that a function $u \in SBV(\Omega)$ is an* absolute $\overline{U}$-minimizer *of the Mumford-Shah functional* (1.2.1) *if the complete graph of u is contained in $\overline{U}$ and $F_{\beta,g}(u) \leq F_{\beta,g}(v)$ for all $v \in SBV(\Omega)$ with complete graph contained in $\overline{U}$, while u is a* $\overline{U}$-Dirichlet minimizer *if we add the requirements that $F_{\beta,g}(u) < +\infty$ and that the competing functions v have in addition the same boundary values as u.*

Given U open subset of $\Omega \times \mathbb{R}$ and satisfying (1.3.1), we shall consider the collection $\mathcal{F}(\overline{U})$ of all bounded vector fields $\phi = (\phi^x, \phi^z) : \overline{U} \to \mathbb{R}^N \times \mathbb{R}$ with the following property: there exists a finite family $(U_i)_{i\in I}$ of pairwise disjoint and Lipschitz open subsets of U whose closures cover $\overline{U}$, and a family $(\phi_i)_{i\in I}$ of vector fields in $Lip(\overline{U_i}, \mathbb{R}^N \times \mathbb{R})$ such that ϕ agrees at any point with one of the ϕ_i.

A *calibration* in $\overline{U}$ (with respect to $F_{\beta,g}$) for $u \in SBV(\Omega)$ with complete graph contained in $\overline{U}$ is a vector field $\phi \in \mathcal{F}(\overline{U})$ which satisfies the following properties:

(a) $\text{div}\phi = 0$ in U_i, for every $i \in I$;

(b) $\nu_{\partial U_i} \cdot \phi^+ = \nu_{\partial U_i} \cdot \phi^- = \nu_{\partial U_i} \cdot \phi$ $\mathcal{H}^N$-a.e in ∂U_i for every $i \in I$, where $\nu_{\partial U_i}(x, z)$ denotes the (unit) normal vector at (x, z) to ∂U_i, while ϕ^+ and ϕ^- denote the two traces of ϕ on the two sides of ∂U_i;

(c) $\dfrac{(\phi^x(x,z))^2}{4} \leq \phi^z(x,z) + \beta(z - g(x))^2$ for $\mathcal{L}^N$-a. e. $x \in \Omega$ and every $z \in [\tau_1(x), \tau_2(x)]$;

(d) $\phi^x(x, u(x)) = 2\nabla u(x)$ and $\phi^z(x, u(x)) = |\nabla u(x)|^2 - \beta(g(x) - u(x))^2$ for $\mathcal{L}^N$-a. e. $x \in \Omega \setminus S_u$;

(e) $\displaystyle\int_{u^-(x)}^{u^+(x)} \phi^x(x,z)\, dz = \nu_u(x)$ for $\mathcal{H}^{N-1}$-a.e. $x \in S_u$, where $\nu_u(x)$ denotes the unit normal vector at x to S_u, which points toward u^+;

(f) $\left|\displaystyle\int_s^t \phi^x(x,z)\, dz\right| \leq 1$ for $\mathcal{H}^{N-1}$-a.e. $x \in \Omega$ and for every s, $t \in [\tau_1(x), \tau_2(x)]$.

If also the following condition is satisfied

(g) $\phi^x(x,z) \cdot \nu(x) = 0$ for $\mathcal{H}^N$-a.e. $(x,z) \in \partial(\Omega \times \mathbb{R}) \cap \partial U$, where $\nu(x)$ denotes the unit normal vector at x to $\partial\Omega$,

then ϕ is called an *absolute calibration* of u in $\overline{U}$.

Remark 1.3.2 Note that conditions **(a)** and **(b)** can be grouped by simply saying that the vectofield ϕ is divergence-free in the sense of distributions.

We are now in a position to state the fundamental theorem on which the calibration method is based.

Theorem 1.3.3 *If there exists a calibration ϕ for u in U, then u is a $\overline{U}$-Dirichlet minimizer of $F_{\beta,g}$. If there exists an absolute calibration then u is an absolute $\overline{U}$-minimizer.*

Remark 1.3.4 Let us underline that the theorem is valid also for the case $\beta = 0$ i.e. for the homogeneous Mumford-Shah functional F_0; of course a calibration with respect to F_0 is a vectorfield satisfying the conditions above with $\beta = 0$, so that conditions **(a)**, **(b)**, **(e)**, **(f)** (and **(g)**) remain unchanged, while **(c)** and **(d)** become simply

(c)' $\dfrac{(\phi^x(x,z))^2}{4} \leq \phi^z(x,z)$ for $\mathcal{L}^N$-a. e. $x \in \Omega$ and every $z \in [\tau_1(x), \tau_2(x)]$;

(d)' $\phi^x(x, u(x)) = 2\nabla u(x)$ and $\phi^z(x, u(x)) = |\nabla u(x)|^2$ for $\mathcal{L}^N$-a. e. $x \in \Omega \setminus S_u$.

It is interesting to see directly that the existence of a calibration for u implies that u satisfies Euler-Lagrange equations. Let A be an open set such that $A \cap S_u$ is of class C^2 and $A \setminus S_u$ is made up of two connected components A^+ and A^-, and let $g : A \to \mathbb{R}$ be of class C^1. Suppose also that there exists a calibration $\phi : A \times \mathbb{R} \to \mathbb{R}^N \times \mathbb{R}$ for u, with respect to $F_{\beta,g}$. We want to prove that u satisfies (1.2.3) and (1.2.11), by assuming for simplicity that ϕ is of class C^1. So fix $x \in A$ and consider the function

$$\psi(z) := \phi^z(x,z) + \beta(z-g)^2 - \frac{|\phi^x(x,z)|^2}{4};$$

conditions **(c)** and **(d)** imply that $z = u(x)$ is a minimum point for ψ. By differentiating we get,

$$\begin{aligned} 0 = \psi'(u(x)) &= \partial_z \phi^z(x,u) + 2\beta(u-g) - \frac{1}{2}\phi^x(x,u)\cdot\partial_z\phi^x(x,u) \\ &= -\mathrm{div}_x \phi^x(x,u) + 2\beta(u-g) - \nabla u \cdot \partial_z \phi^x(x,u), \end{aligned} \tag{1.3.2}$$

where, in the last equality, we have used the fact that ϕ is divergence-free and condition **(d)**. Since $\phi^x(x,u) = 2\nabla u$ (by **(d)**), it turns out that $2\Delta u = \mathrm{div}_x\phi^x(x,u) + \partial_z\phi^x(x,u)\cdot\nabla u$; substituting in (1.3.2), we get

$$\Delta u = \beta(u-g). \tag{1.3.3}$$

Denote by $u^\pm$ te restriction of u to $A^\pm$; by extension, we can suppose that u^+ and u^- are defined in the whole A and of class C^2. Fix $x_0 \in S_u \cap A$ and consider the function

$$\chi(t) := \left| \int_{u^-(x_0)}^{t} \phi^x(x,z)\,dz \right|^2 ;$$

conditions **(e)** and **(f)** imply that $t = u^+(x_0)$ is a minimum point of χ. Therefore, by differentiating, we have

$$0 = \chi'(u^+(x_0)) = 2\left(\int_{u^-(x_0)}^{u^+(x_0)} \phi^x(x,z)\,dz\right)\cdot\phi^x(x,u^+(x_0)) = 4\nu_u(x_0)\cdot\nabla u^+(x_0) = 4\partial_{\nu_u}u^+(x_0),$$

where, in the third equality we have used conditions **(d)** and **(e)**; the same can be proved for u^- and therefore, recalling (1.3.3), we have that u solves (1.2.3).

Consider now the function

$$\Psi(x) := \int_{u^-(x)}^{u^+(x)} \phi^x(x,z)\,dz$$

and note that again by **(e)** and **(f)** each point of $S_u \cap A$ is a maximum point for $|\Psi|^2$. Fix $x_0 \in S_u\cap A$ and compute the derivative of $|\Psi|^2$ in the normal direction $\nu_u(x_0)$ (to simplify the notation we will write ν instead of $\nu_u(x_0)$):

$$0 = 2\Psi(x_0)\cdot\partial_\nu\Psi(x_0) = 2\nu\cdot\partial_\nu\Psi(x_0),$$

where we have used the fact that $\Psi(x_0) = \nu$ (by **(e)**); using formula (1.2.5), we can go further and write

$$0 = \partial_\nu\Psi(x_0)\cdot\nu = \mathrm{div}\Psi(x_0) - \mathrm{div}^{S_u}\Psi(x_0) = \mathrm{div}\Psi(x_0) - H_{S_u}(x_0), \tag{1.3.4}$$

where the last equality follows from the fact that $\Psi|_{S_u} \equiv \nu_u$ and from Definition 1.2.5. Using condition **(d)**, we can finally compute

$$\begin{aligned} \mathrm{div}\Psi(x_0) &= \int_{u^-(x_0)}^{u^+(x_0)} \mathrm{div}_x\phi^x(x_0,z)\,dz + \phi^x(x_0,u^+(x_0))\nabla u^+(x_0) - \phi^x(x_0,u^-(x_0))\nabla u^-(x_0) \\ &= -\int_{u^-(x_0)}^{u^+(x_0)} \partial_z\phi^z(x_0,z)\,dz + 2(|\nabla u^+(x_0)|^2 - |\nabla u^-(x_0)|^2) \\ &= -\Big(\phi^z(x_0,u^+(x_0)) - \phi^z(x_0,u^-(x_0))\Big) + 2(|\nabla u^+(x_0)|^2 - |\nabla u^-(x_0)|^2) \\ &= |\nabla u^+(x_0)|^2 + \beta(u^+(x_0)-g(x_0))^2 - \Big(|\nabla u^-(x_0)|^2 + \beta(u^-(x_0)-g(x_0))^2\Big), \end{aligned}$$

which, by substitution in (1.3.4), gives (1.2.11).

We conclude the section by showing how the calibration method can be used to provide easy and short proofs of some natural minimality results. All the examples we are going to describe are taken from [2] and deal with minimizers presenting either a vanishing gradient or an empty discontinuity set; the simple structure of such minimizers allows quite simple constructions. In the thesis we will face the (more complicated) problem of constructing a calibration also for candidate functions presenting both non-vanishing gradient and nonempty discontinuity set.

Example 1.3.5 (Harmonic function) Let u be a harmonic function on Ω; the u Dirichlet minimizer of F_0 if

$$\operatorname*{osc}_{\Omega} u \, \sup_{\Omega} |\nabla u| \leq 1 \tag{1.3.5}$$

where $\operatorname{osc}_\Omega u := \sup_\Omega u - \inf_\Omega u$. A calibration of u is given by

$$\phi(x,z) := \begin{cases} (2\nabla u(x), |\nabla u(x)|^2) & \text{if } \frac{1}{2}(u(x)+m) \leq z \leq \frac{1}{2}(u(x)+M), \\ (0,0) & \text{otherwise,} \end{cases} \tag{1.3.6}$$

where $m := \inf_\Omega u$ and $M := \sup_\Omega u$. (see [2] for the details). If (1.3.5) is not satisfied, u is still is a Dirichlet $\overline{U}$-Dirichlet minimizer of F_0, for

$$U := \left\{ (x,z) \in \Omega \times \mathbf{R} : u(x) - \frac{1}{4}|\nabla u(x)|^{-1} < z < u(x) + \frac{1}{4}|\nabla u(x)|^{-1} \right\},$$

and a calibration is given by $\phi(x,z) := \big(2\nabla u(x), |\nabla u(x)|^2\big)$.

Example 1.3.6 (Pure jump) Let $N \geq 2$ and let $\Omega :=]0,a[\times V$, where V is a bounded domain in $\mathbf{R}^{N-1}$ with Lipschitz boundary. Denoting the first coordinate of x by x_1, let $u(x) := 0$ for $0 < x_1 < c$, and $u(x) := h$ for $c < x_1 < a$, with $0 < c < a$ and $h > 0$. Then u is a Dirichlet minimizer of F_0 if $a \leq h^2$. A calibration is given by

$$\phi(x,z) = (\phi^x(x,z), \phi^z(x,z)) := \begin{cases} \left(\frac{2}{\sqrt{a}} e_1, \frac{1}{a}\right) & \text{if } \frac{1}{2\sqrt{a}} x_1 \leq z \leq \frac{1}{2\sqrt{a}}(x_1 + a), \\ (0,0) & \text{otherwise.} \end{cases}$$

Example 1.3.7 (Triple junction) Let $N := 2$, let $\Omega := B(0,r)$ be the open ball with radius $r > 0$ centered at the origin, and let u be given, in polar coordinates, by $u(\rho,\theta) := a$ for $0 \leq \theta < \frac{2}{3}\pi$, $u(\rho,\theta) := b$ for $\frac{2}{3}\pi \leq \theta < \frac{4}{3}\pi$, and $u(\rho,\theta) := c$ for $\frac{4}{3}\pi \leq \theta < 2\pi$, where a, b, and c are distinct constants. Thus S_u is given by three line segments meeting at the origin with equal angles. If

$$2r \leq \min\{|a-b|^2, |b-c|^2, |c-a|^2\}, \tag{1.3.7}$$

then u is a Dirichlet minimizer of F_0. To construct a calibration, it is not restrictive to assume $a < b = 0 < c$. We denote $e_\pm := (\pm\sqrt{3}/2, -1/2)$, and $\lambda > 0$ such that $\frac{\lambda r}{2} + \frac{1}{\lambda} \leq \min\{-a, c\}$ (which is possible by (1.3.7)), and we define the calibration by

$$\phi(x,z) := \begin{cases} (\lambda e_+, \lambda^2/4) & \text{if } \frac{\lambda}{4}(r + x\cdot e_+) \le z \le \frac{\lambda}{4}(r + x\cdot e_+) + \frac{1}{\lambda}, \\ (\lambda e_-, \lambda^2/4) & \text{if } \frac{\lambda}{4}(-r + x\cdot e_-) - \frac{1}{\lambda} \le z \le \frac{\lambda}{4}(-r + x\cdot e_-), \\ (0,0) & \text{otherwise.} \end{cases}$$

If r is much larger than $\min\{|a-b|^2, |b-c|^2, |c-a|^2\}$, it is easy to construct a comparison function v with the same boundary values as u and such that $F_0(v) < F_0(u)$. This shows that in this case u is not a Dirichlet minimizer.

We consider now the functional $F_{\beta,g}$, with $\beta > 0$.

In the next examples we construct a calibration for $F_{\beta,g}$ when the parameter β is large enough.

Example 1.3.8 (Smooth g and large β) Let Ω be a bounded open set in $\mathbf{R}^N$ with smooth boundary, and let $g \in C^2(\overline{\Omega})$. There exists a constant $\beta_0 \ge 0$, depending on g and α, such that for every $\beta > \beta_0$ the solution u of the Neumann problem (see (1.2.3))

$$\begin{cases} \Delta u = \beta(u-g) & \text{in } \Omega, \\ \partial_\nu u + 0 & \text{on } \partial\Omega, \end{cases}$$

is the unique absolute minimizer of $F_{\beta,g}$. We are going to describe how to construct an absolute calibration. To begin with, we fix a smooth function $\sigma : \mathbb{R} \to [0,1]$, with compact support which satisfies $\sigma(z) = 1$ for $|z| \le 3\|u-g\|_\infty$. Then we set

$$\phi^x(x,z) := 2\sigma(z - u(x))\, \nabla u(x)\ .$$

In particular ϕ has vanishing normal component at the boundary of $\Omega \times \mathbb{R}$, and $\phi^x = 2\nabla u$ on the graph of u. We set

$$\phi^z(x,u(x)) := |\nabla u|^2 - \beta(u-g)^2 \quad \text{for all } x \in \Omega. \tag{1.3.8}$$

We impose now that ϕ is divergence-free, which reduces to

$$\begin{aligned} \partial_z \phi^z(x,z) = -\mathrm{div}_x \phi^x &= -2\sigma\, \Delta u + 2\dot\sigma\, |\nabla u|^2 \\ &= -2\beta\sigma\, (u-g) + 2\dot\sigma\, |\nabla u|^2 \end{aligned} \tag{1.3.9}$$

Identity (1.3.9) together with (1.3.8) determine ϕ^z everywhere. Using well-known estimates on the solutions of the Neumann problem, it can be proven that ϕ is a calibration for β large enough. We refer to [2] for the details.

Example 1.3.9 (Characteristic functions of regular sets) Let Ω be an open set in $\mathbf{R}^N$ and let E be a compact set contained in Ω with boundary of class C^2. Let $g(x) := \chi_E(x)$. Then, there exists a constant $\beta_0 \ge 0$, depending on E, such that for every $\beta > \beta_0$ the function $u := g$ is the unique minimizer of $F_{\beta,g}$. We take a C^1 vectorfield $\nu : \Omega \to \mathbf{R}^N$ with compact support in Ω such that $|\nu(x)| \le 1$ for every $x \in \Omega$ and $\nu(x)$ is the outer unit normal to ∂E for every $x \in \partial E$. Then we set $\phi^x(x,z) = \sigma(z)\nu(x)$, where σ is a fixed positive smooth function with integral equal to 1 and support contained in $]0,1[$. We are now forced to set $\phi^z(x,z)=0$ for $z=g(x)$ (see (d)) and in order to have a divergence-free vectorfield we require $\partial_z\phi^z(x,z) = -\sigma(z)\mathrm{div}_x\nu(x)$. These two conditions determine $\phi^z(x,z)$ at every point (x,z). It is then easy to see that ϕ is an absolute calibration, for β large enough.

The example shows that, in agreement with our expectation, the contours of "regular" objects are exactly reconstructed by the Mumford-Shah functional if the fidelity parameter β is large enough (see the Introduction). We will generalize this example in Chapter 4.

1.4 Γ-convergence

We recall here the definition and the main properties of Γ-convergence: for the general theory we refer to [25].

Definition 1.4.1 *Let (X,d) be a metric space and let $F_h : X \to \mathbb{R}\cup\{+\infty\}$ be a sequence of functions. We set*

$$\Gamma\text{-}\liminf_{h\to\infty} F_h(x) := \inf\left\{\liminf_{h\to\infty} F_h(x_h) : \ x_h \to x\right\}$$

and

$$\Gamma\text{-}\limsup_{h\to\infty} F_h(x) := \inf\left\{\limsup_{h\to\infty} F_h(x_h) : \ x_h \to x\right\}.$$

We say that the sequence $(F_h)_{h\in\mathbb{N}}$ Γ-converges if

$$\Gamma\text{-}\liminf_{h\to\infty} F_h(x) = \Gamma\text{-}\limsup_{h\to\infty} F_h(x) \qquad \forall x\in X.$$

The common value is called Γ-limit *and is denoted by* $\Gamma\text{-}\lim_{h\to\infty} F_h$.

Definition 1.4.2 *We say that the maps $F_h : X \to \mathbb{R}\cup\{+\infty\}$ are equicoercive if for every $t\in\mathbb{R}$ there exists a compact subset $K_t \subseteq X$ such that*

$$\{x\in X : F_h(x)\le t\} \subseteq K_t \qquad \forall h\in\mathbb{N}.$$

The following theorem explains the variational meaning of this kind of convergence.

Theorem 1.4.3 *Let $(F_h)_h$ be a sequence of equicoercive maps which Γ-converges to F. Then, if $(x_h)_h$ is a sequence such that*

$$\lim_{h\to\infty} F_h(x_h) = \lim_{h\to\infty} \inf_X F_h,$$

then x_h is precompact and any cluster point is a minimizer of F.

We finally recall that given $F : X \to \mathbb{R}\cup\{+\infty\}$, the relaxed functional $\overline{F}$ can be characterized as the Γ-limit of the constant sequence $F_n = F$ for every $n\in\mathbb{N}$.

Part I

Calibration of solutions

Chapter 2

Calibration of solutions with a rectilinear discontinuity set

The following question arises naturally: given a function $u \in SBV(\Omega)$ which satisfies all Euler conditions for the homogeneous Mumford-Shah functional F_0 (see (1.2.2)), does it enjoy any minimality property? In this chapter, using the calibration method, we give a first partial answer, in two dimensions, considering the special case of extremals with a rectlinear discontinuity set. We present the first examples of calibrations for discontinuous functions which are not locally constant.

2.1 Notations and preliminary results

Let Ω be a bounded open subset of $\mathbb{R}^2$ with Lipschitz boundary and set

$$\Omega_0 = \{(x, y) \in \mathbb{R}^2 : y \neq 0\}, \qquad S = \{(x, y) \in \Omega : y = 0\}.$$

Suppose that w is a Dirichlet-minimizer (according to Definition (1.2.1)) in Ω and $S_w = S$, then it must satisfies the following conditions (recall (1.2.3) and (1.2.11) with $\beta = 0$):

i) w is harmonic on Ω_0;

ii) the normal derivative of w vanishes on both sides of S;

iii) the squares of the tangential gradients of w on the two sides of S are equal.

If Ω is a circle with centre on the x-axis, and $w \in C^1(\Omega_0)$ with $\int_{\Omega_0} |\nabla w|^2 dx\, dy < +\infty$, then w satisfies the Euler conditions **(i)**, **(ii)**, and **(c)** if and only if w has one of the following forms:

$$w(x, y) = \begin{cases} u(x, y) & \text{if } y > 0, \\ -u(x, y) + c_1 & \text{if } y < 0, \end{cases} \tag{2.1.1}$$

or

$$w(x, y) = \begin{cases} u(x, y) + c_2 & \text{if } y > 0, \\ u(x, y) & \text{if } y < 0, \end{cases} \tag{2.1.2}$$

where $u \in C^1(\Omega)$ is harmonic with normal derivative vanishing on S and c_1, c_2 are real constants. For our purposes, it is enough to consider the case $c_1 = 0$ in (2.1.1) and $c_2 = 1$ in (2.1.2). We are going to rewrite the definition of calibration, given in Section 1.3, using a slightly different notation, more convenient for $N = 2$.

For every vectorfield $\varphi : \Omega \times \mathbb{R} \to \mathbb{R}^2$ we define the maps φ^x, φ^y, $\varphi^z : \Omega \times \mathbb{R} \to \mathbb{R}$ by

$$\varphi(x,y,z) = (\varphi^x(x,y,z), \varphi^y(x,y,z), \varphi^z(x,y,z)).$$

We shall consider the collection $\mathcal{F}(\Omega \times \mathbb{R})$ of all bounded vectorfields $\varphi : \Omega \times \mathbb{R} \to \mathbb{R}^2 \times \mathbb{R}$ with the following property: there exists a finite family $(U_i)_{i \in I}$ of pairwise disjoint and Lipschitz open subsets of $\Omega \times \mathbb{R}$ whose closures cover $\overline{\Omega} \times \mathbb{R}$, and a family $(\varphi_i)_{i \in I}$ of vectorfields in $Lip(\overline{U_i}, \mathbb{R}^2 \times \mathbb{R})$ such that φ agrees at any point with one of the φ_i.

A calibration in $\Omega \times \mathbb{R}$ (with respect to F_0) for $w \in SBV(\Omega)$ is a vectorfield $\varphi \in \mathcal{F}(\Omega \times \mathbb{R})$ which satisfies the following properties:

(a) $\text{div}\varphi = 0$ in U_i, for every $i \in I$;

(b) $\nu_{\partial U_i} \cdot \varphi^+ = \nu_{\partial U_i} \cdot \varphi^- = \nu_{\partial U_i} \cdot \varphi$ $\mathcal{H}^2$-a.e in ∂U_i for every $i \in I$, where $\nu_{\partial U_i}(x,y,z)$ denotes the (unit) normal vector at (x,y,z) to ∂U_i, while φ^+ and φ^- denote the two traces of φ on the two sides of ∂U_i;

(c) $(\varphi^x(x,y,z))^2 + (\varphi^y(x,y,z))^2 \leq 4\varphi^z(x,y,z)$ for almost every (x,y) in Ω and for every $z \in \mathbb{R}$;

(d) $(\varphi^x, \varphi^y)(x,y,w(x,y)) = 2\nabla w(x,y)$ and $\varphi^z(x,y,w(x,y)) = |\nabla w(x,y)|^2$ for almost every $(x,y) \in \Omega_0$;

(e) $\displaystyle\int_{w^-(x,0)}^{w^+(x,0)} \varphi^x(x,0,z)\,dz = 0$ and $\displaystyle\int_{w^-(x,0)}^{w^+(x,0)} \varphi^y(x,0,z)\,dz = 1$ for $\mathcal{H}^1$-almost every $(x,0) \in S$;

(f) $\displaystyle\left(\int_{t_1}^{t_2} \varphi^x(x,y,z)\,dz\right)^2 + \left(\int_{t_1}^{t_2} \varphi^y(x,y,z)\,dz\right)^2 \leq 1$ for $\mathcal{H}^1$-almost every $(x,y) \in \Omega$ and for every $t_1, t_2 \in \mathbb{R}$.

2.2 A model case

In this section we consider in (2.1.1) and in (2.1.2) the particular function $u(x,y) = x$ and we deal with the minimality of the functions

$$w(x,y) := \begin{cases} x & \text{if } y > 0, \\ -x & \text{if } y < 0, \end{cases} \tag{2.2.1}$$

and

$$w(x,y) := \begin{cases} x+1 & \text{if } y > 0, \\ x & \text{if } y < 0. \end{cases} \tag{2.2.2}$$

The aim of the study of these simpler cases (but we will see that they involve the main difficulties) is to clarify the ideas of the general construction.

Theorem 2.2.1 *Let $w : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by*

$$w(x,y) := \begin{cases} x & \text{if } y > 0, \\ -x & \text{if } y < 0. \end{cases}$$

Then every point $(x_0, y_0) \neq (0,0)$ has an open neighbourhood U such that w is a Dirichlet minimizer in U of the Mumford-Shah functional (1.2.2).

Proof. The result follows by Theorem 4.1 of [1] if $y_0 \neq 0$. We consider now the case $y_0 = 0$, assuming for simplicity that $x_0 > 0$. We will construct a local calibration of w near $(x_0, 0)$ and then we will conclude thanks to Theorem 1.3.3. Let us fix $\varepsilon > 0$ such that

$$0 < \varepsilon < \frac{x_0}{10}, \qquad 0 < \varepsilon < \frac{1}{32}. \tag{2.2.3}$$

For $0 < \delta < \varepsilon$ we consider the open rectangle

$$U := \{(x,y) \in \mathbb{R}^2 : |x - x_0| < \varepsilon, |y| < \delta\}$$

and the following subsets of $U \times \mathbb{R}$ (see Fig. 2.1)

$$\begin{aligned} A_1 &:= \{(x,y,z) \in U \times \mathbb{R} : x - \alpha(y) < z < x + \alpha(y)\}, \\ A_2 &:= \{(x,y,z) \in U \times \mathbb{R} : b + \kappa(\lambda)\, y < z < b + \kappa(\lambda)\, y + h\}, \\ A_3 &:= \{(x,y,z) \in U \times \mathbb{R} : -h < z < h\}, \\ A_4 &:= \{(x,y,z) \in U \times \mathbb{R} : -b + \kappa(\lambda)\, y - h < z < -b + \kappa(\lambda)\, y\}, \\ A_5 &:= \{(x,y,z) \in U \times \mathbb{R} : -x - \alpha(-y) < z < -x + \alpha(-y)\}, \end{aligned}$$

where

$$\alpha(y) := \sqrt{4\varepsilon^2 - (\varepsilon - y)^2},$$

$$h := \frac{x_0 - 3\varepsilon}{4}, \qquad \kappa(\lambda) := \frac{\lambda}{4} - \frac{1}{\lambda}, \qquad b := 2h + \kappa(\lambda)\,\delta, \qquad \lambda := \frac{1 - 4\varepsilon}{2h}.$$

We will assume that

$$\delta < \frac{x_0 - 3\varepsilon}{8\,|\kappa(\lambda)|}, \tag{2.2.4}$$

so that the sets $A_1, \ldots, A_5$ are pairwise disjoint.

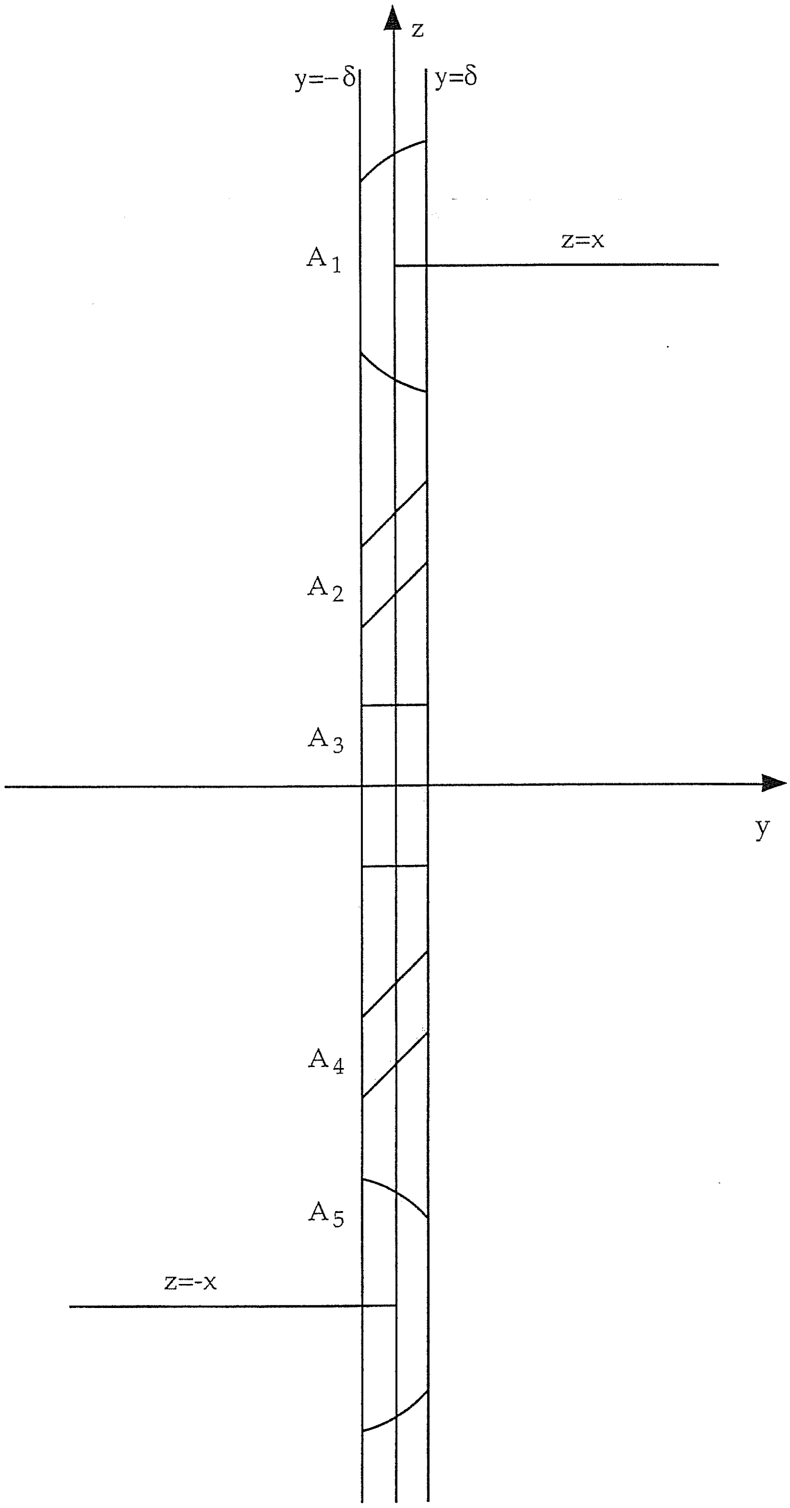

Figure 2.1: Section of the sets $A_1, \ldots, A_5$ at $x =$ constant.

For every $(x,y,z) \in U\times\mathbb{R}$, let us define the vector $\varphi(x,y,z) = (\varphi^x, \varphi^y, \varphi^z)(x,y,z) \in \mathbb{R}^3$ as follows:

$$\begin{cases} \left(\dfrac{2(\varepsilon-y)}{\sqrt{(\varepsilon-y)^2+(z-x)^2}}, \dfrac{-2(z-x)}{\sqrt{(\varepsilon-y)^2+(z-x)^2}}, 1\right) & \text{if } (x,y,z)\in A_1, \\ \left(0,\lambda,\dfrac{\lambda^2}{4}\right) & \text{if } (x,y,z)\in A_2, \\ (f(y),0,1) & \text{if } (x,y,z)\in A_3, \\ \left(0,\lambda,\dfrac{\lambda^2}{4}\right) & \text{if } (x,y,z)\in A_4, \\ \left(\dfrac{-2(\varepsilon+y)}{\sqrt{(\varepsilon+y)^2+(z+x)^2}}, \dfrac{2(z+x)}{\sqrt{(\varepsilon+y)^2+(z+x)^2}}, 1\right) & \text{if } (x,y,z)\in A_5, \\ (0,0,1) & \text{otherwise,} \end{cases}$$

where

$$f(y) := -\frac{1}{h}\left(\int_0^{\alpha(y)} \frac{\varepsilon-y}{\sqrt{t^2+(\varepsilon-y)^2}}\,dt - \int_0^{\alpha(-y)} \frac{\varepsilon+y}{\sqrt{t^2+(\varepsilon+y)^2}}\,dt\right).$$

Note that $A_1 \cup A_5$ is an open neighbourhood of $\operatorname{graph} w \cap (U\times\mathbb{R})$. The purpose of the definition of φ in A_1 and A_5 (see Fig. 2.2) is to provide a divergence-free vectorfield satisfying condition **(d)**

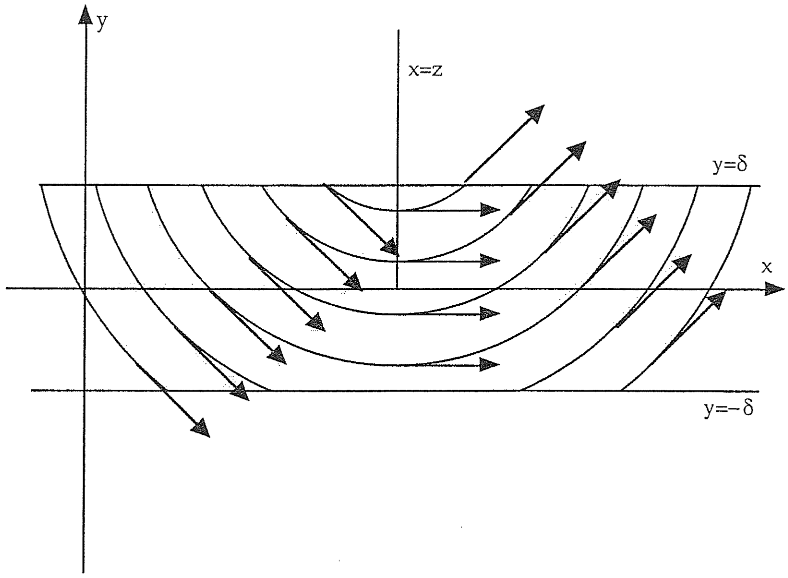

Figure 2.2: Section of the set A_1 at $z =$ constant.

of Section 2.1 and such that

$$\varphi^y(x,0,z) > 0 \qquad \text{for } |z| < x,$$
$$\varphi^y(x,0,z) < 0 \qquad \text{for } |z| > x.$$

These properties are crucial in order to obtain **(e)** and **(f)** simultaneously.

The rôle of A_2 and A_4 is to give the main contribution to the integral in **(e)**. To explain this fact, suppose, for a moment, that $\varepsilon = 0$; in this case we would have $A_1 = A_5 = \emptyset$ and

$$\int_{-x}^{x} \varphi^y(x,0,z)\,dz = 1,$$

so that the y-component of equality **(e)** would be satisfied.

The purpose of the definition of φ in A_3 is to correct the x-component of φ, in order to obtain **(f)**.

We shall prove that, for a suitable choice of δ, the vectorfield φ is a calibration for w in the rectangle U.

Note that for a given $z \in \mathbb{R}$ we have

$$\partial_x \varphi^x(x,y,z) + \partial_y \varphi^y(x,y,z) = 0 \qquad (2.2.5)$$

for every (x,y) such that $(x,y,z) \in A_1 \cup A_5$. This implies φ is divergence free in $A_1 \cup A_5$. Moreover $\text{div}\varphi = 0$ in the other sets A_i, and the normal component of φ is continuous across ∂A_i: the choice of $\kappa(\lambda)$ ensures that this property holds for ∂A_2 and ∂A_4 (see Fig. 2.3). Therefore φ satisfies conditions

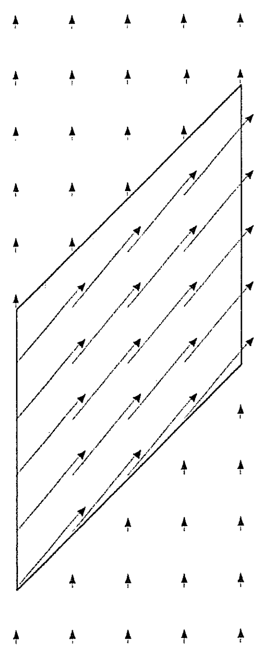

Figure 2.3: Section of the set A_2 at $x =$ constant.

(**a**) and (**b**), i.e. is divergence-free in the sense of distributions in $U\times\mathbb{R}$.

On the graph of w we have

$$\varphi(x,y,w(x,y)) = \begin{cases} (\ \ 2,0,1) & \text{if } y>0, \\ (-2,0,1) & \text{if } y<0, \end{cases}$$

so condition (**d**) is satisfied.

Inequality (**c**) is clearly satisfied in all regions: the only non trivial case is A_3, where we have, using (2.2.3),

$$|f(y)| \le \frac{4\,(\alpha(y)+\alpha(-y))}{x_0-3\varepsilon} \le \frac{8\sqrt{3}\varepsilon}{x_0-3\varepsilon} < 2.$$

We now compute

$$\int_{-x}^{x} \varphi^y(x,y,z)\,dz. \tag{2.2.6}$$

Let us fix y with $|y|<\delta$. Since $\varphi^y(x,y,z)$ depends on $z-x$, we have

$$\int_{x-\alpha(y)}^{x} \varphi^y(x,y,z)\,dz = \int_{x}^{x+\alpha(y)} \varphi^y(\xi,y,x)\,d\xi. \tag{2.2.7}$$

Using (2.2.5) and applying the divergence theorem to the curvilinear triangle

$$T=\{(\xi,\eta)\in\mathbb{R}^2 : \xi>x,\ \eta<y,\ (\varepsilon-\eta)^2+(x-\xi)^2<4\varepsilon^2\}$$

(see Fig. 2.4), we obtain

$$\int_{x}^{x+\alpha(y)} \varphi^y(\xi,y,x)\,d\xi = \int_{-\varepsilon}^{y} \varphi^x(x,\eta,x)\,d\eta = 2(y+\varepsilon). \tag{2.2.8}$$

From (2.2.7) and (2.2.8), we get

$$\int_{x-\alpha(y)}^{x} \varphi^y(x,y,z)\,dz = 2(y+\varepsilon). \tag{2.2.9}$$

Similarly we can prove that

$$\int_{-x}^{-x+\alpha(-y)} \varphi^y(x,y,z)\,dz = 2(-y+\varepsilon). \tag{2.2.10}$$

Using the definition of φ in A_2, A_3, A_4, we obtain

$$\int_{-x}^{x} \varphi^y(x,y,z)\,dz = 1. \tag{2.2.11}$$

On the other hand, by the definition of f, we have immediately that

$$\int_{-x}^{x} \varphi^x(x,y,z)\,dz = 0. \tag{2.2.12}$$

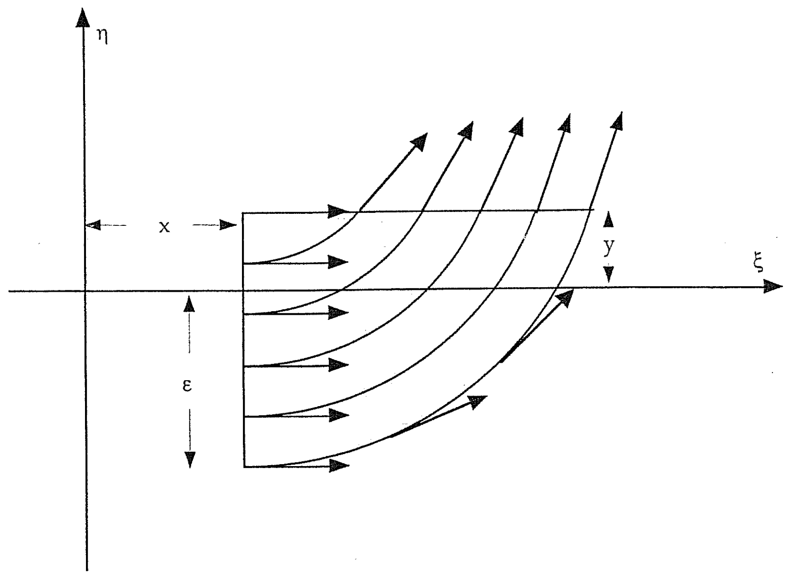

Figure 2.4: The curvilinear triangle T.

From these equalities it follows in particular that condition **(e)** is satisfied on the jump set $S_w \cap U = \{(x,y) \in U : y = 0\}$.

Let us begin now the proof of **(f)**. Let us fix $(x,y) \in U$. For every $t_1 < t_2$ we set

$$I(t_1, t_2, :) = \int_{t_1}^{t_2} (\varphi^x, \varphi^y)(x, y, z)\, dz.$$

It is enough to consider the case $-x - \alpha(-y) \leq t_1 \leq t_2 \leq x - \alpha(y)$. We can write

$$\begin{aligned} I(t_1, t_2) &= I(t_1, -x) + I(-x, x) + I(x, t_2), \\ I(t_1, -x) &= I(t_1 \wedge (-x + \alpha(-y)), -x) + I(t_1 \vee (-x + \alpha(-y)), -x + \alpha(-y)), \\ I(x, t_2) &= I(x, t_2 \vee (x - \alpha(y))) + I(x - \alpha(y), t_2 \wedge (x - \alpha(y))). \end{aligned}$$

Therefore

$$I(t_1, t_2) = I(-x, x) + I(t_1 \wedge (-x + \alpha(-y)), -x) + I(x, t_2 \vee (x - \alpha(y)))$$

$$+ I(t_1 \vee (-x + \alpha(-y)), t_2 \wedge (x - \alpha(y))) - I(-x + \alpha(-y), x - \alpha(y)). \tag{2.2.13}$$

Let B be the ball of radius 4ε centred at $(0, -4\varepsilon)$. We want to prove that

$$I(x, t) \in \overline{B} \tag{2.2.14}$$

for every t with $x - \alpha(y) \leq t \leq x + \alpha(y)$. Let us denote the components of $I(x,t)$ by a^x and a^y. Arguing as in the proof of (2.2.9), we get the identity

$$a^y = 2(\varepsilon - y) - 2\sqrt{(t - x)^2 + (\varepsilon - y)^2} \leq 0. \tag{2.2.15}$$

As $|\varphi^x| \leq 2$, we have also

$$(a^x)^2 \leq 4(t-x)^2 = (2(\varepsilon - y) - a^y)^2 - 4(\varepsilon - y)^2.$$

From these estimates it follows that

$$(a^x)^2 + (a^y + 4\varepsilon)^2 \leq 16\varepsilon^2,$$

which proves (2.2.14). In the same way we can prove that

$$I(t, -x) \in \overline{B} \tag{2.2.16}$$

for every t with $-x - \alpha(-y) \leq t \leq -x + \alpha(-y)$.

If $f(y) \geq 0$, we define

$$C := ([0, 2hf(y)] \times [0, \tfrac{1}{2} - 2\varepsilon]) \cup (\{2hf(y)\} \times [0, 1 - 4\varepsilon]);$$

if $f(y) \leq 0$, we simply replace $[0, 2hf(y)]$ by $[2hf(y), 0]$. From the definition of φ in A_2, A_3, A_4, it follows that

$$I(-x + \alpha(-y), x - \alpha(y)) = (2hf(y), 1 - 4\varepsilon) \tag{2.2.17}$$

and

$$I(s_1, s_2) \in C \tag{2.2.18}$$

for $-x + \alpha(-y) \leq s_1 \leq s_2 \leq x - \alpha(y)$. Let $D := C - (2hf(y), 1 - 4\varepsilon)$, i.e.,

$$D = ([-2hf(y), 0] \times [-1 + 4\varepsilon, -\tfrac{1}{2} + 2\varepsilon]) \cup (\{0\} \times [-1 + 4\varepsilon, 0]),$$

for $f(y) \geq 0$; the interval $[-2hf(y), 0]$ is replaced by $[0, -2hf(y)]$ when $f(y) \leq 0$. From (2.2.13), (2.2.11), (2.2.12), (2.2.14), (2.2.16), (2.2.17) and (2.2.18) we obtain

$$I(t_1, t_2) \in (0, 1) + 2\overline{B} + D. \tag{2.2.19}$$

As $f(0) = 0$, we can choose δ so that (2.2.4) is satisfied and

$$|2hf(y)| = \frac{x_0 - 3\varepsilon}{2} |f(y)| \leq \varepsilon \tag{2.2.20}$$

for $|y| < \delta$. It is then easy to see that, by (2.2.3), the set $(0, 1) + 2\overline{B} + D$ is contained in the unit ball centred at $(0, 0)$. So that (2.2.19) implies **(f)**. □

Remark 2.2.2 The assumption $(x_0, y_0) \neq (0, 0)$ in Theorem 2.2.1 cannot be dropped. Indeed, there is no neighbourhood U of $(0, 0)$ such that w is a Dirichlet minimizer of the Mumford-Shah functional in U.

To see this fact, let ψ be a function defined on the square $Q = (-1, 1) \times (-1, 1)$ satisfying the boundary condition $\psi = w$ on ∂Q and such that $S_\psi = ((-1, -1/2) \cup (1/2, 1)) \times \{0\}$. For every ε, let ψ_ε be the function defined on $Q_\varepsilon = \varepsilon Q$ by $\psi_\varepsilon(x, y) := \varepsilon \psi(x/\varepsilon, y/\varepsilon)$. Note that ψ_ε satisfies the boundary condition $\psi_\varepsilon = w$ on ∂Q_ε. Let us compute the Mumford-Shah functional for ψ_ε on Q_ε:

$$\int_{Q_\varepsilon} |\nabla \psi_\varepsilon|^2 dx\, dy + \mathcal{H}^1(S_{\psi_\varepsilon}) = \varepsilon^2 \int_Q |\nabla \psi|^2 dx\, dy + \varepsilon.$$

Since

$$\int_{Q_\varepsilon} |\nabla w|^2 dx\, dy + \mathcal{H}^1(S_w) = 4\varepsilon^2 + 2\varepsilon,$$

we have

$$\int_{Q_\varepsilon} |\nabla \psi_\varepsilon|^2 dx\, dy + \mathcal{H}^1(S_{\psi_\varepsilon}) < \int_{Q_\varepsilon} |\nabla w|^2 dx\, dy + \mathcal{H}^1(S_w)$$

for ε sufficiently small.

The construction shown in the proof of Theorem 2.2.1 can be easily adapted to define a calibration for the function w in (2.2.2).

Theorem 2.2.3 *Let $w : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by*

$$w(x,y) := \begin{cases} x+1 & \text{if } y > 0, \\ x & \text{if } y < 0. \end{cases}$$

Then every point $(x_0, y_0) \in \mathbb{R}^2$ has an open neighbourhood U such that w is a Dirichlet minimizer in U of the Mumford-Shah functional (1.2.2).

Proof. The result follows by Theorem 4.1 of [1] if $y_0 \neq 0$. We consider now the case $y_0 = 0$; we will construct a local calibration of w near $(x_0, 0)$, using the same technique as in Theorem 2.2.1. We give only the new definitions of the sets $A_1, \ldots, A_5$ and of the function φ, and leave to the reader the verification of the fact that this function is a calibration for suitable values of the involved parameters.

Let us fix $\varepsilon > 0$ such that

$$0 < \varepsilon < \frac{1}{24}, \qquad 0 < \varepsilon < \frac{1}{32}. \tag{2.2.21}$$

For $0 < \delta < \varepsilon$ we consider the open rectangle

$$U := \{(x,y) \in \mathbb{R}^2 : |x - x_0| < \varepsilon, |y| < \delta\}$$

and the following subsets of $U \times \mathbb{R}$

$$\begin{aligned}
A_1 &:= \{(x,y,z) \in U\times\mathbb{R} : x + 1 - \alpha(y) < z < x + 1 + \alpha(y)\}, \\
A_2 &:= \{(x,y,z) \in U\times\mathbb{R} : b + \kappa(\lambda)\, y + 3h < z < b + \kappa(\lambda)\, y + 4h\}, \\
A_3 &:= \{(x,y,z) \in U\times\mathbb{R} : x_0 + 3\varepsilon + 2h < z < x_0 + 3\varepsilon + 3h\}, \\
A_4 &:= \{(x,y,z) \in U\times\mathbb{R} : b + \kappa(\lambda)\, y < z < b + \kappa(\lambda)\, y + h\}, \\
A_5 &:= \{(x,y,z) \in U\times\mathbb{R} : x - \alpha(-y) < z < x + \alpha(-y)\},
\end{aligned}$$

where

$$\alpha(y) := \sqrt{4\varepsilon^2 - (\varepsilon - y)^2},$$

$$h := \frac{1 - 6\varepsilon}{5}, \qquad \kappa(\lambda) := \frac{\lambda}{4} - \frac{1}{\lambda}, \qquad b := x_0 + 3\varepsilon + \kappa(\lambda)\, \delta, \qquad \lambda := \frac{1 - 4\varepsilon}{2h}.$$

We will assume that

$$\delta < \frac{1 - 6\varepsilon}{10|\kappa(\lambda)|}, \tag{2.2.22}$$

so that the sets $A_1, \dots, A_5$ are pairwise disjoint. The family $(U_i)_{i\in I}$ appearing in the definition of $\mathcal{F}(\Omega \times \mathbb{R})$ is given by the subsets A_i and by the connected components of the complement in $U \times \mathbb{R}$ of $\cup_i A_i$. For every $(x,y,z) \in U\times\mathbb{R}$, let us define the vector $\varphi(x,y,z) \in \mathbb{R}^3$ as follows:

$$\begin{cases} \left(\dfrac{2(\varepsilon - y)}{\sqrt{(\varepsilon-y)^2+(z-x-1)^2}}, \dfrac{-2(z-x-1)}{\sqrt{(\varepsilon-y)^2+(z-x-1)^2}}, 1\right) & \text{if } (x,y,z) \in A_1, \\ \left(0, \lambda, \dfrac{\lambda^2}{4}\right) & \text{if } (x,y,z) \in A_2, \\ (f(y), 0, 1) & \text{if } (x,y,z) \in A_3, \\ \left(0, \lambda, \dfrac{\lambda^2}{4}\right) & \text{if } (x,y,z) \in A_4, \\ \left(\dfrac{2(\varepsilon + y)}{\sqrt{(\varepsilon+y)^2+(z-x)^2}}, \dfrac{2(z-x)}{\sqrt{(\varepsilon+y)^2+(z-x)^2}}, 1\right) & \text{if } (x,y,z) \in A_5, \\ (0,0,1) & \text{otherwise,} \end{cases}$$

where

$$f(y) := -\frac{2}{h}\left(\int_0^{\alpha(y)} \frac{\varepsilon - y}{\sqrt{t^2+(\varepsilon-y)^2}}\,dt + \int_0^{\alpha(-y)} \frac{\varepsilon + y}{\sqrt{t^2+(\varepsilon+y)^2}}\,dt\right)$$

for every $|y| < \delta$. □

2.3 The general case

In this section we denote by Ω a ball in $\mathbb{R}^2$ centred at $(0,0)$ and we consider as u in (2.1.1) and in (2.1.2) a generic harmonic function with normal derivative vanishing on S. We add the technical assumption that the first and second order tangential derivatives of u are not zero on S.

Theorem 2.3.1 *Let $u : \Omega \to \mathbb{R}$ be a harmonic function such that $\partial_y u(x,0) = 0$ for $(x,0) \in \Omega$, and let $w : \Omega \to \mathbb{R}$ be the function defined by*

$$w(x,y) := \begin{cases} u(x,y) & \text{for } y > 0, \\ -u(x,y) & \text{for } y < 0. \end{cases}$$

Assume that $u_0 := u(0,0) \neq 0$, $\partial_x u(0,0) \neq 0$, and $\partial^2_{xx} u(0,0) \neq 0$. Then there exists an open neighbourhood U of $(0,0)$ such that w is a Dirichlet minimizer in U of the Mumford-Shah functional (1.2.2).

Proof. We may assume $u(0,0) > 0$ and $\partial_x u(0,0) > 0$. We shall give the proof only for $\partial^2_{xx} u(0,0) > 0$, and we shall explain at the end the modification needed for $\partial^2_{xx} u(0,0) < 0$. Let

$v : \Omega \to \mathbb{R}$ be the harmonic conjugate of u that vanishes on $y = 0$, i.e., the function satisfying $\partial_x v(x,y) = -\partial_y u(x,y)$, $\partial_y v(x,y) = \partial_x u(x,y)$, and $v(x,0) = 0$.

Consider a small neighbourhood U of $(0,0)$ such that the map $\Phi(x,y) := (u(x,y), v(x,y))$ is invertible on U and $\partial_x u > 0$ on U. We call Ψ the inverse function $(u,v) \mapsto (\xi(u,v), \eta(u,v))$, which is defined in the neighbourhood $V := \Phi(U)$ of $(u_0, 0)$. Note that, if U is small enough, then $\eta(u,v) = 0$ if and only if $v = 0$. Moreover,

$$D\Psi = \begin{pmatrix} \partial_u \xi & \partial_v \xi \\ \partial_u \eta & \partial_v \eta \end{pmatrix} = \frac{1}{|\nabla u|^2} \begin{pmatrix} \partial_x u & \partial_x v \\ \partial_y u & \partial_y v \end{pmatrix}, \tag{2.3.1}$$

where, in the last formula, all functions are computed at $(x,y) = \Psi(u,v)$, and so $\partial_u \xi = \partial_v \eta$, $\partial_v \xi = -\partial_u \eta$ and $\partial_u \eta(u,0) = 0$, $\partial_v \eta(u,0) > 0$. In particular, ξ and η are harmonic, and

$$\partial^2_{uu} \eta(u,0) = 0, \qquad \partial^2_{vv} \eta(u,0) = 0. \tag{2.3.2}$$

On U we will use the coordinate system (u,v) given by Φ. By (2.3.1) the canonical basis of the tangent space to U at a point (x,y) is given by

$$\tau_u = \frac{\nabla u}{|\nabla u|^2}, \qquad \tau_v = \frac{\nabla v}{|\nabla v|^2}. \tag{2.3.3}$$

For every $(u,v) \in V$, let $G(u,v)$ be the matrix associated with the first fundamental form of U in the coordinate system (u,v), and let $g(u,v)$ be its determinant. By (2.3.1) and (2.3.3),

$$g = ((\partial_u \eta)^2 + (\partial_v \eta)^2)^2 = \frac{1}{|\nabla u(\Psi)|^4}. \tag{2.3.4}$$

We set $\gamma(u,v) := \sqrt[4]{g(u,v)}$.

The calibration $\varphi(x,y,z)$ on $U \times \mathbb{R}$ will be written as

$$\varphi(x,y,z) = \frac{1}{\gamma^2(u(x,y), v(x,y))} \phi(u(x,y), v(x,y), z). \tag{2.3.5}$$

We will adopt the following representation for $\phi : V \times \mathbb{R} \to \mathbb{R}^3$:

$$\phi(u,v,z) = \phi^u(u,v,z)\tau_u + \phi^v(u,v,z)\tau_v + \phi^z(u,v,z)e_z, \tag{2.3.6}$$

where e_z is the third vector of the canonical basis of $\mathbb{R}^3$, and τ_u, τ_v are computed at the point $\Psi(u,v)$. We now reformulate the conditions of Section 2.1 in this new coordinate system. It is known from Differential Geometry (see, e.g., [22, Proposition 3.5]) that, if $X = X^u \tau_u + X^v \tau_v$ is a vectorfield on U, then the divergence of X is given by

$$\mathrm{div} X = \frac{1}{\gamma^2}(\partial_u(\gamma^2 X^u) + \partial_v(\gamma^2 X^v)). \tag{2.3.7}$$

Using (2.3.3), (2.3.4), (2.3.5), (2.3.6), and (2.3.7) it turns out that φ is a calibration if we can find a Lipschitz decomposition $(V_i)_{i \in I}$ of $V \times \mathbb{R}$ such that

(a) $\mathrm{div}\phi = 0$ in V_i, for every $i \in I$;

(b) $\nu_{\partial V_i} \cdot \varphi^+ = \nu_{\partial V_i} \cdot \phi^- = \nu_{\partial V_i} \cdot \phi$ $\mathcal{H}^2$-a.e in ∂V_i for every $i \in I$, where $\nu_{\partial V_i}(u,v,z)$ denotes the (unit) normal vector at (u,v,z) to ∂V_i, while ϕ^+ and ϕ^- denote the two traces of ϕ on the two sides of ∂V_i;

(c) $(\varphi^u(u,v,z))^2 + (\varphi^v(u,v,z))^2 \le 4\varphi^z(u,v,z)$ for almost every (u,v) in V and for every $z \in \mathbb{R}$;

(d) $(\varphi^u(u,v,\pm u) = \pm 2,\ \phi^v(u,v,\pm u) = 0$, and $\phi^z(u,v,\pm u) = 1$, for almost every $(u,v) \in V$;

(e) $\displaystyle\int_{-u}^{u} \phi^u(u,0,z)\,dz = 0$ and $\displaystyle\int_{-u}^{u} \phi^v(u,0,z)\,dz = \gamma(u,0)$ for $\mathcal{H}^1$-almost every $(u,0) \in V$;

(f) $\displaystyle\Big(\int_{t_1}^{t_2} \phi^u(u,v,z)\,dz\Big)^2 + \Big(\int_{t_1}^{t_2} \phi^v(u,v,z)\,dz\Big)^2 \le \gamma^2(u,v)$ for almost every $(u,v) \in V$ and for every $t_1, t_2 \in \mathbb{R}$.

Given suitable parameters $\varepsilon > 0$, $h > 0$, $\lambda > 0$, that will be chosen later, and assuming

$$V = \{(u,v) : |u - u_0| < \delta, |v| < \delta\}, \tag{2.3.8}$$

with $\delta < \varepsilon$, we consider the following subsets of $V \times \mathbb{R}$

$$\begin{aligned}
A_1 &:= \{(u,v,z) \in V\times\mathbb{R} : u - \alpha(v) < z < u + \alpha(v)\},\\
A_2 &:= \{(u,v,z) \in V\times\mathbb{R} : 3h + \beta(u,v) < z < 3h + \beta(u,v) + 1/\lambda\},\\
A_3 &:= \{(u,v,z) \in V\times\mathbb{R} : -h < z < h\},\\
A_4 &:= \{(u,v,z) \in V\times\mathbb{R} : -3h + \beta(u,v) - 1/\lambda < z < -3h + \beta(u,v)\},\\
A_5 &:= \{(u,v,z) \in V\times\mathbb{R} : -u - \alpha(-v) < z < -u + \alpha(-v)\},
\end{aligned}$$

where

$$\alpha(v) := \sqrt{4\varepsilon^2 - (\varepsilon - v)^2},$$

and β is a suitable smooth function satisfying $\beta(u,0) = 0$, which will be defined later. It is easy to see that, if ε and h are sufficiently small, while λ is sufficiently large, then the sets $A_1, \dots, A_5$ are pairwise disjoint, provided δ is small enough. Moreover, since $\gamma(u,0) = \partial_v \eta(u,0) > 0$, by continuity we may assume that

$$\gamma(u,v) > 128\varepsilon \qquad \text{and} \qquad \partial_v \eta(u,v) > 8\varepsilon \tag{2.3.9}$$

for every $(u,v) \in V$.

For $(u,v) \in V$ and $z \in \mathbb{R}$ the vector $\phi(u,v,z)$ introduced in (2.3.5) is defined as follows:

$$\begin{cases}
\dfrac{2(\varepsilon - v)}{\sqrt{(\varepsilon - v)^2 + (z - u)^2}}\tau_u - \dfrac{2(z - u)}{\sqrt{(\varepsilon - v)^2 + (z - u)^2}}\tau_v + e_z & \text{in } A_1,\\[2ex]
-\lambda\sigma(u,v)\dfrac{v}{\sqrt{(u-a)^2 + v^2}}\tau_u + \lambda\sigma(u,v)\dfrac{u - a}{\sqrt{(u-a)^2 + v^2}}\tau_v + \mu e_z & \text{in } A_2,\\[2ex]
f(v)\tau_u + e_z & \text{in } A_3,\\[2ex]
-\lambda\sigma(u,v)\dfrac{v}{\sqrt{(u-a)^2 + v^2}}\tau_u + \lambda\sigma(u,v)\dfrac{u - a}{\sqrt{(u-a)^2 + v^2}}\tau_v + \mu e_z & \text{in } A_4,\\[2ex]
-\dfrac{2(\varepsilon + v)}{\sqrt{(\varepsilon + v)^2 + (z + u)^2}}\tau_u + \dfrac{2(z + u)}{\sqrt{(\varepsilon + v)^2 + (z + u)^2}}\tau_v + e_z & \text{in } A_5,\\[2ex]
e_z & \text{otherwise,}
\end{cases}$$

where

$$a < u_0 - 11\delta, \qquad \mu > 0 \tag{2.3.10}$$

$$f(v) := -\frac{1}{h}\left(\int_0^{\alpha(v)} \frac{(\varepsilon - v)}{\sqrt{t^2 + (\varepsilon - v)^2}}\, dt - \int_0^{\alpha(-v)} \frac{(\varepsilon + v)}{\sqrt{t^2 + (\varepsilon + v)^2}}\, dt\right),$$

$$\sigma(u,v) := \tfrac{1}{2}\gamma(a + \sqrt{(u-a)^2 + v^2}, 0) - 2\varepsilon. \tag{2.3.11}$$

We choose β as the solution of the Cauchy problem

$$\begin{cases} \lambda\sigma(u,v)(-v\,\partial_u\beta + (u-a)\partial_v\beta) = (\mu - 1)\sqrt{(u-a)^2 + v^2}, \\ \beta(u,0) = 0. \end{cases} \tag{2.3.12}$$

Since the line $v = 0$ is not characteristic for the equation near $(u_0, 0)$, there exists a unique solution $\beta \in C^\infty(V)$, provided V is small enough.

In the coordinate system (u,v) the definition of the field ϕ in A_1, A_3, and A_5 is the same as the definition of φ in the proof of Theorem 2.2.1. The crucial difference is in the definition on the sets A_2 and A_4, where now we are forced to introduce two new parameters a and μ. Note that the definition given in Theorem 2.2.1 can be regarded as the limiting case as a tends to $+\infty$.

By direct computations it is easy to see that ϕ satisfies condition **(a)** on A_1 and A_5. Similarly, the vectorfield

$$\left(-\frac{v}{\sqrt{(u-a)^2 + v^2}}, \frac{u-a}{\sqrt{(u-a)^2 + v^2}}\right)$$

is divergence free; since $(u-a)^2 + v^2$ is constant along the integral curves of this field, by construction the same property holds for σ, so that ϕ satisfies condition **(a)** in A_2 and A_4.

In A_3, condition **(a)** is trivially satisfied.

Note that the normal component of ϕ is continuous across each ∂A_i: for the region A_3 this continuity is guaranteed by our choice of β. This implies that also conditions **(b)** is fulfilled.

In order to satisfy condition **(c)**, it is enough to take the parameter μ such that

$$\frac{\lambda^2}{4}\sigma^2(u,v) \le \mu$$

for every $(u,v) \in V$, and require that

$$|f(v)| \le 2. \tag{2.3.13}$$

Since

$$|f(v)| \le \frac{\alpha(v) + \alpha(-v)}{h} \le \frac{4\,\varepsilon}{h}, \tag{2.3.14}$$

inequality (2.3.13) is true if we impose

$$2\,\varepsilon \le h.$$

Looking at the definition of ϕ on A_1 and A_5, one can check that condition **(d)** is satisfied.

Arguing as in the proof of (2.2.9), (2.2.10), (2.2.12) in Theorem 2.2.1, we find that for every $(u,v) \in V$

$$\int_{-u}^{-u+\alpha(-v)} \phi^u(u,v,z)\,dz + \int_{-h}^{h} \phi^u(u,v,z)\,dz + \int_{u-\alpha(v)}^{u} \phi^u(u,v,z)\,dz = 0,$$

$$\int_{-u}^{-u+\alpha(-v)} \phi^v(u,v,z)\,dz + \int_{-h}^{h} \phi^v(u,v,z)\,dz + \int_{u-\alpha(v)}^{u} \phi^v(u,v,z)\,dz = 4\varepsilon.$$

Now, it is easy to see that

$$\int_{-u}^{u} \phi^u(u,v,z)\,dz = -2\sigma(u,v)\frac{v}{\sqrt{(u-a)^2+v^2}}, \tag{2.3.15}$$

$$\int_{-u}^{u} \phi^v(u,v,z)\,dz = 4\varepsilon + 2\sigma(u,v)\frac{u-a}{\sqrt{(u-a)^2+v^2}}; \tag{2.3.16}$$

since for $v=0$ we have

$$\sigma(u,0) = \frac{1}{2}\gamma(u,0) - 2\varepsilon,$$

condition **(e)** is satisfied.

By continuity, if δ is small enough, we have

$$\int_{-u}^{u} \phi^v(u,v,z)\,dz > \frac{7}{8}\gamma(u,v) \tag{2.3.17}$$

for every $(u,v) \in V$.

From now on, we regard the pair (ϕ^u, ϕ^v) as a vector in $\mathbb{R}^2$. To prove condition **(f)** we set

$$I_{\varepsilon,a}(u,v,s,t) := \int_s^t (\phi^u,\phi^v)(u,v,z)\,dz$$

for every $(u,v) \in V$, and for every $s,t \in \mathbb{R}$. We want to compare the behaviour of the functions $|I_{\varepsilon,a}|^2$ and γ^2; to this aim, we define the function

$$d_{\varepsilon,a}(u,v,s,t) := |I_{\varepsilon,a}(u,v,s,t)|^2 - \gamma^2(u,v).$$

We have already shown (condition **(e)**) that

$$d_{\varepsilon,a}(u,0,-u,u) = 0. \tag{2.3.18}$$

We start by proving that, if V is sufficiently small, condition **(f)** holds for every $(u,v) \in V$, for t_1 close to $-u$ and t_2 close to u. Using the definition of $\phi(u,v,z)$ on A_1 and A_5, one can compute explicitly $d_{\varepsilon,a}(u,v,s,t)$ for $|s+u| \le \alpha(-v)$ and for $|t-u| \le \alpha(v)$. By direct computations one obtains

$$\nabla_{vst}\, d_{\varepsilon,a}(u,0,-u,u) = 0 \tag{2.3.19}$$

for $(u,0) \in V$.

We now want to compute the hessian matrix $\nabla^2_{vst}\, d_{\varepsilon,a}$ at the point $(u_0,0,-u_0,u_0)$. By (2.3.11) and (2.3.4), after some easy computations, we get

$$\partial^2_{vv}\sigma(u,0) = \frac{1}{2(u-a)}\partial_u\gamma(u,0) = \frac{1}{2(u-a)}\partial^2_{vu}\eta(u,0).$$

Using this equality and the explicit expression of $d_{\varepsilon,a}$ near $(u_0, 0, -u_0, u_0)$, we obtain

$$\begin{aligned}\partial^2_{vv} d_{\varepsilon,a}(u_0, 0, -u_0, u_0) &= -\frac{8\varepsilon}{(u_0-a)^2}(\partial_v \eta(u_0, 0) - 4\varepsilon) + \\ &+ \frac{2}{u_0 - a}\partial_v \eta(u_0, 0)\, \partial^2_{uv}\eta(u_0, 0) - \partial^2_{vv}(\gamma^2)(u_0, 0).\end{aligned}$$

Since η and γ do not depend on a and ε, for every ε satisfying (2.3.9) we can find a so close to u_0 that

$$\partial^2_{vv} d_{\varepsilon,a}(u_0, 0, -u_0, u_0) < 0. \tag{2.3.20}$$

Moreover, we easily obtain that

$$\begin{gathered}\partial^2_{tt} d_{\varepsilon,a}(u_0, 0, -u_0, u_0) = \partial^2_{ss} d_{\varepsilon,a}(u_0, 0, -u_0, u_0) = 8 - \frac{4}{\varepsilon}\partial_v \eta(u_0, 0), \\ \partial^2_{tv} d_{\varepsilon,a}(u_0, 0, -u_0, u_0) = \partial^2_{vs} d_{\varepsilon,a}(u_0, 0, -u_0, u_0) = -\frac{4}{u_0 - a}(\partial_v \eta(u_0, 0) - 4\varepsilon), \\ \partial^2_{st} d_{\varepsilon,a}(u_0, 0, -u_0, u_0) = 8.\end{gathered}$$

By the above expressions above, it follows that

$$\begin{aligned}&\det \begin{pmatrix} \partial^2_{vv} d_{\varepsilon,a} & \partial^2_{tv} d_{\varepsilon,a} \\ \partial^2_{tv} d_{\varepsilon,a} & \partial^2_{tt} d_{\varepsilon,a} \end{pmatrix} (u_0, 0, -u_0, u_0) = \\ &= \frac{16}{(u_0 - a)^2}\partial_v \eta(u_0, 0)(\partial_v \eta(u_0, 0) - 4\varepsilon) + \frac{c_1(\varepsilon)}{u_0 - a} + c_2(\varepsilon),\end{aligned}$$

where $c_1(\varepsilon)$, $c_2(\varepsilon)$ are two constants depending only on ε. Then, if ε satisfies (2.3.9), a can be chosen so close to u_0 that

$$\det \begin{pmatrix} \partial^2_{vv} d_{\varepsilon,a} & \partial^2_{tv} d_{\varepsilon,a} \\ \partial^2_{tv} d_{\varepsilon,a} & \partial^2_{tt} d_{\varepsilon,a} \end{pmatrix} (u_0, 0, -u_0, u_0) > 0. \tag{2.3.21}$$

At last, the determinant of the hessian matrix of $d_{\varepsilon,a}$ at $(u_0, 0, -u_0, u_0)$ is given by

$$\det \nabla^2_{vst}\, d_{\varepsilon,a}(u_0, 0, -u_0, u_0) = \frac{1}{u_0 - a}(\partial_v \eta(u_0, 0))^2 \partial^2_{uv}\eta(u_0, 0)(\partial_v \eta(u_0, 0) - 4\varepsilon)\frac{32}{\varepsilon^2} + c_3(\varepsilon),$$

where $c_3(\varepsilon)$ is a constant depending only on ε. Since, by (2.3.1),

$$\partial^2_{uv}\eta(u_0, 0) = -\frac{\partial^2_{xx}u(0, 0)}{(\partial_x u(0, 0))^3},$$

given ε satisfying (2.3.9), we can choose a so close to u_0 that

$$\det \nabla^2_{vst}\, d_{\varepsilon,a}(u_0, 0, -u_0, u_0) < 0. \tag{2.3.22}$$

By (2.3.20), (2.3.21), and (2.3.22), we can conclude that, by a suitable choice of the parameters, the hessian matrix of $d_{\varepsilon,a}$ (with respect to v, s, t) at $(u_0, 0, -u_0, u_0)$ is negative definite. This fact, with (2.3.18) and (2.3.19), allows us to state the existence of a constant $\tau > 0$ such that

$$d_{\varepsilon,a}(u, v, s, t) < 0 \tag{2.3.23}$$

for $|s+u_0|<\tau$, $|t-u_0|<\tau$, $(u,v)\in V$, $v\neq 0$, provided V is sufficiently small. So, condition **(f)** is satisfied for $|t_1+u_0|<\tau$ and $|t_2-u_0|<\tau$. We can assume $\delta<\tau<\alpha(v)$ for every $(u,v)\in V$.

From now on, since at this point the parameters ε, a have been fixed, we simply write I instead of $I_{\varepsilon,a}$. We now study the more general case $|t_1+u|<\alpha(-v)$ and $|t_2-u|<\alpha(v)$.

Let us set

$$m_1(u,v):=\max\{|I(u,v,s,t)|:\ |s+u|\le\alpha(-v),\ |t-u|\le\alpha(v),\ |t-u_0|\ge\tau\}.$$

By the definition of $A_1,\ldots,A_5$, for $\rho=\alpha(\delta)+\delta$ we have $(\phi^u,\phi^v)=0$ on $(V\times[u_0-\rho,u_0+\rho])\setminus A_1$ and $(V\times[-u_0-\rho,-u_0+\rho])\setminus A_5$. This implies that

$$m_1(u,v):=\max\{|I(u,v,s,t)|:\ |s+u_0|\le\rho,\ \tau\le|t-u_0|\le\rho\}$$

for $(u,v)\in V$. The function m_1, as supremum of a family of continuous functions, is lower semicontinuous. Moreover, m_1 is also upper semicontinuous; indeed, suppose, by contradiction, that there exist two sequences (u_n), (v_n) converging respectively to u, v, such that $(m_1(u_n,v_n))$ converges to a limit $l>m_1(u,v)$; then, there exist (s_n), (t_n) such that

$$|s_n+u_n|\le\alpha(-v_n),\qquad |t_n-u_n|\le\alpha(v_n),\qquad |t_n-u_0|\ge\tau, \tag{2.3.24}$$

and $m_1(u_n,v_n)=|I(u_n,v_n,s_n,t_n)|$. Up to subsequences, we can assume that (s_n), (t_n) converge respectively to s, t such that, by (2.3.24),

$$|s+u|\le\alpha(-v),\qquad |t-u|\le\alpha(v),\qquad |t-u_0|\ge\tau;$$

hence, we have that

$$m_1(u,v)\ge|I(u,v,s,t)|=\lim_{n\to\infty}|I(u_n,v_n,s_n,t_n)|=l>m_1(u,v),$$

which is impossible. Therefore, m_1 is continuous.

Let B be the open ball of radius 4ε centred at $(0,-4\varepsilon)$. Arguing as in (2.2.14), we can prove that

$$I(u,v,u,t)\in B \tag{2.3.25}$$

whenever $0<|t-u|\le\alpha(v)$. In the same way we can prove that

$$I(u,v,s,-u)\in B \tag{2.3.26}$$

for $0<|s+u|\le\alpha(-v)$. We can write

$$I(u,v,s,t)=I(u,v,s,-u)+I(u,v,-u,u)+I(u,v,u,t). \tag{2.3.27}$$

So, for $|s+u|\le\alpha(-v)$, $|t-u|\le\alpha(v)$, and $|t-u_0|\ge\tau$, by (2.3.26), (2.3.15), (2.3.16), and (2.3.25), we obtain that

$$I(u,0,s,t)\in(0,\gamma(u,0))+B+\overline{B},$$

hence, by (2.3.9), $I(u,0,s,t)$ belongs to the open ball of radius $\gamma(u,0)$ centred at $(0,0)$, and so, $m_1(u,0)<\gamma(u,0)$. By continuity, if V is small enough,

$$m_1(u,v)<\gamma(u,v) \tag{2.3.28}$$

for every $(u,v)\in V$.

Analogously, we define

$$m_2(u,v) := \max\{|I(u,v,s,t)| : |s+u| \le \alpha(-v), |s+u_0| \ge \tau, |t-u| \le \alpha(v),\}.$$

Arguing as in the case of m_1, we can prove that, if V is small enough,

$$m_2(u,v) < \gamma(u,v) \tag{2.3.29}$$

for every $(u,v) \in V$.

By (2.3.28), (2.3.29), and (2.3.23), we can conclude that $I(u,v,t_1,t_2)$ belongs to the ball centred at $(0,0)$ with radius $\gamma(u,v)$, for $|t_1+u| \le \alpha(-v)$ and $|t_2-u| \le \alpha(v)$. More precisely, let $E(u,v)$ be the intersection of this ball with the upper half plane bounded by the horizontal straight line passing through the point $(0, \frac{3}{4}\gamma(u,v))$: by (2.3.27), (2.3.17), (2.3.25), (2.3.26), and (2.3.9), we deduce that

$$I(u,v,t_1,t_2) \in E(u,v) \tag{2.3.30}$$

for $|t_1+u| \le \alpha(-v)$ and $|t_2-u| \le \alpha(v)$.

We can now conclude the proof of **(f)**. It is enough to consider the case $-u-\alpha(-v) \le t_1 \le t_2 \le u+\alpha(v)$. We can write

$$\begin{aligned} I(u,v,t_1,t_2) &= I(u,v,t_1 \wedge (-u+\alpha(-v)), t_2 \vee (u-\alpha(v))) + \\ &\quad + I(u,v,t_1 \vee (-u+\alpha(-v)), t_2 \wedge (u-\alpha(v))) - \\ &\quad - I(u,v,-u+\alpha(-v), u-\alpha(v)). \end{aligned} \tag{2.3.31}$$

By (2.3.30), it follows that

$$I(u,v,t_1 \wedge (-u+\alpha(-v)), t_2 \vee (u-\alpha(v))) \in E(u,v). \tag{2.3.32}$$

Let $C_1(u,v)$ be the parallelogram having three consecutive vertices at the points

$$(2hf(v),0), \qquad (0,0), \qquad \sigma(u,v)\frac{(-v,u-a)}{\sqrt{(u-a)^2+v^2}},$$

let $C_2(u,v)$ be the segment with endpoints

$$(2hf(v),0), \qquad (2hf(v),0) + 2\sigma(u,v)\frac{(-v,u-a)}{\sqrt{(u-a)^2+v^2}},$$

and let $C(u,v) := C_1(u,v) \cup C_2(u,v)$.

From the definition of φ in A_2, A_3, A_4, it follows that

$$I(u,v,-u+\alpha(-v), u-\alpha(v)) = (2hf(v),0) + 2\sigma(u,v)\frac{(-v,u-a)}{\sqrt{(u-a)^2+v^2}} \tag{2.3.33}$$

and

$$I(u,v,s_1,s_2) \in C(u,v) \tag{2.3.34}$$

for $-u+\alpha(-v) \le s_1 \le s_2 \le u-\alpha(v)$. Let

$$D(u,v) := C(u,v) - (2hf(v),0) - 2\sigma(u,v)\frac{(-v,u-a)}{\sqrt{(u-a)^2+v^2}}.$$

From (2.3.31), (2.3.32), (2.3.33), and (2.3.34) we obtain

$$I(u,v,t_1,t_2) \in E(u,v) + D(u,v). \tag{2.3.35}$$

As $|v| < \delta < 10\delta < u - a$ by (2.3.10), the angle that the segment $C_2(u,v)$ forms with the vertical is less than $\arctan(1/10)$. Moreover, we may assume that the length $2\sigma(u,v)$ of the segment $C_2(u,v)$ is less than $\gamma(u,v)$; indeed, this is true for $v = 0$ and, by continuity, it remains true if δ is small enough. By (2.3.9) and (2.3.14), we have also that $|2hf(v)| \le \gamma(u,v)/16$. Using these properties and simple geometric considerations, it is possible to prove that $E(u,v) + D(u,v)$ is contained in the ball with centre $(0,0)$ and radius $\gamma(u,v)$. This concludes the proof of **(f)**.

If $\partial^2_{xx}u(0,0) < 0$, it is enough to change the definition of ϕ in the sets A_2 and A_4, as follows:

$$\lambda\sigma(u,v)\frac{v}{\sqrt{(a-u)^2+v^2}}\tau_u + \lambda\sigma(u,v)\frac{a-u}{\sqrt{(a-u)^2+v^2}}\tau_v + \mu e_z,$$

where $a > u_0 + 11\delta$ and

$$\sigma(u,v) := \frac{1}{2}\gamma(a - \sqrt{(a-u)^2+v^2}, 0) - 2\varepsilon.$$

□

Theorem 2.3.2 *Let $u : \Omega \to \mathbb{R}$ be a harmonic function such that $\partial_y u(x,0) = 0$ for $(x,0) \in \Omega$, and let $w : \Omega \to \mathbb{R}$ be the function defined by*

$$w(x,y) := \begin{cases} u(x,y) + 1 & \text{for } y > 0, \\ u(x,y) & \text{for } y < 0. \end{cases}$$

Assume that $\partial_x u(0,0) \neq 0$ and $\partial^2_{xx}u(0,0) \neq 0$. Then there exists an open neighbourhood U of $(0,0)$ such that w is a Dirichlet minimizer in U of the Mumford-Shah functional (1.2.2).

Proof. We will write the calibration φ as in (2.3.5) and we will adopt the representation (2.3.6) for ϕ. We will use the same technique as in Theorem 2.3.1. We give only the new definitions of the sets $A_1, \ldots, A_5$ and of the function ϕ when $\partial_x u(0,0) > 0$ and $\partial^2_{xx}u(0,0) > 0$, and leave to the reader the verification of the fact that this function is a calibration for suitable values of the involved parameters. The case $\partial^2_{xx}u(0,0) < 0$ can be treated by the changes introduced at the end of Theorem 2.3.1.

Let $u_0 := u(0,0)$. Given $\varepsilon > 0$, $h > 0$, $\lambda > 0$, and assuming

$$V := \{(u,v) : |u - u_0| < \delta, |v| < \delta\},$$

we consider the following subsets of $V \times \mathbb{R}$

$$\begin{aligned}
A_1 &:= \{(u,v,z) \in V\times\mathbb{R} : u + 1 - \alpha(v) < z < u + 1 + \alpha(v)\}, \\
A_2 &:= \{(u,v,z) \in V\times\mathbb{R} : 5h + \beta(u,v) < z < 5h + \beta(u,v) + 1/\lambda\}, \\
A_3 &:= \{(u,v,z) \in V\times\mathbb{R} : 2h < z < 4h\}, \\
A_4 &:= \{(u,v,z) \in V\times\mathbb{R} : h + \beta(u,v) < z < h + \beta(u,v) + 1/\lambda\}, \\
A_5 &:= \{(u,v,z) \in V\times\mathbb{R} : u - \alpha(-v) < z < u + \alpha(-v)\},
\end{aligned}$$

where

$$\alpha(v) := \sqrt{4\varepsilon^2 - (\varepsilon - v)^2},$$

and β is a suitable smooth function satisfying $\beta(u,0) = 0$, which will be defined later. For $(u,v) \in V$ and $z \in \mathbb{R}$ the vector $\phi(u,v,z)$ is defined as follows:

$$\begin{cases} \dfrac{2(\varepsilon - v)}{\sqrt{(\varepsilon - v)^2 + (z-u-1)^2}}\tau_u - \dfrac{2(z-u-1)}{\sqrt{(\varepsilon - v)^2 + (z-u-1)^2}}\tau_v + e_z & \text{in } A_1, \\ -\lambda\sigma(u,v)\dfrac{v}{\sqrt{(u-a)^2+v^2}}\tau_u + \lambda\sigma(u,v)\dfrac{u-a}{\sqrt{(u-a)^2+v^2}}\tau_v + \mu e_z & \text{in } A_2, \\ f(v)\tau_u + e_z & \text{in } A_3, \\ -\lambda\sigma(u,v)\dfrac{v}{\sqrt{(u-a)^2+v^2}}\tau_u + \lambda\sigma(u,v)\dfrac{u-a}{\sqrt{(u-a)^2+v^2}}\tau_v + \mu e_z & \text{in } A_4, \\ \dfrac{2(\varepsilon + v)}{\sqrt{(\varepsilon + v)^2 + (z-u)^2}}\tau_u + \dfrac{2(z-u)}{\sqrt{(\varepsilon + v)^2 + (z-u)^2}}\tau_v + e_z & \text{in } A_5, \\ e_z & \text{otherwise,} \end{cases}$$

where $a < u_0 - 11\delta$, $\mu > 0$,

$$f(v) := -\frac{1}{h}\left(\int_0^{\alpha(v)} \frac{(\varepsilon - v)}{\sqrt{t^2 + (\varepsilon - v)^2}}\,dt + \int_0^{\alpha(-v)} \frac{(\varepsilon + v)}{\sqrt{t^2 + (\varepsilon + v)^2}}\,dt\right),$$

$$\sigma(u,v) := \tfrac{1}{2}\gamma(a + \sqrt{(u-a)^2 + v^2}, 0) - 2\varepsilon,$$

and β is the solution of the Cauchy problem (2.3.12). □

Chapter 3

Calibration of solutions with a regular discontinuity set

By means of a new technique in the construction of the calibration, we generalize the results of the previous chapter to the case when the discontinuity set of the candidate u (defined in a two-dimensional domain) is any analytic curve joining two boundary points.

3.1 Statement of the main results

Let Ω be an open bounded subset of $\mathbb{R}^2$ and let Γ be a regular curve joining two boundary points of $\partial\Omega$. By (1.2.3) and (1.2.11) we have that $u \in SBV(\Omega)$, with $S_u = \Gamma$, satisfies the Euler-Lagrange conditions for the homogeneous Mumford-Shah functional F_0 (defined in (1.2.2)) if

i) u is harmonic in $\Omega \setminus \Gamma$ and $u \in H^1(\Omega \setminus \Gamma)$,

ii) $\partial_\nu u = 0$ on Γ,

iii) $\left[|\nabla u|^2\right]^{\pm} = \operatorname{curv} \Gamma$ at every point of Γ,

where $\nabla u^{\pm}$ denote the traces of ∇u on Γ. The main result of the chapter is stated in the following theorem.

Theorem 3.1.1 *Let Γ be a simple analytic curve and let Ω_0 be a (connected) open subset of $\mathbb{R}^2$ such that $\Gamma \cap \overline{\Omega}$ connects two points of $\partial\Omega_0$. Let u be a function in $H^1(\Omega_0 \setminus \Gamma)$ with $S_u = \Gamma$, with different traces at every point of Γ, and satisfying the Euler conditions* **i)**, **ii)**, *and* **iii)** *in Ω_0. Then, for every subarc $\Gamma' \subset \Gamma$ compactly contained in Ω_0, there exists a uniform open neighbourhood U of Γ' contained in Ω_0 such that u is a Dirichlet-minimizer in U of the Mumford-Shah functional (1.2.2).*

As we already said in the Introduction, we want investigate also a different kind of minimality, by considering as competitors functions with complete graph (see Subsection 1.1.1) contained in a small neighbourhood of the complete graph of the candidate u. Accordingly we give the following definition.

Definition 3.1.2 *A function $u \in SBV(\Omega)$ is a local graph-minimizer in Ω if there exists a neighbourhood U of the complete graph Γ_u of u such that*

$$\int_\Omega |\nabla u(x,y)|^2 dx\, dy + \mathcal{H}^1(S_u) \le \int_\Omega |\nabla v(x,y)|^2 dx\, dy + \mathcal{H}^1(S_v)$$

for every $v \in SBV(\Omega)$ *with the same trace as* u *on* $\partial\Omega$ *and whose complete graph* Γ_v *is contained in* U.

Given an open set A (with Lipschitz boundary) and a portion Γ of ∂A (with nonempty relative interior in ∂A), we define

$$K(\Gamma, A) := \inf\left\{ \int_A |\nabla v(x,y)|^2 dx\, dy : \ v \in H^1(A),\ \int_\Gamma v^2 d\mathcal{H}^1 = 1,\ \text{and } v = 0 \text{ on } \partial A \setminus \Gamma \right\}. \quad (3.1.1)$$

It is easy to see that in the problem above the infimum is attained; moreover, the notation is well chosen since $K(\Gamma, A)$ is a quantity depending only on Γ and A, which describes a kind of "capacity" of the prescribed portion of the boundary with respect to the whole open set. It is convenient to give the following definition.

Definition 3.1.3 *Given a simple analytic curve* Γ, *we say that an open set* Ω *is* Γ*-admissible if it is bounded,* $\Gamma \cap \overline{\Omega}$ *connects two points of* $\partial\Omega$, *and* $\Omega \setminus \Gamma$ *has two connected components, with Lipschitz boundaries.*

We will prove the following theorem which gives a sufficient condition for the graph-minimality in terms of $K(\Gamma, \Omega)$ and of the geometrical properties of the curve, i.e. the length of Γ denoted by $l(\Gamma)$, and the L^∞-norm of $\mathrm{curv}\,\Gamma$ denoted by $k(\Gamma)$.

Theorem 3.1.4 *Let* Ω_0, u, *and* $\Gamma = S_u$ *satisfy the same assumptions as in Theorem 3.1.1 and let* Ω *be a* Γ*-admissible open set compactly contained in* Ω_0. *Denote by* Ω_1 *and* Ω_2 *the two connected components of* $\Omega \setminus \Gamma$, *by* u_i *the restriction of* u *to* Ω_i, *and by* $\partial_\tau u_i$ *its tangential derivative on* Γ. *There exists an absolute constant* $c > 0$ *(independent of* Ω_0, Ω, Γ, *and* u*) such that if*

$$\frac{\min_{i=1,2} K(\Gamma \cap \Omega, \Omega_i)}{1 + l^2(\Gamma \cap \Omega) + l^2(\Gamma \cap \Omega) k^2(\Gamma \cap \Omega)} > c \sum_{i=1}^{2} \|\partial_\tau u_i\|^2_{C^1(\Gamma\cap\Omega)}, \quad (3.1.2)$$

then u *is a local graph-minimizer on* Ω.

3.2 Proof of Theorem 3.1.1

Lemma 3.2.1 *Let* U *be an open subset of* $\mathbb{R}^2$ *and* I, J *be two real intervals. Let* $u : U \times J \to I$ *be a function of class* C^1 *such that*

- $u(\cdot, \cdot; s)$ *is harmonic for every* $s \in J$;
- *there exists a* C^1 *function* $t : U \times I \to J$ *such that* $u(x, y; t(x, y; z)) = z$.

Then, if we define in $U \times I$ *the vectorfield*

$$\phi(x, y, z) := (2\nabla u(x, y; t(x, y; z)), |\nabla u(x, y; t(x, y; z))|^2),$$

where $\nabla u(x, y; t(x, y; z))$ *denotes the gradient of* u *with respect to the variables* (x, y) *computed at* $(x, y; t(x, y; z))$, ϕ *is divergence free in* $U \times I$.

PROOF OF THE LEMMA. Let us compute the divergence of ϕ:

$$\begin{aligned}\mathrm{div}\phi(x,y,z) &= 2\Delta u(x,y;t(x,y;z)) + 2\partial_s \nabla u(x,y;t(x,y;z)) \cdot \nabla t(x,y;z) \\ &\quad +2\partial_z t(x,y;z)\, \nabla u(x,y;t(x,y;z)) \cdot \partial_s \nabla u(x,y;t(x,y;z)),\end{aligned} \tag{3.2.1}$$

where $\Delta u(x,y;t(x,y;z))$ denotes the laplacian of u with respect to (x,y) computed at $(x,y;t(x,y;z))$, and $\nabla t(x,y;z)$ denotes the gradient of t with respect to (x,y). By differentiating the identity verified by the function t first with respect to z and with respect to (x,y), we derive that

$$\partial_s u(x,y;t(x,y;z))\, \partial_z t(x,y;z) = 1, \qquad \nabla u(x,y;t(x,y;z)) + \partial_s u(x,y;t(x,y;z))\, \nabla t(x,y;z) = 0.$$

Using these identities and substituting in (3.2.1), we finally obtain

$$\mathrm{div}\phi(x,y,z) = 2\Delta u(x,y;t(x,y;z)) = 0,$$

since by assumption u is harmonic with respect to (x,y). □

PROOF OF THEOREM 3.1.1. Again the proof will be achieved by constructing a calibration.

Without loss of generality we can suppose that Γ' is connected. For the sake of notation, in the sequel, we will write Γ instead of Γ'. Let

$$\Gamma : \begin{cases} x = x(s) \\ y = y(s) \end{cases}$$

be a parameterization by the arc-length, where s varies in $[0, l(\Gamma)]$; we choose as orientation the normal vectorfield $\nu(s) = (-\dot{y}(s), \dot{x}(s))$.

By Cauchy-Kowalevski Theorem (see [36]) there exist an open neighbourhood U of Γ contained in Ω_0 and a harmonic function ξ defined on U such that

$$\xi(\Gamma(s)) = s \qquad \text{and} \qquad \partial_\nu \xi(\Gamma(s)) = 0.$$

We can suppose that U is simply connected. Let $\eta : U \to \mathbb{R}^2$ be the harmonic conjugate of ξ that vanishes on Γ, i.e., the function satisfying $\partial_x \eta(x,y) = -\partial_y \xi(x,y)$, $\partial_y \eta(x,y) = \partial_x \xi(x,y)$, and $\eta(\Gamma(s)) = 0$.

Taking U smaller if needed, we can suppose that the map $\Phi(x,y) := (\xi(x,y), \eta(x,y))$ is invertible on U. We call Ψ the inverse function $(\xi,\eta) \mapsto (\tilde{x}(\xi,\eta), \tilde{y}(\xi,\eta))$, which is defined in the open set $V := \Phi(U)$. Note that, if U is small enough, then $(\tilde{x}(\xi,\eta), \tilde{y}(\xi,\eta))$ belongs to Γ if and only if $\eta = 0$. Moreover,

$$D\Psi = \begin{pmatrix} \partial_\xi \tilde{x} & \partial_\eta \tilde{x} \\ \partial_\xi \tilde{y} & \partial_\eta \tilde{y} \end{pmatrix} = \frac{1}{|\nabla \xi|^2} \begin{pmatrix} \partial_x \xi & \partial_x \eta \\ \partial_y \xi & \partial_y \eta \end{pmatrix}, \tag{3.2.2}$$

where, in the last formula, all functions are computed at $(x,y) = \Psi(\xi,\eta)$, and so

$$\partial_\xi \tilde{x} = \partial_\eta \tilde{y} \qquad \text{and} \qquad \partial_\eta \tilde{x} = -\partial_\xi \tilde{y}. \tag{3.2.3}$$

In particular, $\tilde{x}$ and $\tilde{y}$ are harmonic.

On U we will use the coordinate system (ξ,η) given by Φ. By (3.2.2) the canonical basis of the tangent space to U at a point (x,y) is given by

$$\tau_\xi = \frac{\nabla\xi}{|\nabla\xi|^2}, \qquad \tau_\eta = \frac{\nabla\eta}{|\nabla\eta|^2}. \tag{3.2.4}$$

For every $(\xi,\eta)\in V$, let $G(\xi,\eta)$ be the matrix associated with the first fundamental form of U in the coordinate system (ξ,η), and let $g(\xi,\eta)$ be its determinant. By (3.2.2) and (3.2.4),

$$g = ((\partial_\xi\tilde{x})^2 + (\partial_\xi\tilde{y})^2)^2 = \frac{1}{|\nabla\xi(\Psi)|^4}. \tag{3.2.5}$$

We set $\gamma(\xi,\eta) = \sqrt[4]{g(\xi,\eta)}$.

From now on we will assume that V is symmetric with respect to $\{(\xi,\eta)\in\Phi(U) : \eta = 0\}$.

Note that we can write the function u in this new coordinate system as

$$u(\xi,\eta) = \begin{cases} u_1(\xi,\eta) & \text{if } (\xi,\eta)\in V,\ \eta<0,\\ u_2(\xi,\eta) & \text{if } (\xi,\eta)\in V,\ \eta>0,\end{cases}$$

where we can suppose that u_1 and u_2 are defined in V (indeed, u_1 is a priori defined only on the set $\{(\xi,\eta)\in V : \eta<0\}$, but it can be extended to V by reflection; an analogous argument applies to u_2), $0 < u_1(\xi,0) < u_2(\xi,0)$ for every $(\xi,0)\in V$, and

i) $\partial^2_{\xi\xi}u_i(\xi,\eta) + \partial^2_{\eta\eta}u_i(\xi,\eta) = 0$ for $i=1,2$;

ii) $\partial_\eta u_1(\xi,0) = \partial_\eta u_2(\xi,0) = 0$;

iii) $(\partial_\xi u_2(\xi,0))^2 - (\partial_\xi u_1(\xi,0))^2 = \operatorname{curv}\Gamma(\xi)$.

The calibration $\varphi(x,y,z)$ on $U\times\mathbb{R}$ will be written as

$$\varphi(x,y,z) = \frac{1}{\gamma^2(\xi(x,y),\eta(x,y))}\phi(\xi(x,y),\eta(x,y),z), \tag{3.2.6}$$

where $\phi : V\times\mathbb{R}\to\mathbb{R}^3$ can be represented by

$$\phi(\xi,\eta,z) = \phi^\xi(\xi,\eta,z)\tau_\xi + \phi^\eta(\xi,\eta,z)\tau_\eta + \phi^z(\xi,\eta,z)e_z, \tag{3.2.7}$$

where e_z is the third vector of the canonical basis of $\mathbb{R}^3$, and τ_ξ, τ_η are computed at the point $\Psi(\xi,\eta)$. We now reformulate the conditions of the calibration in this new coordinate system. It is known from Differential Geometry (see, e.g., [22, Proposition 3.5]) that, if $X = X^\xi\tau_\xi + X^\eta\tau_\eta$ is a vectorfield on U, then the divergence of X is given by

$$\operatorname{div}X = \frac{1}{\gamma^2}(\partial_\xi(\gamma^2X^\xi) + \partial_\eta(\gamma^2X^\eta)). \tag{3.2.8}$$

Using (3.2.4), (3.2.5), (3.2.6), (3.2.7), (3.2.8), and arguing as in Section 2.3, it turns out that a vectorfield $\varphi\in\mathcal{F}(V\times\mathbb{R})$ (see Section 1.3 or 2.1 for the definition of $\mathcal{F}(V\times\mathbb{R})$) is a calibration if, for the associated Lipschitz decomposition $(A_i)_{i\in I}$ of $V\times\mathbb{R}$, the following conditions are satisfied:

(a) $\operatorname{div}\phi = 0$ in A_i, for every $i\in I$;

(b) $\nu_{\partial A_i} \cdot \varphi^+ = \nu_{\partial A_i} \cdot \phi^- = \nu_{\partial A_i} \cdot \phi$ $\mathcal{H}^2$-a.e in ∂A_i for every $i \in I$, where $\nu_{\partial A_i}(\xi,\eta,z)$ denotes the (unit) normal vector at (ξ,η,z) to ∂A_i, while ϕ^+ and ϕ^- denote the two traces of ϕ on the two sides of ∂A_i;

(c) $(\phi^\xi(\xi,\eta,z))^2 + (\phi^\eta(\xi,\eta,z))^2 \leq 4\phi^z(\xi,\eta,z)$ for almost every $(\xi,\eta) \in V$ and every $z \in \mathbb{R}$;

(d) $\phi^\xi(\xi,\eta,u(\xi,\eta)) = 2\partial_\xi u(\xi,\eta)$, $\phi^\eta(\xi,\eta,u(\xi,\eta)) = 2\partial_\eta u(\xi,\eta)$, and $\phi^z(\xi,\eta,u(\xi,\eta)) = (\partial_\xi u(\xi,\eta))^2 + (\partial_\eta u(\xi,\eta))^2$ for almost every $(\xi,\eta) \in V$;

(e) $\displaystyle\int_{u_1}^{u_2} \phi^\xi(\xi,0,z)\,dz = 0$ and $\displaystyle\int_{u_1}^{u_2} \phi^\eta(\xi,0,z)\,dz = \gamma(\xi,0) = 1$ for $\mathcal{H}^1$-almost every $(\xi,0) \in V$;

(f) $\displaystyle\left(\int_s^t \phi^\xi(\xi,\eta,z)\,dz\right)^2 + \left(\int_s^t \phi^\eta(\xi,\eta,z)\,dz\right)^2 \leq \gamma^2(\xi,\eta)$ for $\mathcal{H}^1$-almost every $(\xi,\eta) \in V$, and for every $s,t \in \mathbb{R}$.

Given suitable parameters $\varepsilon > 0$ and $\lambda > 0$, that will be chosen later, we consider the following subsets of $V \times \mathbb{R}$

$$\begin{aligned}
A_1 &:= \{(\xi,\eta,z) \in V\times\mathbb{R} : z < u_1(\xi,\eta) - \varepsilon\},\\
A_2 &:= \{(\xi,\eta,z) \in V\times\mathbb{R} : u_1(\xi,\eta) - \varepsilon < z < u_1(\xi,\eta) + \varepsilon\},\\
A_3 &:= \{(\xi,\eta,z) \in V\times\mathbb{R} : u_1(\xi,\eta) + \varepsilon < z < \beta_1(\xi,\eta)\},\\
A_4 &:= \{(\xi,\eta,z) \in V\times\mathbb{R} : \beta_1(\xi,\eta) < z < \beta_2(\xi,\eta) + 1/\lambda\},\\
A_5 &:= \{(\xi,\eta,z) \in V\times\mathbb{R} : \beta_2(\xi,\eta) + 1/\lambda < z < u_2(\xi,\eta) - \varepsilon\},\\
A_6 &:= \{(\xi,\eta,z) \in V\times\mathbb{R} : u_2(\xi,\eta) - \varepsilon < z < u_2(\xi,\eta) + \varepsilon\},\\
A_7 &:= \{(\xi,\eta,z) \in V\times\mathbb{R} : z > u_2(\xi,\eta) + \varepsilon\},
\end{aligned}$$

where β_1 and β_2 are suitable smooth function such that $u_1(\xi,0) < \beta_1(\xi,0) = \beta_2(\xi,0) < u_2(\xi,0)$, which will be defined later. It is clear that if ε is small enough and λ is sufficiently large, then the sets $A_1,\dots,A_7$ are nonempty and disjoint, provided V is sufficiently small.

The vector $\phi(\xi,\eta,z)$ introduced in (3.2.6) will be written as

$$\phi(\xi,\eta,z) = (\phi^{\xi\eta}(\xi,\eta,z), \phi^z(\xi,\eta,z)),$$

where $\phi^{\xi\eta}$ is the two-dimensional vector given by the pair (ϕ^ξ,ϕ^η). For $(\xi,\eta) \in V$ and $z \in \mathbb{R}$ we define $\phi(\xi,\eta,z)$ as follows:

$$\begin{cases}
(0,\omega_1(\xi,\eta)) & \text{in } \overline{A}_1 \cup \overline{A}_3,\\[2ex]
\left(2\nabla u_1 - 2\dfrac{u_1 - z}{v_1}\nabla v_1, \left|\nabla u_1 - \dfrac{u_1 - z}{v_1}\nabla v_1\right|^2\right) & \text{in } A_2,\\[2ex]
(\lambda\sigma(\xi,\eta)\nabla w, \mu) & \text{in } A_4,\\[2ex]
(0,\omega_2(\xi,\eta)) & \text{in } \overline{A}_5 \cup \overline{A}_7,\\[2ex]
\left(2\nabla u_2 - 2\dfrac{u_2 - z}{v_2}\nabla v_2, \left|\nabla u_2 - \dfrac{u_2 - z}{v_2}\nabla v_2\right|^2\right) & \text{in } A_6,
\end{cases}$$

where ∇ denotes the gradient with respect to the variables (ξ, η), the functions v_i are defined by

$$v_1(\xi,\eta) := \varepsilon + M\eta, \ v_2(\xi,\eta) := \varepsilon - M\eta,$$

and M and μ are positive parameters which will be fixed later, while

$$\omega_i(\xi,\eta) := \frac{\varepsilon^2 M^2}{v_i^2(\xi,\eta)} - (\partial_\xi u_i(\xi,\eta))^2 - (\partial_\eta u_i(\xi,\eta))^2 \tag{3.2.9}$$

for $i = 1, 2$, and for every $(\xi, \eta) \in V$. We choose w as the solution of the Cauchy problem

$$\begin{cases} \Delta w = 0, \\ w(\xi,0) = -\dfrac{2\varepsilon}{1-2\varepsilon M}\displaystyle\int_0^\xi n(s)(\partial_\xi u_1(s,0) + \partial_\xi u_2(s,0))\,ds, \\ \partial_\eta w(\xi,0) = n(\xi), \end{cases} \tag{3.2.10}$$

where n is a positive analytic function that will be chosen later in a suitable way (if V is sufficiently small, w is defined in V). To define σ, we need some further explanations: we call $p(\xi,\eta)$ the solution of the problem

$$\begin{cases} \partial_\eta p(\xi,\eta) = \dfrac{\partial_\xi w}{\partial_\eta w}(p(\xi,\eta),\eta), \\ p(\xi,0) = \xi, \end{cases} \tag{3.2.11}$$

which is defined in V, provided V is small enough. By applying the Implicit Function Theorem, it is easy to see that there exists a function q defined in V (take V smaller, if needed) such that

$$p(q(\xi,\eta),\eta) = \xi. \tag{3.2.12}$$

At last, we define

$$\sigma(\xi,\eta) := \frac{1}{n(q(\xi,\eta))}(1-2\varepsilon M).$$

We choose β_i, for $i = 1, 2$, as the solution of the Cauchy problem

$$\begin{cases} \lambda\sigma(\xi,\eta)\partial_\xi w(\xi,\eta)\partial_\xi\beta_i(\xi,\eta) + \lambda\sigma(\xi,\eta)\partial_\eta w(\xi,\eta)\partial_\eta\beta_i(\xi,\eta) - \mu = -\omega_i(\xi,\eta), \\ \beta_i(\xi,0) = \dfrac{1}{2}(u_1(\xi,0) + u_2(\xi,0)). \end{cases} \tag{3.2.13}$$

Since the line $\eta = 0$ is not characteristic, there exists a unique solution $\beta_i \in C^\infty(V)$, provided V is small enough.

The purpose of the definition of ϕ in A_2 and A_6 is to provide a divergence free vectorfield satisfying condition **(d)** and such that

$$\phi^\eta(\xi,0,z) \geq 0 \quad \text{for } u_1 < z < u_2,$$
$$\phi^\eta(\xi,0,z) \leq 0 \quad \text{for } z < u_1 \text{ and } z > u_2.$$

These properties are crucial in order to obtain **(e)** and **(f)** simultaneously.

The role of A_4 is to give the main contribution to the integral in **(e)**. The idea of the construction is to start from the gradient field of a harmonic function w whose normal derivative is positive on the line $\eta = 0$, while the tangential derivative is chosen in order to annihilate the ξ-component of ϕ, as required in **(e)**. Then, we multiply the field by a function σ which is defined first on $\eta = 0$ in order to make **(e)** true, and then in a neighbourhood of $\eta = 0$ by assuming σ constant along the integral curves of the gradient field, so that $\sigma\nabla w$ remains divergence free.

The other sets A_i are simply regions of transition, where the field is taken purely vertical.

Let us prove conditions **(a)** and **(b)**. By Lemma 3.2.1 it follows that ϕ is divergence free in $A_2 \cup A_6$, noting that it is constructed starting from the family of harmonic functions $u_i(\xi,\eta) - tv_i(\xi,\eta)$.

In A_4 condition **(a)** is true since, as remarked above, ϕ is the product of ∇w with the function σ which, by construction, is constant along the integral curves of ∇w.

In the other sets, condition **(a)** is trivially satisfied.

Note that the normal component of ϕ is continuous across each ∂A_i: for the regions A_2, A_6, and for A_4, this continuity is guaranteed by our choice of ω_i and β_i respectively, so that also condition **(b)** is satisfied.

Since

$$\omega_i(\xi, 0) = M^2 - (\partial_\xi u_i(\xi, 0))^2,$$

condition **(c)** is satisfied in $A_1 \cup A_3$ and in $A_5 \cup A_7$ if we require that

$$M > \sup\{|\partial_\xi u_i(\xi, 0)| : (\xi, 0) \in V, \ i = 1, 2\},$$

provided V is small enough.

Arguing in a similar way, if we impose that

$$\mu > \sup\left\{\frac{\lambda^2}{4}(1 - 2\varepsilon M)^2 \left(1 + \frac{4\varepsilon^2}{(1 - 2\varepsilon M)^2}(\partial_\xi u_1(\xi, 0) + \partial_\xi u_2(\xi, 0))^2\right) : (\xi, 0) \in V\right\},$$

condition **(c)** holds in A_4, provided V is sufficiently small.

In the other cases, **(c)** is trivial.

Looking at the definition of ϕ on A_2 and A_6, one can check that condition **(d)** is satisfied.

By direct computations we find that

$$\int_{u_1}^{u_2} \phi^\xi \, dz = 2\varepsilon\partial_\xi u_1 + 2\varepsilon\partial_\xi u_2 + \lambda\left(\beta_2 - \beta_1 + \frac{1}{\lambda}\right)\sigma\partial_\xi w, \tag{3.2.14}$$

$$\int_{u_1}^{u_2} \phi^\eta \, dz = 2\varepsilon\partial_\eta u_1 + 2\varepsilon\partial_\eta u_2 + M\frac{\varepsilon^2}{\varepsilon + M\eta} + M\frac{\varepsilon^2}{\varepsilon - M\eta} + \lambda\left(\beta_2 - \beta_1 + \frac{1}{\lambda}\right)\sigma\partial_\eta w, \tag{3.2.15}$$

for every $(\xi, \eta) \in V$.

By using (3.2.10) and the definition of σ, we obtain

$$\int_{u_1(\xi,0)}^{u_2(\xi,0)} \phi^\xi(\xi, 0, z) \, dz = 0 \tag{3.2.16}$$

and

$$\int_{u_1(\xi,0)}^{u_2(\xi,0)} \phi^\eta(\xi, 0, z) \, dz = 1, \tag{3.2.17}$$

so condition **(e)** is satisfied.

The proof of condition **(f)** will be split in two steps: we first prove that condition **(f)** holds if s and t belong to a suitable neighbourhood of $u_1(\xi,\eta)$ and $u_2(\xi,\eta)$ respectively, whose width is uniform with respect to (ξ,η) in V; then, by a quite simple continuity argument we show that condition **(f)** is true if s or t are outside that neighbourhood.

For $(\xi,\eta)\in V$ and $s,t\in\mathbb{R}$, we set

$$I(\eta,s,t):=\int_s^t \phi^{\xi\eta}(\xi,\eta,z)\,dz$$

and we denote by I^ξ and I^η its components.

STEP 1. For a suitable choice of ε and of the function n (see (3.2.10)) there exists $\delta>0$ such that condition **(f)** holds for $|s-u_1(\xi,\eta)|<\delta$, $|t-u_2(\xi,\eta)|<\delta$, and $(\xi,\eta)\in V$, provided V is small enough.

To estimate the vector whose components are given by (3.2.14) and (3.2.15), we use suitable polar coordinates. If V is small enough, for every $(\xi,\eta)\in V$ there exist $\rho_{\varepsilon,n}(\xi,\eta)>0$ and $-\pi/2<\theta_{\varepsilon,n}(\xi,\eta)<\pi/2$ such that

$$I^\xi(\xi,\eta,u_1(\xi,\eta),u_2(\xi,\eta)) = \rho_{\varepsilon,n}(\xi,\eta)\sin\theta_{\varepsilon,n}(\xi,\eta), \tag{3.2.18}$$

$$I^\eta(\xi,\eta,u_1(\xi,\eta),u_2(\xi,\eta)) = \rho_{\varepsilon,n}(\xi,\eta)\cos\theta_{\varepsilon,n}(\xi,\eta). \tag{3.2.19}$$

In the notation above we have made explicit the dependence on the parameter ε and on the function n which appears in the definition of w (see (3.2.10)).

In order to prove condition **(f)**, we want to compare the behaviour of the functions $\rho_{\varepsilon,n}$ and γ for $|\eta|$ small. We have already proved that $\rho_{\varepsilon,n}(\xi,0)=\gamma(\xi,0)=1$; we start computing the first derivative of γ and of $\rho_{\varepsilon,n}$ with respect to the variable η.

Claim 1. $\partial_\eta(|\nabla_{xy}\xi(\Psi)|^2)(\xi,0)=-2\,\mathrm{curv}\,\Gamma(\xi)$.

PROOF OF THE CLAIM. By (3.2.5) we obtain

$$|\nabla_{xy}\xi(\Psi)|^2=\frac{1}{(\partial_\xi\tilde{x})^2+(\partial_\xi\tilde{y})^2},$$

hence

$$\partial_\eta(|\nabla_{xy}\xi(\Psi)|)^2=-[(\partial_\xi\tilde{x})^2+(\partial_\xi\tilde{y})^2]^{-2}(2\partial_\xi\tilde{x}\,\partial^2_{\xi\eta}\tilde{x}+2\partial_\xi\tilde{y}\,\partial^2_{\xi\eta}\tilde{y}). \tag{3.2.20}$$

Using the fact that $(\partial_\xi\tilde{x})^2+(\partial_\xi\tilde{y})^2$ is equal to 1 at $(\xi,0)$, and the equalities in (3.2.3), we finally get

$$\partial_\eta(|\nabla_{xy}\xi(\Psi)|^2)(\xi,0)=-2(-\partial_\xi\tilde{x}\,\partial^2_{\xi\xi}\tilde{y}+\partial_\xi\tilde{y}\,\partial^2_{\xi\xi}\tilde{x})=-2\,\mathrm{curv}\,\Gamma(\xi),$$

where the last equality follows from (1.2.7): therefore the claim is proved.

Since $\gamma=(|\nabla_{xy}\xi(\Psi)|^2)^{-\frac{1}{2}}$, one has that $\partial_\eta\gamma=-\frac{1}{2}(|\nabla_{xy}\xi(\Psi)|^2)^{-\frac{3}{2}}\partial_\eta(|\nabla_{xy}\xi(\Psi)|^2)$; using the previous claim we can conclude that

$$\partial_\eta(\gamma)(\xi,0)=-\frac{1}{2}\partial_\eta(|\nabla_{xy}\xi(\Psi)|^2)(\xi,0)=\mathrm{curv}\,\Gamma(\xi).$$

Using the equality

$$\rho^2_{\varepsilon,n}(\xi,\eta)=\Big(I^\xi(\xi,\eta,u_1(\xi,\eta),u_2(\xi,\eta))\Big)^2+(I^\eta(\xi,\eta,u_1(\xi,\eta),u_2(\xi,\eta)))^2,$$

we obtain

$$\partial_\eta(\rho_{\varepsilon,n}) = \frac{1}{\rho_{\varepsilon,n}}\partial_\eta\left(I^\xi(\xi,\eta,u_1,u_2)\right) I^\xi(\xi,\eta,u_1,u_2) + \frac{1}{\rho_{\varepsilon,n}}\partial_\eta\left(I^\eta(\xi,\eta,u_1,u_2)\right) I^\eta(\xi,\eta,u_1,u_2).$$

By (3.2.16) it follows that the first addend in the expression above is equal to zero at $(\xi,0)$, while by (3.2.17) it turns out that $I^\eta(\xi,0,u_1,u_2) = \rho_{\varepsilon,n}(\xi,0) = 1$; therefore,

$$\partial_\eta(\rho_{\varepsilon,n})(\xi,0) = \partial_\eta\left(I^\eta(\xi,0,u_1,u_2)\right). \tag{3.2.21}$$

By (3.2.15) it follows that

$$\begin{aligned}\partial_\eta\left(I^\eta(\xi,\eta,u_1,u_2)\right) = {} & 2\varepsilon\partial^2_{\eta\eta}u_1 + 2\varepsilon\partial^2_{\eta\eta}u_2 - \frac{\varepsilon^2}{(\varepsilon+M\eta)^2}M^2 + \frac{\varepsilon^2}{(\varepsilon-M\eta)^2}M^2 + \lambda(\partial_\eta\beta_2 - \partial_\eta\beta_1)\sigma\partial_\eta w + \\ & + \lambda(\beta_2-\beta_1+1/\lambda)\partial_\eta(\sigma\partial_\eta w).\end{aligned} \tag{3.2.22}$$

From (3.2.13) and the Euler condition **iii)**, we have that

$$\begin{aligned}\lambda(\partial_\eta\beta_2(\xi,0) - \partial_\eta\beta_1(\xi,0))\sigma(\xi,0)\partial_\eta w(\xi,0) &= -\omega_2(\xi,0)+\omega_1(\xi,0) \\ &= (\partial_\xi u_2(\xi,0))^2 - (\partial_\xi u_1(\xi,0))^2 \\ &= \operatorname{curv}\Gamma(\xi),\end{aligned} \tag{3.2.23}$$

while

$$\partial_\eta(\sigma\partial_\eta w)(\xi,0) = -\partial_\xi(\sigma\partial_\xi w)(\xi,0) = \partial_\xi(2\varepsilon\partial_\xi u_1(\xi,0) + 2\varepsilon\partial_\xi u_2)(\xi,0),$$

where we have used the fact that $\sigma\nabla w$ is divergence free and the definition of σ and w. Putting this last fact together with (3.2.22), (3.2.23), and the harmonicity of u_i, we finally get

$$\partial_\eta(\rho_{\varepsilon,n})(\xi,0) = \operatorname{curv}\Gamma(\xi) = \partial_\eta(\gamma)(\xi,0). \tag{3.2.24}$$

Claim 2. $\partial^2_{\eta\eta}(|\nabla_{xy}\xi(\Psi)|^2)(\xi,0) = 4\,[\operatorname{curv}\Gamma(\xi)]^2$.

PROOF OF THE CLAIM. By differentiating with respect to η the expression in (3.2.20) and by (3.2.3), we obtain

$$\begin{aligned}\partial^2_{\eta\eta}(|\nabla_{xy}\xi(\Psi)|^2) &= -2[(\partial_\xi\tilde{x})^2 + (\partial_\xi\tilde{y})^2]^{-2}[(\partial^2_{\xi\eta}\tilde{x})^2 + \partial_\xi\tilde{x}\,\partial^3_{\xi\eta\eta}\tilde{x} + (\partial^2_{\xi\eta}\tilde{y})^2 + \partial_\xi\tilde{y}\,\partial^3_{\xi\eta\eta}\tilde{y}] + \\ &\quad +8[(\partial_\xi\tilde{x})^2 + (\partial_\xi\tilde{y})^2]^{-3}(\partial_\xi\tilde{x}\,\partial^2_{\xi\eta}\tilde{x} + \partial_\xi\tilde{y}\,\partial^2_{\xi\eta}\tilde{y})^2 \\ &= -2[(\partial_\xi\tilde{x})^2 + (\partial_\xi\tilde{y})^2]^{-2}[(\partial^2_{\xi\xi}\tilde{y})^2 + (\partial^2_{\xi\xi}\tilde{x})^2 - \partial_\xi\tilde{x}\,\partial^3_{\xi\xi\xi}\tilde{x} - \partial_\xi\tilde{y}\,\partial^3_{\xi\xi\xi}\tilde{y}] + \\ &\quad +8[(\partial_\xi\tilde{x})^2 + (\partial_\xi\tilde{y})^2]^{-3}(-\partial_\xi\tilde{x}\,\partial^2_{\xi\xi}\tilde{y} + \partial_\xi\tilde{y}\,\partial^2_{\xi\xi}\tilde{x})^2.\end{aligned}$$

Note that

$$-\partial_\xi\tilde{x}\,\partial^3_{\xi\xi\xi}\tilde{x} - \partial_\xi\tilde{y}\,\partial^3_{\xi\xi\xi}\tilde{y} = (\partial^2_{\xi\xi}\tilde{y})^2 + (\partial^2_{\xi\xi}\tilde{x})^2 - \frac{1}{2}\partial^2_{\xi\xi}((\partial_\xi\tilde{x})^2 + (\partial_\xi\tilde{y})^2).$$

Using (1.2.7), (1.2.8), and the fact that $(\partial_\xi\tilde{x})^2 + (\partial_\xi\tilde{y})^2$ is equal to 1 at $(\xi,0)$, we obtain the claim.

By using Claims 1 and 2, we can conclude that

$$\begin{aligned}\partial^2_{\eta\eta}(\gamma)(\xi,0) &= \left[\frac{3}{4}(|\nabla_{xy}\xi(\Psi)|^2)^{-\frac{5}{2}}[\partial_\eta(|\nabla_{xy}\xi(\Psi)|^2)]^2 - \frac{1}{2}(|\nabla_{xy}\xi(\Psi)|^2)^{-\frac{3}{2}}\partial^2_{\eta\eta}(|\nabla_{xy}\xi(\Psi)|^2)\right]\Bigg|_{(\xi,0)} \\ &= [\operatorname{curv}\Gamma(\xi)]^2.\end{aligned} \tag{3.2.25}$$

The second derivative of $\rho_{\varepsilon,n}$ with respect to η is given by

$$\begin{aligned}\partial^2_{\eta\eta}(\rho_{\varepsilon,n}) &= \frac{1}{\rho_{\varepsilon,n}}\Big\{\Big[\partial_\eta\left(I^\xi(\xi,\eta,u_1,u_2)\right)\Big]^2 + \partial^2_{\eta\eta}\left(I^\xi(\xi,\eta,u_1,u_2)\right)I^\xi(\xi,\eta,u_1,u_2)+\\ &\quad +[\partial_\eta\left(I^\eta(\xi,\eta,u_1,u_2)\right)]^2 + \partial^2_{\eta\eta}\left(I^\eta(\xi,\eta,u_1,u_2)\right)I^\eta(\xi,\eta,u_1,u_2)\Big\} -\\ &\quad -\frac{1}{\rho_{\varepsilon,n}}[\partial_\eta(\rho_{\varepsilon,n})]^2.\end{aligned}$$

By the equalities (3.2.16), (3.2.17), and (3.2.21), the expression above computed at $(\xi,0)$ reduces to

$$\partial^2_{\eta\eta}(\rho_{\varepsilon,n})(\xi,0) = \left[\partial_\eta\left(I^\xi(\xi,\eta,u_1,u_2)\right)\Big|_{(\xi,0)}\right]^2 + \partial^2_{\eta\eta}\left(I^\eta(\xi,\eta,u_1,u_2)\right)|_{(\xi,0)}\,. \tag{3.2.26}$$

By differentiating (3.2.14) and (3.2.22) with respect to η, we obtain that

$$\partial_\eta\left(I^\xi(\xi,\eta,u_1,u_2)\right)(\xi,0) = [\lambda(\partial_\eta\beta_2 - \partial_\eta\beta_1)\sigma\partial_\xi w + \partial_\eta\sigma\,\partial_\xi w + \sigma\partial^2_{\xi\eta}w]|_{(\xi,0)}, \tag{3.2.27}$$

and

$$\begin{aligned}\partial^2_{\eta\eta}\left(I^\eta(\xi,\eta,u_1,u_2)\right)(\xi,0) &= \frac{4}{\varepsilon}M^3 + \lambda[\partial^2_{\eta\eta}\beta_2(\xi,0) - \partial^2_{\eta\eta}\beta_1(\xi,0)]\sigma(\xi,0)\partial_\eta w(\xi,0) +\\ &\quad +2\lambda[\partial_\eta\beta_2(\xi,0) - \partial_\eta\beta_1(\xi,0)]\partial_\eta(\sigma\partial_\eta w)(\xi,0) + \partial^2_{\eta\eta}\sigma(\xi,0)\partial_\eta w(\xi,0) +\\ &\quad +2\partial_\eta\sigma(\xi,0)\partial^2_{\eta\eta}w(\xi,0) + \sigma(\xi,0)\partial^3_{\eta\eta\eta}w(\xi,0),\end{aligned} \tag{3.2.28}$$

while, by using the equation (3.2.13),

$$\begin{aligned}&[\lambda(\partial^2_{\eta\eta}\beta_2 - \partial^2_{\eta\eta}\beta_1)\sigma\partial_\eta w]|_{(\xi,0)} =\\ &= [\partial_\eta\omega_1 - \partial_\eta\omega_2 - \lambda\partial_\eta(\partial_\xi\beta_2 - \partial_\xi\beta_1)\sigma\partial_\xi w - \lambda\partial_\eta(\sigma\partial_\eta w)(\partial_\eta\beta_2 - \partial_\eta\beta_1)]|_{(\xi,0)}\\ &= [-\frac{4}{\varepsilon}M^3 - \lambda\partial_\xi(\partial_\eta\beta_2 - \partial_\eta\beta_1)\sigma\partial_\xi w + \lambda\partial_\xi(\sigma\partial_\xi w)(\partial_\eta\beta_2 - \partial_\eta\beta_1)]|_{(\xi,0)}.\end{aligned}$$

Since by (3.2.23) and by the definition of σ we have that

$$\lambda[\partial_\eta\beta_2(\xi,0) - \partial_\eta\beta_1(\xi,0)] = \frac{\operatorname{curv}\Gamma(\xi)}{1-2\varepsilon M},$$

and moreover,

$$\sigma(\xi,0)\partial_\xi w(\xi,0) = -2\varepsilon(\partial_\xi u_1(\xi,0) + \partial_\xi u_2(\xi,0)),$$

we obtain that

$$\begin{aligned}&[\lambda(\partial^2_{\eta\eta}\beta_2 - \partial^2_{\eta\eta}\beta_1)\sigma\partial_\eta w + 2\lambda(\partial_\eta\beta_2 - \partial_\eta\beta_1)\partial_\eta(\sigma\partial_\eta w)]|_{(\xi,0)} =\\ &\qquad = -\frac{4}{\varepsilon}M^3 + \frac{2\varepsilon}{1-2\varepsilon M}\partial_\xi((\partial_\xi u_1 - \partial_\xi u_2)\operatorname{curv}\Gamma)(\xi,0).\end{aligned}$$

By using the definition of σ, we can write

$$\begin{aligned}\partial_\eta\sigma &= -(1-2\varepsilon M)\frac{n'(\xi)}{n^2(\xi)}\partial_\eta q,\\ \partial^2_{\eta\eta}\sigma &= -(1-2\varepsilon M)\left[-2\frac{(n'(\xi))^2}{n^3(\xi)}(\partial_\eta q)^2 + \frac{n''(\xi)}{n^2(\xi)}(\partial_\eta q)^2 + \frac{n'(\xi)}{n^2(\xi)}\partial^2_\eta q\right].\end{aligned}$$

In order to compute the derivatives of q, we differentiate the equality (3.2.12) with respect to η:

$$\begin{aligned}
\partial_\eta q(\xi,0) &= -\partial_\eta p(\xi,0) = \frac{2\varepsilon}{1-2\varepsilon M}(\partial_\xi u_1(\xi,0)+\partial_\xi u_2(\xi,0)),\\
\partial^2_{\eta\eta} q(\xi,0) &= -2\partial^2_{\xi\eta}p(\xi,0)\partial_\eta q(\xi,0) - \partial^2_{\eta\eta}p(\xi,0)\\
&= \left[-\frac{(\partial_\xi w)^2}{(\partial_\eta w)^3}\partial^2_{\xi\eta}w - \frac{1}{\partial_\eta w}\partial^2_{\xi\eta}w\right](\xi,0).
\end{aligned}$$

By the definition of w, we obtain

$$\partial^2_\eta q(\xi,0) = -\frac{n'(\xi)}{n(\xi)} - \frac{n'(\xi)}{n(\xi)}\frac{4\varepsilon^2}{(1-2\varepsilon M)^2}(\partial_\xi u_1(\xi,0)+\partial_\xi u_2(\xi,0))^2.$$

Finally, we have

$$\begin{aligned}
\partial^2_{\eta\eta} w(\xi,0) &= -\partial^2_{\xi\xi}w(\xi,0) = \frac{2\varepsilon}{1-2\varepsilon M}[n'(\partial_\xi u_1+\partial_\xi u_2)+n(\partial^2_{\xi\xi}u_1+\partial^2_{\xi\xi}u_2)]|_{(\xi,0)},\\
\partial^3_{\eta\eta\eta} w(\xi,0) &= -\partial^2_{\xi\xi}\partial_\eta w(\xi,0) = -n''(\xi).
\end{aligned}$$

By substituting all information above in (3.2.27) and in (3.2.28), and by using (3.2.26), we finally obtain that

$$\begin{aligned}
\partial^2_{\eta\eta}(\rho_{\varepsilon,n})(\xi,0) &= -a_\varepsilon(\xi)\frac{n''(\xi)}{n(\xi)} + h_\varepsilon\left(\xi,\frac{n'(\xi)}{n(\xi)}\right)\\
&= -a_\varepsilon(\xi)\left(\frac{n'(\xi)}{n(\xi)}\right)' + h_\varepsilon\left(\xi,\frac{n'(\xi)}{n(\xi)}\right) - a_\varepsilon(\xi)\left(\frac{n'(\xi)}{n(\xi)}\right)^2,
\end{aligned}\tag{3.2.29}$$

where

$$\begin{aligned}
a_\varepsilon(\xi) &\to 1 && \text{uniformly in } [0,l(\Gamma)],\\
h_\varepsilon(\xi,\tau) &\to 2\tau^2 && \text{uniformly on the compact sets of } [0,l(\Gamma)]\times\mathbb{R},
\end{aligned}\tag{3.2.30}$$

as $\varepsilon \to 0$.

Claim 3. *There exists $\overline{\varepsilon} > 0$ such that for every $\varepsilon \in (0,\overline{\varepsilon})$, we can find an analytic function $n : [0,l(\Gamma)] \to (0,+\infty)$ satisfying*

$$\partial^2_{\eta\eta}(\rho_{\varepsilon,n}-\gamma)(\xi,0) = -\frac{\pi^2}{16\,l^2(\Gamma)} \qquad \textit{and} \qquad \left|\frac{n'(\xi)}{n(\xi)}\right| \le N \quad \forall\xi\in[0,l(\Gamma)],\tag{3.2.31}$$

where $N := 1+\max\left\{\frac{\pi}{4\,l(\Gamma)},k(\Gamma)\right\}$ *and* $k(\Gamma) = \|\operatorname{curv}\Gamma\|_\infty$.

Proof of the claim. Set $\tau := n'/n$; in order to prove the claim, by (3.2.29) and (3.2.25) we study the Cauchy problem

$$\begin{cases}
-a_\varepsilon(\xi)\tau' + h_\varepsilon(\xi,\tau) - \tau^2 - [\operatorname{curv}\Gamma(\xi)]^2 = -\dfrac{\pi^2}{16\,l^2(\Gamma)},\\
\tau(0) = 0,
\end{cases}\tag{3.2.32}$$

and we investigate for which values of ε it admits a solution defined in the whole interval $[0, l(\Gamma)]$, with L^∞-norm less than N. As $\varepsilon \to 0$, by (3.2.30) we obtain the limit problem

$$\begin{cases} -\tau' + \tau^2 - (\operatorname{curv}\Gamma)^2 = -\dfrac{\pi^2}{16\, l^2(\Gamma)}, \\ \tau(0) = 0. \end{cases} \tag{3.2.33}$$

By comparing with the solutions τ_1 and τ_2 of the Cauchy problems

$$\begin{cases} -\tau_1' + \tau_1^2 = -\dfrac{\pi^2}{16\, l^2(\Gamma)}, \\ \tau_1(0) = 0, \end{cases} \qquad \begin{cases} -\tau_2' + \tau_2^2 - k^2(\Gamma) = -\dfrac{\pi^2}{16\, l^2(\Gamma)}, \\ \tau_2(0) = 0, \end{cases} \tag{3.2.34}$$

one easily sees that the solution of (3.2.33) is defined in $[0, l(\Gamma)]$, with L^∞-norm less than the maximum between $\|\tau_1\|_\infty$ and $\|\tau_2\|_\infty$, which is, by explicit computation, less than $\max\{\pi/(4l(\Gamma)), k(\Gamma)\}$. By the continuous dependence on the coefficients (see [34]), we can find $\overline{\varepsilon}$ such that, for every $\varepsilon \in (0, \overline{\varepsilon})$, the solution of (3.2.32) is defined in $[0, l(\Gamma)]$ with L^∞-norm less than N.

For every $\varepsilon \in (0, \overline{\varepsilon})$, we set

$$n_\varepsilon(\xi) := e^{\int_0^\xi \tau_\varepsilon(s)\, ds}, \tag{3.2.35}$$

where τ_ε is the solution of (3.2.32).

From now on we will simply write ρ_ε and θ_ε instead of $\rho_{\varepsilon,n_\varepsilon}$ and $\theta_{\varepsilon,n_\varepsilon}$.

We now want to estimate the angle $\theta_\varepsilon(\xi, \eta)$ by a quantity which is independent of ε. Since by (3.2.14) and (3.2.15)

$$\tan\theta_\varepsilon = \frac{2\varepsilon\partial_\xi u_1 + 2\varepsilon\partial_\xi u_2 + \lambda\left(\beta_2 - \beta_1 + \frac{1}{\lambda}\right)\sigma\partial_\xi w}{2\varepsilon\partial_\eta u_1 + 2\varepsilon\partial_\eta u_2 + M\varepsilon^2(\varepsilon + M\eta)^{-1} + M\varepsilon^2(\varepsilon - M\eta)^{-1} + \lambda\left(\beta_2 - \beta_1 + \frac{2}{\lambda}\right)\sigma\partial_\eta w},$$

we have

$$\partial_\eta\theta_\varepsilon(\xi, 0) = -\frac{2\varepsilon}{1 - 2\varepsilon M}(\partial_\xi u_1 + \partial_\xi u_2)\left(\operatorname{curv}\Gamma - 2\varepsilon(\partial_\xi u_1 + \partial_\xi u_2)\frac{n_\varepsilon'(\xi)}{n_\varepsilon(\xi)}\right) + (1 - 2\varepsilon M)\frac{n_\varepsilon'(\xi)}{n_\varepsilon(\xi)},$$

and so, by Claim 3, if ε is sufficiently small,

$$|\partial_\eta\theta_\varepsilon(\xi, 0)| < N \qquad \forall \xi \in [0, l(\Gamma)]. \tag{3.2.36}$$

Let $\tilde{\theta}(\eta)$ be an arbitrary continuous function with

$$\tilde{\theta}(0) = 0 \qquad \text{and} \qquad \tilde{\theta}'(0) = N; \tag{3.2.37}$$

by (3.2.36), it follows that

$$|\theta_\varepsilon(\xi, \eta)| < \tilde{\theta}(\eta)\operatorname{sign}\eta \tag{3.2.38}$$

for every $(\xi, \eta) \in V$, provided V is sufficiently small.

Given $h > 0$, we consider the vectors

$$\begin{aligned} b_1^h(\xi, \eta, s) &:= \left(0, -2(s - u_1(\xi, \eta))\partial_\eta u_1(\xi, \eta) - h(s - u_1(\xi, \eta))^2\right), \\ b_2^h(\xi, \eta, t) &:= \left(0, 2(t - u_2(\xi, \eta))\partial_\eta u_2(\xi, \eta) - h(t - u_2(\xi, \eta))^2\right) \end{aligned}$$

for $(\xi,\eta)\in V$ and $s,t\in\mathbb{R}$. We denote by $B(r)$ the open ball centred at $(0,-r)$ with radius r.

Let us define $r^h_\varepsilon(\xi,\eta,s,t)$ as the maximum radius r such that the set

$$(\rho_\varepsilon(\xi,\eta)\sin\tilde\theta(\eta),\rho_\varepsilon(\xi,\eta)\cos\tilde\theta(\eta))+b^h_1(\xi,\eta,s)+b^h_2(\xi,\eta,t)+B(r)$$

is contained in the ball centred at $(0,0)$ with radius $\gamma(\xi,\eta)$.

Claim 4. *Setting*

$$d:=\frac{1}{1+16\,l^2(\Gamma)N^2/\pi^2},\tag{3.2.39}$$

where N is the constant introduced in the previous claim, there exists $h>0$ such that for every $\varepsilon\in(0,\overline{\varepsilon})$ (see Claim 3), there exists $\delta\in(0,\varepsilon)$ so that, if V is small enough,

$$\inf\Big\{2\,r^h_\varepsilon(\xi,\eta,s,t):(\xi,\eta)\in V,\ |s-u_1(\xi,\eta)|\le\delta,\ |t-u_2(\xi,\eta)|\le\delta\Big\}>\frac{d}{2}.\tag{3.2.40}$$

Proof of the claim. Let $\overline{\rho}^h_\varepsilon(\xi,\eta,s,t)>0$ and $-\pi/2<\overline{\theta}^h_\varepsilon(\xi,\eta,s,t)<\pi/2$ be such that

$$\Big(\rho_\varepsilon(\xi,\eta)\sin\tilde\theta(\eta),\rho_\varepsilon(\xi,\eta)\cos\tilde\theta(\eta)\Big)+b^h_1(\xi,\eta,s)+b^h_2(\xi,\eta,t)=$$
$$=\Big(\overline{\rho}^h_\varepsilon(\xi,\eta,s,t)\sin\overline{\theta}^h_\varepsilon(\xi,\eta,s,t),\overline{\rho}^h_\varepsilon(\xi,\eta,s,t)\cos\overline{\theta}^h_\varepsilon(\xi,\eta,s,t)\Big).\tag{3.2.41}$$

To prove Claim 4, it is enough to show that, for every $\varepsilon\in(0,\overline{\varepsilon})$, there exists $\delta\in(0,\varepsilon)$ with the property that

$$\left(1-\frac{d}{2}\cos\overline{\theta}^h_\varepsilon(\xi,\eta,s,t)\right)\overline{\rho}^h_\varepsilon(\xi,\eta,s,t)<\left(1-\frac{d}{2}\right)\gamma(\xi,\eta)\tag{3.2.42}$$

for $|s-u_1(\xi,\eta)|\le\delta$, $|t-u_2(\xi,\eta)|\le\delta$, and $(\xi,\eta)\in V$ with $\eta\ne0$, provided V is sufficiently small. Indeed, if (3.2.42) holds, it follows in particular that $\overline{\rho}^h_\varepsilon(\xi,\eta,s,t)<\gamma(\xi,\eta)$, and this inequality with some easy geometric computations implies that

$$2\,r^h_\varepsilon(\xi,\eta,s,t)=\frac{\gamma^2(\xi,\eta)-(\overline{\rho}^h_\varepsilon(\xi,\eta,s,t))^2}{\gamma-\overline{\rho}^h_\varepsilon(\xi,\eta,s,t)\cos\overline{\theta}^h_\varepsilon(\xi,\eta,s,t)};$$

at this point, it is easy to see that, if V is small enough, inequality (3.2.42) implies that $2\,r^h_\varepsilon(\xi,\eta,s,t)>d/2$, that is Claim 4. So let us prove (3.2.42).

We set

$$f^{d,h}(\xi,\eta,s,t):=\left(1-\frac{d}{2}\cos\overline{\theta}^h_\varepsilon(\xi,\eta,s,t)\right)\overline{\rho}^h_\varepsilon(\xi,\eta,s,t)-\left(1-\frac{d}{2}\right)\gamma(\xi,\eta)$$

and we note that $f^{d,h}(\xi,0,u_1(\xi,0),u_2(\xi,0))=0$. We will show that

1. $\nabla_{\eta st}f^{d,h}(\xi,0,u_1(\xi,0),u_2(\xi,0))=0$ if $(\xi,0)\in V$,
2. $\nabla^2_{\eta st}f^{d,h}(\xi,0,u_1(\xi,0),u_2(\xi,0))$ is negative definite if $(\xi,0)\in V$,

where $\nabla_{\eta st} f^{d,h}$ and $\nabla^2_{\eta st} f^{d,h}$ denote respectively the gradient and the hessian matrix of $f^{d,h}$ with respect to the variables (η, s, t). Equality 1 follows by direct computations and by (3.2.24). Using (3.2.41), the equality in (3.2.31), and (3.2.37), we obtain

$$\partial^2_{\eta\eta} f^{d,h}(\xi, 0, u_1(\xi,0), u_2(\xi,0)) = -\frac{\pi^2}{16\, l^2(\Gamma)}\left(1-\frac{d}{2}\right) + \frac{d}{2}N^2;$$

then by the definition of d,

$$\partial^2_{\eta\eta} f^{d,h}(\xi, 0, u_1(\xi,0), u_2(\xi,0)) = -\frac{\pi^2}{32\, l^2(\Gamma)} < 0. \tag{3.2.43}$$

Moreover we easily obtain that

$$\partial^2_{tt} f^{d,h}(\xi, 0, u_1(\xi,0), u_2(\xi,0)) = \partial^2_{ss} f^{d,h}(\xi, 0, u_1(\xi,0), u_2(\xi,0)) = -2h\left(1-\frac{d}{2}\right),$$

$$\partial^2_{s\eta} f^{d,h}(\xi, 0, u_1(\xi,0), u_2(\xi,0)) = -2\left(1-\frac{d}{2}\right)\partial^2_{\eta\eta} u_1(\xi,0),$$

$$\partial^2_{t\eta} f^{d,h}(\xi, 0, u_1(\xi,0), u_2(\xi,0)) = 2\left(1-\frac{d}{2}\right)\partial^2_{\eta\eta} u_2(\xi,0),$$

$$\partial^2_{ts} f^{d,h}(\xi, 0, u_1(\xi,0), u_2(\xi,0)) = 0.$$

By the expressions, it follows that

$$\det\begin{pmatrix} \partial^2_{\eta\eta} f^{d,h} & \partial^2_{s\eta} f^{d,h} \\ \partial^2_{s\eta} f^{d,h} & \partial^2_{ss} f^{d,h} \end{pmatrix}(\xi, 0, u_1(\xi,0), u_2(\xi,0)) = h(2-d)\frac{\pi^2}{32\, l^2(\Gamma)} - (2-d)^2[\partial^2_{\eta\eta} u_1(\xi,0)]^2,$$

and that the determinant of the hessian matrix of $f^{d,h}$ at $(\xi, 0, u_1(\xi,0), u_2(\xi,0))$ is given by

$$\det \nabla^2_{\eta st} f^{d,h}(\xi, 0, u_1(\xi,0), u_2(\xi,0)) = -h^2(2-d)^2\frac{\pi^2}{32\, l^2(\Gamma)} + h(2-d)^3[(\partial^2_{\eta\eta} u_1(\xi,0))^2 + (\partial^2_{\eta\eta} u_2(\xi,0))^2].$$

By the definition of d, if h satisfies

$$h > \frac{32}{\pi^2}(2-d)l^2(\Gamma)\sum_{i=1}^{2}\|\partial^2_{\eta\eta} u_i\|^2_{L^\infty(\Gamma)}, \tag{3.2.44}$$

then for every $(\xi, 0) \in V$ we have

$$\det\begin{pmatrix} \partial^2_{\eta\eta} f^{d,h} & \partial^2_{s\eta} f^{d,h} \\ \partial^2_{s\eta} f^{d,h} & \partial^2_{ss} f^{d,h} \end{pmatrix}(\xi, 0, u_1(\xi,0), u_2(\xi,0)) > 0, \tag{3.2.45}$$

and

$$\det \nabla^2_{\eta st} f^{d,h}(\xi, 0, u_1(\xi,0), u_2(\xi,0)) < 0. \tag{3.2.46}$$

By (3.2.43), (3.2.45), and (3.2.46), we can conclude that the hessian matrix of $f^{d,h}$ is negative definite at $(\xi, 0, u_1(\xi,0), u_2(\xi,0))$: both (3.2.42) and Claim 4 are proved.

Claim 5. *For every $r > 0$ and $h > 0$, there exists $\tilde{\varepsilon} > 0$ with the property that, if $\varepsilon \in (0, \tilde{\varepsilon})$, one can find $\delta \in (0, \varepsilon)$ such that*

$$I(\xi, \eta, u_2(\xi, \eta), t) \in B(r) + b_2^h(\xi, \eta, t),$$
$$I(\xi, \eta, s, u_1(\xi, \eta)) \in B(r) + b_1^h(\xi, \eta, s),$$

for every $(\xi, \eta) \in V$ and $|t - u_2(\xi, \eta)| \leq \delta$, $|s - u_1(\xi, \eta)| \leq \delta$, provided V is small enough.

PROOF OF THE CLAIM. By the definition of ϕ in A_6, we obtain that

$$I^{\xi}(\xi, \eta, u_2(\xi, \eta), t) = 2(t - u_2(\xi, \eta))\partial_{\xi} u_2(\xi, \eta),$$

$$I^{\eta}(\xi, \eta, u_2(\xi, \eta), t) = 2(t - u_2(\xi, \eta))\partial_{\eta} u_2(\xi, \eta) - M(\varepsilon - M\eta)^{-1}(t - u_2(\xi, \eta))^2.$$

To get the claim, we need to prove that

$$(2(t - u_2)\partial_{\xi} u_2)^2 + \left(-M(\varepsilon - M\eta)^{-1}(t - u_2)^2 + h(t - u_2)^2 + r\right)^2 < r^2,$$

which is equivalent to

$$(2(t - u_2)\partial_{\xi} u_2)^2 + \left(-M(\varepsilon - M\eta)^{-1} + h\right)^2 (t - u_2)^4 + 2r\left(-M(\varepsilon - M\eta)^{-1} + h\right)(t - u_2)^2 < 0.$$

The conclusion follows by remarking that, if V is small enough, the left-hand side is less than

$$\left(4(\partial_{\xi} u_2)^2 + 2hr - \frac{2Mr}{3\varepsilon}\right)\delta^2 + o(\delta^2),$$

which is negative if ε is sufficiently small. The proof for u_1 is completely analogous.

Let us conclude the proof of the step. By Claim 4, we can find $h > 0$ such that (3.2.40) is satisfied for $\varepsilon \in (0, \bar{\varepsilon})$. If we choose r such that $2r < d/4$, by Claim 5 there exists $\tilde{\varepsilon} > 0$ such that for every $\varepsilon \in (0, \tilde{\varepsilon})$ there is $\delta \in (0, \varepsilon)$ so that

$$I(\eta, s, u_1(\xi, \eta)) + I(\eta, u_2(\xi, \eta), t) \in B(2r) + b_1^h(\xi, \eta, s) + b_2^h(\xi, \eta, t) \tag{3.2.47}$$

for every $|s - u_1(\xi, \eta)| < \delta$, $|t - u_2(\xi, \eta)| < \delta$, and $(\xi, \eta) \in V$. If we take $\varepsilon \leq \min\{\tilde{\varepsilon}, \bar{\varepsilon}\}$, then by Claim 4 we have that the set

$$B(2r) + (\rho_{\varepsilon}(\xi, \eta)\sin\tilde{\theta}(\eta), \rho_{\varepsilon}(\xi, \eta)\cos\tilde{\theta}(\eta)) + b_1^h(\xi, \eta, s) + b_2^h(\xi, \eta, t)$$

is contained in the ball centred at $(0, 0)$ with radius $\gamma(\xi, \eta)$. Some easy geometric considerations show that the relation between θ_{ε} and $\tilde{\theta}$ (see (3.2.38)) implies that also the set

$$B(2r) + (\rho_{\varepsilon}(\xi, \eta)\sin\theta_{\varepsilon}(\eta), \rho_{\varepsilon}(\xi, \eta)\cos\theta_{\varepsilon}(\eta)) + b_1^h(\xi, \eta, s) + b_2^h(\xi, \eta, t) \tag{3.2.48}$$

is contained in the ball centred at $(0, 0)$ with radius $\gamma(\xi, \eta)$, if the condition

$$|b_1^h(\xi, \eta, s) + b_2^h(\xi, \eta, t)| < 2r$$

holds (to make this true, take δ and V smaller if needed). Since

$$I(\eta, s, t) = I(\eta, s, u_1(\xi, \eta)) + I(\eta, u_1(\xi, \eta), u_2(\xi, \eta)) + I(\eta, u_2(\xi, \eta), t),$$

by (3.2.47), (3.2.18), and (3.2.19), it follows that $I(\eta, s, t)$ belongs to the set (3.2.48), and then to the ball centred at $(0,0)$ with radius $\gamma(\xi,\eta)$ for every $|s-u_1(\xi,\eta)|<\delta$, $|t-u_2(\xi,\eta)|<\delta$, and $(\xi,\eta)\in V$. This concludes the proof of Step 1.

STEP 2. If ε is sufficiently small and $\delta\in(0,\varepsilon)$, condition **(f)** holds for $|s-u_1(\xi,\eta)|\geq\delta$ or $|t-u_2(\xi,\eta)|\geq\delta$, and $(\xi,\eta)\in V$, provided V is small enough.

Let us fix $\delta\in(0,\varepsilon)$ and set

$$m_1(\xi,\eta) := \max\{|I(\eta,s,t)| : u_1(\xi,\eta)-\varepsilon\leq s\leq t\leq u_2(\xi,\eta)+\varepsilon,\ |t-u_2(\xi,\eta)|\geq\delta\}.$$

It is easy to see that the function m_1 is continuous. Let us prove that $m_1(\xi,0)<\gamma(\xi,0)=1$.

Fixed $(\xi,0)\in V$, $u_1(\xi,0)-\varepsilon\leq s\leq t\leq u_2(\xi,0)+\varepsilon$, with $|t-u_2(\xi,0)|\geq\delta$, we can write

$$I(0,s,t)=I(0,s,u_1(\xi,0))+I(0,u_1(\xi,0),u_2(\xi,0))+I(0,u_2(\xi,0),t). \tag{3.2.49}$$

Claim 6. *For every $r>0$ there exists $\varepsilon>0$ such that*

$$I(0,u_2(\xi,0),t)\in B(r), \qquad I(0,s,u_1(\xi,0))\in B(r)$$

for $0<|s-u_1(\xi,0)|\leq\varepsilon$, $0<|t-u_2(\xi,0)|\leq\varepsilon$, and $(\xi,0)\in V$.

PROOF OF THE CLAIM. See the similar proof of Claim 5 above.

By (3.2.49), (3.2.16), (3.2.17), and Claim 6, it follows that

$$I(0,s,t)\in(0,1)+\overline{B(r)}+B(r)=(0,1)+B(2r) \tag{3.2.50}$$

for $0<|s-u_1(\xi,0)|\leq\varepsilon$, $\delta\leq|t-u_2(\xi,0)|\leq\varepsilon$. If $r<1/4$, the set $(0,1)+B(2r)$ is contained in the open ball centred at $(0,0)$ with radius 1.

It remains to study the case $|s-u_1|\geq\varepsilon$ and the case $|t-u_2|\geq\varepsilon$. Let us consider the latter; the former would be completely analogous. We can write

$$\begin{aligned} I(0,s,u_1(\xi,0)) &= I(0,s\wedge(u_1(\xi,0)+\varepsilon),u_1(\xi,0))+I(0,s\vee(u_1(\xi,0)+\varepsilon),u_1(\xi,0)+\varepsilon),\\ I(0,u_2(\xi,0),t) &= I(0,u_2(\xi,0),u_2(\xi,0)-\varepsilon)+I(0,u_2(\xi,0)-\varepsilon,t). \end{aligned}$$

Therefore, by (3.2.49)

$$\begin{aligned} I(0,s,t) &= I(0,u_1(\xi,0),u_2(\xi,0))+I(0,s\wedge(u_1(\xi,0)+\varepsilon),u_1(\xi,0))+\\ &\quad +I(0,u_2(\xi,0),u_2(\xi,0)-\varepsilon)+I(0,s\vee(u_1(\xi,0)+\varepsilon),t)-\\ &\quad -I(0,u_1(\xi,0)+\varepsilon,u_2(\xi,0)-\varepsilon). \end{aligned} \tag{3.2.51}$$

If $-2\varepsilon(\partial_\xi u_1(\xi,0)+\partial_\xi u_2(\xi,0))\geq 0$, we define

$$C := [0,-2\varepsilon(\partial_\xi u_1(\xi,0)+\partial_\xi u_2(\xi,0))]\times[0,1-2\varepsilon M];$$

if $-2\varepsilon(\partial_\xi u_1(\xi,0)+\partial_\xi u_2(\xi,0))<0$, we replace $[0,-2\varepsilon(\partial_\xi u_1(\xi,0)+\partial_\xi u_2(\xi,0))]$ by $[-2\varepsilon(\partial_\xi u_1(\xi,0)+\partial_\xi u_2(\xi,0)),0]$. From the definition of ϕ in $A_3\cup A_4\cup A_5$, it follows that

$$I(0,u_1(\xi,0)+\varepsilon,u_2(\xi,0)-\varepsilon)=(-2\varepsilon(\partial_\xi u_1(\xi,0)+\partial_\xi u_2(\xi,0)),1-2\varepsilon M) \tag{3.2.52}$$

and

$$I(0,s,t) \in C \tag{3.2.53}$$

for $u_1(\xi,0)+\varepsilon \le s \le t \le u_2(\xi,0)-\varepsilon$. Let $D := C - (-2\varepsilon(\partial_\xi u_1(\xi,0)+\partial_\xi u_2(\xi,0)), 1-2\varepsilon M)$. Since $I^\eta(\xi,0,u_2(\xi,0),u_2(\xi,0)-\varepsilon) = -M\varepsilon$, from (3.2.51), (3.2.16), (3.2.17), Claim 6, (3.2.52), and (3.2.53), we obtain

$$\begin{aligned} I(0,s,t) &\in [(0,1)+\overline{B(r)}+B(r)] \cap \{(x,y)\in\mathbb{R}^2 : y<1-\varepsilon M\} + D \\ &= [(0,1)+B(2r)] \cap \{(x,y)\in\mathbb{R}^2 : y<1-\varepsilon M\} + D. \end{aligned}$$

If $r<1/4$ and if ε is sufficiently small, the set $[(0,1)+B(2r)] \cap \{(x,y)\in\mathbb{R}^2 : y<1-\varepsilon M\} + D$ is contained in the open ball centred at $(0,0)$ with radius 1 and this means that $m_1(\xi,0) < \gamma(\xi,0)$.

Analogously we define

$$m_2(\xi,\eta) := \max\{|I(\eta,s,t)| : u_1(\xi,\eta)-\varepsilon \le s \le t \le u_2(\xi,\eta)+\varepsilon,\ |s-u_1(\xi,\eta)| \ge \delta\}.$$

Arguing as in the case of m_1, we can prove that m_2 is continuous and $m_2(\xi,0) < \gamma(\xi,0)$. By continuity, if V is small enough, $m_1(\xi,\eta) < \gamma(\xi,\eta)$ and $m_2(\xi,\eta) < \gamma(\xi,\eta)$, for every $(\xi,\eta)\in V$: Step 2 is proved.

By Step 1 and Step 2, we conclude that, choosing ε sufficiently small and $n = n_\varepsilon$ (see (3.2.35)), condition (f) is true for $u_1(\xi,\eta)-\varepsilon \le s,t \le u_2(\xi,\eta)+\varepsilon$ and in fact for every $s,t\in\mathbb{R}$, from the definition of ϕ in A_1 and A_7. □

3.3 The graph-minimality

We start this section with a negative result: if the domain Ω is too large, the Euler conditions do not guarantee the graph-minimality introduced in Definition 3.1.2, as the following counterexample (suggested by Gianni Dal Maso) shows.

Proposition 3.3.1 *Let R be the rectangle $(1,1+4l)\times(-l,l)$ and let*

$$u(x,y) := \begin{cases} x & \text{if } y \ge 0, \\ -x & \text{if } y < 0. \end{cases}$$

Then, u satisfies the Euler conditions for the Mumford-Shah functional in R, but it is not a local graph-minimizer in R for l large enough.

PROOF. The Euler conditions are obviously satisfied by u in R.

Let R_0 be the rectangle $(0,4)\times(-1,0)$ and let w be any function in $H^1(R_0)$ such that $w(x,0)=x$ for $x\in(0,2)$, and $w(x,y)=0$ for $(x,y)\in\partial R_0\setminus((0,4)\times\{0\})$.

The idea is to perturb u by the rescaled function $v(x,y) := lw(\frac{x-1}{l},\frac{y}{l})$. We define the perturbed function

$$\tilde{u}(x,y) := \begin{cases} x & \text{on } R_1\setminus T_\varepsilon, \\ -x+\eta\,(x-1) & \text{on } T_\varepsilon, \\ -x+\eta\, v(x,y) & \text{on } R_2, \end{cases}$$

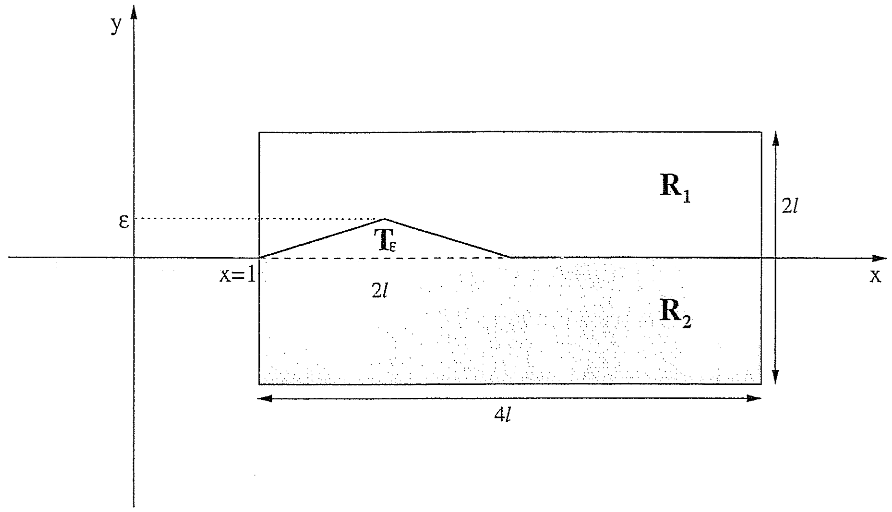

Figure 3.1: the regions R_1, R_2 and T_ε.

where η is a positive parameter and the rectangles R_1, R_2, and the triangle T_ε are indicated in Fig. 3.1. We want to show that, if we set $c := \int_{R_0} |\nabla w(x,y)|^2 dx\, dy$, for every $l > c$ and for every ε_0, $\eta_0 > 0$ there exist $\varepsilon < \varepsilon_0$ and $\eta < \eta_0$ such that

$$\int_R |\nabla u(x,y)|^2 dx\, dy + \mathcal{H}^1(S_u) > \int_R |\nabla \tilde{u}(x,y)|^2 dx\, dy + \mathcal{H}^1(S_{\tilde{u}}).$$

By definition, $\tilde{u}$ satisfies the boundary conditions. Since by the construction of v the function $\tilde{u}$ is continuous on the interface between T_ε and R_2, then

$$\mathcal{H}^1(S_u) - \mathcal{H}^1(S_{\tilde{u}}) = 2l - 2\sqrt{l^2 + \varepsilon^2} = -\frac{\varepsilon^2}{l} + o(\varepsilon^2). \tag{3.3.1}$$

On the triangle T_ε, we obtain

$$\int_{T_\varepsilon} |\nabla u(x,y)|^2 dx\, dy - \int_{T_\varepsilon} |\nabla \tilde{u}(x,y)|^2 dx\, dy = 2l\varepsilon\eta - l\varepsilon\eta^2. \tag{3.3.2}$$

Finally, since we have that $|\nabla \tilde{u}|^2 = 1 + \eta^2 |\nabla v|^2 - 2\eta\, \partial_x v$ in R_2, taking into account the boundary conditions of v, we get

$$\begin{aligned}\int_{R_2} |\nabla u(x,y)|^2 dx\, dy - \int_{R_2} |\nabla \tilde{u}(x,y)|^2 dx\, dy &= -\eta^2 \int_{R_2} |\nabla v(x,y)|^2 dx\, dy \\ &= -l^2\eta^2 \int_{R_0} |\nabla w(x,y)|^2 dx\, dy. \end{aligned} \tag{3.3.3}$$

In order to conclude, by (3.3.1), (3.3.2), and (3.3.3), we have to show that for l large we can choose ε and η arbitrarily close to 0 such that

$$-\frac{\varepsilon^2}{l} - cl^2\eta^2 + 2l\varepsilon\eta - l\varepsilon\eta^2 + o(\varepsilon^2) > 0.$$

If we choose $\eta = \varepsilon/(cl)$, then the equality above reduces to

$$-\frac{\varepsilon^2}{l} + \frac{\varepsilon^2}{c} + o(\varepsilon^2) > 0,$$

which is true if $l > c$. □

3.3.1 Proof of Theorem 3.1.4

From the definition of d and N (see (3.2.39) and Claim 3 in the proof of Theorem 3.1.1) it follows that there is an absolute constant $\tilde{c} > 0$ (independent of Ω_0, Ω, Γ, and u) such that

$$\tilde{c}\,(1 + l^2(\Gamma)k^2(\Gamma)) > \frac{16}{d}. \tag{3.3.4}$$

The absolute constant c, which appears in (3.1.2), is defined by

$$c := \max\left\{\tilde{c}, \frac{64}{\pi^2}\right\}. \tag{3.3.5}$$

Actually, to avoid problems of boundary regularity, we shall work not exactly in Ω, but in a little bit larger set. Let Ω' be a Γ-admissible set such that $\Omega \subset\subset \Omega' \subset\subset \Omega_0$, and

$$\frac{\min_{i=1,2} K(\Gamma \cap \Omega', \Omega_i')}{1 + l^2(\Gamma \cap \Omega') + l^2(\Gamma \cap \Omega')k^2(\Gamma \cap \Omega')} > c \sum_{i=1}^{2} \|\partial_\tau u_i\|^2_{C^1(\Gamma\cap\Omega')},$$

where Ω_i' denote the connected components of $\Omega' \setminus \Gamma$. This is possible by (3.1.2) and by the continuity properties of K.

The idea of the proof is to construct first a calibration φ in a cylinder with base an open neighbourhood of $\Gamma \cap \Omega'$, and then to extend φ in a tubular neighbourhood of graph u.

- *Construction of the calibration around* Γ.

We essentially recycle the construction of Theorem 3.1.1, but we need to slightly modify the definition around the graph of u, in order to exploit condition (3.1.2) and get the extendibility.

To define the calibration $\varphi(x, y, z)$ we use the same notation and the coordinate system (ξ, η) on U (open neighbourhood of $\Gamma \cap \Omega'$) introduced in the proof of Theorem 3.1.1. The vectorfield will be written as

$$\varphi(x, y, z) = \frac{1}{\gamma^2(\xi(x, y), \eta(x, y))}\phi(\xi(x, y), \eta(x, y), z), \tag{3.3.6}$$

where ϕ can be represented by

$$\phi(\xi, \eta, z) = \phi^\xi(\xi, \eta, z)\tau_\xi + \phi^\eta(\xi, \eta, z)\tau_\eta + \phi^z(\xi, \eta, z)e_z.$$

As in the previous section, we can suppose without loss of generality that $0 < u_1 < u_2$. Given $\varepsilon > 0$ and $\lambda > 0$, we consider the following subsets of $V\times\mathbb{R}$

$$\begin{aligned}
A_1 &:= \{(\xi,\eta,z) \in V\times\mathbb{R} : u_1(\xi,\eta) - \varepsilon\, v_1(\xi,\eta) < z \le u_1(\xi,\eta) + \varepsilon\, v_1(\xi,\eta)\},\\
A_2 &:= \{(\xi,\eta,z) \in V\times\mathbb{R} : u_1(\xi,\eta) + \varepsilon\, v_1(\xi,\eta) < z < u_1(\xi,\eta) + 2\varepsilon\},\\
A_3 &:= \{(\xi,\eta,z) \in V\times\mathbb{R} : u_1(\xi,\eta) + 2\varepsilon < z < \beta_1(\xi,\eta)\},\\
A_4 &:= \{(\xi,\eta,z) \in V\times\mathbb{R} : \beta_1(\xi,\eta) < z < \beta_2(\xi,\eta) + 1/\lambda\},\\
A_5 &:= \{(\xi,\eta,z) \in V\times\mathbb{R} : \beta_2(\xi,\eta) + 1/\lambda < z < u_2(\xi,\eta) - 2\varepsilon\},\\
A_6 &:= \{(\xi,\eta,z) \in V\times\mathbb{R} : u_2(\xi,\eta) - 2\varepsilon < z < u_2(\xi,\eta) - \varepsilon\, v_2(\xi,\eta)\},\\
A_7 &:= \{(\xi,\eta,z) \in V\times\mathbb{R} : u_2(\xi,\eta) - \varepsilon\, v_2(\xi,\eta) \le z < u_2(\xi,\eta) + \varepsilon\, v_2(\xi,\eta)\},
\end{aligned}$$

where the functions v_i are defined as

$$v_1(\xi,\eta) := 1 + M\eta,\ v_2(\xi,\eta) := 1 - M\eta$$

with M positive parameter such that

$$c\,(1 + l^2(\Gamma\cap\Omega') + l^2(\Gamma\cap\Omega')k^2(\Gamma\cap\Omega'))\sum_{j=1}^{2}\|\partial_\tau u_j\|^2_{C^1(\Gamma\cap\Omega')} < M < \min_{j=1,2} K(\Gamma\cap\Omega',\Omega_i'), \tag{3.3.7}$$

while β_1 and β_2 are the solutions of the Cauchy problems (3.2.13). Again, if ε is small enough and λ is sufficiently large, the sets $A_1, \dots, A_7$ are nonempty and disjoint, provided V is sufficiently small.

The vector $\phi(\xi,\eta,z)$ introduced in (3.3.6) will be written as

$$\phi(\xi,\eta,z) = (\phi^{\xi\eta}(\xi,\eta,z), \phi^z(\xi,\eta,z)),$$

where $\phi^{\xi\eta}$ is the two-dimensional vector given by the pair (ϕ^ξ, ϕ^η). We define $\phi(\xi,\eta,z)$ as follows:

$$\begin{cases}
\left(2\nabla u_1 - 2\dfrac{u_1 - z}{v_1}\nabla v_1,\ \left|\nabla u_1 - \dfrac{u_1 - z}{v_1}\nabla v_1\right|^2\right) & \text{in } A_1,\\[2ex]
\left(2\nabla(u_1 + \varepsilon v_1) - 2\dfrac{u_1 + \varepsilon v_1 - z}{\tilde v_1}\nabla\tilde v_1,\ \left|\nabla(u_1 + \varepsilon v_1) - \dfrac{u_1 + \varepsilon v_1 - z}{\tilde v_1}\nabla\tilde v_1\right|^2\right) & \text{in } A_2,\\[2ex]
(0,\ \omega_1(\xi,\eta)) & \text{in } A_3,\\[2ex]
(\lambda\sigma(\xi,\eta)\nabla w,\ \mu) & \text{in } A_4,\\[2ex]
(0,\ \omega_2(\xi,\eta)) & \text{in } A_5,\\[2ex]
\left(2\nabla(u_2 - \varepsilon v_2) - 2\dfrac{u_2 - \varepsilon v_2 - z}{\tilde v_2}\nabla\tilde v_2,\ \left|\nabla(u_2 - \varepsilon v_2) - \dfrac{u_2 - \varepsilon v_2 - z}{\tilde v_2}\nabla\tilde v_2\right|^2\right) & \text{in } A_6,\\[2ex]
\left(2\nabla u_2 - 2\dfrac{u_2 - z}{v_2}\nabla v_2,\ \left|\nabla u_2 - \dfrac{u_2 - z}{v_2}\nabla v_2\right|^2\right) & \text{in } A_7,
\end{cases}$$

where ∇ denotes the gradient with respect to the variables (ξ,η), the functions $\tilde{v}_i$ are defined by

$$\tilde{v}_1(\xi,\eta) := 2\varepsilon + M'\eta, \ \tilde{v}_2(\xi,\eta) := 2\varepsilon - M'\eta$$

while

$$\omega_i(\xi,\eta) := \varepsilon^2\left(M + M'\frac{v_i(\xi,\eta)}{\tilde{v}_i(\xi,\eta)}\right)^2 - (\partial_\xi u_i(\xi,\eta))^2 - (\partial_\eta u_i(\xi,\eta))^2$$

for $i = 1,2$, and for every $(\xi,\eta) \in V$; we take the constant μ sufficiently large in order to get the required inequality between the horizontal and the vertical components of the field, and M' so large that ω_i is positive in V, provided V is small enough. We define w as the solution of the Cauchy problem

$$\begin{cases} \Delta w = 0, \\ w(\xi,0) = -\dfrac{4\varepsilon}{1-\varepsilon M' - 6\varepsilon^2 M}\displaystyle\int_0^\xi n(s)(\partial_\xi u_1(s,0) + \partial_\xi u_2(s,0))\,ds, \\ \partial_\eta w(\xi,0) = n(\xi), \end{cases} \tag{3.3.8}$$

where n is a positive analytic function that must be chosen in a suitable way. We define

$$\sigma(\xi,\eta) := \frac{1}{n(q(\xi,\eta))}(1 - \varepsilon M' - 6\varepsilon^2 M),$$

where the function q is constructed in the same way as in (3.2.12).

Let us prove that for a suitable choice of the involved parameters the vectorfield is a calibration in a suitable neighbourhood U of $\Gamma \cap \Omega'$, which is equivalent to prove that ϕ satisfies **(a)**, **(b)**, **(c)**, **(d)**, **(e)**, and **(f)** of page 54. The proof of conditions **(a)**, **(b)**, **(c)**, **(d)**, and **(e)** is the same as in Theorem 3.1.1. The proof of **(f)** is split again in two steps.

STEP 1. For a suitable choice of ε and of the function n (see (3.3.8)) there exists $\delta > 0$ such that condition **(f)** holds for $|s - u_1(\xi,\eta)| < \delta$, $|t - u_2(\xi,\eta)| < \delta$, and $(\xi,\eta) \in V$, provided V is small enough.

We essentially repeat the proof given in Theorem 3.1.1: Claims 1, 2, 3, and 4 are still valid with the same proof (up to the obvious changes due to the different definition of ϕ). Claim 5 must be modified as follows.

Claim 5. *For $h = \frac{64}{\pi^2} l^2(\Gamma)\sum_{i=1}^2 \|\partial_\xi u_i\|^2_{C^1(\Gamma\cap\Omega')}$, there exist $r \in (0, d/8)$ and $\tilde{\delta} > 0$ such that for every $\delta \in (0,\tilde{\delta})$*

$$I(\xi,\eta,u_2(\xi,\eta),t) \in B(r) + b_2^h(\xi,\eta,t),$$
$$I(\xi,\eta,s,u_1(\xi,\eta)) \in B(r) + b_1^h(\xi,\eta,s),$$

provided V is small enough, for every $|t - u_2(\xi,\eta)| \le \delta$, $|s - u_1(\xi,\eta)| \le \delta$.

PROOF OF THE CLAIM. Using the definition of ϕ in A_7, the claim is equivalent to prove

$$(2(t-u_2)\partial_\xi u_2)^2 + \left(-M(1-M\eta)^{-1} + h\right)^2 (t-u_2)^4 + 2r\left(-M(1-M\eta)^{-1} + h\right)(t-u_2)^2 < 0;$$

note that for $a_1 \in (0,1)$ the left-hand side is less than

$$\left(4\sum_{i=1}^{2}\|\partial_\xi u_i\|^2_{C^1(\Gamma\cap\Omega')} + 2hr - \frac{2r}{1+a_1}M\right)\delta^2 + o(\delta^2),$$

provided V is small enough. To obtain the claim, it is sufficient to prove that

$$\frac{2}{r}\sum_{i=1}^{2}\|\partial_\xi u_i\|^2_{C^1(\Gamma\cap\Omega')} < \frac{1}{1+a_1}M - h. \tag{3.3.9}$$

Since by (3.3.7), (3.3.4), and (3.3.5) we can write

$$M = \left(\frac{16+a_2}{d} + \frac{64}{\pi^2}l^2(\Gamma\cap\Omega')\right)\sum_{i=1}^{2}\|\partial_\xi u_i\|^2_{C^1(\Gamma\cap\Omega')},$$

with $a_2 > 0$, the inequality (3.3.9) is equivalent to

$$\frac{2}{r} < \left(\frac{1}{1+a_1} - 1\right)\frac{64}{\pi^2}l^2(\Gamma\cap\Omega') + \frac{16+a_2}{d}\frac{1}{1+a_1},$$

which is true if a_1 is sufficiently small and r is sufficiently close to $d/8$. The proof for u_1 is completely analogous.

To conclude the proof of the step, let r and h be as in Claim 5. If we choose $\varepsilon < \bar{\varepsilon}$ and $\delta \leq \min\{\tilde{\delta}, \varepsilon\}$, by Claim 5 we have that

$$I(\eta, s, u_1(\xi,\eta)) + I(\eta, u_2(\xi,\eta), t) \in B(2r) + b_1^h(\xi,\eta,s) + b_2^h(\xi,\eta,t) \tag{3.3.10}$$

for every $|s - u_1(\xi,\eta)| < \delta$, $|t - u_2(\xi,\eta)| < \delta$, and $(\xi,\eta) \in V$; since h satisfies (3.2.44) and $2r < d/4$, we can apply Claim 4 to deduce that the set

$$B(2r) + (\rho_\varepsilon(\xi,\eta)\sin\tilde{\theta}(\eta), \rho_\varepsilon(\xi,\eta)\cos\tilde{\theta}(\eta)) + b_1^h(\xi,\eta,s) + b_2^h(\xi,\eta,t)$$

is contained in the ball centred at $(0,0)$ with radius $\gamma(\xi,\eta)$. Some easy geometric considerations show that the relation between θ_ε and $\tilde{\theta}$ (see (3.2.38)) implies that also the set

$$B(2r) + (\rho_\varepsilon(\xi,\eta)\sin\theta_\varepsilon(\eta), \rho_\varepsilon(\xi,\eta)\cos\theta_\varepsilon(\eta)) + b_1^h(\xi,\eta,s) + b_2^h(\xi,\eta,t) \tag{3.3.11}$$

is contained in the ball centred at $(0,0)$ with radius $\gamma(\xi,\eta)$, if the condition

$$|b_1^h(\xi,\eta,s) + b_2^h(\xi,\eta,t)| < 2r$$

holds (to make this true, take δ and V smaller if needed). Since

$$I(\eta,s,t) = I(\eta, s, u_1(\xi,\eta)) + I(\eta, u_1(\xi,\eta), u_2(\xi,\eta)) + I(\eta, u_2(\xi,\eta), t),$$

by (3.2.47), it follows that $I(\eta,s,t)$ belongs to the set (3.3.11), and then to the ball centred at $(0,0)$ with radius $\gamma(\xi,\eta)$ for every $|s - u_1(\xi,\eta)| < \delta$, $|t - u_2(\xi,\eta)| < \delta$, and $(\xi,\eta) \in V$. This concludes the proof of Step 1.

STEP 2. If ε is sufficiently small and $\delta \in (0,\varepsilon)$, condition **(f)** holds for $|s-u_1(\xi,\eta)| \geq \delta$ or $|t-u_2(\xi,\eta)| \geq \delta$, and $(\xi,\eta) \in V$, provided V is small enough.

By using condition (3.3.7), arguing as in the proof of Claim 5, we can prove the following claim.

Claim 6. *There exist $r < 1/4$ and $\varepsilon > 0$ such that*

$$I(0, u_2(\xi,0), t) \in B(r), \qquad I(0, s, u_1(\xi,0)) \in B(r)$$

for $0 < |s-u_1(\xi,0)| \leq \varepsilon$, $0 < |t-u_2(\xi,0)| \leq \varepsilon$, and $(\xi,0) \in V$.

We can conclude the proof of Step 2 in the same way as in Theorem 3.1.1, with the minor changes due to the different definition of the field.

By Step 1 and Step 2, we conclude that, choosing ε sufficiently small and n in a suitable way, condition **(f)** is true for $u_1(\xi,\eta)-\varepsilon \leq s,t \leq u_2(\xi,\eta)+\varepsilon$. So, φ is a calibration.

- *Construction of the calibration around the graph of u.*

Now the matter is to extend the field in a tubular neighbourhood of the graph of u. From now on, we reintroduce the Cartesian coordinates.

Let Γ_i be the curve $\eta = (-1)^i k$, where $k > 0$. If k is sufficiently small, for $i = 1,2$ the curve Γ_i connects two points of $\partial\Omega_i'$, divides Ω_i' (and then Ω) in two connected components, and the normal vector ν_i to Γ_i which points towards Γ coincides with $(-1)^{i+1}\nabla\eta/|\nabla\eta|$. Set $U' := U \cap \{(x,y) \in \Omega' : |\eta(x,y)| < k\}$ and $U'' := U' \cap \Omega$. Since $\|\nabla\eta\| = 1$ on Γ, by (3.3.7) we can suppose that

$$\frac{M}{1-Mk}\max_{i=1,2}\|\nabla\eta\|_{L^\infty(\Gamma_i)} < \min_{i=1,2} K(\Gamma_i, \Omega_i' \setminus \overline{U'}). \tag{3.3.12}$$

Chosen δ so small that $(\operatorname{graph} u)_\delta \cap ((U'' \cap \Omega_1) \times \mathbb{R}) \subset A_1$ and $(\operatorname{graph} u)_\delta \cap ((U'' \cap \Omega_2) \times \mathbb{R}) \subset A_7$, we define the vectorfield

$$\hat\varphi(x,y,z) = (\hat\varphi^{xy}(x,y,z), \hat\varphi^z(x,y,z)) \in \mathbb{R}^3,$$

as follows:

$$\begin{cases} \varphi(x,y,z) & \text{in } \{(x,y,z) : (x,y) \in U'',\ u_1-\delta < z < u_2+\delta\}, \\[2ex] \left(2\nabla u - 2\dfrac{u-z}{\hat v_1}\nabla\hat v_1, \left|\nabla u - \dfrac{u-z}{\hat v_1}\nabla\hat v_1\right|^2\right) & \text{in } (\operatorname{graph} u)_\delta \cap (\Omega_1 \setminus U'')\times\mathbb{R}, \\[2ex] \left(2\nabla u - 2\dfrac{u-z}{\hat v_2}\nabla\hat v_2, \left|\nabla u - \dfrac{u-z}{\hat v_2}\nabla\hat v_2\right|^2\right) & \text{in } (\operatorname{graph} u)_\delta \cap (\Omega_2 \setminus U'')\times\mathbb{R}. \end{cases}$$

The function $\hat v_i$ is the solution of the problem

$$\min\left\{\int_{\Omega_i'\setminus\overline{U'}} |\nabla v|^2 dx\,dy - \frac{M}{1-Mk}\int_{\Gamma_i} |\nabla\eta|\, v^2 d\mathcal{H}^1 : v \in H^1(\Omega_i' \setminus \overline{U'}),\ v|_{\partial(\Omega_i'\setminus\overline{U'})\setminus\Gamma_i} = 1\right\}. \tag{3.3.13}$$

Let us show that the problem (3.3.13) admits a solution. If $\{v_n\}$ is a minimizing sequence, then

$$\sup_n \int_{\Omega_i'\setminus\overline{U'}} |\nabla v_n|^2 dx\, dy - \frac{M}{1-Mk}\int_{\Gamma_i} |\nabla\eta|\, v_n^2\, d\mathcal{H}^1 < +\infty. \tag{3.3.14}$$

We have only to show that $\{v_n\}$ is bounded in $H^1(\Omega_i'\setminus\overline{U'})$. If we put $\overline{v}_n := v_n - 1$, by (3.1.1) for every $\tau\in(0,1)$ we have

$$\begin{aligned}
\int_{\Omega_i'\setminus\overline{U'}} |\nabla v_n|^2 dx\, dy &= \int_{\Omega_i'\setminus\overline{U'}} |\nabla \overline{v}_n|^2 dx\, dy \\
&= \left(\int_{\Gamma_i} \overline{v}_n^2 d\mathcal{H}^1\right)\int_{\Omega_i'\setminus\overline{U'}} \left|\nabla\left(\frac{\overline{v}_n}{(\int_{\Gamma_i}\overline{v}_n^2 d\mathcal{H}^1)^{\frac{1}{2}}}\right)\right|^2 dx\, dy \\
&\geq \left(\int_{\Gamma_i}(v_n-1)^2 d\mathcal{H}^1\right) K(\Gamma_i,\Omega_i'\setminus\overline{U'}) \\
&\geq (1-\tau)K(\Gamma_i,\Omega_i'\setminus\overline{U'})\int_{\Gamma_i} v_n^2 d\mathcal{H}^1 + K(\Gamma_i,\Omega_i'\setminus\overline{U'})\left(1-\frac{1}{\tau}\right)\mathcal{H}^1(\Gamma_i),
\end{aligned} \tag{3.3.15}$$

where we used Cauchy Inequality. By (3.3.12), we can choose τ so small that

$$(1-\tau)K(\Gamma_i,\Omega_i'\setminus\overline{U'}) > \frac{M}{1-Mk}\|\nabla\eta\|_{L^\infty(\Gamma_i)},$$

and substituting (3.3.15) in (3.3.14), we obtain

$$\sup_n \int_{\Gamma_i} v_n^2\, d\mathcal{H}^1 < +\infty.$$

Using again (3.3.14) and Poincaré Inequality, we conclude that $\{v_n\}$ is actually bounded in $H^1(\Omega_i'\setminus\overline{U'})$.

The solution of (3.3.13) satisfies

$$\begin{cases}
\Delta\hat{v}_i = 0 & \text{in } \Omega_i'\setminus\overline{U'}, \\
\partial_\nu\hat{v}_i = \dfrac{M}{1-Mk}|\nabla\eta|\hat{v}_i & \text{on } \Gamma_i, \\
\hat{v}_i = 1 & \text{on } \partial(\Omega_i'\setminus\overline{U'})\setminus\Gamma_i,
\end{cases} \tag{3.3.16}$$

and so, in particular, belongs to $C^\infty(\overline{\Omega_i\setminus U''})$. By a truncation argument, it is easy to see that $\hat{v}_i \geq 1$, so $\hat{\varphi}$ is well defined.

Since $\hat{\varphi}$ is a calibration in $\{(x,y,z) : (x,y)\in U'',\ u_1(x,y)-\delta < z < u_2(x,y)+\delta\}$, it remains to prove only that the field is globally divergence free in the sense of distributions and that conditions **(c)**, **(d)** and **(f)** are verified in the regions $(\operatorname{graph} u)_\delta \cap (\Omega_i\setminus U'')\times\mathbb{R}$. First of all, note that by Lemma 3.2.1 the field $\hat{\varphi}$ is divergence free in the regions $(\operatorname{graph} u)_\delta\cap(\Omega_i\setminus U'')\times\mathbb{R}$, since it is constructed starting from the family of harmonic functions $u(x,y) - t\hat{v}_i(x,y)$. To complete the proof, we need to check that the normal components of the traces of φ and of the extension field are equal on the surface of separation, i.e.,

$$\varphi^{xy}\cdot\nu_i = \left(2\nabla u - 2\frac{u-z}{\hat{v}_i}\nabla\hat{v}_i\right)\cdot\nu_i \qquad \text{on } \Gamma_i, \tag{3.3.17}$$

where $\nu_i = (-1)^{i+1}\nabla\eta/|\nabla\eta|$. Using the definition of φ, we obtain that

$$\varphi^{xy}\cdot\nu_i = \left((-1)^{i+1}\partial_\eta u - \frac{u-z}{1-Mk}M\right)|\nabla\eta|;$$

since $\nabla u\cdot\nu_i = (-1)^{i+1}\partial_\eta u|\nabla\eta|$, the equality (3.3.17) is equivalent to

$$\frac{M}{1-Mk}|\nabla\eta| = \frac{1}{\hat{v}_i}\nabla\hat{v}_i\cdot\nu_i,$$

which is true by (3.3.16).

Conditions **(c)** and **(d)** are obviously satisfied, while condition **(f)** is true if we take δ satisfying

$$\delta \leq \sup\left\{\left(4|\nabla u| + 2\frac{|\nabla\hat{v}_i|}{\hat{v}_i}\right)^{-1} : (x,y)\in\Omega_i\setminus U'',\, i=1,2\right\}.$$

Therefore, with this choice of δ, the vectorfield $\hat{\varphi}$ is a calibration. □

3.3.2 Some properties of $K(\Gamma, A)$

In this subsection we investigate some qualitative properties of the quantity $K(\Gamma, A)$ and we shall compute it explicitly in a very particular case. Let us start by a very simple result.

Proposition 3.3.2 *Let Γ be a simple analytic curve and $\tilde{\Gamma}$ an extension of Γ, whose endpoints do not coincide with the endpoints of Γ. If $\Gamma_\delta^\pm$ are the two connected components of $\Gamma_\delta\setminus\tilde{\Gamma}$ (which are well defined if δ is sufficiently small), then*

$$\lim_{\delta\to 0^+} K(\Gamma, \Gamma_\delta^\pm) = +\infty.$$

PROOF. For convenience we set

$$W^\pm(\delta) := \left\{v\in H^1(\Gamma_\delta^\pm) : \int_\Gamma v^2 d\mathcal{H}^1 = 1,\ v = 0 \text{ on } \partial(\Gamma_\delta^\pm)\setminus\Gamma\right\}.$$

Suppose by contradiction that there exists a sequence $\{\delta_n\}$ decreasing to 0 such that $\sup_n K(\Gamma,\Gamma_{\delta_n}^+) = c < +\infty$; this implies the existence of a sequence $\{v_n\}$ such that

$$v_n \in W^+(\delta_n) \qquad \text{and} \qquad \int_{\Gamma_{\delta_n}^+} |\nabla v_n(x,y)|^2 dx\,dy \leq c$$

for every integer n. From now on, we regard v_n as a function belonging to $H^1(\Gamma_{\delta_1}^+)$ which vanishes on $\Gamma_{\delta_1}^+\setminus\Gamma_{\delta_n}^+$. By Poincaré Inequality it follows immediately that $\{v_n\}$ is bounded in $H^1(\Gamma_{\delta_1}^+)$, and so admits a weakly convergent subsequence $\{v_{n_k}\}$. Let us call v the limit of the subsequence; since for every k, v_{n_k} vanishes on $\Gamma_{\delta_1}^+\setminus\Gamma_{\delta_{n_k}}^+$, then v must vanish a.e.; on the other hand, since $\int_\Gamma v_{n_k}^2 d\mathcal{H}^1 = 1$, by the compactness of the trace operator, we have that $\int_\Gamma v^2 d\mathcal{H}^1 = 1$, and this is clearly impossible. □

We remark that by Theorem 3.1.4 and Proposition 3.3.2, if U_0 is a neighbourhood of Γ and $u\in SBV(U_0)$ satisfies the Euler conditions in U_0 with $S_u = \Gamma$, then there exists a neighbourhood U of Γ contained in U_0 such that u is a local graph-minimizer in U. Actually, taking U smaller if needed, by Theorem 3.1.1 we get also the Dirichlet minimality.

Proposition 3.3.3 *(Characterization of $K(\Gamma, A)$.) Let A be an open set with Lipschitz boundary and Γ be a subset of ∂A with nonempty relative interior in ∂A. The constant $K(\Gamma, A)$ is the first eigenvalue of the problem*

$$\begin{cases} \Delta u = 0 & \text{on } A, \\ \partial_\nu u = \lambda u & \text{on } \Gamma, \\ u = 0 & \text{on } \partial A \setminus \Gamma. \end{cases} \tag{3.3.18}$$

Moreover, it is the unique eigenvalue with a positive eigenfunction.

PROOF. If u is a solution of (3.1.1), then it is harmonic and there exists a Lagrange multiplier λ such that

$$2\int_A \nabla u \cdot \nabla \varphi \, dx\, dy = \lambda \int_\Gamma u\varphi \, d\mathcal{H}^1 \qquad \forall \varphi \in C^\infty(A) : \ \varphi = 0 \text{ on } \partial A \setminus \Gamma, \tag{3.3.19}$$

which means, by Green Formula, that $\partial_\nu u = \lambda u$ on Γ. Using (3.3.19), one can easily see that $K(\Gamma, A)$ is in fact the minimal eigenvalue of (3.3.18) and that it has a positive eigenfunction (indeed, if u is a solution also $|u|$ is). Let u be a positive function belonging to the eigenspace of $K(\Gamma, A)$ and v another positive eigenfunction associated with the eigenvalue μ; by Green Formula we have

$$\int_\Gamma v\partial_\nu u \, d\mathcal{H}^1 - \int_\Gamma u\partial_\nu v \, d\mathcal{H}^1 = 0,$$

therefore

$$(K(\Gamma, A) - \mu)\int_\Gamma uv \, d\mathcal{H}^1 = 0.$$

Since both u and v are positive, from the last equality it follows that $\mu = K(\Gamma, A)$. □

Proposition 3.3.4 *If $A = (0, a)\times(0, b)$ and $\Gamma = (0, a)\times\{0\}$, then*

$$K(\Gamma, A) = \frac{\pi}{a\tanh\left(\frac{\pi b}{a}\right)}. \tag{3.3.20}$$

PROOF. The function

$$v(x, y) = \sin\left(\frac{\pi}{a}x\right)\sinh\left(\frac{\pi}{a}(b - y)\right)$$

is positive and satisfies (3.3.18) with $\lambda = \dfrac{\pi}{a\tanh\left(\frac{\pi b}{a}\right)}$. Then, by Proposition 3.3.3, this quantity coincides with $K(\Gamma, A)$. □

Proposition 3.3.5 *Let $g : [0, a_0] \to [0, +\infty)$ be a Lipschitz function and denote the graph of g by Γ. Given $0 \le a_1 < a_2 \le a_0$ and $b > 0$, if we set $\Gamma(a_1, a_2) := \operatorname{graph} g|_{(a_1,a_2)}$ and*

$$R(a_1, a_2, b) := \{(x, y) : x \in (a_1, a_2), \ y \in (g(x), g(x) + b)\},$$

then

$$\lim_{|a_2 - a_1| \to 0} K\left(\Gamma(a_1, a_2), R(a_1, a_2, b)\right) = +\infty \qquad \text{uniformly with respect to } b.$$

PROOF. The idea is to transform the region $R(a_1, a_2, b)$ into the rectangle $(0, a_2 - a_1)\times(0, b)$ by a suitable diffeomorphism in order to use (3.3.20).

Let $\psi : (0, a_2 - a_1)\times(0, b) \to R(a_1, a_2, b)$ be the map defined by $\psi(x, y) = (x + a_1, y + g(x + a_1))$. Let $v \in H^1(R(a_1, a_2, b))$ be such that $v = 0$ on $\partial R(a_1, a_2, b) \setminus \Gamma(a_1, a_2)$ and

$$\int_{\Gamma(a_1,a_2)} v^2 d\mathcal{H}^1 = \int_0^{a_2-a_1} v^2(\psi(x, 0))\sqrt{1 + (g'(x))^2}\, dx = 1. \tag{3.3.21}$$

If we call $\tilde{v}(x, y) := v(\psi(x, y))$, then $\tilde{v} \in H^1((0, a_2 - a_1)\times(0, b))$, $\tilde{v} = 0$ on the boundary of the rectangle except $(0, a_2 - a_1)\times\{0\}$, and by (3.3.21) there exists $\lambda > 0$ such that $\lambda^2 \leq \sqrt{1 + \|g'\|^2_\infty}$ and

$$\lambda^2 \int_0^{a_2-a_1} \tilde{v}^2(x, 0)\, dx = 1.$$

Therefore, since $J\psi \equiv 1$,

$$\begin{aligned}
\int_{R(a_1,a_2,b)} |\nabla v(x, y)|^2 dx\, dy &= \int_{(0,a_2-a_1)\times(0,b)} |\nabla v(\psi(x, y))|^2 dx\, dy \\
&\geq (1 + \|g'\|_\infty + \|g'\|^2_\infty)^{-1} \int_{(0,a_2-a_1)\times(0,b)} |\nabla\tilde{v}(x, y)|^2 dx\, dy \\
&\geq \lambda^{-2}(1 + \|g'\|_\infty + \|g'\|^2_\infty)^{-1} K\Big((0, a_2 - a_1)\times\{0\}, (0, a_2 - a_1)\times(0, b)\Big) \\
&\geq (1 + \|g'\|^2_\infty)^{-3/2} \frac{\pi}{2(a_2 - a_1)\tanh\left(\frac{\pi b}{a_2-a_1}\right)},
\end{aligned}$$

where the last inequality follows by the estimate on λ and by (3.3.20). Since v is arbitrary, using the fact that $0 < \tanh t \leq 1$ for every $t > 0$, we obtain that

$$K(\Gamma(a_1, a_2), R(a_1, a_2, b)) \geq (1 + \|g'\|_\infty)^{-3/2} \frac{\pi}{2(a_2 - a_1)};$$

so far, the conclusion is clear. □

We have already remarked (see Proposition 3.3.2) that the graph-minimality is guaranteed in small neighbourhoods of the discontinuity set Γ. As consequence of Proposition 3.3.5, we obtain that the graph-minimality holds also in the open sets, which are narrow along the direction parallel to Γ and may be very large along the normal direction. This is made precise by the following corollary.

Corollary 3.3.6 *Let g be a positive function, analytic on $[0, a_0]$ (i.e. g admits an analytic extension outside the interval), and denote the graph of g by Γ. For every $M > 0$ there exists $h = h(M, \Gamma)$ such that, if Ω is Γ-admissible (see Definition 3.1.3) and $\Omega \subset (a_1, a_1 + h)\times\mathbb{R}$ with $a_1 \in [0, a_0 - h]$, and if u is a function in $SBV(\Omega)$ with $S_u = \Gamma \cap \Omega$, with different traces at every point of $\Gamma \cap \Omega$, satisfying the Euler conditions in Ω, and $\sum_{i=1}^2 \|\partial_\tau u_i\|_{C^1(\Gamma\cap\Omega)} \leq M$ (where u_i is as above the restriction of u to the connected component Ω_i of $\Omega \setminus \Gamma$), then u is a local graph-minimizer in Ω. (see Fig. 3.2)*

PROOF. By Proposition 3.3.5 there exists $h > 0$ such that

$$\frac{K(\Gamma(a_1, a_2), R(a_1, a_2, b))}{1 + l^2(\Gamma) + l^2(\Gamma)k^2(\Gamma)} > cM^2,$$

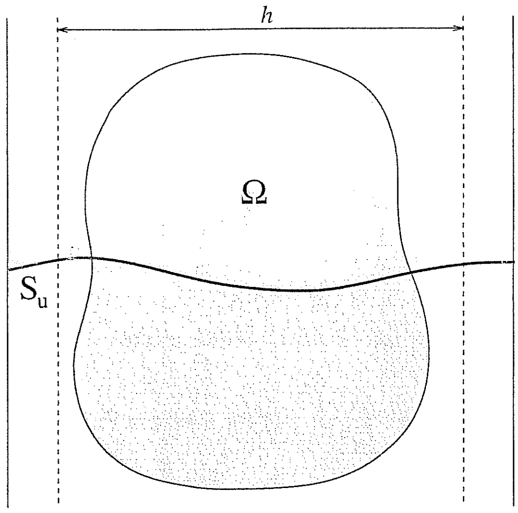

Figure 3.2: if the thickness of Ω is less than h, then u is a local graph-minimizer in Ω.

for every $a_1, a_2 \in [0, a_0]$ with $0 < a_2 - a_1 \leq h$ and for every $b > 0$. If $\Omega \subset (a_1, a_1 + h) \times \mathbb{R}$ we can choose $b > 0$ so large that $\Omega_1 \subset R(a_1, a_1 + h, b)$ (assuming that Ω_1 is the upper component). Then by the monotonicity properties of $K(\Gamma, A)$, it follows that

$$\frac{K(\Gamma \cap \Omega, \Omega_1)}{1 + l^2(\Gamma) + l^2(\Gamma)k^2(\Gamma)} > c\,M^2 \geq c \sum_{i=1}^{2} \|\partial_\tau u_i\|^2_{C^1(\Gamma \cap \Omega)}.$$

Applying the same argument to Ω_2, the conclusion follows from Theorem 3.1.4. $\square$

Chapter 4

Global calibrations for the non-homogeneous functional

In this chapter we are interested in the behaviour of global minimizers of the non-homogeneous Mumford-Shah functional $F_{\beta,g}$ (defined in (1.2.1)) as β becomes large.

4.1 Preliminary Results

We give here some definitions and state some technical results that will be used in the sequel.

For fixed $R > 0$, we introduce the following class of sets:

$$\mathcal{U}_R = \{\, E \subset \mathbb{R}^N, E \text{ open} : \forall p \in \partial E \ \exists p', p'' : \\ p \in \partial B(p', R) \cap \partial B(p'', R),\ B(p', R) \subset E,\ B(p'', R) \subset \mathcal{C}E \,\}, \quad (4.1.1)$$

and

$$\mathcal{U}_R(\Omega) = \{E \in \mathcal{U}_R : \ E \subset \Omega,\ \mathrm{dist}(E, \partial\Omega) \geq R\}. \quad (4.1.2)$$

If E belongs to $\mathcal{U}_R$ and $p \in \partial E$, we denote the centers of the interior and exterior balls associated with p by p' and p'' respectively; moreover, we call $\mathcal{S}_E^p$ the class of all coordinate systems centred at p such that the vector $\frac{1}{2R}(p'' - p')$ coincides with the N-th vector of the coordinate basis. The following proposition is proved in ([38])

Proposition 4.1.1 *There exists a constant $\rho > 0$ (depending only on R), such that for every $E \in \mathcal{U}_R(\Omega)$ and for every $p_0 \in \partial E$, if we call C the cylinder $\{x \in \mathbb{R}^{N-1} : |x| < \rho\} \times]-R, R[$ expressed with respect to a coordinate system belonging to $\mathcal{S}_E^{p_0}$, then $\partial E \cap C$ is the subgraph of a function f belonging to $W^{2,\infty}(\{x \in \mathbb{R}^{N-1} : |x| < \rho\})$. Moreover, the $W^{2,\infty}$-norm of f is bounded by a constant depending only on R (independent of p_0, of E and of the choice of the coordinate system in $\mathcal{S}_E^{p_0}$).*

Remark 4.1.2 Note that if Ω is bounded and of class C^2 then there exists $R > 0$ such that $\Omega \in \mathcal{U}_R$.

For $E \subset \mathbb{R}^N$, we define the signed distance function

$$d_E(x) = \mathrm{dist}(x, E) - \mathrm{dist}(x, \mathcal{C}E).$$

Now we are going to state some basic properties of that function; for a proof see, for example, [28].

Lemma 4.1.3 *i) Let x be a point of $\mathbb{R}^N$. Then $d_E(x)$ is differentiable at x if and only if there exists a unique $y \in \partial E$ such that $|d_E(x)| = |x-y|$. In this case, we have*

$$\nabla d_E(x) = \frac{x-y}{d_E(x)}$$

and we can define the projection on ∂E $\pi_E(x) := y$.

ii) Let ∂E be a hypersurface of class C^k, $k \geq 2$. Then, for every $x \in \partial E$, there exists a neighbourhood V of x such that $d_E \in C^k(V)$ and $\pi_E \in C^{k-1}(V)$.

Lemma 4.1.4 *Let $E \subset \mathbb{R}^N$ be an open set whose boundary is a hypersurface of class $W^{2,\infty}$. Then for every $x \in \partial\Omega$, there exists a neighbourhood V of x where π_E is well defined and such that $d_E \in W^{2,\infty}(V(x))$. Moreover, denoting by $\lambda_1 \leq \cdots \leq \lambda_n$ the eigenvalues of $\nabla^2 d_E$ and by $k_1(y) \leq \cdots \leq k_{n-1}(y)$ the principal curvatures of ∂E at $\pi(y)$, we have*

$$\lambda_i := \begin{cases} 0 & \text{if } i = 1 \\ \frac{k_{i-1}(y)}{1+d_E(y)k_{i-1}(y)} & \text{if } i > 1. \end{cases}$$

Lemma 4.1.5 *Let E be an open set belonging to $\mathcal{U}_R$, for some $R > 0$. Then the projection π_E is well defined and of class $W^{1,\infty}$ in the $(R/2)$-neighbourhood of ∂E, and therefore d_E is of class $W^{2,\infty}$ in that neighbourhood. Moreover we have:*

$$\|d_E\|_{W^{2,\infty}} \leq C \qquad \text{and} \qquad \|\pi_E\|_{W^{1,\infty}} \leq C,$$

where C is a positive constant depending only on R.

PROOF. The fact that π_E is well defined in the $(R/2)$-neighbourhood of ∂E (denoted by $(\partial E)_{R/2}$) is an easy consequence of the definition of $\mathcal{U}_R$: indeed let x be a point of $(\partial E)_{R/2} \cap CE$ and let $p \in \partial E$ such that $d_E(x) = |x-p|$. We claim that such a p is unique. Indeed let $B(p'', R) \subset CE$ be the exterior ball associated with p (see the definition (4.1.1)); since the vector $p''-p$ is parallel to $x-p$ (indeed both vectors are normal to ∂E at p), it is clear that $\overline{B(x, d_E(x))} \setminus \{p\} \subset B(p'', R) \subset CE$ and so p is the unique minimum point.

Concerning the smoothness, it is enough to prove that d_E is of class $W^{2,\infty}$, then we conclude by the equality

$$\pi_E(x) = x - d_E(x)\nabla d_E(x).$$

Exploiting the definition of $\mathcal{U}_R$ in a way similar to the one we did above, we can easily see that, for every $\varepsilon \in (0, R/2)$,

$$(E)_\varepsilon \in \mathcal{U}_{R-\varepsilon} \qquad \text{and} \qquad d_{(E)_\varepsilon} = d_E - \varepsilon, \tag{4.1.3}$$

implying that $\partial((E)_\varepsilon)$ is in turn of class $W^{2,\infty}$. So if $x \in (\partial E)_{R/2}$, then $x \in \partial((E)_\varepsilon)$ for $\varepsilon = d_E(x)$. By Lemma 4.1.4 there exists a neighbourhood V of x where $d_{(E)_\varepsilon}$ is of class $W^{2,\infty}$ and $\|d_{(E)_\varepsilon}\|_{W^{2,\infty}} \leq C$, with C depending only on R. Recalling (4.1.3), we are done. □

For the proof of the announced estimates on the norm of the solutions of (9), we will use some technical results coming from sectorial operators theory and from interpolation theory.

First let us recall what a sectorial operator is.

Let X a complex Banach space and $A : D(A) \to X$ a closed linear operator with not necessarily dense domain; call $\rho(A)$ the resolvent set of A and for $\lambda \in \rho(A)$ denote by $R(\lambda, A)$ the resolvent operator $(\lambda I - A)^{-1}$ belonging to $L(X)$.

Definition 4.1.6 *A is said to be sectorial (in X) if the following two conditions are satisfied:*

i) there exist $\omega \in \mathbb{R}$ *and* $\theta \in (\frac{\pi}{2}, \pi)$ *such that*

$$S_{\theta,\omega} := \{\lambda \in \mathbb{C} : |\arg(\lambda - \omega)| \leq \theta\} \subset \rho(A);$$

ii) there exists a positive constant M such that, for every $\lambda \in S_{\theta,\omega}$*, there holds*

$$\|R(\lambda, A)\|_{L(x)} \leq \frac{M}{|\lambda - \omega|}.$$

We recall that $D(A)$, endowed with the norm

$$\|x\|_{D(A)} = \|x\|_X + \|Ax\|_X$$

is a Banach space continuously embedded in X.

Let Ω be either $\mathbb{R}^N$ or $\mathbb{R}^N_+$ and let $A : \Omega \to \mathbb{R}^{N\times N}$ be a matrix with coefficients belonging to $W^{1,\infty}(\Omega)$ and uniformly elliptic, i.e., satisfying

$$A(x)\xi \cdot \xi \geq \lambda|\xi|^2 \qquad \forall x \in \Omega,\ \forall \xi \in \mathbb{R}^N,$$

where λ_0 is a suitable positive constant; set

$$D(\mathcal{A}_0) := \left\{ u \in L^\infty(\Omega) : u \in \bigcap_{p\geq 1} W^{2,p}_{loc}(\Omega),\ \operatorname{div}(A\nabla u) \in L^\infty(\Omega) \text{ and } A\nabla u \cdot \nu = 0 \text{ on } \partial\Omega \right\}$$

$$D(\mathcal{A}_1) := \left\{ u \in D(\mathcal{A}_0) : \operatorname{div}(A\nabla u) \in W^{1,\infty}(\Omega) \text{ and } A\nabla u \cdot \nu = 0 \text{ on } \partial\Omega \right\},$$

where $\nu(x)$ denotes the outer unit normal vector at x to Ω, and define the operators

$$\begin{array}{rcl} \mathcal{A}_0 : D(\mathcal{A}_0) & \to & L^\infty(\Omega) \\ u & \mapsto & f\operatorname{div}(A\nabla u), \end{array} \tag{4.1.4}$$

and

$$\begin{array}{rcl} \mathcal{A}_1 : D(\mathcal{A}_1) & \to & W^{1,\infty}(\Omega) \\ u & \mapsto & f\operatorname{div}(A\nabla u), \end{array} \tag{4.1.5}$$

where $f : \Omega \to (0, +\infty)$ is a positive function of class $W^{1,\infty}$ satisfying:

$$f(x) \geq \lambda_1 > 0 \qquad \forall x \in \Omega.$$

The following fact is proved in [35] (see Theorem 3.1.6, page 77, Theorem 3.1.7, page 78, and 3.1.26, page 103).

Theorem 4.1.7 *The operators* $\mathcal{A}_0$ *and* $\mathcal{A}_1$ *are sectorial in* $L^\infty(\Omega)$ *and* $W^{1,\infty}(\Omega)$ *respectively. In particular there exist two positive constants* β_0 *and K, depending on the constants* λ_0, λ_1, *on* $W^{1,\infty}$-*norm of A and f, such that the problem*

$$\begin{cases} f\operatorname{div}(A\nabla u) = \beta(u - g) & \text{in } \Omega, \\ A\nabla u \cdot \nu = 0 & \text{in } \partial\Omega, \end{cases} \tag{4.1.6}$$

admits a unique solution $u \in D(\mathcal{A})$, *for every* $\beta \geq \beta_0$ *and for every* $g \in L^\infty(\Omega)$. *Moreover* u *satisfies*

$$\|u\|_\infty + \beta^{-\frac{1}{2}}\|\nabla u\|_\infty \leq K\|g\|_\infty; \tag{4.1.7}$$

if g *belongs to* $W^{1,\infty}(\Omega)$ *then the following estimate actually holds*

$$\|u\|_{W^{1,\infty}} + \beta^{-\frac{1}{2}}\|f\mathrm{div}(A\nabla u)\|_\infty + \sup_{x_0\in\Omega} \beta^{\frac{N}{2p}-1}\|\nabla^2 u\|_{L^p\left(B(x_0,\frac{1}{\sqrt{\beta}})\cap\Omega\right)} \leq K\|g\|_{W^{1,\infty}}. \tag{4.1.8}$$

Given a sectorial operator $A : D(A) \to X$ there is a natural way to construct a family of intermediate spaces between $D(A)$ and X, by setting for $\theta \in (0,1)$

$$D(A,\theta,\infty) = \left\{x \in X : \sup_{t>2\omega\vee 1}\left(t^\theta\|AR(t,A)x\|_{L(X)}\right) < +\infty\right\},$$

where ω is the real number appearing in i) of Definition 4.1.6. Setting

$$[x]_{D(A,\theta,\infty)} = \sup_{t>2\omega\vee 1}\left(t^\theta\|AR(t,A)x\|_{L(X)}\right), \tag{4.1.9}$$

one sees that $[x]_{D(A,\theta,\infty)}$ is a seminorm and $D(A,\theta,\infty)$ endowed with the norm

$$\|x\|_{D(A,\theta,\infty)} = \|x\|_X + [x]_{D(A,\theta,\infty)} \tag{4.1.10}$$

is a Banach space. Moreover, for $0 \leq \theta_1 < \theta_2 \leq 1$,

$$Y \subseteq D(A,\theta_2,\infty) \subset D(A,\theta_1,\infty) \subseteq X,$$

with continuous embeddings. An important fact is stated in the following proposition

Proposition 4.1.8 *(see Proposition 2.2.7, page 50 of [35])*

$$\begin{array}{rcl} A_\theta : D(A,\theta+1,\infty) := \{x \in D(A) :\ Ax \in D(A,\theta,\infty)\} & \to & D(A,\theta,\infty) \\ x & \mapsto & Ax, \end{array}$$

is sectorial in $D(A,\theta,\infty)$; *moreover*

$$\|R(\lambda,A_\theta)\|_{L(D(A,\theta,\infty))} \leq \|R(\lambda,A)\|_{L(X)}. \tag{4.1.11}$$

Next theorem gives a useful characterization of the intermediate spaces $D(A,\theta,\infty)$ in the case of elliptic operators.

Theorem 4.1.9 *(see Theorem 3.1.30, page 108 of [35]) Let* $\mathcal{A}_0$ *be the operator defined in* (4.1.4). *Then for every* $\theta \in (0,\frac{1}{2})$,

$$D(\mathcal{A}_0,\theta,\infty) = C^{0,2\theta}(\overline{\Omega}),$$

with equivalence of the respective norms. In particular there exists two constants C_1 *and* C_2 *depending only on the* $W^{1,\infty}$*-norm of* A *and* f *and on the constants* λ_0 *and* λ_1, *such that*

$$C_1\|g\|_{D(\mathcal{A}_0,\theta,\infty)} \leq \|g\|_{C^{0,2\theta}(\overline{\Omega})} \leq C_2\|g\|_{D(\mathcal{A}_0,\theta,\infty)}. \tag{4.1.12}$$

Let us recall now the definition of gradient flow for the homogeneous Mumford-Shah functional (1.2.2) via minimizing movements (see for instance [21] or [8]). Let Ω be a bounded open subset of $\mathbb{R}^N$ and consider an initial datum $u_0 \in L^\infty(\Omega)$. For fixed $\delta > 0$ (which is the time discretization parameter) we can define the *δ-approximate evolution* $u_\delta(\cdot) : [0, +\infty) \to SBV(\Omega)$ as the affine interpolation of the discrete function

$$\begin{array}{rcl} \delta\mathbb{N} & \to & SBV(\Omega) \\ \delta i & \mapsto & u_{\delta,i}, \end{array}$$

where $u_{\delta,i}$ is inductively defined as follows: $u_{\delta,0} = u_0$ and $u_{\delta,i}$ is a solution of

$$\min_{v \in SBV(\Omega)} \int_\Omega |\nabla v|^2 \, dx + \mathcal{H}^{n-1}(S_v) + \frac{1}{\delta} \int_\Omega |v - u_{\delta,i-1}|^2 \, dx.$$

The existence of a solution of the problem above is guaranteed by the Ambrosio theorem (see [6]). We call *minimizing movement* for F_0 with initial datum u_0, the set of all functions $v : [0, +\infty) \to SBV(\Omega)$ such that, for a suitable subsequence $\delta_n \downarrow 0$, $u_{\delta_n}(t) \to v(t)$ in $L^2(\Omega)$, for every $t > 0$.

4.2 Technical Estimates

4.2.1 Estimates in smooth domains

Given a hypersurface Γ of class $C^{2,\alpha}$ we can define

$$\Lambda^\alpha(\Gamma) := \sup_{x,y \in \Gamma} \frac{|\nabla_\tau \nu(x) - \nabla_\tau \nu(y)|}{|x-y|^\alpha}, \tag{4.2.1}$$

where ν is a smooth unit normal vectorfield to Γ and ∇_τ denotes the tangential gradient along Γ.

Lemma 4.2.1 *Let Ω be either $\mathbb{R}^N$ or $\mathbb{R}^N_+$ and $\mathcal{A}_0$ be the operator defined in (4.1.4). Then for every $\gamma \in (0, \frac{1}{2})$ there exist two positive constants K_0 and β_0, depending only on the constants of ellipticity λ_0, λ_1, on γ, and on the $W^{1,\infty}$-norm of the matrix A and of the function f, such that for every $\beta \geq \beta_0$ and for every $g \in C^{0,1-\gamma}(\overline{\Omega})$ the solution u of (4.1.6) satisfies*

$$\beta^{\frac{1}{2}-\gamma} \|u - g\|_{C^{0,\gamma}(\overline{\Omega})} \leq K_0 \|g\|_{C^{0,1-\gamma}(\overline{\Omega})}. \tag{4.2.2}$$

PROOF. Recall that $u - g = \mathcal{A}_0 R(\beta, \mathcal{A}_0) g$: in order to obtain the thesis we have to estimate the quantity $\beta^{\frac{1}{2}-\gamma} \|\mathcal{A}_0 R(\beta, \mathcal{A}_0) g\|_{C^{0,\gamma}(\overline{\Omega})}$. By Theorems 4.1.7 and 4.1.9, by (4.1.9) and (4.1.10), there exist $C_0 > 0$, $C_1 > 0$, and $\beta_0 > 0$, depending only on λ_0, λ_1, on γ, and on the $W^{1,\infty}$-norm of A and f, such that

$$\begin{aligned} \|\mathcal{A}_0 R(\beta, \mathcal{A}_0) g\|_{C^{0,\gamma}(\overline{\Omega})} &\leq C_0 \|\mathcal{A}_0 R(\beta, \mathcal{A}_0) g\|_{D(\mathcal{A}_0, \frac{\gamma}{2}, \infty)} \\ &= C_0 \left(\|\mathcal{A}_0 R(\beta, \mathcal{A}_0) g\|_\infty + \sup_{t \geq 2\beta_0 \vee 1} t^{\frac{\gamma}{2}} \|\mathcal{A}_0 R(t, \mathcal{A}_0) \mathcal{A}_0 R(\beta, \mathcal{A}_0) g\|_\infty \right) \end{aligned} \tag{4.2.3}$$

and

$$\sup_{2\beta_0 \vee 1 \leq t} t^{\frac{1-\gamma}{2}} \|\mathcal{A}_0 R(t, \mathcal{A}_0) g\|_\infty \leq C_1 \|g\|_{C^{0,1-\gamma}(\overline{\Omega})}. \tag{4.2.4}$$

We observe that (4.1.7) implies the existence of two positive constants β_0 and C_2, depending in turn on λ_0, λ_1 and on the $W^{1,\infty}$-norm of A and f, such that

$$\|\beta R(\beta, \mathcal{A}_0)\|_{L(L^\infty(\Omega))} \leq C_2, \tag{4.2.5}$$

for every $\beta \geq \beta_0$. Using (4.2.5) and (4.2.4), we can estimate

$$\begin{aligned}
&\sup_{2\beta_0 \vee 1 \leq \beta \leq t} \beta^{\frac{1}{2}-\gamma} t^{\frac{\gamma}{2}} \|\mathcal{A}_0 R(t, \mathcal{A}_0)\mathcal{A}_0 R(\beta, \mathcal{A}_0) g\|_\infty = \qquad (4.2.6)\\
&\quad = \sup_{2\beta_0 \vee 1 \leq \beta \leq t} \beta^{\frac{1}{2}-\gamma} t^{\frac{\gamma}{2}} \|\mathcal{A}_0 R(\beta, \mathcal{A}_0)\mathcal{A}_0 R(t, \mathcal{A}_0) g\|_\infty\\
&\quad = \sup_{2\beta_0 \vee 1 \leq \beta \leq t} \left(\frac{\beta}{t}\right)^{\frac{1}{2}-\gamma} t^{\frac{1-\gamma}{2}} \|(\beta R(\beta, \mathcal{A}_0) - I)\mathcal{A}_0 R(t, \mathcal{A}_0) g\|_\infty\\
&\quad \leq (C_2 + 1) \sup_{2\beta_0 \vee 1 \leq t} t^{\frac{1-\gamma}{2}} \|\mathcal{A}_0 R(t, \mathcal{A}_0) g\|_\infty\\
&\quad \leq (C_2 + 1) C_1 \|g\|_{C^{0,1-\gamma}}, \qquad (4.2.7)
\end{aligned}$$

and analogously

$$\sup_{2\beta_0 \vee 1 \leq t \leq \beta} \beta^{\frac{1}{2}-\gamma} t^{\frac{\gamma}{2}} \|\mathcal{A}_0 R(t, \mathcal{A}_0)\mathcal{A}_0 R(\beta, \mathcal{A}_0) g\|_\infty \leq (C_2 + 1) C_1 \|g\|_{C^{0,1-\gamma}}. \tag{4.2.8}$$

Combining (4.2.7), (4.2.8), (4.2.3), and using again (4.2.4), we finally obtain

$$\begin{aligned}
\sup_{\beta \geq 2\beta_0 \vee 1} \beta^{\frac{1}{2}-\gamma} \|\mathcal{A}_0 R(\beta, \mathcal{A}_0) g\|_{C^{0,\gamma}(\overline{\Omega})} &\leq C_0 \Bigg(\sup_{\beta \geq 2\beta_0 \vee 1} \beta^{\frac{1}{2}-\gamma} \|\mathcal{A}_0 R(\beta, \mathcal{A}_0) g\|_\infty\\
&\qquad + \sup_{\beta, t \geq 2\beta_0 \vee 1} \beta^{\frac{1}{2}-\gamma} t^{\frac{\gamma}{2}} \|\mathcal{A}_0 R(t, \mathcal{A}_0)\mathcal{A}_0 R(\beta, \mathcal{A}_0) g\|_\infty \Bigg)\\
&\leq C_0 (C_1 + C_2 + 1) \|g\|_{C^{0,1-\gamma}}.
\end{aligned}$$

□

The following theorem provides the announced estimate on the Hessian $\nabla^2 u$ of the function u which solves (9); we recall that $(\partial\Omega')_R$ denotes the R-neighbourhood of $\partial\Omega'$.

Theorem 4.2.2 *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain of class $C^{1,1}$.*

i) For every $R > 0$, we can find two positive constants $\beta_0 = \beta_0(R)$ and $K = K(R)$ with the property that if Ω' is a domain belonging to $\mathcal{U}_R(\Omega)$, then for every $\beta \geq \beta_0$ and for every $g \in W^{1,\infty}\left(\Omega \setminus \overline{\Omega'}\right)$ the solution u of

$$\begin{cases} \Delta u = \beta(u - g) & \text{in } \Omega \setminus \overline{\Omega'},\\ \partial_\nu u = 0 & \text{on } \partial\left(\Omega \setminus \overline{\Omega'}\right), \end{cases} \tag{4.2.9}$$

satisfies

$$\|\nabla u\|_\infty + \beta^{-\frac{1}{2}} \|\Delta u\|_\infty + \beta^{\frac{N}{2p}-1} \sup_{x_0 \in \Omega \setminus \overline{\Omega'}} \|\nabla^2 u\|_{L^p\left(B(x_0, \frac{1}{\sqrt{\beta}}) \cap \Omega \setminus \overline{\Omega'}\right)} \leq K \|g\|_{W^{1,\infty}}. \tag{4.2.10}$$

A similar conclusion holds for the solution of

$$\begin{cases} \Delta u = \beta(u-g) & \text{in } \Omega', \\ \partial_\nu u = 0 & \text{on } \partial\Omega'. \end{cases} \tag{4.2.11}$$

ii) For every $R > 0$, for every $\overline{\Lambda} > 0$, and for every $\gamma \in (0,\alpha)$ (with $\alpha \in (0,1)$), there exist two positive constants $\beta_0 = \beta_0(R, \overline{\Lambda}, \gamma)$ and $K = K(R, \overline{\Lambda}, \gamma)$ with the property that if Ω' is a domain of class $C^{2,\alpha}$ belonging to $\mathcal{U}_R(\Omega)$, and $\Lambda^\alpha(\partial\Omega') \leq \overline{\Lambda}$, then, for every $\beta \geq \beta_0$ and for every $g \in W^{1,\infty}\left(\Omega \setminus \overline{\Omega'}\right)$, the solution u of (4.2.9) *satisfies*

$$\|\nabla^2 u\|_{L^\infty((\partial\Omega')_R \cap (\Omega\setminus\overline{\Omega'}))} \leq K\beta^{\frac{1}{2}+\gamma}\|g\|_{W^{1,\infty}}.$$

A similar conclusion holds for the solution of problem (4.2.11).

PROOF. We will prove in details only *ii)*. Fix $p \in \partial\Omega'$. By Proposition 4.1.1 there exist two positive constants η and M_1, the former depending only on R while the latter also on $\Lambda^\alpha(\partial\Omega)$, such that the cylinder $C^\eta := \{x \in \mathbb{R}^{N-1} : |x| < \eta\}\times]-R,R[$ (expressed with respect to a coordinate system belonging to $\mathcal{S}^p_{\Omega'}$), intersected with Ω' is the subgraph of a function f belonging to $C^{2,\alpha}(S)$ ($S := C^\eta \cap \{x_n = 0\}$) and satisfying

$$\|f\|_{C^{2,\alpha}} \leq M_1. \tag{4.2.12}$$

Let $\theta \in C_0^{2,\alpha}(C^\eta)$, $0 \leq \theta \leq 1$ and $\theta \equiv 1$ in $2^{-1}C^\eta$, such that

$$\partial_\nu\theta = 0 \text{ on } \partial\Omega' \cap C^\eta \qquad \text{and} \qquad \|\theta\|_{C^{2,\alpha}} \leq M_2, \tag{4.2.13}$$

where M_2 depends only on R.

Set $v = \theta u$ and note that v solves

$$\begin{cases} \Delta v = \beta(v-h) & \text{in } \Omega' \cap C^\eta, \\ \partial_\nu v = 0 & \text{on } \partial(\Omega' \cap C^\eta), \end{cases}$$

where $h := \theta g + \beta^{-1}(\Delta\theta u + 2\nabla u \nabla\theta)$; finally, denoting by ψ the map

$$\begin{array}{ccc} C^\eta & \to & \psi(C^\eta) \\ (x_1,\ldots,x_{n-1},x_n) & \mapsto & (x_1,\ldots,x_{n-1},x_n - f(x_1,\ldots,x_{n-1})), \end{array}$$

and setting $\tilde v := v \circ \psi^{-1}$ and $\tilde h := h \circ \psi^{-1}$, one sees that (recall that $\tilde v$ and $\tilde h$ have compact support in $\psi(C^\eta)$)

$$\begin{cases} \tilde f \operatorname{div}(\tilde A \nabla \tilde v) = \beta(\tilde v - \tilde h) & \text{in } \mathbb{R}^N_+, \\ \tilde A \nabla\tilde v \cdot \nu = 0 & \text{on } \partial(\mathbb{R}^N_+), \end{cases}$$

where $\tilde A$ and $\tilde f$ are $W^{1,\infty}$-extensions to $\mathbb{R}^N_+$ of the matrix-valued function $A := \left[\frac{D\psi(D\psi)^*}{|\det\psi|}\right] \circ \psi^{-1}$ and of the function $f := |\det\psi| \circ \psi^{-1}$ respectively , satisfying

$$\|\tilde A\|_{W^{1,\infty}(\mathbb{R}^N_+)} = \|A\|_{W^{1,\infty}(\psi(C^\eta))}, \qquad \|\tilde f\|_{W^{1,\infty}(\mathbb{R}^N_+)} = \|f\|_{W^{1,\infty}(\psi(C^\eta))}$$

and

$$\tilde A(x)\xi\cdot\xi \geq \frac{1}{2}|\xi|^2 \;\forall x \in \mathbb{R}^N_+,\; \forall\xi \in \mathbb{R}^N, \qquad \tilde f(x) \geq \frac{1}{2}\;\forall x \in \mathbb{R}^N_+$$

(since $A(0) = I$ and $f(0) = 1$, by (4.2.12), we can choose η depending only on R such that the property above holds true in $\psi(C^\eta)$).

The solution $\tilde{v}$ can be suitably decomposed as $\tilde{v} = \tilde{v}_1 + \tilde{v}_2 + \tilde{v}_3$ in the following way: set $h_1 = \theta g$, $h_2 = \beta^{-1}\nabla u \nabla\theta$, $h_3 := \beta^{-1}\Delta\theta u$, and $\tilde{h}_i = h_i \circ \psi^{-1}$ $(i = 1, 2, 3)$ and choose $\tilde{v}_i$ as the solution of

$$\begin{cases} \operatorname{div}(\tilde{A}\nabla\tilde{v}_i) = \beta(\tilde{v}_i - \tilde{h}_i) & \text{in } \mathbb{R}^N_+, \\ \tilde{A}\nabla\tilde{v}_i \cdot \nu = 0 & \text{on } \partial(\mathbb{R}^N_+), \end{cases}$$

for $i = 1, 2, 3$.

Applying Lemma 4.2.1 we have, for $i = 1, 2, 3$,

$$\beta^{\frac{1}{2}-\gamma}\|\tilde{v}_i - \tilde{h}_i\|_{C^{0,\gamma}} \leq K_0\|g\|_{C^{0,1-\gamma}}, \tag{4.2.14}$$

where K_0 is a constant depending only on γ and on the norm of $\tilde{A}$, therefore (by definition of A and by (4.2.12)) only on γ, R, and $\overline{\Lambda}$.

Estimate for $\tilde{v}_1$. ¿From (4.2.14), (4.2.12), (4.2.13), and the definition of $\tilde{h}_1$ we deduce

$$\beta^{\frac{1}{2}-\gamma}\|\tilde{v}_1 - \tilde{h}_1\|_{C^{0,\gamma}} \leq K_0K_1(\|g\|_{C^{0,1-\gamma}} + \beta^{-1}\|u\|_{C^{0,1-\gamma}}),$$

where K_1 depends only on R, and therefore, since by (4.1.11) and (4.1.12), we have

$$\|u\|_{C^{0,1-\gamma}} \leq K_2\|g\|_{C^{0,1-\gamma}},$$

we obtain

$$\beta^{\frac{1}{2}-\gamma}\|\tilde{v}_1 - \tilde{h}_1\|_{C^{0,\gamma}} \leq K_0K_1K_2\|g\|_{C^{0,1-\gamma}},$$

where K_2 depends only on R. Combining the above inequality with the well known Schauder estimate, we finally obtain

$$\|\nabla^2\tilde{v}_1\|_\infty \leq K_3\|\tilde{f}\operatorname{div}(\tilde{A}\nabla\tilde{v}_1)\|_{C^{0,\gamma}} = K_3\beta^{\frac{1}{2}+\gamma}\beta^{\frac{1}{2}-\gamma}\|\tilde{v}_1 - \tilde{h}_1\|_{C^{0,\gamma}} \leq K_3K_0K_1K_2\beta^{\frac{1}{2}+\gamma}\|g\|_{C^{0,1-\gamma}}, \tag{4.2.15}$$

where K_3 depends only on $C^{1,\gamma}$-norm of A and f and therefore only on R and $\overline{\Lambda}$.

Estimate for $\tilde{v}_2$. Arguing exactly as in the previous point, we obtain

$$\beta^{\frac{1}{2}-\gamma}\|\tilde{v}_2 - \tilde{h}_2\|_{C^{0,\gamma}} \leq K_0K_1\beta^{-1}\|\nabla u\|_{C^{0,1-\gamma}}. \tag{4.2.16}$$

By the Sobolev Embedding Theorem and by estimate (4.2.10) (with $p = \frac{N}{\gamma}$) we have, for $\beta \geq \beta_0$ and for every $x \in \Omega \setminus \overline{\Omega'}$,

$$[\nabla u]_{C^{0,1-\gamma}\left((\Omega\setminus\overline{\Omega'})\cap B\left(x,\beta^{-\frac{1}{2}}\right)\right)} \leq Q_0\|\nabla^2 u\|_{L^{\frac{N}{\gamma}}(\Omega\setminus\overline{\Omega'}\cap B(x,\beta^{-\frac{1}{2}}))} \leq Q_0Q_1\beta^{1-\frac{\gamma}{2}}\|g\|_{W^{1,\infty}}, \tag{4.2.17}$$

and

$$\|\nabla u\|_\infty \leq Q_1\|g\|_{W^{1,\infty}}, \tag{4.2.18}$$

where Q_0 is the constant of Sobolev Embedding and depends only on γ while Q_1 depends only on R. If $|x - y| \geq \beta^{-\frac{1}{2}}$, then, by (4.2.18), we infer

$$\frac{|\nabla u(x) - \nabla u(y)|}{|x - y|^{1-\gamma}} \leq \beta^{\frac{1-\gamma}{2}}2\|\nabla u\|_\infty \leq 2Q_1\beta^{\frac{1-\gamma}{2}}\|g\|_{W^{1,\infty}}. \tag{4.2.19}$$

Combining (4.2.17), (4.2.18), and (4.2.19), we get

$$\|\nabla u\|_{C^{0,1-\gamma}} \leq Q_1(Q_0+1)\beta^{1-\frac{\gamma}{2}}\|g\|_{W^{1,\infty}},$$

and substitution in (4.2.16), together with Schauder's Estimate, yields

$$\|\nabla^2\tilde{v}_2\|_\infty \leq K_3\|\tilde{f}\mathrm{div}(\tilde{A}\nabla\tilde{v}_2)\|_{C^{0,\gamma}} = K_3\beta^{\frac{1}{2}+\gamma}\beta^{\frac{1}{2}-\gamma}\|\tilde{v}_2-\tilde{h}_2\|_{C^{0,\gamma}} \leq K_3K_0K_1Q_1(Q_0+1)\beta^{\frac{1+\gamma}{2}}\|g\|_{W^{1,\infty}}. \tag{4.2.20}$$

Estimate for $\tilde{v}_3$. First we note that, by (4.1.11) and (4.1.12),

$$\|\tilde{v}_3\|_{C^{0,\gamma}} \leq K_4\|\tilde{h}_3\|_{C^{0,\gamma}},$$

with K_4 depending only on R; so we can estimate

$$\begin{aligned}\|\tilde{v}_3-\tilde{h}_3\|_{C^{0,\gamma}} &\leq \|\tilde{v}_3\|_{C^{0,\gamma}}+\|\tilde{h}_3\|_{C^{0,\gamma}}\\ &\leq (K_4+1)\|\tilde{h}_3\|_{C^{0,\gamma}} \leq \beta^{-1}(K_4+1)M\|u\|_{C^{0,\gamma}} \leq \beta^{-1}(K_4+1)K_4M\|g\|_{W^{1,\infty}}.\end{aligned}$$

By Schauder's Estimate we finally obtain,

$$\|\nabla^2\tilde{v}_3\|_\infty \leq K_3(K_4+1)K_4\|g\|_{W^{1,\infty}}. \tag{4.2.21}$$

By (4.2.12) and again (4.2.10) we have

$$\begin{aligned}\|\nabla^2 u\|_{L^\infty(2^{-1}C^\eta)} &\leq C\left(\|\nabla^2\tilde{v}\|_{L^\infty(\mathbb{R}^N_+)}+\|\tilde{v}\|_{W^{1,\infty}(\mathbb{R}^N_+)}\right)\\ &\leq CC'\left(\|\nabla^2\tilde{v}_1\|_{L^\infty(\mathbb{R}^N_+)}+\|\nabla^2\tilde{v}_2\|_{L^\infty(\mathbb{R}^N_+)}+\|\nabla^2\tilde{v}_3\|_{L^\infty(\mathbb{R}^N_+)}+\|g\|_{W^{1,\infty}(\mathbb{R}^N_+)}\right),\end{aligned}$$

where C and C' depend only on R. Using (4.2.15), (4.2.20), and (4.2.21), we finally deduce for $\beta \geq \beta_0 \vee 1$

$$\|\nabla^2 u\|_{L^\infty(2^{-1}C^\eta)} \leq CC'C''\left(\beta^{\frac{1}{2}+\gamma}\|g\|_{W^{1,\infty}(\mathbb{R}^N_+)}+\|g\|_{W^{1,\infty}(\mathbb{R}^N_+)}\right) \leq 2CC'C''\beta^{\frac{1}{2}+\gamma}\|g\|_{W^{1,\infty}(\mathbb{R}^N_+)},$$

where C'' depends only on γ, R, and $\overline{\Lambda}$. Repeating all the above argument for every $p \in \partial\Omega'$ we get *ii)*.

The proof of statement *i)* can be done in a similar way: by localizing, straightening the boundary, and using Theorem 4.1.7. □

4.2.2 Estimates in domains with angles

In the following $\Omega \subset \mathbb{R}^2$ will denote a curvilinear polygon which means that $\partial\Omega$ is given by the union of a finite number of simple connected curves $\tau_1, \ldots, \tau_k$ of class C^3 (up to their endpoints) meeting at corners with different angles $\alpha_j \in (0,\pi)$ $(j=1,\ldots,k)$. Finally we will denote by $\mathcal{S}$ the set of the vertices, i.e. the set of the singular points of $\partial\Omega$.

Proposition 4.2.3 *Let* Ω *be as above. Then there exists* $\beta_0 > 0$ *and* $K > 0$ *such that for every* $\beta > \beta_0$ *and for every* $g \in L^\infty(\Omega)$, *the solution* u *of*

$$\begin{cases}\Delta u = \beta(u-g) & \text{in } \Omega\\ \partial_\nu u = 0 & \text{on } \partial\Omega,\end{cases} \tag{4.2.22}$$

satisfies

$$\|u\|_\infty + \beta^{-\frac{1}{2}}\|\nabla u\|_\infty \leq K\|g\|_\infty. \tag{4.2.23}$$

PROOF. The estimate is proved in [32] for the corresponding Dirichlet problem in a polygon, but one easily sees that the same proof actually works also in our case: indeed the change of boundary conditions does not affect the argument, and the main tool, which is a Calderon-Zygmund type inequality, proved in [33], is actually available also for curvilinear polygon, as shown, for example, in [45]. □

The following proposition is proved in [45].

Proposition 4.2.4 *Let Ω be as above. Then there exists $K > 0$ such that for every $\beta > 0$ $\beta > 0$ and for every $g \in W^{1,\infty}(\Omega)$, the function u solution of (4.2.22), satisfies:*

$$\beta^{\frac{1}{2}}\|u - g\|_\infty \le K\|\nabla g\|_\infty. \tag{4.2.24}$$

Proposition 4.2.5 *Let Ω be as above. Then there exists a positive constant K such that for every $\beta \ge 1$ and for every $g \in W^{1,\infty}(\Omega)$, the solution u of (4.2.22) satisfies:*

$$\|\nabla u\|_\infty \le K\|g\|_{W^{1,\infty}}\beta^{\frac{1}{4}}. \tag{4.2.25}$$

PROOF. Fix $\beta \ge 1$; by Proposition 4.2.3 there exists $\lambda_0 > 0$ independent of β such that, setting $g_\lambda = \frac{\Delta u - \lambda u}{\lambda}$, for $\lambda \ge \lambda_0$ we have

$$\|\nabla u\|_\infty \le K\sqrt{\lambda}\|g_\lambda\|_\infty \le K\sqrt{\lambda}\left(\frac{\|\Delta u\|_\infty}{\lambda} + \|u\|_\infty\right) = K\left(\frac{\|\Delta u\|_\infty}{\sqrt{\lambda}} + \sqrt{\lambda}\|u\|_\infty\right). \tag{4.2.26}$$

Now set $\lambda_{\min} := \frac{\|\Delta u\|_\infty}{\|u\|_\infty}$ and suppose that $\|\Delta u\|_\infty \ge \lambda_0\|g\|_\infty$. It follows that $\lambda_{\min} \ge \lambda_0$ (recall that $\|u\|_\infty \le \|g\|_\infty$): therefore, taking $\lambda = \lambda_{\min}$ in (4.2.26), we obtain

$$\|\nabla u\|_\infty \le 2K\|\Delta u\|_\infty^{\frac{1}{2}}\|u\|_\infty^{\frac{1}{2}}$$

and therefore, by Proposition 4.2.4,

$$\|\nabla u\|_\infty \le 2K\|g\|_\infty^{\frac{1}{2}}\left(K'\beta^{\frac{1}{2}}\|\nabla g\|_\infty\right)^{\frac{1}{2}} \le K''\|g\|_{W^{1,\infty}}\beta^{\frac{1}{4}},$$

where K'' is independent of β.

If $\|\Delta u\|_\infty < \lambda_0\|g\|_\infty$, then we simply use the Calderon-Zygmund type estimate proved in [45] (it is crucial here the hypothesis that all the angles are less than π) to get the existence of a constant $C > 0$, depending only on Ω, such that

$$\|u\|_{W^{2,p}} \le C\|g\|_\infty \le C\|g\|_\infty\beta^{\frac{1}{4}}.$$

We conclude by applying the Sobolev Embedding Theorem. □

Proposition 4.2.6 *Let Ω and $\mathcal{S}$ be as above and let Γ be a simple connected curve in Ω joining two points x_1 and x_2 belonging to $\partial\Omega \setminus \mathcal{S}$. Suppose in addition that Γ is of class C^3 up to x_1 and x_2 (actually it would be enough to take Γ of class C^3 in two neighbourhoods U_1 and U_2 of x_1 and x_2 respectively, and of class $C^{2,\alpha}$, for some $\alpha > 0$, outside those neighbourhood). Let us call Ω_1 and Ω_2 the two connected components of $\Omega \setminus \Gamma$. Finally set $\overline{d} := \operatorname{dist}(x_1, \mathcal{S}) \wedge \operatorname{dist}(x_2, \mathcal{S})$. Then for every*

$\delta < \overline{d}$ and $\gamma \in \left(0, \frac{1}{2}\right)$, there exist two positive constants β_0 and K depending on δ, γ, and Γ, such that, for every $\beta \geq \beta_0$ and for every $g \in W^{1,\infty}(\Omega_i)$ $(i = 1, 2)$, the solution u_i of

$$\begin{cases} \Delta u_i = \beta(u_i - g) & \text{in } \Omega_i \\ \partial_\nu u_i = 0 & \text{on } \partial\Omega_i, \end{cases} \tag{4.2.27}$$

satisfies

$$\|\nabla u_i\|_{L^\infty((\Gamma)_\delta \cap \Omega_i)} + \beta^{-\left(\frac{1}{2}+\gamma\right)} \|\nabla^2 u_i\|_{L^\infty((\Gamma)_\delta \cap \Omega_i)} \leq K \|g\|_{W^{1,\infty}}. \tag{4.2.28}$$

PROOF. The estimate can be performed by a localization procedure as for Theorem 4.2.2 and in fact we have only to look at what happens in a neighbourhood of x_1 and x_2. We will look only at x_1 considered as a point of $\partial\Omega_1$, the other cases being analogous.

First of all, as in [45], we can find a neighbourhood $U = B(x_1, r) \cap \Omega_1$ of x_1, for a suitable $r \leq \delta$, and a diffeomorphism which transforms U into a right angle, more precisely we can construct a one-to-one map $\Phi = (\Phi_1, \Phi_2) : U \cap \Omega_1 \to \Phi(U \cap \Omega_1)$ of class $C^{1,1}$ such that $\nabla\Phi(0,0) = I$ and $\Phi(U) = \{w = (w_1, w_2) \in \mathbb{R}^2 : w_1 > 0,\ w_2 > 0\} \cap V$, where V is a neighbourhood of the origin; we can endow Φ with the further property that if v is a function defined in U with normal derivative vanishing on $\partial\Omega \cap \overline{U}$, then $v \circ \Phi^{-1}$ has normal derivative vanishing on $\Phi(\partial\Omega \cap \overline{U})$ and vice-versa. It follows, in particular, that $\Phi_2(x)$ has the following properties:

- $\Phi_2(x) = 0$ for every $x \in \Gamma \cap U$;
- $\partial_\nu \Phi_2 = 0$ on $\partial\Omega \cap \overline{U}$.

It is easy to see that we can choose a positive convex function f such that

$$f(0) = 0,\ f'(0) = 0, \text{ and } \Delta(f \circ \Phi_2) \geq 0 \text{ on } U' := B(x_1, r') \cap \Omega_1,$$

with $r' \leq r$, if needed. Thus we see that $f \circ \Phi_2$ is a subsolution of

$$\begin{cases} \Delta u = 0 & \text{in } U' \\ u = 0 & \text{on } \Gamma \cap \overline{U'} \\ \partial_\nu u = 0 & \text{on } \partial\Omega \cap \overline{U'} \\ u = f \circ \Phi_2 & \text{on } \partial U' \setminus (\partial\Omega \cup \Gamma) \end{cases}$$

and therefore $f \circ \Phi_2 \leq u$ in U'. By Theorem 5.1.3.1 of [33] (actually it is stated only for polygons, but it can be extended to curvilinear polygons, by the continuity method used, for example, in [45]) and the Sobolev Embedding Theorem, u is in $C^2(\overline{U''})$, where $U'' = B(x_1, r'') \cap \Omega$, with $r'' < r'$. Therefore, since $\nabla(f \circ \Phi_2)(x_1) \neq 0$, and so $\nabla u(x_1) \neq 0$, we can say that the map $\Psi := (v, u)$, where v is the harmonic anticonjugate of u, is conformal in a neighbourhood $U''' := B(x_1, r''') \cap \Omega_1$, with $r''' \leq r''$, it belongs to $C^2(\overline{U'''})$ and $\Psi(U''') = \{w = (w_1, w_2) \in \mathbb{R}^2 : w_1 > 0,\ w_2 > 0\} \cap V$, where V is a neighbourhood of the origin. Now take a cut-off function θ of class C^3 such that $\theta \equiv 1$ on $B(x_1, r'''/2) \cap \Omega_1$, $\theta(x) = 0$ for $|x| \geq (2/3)r'''$, and $\partial_\nu \theta = 0$ on $\partial\Omega \cup \Gamma \cap \overline{U'''}$; note that $v_1 := (\theta u_1) \circ \Psi^{-1}$ solves

$$\begin{cases} A(w)\Delta v_1 = \beta(v_1 - h) & \text{in } \Psi(U''') \\ \partial_\nu v_1 = 0 & \text{on } \{w_1 = 0\} \cup \{w_2 = 0\} \cap \overline{\Psi(U''')}, \end{cases}$$

where $h := [\theta g + \beta^{-1}(\Delta\theta u + 2\nabla u \nabla\theta)] \circ \Psi^{-1}$ and $A := |\nabla u|^2 \circ \Psi^{-1}$.

Moreover we have that $\partial_\nu A = 0$ on $\{w_1 = 0\} \cap \overline{\Psi(U''')}$, indeed, in view of the conformality of Ψ, this is equivalent to say that $\partial_\nu |\nabla u|^2 = 0$ on $\partial\Omega \cap \overline{U'''}$, which is true by the following computation

$$\partial_\nu |\nabla u|^2 = \partial_\nu (\partial_\tau u)^2 = 2\partial_\tau u \partial^2_{\nu\tau} u = 0,$$

where we used the fact that $u \in C^2(\overline{U'''})$ and $\partial_\nu u \equiv 0$ on $\partial\Omega \cap U'''$. As a consequence, the function

$$\tilde{A} := \begin{cases} A(w_1, w_2) & \text{if } w_1 > 0 \text{ and } (w_1, w_2) \in \Psi(U''') \\ A(-w_1, w_2) & \text{if } w_1 < 0 \text{ and } (-w_1, w_2) \in \Psi(U''') \end{cases}$$

turns out to be of class C^1 up to the boundary; in particular it can be extended to a function, still denoted by $\tilde{A}$, belonging to $C^1(\overline{\mathbb{R}^2_+}) \cap W^{1,\infty}(\mathbb{R}^2_+)$. Now it is easy to check that, denoting by $\tilde{v}_1$ and $\tilde{h}$ the extensions by reflection of v_1 and h respectively,

$$\begin{cases} \tilde{A}(w)\Delta \tilde{v}_1 = \beta(\tilde{v}_1 - \tilde{h}) & \text{in } \mathbb{R}^2_+ \\ \partial_\nu \tilde{v}_1 = 0 & \text{on } \{w_2 = 0\}; \end{cases}$$

at this point we are in a position to apply the regularity theorems stated in Subsection 4.2.1, obtaining the desired estimate for $\tilde{v}_1$. To complete the proof we can now proceed exactly as we did for Theorem 4.2.2. □

4.3 The main result

4.3.1 The regular case

Let $\Omega \subset \mathbb{R}^N$ be a bounded open subset of class $C^{1,1}$ and let $\Omega_1 \subset \Omega$ be an open set belonging to $\mathcal{U}_R(\Omega)$ (see (4.1.2)). We set $\Omega_2 := \Omega \setminus \overline{\Omega_1}$, $\Gamma := \partial\Omega_1$, and, for every $x \in \Gamma$, we denote the unit outer normal to $\partial\Omega_1$ at x by $\nu(x)$.

Lemma 4.3.1 *There exist two positive constants c and β_0, depending only on R, such that, for every $\beta \geq \beta_0$, we can find two functions $z_{1,\beta} : \Omega_1 \to \mathbb{R}$ and $z_{2,\beta} : \Omega_2 \to \mathbb{R}$ of class $W^{2,\infty}$ with the following properties:*

i) $\frac{1}{2} \leq z_{i,\beta} \leq 1$ *in* Ω_i*, for* $i = 1, 2$ *and* $z_{2,\beta} \equiv \frac{1}{2}$ *in a neighbourhood of* $\partial\Omega$*;*

ii) $\Delta z_{i,\beta} \leq c\beta z_{i,\beta}$ *in* Ω_i*, for* $= 1, 2$*;*

iii) $z_{1,\beta}(x) = z_{2,\beta}(x) = 1$ *and* $\partial_\nu z_{1,\beta}(x) = -\partial_\nu z_{2,\beta}(x) \geq \sqrt{\beta}$ *for every* $x \in \Gamma$*;*

iv) $\|\nabla z_{i,\beta}\|_\infty \leq c\sqrt{\beta}$ *and* $\|\nabla^2 z_{i,\beta}\|_\infty \leq c\beta$.

Proof. Let us denote the signed distance function from Ω_1 by d and let π the projection on Γ which, by Lemma 4.1.5, is well defined in $(\Gamma)_{\frac{R}{2}}$; we begin by constructing $z_{2,\beta}$. Let $w_\beta : [0, +\infty) \to (0, +\infty)$ be the solution of the following problem

$$\begin{cases} w''_\beta = 16\beta w_\beta, \\ w_\beta(0) = 1/2, \\ w'_\beta(R/2) = 0, \end{cases}$$

which can be explicitly computed and it is given by

$$w_\beta(t) = \frac{1}{2}\frac{e^{-4\sqrt{\beta}\frac{R}{2}}}{e^{4\sqrt{\beta}\frac{R}{2}} + e^{-4\sqrt{\beta}\frac{R}{2}}} e^{4\sqrt{\beta}t} + \frac{1}{2}\frac{e^{4\sqrt{\beta}\frac{R}{2}}}{e^{4\sqrt{\beta}\frac{R}{2}} + e^{-4\sqrt{\beta}\frac{R}{2}}} e^{-4\sqrt{\beta}t}, \tag{4.3.1}$$

and let $\theta : [0, +\infty) \to [0,1]$ be a C^∞ function such that

$$\theta \equiv 1 \quad \text{in } [0, R/4] \qquad \theta \equiv 0 \quad \text{in } [R/2, +\infty) \qquad \text{and} \qquad \|\theta\|_{C^2} \leq c_0, \tag{4.3.2}$$

with c_0 depending only on R. We are now ready to define $z_{2,\beta} : \Omega_2 \to \mathbb{R}$ as

$$z_{2,\beta}(x) := \begin{cases} \theta(d(x))((w_\beta(d(x)) + 1/2) + (1 - \theta(d(x)))1/2 & \text{if } 0 < d(x) \leq R/2, \\ 1/2 & \text{otherwise in } \Omega_2. \end{cases}$$

First of all note that, as it is a convex combination of two functions with range contained in $[1/2, 1]$, $z_{2,\beta}$ itself has range in $[1/2, 1]$. Using the expression in (4.3.1) it is easy to see that there exist $\beta_0 > 1$ and $c_1 > 1$ depending only on R such that

$$w'_\beta(0) \leq -\sqrt{\beta} \qquad |w'_\beta| \leq c_1\sqrt{\beta} \quad \text{in } [0, R/2] \qquad \text{and} \qquad |w''_\beta| \leq c_1\beta \quad \text{in } [0, R/2], \tag{4.3.3}$$

for every $\beta \geq \beta_0$. From the first inequality we obtain immediately *iii)* for $z_{2,\beta}$. Moreover, by (4.3.2) and (4.3.3), we can estimate

$$\begin{aligned} |\nabla z_{2,\beta}| &= |\theta(d) w'_\beta(d) \nabla d + \theta' \nabla d \, w_\beta(d)| \\ &\leq |w'_\beta| + |\theta'| \leq c_1\sqrt{\beta} + c_0 \leq c\sqrt{\beta}, \end{aligned}$$

with c depending only on R. Finally, using again (4.3.2),(4.3.3), and Lemma 4.1.5, we have

$$\begin{aligned} |\nabla^2 z_{2,\beta}| &\leq |w'_\beta||\nabla^2 d| + |w'_\beta||\theta'| + |w''_\beta| + |\theta''| + |\theta'||\nabla^2 d| + |\theta'||w'_\beta| \\ &\leq c_1 c_2\sqrt{\beta} + c_0 c_1\sqrt{\beta} + c_1\beta + c_0 + c_0 c_2 + c_0 c_1\sqrt{\beta}, \end{aligned}$$

where all the constants depend only on R so that we can state the existence of $c > 0$, still depending only on R, such that

$$|\nabla^2 z_{2,\beta}| \leq c\beta \qquad \forall \beta \geq \beta_0.$$

To conclude, we define $z_{1,\beta} : \Omega_1 \to \mathbb{R}$ as follows:

$$z_{1,\beta}(x) := \begin{cases} \theta(-d(x))((w_\beta(-d(x)) + 1/2) + (1 - \theta(-d(x)))1/2 & \text{if } 0 > d(x) \geq -R/2, \\ 1/2 & \text{otherwise in } \Omega_1 \end{cases}$$

and we can conclude as in the other case. □ Our main result is given by the following theorem.

Theorem 4.3.2 *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set of class $C^{1,1}$ and let $\Omega_1 \subset \Omega$ be an open set of class $C^{2,\alpha}$ for some $\alpha \in (0,1)$ and compactly contained in Ω. Let $R > 0$ such that $\Omega_1 \in \mathcal{U}_R(\Omega)$ (see (4.1.2) and Remark 4.1.2)and set $\Gamma := \partial\Omega_1$. Then for every function g belonging $W^{1,\infty}(\Omega \setminus \Gamma)$, discontinuous along Γ (i.e., $S_g = \Gamma$) and such that $g^+(x) - g^-(x) > S > 0$ for every $x \in \Gamma$, there exists $\beta_0 > 0$ depending on R, S, $\Lambda^\alpha(\Gamma)$ (see (4.2.1)), and $\|g\|_{W^{1,\infty}}$, such that for $\beta \geq \beta_0$ the solution u_β of*

$$\begin{cases} \Delta u_\beta = \beta(u_\beta - g) & \text{in } \Omega \setminus \Gamma, \\ \partial_\nu u_\beta = 0 & \text{on } \partial\Omega \cup \Gamma, \end{cases} \tag{4.3.4}$$

is discontinuous along Γ ($S_{u_\beta} = \Gamma$) and it is the unique absolute minimizer of $F_{\beta,g}$ over $SBV(\Omega)$.

PROOF. In the sequel we will denote the signed distance from Ω_1 by d and the projection on Γ by π: by Lemma 4.1.5, the two functions are well defined in $(\Gamma)_{R/2}$. Moreover, in that neighbourhood, d and π are at least of class $W^{2,\infty}$ and $W^{1,\infty}$ respectively.

As announced in the Introduction, the proof will be performed by constructing a calibration ϕ; adopting the notation introduced in Section 1.3, the vectorfield ϕ will be written as

$$\phi(x,z) = (\phi^x(x,z), \phi^z(x,z)),$$

where $\phi^x(x,z)$ is a n-dimensional "horizontal" component, while ϕ^z is the (one dimensional) "vertical" component.

- *Preparation.*

Without loss of generality we can suppose that g^+ coincides with the trace on Γ of g from Ω_1, while g^- is trace from Ω_2. First of all let us choose β', depending only on R, S, and $\|g\|_{W^{1,\infty}}$ and G depending on R, such that, for $\beta \geq \beta'$,

$$\|u_\beta - g\|_\infty \leq \frac{S}{16} \quad \text{and} \quad \sqrt{\beta}\|u_\beta - g\|_\infty \leq G\|g\|_{W^{1,\infty}} \quad i = 1,2: \tag{4.3.5}$$

this is possible by virtue of Theorem 4.2.2.

As a second step, it is convenient to extend the restriction of u_β to Ω_i $(i = 1,2)$ to a $C^{1,1}$ function $u_{i,\beta}$ defined in the whole Ω, in such a way that

$$u_{i,\beta}(x) = u_\beta(x) \text{ for } x \in \Omega_i, \quad \|u_{i,\beta}\|_{W^{2,\infty}} \leq c\|u_\beta\|_{W^{2,\infty}}, \quad \text{and} \quad u_{1,\beta} - u_{2,\beta} \geq \frac{3}{4}S \quad \text{for every } x \in \Omega, \tag{4.3.6}$$

where c is a positive constant depending only on R: this operation can be performed in many ways, for example, to construct $u_{2,\beta}$ we can extend the restriction of u_β to Ω_2 in a neighbourhood of Γ by a standard localization procedure and then we can make a convex combination through a cut-off function with $u_\beta - (3/4)S$ (recall that by definition of S and by (4.3.5), we have $u_\beta^+ - u_\beta^- > (3/4)S$ on Γ); it is clear that all can be done in such a way that the constant c depends only on the "$C^{1,1}$-norm" of Γ and therefore only on R. We require also that

$$\partial_\nu u_{1,\beta} = 0 \qquad \text{on } \partial\Omega.$$

By (4.2.10) and (4.3.6), we can state the existence of two positive constants K and β'' depending only on R such that

$$\|\nabla u_{i,\beta}\|_\infty \leq K\|g\|_{W^{1,\infty}} \qquad i = 1,2, \tag{4.3.7}$$

for every $\beta \geq \beta''$.

Let $\beta''' > 0$ satisfying

$$\frac{1}{6}\sqrt{\beta'''} = \max\left\{4(K\|g\|_{W^{1,\infty}})^2, 64/S^2, \beta', \beta'', \beta_0\right\} + 1, \tag{4.3.8}$$

where β_0 is the constant appearing in Lemma 4.3.1. Let $z_{1,\beta'''}$ and $z_{2,\beta'''}$ be the two functions constructed in Lemma 4.3.1 with $\lambda = \beta'''$ and define v_1, v_2 as follows

$$v_1(x) = \begin{cases} z_{1,\beta'''}(x) & \text{if } x \in \overline{\Omega_1} \\ 2 - z_{2,\beta'''}(x) & \text{if } x \in \Omega_2 \end{cases}$$

and

$$v_2(x) = \begin{cases} z_{2,\beta'''}(x) & \text{if } x \in \overline{\Omega_2} \\ 2 - z_{1,\beta'''}(x) & \text{if } x \in \Omega_1. \end{cases}$$

From the properties of $z_{i,\beta}$ ($i = 1, 2$), as stated in Lemma 4.3.1, it follows immediately that $v_i \in W^{2,\infty}(\Omega)$ and

$$\|\nabla v_i\|_\infty \le K_1\sqrt{\beta'''}, \qquad \|\nabla^2 v_i\|_\infty \le K_1\beta''' \quad i = 1, 2 \tag{4.3.9}$$

where K_1 is a positive constant depending only on R. Note that $\nabla v_1(x) = -\nabla v_2(x)$ for every $x \in \Omega$. We remark also that, for $x \in \Gamma$, by construction,

$$\frac{\nabla v_1(x)}{|\nabla v_1(x)|} = -\frac{\nabla v_2(x)}{|\nabla v_2(x)|} = \nu(x), \tag{4.3.10}$$

where $\nu(x)$ denotes the unit normal vector at x to Γ (outer with respect to Ω_1). We set

$$\tilde{h}(x) = \frac{1}{\sqrt{2}}|\nabla v_1|^{-\frac{1}{2}} = \frac{1}{\sqrt{2}}|\nabla v_2|^{-\frac{1}{2}} \tag{4.3.11}$$

for every $x \in \Gamma$. Moreover, using (4.3.9) and *iii)* of Lemma 4.3.1, we can find a positive constant $D \le R/2$, depending only on R, S, and $\|g\|_{W^{1,\infty}}$, such that

$$|\nabla v_i(x)| \ge \frac{1}{2}, \quad \tilde{h}^2(\pi(x))\frac{|\nabla v_i(x)|}{v_i(x)} < 1 - \frac{25}{32}\frac{1}{\sqrt{3}} \qquad \text{if } |d(x)| \le D,\ i = 1, 2. \tag{4.3.12}$$

Applying *iii)* of Lemma 4.3.1, we get

$$|\nabla v_i(x)|^{\frac{1}{2}} \ge \sqrt[4]{\beta'''} \ge \max\{8/S, 1\} \qquad i = 1, 2, \tag{4.3.13}$$

where the last inequality follows directly from (4.3.8).

Moreover, combining Lemma 4.3.1, (4.3.7), and (4.3.8), we deduce

$$4|\nabla u_{i,\beta}(x)|^2 - \frac{1}{6}|\nabla v_i(x)| \le 4(K\|g\|_{W^{1,\infty}})^2 - \frac{1}{6}\sqrt{\beta'''} \le -1 \tag{4.3.14}$$

and analogously

$$\frac{1}{\sqrt{2}}|\nabla v_i(x)|^{-\frac{1}{2}}\|\nabla u_{i,\beta}\|_\infty < \frac{1}{4\sqrt{3}} \qquad i = 1, 2, \tag{4.3.15}$$

for every $x \in \Gamma$ and for every $\beta \ge \beta'''$.

Let $\varepsilon \in (0, 1)$ be such that

$$6\varepsilon\|\nabla u_{i,\beta}\|_\infty + 4\varepsilon^2\|\nabla v_i\|_\infty \le \frac{1}{4} \quad \text{for } i = 1, 2 \text{ and } \beta \ge \beta'''; \tag{4.3.16}$$

by (4.3.7) and (4.3.9) (and the definition of β''') we see that ε can be chosen depending only R, S and $\|g\|_{W^{1,\infty}}$. By (4.3.11), it follows, for every $x \in \Gamma$,

$$4(\tilde{h})^2\|\nabla v_i\|_\infty \ge 4(\tilde{h})^2(-1)^{i+1}\partial_\nu v_i = 4 \cdot \frac{1}{2} > \frac{1}{4},$$

therefore, by (4.3.16),

$$\varepsilon < \tilde{h}(x) \quad \forall x \in \Gamma. \tag{4.3.17}$$

Let γ be a fixed constant belonging to $(0, \frac{1}{2} \wedge \alpha)$: by applying *ii)* of Theorem 4.2.2, we can find two positive constants β^{IV} and K_2 depending only on R and $\Lambda^\alpha(\Gamma)$ (and γ) such that

$$\|\nabla^2 u_\beta\|_{L^\infty\left((\Gamma)_{\frac{R}{2}}\right)} \leq K_2 \beta^{\frac{1}{2}+\gamma} \|g\|_{W^{1,\infty}}, \tag{4.3.18}$$

for every $\beta \geq \beta^{\text{IV}}$.

We can define, for $\beta > 0$,

$$h_\beta(x) = \begin{cases} \left(\tilde{h}(\pi(x)) - \beta^{\frac{1}{2}+\gamma_1}|d(x)|\right) \vee \varepsilon & \text{if } |d(x)| \leq D \\ \varepsilon & \text{if } |d(x)| > D, \end{cases}$$

where γ_1 is a fixed constant belonging to $(\gamma, \frac{1}{2})$. It is easy to see that there exists $\beta^{\text{V}} > 0$ depending on D (and therefore only on R, S, and $\|g\|_{W^{1,\infty}}$) such that h_β is continuous (in fact Lipschitz) for $\beta > \beta^{\text{V}}$.

Using (4.3.13), (4.3.11), (4.3.9), and Lemma 4.1.5, we have

$$\|\nabla h_\beta\|_\infty \leq C' \left(\frac{1}{\sqrt{2}} \left(\frac{S}{8} + 1\right)^4 \|\nabla^2 v_i\|_\infty \|\nabla \pi\|_\infty + \beta^{\frac{1}{2}+\gamma_1} \|\nabla d\|_\infty \right) \leq K_3 \beta^{\frac{1}{2}+\gamma_1}, \tag{4.3.19}$$

where K_3 is a positive constant depending on R, S, and $\|g\|_{W^{1,\infty}}$.

Finally we set

$$\beta_1 = \max\{\beta'', \beta''', \beta^{\text{IV}}, \beta^{\text{V}}, 1\} \tag{4.3.20}$$

and

$$\mu_i(x) = \frac{\Delta v_i(x)}{v_i(x)}; \tag{4.3.21}$$

notice that by (4.3.9) we get

$$\mu_i(x) \leq \frac{K_1 \beta'''}{v_i(x)} \leq 2K_1\beta_1 \qquad \text{for every } x \in \Omega\,. \tag{4.3.22}$$

- *Definition of the calibration.*

From now on we will assume $\beta \geq \beta_1$. Let us consider the following sets

$$A_i := \{(x,z) \in \Omega \times \mathbb{R} : \; u_{i,\beta}(x) - h_\beta(x) \leq z \leq u_{i,\beta}(x) + h_\beta(x)\}, \quad i = 1, 2. \tag{4.3.23}$$

Since, by (4.3.6), $u_{1,\beta}(x) - u_{2,\beta}(x) \geq \frac{3}{4}S$ everywhere, noting that $h_\beta \leq S/8$ everywhere (by (4.3.13) and (4.3.11)), we see that

$$\operatorname{dist}(A_1, A_2) \geq \frac{S}{2} \qquad \text{for } \beta \geq \beta_1\,.$$

The crucial point is in constructing the vectorfield around the graph of u_β, i.e. in $A_i \cap (\Omega_i \times \mathbb{R})$: here we have to provide a divergence free vectorfield satisfying condition **(d)** of Section 1.3 and such that

$$\begin{aligned} \phi^x(x,z)\cdot \nu_{u_\beta} \geq 0 \qquad & \text{for } x\in\Gamma \text{ and } u_{2,\beta} < z < u_{1,\beta}, \\ \phi^x(x,z)\cdot \nu_{u_\beta} \leq 0 \qquad & \text{for } x\in\Gamma,\ z<u_{2,\beta} \text{ or } z>u_{1,\beta}. \end{aligned}$$

These properties are crucial in order to obtain **(e)** and **(f)** simultaneously.

The remaining work is a matter of finding a suitable extension which preserves all the properties of calibrations.

We start by giving the global definition of the horizontal component ϕ^x:

$$\phi^x(x,z) := \begin{cases} 2\nabla u_{i,\beta} - 2\frac{u_{i,\beta}-z}{v_i}\nabla v_i - \frac{16}{h_\beta}\left((-1)^i(z-u_{i,\beta}) - \frac{h_\beta}{2}\right)^+ \nabla u_{i,\beta} & \text{if } (x,z)\in A_i,\ i=1,2, \\ 0 & \text{otherwise in } \Omega\times\mathbb{R}. \end{cases} \tag{4.3.24}$$

Concerning ϕ^z, we begin by defining it in $A_i \cap \left(\overline{\Omega}_i \times \mathbb{R}\right)$:

$$\phi^z_i(x,z) := \left|\nabla u_\beta - \frac{u_\beta - z}{v^i}\nabla v^i\right|^2 - \beta(z-g)^2 + (\beta-\mu_i)(u_\beta - z)^2 + \Psi_i(x,z)\,\forall (x,z)\in A_i\cap\left(\overline{\Omega}_i\times\mathbb{R}\right), \tag{4.3.25}$$

where μ_i is the function defined in (4.3.21) and

$$\Psi_i(x,z) := \int_{u_{i,\beta}}^z \operatorname{div}_x\left[\frac{16}{h_\beta}\left((-1)^i(t-u_{i,\beta}) - \frac{h_\beta}{2}\right)^+ \nabla u_{i,\beta}\right] dt.$$

Let us clarify that in the formulas above $(\cdot)^+$ stands for $(\cdot)\vee 0$.

For $x\in\Omega_i$ and $-h_\beta < (-1)^i(z-u_\beta) < \frac{h_\beta}{2}$, the field ϕ reduces to

$$\left(2\nabla u_\beta - 2\frac{u_\beta - z}{v_i}\nabla v_i,\ \left|\nabla u_\beta - \frac{u_\beta - z}{v_i}\nabla v_i\right|^2 - \beta(z-g)^2 + (\beta-\mu_i)(u_\beta-z)^2\right) \tag{4.3.26}$$

and so, by some easy computation and using the definition of u_β and μ_i, we have

$$\begin{aligned} \operatorname{div}\phi(x,z) &= 2\left(\Delta u_\beta - \frac{u_\beta - z}{v_i}\Delta v_i\right) - 2\beta(z-g) - 2(\beta-\mu_i)(u_\beta - z) \\ &= 2\beta(u_\beta - g) - 2\mu_i(u_\beta - z) - 2\beta(z-g) - 2(\beta-\mu_i)(u_\beta-z) = 0. \end{aligned}$$

For $x\in\Omega_i$ and $\frac{h_\beta}{2} < (-1)^i(z-u_\beta) < h_\beta$, ϕ is the sum of the field in (4.3.26) and

$$\left(-\frac{16}{h_\beta}\left((-1)^i(z-u_\beta) - \frac{h_\beta}{2}\right)^+ \nabla u_\beta,\ \Psi_i(x,z)\right),$$

which is clearly divergence free by the definition of Ψ_i. Eventually we have,

$$\operatorname{div}\phi = 0 \quad \text{in } (\Omega_i\times\mathbb{R})\cap A_i. \tag{4.3.27}$$

It is time now to extend the definition of ϕ^z. Before writing the explicit expression, we remark that conditions **(a)** and **(b)** of Section 1.3 imply that such extension is essentially unique. More precisely, if $(U_j)_{j=1,\dots,10}$ is the family of all connected components of $(\Omega\times\mathbb{R})\setminus(\partial A_1\cup\partial A_2\cup(\Gamma\times\mathbb{R}))$, it easy to see that ϕ^z is uniquely determined on $(\Omega\setminus\Gamma)\times\mathbb{R} = \cup_{j=1}^{10} U_j$ by (4.3.24), (4.3.25), and the two following necessary conditions:

- $\partial_z \phi^z = -\text{div}_x \phi^x$ in U_j for $j = 1, \dots, 10$ (which ensures condition **(a)** of Section 1.3),
- $\phi^+ \cdot \nu_{\partial U_j} = \phi^- \cdot \nu_{\partial U_j}$ on ∂U_j for every $j = 1, \dots, 10$, where ϕ^+ and ϕ^- denote the traces of ϕ on the two sides of ∂U_j.

The only freedom is in the choice of ϕ^z on ∂U_j according to the condition

$$\phi \cdot \nu_{\partial U_j} = \phi^+ \cdot \nu_{\partial U_j} = \phi^- \cdot \nu_{\partial U_j}.$$

We are now ready to give the complete the definition of ϕ^z; for $(x, z) \in (\Omega_1 \times \mathbb{R}) \setminus A_1$ we define $\phi^z(x, z)$ as follows:

$$\begin{cases} \phi^x(x, u_\beta + h_\beta) \cdot (-\nabla u_\beta - \nabla h_\beta) + \phi^z(x, u_\beta + h_\beta) & \text{if } z > u_\beta + h_\beta, \\ \phi^x(x, u_\beta - h_\beta) \cdot (-\nabla u_\beta + \nabla h_\beta) + \phi^z(x, u_\beta - h_\beta) & \text{if } u_\beta - h_\beta > z \geq u_{2,\beta} + h_\beta, \\ \chi_1(x,z) + \phi^z(x, u_{2,\beta} + h_\beta) + \phi^x(x, u_{2,\beta} + h_\beta) \cdot (\nabla u_{2,\beta} + \nabla h_\beta) & \text{if } u_{2,\beta} + h_\beta > z \geq u_{2,\beta} - h_\beta, \\ \phi^x(x, u_{2,\beta} - h_\beta) \cdot (-\nabla u_{2,\beta} + \nabla h_\beta) + \phi^z(x, u_{2,\beta} - h_\beta) & \text{if } u_{2,\beta} - h_\beta > z, \end{cases} \tag{4.3.28}$$

where

$$\chi_1(x, z) = \int_z^{u_{2,\beta} + h_\beta} \text{div}_x \phi^x(x, t)\, dt.$$

We remark that in first and in the second line we used the definition of ϕ^z already given in (4.3.25), in the third line we used the definition of $\phi^z(x, u_{2,\beta} + h_\beta)$ given in the second one, and finally in the last line we exploited the definition $\phi^z(x, u_{2,\beta} - h_\beta)$ given in the previous one.

Analogously, for $(x, z) \in (\Omega_2 \times \mathbb{R}) \setminus A_2$ we define $\phi^z(x, z)$ as follows:

$$\begin{cases} \phi^x(x, u_\beta - h_\beta) \cdot (-\nabla u_\beta + \nabla h_\beta) + \phi^z(x, u_\beta - h_\beta) & \text{if } z < u_\beta - h_\beta, \\ \phi^x(x, u_\beta + h_\beta) \cdot (-\nabla u_\beta - \nabla h_\beta) + \phi^z(x, u_\beta + h_\beta) & \text{if } u_\beta + h_\beta < z \leq u_{1,\beta} - h_\beta, \\ \chi_2(x,z) + \phi^z(x, u_{1,\beta} - h_\beta) + \phi^x(x, u_{1,\beta} - h_\beta) \cdot (\nabla u_{1,\beta} - \nabla h_\beta) & \text{if } u_{1,\beta} - h_\beta < z \leq u_{1,\beta} + h_\beta, \\ \phi^x(x, u_{1,\beta} + h_\beta) \cdot (-\nabla u_{1,\beta} - \nabla h_\beta) + \phi^z(x, u_{1,\beta} + h_\beta) & \text{if } u_{1,\beta} + h_\beta < z, \end{cases} \tag{4.3.29}$$

where

$$\chi_2(x, z) = \int_z^{u_{1,\beta} - h_\beta} \text{div}_x \phi^x(x, t)\, dt.$$

Finally we set

$$\phi^z(x, z) = 0 \qquad \text{on } (\Gamma \cap \mathbb{R}) \setminus (A_1 \cup A_2);$$

this concludes the definition of ϕ which, by construction (and recalling (4.3.27)), satisfies conditions **(a)** and **(b)** of Section 1.3.

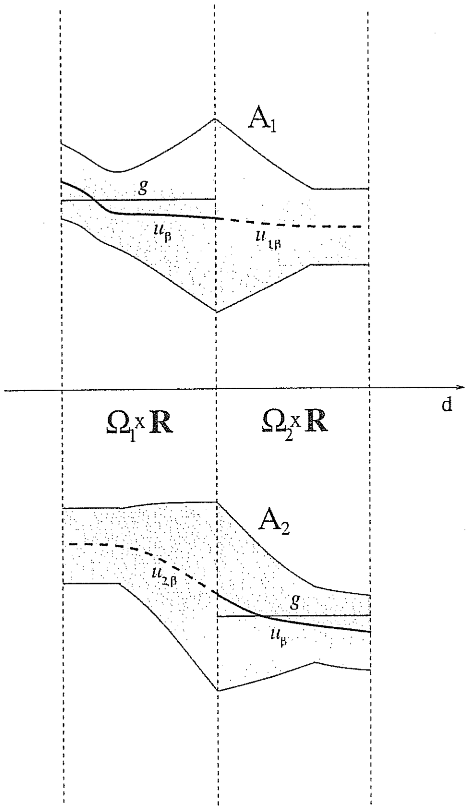

Figure 4.1: A cross section of the sets A_1 and A_2: the vectorfield ϕ is purely vertical out of the shaded regions.

- $\phi^z + \beta(z-g)^2 > |\phi^x|^2/4$ *for almost every* $(x,z) \in \Omega \times \mathbb{R}$ *with* $z \neq u(x)$.

We first prove the condition above in $A_i \cap (\Omega_i \times \mathbb{R})$, and then in the remaining. For $x \in \Omega_i$ and $-h_\beta \leq (-1)^i(z-u_\beta) \leq \frac{h_\beta}{2}$, by (4.3.26), we have that

$$\phi^z + \beta(z-g)^2 = \frac{|\phi^x|^2}{4} + (\beta - \mu_i)(u_\beta - z)^2 > \frac{|\phi^x|^2}{4},$$

so condition **(c)** of Section 1.3 is trivially satisfied, with strict inequality.

For $x \in \Omega_i$ and $\frac{h_\beta}{2} < (-1)^i(z-u_\beta) \leq h_\beta$, using the definition of ϕ we see that (c) is equivalent to

$$\begin{aligned}(1) := (\beta-\mu_i)(u_\beta - z)^2 + \Psi_i(x,z) \quad &> \quad \frac{(16)^2}{4(h_\beta)^2}|\nabla u_\beta|^2([\cdots])^2 \\ &\quad -\frac{16}{h_\beta}[\cdots]\left(\nabla u_\beta - \frac{u_\beta - z}{v_i}\nabla v_i\right)\nabla u_\beta =: (2), \quad (4.3.30)\end{aligned}$$

where we wrote $[\cdots]$ instead of $\left[((-1)^i(z-u_\beta)-\frac{h_\beta}{2})^+\right]$; by (4.3.17), (4.3.5), (4.3.7), and (4.3.19), we have

$$\begin{aligned}\Psi_i(x,z) &\geq \int_{u_\beta}^{z}\left(-\frac{16}{h_\beta}[\cdots]|\Delta u_\beta|-|\nabla u_\beta|\cdot\left|\nabla\left(\frac{16}{h_\beta}[\cdots]\right)\right|\right)\,dt\\ &\geq -\frac{16}{\varepsilon}S^2|\beta(u_\beta-g)|-S|\nabla u_\beta|\left(\frac{16}{\varepsilon}(|\nabla u_\beta|+|\nabla h_\beta|)+\frac{16S}{\varepsilon^2}|\nabla h_\beta|\right)\\ &\geq -\frac{16}{\varepsilon}S^2\sqrt{\beta}G\|g\|_{W^{1,\infty}}-SK\|g\|_{W^{1,\infty}}\left(\frac{16}{\varepsilon}(K\|g\|_{W^{1,\infty}}+K_3\beta^{\frac{1}{2}+\gamma_1})+\frac{16S}{\varepsilon^2}K_3\beta^{\frac{1}{2}+\gamma_1}\right),\end{aligned}$$

therefore, recalling that the all the constants appearing in the last expression depend only on R, S, and $\|g\|_{W^{1,\infty}}$, there exists a positive constant C depending on the same quantities such that

$$\Psi_i(x,z)\geq -C\beta^{\frac{1}{2}+\gamma_1}; \tag{4.3.31}$$

recalling that $|u_\beta - z|\geq \frac{h_\beta}{2}\geq\frac{\varepsilon}{2}$ we finally obtain

$$(1)\geq(\beta-\mu_i)\frac{\varepsilon^2}{4}-C\beta^{\frac{1}{2}+\gamma_1}\qquad\text{for } \beta \text{ large enough}. \tag{4.3.32}$$

Analogously exploiting (4.3.16), (4.3.17), (4.3.7), and (4.3.9), we discover that

$$(2)\leq C_1, \tag{4.3.33}$$

where C_1 depends on R, S, and $\|g\|_{W^{1,\infty}}$; combining (4.3.32), (4.3.33), and recalling (4.3.22), we finally obtain that there exists $b_0>\beta_1$ depending only on R, S, and $\|g\|_{W^{1,\infty}}$ such that (4.3.30) holds true for $\beta\geq b_0$.

Before proceeding let us observe that arguing as for estimate (4.3.33), we easily obtain

$$|\phi_i^x(x,z)|\leq C_2(\|\nabla u_{i,\beta}\|_\infty+\|\nabla v_i\|_\infty)\leq C_3\qquad\text{for every } (x,z)\in A_i, \tag{4.3.34}$$

where C_3 depends only on R, S, and $\|g\|_{W^{1,\infty}}$. For $(x,z)\in(\Omega_i\times\mathbb{R})\cap A_j$ $(i\neq j)$, by the definition of ϕ^x and, by (4.3.17), we have

$$|\mathrm{div}_x\phi^x|\leq C_4\left(\|\nabla^2u_{j,\beta}\|_\infty+\|\nabla^2 v_j\|_\infty+\|\nabla u_{j,\beta}\|_\infty^2+\|\nabla v_j\|_\infty^2+\|\nabla u_{j,\beta}\|_\infty\|\nabla h_\beta\|_\infty\right), \tag{4.3.35}$$

where C_4 depend only on R, and S; by using (4.3.6), (4.3.7), (4.3.9), (4.3.18), (4.3.19), and recalling that $\gamma_1>\gamma$, we deduce, from (4.3.35), that

$$\begin{aligned}|\chi_j| &\leq SC_4\left(C_5\|g\|_{W^{1,\infty}}\beta^{\frac{1}{2}+\gamma}+C_5+K_1\beta_1+(K\|g\|_{W^{1,\infty}})^2+K_1^2\beta_1+K\|g\|_{W^{1,\infty}}K_3\beta^{\frac{1}{2}+\gamma_1}\right)\\ &\leq C_6\beta^{\frac{1}{2}+\gamma_1},\end{aligned} \tag{4.3.36}$$

where C_6 depends only on R, S, $\Lambda^\alpha(\Gamma)$, and $\|g\|_{W^{1,\infty}}$.

Using the definition (4.3.28) for $(x,z)\in(\Omega_1\times\mathbb{R})\cap A_2$, we have

$$\phi^z(x,z)\geq\chi_1-2\|\phi^x\|(\|\nabla u_{2,\beta}\|_\infty+\|\nabla h_\beta\|_\infty)+\phi^z(x,u_\beta-h_\beta), \tag{4.3.37}$$

where, by (4.3.25),

$$\phi^z(x,u_\beta-h_\beta)\geq-\beta(u_\beta-h_\beta-g)^2+\Psi_1(x,u_\beta-h_\beta). \tag{4.3.38}$$

Therefore, for $(x,z) \in (\Omega_1 \times \mathbb{R}) \cap A_2$, combining (4.3.37) and (4.3.38), and using (4.3.7), (4.3.19), (4.3.31), (4.3.34), and (4.3.36), we obtain

$$\begin{aligned} \phi^z(x,z) + \beta(z-g)^2 - \frac{|\phi^x|^2}{4} &\geq \beta\left[(z-g)^2 - (u_\beta - h_\beta - g)^2\right] - |\chi_i| - 2\|\phi^x\|_\infty(\|\nabla h_\beta\|_\infty \\ &\quad + \|\nabla u_{2,\beta}\|_\infty) + \Psi_1(x, u_\beta - h_\beta) - \frac{\|\phi^x\|_\infty^2}{4} \\ &\geq \beta[(7/16)^2 S^2 - (3/16)^2 S^2] - C_5 \beta^{\frac{1}{2}+\gamma_1} \\ &\quad - C_3(K_3 \beta^{\frac{1}{2}+\gamma_1} + K\|g\|_{W^{1,\infty}}) - C\beta^{\frac{1}{2}+\gamma_1} - \frac{(C_3)^2}{4}, \end{aligned}$$

where we used also the fact that that $|z-g| \geq |z-u_\beta| - |u_\beta - g| \geq S/2 - S/16 = (7/16)S$ and, analogously, that $|u_\beta - h_\beta - g| \leq S/16 + S/8 = (3/16)S$ (see (4.3.5)); as $\frac{1}{2} + \gamma_1 < 1$, there exists $b_1 > 0$ depending only on R, S, $\Lambda^\alpha(\Gamma)$, and $\|g\|_{W^{1,\infty}}$ such that

$$\phi^z(x,z) + \beta(z-g)^2 - \frac{|\phi^x|^2}{4} > 0, \tag{4.3.39}$$

for $\beta \geq b_1$ and for $(x,z) \in (\Omega_1 \times \mathbb{R}) \cap A_2$. Analogously we can prove the existence of a constant $b_2 > 0$ depending on the same quantities such that (4.3.39) holds for $\beta \geq b_2$ and for $(x,z) \in (\Omega_2 \times \mathbb{R}) \cap A_1$. Arguing exactly in the same way (in fact exploiting the same estimates), one can finally check that there exists $b_3 > 0$ depending on R, S, $\Lambda^\alpha(\Gamma)$, and $\|g\|_{W^{1,\infty}}$, such that (4.3.39) is true for $(x,z) \in (\Omega \times \mathbb{R}) \setminus (A_1 \cup A_2)$ and $\beta \geq b_3$. If we call $\beta_2 := \max\{b_0, b_1, b_2, b_3\}$ we have that for $\beta \geq \beta_2$ condition **(c)** of Section 1.3 is satisfied for almost every (x,z) in $\Omega \times \mathbb{R}$ with strict inequality if $z \neq u(x)$.

- $\phi(x, u_\beta) = (2\nabla u_\beta, |\nabla u_\beta|^2 - \beta(u_\beta - g)^2)$ *everywhere in* $\Omega \setminus \Gamma$.

Condition **(d)** of Section 1.3 is trivially satisfied, as one can see directly from the definition of ϕ.

- $\displaystyle\int_{u_{2,\beta}(x)}^{u_{1,\beta}(x)} \phi^x(x,t)\,dt = \nu_{u_\beta} = -\nu$ $\mathcal{H}^{N-1}$*-a.e. on* Γ.

By direct computation, for $x \in \Gamma$, we have

$$\int_{u_{2,\beta}}^{u_{1,\beta}} \phi^x(x,z)\,dz = (h_\beta)^2 \frac{\nabla v_2}{v_2} - (h_\beta)^2 \frac{\nabla v_1}{v_1}. \tag{4.3.40}$$

Using (4.3.10), (4.3.11), and the fact that $v_i \equiv 1$ on Γ, we obtain

$$\int_{u_{2,\beta}}^{u_{1,\beta}} \phi^x(x,z)\,dz = +\frac{1}{2}\frac{\nabla v_2}{|\nabla v_2|} - \frac{1}{2}\frac{\nabla v_1}{|\nabla v_1|} = -\nu,$$

so that condition **(e)** of Section 1.3 is satisfied.

- $\displaystyle\left|\int_{t_1}^{t^2} \phi^x(x,z)\,dz\right| \leq 1$ *for every* t_1, $t_2 \in \mathbb{R}$ *and for every* $x \in \Omega$.

It is convenient to introduce the following notation: for every $x \in \Omega$ and for every s, $t \in \mathbb{R}$, we set

$$I(x,[s,t]) := \int_s^t \phi^x(x,z)\,dz,$$

where, with a slight abuse of notation, $[s,t]$ stands for the interval $[s \wedge t, s \vee t]$ positively oriented if $s \le t$, negatively oriented otherwise. We define

$$d_\beta(\pi(x)) := \frac{\tilde{h}(\pi(x)) - \varepsilon}{\beta^{\frac{1}{2}+\gamma_1}}.$$

If $|d(x)| > d_\beta(\pi(x))$, recalling that, by definition, $h_\beta(x) = \varepsilon$ we have

$$\begin{aligned} |I(x,t_1,t_2)| \ \le\ & \int_{u_{1,\beta}-\varepsilon}^{u_{1,\beta}+\varepsilon} \left(2\|\nabla u_{1,\beta}\|_\infty + \tfrac{16}{\varepsilon}\|\nabla u_{1,\beta}\|_\infty \left(u_{1,\beta} - \tfrac{\varepsilon}{2} - z\right)^+ + 4|u_{1,\beta} - z|\|\nabla v_1\|_\infty\right) dz \\ & + \int_{u_{2,\beta}-\varepsilon}^{u_{2,\beta}+\varepsilon} \left(2\|\nabla u_{2,\beta}\|_\infty + \tfrac{16}{\varepsilon}\|\nabla u_{2,\beta}\|_\infty \left(-u_{2,\beta} - \tfrac{\varepsilon}{2} + z\right)^+ + 4|u_{2,\beta} - z|\|\nabla v_2\|_\infty\right) dz \\ \le\ & 6\varepsilon\|\nabla u_{1,\beta}\|_\infty + 4\varepsilon^2\|\nabla v_1\|_\infty + 6\varepsilon\|\nabla u_{2,\beta}\|_\infty + 4\varepsilon^2\|\nabla v_2\|_\infty \le \frac{1}{2}, \end{aligned} \tag{4.3.41}$$

where the last inequality is due to (4.3.16), therefore condition **(f)** is satisfied.

Let us consider now the case of a point x where $|d(x)| \le d_\beta(\pi(x))$. For $x \in \Omega_i \cup \Gamma$ we set

$$n(x) := -\frac{\nabla v_1}{|\nabla v_1|} = \frac{\nabla v_2}{|\nabla v_2|};$$

note that $n(x) = \nu_{u_\beta}(x)$ for every $x \in \Gamma$. Given any vector valued function $\xi : \Omega \to \mathbb{R}^N$, we call $\xi^\perp$ and $\xi^{\|}$ the vector valued functions such that $\xi^\perp(x)$ and $\xi^{\|}(x)$ are equal to the projections of $\xi(x)$ on the orthogonal space and on the space generated by $n(x)$, respectively. We denote the open unit sphere of $\mathbb{R}^N$ centred at the origin by B and the open ball of $\mathbb{R}^N$ centred at the point $-rn(x)$ with radius r, by $A(x,r)$. Finally, for $x \in \Omega$ and $t \in \mathbb{R}$ we introduce the following vector

$$b_i(x,t) := (-1)^i(2(t-u_{i,\beta}) - j_i(x,t))\,(\nabla u_{i,\beta})^{\|},$$

where j_i is defined by

$$j_i(x,t) := \frac{16}{h_\beta}\int_{u_{i,\beta}}^t \left((-1)^i(u_{i,\beta} - z) - \tfrac{h_\beta}{2}\right)^+ dz.$$

CLAIM 1. There exists a positive constant $c_0 > 0$, depending on R, S, $\Lambda^\alpha(\Gamma)$, and $\|g\|_{W^{1,\infty}}$, with the property that for every $x \in \Omega$ such that $|d(x)| \le d_\beta(\pi(x))$, for every $t \in \mathbb{R}$ such that $|t - u_{i,\beta}(x)| \le h_\beta(x)$, and for $\beta \ge c_0$, we have

$$(-1)^{i+1} I(x,[u_{i,\beta},t]) + b_i(x,t) \in A(x,1/3). \tag{4.3.42}$$

A straightforward computation gives

$$\begin{aligned} (-1)^{i+1}I(x,[u_{i,\beta}(x),t]) + b_i(x,t) &= 2(-1)^{i+1}\nabla u_{i,\beta}(t-u_{i,\beta}) + (-1)^i j_i(x,t)\nabla u_{i,\beta} + (-1)^{i+1}\frac{\nabla v_i}{v_i}(t-u_{i,\beta})^2 \\ &\quad + 2(-1)^i(t-u_{i,\beta})\,(\nabla u_{i,\beta})^{\|} + (-1)^{i+1}j_i(x,t)\,(\nabla u_{i,\beta})^{\|} = \\ &= (-1)^{i+1}(2(t-u_{i,\beta}) - j_i(x,t))(\nabla u_{i,\beta})^\perp - \frac{|\nabla v_i|}{v_i}(t-u_{i,\beta})^2 n(x) \end{aligned}$$

and so the claim is equivalent to prove that

$$(2 - j_i(x,t)(t-u_{i,\beta})^{-1})^2[(\nabla u_{i,\beta})^\perp]^2(t-u_{i,\beta})^2 + \frac{|\nabla v_i|^2}{v_i^2}(t-u_{i,\beta})^4 - \frac{2}{3}\frac{|\nabla v_i|}{v_i}(t-u_{i,\beta})^2 < 0;$$

as $0 \le 2 - j_i(x,t)(t-u_{i,\beta})^{-1} \le 2$ everywhere, it is sufficient to prove that

$$(*) := 4\left|(\nabla u_{i,\beta})^{\perp}\right|^2 + h_\beta^2 \frac{|\nabla v_i|^2}{v_i^2} - \frac{2}{3}\frac{|\nabla v_i|}{v_i} < 0. \tag{4.3.43}$$

Since, by (4.3.11), $h_\beta^2 \frac{|\nabla v_i|}{v_i} = \frac{1}{2}$ for $x \in \Gamma$, we can estimate

$$(*) = 4[(\nabla u_{i,\beta})^{\perp}]^2 - \frac{1}{6}\frac{|\nabla v_i|}{v_i} < 0 \qquad \text{on } \Gamma, \tag{4.3.44}$$

where the last inequality follows from (4.3.14). In the following we denote by $\partial_{|d|}$ the differential operator

$$\partial_{|d|} f(x) = \nabla f(x) \cdot \nabla |d(x)|,$$

defined for $x \in (\Gamma)_{\frac{R}{2}} \setminus \Gamma$; noting that, by the estimates (4.3.9), we have

$$\left|\partial_{|d|}\frac{|\nabla v_i|^2}{v_i^2}\right| \le C \qquad \left|\partial_{|d|}\frac{|\nabla v_i|}{v_i}\right| \le C,$$

where C depends only on R, S, and $\|g\|_{W^{1,\infty}}$, and using (4.3.12), (4.3.16), (4.3.18), (4.3.6), and (4.3.19), one sees that

$$\begin{aligned}\partial_{|d|}((*)) &= 8(\nabla u_{i,\beta})^{\perp} \cdot \partial_{|d|}(\nabla u_{i,\beta})^{\perp} + 2\frac{|\nabla v_i|^2}{v_i^2} h_\beta \partial_{|d|} h_\beta + h_\beta^2 \partial_{|d|}\frac{|\nabla v_i|^2}{v_i^2} - \frac{2}{3}\partial_{|d|}\frac{|\nabla v_i|}{v_i} \\ &\le 8cKK_2\beta^{\frac{1}{2}+\gamma}\|g\|_{W^{1,\infty}}^2 + C_1 - \frac{\varepsilon}{2}K_3\beta^{\frac{1}{2}+\gamma_1} + S^2C + C;\end{aligned}$$

as $\gamma_1 > \gamma$ and since all the constants appearing in the last inequality depend only on R, S, $\Lambda^\alpha(\Gamma)$, and $\|g\|_{W^{1,\infty}}$, it is clear that there exists $c_0 > 0$ depending on the same quantities such that $\partial_{|d|}((*)) < 0$, for $x \notin \Gamma$ such that $|d(x)| \le d_\beta(\pi(x))$ and for $\beta \ge c_0$. Therefore, taking into account (4.3.44), (4.3.43) follows immediately: Claim 1 is proved.

CLAIM 2. There exists a positive constant c_1, depending only on R, S, $\Lambda^\alpha(\Gamma)$, and $\|g\|_{W^{1,\infty}}$, such that for every $x \in \Omega$, t_1, $t_2 \in \mathbb{R}$, with $|d(x)| \le d_\beta(\pi(x))$, $|t_1 - u_{1,\beta}| \le h_\beta$, $|t_2 - u_{2,\beta}| \le h_\beta$, and for every $\beta \ge c_1$, we have

$$I(x,[u_{2,\beta},u_{1,\beta}]) - b_1(x,t_1) - b_2(x,t_2) = b(x,t_1,t_2)n(x), \tag{4.3.45}$$

with $b(x,t_1,t_2) < 1$.

First of all observe that for every $x \in \Omega$ $I(x,u_{2,\beta},u_{1,\beta})$ is a vector parallel to $n(x)$, by (4.3.40); it is also clear that

$$\begin{aligned}|I(x,[u_{2,\beta},u_{1,\beta}]) - b_1(x,t) - b_2(x,t)| &\le |I(x,[u_{2,\beta},u_{1,\beta}])| + \left|(\nabla u_{1,\beta})^{\|}\right| |(2h_\beta + j_1(x,h_\beta)) \\ &\quad + \left|(\nabla u_{2,\beta})^{\|}\right| (2h_\beta + j_2(x,h_\beta)) \\ &\le |I(x,[u_{2,\beta},u_{1,\beta}])| + 4h_\beta\left(\left|(\nabla u_{1,\beta})^{\|}\right| + \left|(\nabla u_{2,\beta})^{\|}\right|\right) \\ &=: m_\beta(x);\end{aligned}$$

therefore it is sufficient to prove that $m_\beta(x) < 1$ for $|d(x)| \le d_\beta(\pi(x))$, if β is large enough. Since $m_\beta(x) = |I(x,u_{2,\beta},u_{1,\beta})| = 1$ for every $x \in \Gamma$, it will be enough to show that $\partial_{|d|}m_\beta(x) < 0$ for x

such that $|d(x)| \leq d_\beta(\pi(x))$. We don't enter all the details, indeed arguing as above, that is using (4.3.7), (4.3.18), (4.3.9), and (4.3.19), one easily sees that the derivative of h_β which is negative and of the same order as $\beta^{\frac{1}{2}+\gamma_1}$, dominates the other terms and so there exists a positive constant $c_1 > 0$ depending on R, S, $\Lambda^\alpha(\Gamma)$, and $\|g\|_{W^{1,\infty}}$, such that $\partial_{|d|} m_\beta(x) < 0$ for $\beta \geq c_1$: Claim 2 is proved.

We set $\beta_3 = \max\{c_0, c_1\}$ and we are going to prove that condition **(f)** of Section 1.3 is satisfied for $\beta \geq \beta_3$. We will check the condition only in $\Omega_1 \times \mathbb{R}$: for $\Omega_2 \times \mathbb{R}$ the argument would be analogous. Let $x \in \Omega_1$ and $t_2 < t_1$ two real numbers such that $|t_2 - u_{2,\beta}(x)| \leq h_\beta(x)$ and $|t_1 - u_\beta(x)| \leq h_\beta(x)$; first of all it is easy to see, by explicit computation, that

$$I(x,[t_2,t_1]) \cdot n(x) \geq 0; \tag{4.3.46}$$

recalling that, by Claim 1,

$$I(x,[u_\beta,t_1]) + b_1(x,t_1) \in A(x,1/3) \qquad \text{and} \qquad I(x,[t_2,u_{2,\beta}]) + b_2(x,t) \in A(x,1/3),$$

we have

$$\begin{aligned} I(x,[t_2,t_1]) &= I(x,[u_{2,\beta},u_\beta]) - b_1(x,t_1) - b_2(x,t_2) + I(x,[t_2,u_{2,\beta}]) + b_2(x,t_2) \\ &\quad + I(x,[u_\beta,t_1]) + b_1(x,t_1) \in I(x,[u_{2,\beta},u_\beta]) - b_1(x,t_1) - b_2(x,t_2) + 2A(x,1/3), \end{aligned}$$

therefore, taking into account (4.3.46),

$$I(x,[t_2,t_1]) \in (I(x,[u_{2,\beta},u_\beta]) - b_1(x,t_1) - b_2(x,t_2) + A(x,2/3)) \cap H^+, \tag{4.3.47}$$

where H^+ is the half-space $\{\xi \in \mathbb{R}^N : \xi \cdot n(x) \geq 0\}$. By elementary geometry it is easy to see that $(bn(x) + A(x,r)) \cap H^+ \subset B$ for $b < 1$ and for $r \in (0,1)$, and hence, invoking Claim 2, we get

$$\begin{aligned} I(x,[t_1,t_2]) &\in (I(x,[u_{2,\beta},u_\beta]) - b_1(x,t_1) - b_2(x,t_2) + A(x,2/3)) \cap H^+ \\ &= (b(x,t_1,t_2)n(x) + A(x,2/3)) \cap H^+ \subset B. \end{aligned} \tag{4.3.48}$$

If (x,t_1) and (x,t_2) belong to A_i it is easy to see, by explicitly computing the integral, that

$$|I(x,[t_1,t_2])| \leq h_\beta^2(x)\frac{|\nabla v_i|}{v_i} + \frac{25}{8}h_\beta|\nabla u_{i,\beta}| < 1 - \frac{25}{32}\frac{1}{\sqrt{3}} + \frac{25}{32}\frac{1}{\sqrt{3}} = 1, \tag{4.3.49}$$

where the last inequality follows from (4.3.12), (4.3.15), and (4.3.11) (we recall that for β large enough $d_\beta(\pi(x)) \leq D$, for every x, being D the constant introduced in (4.3.12)).

We now consider the general case. Let $x \in \Omega_1$, t_1, $t_2 \in \mathbb{R}$ with $t_1 < t_2$; since ϕ^x vanishes out of the regions A_1 and A_2, we have

$$I(x,[t_1,t_2]) = I(x,[t_1,t_2] \cap [u_{2,\beta} - h_\beta, u_{2,\beta} + h_\beta]) + I(x,[t_1,t_2] \cap [u_\beta - h_\beta, u_\beta + h_\beta]);$$

by (4.3.49), each integral in the expression above has modulus less than 1, so that if one of the two is vanishing condition **(f)** is verified. If both are non-vanishing, then

$$[t_1,t_2] \cap [u_{2,\beta} - h_\beta, u_\beta + h_\beta] = [s_1,s_2],$$

with $|s_1 - u_{2,\beta}| \leq h_\beta$ and $|s_2 - u_\beta| \leq h_\beta$, so that, again taking into account the fact that ϕ^x vanishes out of the regions A_1 and A_2,

$$|I(x,[t_1,t_2])| = |I(x,[t_1,t_2] \cap [u_{2,\beta} - h_\beta, u_\beta + h_\beta])| = |I(x,[s_1,s_2])| < 1,$$

where the last inequality follows from (4.3.48): condition (f) of Section 1.3 is proved.

Set $\overline{\beta} := \max\{\beta_1, \beta_2, \beta_3\}$; since, by construction, ϕ has vanishing normal component on $\partial\Omega \times \mathbb{R}$, we have that conditions of Section 1.3 are all satisfied for $\beta \geq \overline{\beta}$: the theorem is proved. □

A similar result holds true also if Γ is made up of several connected components, as the following theorem states: we omit the proof, since it is essentially the same as the previous one.

Theorem 4.3.3 *Let Ω as above and let $\Omega_1, \ldots, \Omega_k$ a family of open disjoint subsets belonging of class $C^{2,\alpha}$ and let $R > 0$ such that $\Omega_i \in \mathcal{U}_R(\Omega)$ for $i = 1, \ldots, k$ and $\operatorname{dist}(\Omega_i, \Omega_j) \geq R$ for every $i \neq j$. Set $\Gamma := \partial\Omega_1 \cup \cdots \cup \partial\Omega_k$. Then for every function g belonging $W^{1,\infty}(\Omega \setminus \Gamma)$, discontinuous along Γ (i.e. $S_g = \Gamma$) and such that $g^+(x) - g^-(x) > S > 0$ for every $x \in \Gamma$, there exists $\beta_0 > 0$ depending on R, S, $\Lambda^\alpha(\Gamma)$ (see (4.2.1)), and $\|g\|_{W^{1,\infty}}$, such that for $\beta \geq \beta_0$ the solution u_β of (4.3.4) is discontinuous along Γ ($S_{u_\beta} = \Gamma$) and it is the unique absolute minimizer of $F_{\beta,g}$ over $SBV(\Omega)$.*

Remark 4.3.4 We remark that refining a little the construction, it is possible to improve the result of Theorem 4.3.2 as follows:
there exist $\delta^* > 0$ and $\beta_0 > 0$ such that, for every $\beta \geq \beta_0$ and for every $g \in W^{1,\infty}(\Omega \setminus \Gamma)$, with $\|g\|_{W^{1,\infty}} \leq \beta^{\delta^*}$ and such that $\inf_\Gamma(g^+ - g^-) > S$, the solution u_β of (4.3.4) is the unique absolute minimizer of $F_{\beta,g}$ over $SBV(\Omega)$.
The main difficulty comes from the fact that instead of (4.3.7) we have the weaker estimate

$$\|\nabla u_\beta\|_\infty \leq K\beta^{\delta^*}.$$

Such a difficulty can be overcome replacing, in the construction above, v_1 and v_2 by $v_{1,\beta}$ and $v_{2,\beta}$ defined as

$$v_{1,\beta}(x) = \begin{cases} z_{1,c\beta^{4\delta^*}}(x) & \text{if } x \in \overline{\Omega_1} \\ 2 - z_{2,c\beta^{4\delta^*}}(x) & \text{if } x \in \Omega_2 \end{cases}$$

and

$$v_{2,\beta}(x) = \begin{cases} z_{2,c\beta^{4\delta^*}}(x) & \text{if } x \in \overline{\Omega_2} \\ 2 - z_{1,c\beta^{4\delta^*}}(x) & \text{if } x \in \Omega_1, \end{cases}$$

where $z_{1,c\beta^{4\delta^*}}$ and $z_{2,c\beta^{4\delta^*}}$ are the two functions constructed in Lemma 4.3.1 with $\lambda = c\beta^{4\delta^*}$. One can check that if δ^* is sufficiently small and c sufficiently large, all the conditions of Section 1.3 are still satisfied for β large enough.

Remark 4.3.5 The results we have just proved remains true if we consider Dirichlet boundary conditions instead of Neumann boundary conditions, i.e. if we take as candidate minimizer the solution of the following problem

$$\begin{cases} \Delta u_\beta = \beta(u_\beta - g) & \text{in } \Omega \setminus \Gamma, \\ \partial_\nu u_\beta = 0 & \text{on } \Gamma, \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{4.3.50}$$

where g is as in Theorem 4.3.2. If g is regular enough on $\partial\Omega$, the solutions of (4.3.50) satisfy the same estimate as the solutions of (4.3.4) and therefore, the same construction of Theorem 4.3.2 provides a vectorfield ϕ which fulfill, for β large enough, all the conditions stated in Section 1.3 except (g), of course; this means that for β large enough the solution of (4.3.50) is the unique minimizer of $F_{\beta,g}$ with respect the given Dirichlet boundary conditions.

4.3.2 The two-dimensional case

As stated in the Introduction, in dimension two we are able to treat the case of Ω with piecewise smooth boundary (curvilinear polygon) and of Γ touching (orthogonally) $\partial\Omega$.

Lemma 4.3.6 *Let Ω, $\mathcal{S}$, and Γ be as in Proposition 4.2.6 and denote by Ω_1, Ω_2 the two connected components of $\Omega \setminus \Gamma$. Then for every $\delta > 0$ there exist two positive constants c and β_0 depending on Γ and δ (and Ω of course) such that, for $\beta \geq \beta_0$, we can find two functions $z_{1,\beta} : \Omega_1 \to \mathbb{R}$ and $z_{2,\beta} : \Omega_2 \to \mathbb{R}$ of class $W^{2,\infty}$ with the following properties:*

i) $\frac{1}{2} \leq z_{i,\beta} \leq 1$ *in* Ω_i, *for* $i = 1,2$ *and* $z_{i,\beta} \equiv \frac{1}{2}$ *in* $\Omega \setminus (\Gamma)_\delta$;

ii) $\Delta z_{i,\beta} \leq c\beta z_{i,\beta}$ *in* Ω_i, *for* $= 1,2$;

iii) $z_{1,\beta}(x) = z_{2,\beta}(x) = 1$ *and* $\partial_\nu z_{1,\beta}(x) = -\partial_\nu z_{2,\beta}(x) \geq \sqrt{\beta}$ *for every* $x \in \Gamma$;

iv) $\|\nabla z_{i,\beta}\|_\infty \leq c\sqrt{\beta}$ *and* $\|\nabla^2 z_{i,\beta}\|_\infty \leq c\beta$.

PROOF. Let us denote by x_1 and x_2 the two intersection points of Γ with $\partial\Omega$. If we are able to find a function $\tilde{d}$ belonging to $W^{2,\infty}((\Gamma)_{\delta'} \cap \Omega)$ (for a suitable $\delta' < \operatorname{dist}(\mathcal{S}, \Gamma)$) such that $\tilde{d}$ is vanishing on Γ, positive in $\Omega_2 \cap (\Gamma)_{\delta'}$, negative in $\Omega_1 \cap (\Gamma)_{\delta'}$, satisfying $\partial_\nu \tilde{d} = 0$ on $\partial\Omega \cap \overline{\Gamma^{\delta'}}$ and $\partial_\nu \tilde{d} \neq 0$ on Γ, we are done: indeed we can proceed exactly as in Lemma 4.3.1 using $\tilde{d}$ in place of d. We briefly describe a possible construction: as in Proposition 4.2.6 we can find a neighbourhood U_i of x_i $(i = 1,2)$ and a $C^{1,1}$ function ψ_i vanishing on $\Gamma \cap U_i$, positive in $\Omega_2 \cap U_i$, negative in $\Omega_1 \cap U_i$ and such that $\partial_\nu \psi_i = 0$ on $\partial\Omega \cap U_i$ and $\partial_\nu \psi_i \neq 0$ in $\Gamma \cap U_i$. Now we can define $\tilde{d} := \theta_1\psi_1 + \theta d + \theta_2\psi_2$, where θ_1, θ_2, and θ_3 are suitable positive cut-off functions such that $\theta_1 + \theta_2 + \theta_3 \equiv 1$, while d is the usual signed distance function from Γ, positive in Ω_2 and negative in Ω_1 (it is well defined in $\Gamma^{\delta'}$ if δ' is small enough). □

Theorem 4.3.7 *Let Ω, Ω_1, Ω_2, and Γ as in the previous Lemma and let g be a function in $W^{1,\infty}(\Omega \setminus \Gamma)$, discontinuous along Γ (i.e. $S_g = \Gamma$) and such that $g^+(x) - g^-(x) > S > 0$ for every $x \in \Gamma$. Then there exists $\beta_0 > 0$ depending on Γ, S, and $\|g\|_{W^{1,\infty}}$, such that for $\beta \geq \beta_0$ the solution u_β of* (4.3.4) *is discontinuous along Γ ($S_{u_\beta} = \Gamma$) and it is the unique absolute minimizer of $F_{\beta,g}$ over $SBV(\Omega)$.*

PROOF. As above, let us denote by $\mathcal{S}$ the set of the singular points of $\partial\Omega$. If Ω is regular (i.e. $\mathcal{S} = \emptyset$) we can recycle exactly the same construction of Theorem 4.3.2. If $\mathcal{S} \neq \emptyset$, an additional difficulty is due to the fact that we are not able to prove that $\|\nabla u_\beta\|_{L^\infty(\Omega)} \leq C$ with C independent of β. Since we can perform such an estimate only in a neighbourhood of Γ which does not intersect $\mathcal{S}$, the idea will be to keep the construction of Theorem 4.3.2 in that neighbourhood and to suitably modify it near the singular points in order to exploit estimate (4.2.25).

Denote by γ_1 and γ_2 the two curvilinear edges of Ω containing the intersection points of Γ with $\partial\Omega$ and choose $\delta > 0$ so small that $(\Gamma)_\delta \cap \mathcal{S} = \emptyset$, $(\Gamma)_\delta \cap \partial\Omega = (\Gamma)_\delta \cap (\gamma_1 \cup \gamma_2)$, and d and π are well defined and smooth (according to Lemma 4.1.3) in that neighbourhood.

Let us choose $\beta' > 0$ and $G > 0$ such that, for $\beta \geq \beta'$,

$$\|u_\beta - g\|_{L^\infty(\Omega)} \leq \frac{S}{16} \quad \text{and} \quad \sqrt{\beta}\|u_\beta - g\|_{L^\infty(\Omega)} \leq G\|g\|_{W^{1,\infty}(\Omega)} \quad i = 1,2,: \tag{4.3.51}$$

this is possible by virtue of Proposition 4.2.4.

Again it is convenient to extend the restriction of u_β to Ω_i ($i=1,2$) to a $C^{1,1}$ function $u_{i,\beta}$ defined in the whole Ω, in such a way that

$$u_{i,\beta}(x)=u_\beta(x) \text{ in } \Omega_i, \quad \|u_{i,\beta}\|_{W^{2,\infty}(\Omega)}\le c\|u_\beta\|_{W^{2,\infty}((\Gamma)_\delta\cap\Omega)}, \quad \text{and} \quad u_{1,\beta}-u_{2,\beta}\ge \frac{3}{4}S \quad \text{everywhere,} \tag{4.3.52}$$

where c is a positive constant independent of β. We require also that

$$\partial_\nu u_{i,\beta}=0 \qquad \text{on } \partial\Omega.$$

By (4.2.28) and (4.3.52), and by (4.2.25), we can state the existence of two positive constants K and β'' depending only on Γ, such that

$$\|\nabla u_{i,\beta}\|_{L^\infty(\Omega_{\hat\imath}\cup(\Gamma)_\delta\cap\Omega)}\le K\|g\|_{W^{1,\infty}(\Omega)} \quad \text{for} \quad i=1,2, \qquad \text{and} \qquad \|\nabla u_\beta\|_{L^\infty(\Omega)}\le \beta^{\frac14}K\|g\|_{W^{1,\infty}(\Omega)} \tag{4.3.53}$$

for every $\beta\ge\beta''$ (above and in the sequel, $\hat\imath$ denotes the complement of i, i.e., $\hat\imath$ is such that $i,\hat\imath=\{1,2\}$).

Let $\beta'''>0$ satisfying

$$\frac{1}{6}\sqrt{\beta'''}=\max\left\{4(K\|g\|_{W^{1,\infty}})^2, 64/S^2, \beta', \beta'', \beta_0\right\}+1, \tag{4.3.54}$$

where β_0 is the constant appearing in Lemma 4.3.6 and $z_{1,\beta'''}$, and let $z_{2,\beta'''}$ be the two functions constructed in Lemma 4.3.6 with $\lambda=\beta'''$. We denote by v_1, v_2 the functions defined as follows:

$$v_1(x)=\begin{cases} z_{1,\beta'''}(x) & \text{if } x\in\overline{\Omega_1}\\ 2-z_{2,\beta'''}(x) & \text{if } x\in\Omega_2\end{cases}$$

$$v_2(x)=\begin{cases} z_{2,\beta'''}(x) & \text{if } x\in\overline{\Omega_2}\\ 2-z_{1,\beta'''}(x) & \text{if } x\in\Omega_1,\end{cases}$$

and we choose $0<D<\delta$ in such a way that

$$|\nabla v_i(x)|\ge\frac12, \quad \tilde h^2(\pi(x))\frac{|\nabla v_i|}{v_i}\le 1-\frac{25}{32}\frac{1}{\sqrt3}, \qquad \text{if } |d(x)|\le D,\ i=1,2,$$

where

$$\tilde h(x)=\frac{1}{\sqrt2}|\nabla v_1(x)|^{-\frac12}=\frac{1}{\sqrt2}|\nabla v_2(x)|^{-\frac12} \qquad \forall x\in\Gamma. \tag{4.3.55}$$

Then we choose $\varepsilon\in(0,1)$ in such a way that

$$12\varepsilon\|\nabla u_{i,\beta}\|_{L^\infty(\Omega_{\hat\imath}\cup(\Gamma)_\delta)\cap\Omega}+4\varepsilon^2\|\nabla v_i\|_{L^\infty(\Omega)}<\frac14 \quad \text{for } i=1,2 \text{ and } \beta\ge\beta'''. \tag{4.3.56}$$

Let γ be a fixed constant belonging to $(0,\frac12\wedge\alpha)$: by Proposition 4.2.6, we can find two positive constants β^{IV} and K_2 such that

$$\|\nabla^2 u_\beta\|_{L^\infty((\Gamma)_\delta\cap\Omega)}\le K_2\beta^{\frac12+\gamma}\|g\|_{W^{1,\infty}(\Omega)}, \tag{4.3.57}$$

for every $\beta \geq \beta^{\text{IV}}$.

Now we can define, for $\beta > 0$,

$$h_\beta(x) = \begin{cases} \left(\tilde{h}(\pi(x)) - \beta^{\frac{1}{2}+\gamma_1}|d(x)|\right) \vee \varepsilon & \text{if } |d(x)| \leq \frac{D}{2} \\ f_\beta(|d(x)|) & \text{if } |d(x)| > \frac{D}{2}, \end{cases}$$

where γ_1 is a fixed constant belonging to $(\gamma, \frac{1}{2})$ and $f_\beta : [\delta, +\infty) \to \mathbb{R}$ is the continuous function satisfying

$$f_\beta\left(\frac{D}{2}\right) = \varepsilon \qquad f_\beta(t) \equiv s_\beta := \left(\beta^{\frac{1}{4}} 8K \|g\|_{W^{1,\infty}(\Omega)}\right)^{-1} \quad \text{for } t \geq D \qquad f_\beta \text{ is affine in } \left[\frac{D}{2}, D\right]. \tag{4.3.58}$$

It is easy to see that there exists $\beta^{\text{V}} > 0$ such that h_β is continuous (in fact Lipschitz) for $\beta > \beta^{\text{V}}$.

Finally we introduce a new function $\hat{u}_{i,\beta}$ which is a modification of $u_{i,\beta}$ in the region where we cannot perform a uniform control of the L^∞-norm of its gradient; such a function must satisfy, for $i = 1, 2$:

$$\hat{u}_{i,\beta}(x) = u_{i,\beta}(x) \quad \text{for } x \in \Omega_i \cup (\Gamma)_{\frac{D}{2}} \cap \Omega \qquad \text{and} \qquad \hat{u}_{i,\beta}(x) = g(x) \quad \text{for } x \in \Omega_i \setminus (\Gamma)_D, \tag{4.3.59}$$

$$\|\nabla \hat{u}_{i,\beta}\|_{L^\infty(\Omega)} \leq c(\|\nabla u_\beta\|_{L^\infty((\Gamma)_D)} \vee \|\nabla g\|_{L^\infty(\Omega)}) \qquad \text{and} \qquad \|\hat{u}_{i,\beta} - g\|_{L^\infty(\Omega)} \leq \|u_\beta - g\|_{L^\infty(\Omega)}, \tag{4.3.60}$$

where $c > 0$ is independent of β: a possible construction is given by

$$\hat{u}_{i,\beta}(x) = \theta\left((-1)^i d(x)\right) u_{i,\beta} + \left[1 - \theta\left((-1)^i d(x)\right)\right] g(x),$$

where θ is a smooth positive function such that $\theta(t) = 1$ for $t \leq D/2$ and $\theta(t) = 0$ for $t \geq D$. Now for $\beta \geq \beta_1 := \max\{\beta'', \beta''', \beta^{\text{IV}}, \beta^{\text{V}}\}$ we consider the sets

$$A_i := \{(x, z) \in \Omega \times \mathbb{R} :\ \hat{u}_{i,\beta}(x) - h_\beta(x) \leq z \leq \hat{u}_{i,\beta}(x) + h_\beta(x)\}, \quad i = 1, 2; \tag{4.3.61}$$

setting

$$\hat{h}_\beta(x) := \left[1 + (2/D)(|d(x)| - D/2)^+\right] h_\beta(x),$$

we can define

$$\phi^x(x, z) := \begin{cases} 2\nabla u_{i,\beta} - 2\frac{u_{i,\beta}-z}{v_i}\nabla v_i - \frac{16}{\hat{h}_\beta}\left((-1)^i(z - u_{i,\beta}) - \frac{\hat{h}_\beta}{2}\right)^+ \nabla u_{i,\beta} & \text{if } (x, z) \in A_i, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\phi^z|_{A_i \cap (\overline{\Omega}_i \times \mathbb{R})} := \left|\nabla u_\beta - \frac{u_\beta - z}{v^i}\nabla v^i\right|^2 - \beta(z - g)^2 + (\beta - \mu_i)(u_\beta - z)^2 + \Psi_i,$$

where the functions Ψ_i and μ_i are defined exactly as in the proof of Theorem 4.3.2. At this point, as in the proof of Theorem 4.3.2, the vertical component ϕ^z can be extended to the whole $\Omega \times \mathbb{R}$ in order to satisfy conditions **(a)** and **(b)** of Section 1.3 (we do not rewrite the explicit expression). First of all observe that in $\left((\Gamma)_{\frac{D}{2}} \cap \Omega\right) \times \mathbb{R}$ the definition of ϕ is the same as in Theorem 4.3.2, then, we can state the existence of a constant $\beta_0' > 0$ depending on Γ, S, and $\|g\|_{W^{1,\infty}}$ such that ϕ satisfies **(a)**, **(b)**, **(c)**, **(d)**, **(e)**, **(f)**, and **(g)** of Section 1.3 in $\left((\Gamma)_{\frac{D}{2}} \cap \Omega\right) \times \mathbb{R}$.

From now on we focus our attention on what happens in $\left(\Omega_i \setminus (\Gamma)_{\frac{D}{2}}\right) \times \mathbb{R}$.

Concerning **(d)**, we have only to check that for β large enough the graph of $u_{i,\beta}$ belongs to A_i, but this follows from the fact that, by (4.3.58), A_i contains the s_β-neighbourhood of the graph of $\hat{u}_{i,\beta}$, where s_β is of order $\beta^{-\frac{1}{4}}$, and from the fact that, by (4.3.51) and (4.3.60), it holds

$$\|u_{i,\beta} - \hat{u}_{i,\beta}\|_\infty \leq \|u_{i,\beta} - g\|_\infty + \|\hat{u}_{i,\beta} - g\|_\infty \leq C\beta^{-\frac{1}{2}}.$$

Concerning condition **(c)**, it is clearly satisfied in A_i, then it remains to check, for β large enough, the inequality $\phi^z(x,z) + \beta(z-g)^2 > 0$ holds true outside A_i. For $x \in \left(\Omega_i \setminus (\Gamma)_{\frac{D}{2}}\right) \cap \Gamma_\delta$ such an estimate can be performed using estimates (4.3.53), (4.3.57), (4.3.60) and arguing as in the proof of Theorem 4.3.2. Now let (x,z) belong to $[(\Omega_i \setminus (\Gamma)_D) \times \mathbb{R}] \setminus A_i$ and suppose also that $\hat{u}_{2,\beta}(x) + h_\beta(x) \leq z \leq \hat{u}_{1,\beta}(x) - h_\beta(x)$ (the other cases would be analogous); since $\phi^z(x,z) = \phi(x, \hat{u}_{i,\beta} + (-1)^i h_\beta) \cdot (-\nabla \hat{u}_{i,\beta} + (-1)^{i+1} \nabla h_\beta, 1)$ and observing that $\phi(x, \hat{u}_{i,\beta} + (-1)^i h_\beta)$ reduces to

$$\left(2\nabla u_\beta,\, |\nabla u_\beta|^2 - \beta(z-g)^2 + \beta(u_\beta - z)^2\right),$$

we obtain

$$\begin{aligned} \phi^z(x,z) + \beta(z-g)^2 &\geq -|\nabla u_\beta||\nabla \hat{u}_{i,\beta}| - 2|\nabla u_\beta||\nabla h_\beta| + \beta(u_\beta - z)^2 \\ &\geq -|\nabla u_\beta||\nabla \hat{u}_{i,\beta}| - 2|\nabla u_\beta||\nabla h_\beta| + \beta s_\beta^2; \end{aligned}$$

in the last expression the positive term βs_β^2, which behaves like $\beta^{\frac{1}{2}}$ (see the definition of s_β) dominates the negative ones, indeed these are either bounded or of the same order of $|\nabla u_\beta|$ which is less or equal to the order of $\beta^{\frac{1}{4}}$, thanks to (4.3.53): therefore for β large enough we get the desired inequality.

About condition **(f)** we first observe that if t_1, $t_2 \in \mathbb{R}$ and $x \in \left((\Gamma)_D \setminus (\Gamma)_{\frac{D}{2}}\right) \cap \Omega$ then we obtain

$$\begin{aligned} &\left|\int_{t_1}^{t_2} \phi^x(x,z)\, dz\right| \leq \\ &\leq \sum_{i=1}^{2} \left[\int_{\hat{u}_{i,\beta}-h_\beta}^{\hat{u}_{i,\beta}+h_\beta} \left(2\|\nabla u_{i,\beta}\|_\infty + \tfrac{16}{\hat{h}_\beta}\|\nabla u_{i,\beta}\|_\infty \left((-1)^i(z - u_{i,\beta}) - \tfrac{\hat{h}_\beta}{2}\right)^+ + 4|u_{i,\beta} - z|\|\nabla v_i\|_\infty\right) dz\right] \\ &\leq \sum_{i=1}^{2} \left[4\|\nabla u_{i,\beta}\|_\infty \varepsilon + 4\varepsilon^2 \|\nabla v_i\|_\infty + \tfrac{16}{\hat{h}_\beta}\|\nabla u_{i,\beta}\|_\infty \left[(u_{i,\beta} - \hat{u}_{i,\beta})^2 + \tfrac{\hat{h}_\beta^2}{4}\right]\right] \\ &\leq \sum_{i=1}^{2} \left[4\|\nabla u_{i,\beta}\|_\infty \varepsilon + 4\varepsilon^2 \|\nabla v_i\|_\infty + \frac{16}{s_\beta}\|\nabla u_{i,\beta}\|_\infty (u_{i,\beta} - \hat{u}_{i,\beta})^2 + 8\|\nabla u_{i,\beta}\|_\infty \varepsilon\right] \\ &\leq \sum_{i=1}^{2} \left[12\|\nabla u_{i,\beta}\|_\infty \varepsilon + 4\varepsilon^2\|\nabla v_i\|_\infty + C\beta^{-\frac{3}{4}}\right] \end{aligned}$$

[the fact that $\frac{16}{s_\beta}\|\nabla u_{i,\beta}\|_\infty (u_{i,\beta} - \hat{u}_{i,\beta})^2 \leq C\beta^{-\frac{3}{4}}$ follows from estimates (4.3.53), (4.3.51), (4.3.60), and the definition of s_β]

$$\leq \frac{1}{2},$$

if β is large enough, thanks to (4.3.56).

If $x \in \Omega_i \setminus (\Gamma)_D$ then we can estimate

$$\left| \int_{t_1}^{t_2} \phi^x(x,z)\, dz \right| \leq 2s_\beta \|\nabla u_{1,\beta}\|_\infty + 2s_\beta \|\nabla u_{2,\beta}\|_\infty \leq \frac{1}{2},$$

by (4.3.53) and the definition of s_β. Also condition **(f)** is proved; since, by construction, ϕ has vanishing normal component along $\partial\Omega \times \mathbb{R}$, the theorem is completely proved. □

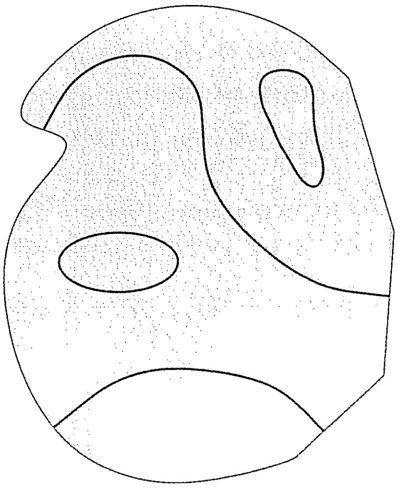

Figure 4.2: An admissible discontinuity set Γ.

Now we can state a theorem which is the analogous of Theorem 4.3.3.

Theorem 4.3.8 *Let Ω as in Proposition 4.2.6 and $\Gamma = \gamma_1 \cup \cdots \cup \gamma_k$ where for every $j = 1, \ldots, k$ γ_j is either a simple, connected, and closed curve of class $C^{2,\alpha}$ contained in Ω or a connected curve with the same regularity outside a neighbourhood of its endpoints (where it is supposed to be of class C^3), which meets orthogonally $\partial\Omega$ in two regular points (see Fig. 4.2); suppose in addition that $\gamma_i \cap \gamma_j = \emptyset$ if $i \neq j$. Then for every $g \in W^{1,\infty}(\Omega \setminus \Gamma)$ discontinuous along Γ and such that $g^+(x) - g^-(x) > S > 0$ for every $x \in \Gamma$, there exists $\beta_0 > 0$ depending on Γ, S, and $\|g\|_{W^{1,\infty}}$, such that for $\beta \geq \beta_0$ the solution u_β of (4.3.4) is discontinuous along Γ ($S_{u_\beta} = \Gamma$) and it is the unique minimizer of $F_{\beta,g}$ over $SBV(\Omega)$.*

Remark 4.3.9 Let Γ be as in Theorem 4.3.8 and let u be harmonic in $\Omega \setminus \Gamma$, discontinuous along Γ with different traces at each point and with vanishing normal derivative on both sides of Γ; then u is a λ-quasi-minimizer (according to Definition 1.2.2) of the homogeneous Mumford-Shah functional F_0, for λ sufficiently large. Indeed, by Remark 4.3.5, u minimizes $F_{\beta,u}$, for β large enough, with respect to its own Dirichlet boundary conditions; this implies that u is a λ-quasi-minimizer for $\lambda = 4\beta\omega_N \|u\|_\infty$.

4.4 Gradient flow for the Mumford-Shah functional

In this section we are going to apply the previous results to the study of the gradient flow of the Mumford-Shah functional by the method of minimizing movements (see Section 4.1) with an initial datum u_0 which is regular outside a regular discontinuity set Γ: we will show that, for an initial interval of time, the discontinuity set does not move while the function evolves according to the heat equation. Our main result is stated in the following theorem.

Theorem 4.4.1 *Let Ω and Γ be either as in Theorem 4.3.3 or as in Theorem 4.3.8. Suppose that u_0 is a function belonging to $W^{2,\infty}(\Omega\setminus\Gamma)$, discontinuous along Γ, and such that $u_0^+(x)-u_0^-(x)>S>0$ for every $x\in\Gamma$ and $\partial_\nu u_0=0$ on $\partial\Omega\cup\Gamma$. Then there exists $T>0$ such that the minimizing movement for the Mumford-Shah functional is unique in $[0,T]$ and it is given by the function $u(x,t)$ satisfying*

$$S_{u(\cdot,t)}=\Gamma \qquad \forall t\in[0,T],$$

and

$$\begin{cases} \partial_t u=\Delta u & \text{in } (\Omega\setminus\Gamma)\times[0,T],\\ \partial_\nu u=0 & \text{on } \partial(\Omega\setminus\Gamma)\times[0,T],\\ u(x,0)=u_0(x) & \text{in } \Omega\setminus\Gamma. \end{cases}$$

Proof. For fixed $\delta>0$, let $v_\delta(t)$ be the affine interpolation of the discrete function

$$\begin{aligned} v_\delta:\delta\mathbb{N} &\to H^1(\Omega\setminus\Gamma)\\ v_\delta(\delta i) &\mapsto v_{\delta,i}, \end{aligned}$$

where $v_{\delta,i}$ is inductively defined as follows:

$$\begin{cases} v_{\delta,0}=u_0,\\ \\ v_{\delta,i} \text{ is the unique solution of}\\ \qquad \min\limits_{z\in H^1(\Omega\setminus\Gamma)}\displaystyle\int_{\Omega\setminus\Gamma}|\nabla z|^2\,dx+\frac{1}{\delta}\int_{\Omega\setminus\Gamma}|z-v_{\delta,i-1}|^2\,dx. \end{cases} \tag{4.4.1}$$

Claim 1. For every $T>0$, we have that

$$v_\delta\to v \qquad \text{in } L^\infty([0,T];L^\infty(\Omega\setminus\Gamma)) \text{ as } \delta\to 0,$$

where v is the solution of

$$\begin{cases} \partial_t v=\Delta v & \text{in } (\Omega\setminus\Gamma)\times[0,T],\\ \partial_\nu v=0 & \text{on } \partial(\Omega\setminus\Gamma)\times[0,T],\\ v(x,0)=u_0(x) & \text{in } \Omega\setminus\Gamma. \end{cases} \tag{4.4.2}$$

We will show that the functions $(v_\delta)_{\delta>0}$ are equibounded in $C^{0,1}([0,T];L^\infty(\Omega\setminus\Gamma))$: since it is well known that, for every $T>0$, $v_\delta\to v$ in $L^\infty([0,T];L^2(\Omega\setminus\Gamma))$ as $\delta\to 0$ (see for example [8]), the a priori estimate in the $C^{0,1}$-norm (via Ascoli-Arzelà Theorem) will give the thesis of Claim 1. First of all we will show that

$$\|\Delta v_{i,\delta}\|_\infty\le\|\Delta u_0\|_\infty \qquad \forall\delta>0,\ \forall i\in\mathbb{N}. \tag{4.4.3}$$

We first prove it for $v_{\delta,1}$: if $\varepsilon\ge\|\Delta u_0\|_\infty/\beta$, then $v_1:=u_0+\varepsilon$ and $v_2:=u_0-\varepsilon$ satisfy:

$$\begin{cases} \Delta v_1\le\beta(v_1-u_0) & \text{in } \Omega\setminus\Gamma\\ \partial_\nu v_1=0 & \text{on } \partial(\Omega\setminus\Gamma), \end{cases} \begin{cases} \Delta v_2\ge\beta(v_2-u_0) & \text{in } \Omega\setminus\Gamma\\ \partial_\nu v_2=0 & \text{on } \partial(\Omega\setminus\Gamma), \end{cases}$$

that is v_1 and v_2 are a supersolution and a subsolution respectively of the problem solved by $v_{1,\delta}$. This implies that

$$\|v_{1,\delta} - u_0\|_\infty \leq \frac{\|\Delta u_0\|_\infty}{\beta}$$

which is equivalent to

$$\|\Delta v_{1,\delta}\|_\infty \leq \|\Delta u_0\|_\infty.$$

By the same argument we can prove that

$$\|\Delta v_{i,\delta}\|_\infty \leq \|\Delta v_{i-1,\delta}\|_\infty \qquad \forall i \geq 2$$

and so (4.4.3) follows by induction on i.

By a standard truncation argument, one can prove also that

$$\|v_{\delta,i}\|_\infty \leq \|u_0\|_\infty \qquad \forall \delta > 0,\ \forall i \in \mathbb{N}. \tag{4.4.4}$$

Then for s, $t > 0$, using Claim 1, we can estimate

$$\|v_\delta(t) - v_\delta(s)\|_\infty \leq \int_s^t \|(v_\delta)'(\xi)\|_\infty \, d\xi \leq \int_s^t \sup_i \|\Delta v_{\delta,i}\|_\infty \, d\xi \leq \|\Delta u_0\|_\infty |t-s|;$$

this, together with (4.4.4) concludes the proof of Claim 1.

As a consequence of (4.4.3), by the well-known Calderon-Zygmund estimates, we get the existence of a constant C such that

$$\|\nabla v_{i,\delta}\|_\infty \leq C\|\Delta v_{i,\delta}\|_\infty \leq C\|\Delta u_0\|_\infty \qquad \forall \delta > 0,\ \forall i \in \mathbb{N}. \tag{4.4.5}$$

It is well known (see, for example, [35]) that

$$v(t) \to u_0 \qquad \text{in } L^\infty(\Omega \setminus \Gamma) \text{ as } t \to 0^+;$$

therefore, by our assumption on u_0, for every $0 < c < S$ we can find $T_c > 0$ such that

$$\inf_{x \in \Gamma} |v^+(x,t) - v^-(x,t)| > c \qquad \forall t \in [0, T_c], \tag{4.4.6}$$

and therefore, by Claim 1, we can choose $\delta_0 > 0$ such that

$$\inf_{x \in \Gamma} |v_\delta^+(t,x) - v_\delta^-(t,x)| > \frac{c}{2} \qquad \forall t \in [0, T_c],\ \forall \delta \leq \delta_0. \tag{4.4.7}$$

We recall now that, by Theorems 4.3.3 and 4.3.8, there exists $\overline{\beta}$ such that, for every function $g \in W^{2,\infty}(\Omega \setminus \Gamma)$ satisfying

$$\|\nabla g\|_\infty \leq C\|\Delta u_0\|_\infty \qquad \inf_{x \in \Gamma} |g^+(x) - g^-(x)| > \frac{c}{2}, \tag{4.4.8}$$

where C is the constant appearing in (4.4.5), and for every $\beta \geq \overline{\beta}$, the function $u_{\beta,g}$ solution of (4.3.4), minimizes the functional $F_{\beta,g}$ over $SBV(\Omega)$.

CLAIM 2. For every $\delta \leq \delta_0 \wedge (\overline{\beta})^{-1}$ the δ-approximate evolution $u_\delta(t)$ (see the end of Section 4.1 for the definition) coincides in the interval $[0, T_c]$ with the function $v_\delta(t)$.

Clearly it is enough to show that

$$v_{\delta,i} = u_{\delta,i} \qquad \text{for } i = 0, \ldots, \left[\frac{T_c}{\delta}\right],$$

and this can be done by induction on i: indeed for $i = 0$ the identity is trivial, and suppose it true for $i-1$ (for $i \leq \left[\frac{T_c}{\delta}\right]$); this means in particular (by (4.4.5) and by (4.4.7)) that $g = u_{\delta,i-1}$ satisfies (4.4.8) and so, being $\frac{1}{\delta} > \overline{\beta}$, we have

$$u_{\delta,i} = u_{\frac{1}{\delta},u_{\delta,i-1}} = v_{\delta,i}.$$

Claim 2 is proved and the thesis of the theorem is now evident. □

Part II

Approximation of solutions

Chapter 5

Approximation via singular perturbations

In the second part of the thesis we deal with the variational approximation of free-discontinuity problems. The main goal of this chapter is to show that a wide class of singularly perturbed functionals generates, as Γ-limit, a functional related to a free-discontinuity problem and to provide a complete description of all possible Γ-limits.

5.1 The main convergence result in the one-dimensional case

Let $f_n : [0,+\infty) \to [0,+\infty)$ be a family of continuous non-decreasing functions and let r_n be an infinitesimal sequence of positive real numbers. For any open bounded subset $I \subset \mathbb{R}$, we define

$$F_n(u) := \begin{cases} \displaystyle\int_I f_n(|u'|)\,dx + (r_n)^3 \int_I |u''|^2\,dx & \text{if } u \in W^{2,2}(I), \\ +\infty & \text{otherwise in } L^1(I). \end{cases} \tag{5.1.1}$$

Moreover, given two functions $b,\ g : [0,+\infty) \to [0,+\infty)$, we set

$$\mathcal{F}_{b,g}(u) := \begin{cases} \displaystyle\int_I g(|u'|)\,dx + \sum_{S_u} \varphi(u^+ - u^-) & \text{if } u \in SBV(I), \\ +\infty & \text{otherwise in } L^1(I), \end{cases} \tag{5.1.2}$$

where

$$\varphi(z) := \inf_{\eta>0} \inf\left\{ \int_0^\eta b(|u'|)\,dx + \int_0^\eta |u''|^2\,dx : \ u \in W^{2,2}(0,\eta), \right.$$
$$\left. u(0) = 0,\ u(\eta) = z, u'(0) = u'(\eta) = 0 \right\}. \tag{5.1.3}$$

If g is convex and b is convex or concave or convex-concave then we can finally define

$$F_{b,g}(u) := \begin{cases} \displaystyle\int_I g_1(|u'|)\,dx + \sum_{S_u} \varphi_1(u^+ - u^-) + (g^\infty(1) \wedge b^0(1))|D^c u| & \text{if } u \in BV(I), \\ +\infty & \text{otherwise in } L^1(I), \end{cases} \tag{5.1.4}$$

where $g_1 := g \triangle b^0 = [g \wedge (b^0 + g(0))]^{**}$ and $\varphi_1 := \varphi \triangle g^\infty = \mathrm{sub}\,(\varphi \wedge g^\infty)$ (g^∞ and b^0 are the recession functions of g and b respectively defined in Subsection 1.1.2).

Remark 5.1.1 Note that if $g^\infty(1) = b^0(1) = +\infty$ then $F_{b,g} = \mathcal{F}_{b,g}(u)$.

Our main result is stated in the following theorem.

Theorem 5.1.2 *Let f_n and r_n be as above and satisfying in addition the following hypotheses:*

i) there exists a non-decreasing function $g : [0, +\infty) \to [0, +\infty)$ such that

$$f_n(t) \to g(t) \qquad \forall t \in [0, +\infty); \tag{5.1.5}$$

ii) there exists a non-decreasing and continuous function $b : (0, +\infty) \to (0, +\infty)$ such that

$$r_n f_n\left(\frac{t}{r_n}\right) \to b(t) \qquad \forall t > 0. \tag{5.1.6}$$

Then

$$\Gamma\text{-}\limsup_{n\to\infty} F_n \le \overline{\mathcal{F}_{b,g}}(u),$$

with respect to the $L^1(I)$-convergence, where F_n are the functionals defined in (5.1.1) *while $\overline{\mathcal{F}_{b,g}}$ denotes the L^1-relaxation of the functional $\mathcal{F}_{b,g}$ introduced in* (5.1.2). *If in addition we assume*

iii) one of the two following structure conditions holds true:

st1) *f_n is convex for every $n \in \mathbb{N}$;*

st2) *there exists a sequence $(x_n) \subset (0, +\infty)$ such that $x_n \to +\infty$ and f_n is convex in $[0, x_n]$ and concave in $[x_n, +\infty)$,*

then

$$\Gamma\text{-}\lim_{n\to\infty} F_n = \overline{\mathcal{F}_{b,g}} = F_{b,g},$$

where $F_{b,g}$ is the functional defined in (5.1.4). *Finally, every sequence u_n such that $\sup_n(F_n(u_n) + \|u_n\|_1) < +\infty$ is strongly precompact in L^p for every $p \ge 1$.*

Remark 5.1.3 If *iii)* holds then g is convex; concerning b, assumption **st1)** implies that it is in turn convex while **st2)** implies that it is either concave or convex-concave. In all these cases the recession function b^0 is well defined. We finally point out that the equality $\overline{\mathcal{F}_{g,b}} = F_{b,g}$ stated in the last part of the theorem is a consequence of Theorem 1.1.3 and of the equality $\varphi^0 = b^0$ which will be proved in the sequel (see Lemma 5.1.6).

Remark 5.1.4 If **st2)** holds with $\limsup_{n\to\infty} x_n r_n = c > 0$, then

$$b(t) \ge g^\infty(1)t = g^\infty(t) \qquad \forall t \in [0, c], \tag{5.1.7}$$

so that, in particular, $b^0(1) \ge g^\infty(1)$. Passing to a subsequence, if needed, we can suppose that $\lim_n x_n r_n = c$; since the functions f_n pointwise converge to g and becomes convex in larger and larger intervals, we have

$$g'(t-) \le \liminf_{n\to\infty} f_n'(t-) \le \limsup_{n\to\infty} f_n'(t+) \le g'(t+), \tag{5.1.8}$$

for every $t > 0$. Suppose that $g^\infty(1) \neq 0$, otherwise the statement is trivial and let $y_k \to +\infty$ such that $g'(y_k) \to g^\infty(1)$. For a fixed k and $\delta \in (0,1)$ there exists $n_{k,\delta}$ such that $x_n \geq y_k$, $f_n(t) \geq g(y_k)/2$, and $f_n'(t) \geq (1-\delta)g'(y_k)$ for every $n \geq n_{k,\delta}$, so that, by convexity,

$$f_n(t) \geq \frac{g(y_k)}{2} + (1-\delta)g'(y_k)(t-y_k) \qquad \forall t \in [y_k, x_n]\ \forall n \geq n_{k,\delta}. \tag{5.1.9}$$

Fix $t < c$; then, by (5.1.9),

$$r_n f_n\left(\frac{t}{r_n}\right) \geq r_n \frac{g(y_k)}{2} + r_n(1-\delta)g'(y_k)\left(\frac{t}{r_n} - y_k\right),$$

for every $n \geq \overline{n}$, where $\overline{n} \geq n_{k,\delta}$ is such that $y_k \leq t/r_n \leq x_n$ for every $n \geq \overline{n}$. Passing to the limit in n in the above inequality and taking into account (5.1.6) we obtain $b(t) \geq (1-\delta)g'(y_k)t$, from which (5.1.7) follows letting k tend to infinity and then δ tend to zero. With the same proof we see that **st1)** implies that $b(t) \geq g^\infty(t)$ for every $t \geq 0$.

Before giving the proof of the theorem we need to state and prove some preparatory lemmas.

Lemma 5.1.5 *Suppose that $b(t) = Mt$ for some $M > 0$ and let φ be the function defined in* (5.1.3). *Then $\varphi(z) = Mz$ for every $z > 0$.*

PROOF. Fix $z > 0$ and let (v,η) be an admissible pair for problem (5.1.3), then

$$\int_0^\eta M|v'|\,dt + \int_0^\eta |v''|^2\,dt \geq \int_0^\eta M|v'|\,dt \geq Mz,$$

and therefore $\varphi(z) \geq Mz$. Let us prove now the reverse inequality. To this aim we construct a sequence of admissible pairs (v_n, η_n) by setting $\eta_n := nz$ and

$$v_n(t) := \begin{cases} \frac{\phi(t)}{n} & \text{if } t \in [0,1) \\ \frac{1}{n} + \frac{1}{n}(t-1) & \text{if } t \in [1, nz-1) \\ z - \frac{\phi(nz-t)}{n} & \text{if } t \in [nz-1, nz], \end{cases}$$

where ϕ is a function belonging to $C^2([0,1])$ and satisfying $\phi(0) = \phi'(0) = 0$, $\phi(1) = \phi'(1) = 1$. We can now estimate

$$\begin{aligned} \varphi(z) &\leq \int_0^{\eta_n} M|v_n'|\,dt + \int_0^{\eta_n} |v_n''|^2\,dt = 2\frac{M}{n}\int_0^1 |\phi'|\,dt + M\frac{nz-2}{n} + 2\frac{1}{n}\int_0^1 |\phi''|^2\,dt \\ &= Mz + O\left(\frac{1}{n}\right) \end{aligned}$$

and therefore, letting $n \to \infty$, we obtain $\varphi(z) \leq Mz$. □

Lemma 5.1.6 *Let b as in Remark 5.1.3. Then the function $\varphi : [0,+\infty) \to [0,+\infty)$ defined in* (5.1.3) *is continuous, non-decreasing, subadditive, and $\varphi^0(1) = b^0(1)$.*

PROOF. The first three properties are easy; let us prove only the last one. We begin with the case

$$b^0(1) = +\infty. \tag{5.1.10}$$

Claim. For every $\varepsilon > 0$ there exists $\delta > 0$ with the following property: if $z < \delta$ and if (η, u) is an *admissible pair* for problem (5.1.3) satisfying

$$\int_0^\eta b(|u'|)\,dx + \int_0^\eta |u''|^2\,dx < (1+\varepsilon)\varphi(z), \tag{5.1.11}$$

then $|u'| \leq \varepsilon$ in $(0, \eta)$.

Suppose by contradiction the existence of $\varepsilon > 0$ and of a sequence $\delta_n \downarrow 0$ such that, for every $n \in \mathbb{N}$, there exist $z_n < \delta_n$ and (η_n, u_n) which satisfies

$$\int_0^{\eta_n} b(|u_n'|)\,dx + \int_0^{\eta_n} |u_n''|^2\,dx < (1+\varepsilon)\varphi(z_n), \tag{5.1.12}$$

and

$$\|u_n'\|_{L^\infty(0,\eta_n)} > \varepsilon. \tag{5.1.13}$$

Note that we can suppose $\eta_n > 1$ for every n (if needed u_n can be extended outside the original interval as the constant function z_n); using Hölder's Inequality we can estimate, for every $x, y \in (0, \eta_n)$

$$|u_n'(x) - u_n'(y)| \leq \int_x^y |u_n''|\,dt \leq \sqrt{|x-y|}\left(\int_0^{\eta_n} |u_n''|^2\,dt\right)^{\frac{1}{2}} \leq C\sqrt{|x-y|},$$

where $C > 0$ is independent of n; by the above estimate and by (5.1.13) we can state the existence of an interval $I_n \subseteq (0, \eta_n)$ such that $|I_n| \geq C'$, with C' independent of n, and $|u_n'| \geq \varepsilon/2$ in I_n. As a consequence we deduce

$$\int_0^\eta b(|u_n'|)\,dx + \int_0^\eta |u_n''|^2\,dx \geq \int_{I_n} b(|u_n'|)\,dx \geq b\left(\frac{\varepsilon}{2}\right) C'$$

which is in contradiction with (5.1.12) since $\varphi(z_n)$, by continuity, tends to 0. The claim is proved. Given $M > 0$, thanks to (5.1.10), we can choose ε such that $b(t)/t \geq M$ for every $t \in (0, \varepsilon]$; if $\delta > 0$ is as in the above Claim, for $0 \leq z < \delta$ we can estimate

$$\begin{aligned}(1+\varepsilon)\varphi(z) &\geq \int_0^\eta b(|u'|)\,dx + \int_0^\eta |u''|^2\,dx \\ &\geq \int_0^\eta \frac{b(|u'|)}{|u'|}|u'|\,dx \geq M\int_0^\eta |u'|\,dx \geq Mz,\end{aligned}$$

where (η, u) is an admissible pair satisfying (5.1.11); this concludes the proof when (5.1.10) holds. Let us suppose now that

$$b^0(1) = C < +\infty. \tag{5.1.14}$$

Fix $\sigma > 0$ and choose $\varepsilon_\sigma > 0$ such that $b(t) < (C+\sigma)t$ for any $t \in (0, \varepsilon_\sigma)$. Consider the sequence of admissible pairs (η_n, v_n) constructed in the previous lemma; for n large we have $\|v_n'\|_\infty \leq \varepsilon_\sigma$ and therefore

$$\begin{aligned}\varphi(z) &\leq \int_0^{\eta_n} b(|v_n'|)\,dt + \int_0^{\eta_n} |v_n''|^2\,dt \leq (C+\sigma)\int_0^{\eta_n} |v_n'|\,dt + \int_0^{\eta_n} |v_n''|^2\,dt \\ &= (C+\sigma)z + O\left(\frac{1}{n}\right).\end{aligned}$$

Letting $n \to \infty$ and $\sigma \to 0$ we obtain

$$\varphi(z) \leq Cz \qquad \forall z > 0. \tag{5.1.15}$$

Finally, arguing exactly as for the other case, we easily obtain $\liminf_{z \to 0^+} \varphi(z)/z \geq C$, which concludes the proof of the lemma. □

Lemma 5.1.7 *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of functions such that $\sup_n F_n(u_n) < +\infty$ and, for a fixed $c > 0$, consider the sets $D_n := \{x \in I : |u_n'(x)| > c/r_n\}$. Then there exists $\overline{n} \in \mathbb{N}$, depending on c, such that*

$$|D_n| \leq \left(\frac{2\sup_n F_n(u_n)}{b(c)}\right) r_n,$$

for every integer $n \geq \overline{n}$.

PROOF. We can estimate

$$F_n(u_n) \geq \int_{D_n} f_n(|u_n'|)\, dx \geq \frac{r_n}{r_n} f_n\left(\frac{c}{r_n}\right) |D_n| \geq \frac{1}{2r_n} b(c)|D_n|,$$

if n is large enough, thanks to (5.1.6). □

Lemma 5.1.8 *Suppose that also iii) of Theorem 5.1.2 holds true and let (u_n) be such that*

$$\sup_n F_n(u_n) < +\infty.$$

Then

$$r_n \|u_n'\|_\infty \leq 2\sup_n F_n(u_n) + 1, \tag{5.1.16}$$

for n large enough. Moreover, if $g \not\equiv 0$, there exists a positive constant C depending only on $|I|$, g, and b such that

$$\operatorname{Var} u_n \leq C\left(\sup_n F_n(u_n) + 1\right)^2, \tag{5.1.17}$$

for n large enough.

PROOF. Take $c = 1$ and consider the sets D_n defined in the previous lemma; since they are open, we can write $D_n = \bigcup_{k=1}^\infty (a_n^k, b_n^k)$. Let y be a point of D_n; therefore there exists $k \in \mathbb{N}$ such that $y \in (a_n^k, b_n^k)$. By Lemma (5.1.7) and using Hölder's Inequality, we have

$$\begin{aligned}
|u_n'(y)| &\leq |u_n'(a_n^k)| + \int_{a_n^k}^{y} |u_n''(t)|\, dt \\
&\leq \frac{1}{r_n} + \frac{|D_n|^{\frac{1}{2}}}{(r_n)^{\frac{3}{2}}} \left((r_n)^3 \int_{a_n^k}^{b_n^k} |u_n''|^2\, dt\right)^{\frac{1}{2}} \\
&\leq \frac{1}{r_n} + 2\frac{(r_n)^{\frac{1}{2}}}{(r_n)^{\frac{3}{2}}} \sup_n F_n(u_n) = \left(2\sup_n F_n(u_n) + 1\right) \frac{1}{r_n},
\end{aligned}$$

so that (5.1.16) is proved. Concerning the second part of the Lemma, we first observe that, by a translation argument, we can suppose that $f_n(0) = g(0) = 0$ for every $n \in \mathbb{N}$. Let x_0 be the last point such that $g(x_0) = 0$ and define

$$\tilde{g}(x) := \begin{cases} 0 & \text{if } x \in [0, x_0], \\ g(x - x_0) & \text{if } x \geq x_0; \end{cases}$$

it is easy to see that $\tilde{g}$ is still convex, $\tilde{g}^\infty = g^\infty$, and, taking into account the fact that $f_n \to g$ uniformly on compact subsets of $[0, +\infty)$ (the uniformity follows from the pointwise convergence and from the monotonicity of f_n),

$$\forall \delta \in (0,1),\ \forall K > 0,\ \exists \overline{n} \text{ s.t. } f_n \geq (1-\delta)\tilde{g} \text{ in } [0, K],\ \forall n \geq \overline{n}. \tag{5.1.18}$$

Fix $\overline{y} > 0$ such that $\tilde{g}'(\overline{y}) > g^\infty(1)/2$; set $k := 2\sup_n F_n(u_n) + 1$ and let $\overline{x}$ be the first point such that $\tilde{g}'(\overline{x}+)/2 \geq \min\{g^\infty(1)/3, b(1)/(3k)\}$. Since either $\overline{x} = 0$ or

$$\frac{\tilde{g}'(\overline{x}-)}{2} \leq \min\left\{\frac{g^\infty(1)}{3}, \frac{b(1)}{3k}\right\} \leq \frac{\tilde{g}'(\overline{x}+)}{2}, \tag{5.1.19}$$

it is clear that $\overline{x} < \overline{y}$. So, by virtue of (5.1.18), (5.1.19), (5.1.8), and (5.1.6), we can find $\overline{n}$ such that

a) $f_n \geq \tilde{g}(x)/2$ in $[0, \overline{x} + 1]$,

b) $f_n'((\overline{x}+1)+) \geq \min\{g^\infty(1)/3, b(1)/(3k)\}$,

c) $f_n(k/r_n)/(k/r_n) \geq b(k)/(3k)$,

for every $n \geq \overline{n}$; we define $a(t) := \tilde{g}(\overline{x})/2 + \min\{g^\infty(1)/3, b(1)/(3k)\}(t - \overline{x})$. Exploiting the convexity of $\tilde{g}$, we observe that, by (5.1.19) and a),

$$a(t) \leq \frac{\tilde{g}(\overline{x})}{2} \leq f_n(t) \qquad \text{in } [0, \overline{x}+1]. \tag{5.1.20}$$

If **st1)** holds, that is if f_n is convex, then, taking into account b) and (5.1.20), we also have

$$a(t) \leq f_n(\overline{x}+1) + f_n'\big((\overline{x}+1)+\big)(t - \overline{x} - 1) \leq f_n(t) \qquad \forall t \geq \overline{x}+1.$$

Suppose now that **st2)** holds; by replacing f_n with

$$\tilde{f}_n(t) := \begin{cases} f_n(t) + (r_n t)^2 & \text{if } t \leq x_n, \\ f_n(t) + (r_n x_n)^2 & \text{if } t > x_n, \end{cases}$$

if needed, we can assume that f_n is strictly convex in $[0, x_n]$ (recall that x_n is the point appearing in condition **st2)**). Arguing as above, we obtain

$$a(t) \leq f_n(t) \qquad \forall t \in [\overline{x}+1, x_n \wedge k/r_n]. \tag{5.1.21}$$

Let us denote by y_n the first strictly positive point such that $f_n(y_n) = [f_n(k/r_n)/(k/r_n)]y_n$; by the strict convexity assumption we have that $0 < y_n \leq k/r_n$. If $x_n < y_n < k/r_n$, we can first observe that, by concavity,

$$f_n'(t\pm) \geq f_n'(y_n-) \geq \frac{r_n}{k} f_n\left(\frac{k}{r_n}\right) \qquad \forall t \in (x_n, y_n), \tag{5.1.22}$$

where the last inequality is a consequence of the following one

$$f_n(t) \geq f_n(y_n) + \frac{r_n}{k} f_n\left(\frac{k}{r_n}\right)(t - y_n) = \frac{r_n}{k} f_n\left(\frac{k}{r_n}\right) t \qquad \text{in } [y_n, k/r_n],$$

where we used again the concavity of f_n in (x_n, y_n). Using (5.1.22) and c) we then have

$$a(t) \leq f_n(x_n) + \min\left\{\frac{g^\infty(1)}{3}, \frac{b(1)}{3k}\right\}(t - x_n) \leq f_n(t) \qquad \text{in } [x_n, y_n] \tag{5.1.23}$$

and therefore

$$a(t) \leq f_n(y_n) + \frac{r_n}{k} f_n\left(\frac{k}{r_n}\right)(t - y_n) \leq f_n(t) \qquad \text{in } [y_n, k/r_n].$$

If $x_n < y_n = k/r_n$ then either

$$f_n(t) > \frac{r_n}{k} f_n\left(\frac{k}{r_n}\right) t \geq \frac{b(k)}{3k} t \qquad \text{in } [0, k/r_n]$$

or

$$f_n(t) < \frac{r_n}{k} f_n\left(\frac{k}{r_n}\right) t.$$

In the first case (5.1.17) follows immediately; in the second case we observe that (5.1.22) and therefore (5.1.23) are still true. Summarizing, we have proved that

$$a(t) \leq f_n(t) \qquad \text{in } [0, k/r_n], \tag{5.1.24}$$

if $x_n < y_n$; arguing in a similar way, we obtain the same estimate also if $y_n \leq x_n$. Using the definition of $a(t)$ and the fact that $\overline{x} < \overline{y}$, from (5.1.24) we easily obtain

$$\min\left\{\frac{g^\infty(1)}{3}, \frac{b(1)}{3}\right\} t \leq k f_n(t) + \frac{k g^\infty(1)}{3} \overline{y} \qquad \forall t \in [0, k/r_n],$$

from which, recalling the definition of k and (5.1.16), the inequality (5.1.17) immediately follows with

$$C := (6 + 2g^\infty(1)\overline{y}|I|)(\min\{g^\infty(1)/3, b(1)/3\})^{-1}.$$

□

Remark 5.1.9 Let us remark that if $u_n \to u$ in L^1 and $\sup_n F_n(u_n) < +\infty$ then $u_n \to u$ in L^p for every $p \geq 1$: indeed from (5.1.17) it easily follows that u_n is equibounded in L^∞. As a consequence we have that in one dimension the functionals F_n Γ-converge with respect to the L^1-norm if and only if they Γ-converge with respect to the L^p-norm, for every $p \geq 1$.

Lemma 5.1.10 *Assume that also condition iii) of Theorem 5.1.2 holds and let $(u_n)_{n\in\mathbb{N}} \subset SBV(I)$ be such that $r_n \|u_n'\|_\infty \to 0$ as $n \to \infty$. Then there exists an increasing sequence $(\psi_i)_{i\in\mathbb{N}}$ of positive convex functions enjoying the following properties:*

i) $\psi_i(t) \uparrow g_1$ for every $t > 0$ as $i \to \infty$ (we recall that g_1 is the function appearing in (5.1.4)*);*

ii) $\psi_i^\infty(1) = g_1^\infty(1) = b^0(1) \wedge g^\infty(1)$ for every i;

iii) passing to a subsequence, still denoted by $(u_n)_n$, we have that for every i there exists n_i such that

$$f_n(|u_n'|) \geq \psi_i(|u_n'|),$$

for every $n \geq n_i$.

PROOF. We can assume that $\min\{g^\infty(1), b^0(1)\} \neq 0$, otherwise the statement is trivial. We will distinguish two cases.
Case 1: $g^\infty(1) > b^0(1)$.

Note that in this case, by Remark 5.1.4, we have that necessarily **st2)** holds true, with $\lim_n x_n r_n = 0$. We can suppose without loss of generality that $f_n(0) = g(0) = 0$ for every $n \in \mathbb{N}$ (otherwise translate). We begin by assuming

$$g'(0+) < b^0(1), \tag{5.1.25}$$

so that, letting $\overline{x}$ be the last point such that $g'(\overline{x}-) \leq b^0(1) \leq g'(\overline{x}+)$ and setting $\overline{y} := \sup\{y \geq 0 : g(t) \leq b^0(1)t, \forall t \in [0,y]\}$, we have $\overline{x} < \overline{y}$. We make also the following assumption:

$$\forall \delta \in (0,1),\ \forall K > 0,\ \exists\, n_{\delta,K} \text{ s.t } f_n(t) \geq (1-\delta)g(t) \quad \forall t \in [0,K] \text{ and } \forall n \geq n_{\delta,K}. \tag{5.1.26}$$

It is clear that we can find $\delta_0 \in (0,1)$ such that for every $0 < \delta \leq \delta_0$ there holds $(1-\delta)g^\infty(1) > b^0(1)$ and $\overline{x}_\delta < \overline{y}$, where $\overline{x}_\delta$ is the last point such that $(1-\delta)g'(\overline{x}_\delta-) \leq b^0(1) \leq (1-\delta)g'(\overline{x}_\delta+)$. In particular, for every $\delta \leq \delta_0$, there exists $x_\delta \in [\overline{x}_\delta, \overline{y})$ satisfying

$$(1-\delta)g'(x_\delta) \geq b^0(1). \tag{5.1.27}$$

Let us choose now a sequence d_n increasing to $+\infty$ with the following properties:

a) $d_n > \|u_n'\|_\infty$ and $d_n > x_n$ for every $n \in \mathbb{N}$, where x_n is the point appearing in assumption **st2)**;

b) $d_n r_n \to 0$ so slowly that $r_n f_n(d_n)/b(d_n r_n) \to 1$ (this is possible thanks to (5.1.6)).

Setting $\tilde{s}_n := f_n(d_n)/d_n$, from b) it easily follows that $\lim_{n\to\infty} \tilde{s}_n = b^0(1)$, so that, passing to a subsequence if needed and denoting $s_n := \tilde{s}_n \wedge b^0(1)$, we have that s_n is a non-decreasing sequence converging to $b^0(1)$. Finally, denoting by y_n the first strictly positive point such that $f_n(y_n) = s_n y_n$, the convergence of s_n to $b^0(1)$ and of f_n to g implies $y_n \to \overline{y}$. Taking into account all this facts and recalling (5.1.8), it is now evident that we can find $\overline{n}_\delta > 0$ such that

*) $f_n(t) \geq (1-\delta)g(t)$, for every $t \in [0, x_\delta]$,

**) $f_n'(x_\delta-) > b^0(1) \geq s_n$ and $x_\delta < y_n$,

for every $n \geq \overline{n}_\delta$. At this point, for $k > \overline{n}_\delta$ we define the function ψ_δ^k by induction in the following way:

$$\psi_\delta^k = [(1-\delta)g \wedge s_k t]^{**} \text{ in } [0, d_k] \quad \text{and} \quad \psi_\delta^k = \psi_\delta^k(d_j) + s_{j+1}(t - d_j) \text{ in } [d_j, d_{j+1}], \text{ for } j \geq k.$$

Recalling that s_n increases to $b^0(1)$ it is easily seen that ψ_δ^k is convex with $(\psi_\delta^k)^\infty(1) = b^0(1)$ and $\psi_\delta^k \uparrow [(1-\delta)g \wedge b^0(1)t]^{**}$ as k tends to infinity. Defining

$$\tilde{f}_n(t) := \begin{cases} f_n(t) & \text{if } t \in [0, x_\delta], \\ f_n(x_\delta) + s_n(t - x_\delta) & \text{otherwise,} \end{cases}$$

by *) and **), we have

$$\psi_\delta^k(t) \le \tilde{f}_n(t) \qquad \text{in } [0, d_n]. \tag{5.1.28}$$

Moreover it turns out

$$\tilde{f}_n(t) \le f_n(t) \qquad \text{in } [0, y_n]: \tag{5.1.29}$$

actually, this is true in $[0, x_n]$ by **) and by convexity (since $x_n \to +\infty$ we have that $x_n > y_n$ provided n is large enough). Exploiting the concave or convex-concave structure of f_n in $[y_n, d_n]$ it is also easy to prove (see Figure 5.1 and the proof of the previous Lemma for the details of the argument) that

$$\tilde{f}_n(t) \le s_n t \le f_n(t) \qquad \text{in } [y_n, d_n]. \tag{5.1.30}$$

Combining (5.1.28), (5.1.29), and (5.1.30), we obtain that $\psi_\delta^k \le f_n$ in $[0, d_n]$ for every $n > k$ and so,

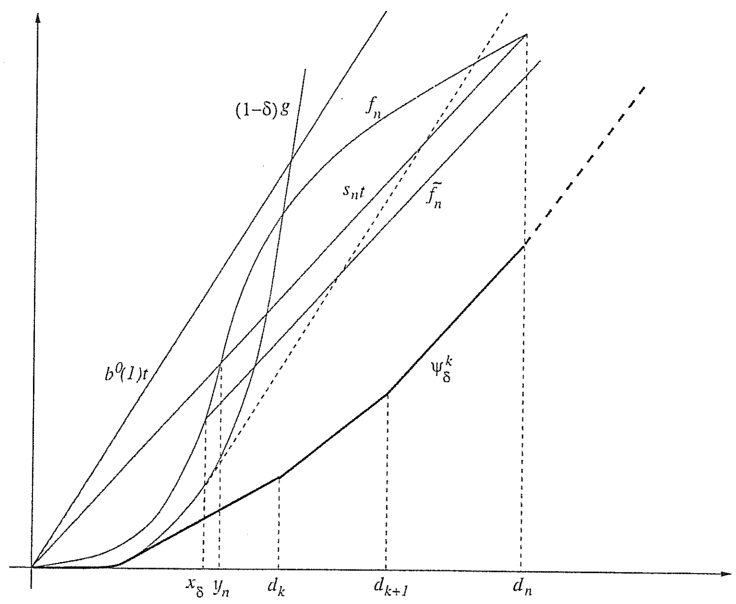

Figure 5.1: The construction of ψ_δ^k in the case $g^\infty(1) > b^0(1)$.

in particular, $f_n(|u_n'|) \ge \psi_\delta^k(|u_n'|)$ almost everywhere for $n > k$. Finally, choosing a sequence $\delta_n \downarrow 0$, by diagonalization, from the family $(\psi_{\delta_n}^k)_{k,n}$ we can extract a subfamily $(\psi_i)_i$ having all the required properties. If g does not satisfy (5.1.26), we can proceed in the following way: let x_0 be the last point where g vanishes and define

$$g_k(x) := \begin{cases} 0 & \text{if } x \in [0, x_0], \\ g\left(x - \frac{1}{k}\right) & \text{if } x \ge x_0; \end{cases}$$

It turns out that $g_k^\infty(1) = g^\infty(1)$, $g_k \uparrow g$ as $k \to \infty$, and g_k satisfies (5.1.26). Hence we can repeat the construction above for every g_k and conclude by diagonalization. If g does not satisfy (5.1.25), then in particular $g(t) \ge b^0(1)t$ for every $t > 0$ and therefore $g_1 = b^0(1)t$; moreover there exists $\overline{n} \in \mathbb{N}$

such that $f_n'(0+) > (1-\delta)b^0(1)$ for every $n \geq \overline{n}$. If (d_n) and (s_n) are as above, for $k > \overline{n}$ we define ψ_δ^k by induction in the following way:

$$\psi_\delta^k(t) = (1-\delta)s_k t \text{ in } [0, d_k]$$

and

$$\psi_\delta^k(t) = \psi_\delta^k(d_{k+j}) + \left(1 - \frac{\delta}{j+1}\right) s_{k+j+1}(t - d_{k+j}) \text{ in } [d_{k+j}, d_{k+j+1}] \text{ for } j \geq 0.$$

Arguing as above, it is easy to see that from the family $(\psi_\delta^k)_{k,\delta}$ we can extract by diagonalization a subfamily satisfying all the requirements.

Case: $b^0(1) \geq g^\infty(1)$. Note that in this case $g_1 = g$. As above it is not restrictive to suppose that $g(0) = f_n(0) = 0$ for every $n \in \mathbb{N}$ and that g satisfies (5.1.26). At first we choose a sequence $\delta_n \downarrow 0$ and, as above, a diverging sequence d_n satisfying $\|u_n'\|_\infty \leq d_n$ for every $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \frac{f_n(d_n)}{d_n} = b^0(1) \geq g^\infty(1).$$

Recalling (5.1.8) we can define for every $i \in \mathbb{N}$

$$n_{0,i} := \inf\left\{ j \in \mathbb{N} : j > i,\ f_n(t) \geq (1-\delta_i)g(t) \text{ in } [0, d_i],\ f_n'(d_i-) > (1-\delta_i)g'(d_i-), \right.$$
$$\left. \text{and } \frac{f_n(d_n)}{d_n} \geq (1-\delta_i)g'(d_i-),\ \forall n \geq j \right\},$$

and, for $h \geq 1$,

$$n_{h,i} := \inf\left\{ j > n_{h-1,i} : f_n(t) \geq (1-\delta_{i+h})g(t) \text{ in } [0, d_{i+h}],\ f_n'(d_{i+h}-) > (1-\delta_{i+h})g'(d_{i+h}-), \right.$$
$$\left. \text{and } \frac{f_n(d_n)}{d_n} \geq (1-\delta_{i+h})g'(d_{i+h}-),\ \forall n \geq j \right\}.$$

We define the function ψ_i by induction on h in the following way:

$$\psi_i(t) := \begin{cases} (1-\delta_i)g(t) & \text{if } t \in [0, d_i], \\ (1-\delta_i)[g(d_i) + g'(d_i-)(t - d_i)] & \text{if } t \in (d_i, d_{n_{1,i}}], \end{cases}$$

and, for $h \geq 1$,

$$\psi_i(t) := \psi_k(d_{n_{h,i}}) + (1-\delta_{i+h})g'(d_{i+h}-)(t - d_{n_{h,i}}) \quad \text{in} \quad (d_{n_{h,i}}, d_{n_{h+1,i}}];$$

clearly $\psi_i^\infty = g^\infty(1)$ for every i and $\psi_i \uparrow g$ as $i \to \infty$. Set, for every $h \geq 0$,

$$\phi_{i+h}(t) := \begin{cases} (1-\delta_{i+h})g(t) & \text{if } t \in [0, d_{i+h}], \\ (1-\delta_{i+h})[g(d_{i+h}) + g'(d_{i+h}-)(t - d_{i+h})] & \text{if } t > d_{i+h}. \end{cases}$$

First of all, taking into account the definition of $n_{h,i}$ and exploiting the structure assumption on f_n exactly as we did before, one can prove that

$$\phi_{i+h} \leq f_n \qquad \forall n \geq n_{h,i}; \tag{5.1.31}$$

moreover we have

$$\psi_i \le \phi_{i+h} \qquad \text{in } [0, d_{n_{h+1,i}}]. \tag{5.1.32}$$

The last inequality is an immediate consequence of the following one

$$\psi_i \le (1 - \delta_{i+h})g \qquad \text{in } [0, d_{i+h}],$$

which can be proved easily by induction on h.

Take $n \ge n_{0,i}$ and let h be such that $n_{h,i} \le n \le n_{h+1,i}$: combining (5.1.31) and (5.1.32) we finally obtain that $\psi_i \le f_n$ in $[0, d_n]$. □

Lemma 5.1.11 *Suppose that also* (5.1.10) *and condition iii) of Theorem 5.1.2 hold and let* $(u_n)_{n\in\mathbb{N}} \subset W^{2,2}(I)$ *be such that* $\sup_n F_n(u_n) < +\infty$. *Then, for every* $\delta > 0$, *there exists a sequence* $(v_n)_{n\in\mathbb{N}} \subseteq SBV(I)$ *such that* $\|u_n - v_n\|_1 \to 0$, $r_n\|v_n'\|_\infty \to 0$ *as* $n \to \infty$, $|v_n'| \le |u_n'|$ *everywhere, and*

$$F_n(u_n) \ge (1-\delta) \sum_{x \in S_{v_n}} \varphi(v_n^+(x) - v_n^-(x)),$$

for n *sufficiently large.*

Proof. By Lemma 5.1.8 there exists $K > 0$ such that

$$r_n\|u_n'\|_\infty \le K; \tag{5.1.33}$$

for every $0 < s \le K$ we define

$$\omega_n(s) := \sup_{t\in[s,K]} \left| r_n f_n\left(\frac{t}{r_n}\right) - b(t)\right|. \tag{5.1.34}$$

Recalling that if a family of monotone functions pointwise converges to a continuous function, then the convergence is actually uniform on compact subsets, by (5.1.6) we have that $\omega_n \to 0$ pointwise. As a first step we choose a sequence $(c_n)_n$ of positive real numbers converging to 0 so slowly that:

a) $\dfrac{r_n}{(c_n)^{\frac{5}{2}}} \to 0$ as $n \to \infty$;

b) $\displaystyle\lim_{n\to\infty} \frac{\omega_n(c_n)}{b(c_n)} = 0$.

We set $D_n := \{x \in I : |u_n'| > c_n/r_n\} = \bigcup_{k=1}^\infty I_n^k$, where (I_n^k) is the collection of the connected components of D_n; we also denote $I_n^k = (a_n^k, b_n^k)$. Arguing as in Lemma 5.1.8 and taking into account condition b), we obtain

$$|D_n| \le \left(\frac{2\sup_n F_n(u_n)}{b(c_n)}\right) r_n, \tag{5.1.35}$$

for n large enough. For every $n \in \mathbb{N}$ we define

$$\tilde{v}_n(x) := \begin{cases} u_n(x) & \text{if } x \in I \setminus D_n, \\ u_n(a_n^k) & \text{if } x \in (a_n^k, b_n^k); \end{cases}$$

clearly $\tilde{v}_n \in L^1(I) \cap SBV(I)$. Moreover we set $w_n := u_n - \tilde{v}_n$; since $w_n' = u_n'$ and $w_n'' = u_n''$ on D_n, we have

$$F_n(u_n, I_n^k) = \int_{I_n^k} f_n(|w_n'|)\, dx + (r_n)^3 \int_{I_n^k} |w_n''|^2\, dx;$$

summing over k and setting $\tilde{z}_n(x) := w_n(r_n x)$ we therefore obtain

$$\begin{aligned} F_n(u_n, D_n) &= \sum_k \left(\int_{I_n^k} f_n(|w_n'|)\, dx + (r_n)^3 \int_{I_n^k} |w_n''|^2\, dx \right) \\ &= \sum_k \left(\int_{I_n^k} f_n\left(\frac{1}{r_n} \left| \tilde{z}_n'\left(\frac{x}{r_n}\right) \right| \right) dx + \frac{1}{r_n} \int_{I_n^k} \left| \tilde{z}_n''\left(\frac{x}{r_n}\right) \right|^2 dx \right) \\ &= \sum_k \left(r_n \int_{\frac{I_n^k}{r_n}} f_n\left(\frac{1}{r_n} |\tilde{z}_n'| \right) dy + \int_{\frac{I_n^k}{r_n}} |\tilde{z}_n''|^2\, dy \right). \end{aligned} \tag{5.1.36}$$

By (5.1.33) we have

$$c_n \le |\tilde{z}_n'| \le K \qquad \text{in } D_n/r_n; \tag{5.1.37}$$

moreover, we can prove that for every $\delta > 0$, there exists $\overline{n}$ such that $r_n f_n\left(\frac{t}{r_n}\right) \ge (1-\delta) b(t)$ for every $t \in [c_n, K]$ and for every $n \ge \overline{n}$ and thus, by (5.1.37),

$$r_n f_n\left(\frac{|\tilde{z}_n'|}{r_n} \right) \ge (1-\delta) b\left(|\tilde{z}_n'|\right) \qquad \text{in } D_n/r_n. \tag{5.1.38}$$

Indeed, by condition b), for every $\delta > 0$ we can find $\overline{n}$ such that $\omega_n(c_n) \le \delta b(c_n)$ for every $n \ge \overline{n}$, so that, recalling (5.1.34),

$$r_n f_n\left(\frac{t}{r_n} \right) \ge b(t) - \omega_n(c_n) \ge b(t) - \delta b(c_n) \ge (1-\delta) b(t) \qquad \forall t \in [c_n, K],$$

where we used the monotonicity of b. Let us define the functions z_n as

$$z_n(x) := \begin{cases} \tilde{z}_n(x) & \text{if } x \in (I \setminus D_n)/r_n, \\ \tilde{z}_n(x) - \tilde{z}_n'\left(\frac{a_k^n}{r_n}\right)\left(x - \frac{a_k^n}{r_n}\right) & \text{in } I_n^k/r_n. \end{cases}$$

By (5.1.36) and (5.1.38), by using the fact that $|z_n'| \le |\tilde{z}_n'|$ everywhere and the monotonicity of b, we have

$$\begin{aligned} F_n(u_n, D_n) &\ge (1-\delta) \sum_k \left(\int_{\frac{I_n^k}{r_n}} b\left(|\tilde{z}_n'|\right) dy + \int_{\frac{I_n^k}{r_n}} |\tilde{z}_n''|^2\, dy \right) \\ &\ge (1-\delta) \sum_k \left(\int_{\frac{I_n^k}{r_n}} b\left(|z_n'|\right) dy + \int_{\frac{I_n^k}{r_n}} |z_n''|^2\, dy \right) \\ &\ge (1-\delta) \sum_k \varphi\left(\left| z_n\left(\frac{b_n^k}{r_n} \right) \right| \right) = (1-\delta) \sum_k \varphi(v_n^+(b_n^k) - v_n^-(b_n^k)) = (**), \end{aligned} \tag{5.1.39}$$

where $v_n(x) := u_n(x) - z_n(x/r_n)$. Using the definition of z_n, it is easy to check that

$$v_n(x) = \begin{cases} u_n(x) & \text{if } x \in I \setminus D_n \\ u_n(a_n^k) + u_n'(a_n^k)(x - a_n^k) & \text{if } x \in (a_n^k, b_n^k) \end{cases}$$

and to see that $(**) = (1-\delta)\sum_{x\in S_{v_n}} \varphi(v_n^+(x) - v_n^-(x))$, which, combined with (5.1.39), gives the thesis of the lemma, once we have shown that

$$\|v_n - u_n\|_1 \to 0 \qquad \text{as } n \to \infty. \tag{5.1.40}$$

If $t \in I_n^k$, by Hölder's Inequality, we have

$$\begin{aligned} |v_n(t) - u_n(t)| &\leq \int_{a_n^k}^t |v_n'(s) - u_n'(s)|\,ds \leq \int_{a_n^k}^t \int_{a_n^k}^s |u_n''(z)|\,dz \\ &\leq \left(\int_{I_n^k} |u_n''|^2\,dz\right)^{\frac{1}{2}} \int_{a_n^k}^t (s - a_n^k)^{\frac{1}{2}}\,ds \\ &= \frac{2}{3}\left(\int_{I_n^k} |u_n''|^2\,dz\right)^{\frac{1}{2}} (t - a_n^k)^{\frac{3}{2}}, \end{aligned}$$

therefore, integrating on I_n^k,

$$\int_{I_n^k} |v_n(t) - u_n(t)|\,dt \leq \frac{4}{15}\left(\int_{I_n^k} |u_n''|^2\,dz\right)^{\frac{1}{2}} |I_n^k|^{\frac{5}{2}};$$

using (5.1.35), we can conclude

$$\begin{aligned} \|u_n - v_n\|_1 &\leq \frac{4}{15}\sum_{k\in\mathbb{N}} \left(\int_{I_n^k} |u_n''|^2\,dz\right)^{\frac{1}{2}} |I_n^k|^{\frac{5}{2}} \\ &\leq \frac{4}{15(r_n)^{\frac{3}{2}}}\left((r_n)^3 \int_{D_n} |u_n''|^2\,dz\right)^{\frac{1}{2}} \left(\sum_{k\in\mathbb{N}} |I_n^k|^5\right)^{\frac{1}{2}} \\ &\leq \frac{4(\sup_n F_n(u_n))^{\frac{1}{2}}}{15(r_n)^{\frac{3}{2}}} |D_n|^{\frac{5}{2}} \leq \frac{4(\sqrt{2})^5(\sup_n F_n(u_n))^3}{15} \frac{r_n}{(b(c_n))^{\frac{5}{2}}}; \end{aligned}$$

recalling condition a) and (5.1.10), we finally get (5.1.40). □

Lemma 5.1.12 *Let $g : [0,\infty) \to [0,\infty)$ be a convex superlinear function and let $u \in SBV(I)$ be such that $\int_I g(|u'|)\,dx + \mathcal{H}^0(S_u) < +\infty$. Then there exists a sequence $(u_n) \in SBV(I)$ such that $S_{u_n} \subseteq S_u$, $u_n \in W^{2,2}(I \setminus S_u)$, $u_n'(t\pm) = 0$ on S_u, $u_n \to u$ in $L^\infty(I)$, $u_n^\pm(t) \to u^\pm(t)$ on S_u, and $\int_I g(|u_n'|)\,dx \to \int_I g(|u'|)\,dx$.*

PROOF. Let $I = (a,b)$ and $S_u = \{x_1, \ldots, x_N\}$ with $x_i < x_{i+1}$ and set $x_0 = a$ and $x_{N+1} = b$. We can construct a family (g_k) of strictly convex and superlinear functions belonging to $C^2([0,+\infty))$ and satisfying

$$g_k'(0) = 0, \qquad g_k \downarrow g, \qquad \text{and} \qquad \lim_{t\to+\infty} \frac{g_k(t)}{g(t)} = 1.$$

For every $k \in \mathbb{N}$ and for every $i \in \{0, \dots, N\}$ let $u^k_{i,j}$ be the solution of the minimum problem

$$\min\left\{\int_I g_k(|v'|)\,dx + j\int_I |v-u|^2\,dx \,:\, v \in W^{1,1}(x_i, x_{i+1})\right\};$$

note that the existence of such a solution is guaranteed by the convexity and the superlinearity of g_k; moreover $u^k_{i,j}$ is a classical solution to the Euler equation $h''_k(w')w'' = j(w-u)$ with the Neumann conditions $w'(x_i) = w'(x_{i+1}) = 0$, where h_k is the function in $C^2(\mathbb{R})$ obtained by reflection of g_k. Therefore, taking into account the regularity and strict convexity assumptions on g_k and the fact that $u \in C([x_i, x_{i+1}])$, we get $u^k_{i,j} \in C^2([x_i, x_{i+1}])$, so that, denoting by u^k_j the function in $SBV(I)$ which coincides with $u^k_{i,j}$ on (x_i, x_{i+1}), we clearly have that the family $(u^k_j)_j$ satisfies all the required conditions except for the last one. Indeed, by construction, we know only $\int_I g_k(|(u^k_j)'|)\,dx \xrightarrow{j} \int_I g_k(|u'|)\,dx$, but recalling that $\int_I g_k(|u'|)\,dx \xrightarrow{k} \int_I g(|u'|)\,dx$, the desired approximating sequence can be obtained by diagonalization. □

We finally state a Lemma which will be useful in the sequel.

Lemma 5.1.13 *Denote by $\mathcal{A}(\Omega)$ the family of all open subsets of Ω and let $\nu : \mathcal{A}(\Omega) \to [0,+\infty)$ be a superadditive set-function. Let λ be a positive measure on Ω and let $(\psi_i)_i$ a family of positive Borel functions such that $\nu(A) \geq \int_A \psi_i\,d\lambda$ for all $A \in \mathcal{A}(\Omega)$ and for all $i \in \mathbb{N}$. Then $\nu(A) \geq \int_A \sup_i \psi_i\,d\lambda$, for all $A \in \mathcal{A}(\Omega)$.*

PROOF. See Proposition 1.16 of [14]. □

PROOF OF THEOREM 5.1.2: the case $b^0(1) = +\infty$.

- Γ-lim sup Inequality

Let us set for notational convenience $F'' := \Gamma\text{-}\limsup_n F_n$. We first remark that it is enough to show that $F''(u) \leq \int_I g(|u'|)\,dx + \sum_{S_u} \varphi(u^+ - u^-)$ for every $u \in SBV(I)$ with $\mathcal{H}^0(S_u) < +\infty$, indeed, once we have this inequality, the thesis follows from the semicontinuity of F'' and the fact that $\overline{\mathcal{F}}_{b,g}$ coincides with the relaxed functional of

$$H(u) := \begin{cases} \displaystyle\int_I g(|u'|)\,dx + \sum_{S_u} \varphi(u^+ - u^-) & \text{if } u \in SBV(I) \text{ and } \mathcal{H}^0(S_u) < +\infty. \\ +\infty & \text{otherwise} \end{cases}$$

Claim 1. Let $u \in SBV(I)$ such that $\mathcal{H}^0(S_u) < +\infty$, $u \in W^{2,2}(I \setminus S_u)$, $F(u) < +\infty$, and $u'(t\pm) = 0$ for every $t \in S_u$. Then

$$F''(u) \leq \int_I g(|u'|)\,dx + \sum_{S_u} \varphi(u^+ - u^-).$$

Since the construction is local, we may assume that $S_u = \{\bar{t}\}$ and $u(t\pm) = u^\pm(\bar{t})$.

Fix $\delta > 0$ and choose an admissible pair (η, v) for problem (5.1.3) (with $z = u^+(\bar{t}) - u^-(\bar{t})$) satisfying:

$$\int_0^\eta b(|v'|)\,dx + \int_0^\eta |v''|^2\,dx < \varphi(u^+(\bar{t}) - u^-(\bar{t})) + \delta.$$

We define the recovery sequence in the following way:

$$u_n(x) := \begin{cases} u(x) & \text{if } x \le \bar{t} \\ v\left(\dfrac{x-\bar{t}}{r_n}\right) + u^-(\bar{t}) & \text{if } x \in (\bar{t}, \bar{t}+r_n\eta) \\ u(x - r_n\eta) + u^+(\bar{t}) & \text{if } x \ge \bar{t} + r_n\eta. \end{cases}$$

Clearly $u_n \to u$ in L^1. We can now compute

$$\begin{aligned} F_n(u_n) &= F_n(u_n, I \setminus (\bar{t}, \bar{t}+r_n\eta)) + \int_{\bar{t}}^{\bar{t}+r_n\eta} f_n\left(\frac{1}{r_n}\left|v'\left(\frac{x-\bar{t}}{r_n}\right)\right|\right) dx \\ &\quad + (r_n)^3 \int_{\bar{t}}^{\bar{t}+r_n\eta} \frac{1}{(r_n)^4}\left|v''\left(\frac{x-\bar{t}}{r_n}\right)\right|^2 dx \\ &= F_n(u_n, I \setminus (\bar{t}, \bar{t}+r_n\eta)) + \underbrace{\int_0^\eta r_n f_n\left(\frac{|v'|}{r_n}\right) dy + \int_0^\eta |v''|^2\, dy}_{=\,(*)_n}. \end{aligned} \tag{5.1.41}$$

Since

$$r_n f_n\left(\frac{|v'|}{r_n}\right) \to b(|v'|) \text{ in } \{x \in I : |v'(x)| \ne 0\}, \qquad r_n f_n\left(\frac{|v'|}{r_n}\right) \le r_n f_n\left(\frac{\|v'\|_\infty}{r_n}\right) \to b(\|v'\|_\infty),$$

by the Dominated Convergence Theorem and the fact that

$$\lim_{n\to\infty} \int_{\{x\in I:\, |v'(x)| \ne 0\}} r_n f_n\left(\frac{|v'|}{r_n}\right) dx = \lim_{n\to\infty} |\{x \in I : |v'(x)| \ne 0\}| r_n f_n(0) = 0,$$

we have

$$\limsup_{n\to\infty} (*)_n \le \int_0^\eta b(|v'|)\, dx + \int_0^\eta |v''|^2\, dx \quad \text{as } n \to \infty;$$

moreover, again by the Dominated Convergence Theorem, we easily see that

$$\lim_{n\to\infty} F_n(u_n, I \setminus (\bar{t}, \bar{t}+r_n\eta)) = \int_I g(|u'|)\, dx.$$

From (5.1.41), we therefore obtain

$$\begin{aligned} \limsup_{n\to\infty} F_n(u_n) &\le \int_I g(|u'|)\, dx + \int_0^\eta b(|v'|)\, dx + \int_0^\eta |v''|^2\, dx \\ &\le \int_I g(|u'|)\, dx + \varphi(u^+(\bar{t}) - u^-(\bar{t})) + \delta. \end{aligned}$$

By the arbitrariness of δ, Claim 1 is proved. By a standard density argument based on the use of Lemma 5.1.12 we recover the same inequality for every $u \in SBV(I)$ with $\mathcal{H}^0(S_u) < +\infty$ and this concludes the proof of the Γ-lim sup inequality, as we remarked above.

- Γ-lim inf Inequality

We are supposing that the structure condition *iii)* holds so that g is convex and b is convex or concave or convex-concave; the functional $F_{b,g}$ is then well defined and, by Theorem 1.1.3, coincides with $\overline{\mathcal{F}_{b,g}}$. We distinguish two cases.

Case 1: $g^\infty(1) = +\infty$ i.e. g is superlinear.

Note that in this case $F_{b,g}(u)$ is finite only if $u \in SBV(I)$ and for such u we have

$$F_{b,g}(u) = \int_I g(|u'|)\,dx + \sum_{S_u} \varphi(u^+ - u^-).$$

Let $u_n \to u$ in L^1 and such that $\sup_n F_n(u_n) < +\infty$, let $(v_n)_n$ be the sequence constructed in Lemma 5.1.11 and $(\psi_i)_i$ the associated family of convex superlinear functions according to Lemma 5.1.10. For δ and $\mu \in (0,1)$, and for every open subset $J \subseteq I$, by Lemmas 5.1.11 and 5.1.10, we have, for n sufficiently large,

$$\begin{aligned} F_n(u_n, J) &= (1-\delta)\int_J f_n(|u_n'|)\,dx + \delta\left[\int_J f_n(|u_n'|)\,dx + ((1/\sqrt[3]{\delta})r_n)^3 \int_J |u_n''|^2\,dx\right] \\ &\geq (1-\delta)\int_J \psi_i(|v_n'|)\,dx + \delta(1-\mu)\sum_{x\in S_u} \varphi(v_n^+ - v_n^-). \end{aligned} \tag{5.1.42}$$

Therefore, by the Ambrosio Semicontinuity Theorem (recall also Lemma 5.1.6), we obtain that $u \in SBV(I)$ and

$$\liminf_{n\to\infty} F_n(u_n, J) \geq (1-\delta)\int_J \psi_i(|u'|)\,dx + \delta(1-\mu)\sum_{x\in S_u}\varphi(u^+ - u^-) \qquad \forall i;$$

letting $i \uparrow \infty$ and $\mu \downarrow 0$, we obtain

$$\begin{aligned} (\Gamma\text{-}\liminf_{n\to\infty} F_n)(u, J) &\geq (1-\delta)\int_J g(|u'|)\,dx + \delta\sum_{x\in S_u}\varphi(u^+ - u^-) \\ &= \int_J h^\delta(x)\,d\lambda \quad \forall \text{ open } J \subseteq I,\ \forall \delta \in (0,1), \end{aligned} \tag{5.1.43}$$

where we have set $\lambda := g(|u'|)\mathcal{L}^1 + \varphi(u^+ - u^-)\mathcal{H}^0$ and $h^\delta := (1-\delta)(1-\chi_{S_u}) + \delta\chi_{S_u}$. Let δ_n be a dense sequence in $(0,1)$; since $\sup_i h^{\delta_i} = 1$, from (5.1.43) we finally deduce

$$(\Gamma\text{-}\liminf_{n\to\infty} F_n)(u) \geq \int_I \sup_i h^{\delta_i}\,d\lambda = \int_I g(|u'|)\,dx + \sum_{x\in S_u}\varphi(u^+ - u^-),$$

where we applied Lemma 5.1.13 (with $\nu := (\Gamma\text{-}\liminf_{n\to\infty} F_n)(u,\cdot)$).

Case 2: $g^\infty(1) < +\infty$. Let v_n be as above: according to Lemma 5.1.10, construct a family $(\psi_i)_i$ of convex functions such that $\psi_i^\infty(1) = g^\infty(1)$ for every $i \in \mathbb{N}$, $\psi_i \uparrow g$ as $i \to \infty$, and $\psi_i(|v_n'|) \leq f_n(|v_n'|)$ for every i and for n sufficiently large. Therefore, by using Lemma 5.1.11, we can write

$$\begin{aligned} F_n(u_n, I) &= F_n(u_n, D_n) + F_n(u_n, I \setminus D_n) \\ &\geq (1-\delta)\left(\int_I \psi_i(|v_n'|)\,dx + \sum_{x\in S_{v_n}}\varphi(v_n^+ - v_n^-)\right) - \int_{D_n}\psi_i(|v_n'|)\,dx; \end{aligned} \tag{5.1.44}$$

using the inequality $\psi_i(t) \le g(0) + g^\infty(1)t$, true for every $t > 0$, and recalling (5.1.35) and the fact that $\lim_n b(c_n)/c_n = +\infty$, we can estimate

$$\int_{D_n} \psi_i(|v_n'|)\,dx = \psi_i\left(\frac{c_n}{r_n}\right)|D_n| \le \left(g(0) + g^\infty(1)\frac{c_n}{r_n}\right)\left(\frac{2\sup_n F_n(u_n)}{b(c_n)}\right) r_n = O(1)$$

hence, invoking the Relaxation Theorem 1.1.3, from (5.1.44), we get

$$\liminf_{n\to\infty} F_n(u_n) \ge (1-\delta)\left(\int_I \psi_i(|u'|)\,dx + \sum_{S_u}(\varphi\Delta g^\infty)(u^+ - u^-) + g^\infty(1)|D^c u|\right).$$

Letting $i \uparrow +\infty$ and $\delta \downarrow 0$ we complete the proof of the Γ-lim inf inequality.

Concerning the last part of the theorem, we first observe that, thanks to (5.1.17) the approximating functionals are equicoercive: the conclusion then follows from Remark 5.1.9. □

To treat the case

$$b^0(1) = \lim_{t\to 0^+} \frac{b(t)}{t} < +\infty,$$

we need first the following lemma.

Lemma 5.1.14 *Let b be satisfy* (5.1.14), *and, for every $\delta > 0$, let $\varphi^\delta : (0,\infty) \to (0,\infty)$ be the function defined by*

$$\varphi^\delta(z) := \inf_{\eta>0} \inf\left\{\int_0^\eta b(|u'|)\,dx + \int_0^\eta |u''|^2\,dx : u \in W^{2,2}(0,\eta),\right.$$
$$\left. u(0) = 0,\ u(\eta) = z, u'(0) = u'(\eta) = \delta\right\}.$$

Then the following properties hold true:

i) $\lim_{\delta\to 0^+} \varphi^\delta(z) = \varphi(z)$, *uniformly in* $[k, +\infty)$, *for every* $k > 0$;

ii) for every $\varepsilon \in (0,1)$, *there exists* $\overline{\delta}$ *such that* $\varphi^\delta(z) \ge (1-\varepsilon)\varphi(z)$ *for every* $\delta \le \overline{\delta}$ *and for every* $z > 0$.

PROOF. Fix $k > 0$ and let $\phi \in C^2([0,1])$ be such that $\phi(0) = \phi'(0) = 0$ and $\phi(1) = \phi'(1) = 1$. Moreover choose $0 < \overline{\delta} < k$ such that

$$\int_0^1 b(\delta|\phi'|)\,dx + \delta^2 \int_0^1 |\phi''|^2\,dx \le \frac{\varepsilon}{4}, \tag{5.1.45}$$

for every $\delta \le \overline{\delta}$. Fix now $\delta \in (0, \overline{\delta})$ and for a given $z \ge k$ set $z' := z - 2\delta$ and take (v, η), admissible pair for the minimum problem defining $\varphi(z')$ such that

$$\int_0^\eta b(|v'|)\,dx + \int_0^\eta |v''|^2\,dx \le \varphi(z') + \frac{\varepsilon}{2} \le \varphi(z) + \frac{\varepsilon}{2}. \tag{5.1.46}$$

We now define $\tilde{\eta} := \eta + 2$ and $\tilde{v} \in W^{2,2}(0,\tilde{\eta})$ by

$$\tilde{v}(t) := \begin{cases} \delta - \delta\phi(1-t) & \text{if } t \in [0,1), \\ v(t-1) + \delta & \text{if } t \in [1, \eta+1), \\ z - \delta + \delta\phi(t - \eta - 1) & \text{if } t \in [\eta+1, \tilde{\eta}]. \end{cases}$$

It is clear that $(\tilde{v}, \tilde{\eta})$ is an admissible pair for the minimum problem defining $\varphi^\delta(z)$, so that we have

$$\begin{aligned}
\varphi^\delta(z) &\leq \int_0^{\tilde{\eta}} b(|\tilde{v}'|)\,dx + \int_0^{\tilde{\eta}} |\tilde{v}''|^2\,dx \\
&= 2\left(\int_0^1 b(\delta|\phi'|)\,dx + \delta^2 \int_0^1 |\phi''|^2\,dx\right) + \int_0^\eta b(|v'|)\,dx + \int_0^\eta |v''|^2\,dx \\
&\leq \varphi(z) + \varepsilon,
\end{aligned} \tag{5.1.47}$$

where the last inequality follows from (5.1.45) and (5.1.46). Let now (v, η) be an admissible pair for $\varphi^\delta(z)$ satisfying

$$\int_0^\eta b(|v'|)\,dx + \int_0^\eta |v''|^2\,dx \leq \varphi^\delta(z) + \frac{\varepsilon}{2}. \tag{5.1.48}$$

We now define $\tilde{\eta} := \eta + 2$ and $\tilde{v}$ by

$$\tilde{v}(t) := \begin{cases} \delta\phi(t) & \text{if } t \in [0,1), \\ v(t-1) + \delta & \text{if } t \in [1, \eta+1), \\ z + 2\delta - \delta\phi(\tilde{\eta} - t) & \text{if } t \in [\eta+1, \tilde{\eta}]. \end{cases}$$

As above, we have

$$\begin{aligned}
\varphi(z) &\leq \varphi(z + 2\delta) \leq \int_0^{\tilde{\eta}} b(|\tilde{v}'|)\,dx + \int_0^{\tilde{\eta}} |\tilde{v}''|^2\,dx \\
&= 2\left(\int_0^1 b(\delta|\phi'|)\,dx + \delta^2 \int_0^1 |\phi''|^2\,dx\right) + \int_0^\eta b(|v'|)\,dx + \int_0^\eta |v''|^2\,dx \\
&\leq \varphi^\delta(z) + \varepsilon,
\end{aligned}$$

thanks to (5.1.45) and (5.1.48); recalling (5.1.47), *i)* is proved.

For the last part we suppose by contradiction that there exist $\varepsilon \in (0,1)$, a sequence $\delta_n \downarrow 0$, and sequence x_n such that

$$\varphi^{\delta_n}(x_n) < (1 - \varepsilon)\varphi(x_n), \tag{5.1.49}$$

for every $n \in \mathbb{N}$. Testing with the pair $(v(t) := \delta t, z/\delta)$, we easily obtain

$$\varphi^\delta(z) \leq \frac{b(\delta)}{\delta} z \leq C'z, \qquad \forall \delta < 1. \tag{5.1.50}$$

Taking into account *i)* we see that (5.1.49) and (5.1.50) imply

$$x_n \to 0 \qquad \text{and} \qquad \varphi^{\delta_n}(x_n) \to 0. \tag{5.1.51}$$

Let (v_n, η_n) be an admissible pair for the minimum problem defining $\varphi^{\delta_n}(x_n)$ such that

$$\int_0^{\eta_n} b(|v_n'|)\,dx + \int_0^{\eta_n} |v_n''|^2\,dx \leq \varphi^{\delta_n}(x_n) + (\varphi^{\delta_n}(x_n))^2; \tag{5.1.52}$$

arguing as in the proof of Lemma 5.1.6 we deduce that $\|v_n'\|_\infty \to 0$. Choose $\sigma > 0$ such that

$$b(t) \geq \left(1 - \frac{\varepsilon}{2}\right) Ct \qquad \forall t \leq \sigma \tag{5.1.53}$$

and let $\overline{n}$ be such that $\|v_n'\|_\infty \leq \sigma$ for every $n \geq \overline{n}$. Then, using (5.1.52), (5.1.53), (5.1.50), (5.1.51) and recalling that $\varphi(z) \leq Cz$ for every $z > 0$ (see (5.1.15)), we estimate

$$\begin{aligned} \varphi^{\delta_n}(x_n) &\geq \int_0^{\eta_n} b(|v_n'|)\,dx - (\varphi^{\delta_n}(x_n))^2 \geq \left(1 - \frac{\varepsilon}{2}\right) Cx_n - (C'x_n)^2 \\ &\geq \left(1 - \frac{3}{4}\varepsilon\right) Cx_n \geq \left(1 - \frac{3}{4}\varepsilon\right) \varphi(x_n), \end{aligned}$$

for n large enough, a contradiction with (5.1.49). □

We are now in a position to conclude the proof of Theorem 5.1.2.

PROOF OF THEOREM 5.1.2: the case $b^0(1) < +\infty$.

The Γ-lim sup inequality can be proved as in the other case. So suppose that the structure condition *iii)* appearing in the statement of the theorem holds so that $F_{b,g}$ is well defined and coincides with $\overline{\mathcal{F}_{b,g}}$. We may also suppose that $g \not\equiv 0$, otherwise the Γ-liminf inequality is trivial. Let $\varepsilon_n \to 0$ and $u_{\varepsilon_n} \to u$ in L^1 and such that $\exists \lim_{n\to\infty} F_{\varepsilon_n}(u_{\varepsilon_n}) < +\infty$. Choose now an infinitesimal sequence c_n with the same properties as in the proof of Lemma 5.1.11; set

$$D_n := \left\{ x \in I : \; |u_{\varepsilon_n}'| > \frac{c_n}{r(\varepsilon_n)} \right\} = \bigcup_{k=1}^{\infty} (a_n^k, b_n^k) = \bigcup_{k=1}^{\infty} I_n^k$$

and define

$$v_{\varepsilon_n}(x) := \begin{cases} u_{\varepsilon_n}(x) & \text{if } x \in I \setminus D_n \\ u_{\varepsilon_n}(a_n^k) & \text{if } x \in (a_n^k, b_n^k) \end{cases}.$$

Finally set $w_{\varepsilon_n} := u_{\varepsilon_n} - v_{\varepsilon_n}$ and $z_{\varepsilon_n}(x) := w_{\varepsilon_n}(r(\varepsilon_n)x)$. For fixed $\delta \in (0,1)$, with exactly the same arguments of Lemma 5.1.11, we obtain

$$\begin{aligned} F_{\varepsilon_n}(u_{\varepsilon_n}, D_n) &\geq (1-\delta) \sum_k \left(\int_{I_n^k/(r(\varepsilon_n))} b(|z_{\varepsilon_n}'|)\,dx + \int_{I_n^k/(r(\varepsilon_n))} |z_{\varepsilon_n}''|^2\,dx \right) \\ &\geq (1-\delta) \sum_k \inf_{\eta>0} \inf \left\{ \int_0^\eta b(|z'|)\,dx + \int_0^\eta |z''|^2\,dx : \; z \in W^{2,2}(0,\eta), \right. \\ &\qquad \left. z(0) = 0,\, z(\eta) = |w_{\varepsilon_n}(b_n^k)|,\; z'(0) = z'(\eta) = c_n \right\} \\ &= (1-\delta) \sum_k \varphi^{c_n}(|w_{\varepsilon_n}(b_n^k)|) = (1-\delta) \sum_{S_{v_{\varepsilon_n}}} \varphi^{c_n}(v_{\varepsilon_n}^+ - v_{\varepsilon_n}^-), \end{aligned} \tag{5.1.54}$$

for n large enough, where φ^{c_n} is the function defined in Lemma 5.1.14 (with $\delta = c_n$). Using *ii)* of Lemma 5.1.14, from (5.1.54) we deduce

$$F_{\varepsilon_n}(u_{\varepsilon_n}, D_n) \geq (1-\delta)^2 \sum_{S_{v_{\varepsilon_n}}} \varphi(v_{\varepsilon_n}^+ - v_{\varepsilon_n}^-),$$

for n large enough. Combining the estimate above with Lemma 5.1.10, we therefore obtain (passing to a subsequence, if needed),

$$\begin{aligned} F_{\varepsilon_n}(u_{\varepsilon_n}) &\geq (1-\delta)^2 \left(\int_{I\setminus D_n} \psi_i(|v'_{\varepsilon_n}|)\,dx + \sum_{S_{v_{\varepsilon_n}}} \varphi(v^+_{\varepsilon_n} - v^-_{\varepsilon_n}) \right) \\ &= (1-\delta)^2 \left(\int_I \psi_i(|v'_{\varepsilon_n}|)\,dx + \sum_{S_{v_{\varepsilon_n}}} \varphi(v^+_{\varepsilon_n} - v^-_{\varepsilon_n}) \right), \end{aligned} \qquad (5.1.55)$$

where, according to Lemma 5.1.10, ψ_i is convex, $\psi_i^\infty(1) = g^\infty(1) \wedge b^0(1)$ and $\psi_i \uparrow g_1$ as $i \to \infty$. Since, by Lemma 5.1.8, we have $\sup_n \operatorname{Var} v_{\varepsilon_n} \leq \sup_n \operatorname{Var} u_{\varepsilon_n} < +\infty$, Rellich's Theorem implies that v_{ε_n} is precompact in L^1 and since $v_{\varepsilon_n} \to u$ in measure (recall that $|D_n| \to 0$), we get $v_{\varepsilon_n} \to u$ in L^1. Applying Theorem 1.1.3 (recall that $b^0(1) = \varphi^0(1)$, by virtue of Lemma 5.1.6), from (5.1.55) we deduce

$$\liminf_{n\to\infty} F_{\varepsilon_n}(u_{\varepsilon_n}) \geq (1-\delta)^2 \left(\int_I \psi_i(|u'|)\,dx + \sum_{S_u} \varphi_1(u^+ - u^-) + (g^\infty(1) \wedge b^0(1))|D^c u| \right);$$

letting $i \uparrow +\infty$ and $\delta \downarrow 0$ we finally obtain the desired Γ-$\liminf$ inequality. □

Remark 5.1.15 Looking carefully at the proof we see that the structure assumption *iii)* of Theorem 5.1.2 can be slightly weakened without changing the result; more precisely it is sufficient to suppose that there exists a family $(g_n^k)_{n,k}$ of positive continuous non-decreasing functions enjoying the following properties:

i) $f_n \geq g_n^k$ for very $n,\ k \in \mathbb{N}$;

ii) for every $k \in \mathbb{N}$ the family $(g_n^k)_n$ satisfies either **st1)** or **st2)**;

iii) $g_n^k(t) \to g^k(t)$ for every $t \geq 0$ and $r_n g_n^k(t/r_n) \to b^k(t)$ for every $t > 0$, as $n \to \infty$, with g^k and b^k satisfying
$$g^k \uparrow g \qquad \text{and} \qquad (b^k)^0(1) \uparrow b^0(1) \qquad \text{as } k \to \infty.$$

Indeed call G_n^k the functional associated with g_n^k; then, for every $k \in \mathbb{N}$, by Theorem 5.1.2 we have

$$\Gamma\text{-}\liminf_{n\to\infty} F_n \geq \Gamma\text{-}\lim_{n\to\infty} G_n^k = F_{b^k,g^k},$$

where $F_{b^k,g^k} \uparrow F_{b,g}$ as $k \to \infty$.

We want now to show that if $g : [0,+\infty) \to [0,+\infty)$ is any superlinear non-decreasing convex function and $b : [0,+\infty) \to [0,+\infty)$ is an arbitrary concave function with $b^0(1) = +\infty$, then $F_{b,g}$ can be reached by functionals of the form (5.1.1).

Theorem 5.1.16 *Let $g : [0,+\infty) \to [0,+\infty)$ be non-decreasing, convex, and superlinear ($g^\infty(1) = +\infty$) and let $b : [0,+\infty) \to [0,+\infty)$ be non-decreasing and concave with $b(0) = 0$ and $b^0(1) = +\infty$. Then there exists a family (f_ε) of positive, continuous, and non-decreasing functions such that the functionals*

$$F_\varepsilon := \begin{cases} \displaystyle\int_I f_\varepsilon(|u'|)\,dx + \varepsilon^3 \int_I |u''|^2\,dx & \text{if } u \in W^{2,2}(I); \\ +\infty & \text{otherwise in } L^1(I), \end{cases}$$

Γ-converge with respect to the L^1-metric to $F_{b,g}$, as $\varepsilon \to 0^+$.

The theorem is an immediate consequence of Theorem 5.1.2 and of the following proposition which is proved in [31] (see Lemmas 6.6 and 6.7).

Proposition 5.1.17 *Let g and b be as in the previous theorem. Then the functions f_ε defined by*

$$f_\varepsilon(t) := \min\left\{g(s) + \frac{1}{\varepsilon}b(\varepsilon(t-s)) \,:\, s \in [0,t]\right\},$$

are continuous, non-decreasing and satisfy the following properties:

i) $f_\varepsilon(t) \to g(t)$ for every $t \geq 0$;

ii) $\varepsilon f(t/\varepsilon) \to b(t)$ for every $t > 0$;

iii) setting $x_\varepsilon := \sup\{x \geq 0 : f_\varepsilon(x) = g(x)\}$, there holds that $f_\varepsilon = g$ in $[0, x_\varepsilon]$ and f_ε is concave in $[x_\varepsilon, +\infty)$; moreover $x_\varepsilon \to +\infty$ as $\varepsilon \to 0^+$.

We conclude this subsection with some considerations on the asymptotic behaviour of the function φ defined in (5.1.3).

Proposition 5.1.18 *i) Let $b(t) = ct^p$ with $c > 0$ and $p \in [0,1)$. Then $\varphi(z) = m(p)c^{\frac{3}{4-p}}z^{\frac{2+p}{4-p}}$, where*

$$m(p) := \min\left\{\left[\left(\frac{3}{1-p}\right)^{\frac{1-p}{4-p}} + \left(\frac{1-p}{3}\right)^{\frac{3}{4-p}}\right]\left(\int_0^1 |v''|^2\,dt\right)^{\frac{1-p}{4-p}}\left(\int_0^1 |v'|^p\,dt\right)^{\frac{3}{4-p}} : \right.$$
$$\left. v \in W^{2,2}(0,1),\ v(0) = 0,\ v(1) = 1,\ v'(0) = v'(1) = 0\right\}. \quad (5.1.56)$$

ii) Let $b : [0,+\infty) \to [0,+\infty)$ be concave with $b^0(1) \neq 0$. Then the function φ defined in (5.1.3) satisfies the growth condition

$$C_1(\sqrt{z} - 1) \leq \varphi(z) \leq C_2(z+1) \qquad \forall z \geq 0, \quad (5.1.57)$$

for suitable C_1, $C_2 > 0$.

iii) For every $\gamma \in [1/2, 1)$ there exists a concave function b satisfying the hypotheses of Theorem 5.1.16 such that the associated φ satisfies

$$\lim_{z\to+\infty} \frac{\varphi(z)}{z^\gamma} = +\infty \qquad \textit{and} \qquad \lim_{z\to+\infty}\frac{\varphi(z)}{z^{\gamma+\varepsilon}} = 0 \qquad \forall \varepsilon > 0. \quad (5.1.58)$$

Proof. *i):* For notational convenience we set

$$S_{\eta,z} := \{u \in W^{2,2}(0,\eta) : \ u(0) = 0,\ u(\eta) = z,\ u'(0) = u'(\eta) = 0\};$$

then, noting that for every $v \in S_{\eta,z}$ we can write $v(\cdot) = w(\cdot/\eta)$ with $w \in S_{1,z}$, we can use the definition of φ to compute

$$\begin{aligned}
\varphi(z) &= \inf_{\eta}\inf_{v\in S_{\eta,z}} \left(c\int_0^\eta |v'|^p\,dt + \int_0^\eta |v''|^2\,dt \right) \\
&= \inf_{w\in S_{1,z}}\inf_{\eta} \left(c\int_0^\eta \frac{1}{\eta^p}\left|w'\left(\frac{t}{\eta}\right)\right|^p dt + \int_0^\eta \frac{1}{\eta^4}\left|w''\left(\frac{t}{\eta}\right)\right|^2 dt \right) \\
&= \inf_{w\in S_{1,z}}\inf_{\eta} \left(c\eta^{1-p}\int_0^1 |w'|^p\,ds + \frac{1}{\eta^3}\int_0^1 |w''|^2\,ds \right) \\
&= \inf_{w\in S_{1,z}} \left\{ \left[\left(\frac{3}{1-p}\right)^{\frac{1-p}{4-p}} + \left(\frac{1-p}{3}\right)^{\frac{3}{4-p}} \right] c^{\frac{3}{4-p}} \left(\int_0^1 |w''|^2\,ds\right)^{\frac{1-p}{4-p}} \left(\int_0^1 |w'|^p\,ds\right)^{\frac{3}{4-p}} \right\} \\
&= \inf_{w\in S_{1,1}} \left\{ \left[\left(\frac{3}{1-p}\right)^{\frac{1-p}{4-p}} + \left(\frac{1-p}{3}\right)^{\frac{3}{4-p}} \right] \left(\int_0^1 |w''|^2\,ds\right)^{\frac{1-p}{4-p}} \left(\int_0^1 |w'|^p\,ds\right)^{\frac{3}{4-p}} \right\} c^{\frac{3}{4-p}} z^{\frac{2+p}{4-p}} \\
&= m(p)c^{\frac{3}{4-p}} z^{\frac{2+p}{4-p}}.
\end{aligned}$$

It is clear also from the computations above that there exists an optimal pair (η, v) for the problem defining φ: let v be a solution of problem (5.1.56), then, setting

$$\eta := \left[\frac{3}{c^p(1-p)}\right]^{\frac{1}{4-p}} \left(\frac{\int_0^1 |v''|^2\,ds}{\int_0^1 |v'|^p\,ds}\right)^{\frac{1}{4-p}} z^{\frac{2+p}{4-p}} \qquad \text{and} \qquad w(t) := zv\left(\frac{t}{\eta}\right), \tag{5.1.59}$$

we have that (η, w) is an optimal pair.

ii): Under our assumptions there exists $C > 0$ such that $b(t) \le C(1+t)$ for every $t \ge 0$. Take (η, v) such that $v \in S_{\eta,z}$, v is non-decreasing and

$$C\eta + \int_0^\eta |v''|^2\,dx = m(0)C^{3/4}\sqrt{z}:$$

this is possible thanks to the previous point. Then

$$\varphi(z) \le C\int_0^\eta |v'|\,dx + C\eta + \int_0^\eta |v''|^2\,dx = Cz + m(0)C^{3/4}\sqrt{z} \le C'(1+z).$$

Concerning the other inequality, since, under our hypotheses, there exist α, $\beta > 0$ such that $b(t) \ge \alpha t \wedge \beta$, it will be enough to prove the following claim.

Claim. Let $b(t) = \alpha t \wedge \beta$ with α, $\beta > 0$. Then

$$\lim_{z\to+\infty} \frac{\varphi(z)}{m(0)\beta^{3/4}\sqrt{z}} = 1.$$

First of all, since $b(t) \le \beta$, by comparison and by the previous point we immediately obtain

$$\varphi(z) \le m(0)\beta^{3/4}\sqrt{z}. \tag{5.1.60}$$

Let $z_n \uparrow +\infty$ and let (η_n, z_n) be an admissible pair for $\varphi(z_n)$ such that v_n is non-decreasing and

$$\int_0^{\eta_n} (\alpha|v_n'| \wedge \beta)\,dx + \int_0^{\eta_n} |v_n''|^2\,dx < \varphi(z_n) + 1. \tag{5.1.61}$$

Let $\sigma_n \in (0,1)$ be such that $\int_{\{x\in I:\,|v_n'|\le\beta/\alpha\}} |v_n'|\,dx = \sigma_n z_n$; since, by (5.1.60) and (5.1.61),

$$m(0))\beta^{3/4}\sqrt{z_n}+1 \ge \int_0^{\eta_n} (\alpha|v_n'|\wedge\beta)\,dx \ge \int_{\{x\in I:\,|v_n'|\le\beta/\alpha\}} \alpha|v_n'|\,dx = \alpha\sigma_n z_n,$$

it follows that $\sigma_n \to 0$. Consider the sets $D_n := \{x\in I:\, |v_n'|>\beta/\alpha\} = \cup_{k=1}^{\infty} I_n^k$, where $(I_n^k)_k$ is the collection of the connected components of D_n. We denote also $I_n^k := (a_n^k, b_n^k)$. Let $\Phi \in C^2([0,1])$ be such that $\Phi(0)=\Phi'(0)=0$, $\Phi(1)=1$, and $\Phi'(1)=\beta/\alpha$ and, for every $t\in[1,1+|D_n|]$ set

$$i_n(t) := \min\left\{k:\ \sum_{j=1}^{k}|I_n^j| \ge t-1\right\} \qquad \tau_n(t) := t-1-\sum_{j=1}^{i_n(t)-1}|I_n^j|.$$

We can now define the new sequence of admissible pairs $(\tilde\eta_n,\tilde v_n)$ by $\tilde\eta_n := |D_n|+2$ and $\tilde v_n(t) = \int_0^t \tilde v_n'(s)\,ds$, where

$$\tilde v_n' := \begin{cases} \Phi'(s) & \text{if } s\in[0,1],\\ v_n'(a_n^{i_n(s)}+\tau_n(s)) & \text{if } s\in[1,\tilde\eta_n-1],\\ \Phi'(\tilde\eta_n - s) & \text{if } s\in[\tilde\eta_n-1,\tilde\eta_n]. \end{cases}$$

Note that $\tilde v_n$ is constructed by gluing together the pieces of v_n defined on the sets I_n^k; since $v_n'(a_n^k) = v'(b_n^k) = \beta/\alpha$ for every k, we have $\tilde v_n \in W^{2,2}(0,\tilde\eta_n)$. Therefore $\tilde v_n \in S_{\tilde\eta_n,\tilde z_n}$ with $\tilde z_n := (1-\sigma_n)z_n + 2$ and, since by construction

$$\int_{D_n}(\alpha|v_n'|\wedge\beta)\,dx + \int_{D_n}|v_n''|^2\,dx = \beta\tilde\eta_n + \int_0^{\tilde\eta_n}|\tilde v_n''|^2\,dx - 2\left(\int_0^1|\Phi'|\,dx + \int_0^1|\Phi''|^2\,dx\right),$$

recalling (5.1.61) and *i)* we can estimate

$$\begin{aligned} \varphi(z_n)+1 &\ge \beta\tilde\eta_n + \int_0^{\tilde\eta_n}|\tilde v_n''|^2\,dx - 2\left(\int_0^1|\Phi'|\,dx+\int_0^1|\Phi''|^2\,dx\right)\\ &\ge \inf_{\eta>0}\inf_{S_{\eta,\tilde z_n}}\left(\beta\eta+\int_0^{\eta}|\tilde v''|^2\,dx\right) - 2\left(\int_0^1|\Phi'|\,dx+\int_0^1|\Phi''|^2\,dx\right)\\ &= m(0)\beta^{3/4}\sqrt{(1-\sigma_n)z_n+2} - 2\left(\int_0^1|\Phi'|\,dx+\int_0^1|\Phi''|^2\,dx\right), \end{aligned}$$

whence, taking into account that $\sigma_n\to 0$,

$$\liminf_{z\to+\infty}\frac{\varphi(z)}{m(0)\beta^{3/4}\sqrt{z}} \ge 1,$$

which combined with (5.1.60), gives the thesis of the claim.

iii): For simplicity we treat in details only the case $\gamma = 1/2$. We take $b(t) := 1+\log(1+t)$ for $t>0$ and $b(0)=0$. Fix $p\in(0,1)$ and take (η,w) with $w\in S_{\eta,z}$ and satisfying

$$\int_0^{\eta}|w'|^p\,dx + \int_0^{\eta}|w''|^2\,dx = m(p)z^{\frac{2-p}{4-p}} \quad\text{and}\quad \eta \le c(p)z^{\frac{2-p}{4-p}}:$$

this is possible by virtue of *i)* (see (5.1.59)). Then, since $b(t)\le 1+t^p$ we have

$$\varphi(z) \le (m(p)+c(p))z^{\frac{2-p}{4-p}}; \tag{5.1.62}$$

since as p varies in $(0,1)$ the exponent $(2-p)/(4-p)$ varies in $(1/2,1)$, from (5.1.62) we deduce that

$$\lim_{z\to+\infty}\frac{\varphi(z)}{z^{(1/2)+\varepsilon}}=0 \qquad \forall\varepsilon>0.$$

Now take two positive sequences (α_n) and (β_n) with $\beta_n\to+\infty$ such that $b(t)\geq b_n(t):=\alpha_n t\wedge\beta_n$, for every $t\geq 0$ and for every $n\in\mathbb{N}$. Calling φ_n the function associated with b_n, by the claim proved above, we have

$$\liminf_{z\to+\infty}\frac{\varphi(z)}{\sqrt{z}}\geq\lim_{z\to+\infty}\frac{\varphi_n(z)}{\sqrt{z}}=m(0)\beta_n^{3/4},$$

for every $n\in\mathbb{N}$; letting $n\to\infty$ we eventually complete the proof of (5.1.58). If γ is any number in $(1/2,1)$, take $b(t)=t^p\log(1+t)$, where p is such that $\gamma=(2-p)/(4-p)$, and argue as above. □

5.2 Some applications

In this subsection we are going to apply the results of the previous one to study the singular perturbations of the one-dimensional functionals of the form

$$G_\varepsilon(u)=\frac{1}{\varepsilon}\int_I f(\varepsilon^{1/q}|u'|)\,dx,$$

where $q\geq 1$. More precisely, given a positive function $p(\varepsilon)$ such that $\lim_{\varepsilon\to 0^+}p(\varepsilon)=0$, we set

$$F_\varepsilon(u)=\begin{cases}\dfrac{1}{\varepsilon}\displaystyle\int_I f(\varepsilon^{1/q}|u'|)\,dx+(p(\varepsilon))^3\int_I|u''|^2\,dx & \text{if } u\in W^{2,2}(I),\\ +\infty & \text{otherwise in } L^1(I),\end{cases} \tag{5.2.1}$$

and we aim to classify all the possible Γ-limits generated by the family (F_ε) depending on the asymptotic behaviour of the "rescaling" function p. Let us begin with the case $q>1$. Let $f:[0,+\infty)\to[0,+\infty)$ be non-decreasing, continuous, and satisfying the following properties:

H1) f is concave in $(x_1,+\infty)$ for some $x_1>0$;

H2) $\displaystyle\lim_{x\to0^+}\frac{f(x)}{x^q}=\alpha>0$, with $q>1$;

H3) $\displaystyle\lim_{x\to+\infty}\frac{f(x)}{x}=0$.

We will show that there exists a unique (up to asymptotic equivalence) rescaling function $r(\varepsilon)$ which generates non-trivial (i.e. non-zero) "free-discontinuity" functionals. Setting $h(x):=f(x)/x$, such a rescaling function is defined as:

$$r(\varepsilon):=\frac{\varepsilon^{1/q}}{h^{-1}\left(\varepsilon^{1/q'}\right)}, \tag{5.2.2}$$

where q is the exponent appearing in *H2)* while q' denotes its Lebesgue conjugate exponent satisfying $1/q+1/q'=1$.

Remark 5.2.1 Note that, for ε small enough, r is well defined, indeed, being f concave at infinity and sublinear, h becomes decreasing for x large enough. Moreover

$$\lim_{n\to\infty} \frac{\sqrt{\varepsilon_n}}{r(\varepsilon_n)} = \lim_{n\to\infty} h^{-1}(\sqrt{\varepsilon_n}) = +\infty$$

since $h \downarrow 0$ as $x \to +\infty$.

Our main result is the following theorem.

Theorem 5.2.2 *Let $I \subset \mathbb{R}$ be a bounded interval and let $f : [0,+\infty) \to [0,+\infty)$ be a non-decreasing continuous function satisfying hypotheses H1), H2), and H3) and $p(\varepsilon)$ be a positive function such that $\lim_{\varepsilon\to 0^+} p(\varepsilon) = 0$. Finally let $(\varepsilon_n)_{n\in\mathbb{N}}$ be an infinitesimal sequence such that*

$$\lim_{n\to\infty} \frac{p(\varepsilon_n)}{r(\varepsilon_n)} = a > 0 \qquad \text{and} \qquad \exists \lim_{n\to+\infty} \frac{f\left(t\frac{\varepsilon_n{}^{1/q}}{r(\varepsilon_n)}\right)}{f\left(\frac{\varepsilon_n{}^{1/q}}{r(\varepsilon_n)}\right)} =: b(t) \qquad \forall t > 0. \tag{5.2.3}$$

If $b^0(1) = +\infty$ then the functionals F_{ε_n} (defined in (5.2.1)) Γ-converge with respect to the L^1-metric to

$$F(u) := \begin{cases} \alpha \displaystyle\int_I |u'|^q\,dx + \sum_{x\in S_u} \varphi^{(a)}(u^+(x) - u^-(x)) & \text{if } u \in SBV(I), \\ +\infty & \text{otherwise in } L^1(I); \end{cases} \tag{5.2.4}$$

where $\varphi^{(a)}$ is defined by (5.1.3) with $b^{(a)}(t) := ab(t/a)$ instead of $b(t)$.
Conversely, if $b^0(1) = C < +\infty$, then $\Gamma\text{-}\lim_{n\to\infty} F_{\varepsilon_n} = F$ with F given by

$$F(u) := \begin{cases} \displaystyle\int_I g(|u'|)\,dx + \sum_{x\in S_u} \varphi^{(a)}(u^+(x) - u^-(x)) + C|D^c u| & \text{if } u \in BV(I), \\ +\infty & \text{if } x \in L^1(I)\setminus BV(I), \end{cases} \tag{5.2.5}$$

where $g := (\alpha x^q \wedge Cx)^{**}$ while $\varphi^{(a)}$ is as above. Moreover, in both cases, every sequence u_n such that $\sup_n(F_n(u_n) + \|u_n\|_1) < +\infty$ is strongly precompact in L^p for every $p \geq 1$.

An easy consequence of the theorem is the fact that, up to asymptotic equivalence, the function r defined in (5.2.2) is the unique nontrivial rescaling function; this is made precise by the following Corollary whose easy proof is left to the reader (see [4]).

Corollary 5.2.3 *Let I, f, and r be as in Theorem 5.1.2. Let $(\varepsilon_n)_{n\in\mathbb{N}}$ and $(a_n)_{n\in\mathbb{N}}$ be two sequences converging to 0 and, for every n, set*

$$F_n(u) = \begin{cases} \dfrac{1}{\varepsilon_n} \displaystyle\int_I f(\varepsilon_n{}^{1/q}|u'|)\,dx + (a_n)^3 \int_I |u''|^2\,dx & \text{if } u \in W^{2,2}(I), \\ +\infty & \text{otherwise in } L^1(I). \end{cases}$$

If $\lim_{n\to\infty} a_n/r(\varepsilon_n) = 0$, then $\Gamma\text{-}\lim_{n\to\infty} F_n = 0$ with respect to the L^1-metric; if $\lim_{n\to\infty} a_n/r(\varepsilon_n) = +\infty$, then the functionals F_n Γ-converge to

$$F(u) := \begin{cases} \alpha \displaystyle\int_I |u'|^q\,dx & \text{if } u \in W^{1,q}(I), \\ +\infty & \text{otherwise in } L^1(I). \end{cases}$$

Remark 5.2.4 Since f is concave for x large, from the previous remark and from the definition (5.2.3), it follows that $b(\cdot)$ is in turn concave. Moreover, again by the sublinearity and the concavity assumption we get the existence of $x_2 \geq x_1$ such that

$$f(a+b) \leq f(a) + f(b) \qquad \forall a, b > x_2; \tag{5.2.6}$$

if f is unbounded, we deduce

$$\begin{aligned} b(t) &\leq \limsup_{x\to+\infty} \frac{f(tx)}{f(x)} \leq \limsup_{x\to+\infty} \frac{f(([t]+1)x)}{f(x)} \\ &\leq \limsup_{x\to+\infty} \frac{([t]+1)f(x)}{f(x)} = [t]+1, \end{aligned} \tag{5.2.7}$$

where $[t]$ denotes the integer part of t; if f is bounded we get trivially $b(t) \equiv 1$. Finally, since $b(1) = 1$, taking into account the concavity it turns out that $b(t) > 0$ for any $t > 0$.

PROOF OF THEOREM 5.2.2. Setting $r_n := p(\varepsilon_n)$ and $f_n(t) := \frac{1}{\varepsilon_n} f\left(\varepsilon_n^{1/q} t\right)$, by *H2)* we get immediately that $f_n(t) \to \alpha t^q$ for every $t \geq 0$; moreover, using the identity $f\left(\frac{\varepsilon_n^{1/q}}{r(\varepsilon_n)}\right) = \frac{r(\varepsilon_n)}{\varepsilon_n}$, which follows easily from the definition of r (see (5.2.2)), for $t > 0$ it turns out

$$r_n f_n\left(\frac{t}{r_n}\right) = \frac{p(\varepsilon_n)}{r(\varepsilon_n)} \frac{r(\varepsilon_n)}{\varepsilon_n} f\left(\frac{\varepsilon_n^{1/q}}{r(\varepsilon_n)} \frac{r(\varepsilon_n)}{p(\varepsilon_n)} t\right) = \frac{p(\varepsilon_n)}{r(\varepsilon_n)} \frac{f\left(\frac{\varepsilon_n{}^{1/q}}{r(\varepsilon_n)} \frac{r(\varepsilon_n)}{p(\varepsilon_n)} t\right)}{f\left(\frac{\varepsilon_n{}^{1/q}}{r(\varepsilon_n)}\right)} \overset{n\to\infty}{\longrightarrow} ab\left(\frac{t}{a}\right) = b^{(a)}(t), \tag{5.2.8}$$

where we used (5.2.3). By the first part of Theorem 5.1.2 we therefore obtain the Γ-lim sup inequality. Concerning the other inequality, by Theorem 5.1.2 and Remark 5.1.15, it will be proved if for every $\delta > 0$ we are able to construct a family of functions (f_n^δ) such that $f_n \geq f_n^\delta$, f_n^δ satisfies the structure condition **st2)** and finally

$$f_n^\delta(t) \to (1-\delta)\alpha t^q \ \forall t \geq 0 \qquad \text{and} \qquad r_n f_n^\delta\left(\frac{t}{r_n}\right) \to b^{(a)}(t) \ \forall t > 0. \tag{5.2.9}$$

It is also clear that if we exhibit a function f^δ verifying

a) $f \geq f^\delta$,

b) $\displaystyle\lim_{t\to 0+} \frac{f^\delta(t)}{t^q} = (1-\delta)\alpha$,

c) $\displaystyle\lim_{t\to+\infty} \frac{f^\delta(t)}{f(t)} = 1$,

d) there exists $\overline{x}$ such that f^δ is convex in $[0, \overline{x}]$ and concave in $[\overline{x}, +\infty)$,

then the family $f_n^\delta(t) := \frac{1}{\varepsilon_n} f^\delta(\varepsilon_n^{1/q} t)$ enjoys all the required conditions. Therefore it remains only to construct such a f^δ. By assumption we know that there exist $x' < x''$ such that $f(t) \geq (1-\delta)\alpha t^q$ for every $t \in [0, x']$ and f is concave in $[x'', +\infty)$. Define $a(t) := (1-\delta)\alpha \frac{(x')^q}{x''} t$, $g := [\min\{(1-\delta)\alpha t^q, a(t)\}]^{**}$, and finally

$$f^\delta(t) := \begin{cases} g(t) & \text{if } t \leq x'', \\ f(t) + g(x'') - f(x'') & \text{if } t \geq x''; \end{cases}$$

it is easy to see that f^δ satisfies all conditions a),..., d) (see Figure 5.2). Finally the equicoerciveness of the family (F_{ε_n}) follows again from Theorem 5.1.2. □

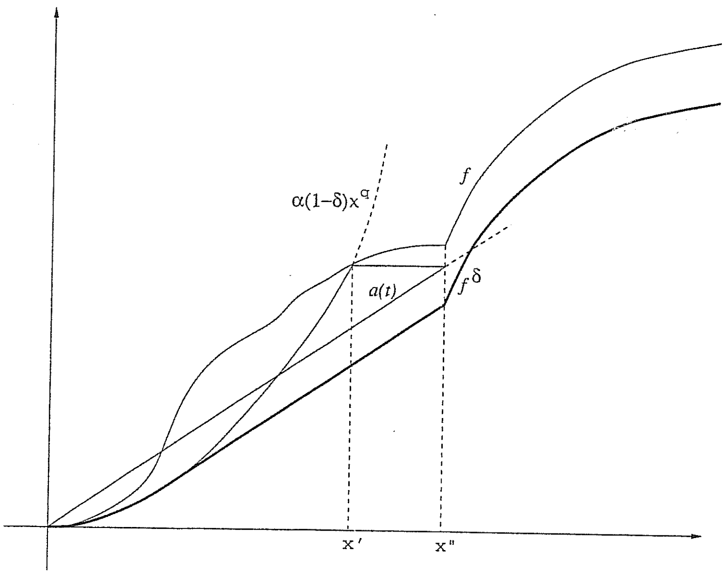

Figure 5.2: The construction of f^δ.

An easy consequence of Theorem 5.2.2 is the following compactness result.

Theorem 5.2.5 *Let I, f, and r as above and consider the family of functionals F_ε defined in* (5.2.1) *with $p(\varepsilon)$ satisfying $0 < \liminf_{\varepsilon\to 0^+} \frac{p(\varepsilon)}{r(\varepsilon)} \le \limsup_{\varepsilon\to 0^+} \frac{p(\varepsilon)}{r(\varepsilon)} < +\infty$. Then, for every infinitesimal sequence $(\varepsilon_n)_n$, there exist a subsequence, still denoted by $(\varepsilon_n)_n$ and a concave, non-decreasing function b such that $\exists\Gamma\text{-}\lim_n F_{\varepsilon_n} = F$ with respect to the L^1-convergence, where F is either as in* (5.2.4) *or as in* (5.2.5).

PROOF. It is sufficient to extract a subsequence such that (5.2.3) holds and then to apply Theorem 5.2.2. The existence of such a subsequence is an easy consequence of Helly's Theorem. □

Proposition 5.2.6 *Let f be a function satisfying the hypotheses H1), H2), and H3) of Theorem 5.2.2 and let us suppose in addition that*

$$\exists \lim_{x\to+\infty} \frac{f(tx)}{f(x)} := b(t) \qquad \forall t > 0. \tag{5.2.10}$$

Let F_ε the functional defined in (5.2.1), *with p satisfying $\lim_{\varepsilon\to 0^+} \frac{p(\varepsilon)}{r(\varepsilon)} = a > 0$, where r is the rescaling function defined in* (5.2.2). *If $b^0(1) = +\infty$, then $\Gamma\text{-}\lim_{\varepsilon\to 0^+} F_\varepsilon = F$ with respect to the L^1-metric, with F given by*

$$F(u) := \begin{cases} \displaystyle\int_I |u'|^q\,dx + m(\gamma) a^{\frac{3(1-\gamma)}{4-\gamma}} \sum_{x\in S_u} (u^+ - u^-)^{\frac{2+\gamma}{4-\gamma}} & \text{if } u \in SBV(I) \\ +\infty & \text{in } L^1(I)\setminus SBV(I), \end{cases}$$

where $\gamma = \log b(\mathrm{e})$ *and* $m(\gamma)$ *is the constant defined in* (5.1.56). *If* $b^0(1) < +\infty$ *the family* F_ε Γ*-converges to the functional* F *given by*

$$F(u) := \begin{cases} \int_I g_\alpha(|u'|)\,dx + |D^s u| & \text{if } u \in BV(I) \\ +\infty & \text{in } L^1(I)\setminus BV(I), \end{cases}$$

where $g_\alpha = (\alpha x^q \wedge x)^{**}$.

Proof. From Theorem 5.2.2, it is clear that $\Gamma\text{-}\lim_{\varepsilon\to 0^+} F_\varepsilon = F$ where F is the functional defined either in (5.2.4) or in (5.2.5). It remains only to prove that

$$\varphi^{(a)}(z) = m(\gamma) a^{\frac{3(1-\gamma)}{4-\gamma}} z^{\frac{2+\gamma}{4-\gamma}} \qquad \forall z > 0, \tag{5.2.11}$$

if (5.2.3) hold true, or

$$\varphi^{(a)}(z) = z \qquad \forall z > 0, \tag{5.2.12}$$

otherwise. First of all note that from (5.2.10) it follows immediately that $b(st) = b(s)b(t)$ for $t,\ s > 0$ and therefore $b(t) = t^\gamma$ for $t > 0$, with $\gamma = \log b(\mathrm{e})$; by remark 5.2.4 (and in particular by (5.2.7)), we have that $\gamma \in [0,1]$. If $\gamma < 1$, (5.2.11) follows from Proposition 5.1.18 since $b^{(a)}(t) ab(t/a) = a^{1-\gamma} t^\gamma$. If $\gamma = 1$, then by Lemma 5.1.5, we get (5.2.12). □

Let us see now some examples. We will use the following notation: given two functions r_1 and r_2 we will write $r_1 \simeq r_2$ if they are asymptotically equivalent, that is if $\lim_{\varepsilon\to 0^+} \frac{r_1(\varepsilon)}{r_2(\varepsilon)} = 1$.

Example 5.2.7 Let γ belong to $[0,1)$ and set $f(x) := \frac{\alpha x^2}{1+x^{2-\gamma}}$; using the definitions (see (5.2.2) and (5.2.3)), it is easy to see that $r(\varepsilon) \simeq \varepsilon^{\frac{2-\gamma}{2-2\gamma}}$ and $b(t) = t^\gamma$, and therefore, setting

$$F_\varepsilon(u) := \begin{cases} \displaystyle\int_I \frac{\alpha |u'|^2}{1 + \varepsilon^{\frac{2-\gamma}{2}} |u'|^{2-\gamma}}\,dx + a^3 \varepsilon^{\frac{6-3\gamma}{2-2\gamma}} \int_I |u''|^2\,dx & \text{if } u \in W^{2,2}(I), \\ +\infty & \text{otherwise in } L^1(I), \end{cases}$$

by Proposition 5.2.6, we have that the functionals F_ε Γ-converge to

$$F^\gamma(u) := \begin{cases} \displaystyle\alpha \int_I |u'|^2\,dx + m(\gamma) a^{\frac{3(1-\gamma)}{4-\gamma}} \sum_{x\in S_u} (u^+ - u^-)^{\frac{2+\gamma}{4-\gamma}} & \text{if } u \in SBV(I) \\ +\infty & \text{in } L^1(I)\setminus SBV(I), \end{cases} \tag{5.2.13}$$

as $\varepsilon \to 0^+$, with respect to the L^1-metric, so that we recover the result of Dubs, Bouchitté & Seppecher (see [13]). Note that as γ varies in $[0,1)$, the exponent $\frac{2+\gamma}{4-\gamma}$ varies in $[\frac{1}{2}, 1)$. Moreover, note that $m(0)$ can be easily computed and it is equal to $2\sqrt{3/2} + \sqrt{2/3}$ (see [4]).

Example 5.2.8 Let $f(x) := (1 + x^\gamma)\log(1 + \alpha x^2)$ with $\gamma \in [0,1)$. We show now that

$$r(\varepsilon) \simeq (1-\gamma)^{\frac{1}{1-\gamma}} \frac{\varepsilon^{\frac{2-\gamma}{2-2\gamma}}}{(\log\frac{1}{\varepsilon})^{\frac{1}{1-\gamma}}}. \tag{5.2.14}$$

Indeed, with the same notations of Theorem 5.1.2, we have

$$\lim_{\varepsilon\to 0^+} r(\varepsilon)\frac{\left(\log\frac{1}{\varepsilon}\right)^{\frac{1}{1-\gamma}}}{\varepsilon^{\frac{2-\gamma}{2-2\gamma}}} = \lim_{\varepsilon\to 0^+}\frac{\sqrt{\varepsilon}}{h^{-1}(\sqrt{\varepsilon})}\frac{\left(\log\frac{1}{\varepsilon}\right)^{\frac{1}{1-\gamma}}}{\varepsilon^{\frac{2-\gamma}{2-2\gamma}}} = \lim_{y\to+\infty}\frac{\left(\log\frac{1}{h^2}\right)^{\frac{1}{1-\gamma}}}{y h^{\frac{1}{1-\gamma}}} = (1-\gamma)^{\frac{1}{1-\gamma}},$$

where we performed the change of variable $y = h^{-1}(\sqrt{\varepsilon})$.

We finally observe that $b(t) = \lim_{x\to+\infty}\frac{f(tx)}{f(x)} = t^\gamma$ for all $t > 0$; therefore, setting

$$F_\varepsilon^\gamma(u) := \begin{cases} \dfrac{1}{\varepsilon}\displaystyle\int_I (1+\varepsilon^{\frac{\gamma}{2}}|u'|^\gamma)\log(1+\varepsilon\alpha|u'|^2)\,dx \\ \qquad\qquad + \left(a^{1-\gamma}(1-\gamma)\dfrac{\varepsilon^{\frac{2-\gamma}{2}}}{\log\frac{1}{\varepsilon}}\right)^{\frac{3}{1-\gamma}}\displaystyle\int_I |u''|^2\,dx & \text{if } u\in W^{2,2}(I), \\ +\infty & \text{otherwise in } L^1(I), \end{cases}$$

by (5.2.14) and by Proposition 5.2.6 we obtain that the sequence F_ε^γ Γ-converges, as $\varepsilon\to 0^+$, to the functional F^γ defined in (5.2.13). In particular, taking $\gamma = 0$, we prove that the singular perturbations of the rescaled Perona-Malik functionals

$$F_\varepsilon(u) := \begin{cases} \dfrac{1}{\varepsilon}\displaystyle\int_I \log(1+\varepsilon\alpha|u'|^2)\,dx + \left(\dfrac{a\varepsilon}{\log\frac{1}{\varepsilon}}\right)^3\displaystyle\int_I |u''|^2\,dx & \text{if } u\in W^{2,2}(I), \\ +\infty & \text{otherwise in } L^1(I), \end{cases}$$

Γ-converge to F^0, as announced in the Introduction.

Remark 5.2.9 Let f_1 and f_2 be two functions satisfying the hypotheses of Theorem 5.1.2 and let r_1 and r_2 be the rescaling functions associated with f_1 and f_2 respectively according to (5.2.2) and, for $\varepsilon > 0$ and $i = 1,2$ denote by $F_{i,\varepsilon}$ the functional

$$F_{i,\varepsilon}(u) = \begin{cases} \dfrac{1}{\varepsilon}\displaystyle\int_I f_i(\sqrt{\varepsilon}|u'|)\,dx + (r_i(\varepsilon))^3\displaystyle\int_I |u''|^2\,dx & \text{if } u\in W^{2,2}(I), \\ +\infty & \text{otherwise in } L^1(I). \end{cases}$$

Suppose in addition that

$$\lim_{x\to+\infty}\frac{f_1(x)\log^\gamma(x)}{f_2(x)} = 1; \tag{5.2.15}$$

then, for every infinitesimal sequence $(\varepsilon_n)_n$, $\Gamma\text{-}\lim_{n\to\infty} F_{1,\varepsilon_n} = F \iff \Gamma\text{-}\lim_{n\to\infty} F_{2,\varepsilon_n} = F$; in other words, functions asymptotically differing by a logarithmic factor generate the same Γ-limits. To prove this fact we pass to a subsequence such that

$$\exists \lim_{n\to+\infty}\frac{f_1\left(t\frac{\sqrt{\varepsilon_n}}{r_1(\varepsilon_n)}\right)}{f_1\left(\frac{\sqrt{\varepsilon_n}}{r_1(\varepsilon_n)}\right)} =: b_1(t) \quad \forall t>0 \qquad\text{and}\qquad \exists \lim_{n\to+\infty}\frac{f_2\left(t\frac{\sqrt{\varepsilon_n}}{r_2(\varepsilon_n)}\right)}{f_2\left(\frac{\sqrt{\varepsilon_n}}{r_2(\varepsilon_n)}\right)} =: b_2(t) \quad \forall t>0,$$

and we observe that, by virtue of (5.2.15), we have $b_1 \equiv b_2$; we conclude by applying Theorem 5.2.2. Note that the results of Example 5.2.8 can be derived from Examples 5.2.7, using the present remark.

Example 5.2.10 Let $f_\alpha(x) := \frac{\alpha x^2}{(1+x)\log(e+x)}$; then, by easy computations, the rescaling function r defined in (5.2.2) satisfies $r(\varepsilon) \simeq \frac{\sqrt{\varepsilon}}{e^{\frac{1}{\sqrt{\varepsilon}}}}$. Moreover $b(t) = \lim_{x\to+\infty} \frac{f(tx)}{f(x)} = t$ for every $t > 0$; therefore, setting

$$F_{\alpha,\varepsilon}(u) := \begin{cases} \frac{1}{\varepsilon}\int_I f_\alpha(\sqrt{\varepsilon}|u'|)\,dx + \left(\frac{\sqrt{\varepsilon}}{e^{\frac{1}{\sqrt{\varepsilon}}}}\right)^3 \int_I |u''|^2\,dx & \text{if } u \in W^{2,2}(I), \\ +\infty & \text{otherwise in } L^1(I), \end{cases}$$

by Proposition 5.2.6 we obtain that the family F_ε Γ-converges, with respect to L^1-metric, to

$$F_\alpha(u) := \begin{cases} \int_I g_\alpha(|u'|)\,dx + |D^s u| & \text{if } u \in BV(I) \\ +\infty & \text{in } L^1(I)\setminus BV(I), \end{cases}$$

where $g_\alpha = (\alpha x^2 \wedge x)^{**}$. Note that $g_\alpha(x) \uparrow x$ as $\alpha \to +\infty$, and therefore

$$\Gamma\text{-}\lim_{\alpha\to+\infty}(\Gamma\text{-}\lim_{\varepsilon\to 0^+} F_{\alpha,\varepsilon}) = \Gamma\text{-}\lim_{\alpha\to+\infty} F_\alpha = G,$$

with G given by

$$G(u) := \begin{cases} |Du| & \text{if } u \in BV(I), \\ +\infty & \text{otherwise in } L^1(I). \end{cases}$$

Remark 5.2.11 The hypothesis *H3)* is in some sense necessary; indeed suppose that f is an increasing function satisfying *H1)*, *H2)*, and $\lim_{x\to+\infty}\frac{f(x)}{x} = C > 0$. Then it is easy to see that the functionals

$$G_\varepsilon(u) := \begin{cases} \frac{1}{\varepsilon}\int_I f(\sqrt{\varepsilon}|u'|)\,dx, & \text{if } u \in C^1(I), \\ +\infty & \text{otherwise in } L^1(I), \end{cases}$$

Γ-converge in the L^1-topology, to the functional

$$G(u) := \begin{cases} \alpha\int_I |u'|^q dx, & \text{if } u \in W^{1,q}(I), \\ +\infty & \text{otherwise in } L^1(I), \end{cases}$$

as $\varepsilon \to 0^+$. To prove this fact, we first observe that, for every $\delta \in (0,1)$, we can find x_1 such that

$$f(x) \geq (1-\delta)\alpha x^q \quad \forall x \in (0,x_1) \qquad \text{and} \qquad f(x) \geq cx \quad \forall x \geq x_1, \tag{5.2.16}$$

for some $c > 0$.

Let $u_\varepsilon \to u$ in $L^1(I)$ and such that $\sup_\varepsilon G_\varepsilon(u_\varepsilon) = K < +\infty$ and set

$$A_\varepsilon := \{x \in I : \sqrt{\varepsilon}|u'_\varepsilon(x)| \leq x_1\};$$

then, for every ε, we have, by (5.2.16),

$$K \geq G_\varepsilon(u_\varepsilon) \geq (1-\delta)\alpha\int_{A_\varepsilon} |u'_\varepsilon|^q\,dx + \frac{c}{\sqrt{\varepsilon}}\int_{I\setminus A_\varepsilon} |u'_\varepsilon|\,dx$$

which implies

$$\lim_{\varepsilon\to 0^+}\int_{I\setminus A_\varepsilon}|u'_\varepsilon|\,dx=0. \tag{5.2.17}$$

Let $\overline{x}$ such that $u_\varepsilon(\overline{x})\to u(\overline{x})$; taking into account (5.2.17), it is easy to see that the functions

$$v_\varepsilon(x):=u(\overline{x})+\int_{\overline{x}}^{x}\left(u'_\varepsilon(s)\vee\left(-\frac{x_1}{\sqrt{\varepsilon}}\right)\right)\wedge\frac{x_1}{\sqrt{\varepsilon}}\,ds$$

still converge to u. Therefore we can estimate

$$K\geq G_\varepsilon(u_\varepsilon)\geq G_\varepsilon(v_\varepsilon)\geq(1-\delta)\int_I|v'_\varepsilon|^q\,dx,$$

which implies $u\in W^{1,q}(I)$ and

$$\liminf_{\varepsilon\to 0^+}G_\varepsilon(u_\varepsilon)\geq\liminf_{\varepsilon\to 0^+}(1-\delta)\alpha\int_I|v'_\varepsilon|^q\,dx\geq(1-\delta)\alpha\int_I|u'|^q\,dx;$$

since δ is arbitrary, we recover the Γ-$\liminf$ inequality. Concerning the Γ-$\limsup$ inequality, if u is smooth we can take $u_\varepsilon=u$ for every ε as a recovery sequence; for a general u the conclusion follows from a standard density argument.

Let us see what happens when in (5.2.1) the function f has a finite strictly positive derivative at the origin so that $q=1$.

Theorem 5.2.12 *Let $f:[0,+\infty)\to[0,+\infty)$ be continuous, non-decreasing , differentiable in 0 with $f'(0)>0$, and concave in $(x_1,+\infty)$ for a suitable $x_1>0$. Then the family*

$$F_\varepsilon:=\begin{cases}\dfrac{1}{\varepsilon}\displaystyle\int_I f(\varepsilon|u'|)\,dx+\varepsilon^3\int_I|u''|^2\,dx & \text{if } u\in W^{2,2}(I),\\ +\infty & \text{otherwise in } L^1(I),\end{cases}$$

Γ-converges with respect to the L^1-norm to the functional

$$F(u):=\begin{cases}f'(0)\displaystyle\int_I|u'|\,dx+\sum_{S_u}\varphi(u^+-u^-)+f'(0)|D^cu| & \text{if } u\in BV(I),\\ +\infty & \text{otherwise in } L^1(I),\end{cases}$$

where φ is the function defined in (5.1.3) with $b=f$. Moreover every sequence u_ε such that $\sup_\varepsilon(F_\varepsilon(u_\varepsilon)+\|u_\varepsilon\|_1)<+\infty$ is strongly precompact in L^p for every $p\geq 1$.

PROOF. Take an infinitesimal sequence (ε_n) and consider the family of functions $f_n:=(1/\varepsilon_n)f(\varepsilon_n\cdot)$: we clearly have that $f_n(t)\to f'(0)t$ for every $t>0$ and $\varepsilon_n f_n(t/\varepsilon_n)=f(t)$ for every $t\geq 0$ and every $n\in\mathbb{N}$, so that (f_n) verifies (5.1.5) and (5.1.6) with $g=f'(0)t$ and $b=f$. Now construct a sequence of functions (f^k) such that $f\geq f^k$ for every k, $(f^k)'(0)\uparrow f'(0)$ as $k\to\infty$, and f^k is linear in $[0,y_k]$ and concave in $[y_k,+\infty)$ for a suitable $y_k>0$ (it is clear that under our assumptions such a construction is possible); then, setting $f_n^k(t):=(1/\varepsilon_n)f^k(\varepsilon_n t)$, we have that the family $(f_n^k)_{n,k}$ satisfies the weaker structure assumption introduced in Remark 5.1.15. At this point we can conclude by applying Theorem 5.1.2. □

The following example is in the spirit of Theorem 5.1.17.

Example 5.2.13 Given a convex non-decreasing positive function g and a concave positive function b satisfying $b^0(1) = g^\infty(1) = C \in (0, +\infty)$, we have that the family

$$F_\varepsilon := \begin{cases} \int_I \left[g(|u'|) \wedge \left(\frac{1}{\varepsilon} b(\varepsilon |u'|) + g(0) \right) \right] dx + \varepsilon^3 \int_I |u''|^2 \, dx & \text{if } u \in W^{2,2}(I), \\ +\infty & \text{otherwise} \end{cases}$$

Γ-converges to the functional $F_{b,g}$ defined in (5.1.4) which, under our assumptions, takes the form

$$\int_I g(|u'|) \, dx + \sum_{S_u} \varphi(u^+ - u^-) + C|D^c u| \qquad u \in BV(I),$$

where φ is the function associated with b according to (5.1.3). It is enough to apply Theorem 5.1.2 to the family $f_\varepsilon := g(t) \wedge \left(\frac{1}{\varepsilon} b(\varepsilon t) + g(0)\right)$, after noting that

$$f_\varepsilon(t) \to g(t) \wedge (b^0(t) + g(0)) = g(t) \qquad \text{and} \qquad \varepsilon f_\varepsilon\left(\frac{t}{\varepsilon}\right) \to b(t) \wedge g^\infty(t) = b(t).$$

5.3 The N-dimensional case

In the section we aim to extend the results of the previous ones to the N-dimensional case. Let us fix first some notations: for $u \in W^{2,2}(\Omega)$, we denote its hessian matrix by $\nabla^2 u$ and, given a square matrix A we consider the norm defined by

$$\|A\| := \sup_{|\xi|=1} A\xi \cdot \xi \, .$$

It is convenient to introduce the following definition.

Definition 5.3.1 *Given $X \subseteq L^1(\Omega)$ we say that the sequence of functionals $F_n : X \to \mathbb{R} \cup \{+\infty\}$* steadily Γ-converges *in X to $F : X \to \mathbb{R} \cup \{+\infty\}$ (and we will write $\Gamma^s\text{-}\lim_{n\to\infty} F_n = F$ or, shortly, $F_n \xrightarrow{\Gamma^s} F$) if, for every $p \geq 1$, $F_n|_{X\cap L^p(\Omega)}$ Γ-converges to $F|_{X\cap L^p(\Omega)}$ with respect to the L^p-convergence. Equivalently we have that $\Gamma^s\text{-}\lim_{n\to\infty} F_n = F$ if and only if the two following conditions are satisfied:*

i) for every $(u_n)_n \subset X$ such that $u_n \to u \in X$ in L^1, we have

$$\liminf_{n\to\infty} F_n(u_n) \geq F(u);$$

ii) for every $u \in X \cap L^p(\Omega)$, there exists a sequence $(u_n)_n \subset X \cap L^p(\Omega)$ such that

$$u_n \to u \text{ in } L^p \qquad \text{and} \qquad \limsup_{n\to\infty} F_n(u_n) \leq F(u).$$

We will also say that G is the steady relaxed functional *of F if G is the Γ^s-limit of the constant sequence $F_n = F$.*

We underline that, thanks to Remark 5.1.9, in the one-dimensional case we have in fact proved the steady Γ-convergence in the whole $L^1(I)$ of the functionals F_n. The main result of the section is the following theorem.

Theorem 5.3.2 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary and let f_n, r_n satisfy hypotheses i), ii), and iii) of Theorem 5.1.2. For every $n \in \mathbb{N}$, consider the following N-dimensional version of the functional F_n defined in* (5.1.1)*:*

$$F_n^N(u) = \begin{cases} \int_\Omega f_n(|\nabla u|)\,dx + (r_n)^3 \int_\Omega \|\nabla^2 u\|^2\,dx & \text{if } u \in W^{2,2}(\Omega), \\ +\infty & \text{otherwise in } L^1(\Omega). \end{cases} \tag{5.3.1}$$

Then

$$\Gamma\text{-}\liminf_{n\to\infty} F_n^N \geq F_{b,g}^N, \tag{5.3.2}$$

with respect to the L^1-convergence, where $F_{b,g}^N$ is the N-dimensional version of $F_{b,g}$ given by

$$F_{b,g}^N(u) := \begin{cases} \int_\Omega g_1(|\nabla u|)\,dx + \int_{S_u} \varphi_1(u^+(x) - u^-(x))\,d\mathcal{H}^{N-1} + (g^\infty(1) \wedge b^0(1))|D^c u| & \text{if } u \in GBV(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Suppose now that, for every n, f_n satisfies the following additional growth conditions:

gr1) *there exists C, $C_0 > 0$ and $q \geq 1$ such that $f_n(t) \leq C(1 + t^q)$ and $C_0 t^q \leq g(t) \leq C(1 + t^q)$ for every $t \geq 0$,*

gr2) *for every $\alpha > 0$ there exists $c(\alpha) > 0$ such that $f_n(\alpha t) \leq c(\alpha) f_n(t)$ for every $t \geq 0$,*

where C, C_0, q, and $c(\alpha)$ are independent of n. Then the sequence F_n^N Γ^s-converges in $GSBV^q$ to $F_{b,g}^N$. Moreover if $g^\infty(1) \wedge b^0(1) < +\infty$, actually we have that $\Gamma^s\text{-}\lim_{n\to\infty} F_n^N = F_{b,g}^N$ in the whole $L^1(\Omega)$.

Remark 5.3.3 Note that, for technical reasons, when $g^\infty(1) \wedge b^0(1) = +\infty$ in the N-dimensional case we are able to represent the Γ-limit only on $GSBV^q(\Omega)$; to complete the result one would have to show that every $u \in GSBV(\Omega)$ can be approximated in L^1 by a sequence $u_j \in GSBV^q(\Omega)$ such that $F_{b,g}^N(u_j) \to F_{b,g}^N(u)$ (see the introduction).

Remark 5.3.4 Note that if g satisfies both **gr1)** and **gr2)**, also the family f_ε constructed in Proposition 5.1.17 verifies the same growth conditions.

PROOF. Let us prove (5.3.2). The inequality will be proved by means of the so called slicing method, which relies on the use of Theorem 1.1.1.

Let us suppose for simplicity that $g^\infty(1) \wedge b^0(1) = +\infty$; the other case can be treated in an analogous way. First of all we observe that, for $\xi \in S^{n-1}$, for $u \in W^{2,2}(\Omega)$, and for $A \in \mathcal{A}(\Omega)$ we have, by Fubini's Theorem and by the monotonicity of f_n,

$$\begin{aligned} F_n^N(u, A) &= \int_{\Pi_\xi} \int_{A_\xi^y} \left(f_n(|\nabla u(y + t\xi)|) + (r_n)^3 \|\nabla^2 u(y + t\xi)\|^2\right)\,dt\,d\mathcal{H}^{N-1}(y) \\ &\geq \int_{\Pi_\xi} \int_{A_\xi^y} \left(f_n(|(u_\xi^y)'|) + (r_n)^3 |(u_\xi^y)''|^2\right)\,dt\,d\mathcal{H}^{N-1}(y) \\ &= \int_{\Pi_\xi} F_n(u_\xi^y, A_\xi^y)\,d\mathcal{H}^{N-1}(y), \end{aligned}$$

where, Π_ξ is the hyperplane orthogonal to ξ while A_ξ^y and u_ξ^y are the one-dimensional sections defined in Subsection 1.1.1. Let $u_n \to u$ in $L^1(A)$ be such that $\sup_n F_n^N < +\infty$ and note that, for every $\xi \in S^{n-1}$ and for almost every $y \in A_\xi^y$, $(u_n)_\xi^y \in W^{2,2}(A_\xi^y)$ and $(u_n)_\xi^y \to (u)_\xi^y$ in $L^1(A_\xi^y)$, hence, recalling that, by our assumptions, $F_n(\cdot, A_\xi^y) \xrightarrow{\Gamma} F_{b,g}(\cdot, A_\xi^y)$ and using Fatou's Lemma,

$$\begin{aligned}\liminf_{n\to\infty} F_n^N(u_n, A) &\geq \int_{\Pi_\xi} \liminf_{n\to\infty} F_n((u_n)_\xi^y, A_\xi^y)\, d\mathcal{H}^{N-1}(y)\\ &\geq \int_{\Pi_\xi}\left(\int_{A_\xi^y} g(|(u_\xi^y)'|) + \sum_{(S_u)_\xi^y\cap A_\xi^y} \varphi((u_\xi^y)^+ - (u_\xi^y)^-)\right)\mathcal{H}^{N-1}(y). \qquad (5.3.3)\end{aligned}$$

From (5.3.3), by virtue of Theorem 1.1.1, we get $u \in GSBV(\Omega)$ and

$$\Gamma\text{-}\liminf_{n\to\infty} F_n^N(u, A) \geq \alpha \int_A g(|\nabla u\cdot\xi|)\, dx + \int_{S_u\cap A} \varphi(u^+ - u^-)|\nu_u\cdot\xi|\, d\mathcal{H}^{N-1} = \int_A \psi_\xi(x)\,\lambda, \qquad (5.3.4)$$

where we have set

$$\lambda := \mathcal{L}^N + \varphi(u^+ - u^-)\mathcal{H}^{N-1}\lfloor S_u,$$

and

$$\psi_\xi := g(|\nabla u\cdot\xi|)(1-\chi_{S_u}) + |\nu_u\cdot\xi|\chi_{S_u};$$

since (5.3.4) holds true for every ξ and for every $A \in \mathcal{A}(\Omega)$, choosing a sequence $(\xi_i)_{i\in\mathbb{N}}$ dense in S^{n-1}, and by applying Lemma 5.1.13 (with $\nu(\cdot) := \Gamma\text{-}\liminf_{n\to\infty} F_n^N(u,\cdot)$), we finally obtain

$$\Gamma\text{-}\liminf_{n\to\infty} F_n^N(u) \geq \int_\Omega \sup_i \psi_{\xi_i}\, d\lambda = \int_\Omega g(|\nabla u|)\, dx + \int_{S_u} \varphi(u^+ - u^-)\, d\mathcal{H}^{N-1},$$

as desired.

Concerning the Γ-lim sup inequality, we adapt the proof given in [5]. In the sequel we will assume that **gr1)** and **gr2)** hold true. For every $p \geq 1$ we denote by $G_p : L^p(\Omega) \times \mathcal{A}(\Omega) \to [0, +\infty]$ the following functional

$$G_p(u, A) := \inf\{\limsup_{n\to\infty} F_n^N(u_n, A) : u_n \to u \text{ in } L^p(A)\};$$

our thesis is then equivalent to prove that $G_p(u,\Omega) \leq F_{b,g}^N(u,\Omega)$ for every $u \in GSBV^q(\Omega)\cap L^p(\Omega)$. It is clear that

$$G_{p_1}(u, A) \leq G_{p_2}(u, A), \qquad (5.3.5)$$

for every $1 \leq p_1 < p_2$, for every $u \in L^{p_2}(\Omega)$, and for every $A \in \mathcal{A}(\Omega)$.
STEP 1. Let Π be an affine hyperplane, and denote by Π^+ and Π^- the two open half-spaces whose union gives $\mathbb{R}^N \setminus \Pi$ and by ν the unit normal vector to Π which points towards Π^+. Then, for every $A \in \mathcal{A}(\Omega)$ and for every $z \in \mathbb{R}$,

$$G_p(z\chi_{\Pi^+}, A) \leq \varphi(|z|)\mathcal{H}^{N-1}(\Pi\cap A) = F_{b,g}^N(z\chi_{\Pi^+}, A) \qquad \forall p \geq 1.$$

First of all, since

$$\lim_{t\to 0} \mathcal{H}^{N-1}(\{x \in A : d(x) = t\}) = \mathcal{H}^{N-1}(\Pi\cap A),$$

for $\delta \in (0,1)$, we can choose $\eta > 0$ such that

$$\sup_{t\in(-\eta,\eta)} \mathcal{H}^{N-1}(\{x \in A: d(x) = t\}) \leq (1+\delta)\mathcal{H}^{N-1}(\Pi \cap A). \tag{5.3.6}$$

Let $u_n \to z\chi_{(0,+\infty)}$ be the one-dimensional recovery sequence constructed in the previous section which satisfies $\|u_n\|_\infty \leq |z|$, $u_n \equiv z\chi_{(0,+\infty)}$ in $\mathbb{R} \setminus (-\eta,\eta)$, and

$$\lim_{n\to\infty} F_n(u_n,(-\eta,\eta)) = F_{b,g}(z\chi_{(0,+\infty)},(-\eta,\eta)) = \varphi(|z|); \tag{5.3.7}$$

we recall also that, for a suitable $K > 0$,

$$r_n\|u_n'\|_\infty \leq K. \tag{5.3.8}$$

We define, for every $x \in \Omega$, $v_n(x) := u_n(d(x))$, where d is the signed distance function from Π, positive in Π^+ and negative in Π^-. Clearly $v_n \in W^{2,2}(\Omega)$ and $v_n \to z\chi_{\Pi^+}$ in $L^p(\Omega)$ for every $p \geq 1$, moreover, using co-area formula (see (1.1.1)), (5.3.6), and (5.3.8) , we can estimate

$$\begin{aligned}
F_n^N(v_n, A) &= \int_{A\cap(\Pi)_\eta} f_n(|u'(d)|)\,dx + (r_n)^3 \int_{A\cap(\Pi)_\eta} \|u_n''(d)\nabla d \otimes \nabla d + u_n'(d)\nabla^2 d\|^2\,dx \\
&\leq \int_{-\eta}^{\eta}\int_{\{x\in A:\, d(x)=t\}} f_n(|u_n'(t)|)\,d\mathcal{H}^{N-1}\,dt \\
&\quad + \int_{-\eta}^{\eta}(r_n)^3\int_{\{x\in A:\, d(x)=t\}} ((1+\varepsilon)|u_n''(t)|^2 + c_\varepsilon|u_n'(t)|^2\|\nabla^2 d\|)\,d\mathcal{H}^{N-1}\,dt \\
&\leq \int_{-\eta}^{\eta} F_n(u_n,(-\eta,\eta))\mathcal{H}^{N-1}(\{x\in A:\, d(x)=t\})\,dt \\
&\quad + c_\varepsilon K r_n\|\nabla^2 d\|_\infty \int_{-\eta}^{\eta} \mathcal{H}^{N-1}(\{x\in A:\, d(x)=t\})\,dt \\
&\leq (1+\varepsilon)(1+\delta)\mathcal{H}^{N-1}(\Pi\cap A)F_n(u_n,(-\eta,\eta)) + c_\varepsilon(1+\delta)\mathcal{H}^{N-1}(\Pi\cap A)2K\eta r_n\|\nabla^2 d\|_\infty,
\end{aligned}$$

where we denoted by $(\Pi)_\eta$ the η-neighbourhood of Π. From the last inequality, taking into account (5.3.7), we deduce

$$\limsup_{n\to\infty} F_n^N(v_n, A) \leq (1+\varepsilon)(1+\delta)\mathcal{H}^{N-1}(\Pi\cap A)\varphi(|z|);$$

since δ and ε are arbitrary, STEP 1 is proved.

STEP 2. Let $u = \sum_{i=1}^k z_i\chi_{E_i}$ with E_i closed polyhedra such that $\overset{\circ}{E_i} \cap \overset{\circ}{E_j} = \emptyset$ for $i \neq j$. Then

$$G_p(u, A) \leq F_{b,g}^N(u, A),$$

for all $A \in \mathcal{A}(\Omega)$ and for every $p \geq 1$.

The proof is based on a standard partition of unity argument, we refer to Proposition 2.6 of [5] for the details.

STEP 3. Let A', A, $B \in \mathcal{A}(I)$ such that $A' \subset\subset A$ and let ϕ be a cut-off function between A' and A. Then there exists a positive constant $C > 0$ such that, for every u, $v \in W^{2,2}(\Omega)\cap L^q(\Omega)$, we have

$$\begin{aligned}
F_n^N(\phi u + (1-\phi)v, A'\cup B) &\leq F_n^N(u, A) + F_n^N(v, B) \\
&\quad + C(F_n^N(u, S) + F_n^N(v, S)) + C\|\nabla\phi\|_\infty^q\|u-v\|_{L^q(S)}^q + C\mathcal{L}^N(S) \\
&\quad + C(r_n)^3(\|\nabla\phi\|_\infty^2\|\nabla u - \nabla v\|_{L^2(S)}^2 + \|\nabla^2\phi\|_\infty^2\|u-v\|_{L^2(S)}^2),
\end{aligned} \tag{5.3.9}$$

where $S := (A \setminus \overline{A'}) \cap B$.

Using the monotonicity of f, we can estimate

$$\begin{aligned}
& F_n^N(\phi u + (1-\phi)v, A' \cup B) \\
\leq \;& F_n^N(u,A) + F_n^N(v,B) + \int_S f_n(|(u-v)\nabla\phi| + \phi|\nabla u| + (1-\phi)|\nabla v|)\,dx \\
& + C_1(r_n)^3 \int_S (|\nabla\phi|^2|\nabla u - \nabla v|^2 + |u-v|^2\|\nabla^2\phi\|^2 + \|\nabla^2 u - \nabla^2 v\|^2)\,dx \\
\leq \;& F_n^N(u,A) + F_n^N(v,B) + \int_S (f_n(3|(u-v)\nabla\phi|) + f_n(3\phi|\nabla u|) + f_n(3(1-\phi)|\nabla v|))\,dx \\
& + C_1(r_n)^3 \int_S (|\nabla\phi|^2|\nabla u - \nabla v|^2 + |u-v|^2\|\nabla^2\phi\|^2 + \|\nabla^2 u\|^2 + \|\nabla^2 v\|^2)\,dx =: (*);
\end{aligned}$$

using **gr2)** we can continue our estimate

$$\begin{aligned}
(*) \;\leq\; & F_n^N(u,A) + F_n^N(v,B) + (C_1 + c(3))(F_n^N(u,S) + F_n^N(v,S)) + \int_S f_n(3|(u-v)\nabla\phi|)\,dx \\
& + C_1(r_n)^3 \int_S (|\nabla\phi|^2|\nabla u - \nabla v|^2 + |u-v|^2\|\nabla^2\phi\|^2)\,dx;
\end{aligned}$$

recalling **gr1)**, from the last inequality we easily get (5.3.9).

STEP 4. Let A', A, B, and S be as in STEP 3 and $p \geq q$. Then for every u, $v \in L^p(\Omega)$ and for every $K \in \mathbb{N}$ there exists a cut-off function ϕ_K between A and A', such that

$$G_p(\phi_K u + (1-\phi_K)v, A' \cup B) \leq \left(1 + \frac{C}{K}\right)(G_p(u,A) + G_p(v,B)) + C\frac{K^{q-1}}{d^q}\|u-v\|^q_{L^q(S)} + \frac{C}{K}\mathcal{L}^N(S), \tag{5.3.10}$$

where $d := \operatorname{dist}(A', \Omega \setminus A)$.

First of all choose u_n, $v_n \in W^{2,2}(\Omega)$ such that $u_n \to u$, $v_n \to v$ in $L^p(\Omega)$ and

$$G_p(u,A) = \lim_{n\to\infty} F_n^N(u_n, A) \qquad \text{and} \qquad G_p(u,B) = \lim_{n\to\infty} F_n^N(v_n, B).$$

For $j \in \{0,1,\ldots,K\}$ we consider the set

$$A_j^K = \left\{x \in A : \operatorname{dist}(x, A') < j\frac{d}{K}\right\};$$

for any $j \in \{0,1,\ldots,K-1\}$ we choose a cut-off function ϕ_j^K between A_j^K and A_{j+1}^K such that

$$\|\nabla\phi_j^K\|_\infty \leq 2\frac{K}{d}; \tag{5.3.11}$$

finally we set $S_j^K := \left(A_{j+1}^K \setminus \overline{A}_j^K\right) \cap B$. By using (5.3.9) and (5.3.11), we get

$$\begin{aligned}
& F_n^N(\phi_j^K u_n + (1-\phi_j^K)v_n, A' \cup B) \\
& \quad \leq F_n^N(u_n,A) + F_n^N(v_n,B) + C(F_n^N(u_n,S_j^K) + F_n^N(v_n,S_j^K)) + C\frac{K^q}{d^q}\|u-v\|^q_{L^q(S_j^K)} \\
& \quad\quad + C(r_n)^3(\|\nabla\phi_j^K\|^2_\infty\|\nabla u_n - \nabla v_n\|^2_{L^2(S_j^K)} + \|\nabla^2\phi_j^K\|^2_\infty\|u_n - v_n\|^2_{L^2(S_j^K)}).
\end{aligned}$$

Passing to a subsequence, if needed, it follows that there exists $j_K \in \{0,1,\dots,K-1\}$ such that

$$\begin{aligned} F_n^N(\phi_{j_K}^K u_n + (1-\phi_{j_K}^K)v_n, A'\cup B) &\le \frac{1}{K}\sum_{j=0}^{K-1} F_n^N(\phi_j^K u_n + (1-\phi_j^K)v_n, A'\cup B) \\ &\le \left(1+\frac{C}{K}\right)(F_n^N(u_n,A) + F_n^N(v_n,B)) + C\frac{K^{q-1}}{d^q}\|u_n - v_n\|_{L^q(S)}^q \\ &\quad + \frac{C}{K}\mathcal{L}^N(S) + C(K)(r_n)^3(\|\nabla u_n - \nabla v_n\|_{L^2(S)}^2 + \|u_n - v_n\|_{L^2(S)}^2), \end{aligned} \tag{5.3.12}$$

for every $n \in \mathbb{N}$. Recall that by Nirenberg inequality (see [46]), there exists $M > 0$ such that for all $u \in W^{2,2}(S)$

$$\|\nabla u\|_{L^2(S)} \le M(\|\nabla^2 u\|_{L^2(S)}^{1/2}\|u\|_{L^2(S)}^{1/2} + \|u\|_{L^2(S)}),$$

therefore from the equiboundedness of

$$(\|u_n - v_n\|_{L^2(S)} + (r_n)^3\|\nabla^2 u_n - \nabla^2 v_n\|_{L^2(S)}^2)_n$$

we get

$$(r_n)^3\|\nabla u_n - \nabla v_n\|_{L^2(S)}^2 \to 0 \qquad \text{as } n \to \infty.$$

Thus (5.3.10) follows letting n tend to $+\infty$ in (5.3.12).
STEP 5. For every $u \in GSBV^q(\Omega)\cap L^\infty(\Omega)$ we have

$$G_p(u,\Omega) \le \int_\Omega g(|\nabla u|)\,dx + \int_{S_u}\varphi(u^+ - u^-)\,d\mathcal{H}^{N-1} \qquad \forall p \ge 1. \tag{5.3.13}$$

We start with $u \in \mathcal{W}(\Omega)$ (see Subsection 1.1.3) and, for every $h \in \mathbb{N}$, we consider the sets

$$B_h := (S_u)_{1/h}\cap\Omega = \left\{x \in \Omega : \operatorname{dist}(x,S_u) < \frac{1}{h}\right\};$$

by the regularity assumptions on S_u we have that $\mathcal{L}^N(B_h) = O(1/h)$ and therefore, setting

$$\rho_h := h^{-\frac{1}{2}}\left(\int_{B_h}|\nabla u|^2\right)^{\frac{1}{4}},$$

we have

$$\lim_{h\to\infty}\frac{1}{\rho_h}\int_{B_h}|\nabla u|\,dx = 0. \tag{5.3.14}$$

By a standard argument based on the use of coarea formula (1.1.1) (see for example [15]) it is possible to find a sequence u_h satisfying the hypotheses of STEP 2 such that

$$\|u - u_h\|_{L^\infty(B_h)} \le \rho_h, \qquad \mathcal{H}^{N-1}((S_{u_h}\cap B_h)\setminus S_u) \le \frac{1}{\rho_h}\int_{B_h}|\nabla u|\,dx + O(1). \tag{5.3.15}$$

We are going to apply STEP 4 with $A = B_h$, $A' = B_{2h}$, $B = \Omega\setminus\overline{B}_{3h}$, obtaining for every $K \in \mathbb{N}$ the existence of a cut-off function ϕ_K^h such that

$$\begin{aligned} G_p(\phi_K^h u_h + (1-\phi_K^h)u,\Omega) &\le \left(1+\frac{C}{K}\right)(G_p(u,\Omega\setminus\overline{S}_u) + G_p(u_h,B_h)) + C\frac{K^{q-1}}{d^q}\|u-u_h\|_{L^q(S)}^q \\ &\quad + \frac{C}{K}\mathcal{L}^N(B_h) \\ &\le \left(1+\frac{C}{K}\right)(G_p(u,\Omega\setminus\overline{S}_u) + G_p(u_h,B_h)) + \left(CK^{q-1}h^q\rho_h^q + \frac{C}{K}\right)\mathcal{L}^N(B_h), \end{aligned} \tag{5.3.16}$$

where $p \geq q$. By STEP 2 we have

$$
\begin{aligned}
G_p(u_h, B_h) &\leq \int_{S_{u_h} \cap B_h} \varphi(u_h^+ - u_h^-)\, d\mathcal{H}^{N-1} \\
&\leq \int_{S_u} \varphi(u^+ - u^-)\, d\mathcal{H}^{N-1} + \varphi(2\|u\|_\infty)\mathcal{H}^{N-1}((S_{u_h} \cap B_h) \setminus S_u) \\
&\quad + \int_{S_u} (\varphi(u_h^+ - u_h^-) - \varphi(u^+ - u^-))\, d\mathcal{H}^{N-1};
\end{aligned}
$$

using (5.3.15) and (5.3.14) and noting that, by the Dominated Convergence Theorem

$$
\lim_{h \to \infty} \int_{S_u} (\varphi(u_h^+ - u_h^-) - \varphi(u^+ - u^-)\, d\mathcal{H}^{N-1} = 0,
$$

we therefore obtain

$$
\limsup_{h \to \infty} G_p(u_h, B_h) \leq \int_{S_u} \varphi(u^+ - u^-)\, d\mathcal{H}^{N-1}. \tag{5.3.17}
$$

Moreover, taking as approximating sequence $u_n = u$ for every $n \in \mathbb{N}$, we discover that

$$
G_p(u, \Omega \setminus \overline{S}_u) \leq \int_\Omega g(|\nabla u|)\, dx; \tag{5.3.18}
$$

combining (5.3.17) and (5.3.18), letting $h \to +\infty$ in (5.3.16), and taking into account the lower semicontinuity of G_p we finally get (5.3.13), for $p \geq q$ and therefore for every $p \geq 1$, by virtue of (5.3.5). For a general $u \in GSBV^q(\Omega) \cap L^\infty(\Omega)$ we conclude by a standard density argument based on Theorem 1.1.5.

We are now in a position to conclude the proof of the Theorem. Take $u \in GSBV^q(\Omega) \cap L^p(\Omega)$ and set $u_k := (-k \vee u) \wedge k$; then by (5.3.13), and the Monotone Convergence Theorem we have

$$
\begin{aligned}
G_p(u, \Omega) &\leq \liminf_{k \to \infty} G_p(u_k, \Omega) \\
&\leq \lim_{k \to \infty} \int_\Omega g(|\nabla u_k|)\, dx + \int_{S_u} \varphi(u_k^+ - u_k^-)\, d\mathcal{H}^{N-1} \\
&= \int_\Omega g(|\nabla u|)\, dx + \int_{S_u} \varphi(u^+ - u^-)\, d\mathcal{H}^{N-1}.
\end{aligned}
$$

So if $g^\infty(1) \wedge b^0(1) = +\infty$ then we are done; if it is not the case, then the conclusion follows by the fact that, thanks to Theorem 1.1.3 and an easy truncation argument, $F_{b,g}^N$ coincides with the steady relaxed functional (see Definition 5.3.1) of

$$
H(u) := \begin{cases} \displaystyle\int_\Omega g(|\nabla u|)\, dx + \int_{S_u} \varphi(u^+ - u^-)\, d\mathcal{H}^{N-1} & \text{if } u \in GSBV^q(\Omega), \\ +\infty & \text{if } u \in L^1(\Omega) \setminus GSBV^q(\Omega). \end{cases}
$$

□

The two following corollaries are an immediate consequence of Theorems 5.3.2, 5.2.2, 5.2.12.

Corollary 5.3.5 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary and let f, r, p as in Theorem 5.2.2. Let $(\varepsilon_n)_n$ be an infinitesimal sequence such that (5.2.3) holds. If $b^0(1) = +\infty$ then the functionals*

$$F_n^N(u) = \begin{cases} \dfrac{1}{\varepsilon_n} \displaystyle\int_\Omega f(\sqrt{\varepsilon_n}|\nabla u|)\,dx + (p(\varepsilon_n))^3 \int_\Omega \|\nabla^2 u\|^2\,dx & \text{if } u \in W^{2,2}(\Omega), \\ +\infty & \text{otherwise in } L^1(\Omega). \end{cases}$$

Γ^s-converge in $GSBV^q(\Omega)$ to

$$F^N(u) := \begin{cases} \alpha \displaystyle\int_\Omega |\nabla u|^q\,dx + \int_{S_u} \varphi^{(a)}(u^+(x) - u^-(x))\,d\mathcal{H}^{N-1} & \text{if } u \in GSBV(\Omega), \\ +\infty & \text{otherwise in } L^1(\Omega), \end{cases}$$

where $\varphi^{(a)}$ is as in Theorem 5.2.2; conversely, if $b^0(1) = C$ then the sequence F_n^N Γ^s-converges in $L^1(\Omega)$ to

$$F^N(u) := \begin{cases} \displaystyle\int_\Omega g(|\nabla u|)\,dx + \int_{S_u} \varphi^{(a)}(u^+(x) - u^-(x))\,d\mathcal{H}^{N-1} + C|D^c u| & \text{if } u \in GBV(\Omega), \\ +\infty & \text{if } u \in L^1(\Omega) \setminus GBV(\Omega), \end{cases}$$

with $\varphi^{(a)}$ still given by (5.1.3) and $g = (\alpha x^q \wedge Cx)^{**}$.

Corollary 5.3.6 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary and let f be as in Theorem 5.2.12. Then the family*

$$F_\varepsilon^N := \begin{cases} \dfrac{1}{\varepsilon} \displaystyle\int_\Omega f(\varepsilon|\nabla u|)\,dx + \varepsilon^3 \int_\Omega \|\nabla^2 u\|^2\,dx & \text{if } u \in W^{2,2}(\Omega), \\ +\infty & \text{otherwise in } L^1(\Omega), \end{cases}$$

Γ^s-converges in $L^1(\Omega)$ to the functional

$$F^N(u) := \begin{cases} f'(0) \displaystyle\int_\Omega |\nabla u|\,dx + \int_{S_u} \varphi(u^+ - u^-)\,d\mathcal{H}^{N-1} + f'(0)|D^c u| & \text{if } u \in BV(\Omega), \\ +\infty & \text{otherwise in } L^1(\Omega), \end{cases}$$

where φ is the function defined in (5.1.3) with $b = f$.

To conclude the N-dimensional analysis it remains to prove the equicoerciveness of the approximating functionals: this is done in the following proposition.

Proposition 5.3.7 *Under the same hypotheses of Theorem 5.3.2, let $(u_n)_n \subset L^1(\Omega)$ be equiintegrable and such that*

$$\sup_n F_n^N(u_n) < M < +\infty;$$

then $(u_n)_n$ is strongly precompact in $L^1(\Omega)$. Suppose in addition that $F_n^N \overset{\Gamma^s}{\to} G$ in $L^1(\Omega)$; then, for every $g \in L^p(\Omega)$ $(p > 1)$ and $\beta > 0$, the solutions u_n of

$$\min\left\{ F_n^N(v) + \beta \int_\Omega |v - g|^p\,dx : v \in W^{2,2}(\Omega) \right\}$$

converge, up to subsequences, in the $L^p(\Omega)$-norm to a solution of

$$\min\left\{G(v)+\beta\int_\Omega |v-g|^p\,dx\,:\,v\in L^1(\Omega)\right\}.$$

PROOF. As at the beginning of the proof of Theorem 5.3.2, we fix $\xi\in S^{n-1}$ and we get

$$M\geq F_n^N(u_n)\geq\int_{\Omega_\xi} g_n(y)\,d\mathcal{H}^{N-1}(y), \tag{5.3.19}$$

where $g_n(y):=F_n((u_n)_\xi^y,\Omega_\xi^y)$. Using the equiintegrability assumption, for fixed $\delta>0$, we find $\sigma_\delta>0$ such that

$$\mathcal{L}^N(B)\leq\sigma_\delta\Rightarrow\int_B |u_n|\,dx<\delta\qquad\forall n\in\mathbb{N}. \tag{5.3.20}$$

Choose $k>0$ such that

$$\frac{M\mathrm{diam}(\Omega)}{k}\leq\sigma_\delta; \tag{5.3.21}$$

set $A_{n,k}:=\{y\in\Omega_\xi\,:\,g_n(y)>k\}$ and denote by P_ξ the orthogonal projection on Π_ξ. We now define the new sequence v_n in the following way

$$v_n(x):=\begin{cases}u_n(x) & \text{if } P_\xi(x)\in\Omega_\xi\setminus A_{n,k}\\ 0 & \text{otherwise.}\end{cases}$$

Note that $\|u_n-v_n\|_{L^1(\Omega)}=\int_{\{x\in\Omega:\,P_\xi(x)\in A_{n,k}\}}|u_n|\,dx$; since by Chebicev Inequality, (5.3.19), and (5.3.21)

$$\begin{aligned}\mathcal{L}^N(\{x\in\Omega\,:\,P_\xi(x)\in A_{n,k}\}) &\leq \mathcal{H}^{N-1}(A_{n,k})\mathrm{diam}(\Omega)\\ &\leq \frac{\|g_n\|_{L^1(\Omega_\xi)}}{k}\mathrm{diam}(\Omega)\leq\frac{M}{k}\mathrm{diam}(\Omega)\leq\sigma_\delta,\end{aligned}$$

recalling (5.3.20) we finally obtain $\|u_n-v_n\|_{L^1(\Omega)}\leq\delta$.

Moreover $F_n((v_n)_\xi^y,\Omega_\xi^y)\leq g_n(y)(1-\chi_{A_{n,k}}(y))\leq k$ and therefore, by the one-dimensional results, $(v_n)_\xi^y$ is precompact in $L^1(\Omega_\xi^y)$ for every $y\in\Omega_\xi$. Since the construction can be repeated for every $\delta>0$ and for every $\xi\in S^{n-1}$, the thesis follows by applying Lemma 1.1.2.

Concerning the second part, we first observe that

$$\sup_n\left(F_n^N(u_n)+\beta\int_\Omega|u_n-g|^p\,dx\right)\leq\sup_n f_n(0)|\Omega|+\beta\int_\Omega|g|^p\,dx<+\infty \tag{5.3.22}$$

and therefore, by the first part of the theorem, there exist $u\in L^1(\Omega)$ and a subsequence, still denoted by u_n, such that $u_n\to u$ in L^1. Note that by (5.3.22) $\sup_n\|u_n\|_{L^p}<+\infty$ which implies that $u_n\rightharpoonup u$ weakly in L^p. Since $F_n^N\overset{\Gamma^s}{\to}G$, there exists $v_n\to v$ in L^p such that $F_n^N(v_n)\to G(u)$ and therefore, by exploiting the minimality of u_n,

$$\begin{aligned}G(u)+\beta\int_\Omega|u-g|^p\,dx &\leq \liminf_{n\to\infty}\left(F_n^N(u_n)+\beta\int_\Omega|u_n-g|^p\,dx\right)\\ &\leq \limsup_{n\to\infty}\left(F_n^N(u_n)+\beta\int_\Omega|u_n-g|^p\,dx\right)\\ &\leq \lim_{n\to\infty}\left(F_n^N(v_n)+\beta\int_\Omega|v_n-g|^p\,dx\right)=G(u)+\beta\int_\Omega|u-g|^p\,dx\end{aligned}$$

whence,

$$\begin{aligned} G(u) + \beta \int_\Omega |u-g|^p\,dx &= \lim_{n\to\infty}\left(F_n^N(u_n) + \beta\int_\Omega |u_n - g|^p\,dx\right) \\ &\geq G(u) + \limsup_{n\to\infty} \beta \int_\Omega |u_n - g|^p\,dx \\ &\geq G(u) + \liminf_{n\to\infty} \beta \int_\Omega |u_n - g|^p\,dx \geq G(u) + \beta\int_\Omega |u-g|^p\,dx. \end{aligned}$$

We deduce $\int_\Omega |u_n - g|^p\,dx \to \int_\Omega |u-g|^p\,dx$ and since $u_n - g \rightharpoonup u - g$ weakly in L^p, we conclude that $u_n \to u$ in L^p. The minimality of u follows now from the properties of Γ-convergence. □

We conclude the section by remarking that all the examples contained in Section 5.2 can be generalized to the N-dimensional case by means of Theorem 5.3.2. In particular let us underline the following ones.

Example 5.3.8 *(Perona-Malik Energy)* By Example 5.2.8 and by Theorem 5.3.2 we obtain that the functionals

$$F_\varepsilon^N(u) := \begin{cases} \dfrac{1}{\varepsilon}\displaystyle\int_\Omega \log(1+\varepsilon\alpha|\nabla u|^2)\,dx + \left(\dfrac{a\varepsilon}{\log\frac{1}{\varepsilon}}\right)^3 \int_\Omega \|\nabla^2 u\|^2\,dx & \text{if } u \in W^{2,2}(\Omega), \\ +\infty & \text{otherwise in } L^1(\Omega), \end{cases}$$

Γ^s-converge, as $\varepsilon \to 0^+$, to

$$F^N(u) := \begin{cases} \alpha\displaystyle\int_\Omega |\nabla u|^2\,dx + m(0)a^{\frac{3}{4}}\int_{S_u}\sqrt{u^+ - u^-}\,d\mathcal{H}^{N-1} & \text{if } u \in GSBV(\Omega) \\ +\infty & \text{in } L^1(\Omega)\setminus GSBV(\Omega), \end{cases}$$

in $GSBV^2(\Omega)$.

Example 5.3.9 Let b and g be as in Example 5.2.13 (and suppose for simplicity $g(0)=0$); then the family

$$F_\varepsilon^N := \begin{cases} \displaystyle\int_\Omega \left(g(|\nabla u|) \wedge \frac{1}{\varepsilon}b(\varepsilon|\nabla u|)\right)dx + \varepsilon^3\int_\Omega \|\nabla^2 u\|^2\,dx & \text{if } u \in W^{2,2}(\Omega), \\ +\infty & \text{otherwise} \end{cases}$$

Γ^s-converges in $L^1(\Omega)$ to the functional

$$\begin{cases} \displaystyle\int_\Omega g(|\nabla u|)\,dx + \int_{S_u}\varphi(u^+ - u^-)\,d\mathcal{H}^{N-1} + C|D^c u| & \text{if } u \in GBV(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

where φ is the function associated with b according to (5.1.3).

Chapter 6

Approximation via discrete Perona-Malik energies

In the main result of the chapter we show that, in the context of finite-differences discretization, a suitable rescaling of the Perona-Malik energy provides a new approximation of the (anisotropic) Mumford-Shah functional. Related numerical experiments are in progress.

6.1 Notations and statement of the main result

Given a vector $\tau \in \mathbb{R}^2$ we set

$$\mathbb{Z}^2_\tau := \{s\tau + t\tau^\perp : (s,t) \in \mathbb{Z}^2\},$$

where $\tau^\perp$ is othogonal to τ and has the same modulus; if $y \in \mathbb{R}^2$ and A is an open subset of $\mathbb{R}^2$, we denote

$$l^1\big((y+\mathbb{Z}^2_\tau)\cap A\big) := \Big\{v : (y+\mathbb{Z}^2_\tau)\cap A \to \mathbb{R} \text{ such that } \sum_{x\in(y+\mathbb{Z}^2_\tau)\cap A} |v(x)| < +\infty\Big\};$$

in the following every function $v \in l^1\big((y+\mathbb{Z}^2_\tau)\cap A\big)$ will be identified with the function $\tilde{v} \in L^1(A)$ which takes the constant value $v(x)$ in the square $x + C_\tau$ if $x \in (y+\mathbb{Z}^2_\tau)\cap A$, where

$$C_\tau := \{s\tau + t\tau^\perp : s,t \in [0,1)\}, \tag{6.1.1}$$

and zero otherwise. So, having in mind this identification, given a sequence $v_\varepsilon \in l^1\big((y_\varepsilon + \mathbb{Z}^2_{\tau_\varepsilon})\cap A\big)$ and a function $v \in L^1(A)$, we will often write, with a slight abuse of notation, $v_\varepsilon \to v$ in $L^1(A)$ instead of $\tilde{v}_\varepsilon \to v$ in $L^1(A)$. Given a vector τ we will denote $\hat{\tau} := \frac{\tau}{|\tau|}$.

Let $\Omega \subset \mathbb{R}^2$ be a bounded open domain with Lipschitz boundary and for every $\varepsilon > 0$ consider the following functional

$$F_\varepsilon(u) := \begin{cases} \varepsilon^2 \displaystyle\sum_{x\in\Omega\cap\varepsilon\mathbb{Z}^2} \sum_{\substack{\xi\in\mathbb{Z}^2\\ x+\varepsilon\xi\in\Omega}} \frac{1}{a_\varepsilon|\xi|}\log\left(1 + a_\varepsilon|\xi|\frac{|u(x+\varepsilon\xi)-u(x)|^2}{\varepsilon^2|\xi|^2}\right)\rho(\xi) & \text{if } u \in l^1(\varepsilon\mathbb{Z}^2\cap\Omega), \\ +\infty & \text{otherwise in } L^1(\Omega), \end{cases} \tag{6.1.2}$$

where $a_\varepsilon = \varepsilon \log \frac{1}{\varepsilon}$ and $\rho : \mathbb{Z}^2 \to [0, +\infty)$ satisfies

$$\sum_{\xi \in \mathbb{Z}^2} \rho(\xi) < +\infty \qquad \text{and} \qquad \rho(\xi) = \rho(\xi^\perp)\ \forall \xi \in \mathbb{Z}^2. \tag{6.1.3}$$

In this chapter we will prove the following theorem.

Theorem 6.1.1 *The functionals F_ε Γ-converge (as $\varepsilon \to 0$) with respect to the L^1-norm to the anisotropic Mumford-Shah functional F given by*

$$F(u) := \begin{cases} c_\rho \int_\Omega |\nabla u|^2 \, dx + \int_{S_u} \Phi(\nu_u) \, d\mathcal{H}^1 & \text{if } u \in GSBV(\Omega), \\ +\infty & \text{otherwise in } L^1(\Omega), \end{cases}$$

where

$$c_\rho := \frac{1}{2} \sum_{\xi \in \mathbb{Z}^2} \rho(\xi) \qquad \text{and} \qquad \Phi(\nu) := \sum_{\xi \in \mathbb{Z}^2} \rho(\xi) |\nu \cdot \hat{\xi}|. \tag{6.1.4}$$

Moreover, every sequence (u_ε) satisfying $\sup_\varepsilon (F_\varepsilon(u_\varepsilon) + \|u_\varepsilon\|_\infty) < +\infty$ is strongly precompact in $L^p(\Omega)$, for every $p \geq 1$.

The proof of the theorem will be split in the next sections.

6.2 Estimate from below of the Γ-limit for $N = 1$

In this section we study the one-dimensional version of the functionals defined above. Given a bounded open subset $I \subset \mathbb{R}$ we define

$$I_\varepsilon := \{x \in I \cap \varepsilon\mathbb{Z} : x + \varepsilon \in I\},$$

and, for every $u : I \cap \varepsilon\mathbb{Z} \to \mathbb{R}$, we define

$$\mathcal{F}_\varepsilon(u, I) := \frac{\varepsilon}{a_\varepsilon} \sum_{x \in I_\varepsilon} \log \left(1 + a_\varepsilon \frac{|u(x+\varepsilon) - u(x)|^2}{\varepsilon^2} \right),$$

where, as above, $a_\varepsilon = \varepsilon \log \frac{1}{\varepsilon}$. As usual we will identify every function $u : I \cap \varepsilon\mathbb{Z} \to \mathbb{R}$ (briefly $u \in l^1(I \cap \varepsilon\mathbb{Z})$) with the piecewise constant function u of $L^1(I)$ given by

$$u(x) := \begin{cases} u\left(\varepsilon \left[\frac{x}{\varepsilon}\right]\right) & \text{if } \varepsilon \left[\frac{x}{\varepsilon}\right] \in I, \\ 0 & \text{otherwise.} \end{cases}$$

Our aim is to prove the following proposition

Proposition 6.2.1 *Let $u_\varepsilon \in l^1(I \cap \varepsilon\mathbb{Z})$ such that $u_\varepsilon \to u$ in $L^1(I)$ as $\varepsilon \to 0^+$ and $\sup_\varepsilon \mathcal{F}_\varepsilon(u_\varepsilon) < +\infty$. Then $u \in SBV(I)$ and*

$$\liminf_{\varepsilon \to 0^+} \mathcal{F}_\varepsilon(u_\varepsilon, I) \geq \int_I |u'|^2 \, dx + \mathcal{H}^0(S_u).$$

We postpone the proof of the proposition after proving some useful lemmas.

Lemma 6.2.2 *Let $p(\varepsilon) > 0$ be such that $\lim_{\varepsilon\to 0^+} p(\varepsilon) = 0$ and*

$$\lim_{\varepsilon\to 0^+}\left(p(\varepsilon)\log\frac{1}{\varepsilon} - \log\log\frac{1}{\varepsilon}\right) = +\infty,$$

and set $c_\varepsilon := \varepsilon^{p(\varepsilon)}$. Then the following properties hold true:

a) $\lim_{\varepsilon\to 0^+} c_\varepsilon = 0$;

b) $\lim_{\varepsilon\to 0^+} \dfrac{\log c_\varepsilon}{\log\frac{1}{\varepsilon}} = 0$;

c) $\lim_{\varepsilon\to 0^+} \dfrac{c_\varepsilon^2}{\varepsilon}\log\dfrac{1}{\varepsilon} = +\infty$;

d) $\lim_{\varepsilon\to 0^+} c_\varepsilon\log\dfrac{1}{\varepsilon} = 0$;

e) $\lim_{\varepsilon\to 0^+} \dfrac{\log\left(1 + a_\varepsilon\frac{c_\varepsilon^2}{\varepsilon^2}\right)}{\log\frac{1}{\varepsilon}} = 1$.

PROOF. Properties *a), b), c)*, and *d)* follow immediately. Let us check only *e)*. Recalling the definition of a_ε, we have

$$\lim_{\varepsilon\to 0^+}\frac{\log\left(1 + a_\varepsilon\frac{c_\varepsilon^2}{\varepsilon^2}\right)}{\log\frac{1}{\varepsilon}} = \lim_{\varepsilon\to 0^+}\frac{\log\left(1 + \frac{\log\frac{1}{\varepsilon}c_\varepsilon^2}{\varepsilon}\right)}{\log\frac{1}{\varepsilon}}$$

$$= \lim_{\varepsilon\to 0^+}\frac{\log\left(\frac{\log\frac{1}{\varepsilon}c_\varepsilon^2}{\varepsilon}\right)}{\log\frac{1}{\varepsilon}} = \lim_{\varepsilon\to 0^+}\left(1 + \frac{\log\log\frac{1}{\varepsilon}}{\log\frac{1}{\varepsilon}} + \frac{2\log c_\varepsilon}{\log\frac{1}{\varepsilon}}\right) = 1,$$

where the second equality follows from *c)* while the last from *b)*. □

Lemma 6.2.3 *Let $u_\varepsilon \in l^1(I\cap\varepsilon\mathbb{Z})$ be such that $\sup_\varepsilon \mathcal{F}_\varepsilon(u_\varepsilon) < K < +\infty$ and let c_ε be as in the previous lemma. Set $b_\varepsilon^2 := \sqrt{c_\varepsilon\log\frac{1}{\varepsilon}}$ and consider the following sets:*

$$D_\varepsilon := \left\{x\in I_\varepsilon : \frac{|u_\varepsilon(x+\varepsilon) - u_\varepsilon(x)|}{\varepsilon} > \frac{b_\varepsilon}{\sqrt{a_\varepsilon}}\right\}.$$

Then

$$\lim_{\varepsilon\to 0^+}\mathcal{H}^0(D_\varepsilon)c_\varepsilon = 0.$$

PROOF. By our assumptions and recalling the definition of a_ε, we have

$$K > \frac{\varepsilon}{a_\varepsilon}\sum_{x\in D_\varepsilon}\log\left(1 + a_\varepsilon\frac{|u(x+\varepsilon) - u(x)|^2}{\varepsilon^2}\right) \geq \frac{\log(1+b_\varepsilon^2)}{\log\frac{1}{\varepsilon}}\mathcal{H}^0(D_\varepsilon)$$

so that, substituting the expression of b_ε^2, if ε is small enough, from *d)* of Lemma 6.2.2 we get

$$\mathcal{H}^0(D_\varepsilon)c_\varepsilon \leq K\frac{c_\varepsilon\log\frac{1}{\varepsilon}}{\log\left(1+\sqrt{c_\varepsilon\log\frac{1}{\varepsilon}}\right)} \leq K'\frac{c_\varepsilon\log\frac{1}{\varepsilon}}{\sqrt{c_\varepsilon\log\frac{1}{\varepsilon}}} = K'\sqrt{c_\varepsilon\log\frac{1}{\varepsilon}}.$$

Again *d)* implies now the thesis. □

Lemma 6.2.4 *Let $v_\varepsilon \in SBV(I)$ such that $\lim_{\varepsilon\to 0^+}\|v'_\varepsilon\|_\infty\sqrt{a_\varepsilon} = 0$. Then, for every $\delta > 0$ there exists $\overline{\varepsilon} > 0$ such that*

$$\frac{1}{a_\varepsilon}\int_I \log(1 + a_\varepsilon|v'_\varepsilon|^2)\,dx \geq (1-\delta)\int_I |v'_\varepsilon|^2\,dx,$$

for every $\varepsilon \leq \overline{\varepsilon}$.

Proof. Fix $\delta > 0$ and note that there exists $T_\delta > 0$ such that

$$\log(1 + a_\varepsilon t^2) \geq (1-\delta)a_\varepsilon t^2 \qquad \forall t \in \left[0, \frac{T_\delta}{\sqrt{a_\varepsilon}}\right];$$

by the assumptions, if ε is small enough we have $\|v'_\varepsilon\|_\infty \leq T_\delta/\sqrt{a_\varepsilon}$ and therefore

$$\frac{1}{a_\varepsilon}\int_I \log(1 + a_\varepsilon|v'_\varepsilon|^2)\,dx \geq (1-\delta)\int_I |v'_\varepsilon|^2\,dx.$$

□

We are now in a position to prove Proposition 6.2.1.

Proof of Proposition 6.2.1. Let b_ε and c_ε be as in Lemma 6.2.3 and set

$$B_\varepsilon(u_\varepsilon) := \left\{x \in I_\varepsilon : \frac{b_\varepsilon}{\sqrt{a_\varepsilon}} < \frac{|u_\varepsilon(x+\varepsilon) - u_\varepsilon(x)|}{\varepsilon} < \frac{c_\varepsilon}{\varepsilon}\right\} = \{x^1_\varepsilon, x^2_\varepsilon, \ldots, x^{m_\varepsilon}_\varepsilon\},$$

where $x^1_\varepsilon < x^2_\varepsilon < \cdots < x^m_\varepsilon$ and $m_\varepsilon := \mathcal{H}^0(B_\varepsilon)$. Now we want to replace the sequence u_ε with a new one $\tilde{u}_\varepsilon$, still converging to u, such that $B_\varepsilon(\tilde{u}_\varepsilon)$ is empty and $\mathcal{F}_\varepsilon(\tilde{u}_\varepsilon) \leq \mathcal{F}_\varepsilon(u_\varepsilon)$. We set $v^0_\varepsilon = u_\varepsilon$ and for $k = 1, \ldots, m_\varepsilon - 1$ we define, by induction,

$$v^{k+1}_\varepsilon(t) := \begin{cases} v^k_\varepsilon(t) & \text{for } t < x^{k+1}_\varepsilon, \\ v^k_\varepsilon(t) - [v^k_\varepsilon(x^{k+1}_\varepsilon) - v^k_\varepsilon(x^k_\varepsilon)] & \text{for } t \geq x^{k+1}_\varepsilon, \end{cases}$$

and finally we set $\tilde{u}_\varepsilon := v^{m_\varepsilon}_\varepsilon$ (see Figure 6.1).

First of all, using the fact that for every $\varepsilon > 0$, and for every $i = 1, \ldots, m_\varepsilon$ we get

$$\int_I |v^i_\varepsilon - v^{i-1}_\varepsilon|\,dx \leq |v^{i-1}_\varepsilon(x^i_\varepsilon) - v^{i-1}_\varepsilon(x^{i-1}_\varepsilon)|\,|I| = |u_\varepsilon(x^i_\varepsilon) - u_\varepsilon(x^{i-1}_\varepsilon)|\,|I| \leq c_\varepsilon\,|I|,$$

then we can estimate

$$\int_I |\tilde{u}_\varepsilon - u_\varepsilon|\,dx \leq \sum_{i=1}^{m_\varepsilon}\int_I |v^i_\varepsilon - v^{i-1}_\varepsilon|\,dx \leq \sum_{i=1}^{m_\varepsilon} c_\varepsilon|I| \leq \mathcal{H}^0(B_\varepsilon(u_\varepsilon))c_\varepsilon|I| \leq \mathcal{H}^0(D_\varepsilon)c_\varepsilon|I|.$$

Therefore, by Lemma 6.2.3, we get $\tilde{u}_\varepsilon \to u$ in $L^1(I)$. Moreover, by construction, we clearly have that $\mathcal{F}_\varepsilon(\tilde{u}_\varepsilon) \leq \mathcal{F}_\varepsilon(u_\varepsilon)$. We set

$$I^\flat_\varepsilon := \left\{x \in I_\varepsilon : \frac{|u_\varepsilon(x+\varepsilon) - u_\varepsilon(x)|}{\varepsilon} \leq \frac{b_\varepsilon}{\sqrt{a_\varepsilon}}\right\} \qquad \text{and} \qquad I^\sharp_\varepsilon := \left\{x \in I_\varepsilon : \frac{|u_\varepsilon(x+\varepsilon) - u_\varepsilon(x)|}{\varepsilon} \geq \frac{c_\varepsilon}{\varepsilon}\right\}$$

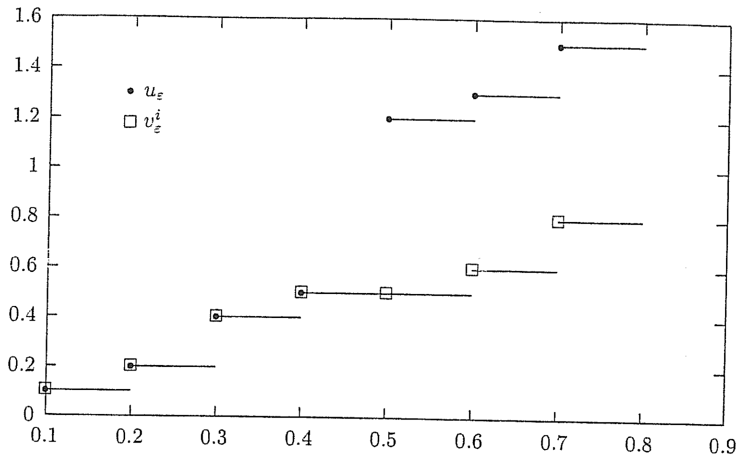

Figure 6.1: The construction of $\tilde{u}_\varepsilon$.

and we call w_ε the function belonging to $SBV(I)$ defined by

$$w_\varepsilon(x) := \begin{cases} \tilde{u}_\varepsilon\left(\varepsilon\left[\frac{x}{\varepsilon}\right]\right) + \dfrac{\tilde{u}_\varepsilon\left(\varepsilon\left[\frac{x}{\varepsilon}\right]+\varepsilon\right) - \tilde{u}_\varepsilon\left(\varepsilon\left[\frac{x}{\varepsilon}\right]\right)}{\varepsilon}\left(x - \varepsilon\left[\frac{x}{\varepsilon}\right]\right) & \text{if } \varepsilon\left[\frac{x}{\varepsilon}\right] \in I^\flat_\varepsilon, \\ \tilde{u}_\varepsilon\left(\varepsilon\left[\frac{x}{\varepsilon}\right]\right) & \text{if } \varepsilon\left[\frac{x}{\varepsilon}\right] \in I^\sharp_\varepsilon \text{ or } \varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon \notin (\varepsilon\mathbb{Z}\cap I), \\ 0 & \text{otherwise.} \end{cases}$$

Roughly speaking w_ε coincides with the affine interpolation of $\tilde{u}_\varepsilon$ in the intervals $(y, y+\varepsilon)$ with $y \in I^\flat_\varepsilon$ while takes the constant value $\tilde{u}_\varepsilon(y)$ in the intervals $(y, y+\varepsilon)$ with $y \in I^\sharp_\varepsilon$; it is clear that

$$w_\varepsilon \to u \text{ in } L^1 \qquad \sqrt{a_\varepsilon}\|w'_\varepsilon\|_\infty \le b_\varepsilon \to 0 \qquad \text{and} \qquad S_{w_\varepsilon} = I^\sharp_\varepsilon + \varepsilon. \tag{6.2.1}$$

Now we can estimate

$$\begin{aligned} \mathcal{F}_\varepsilon(\tilde{u}_\varepsilon, I) &\ge \frac{\varepsilon}{a_\varepsilon}\sum_{x\in I^\flat_\varepsilon}\log\left(1 + a_\varepsilon\frac{|\tilde{u}_\varepsilon(x+\varepsilon) - \tilde{u}_\varepsilon(x)|^2}{\varepsilon^2}\right) + \frac{\varepsilon}{a_\varepsilon}\sum_{x\in I^\sharp_\varepsilon}\log\left(1 + a_\varepsilon\frac{|\tilde{u}_\varepsilon(x+\varepsilon) - \tilde{u}_\varepsilon(x)|^2}{\varepsilon^2}\right) \\ &\ge \frac{1}{a_\varepsilon}\int_I \log(1 + a_\varepsilon|w'_\varepsilon|^2)\,dx + \mathcal{H}^0(I^\sharp_\varepsilon)\frac{\varepsilon}{a_\varepsilon}\log\left(1 + a_\varepsilon\frac{c^2_\varepsilon}{\varepsilon^2}\right). \end{aligned} \tag{6.2.2}$$

Fix $\delta \in (0,1)$; recalling (6.2.1) and the definition of a_ε, by Lemma 6.2.4 and by *e)* of Lemma 6.2.2, from (6.2.2) we deduce the existence of $\bar{\varepsilon}$ such that

$$\mathcal{F}_\varepsilon(\tilde{u}_\varepsilon, I) \ge (1-\delta)\left(\int_I |w'_\varepsilon|^2\,dx + \mathcal{H}^0(S_{w_\varepsilon})\right);$$

by the Ambrosio Semicontinuity Theorem we therefore obtain that $u \in SBV(I)$ and

$$\liminf_{\varepsilon\to0^+}\mathcal{F}_\varepsilon(u_\varepsilon, I) \ge \liminf_{\varepsilon\to0^+}\mathcal{F}_\varepsilon(\tilde{u}_\varepsilon, I) \ge (1-\delta)\left(\int_I |u'|^2\,dx + \mathcal{H}^0(S_u)\right),$$

which conclude the proof of the proposition since δ is arbitrary. □

We conclude this section with a remark that will be useful in the sequel.

Remark 6.2.5 Fix $t \in \mathbb{R}$ and for $u \in l^1(\varepsilon\mathbb{Z} \cap I)$ define

$$\mathcal{F}^t_\varepsilon(u, I) := \frac{\varepsilon}{a_\varepsilon} \sum_{x \in I^t_\varepsilon} \log\left(1 + a_\varepsilon \frac{|u(x+\varepsilon) - u(x)|^2}{\varepsilon^2}\right),$$

where

$$I^t_\varepsilon := \{x \in I \cap \varepsilon(t + \mathbb{Z}) : x + \varepsilon \in I\};$$

then we have that Proposition 6.2.1 is still valid with $\mathcal{F}^t_\varepsilon$ instead of $\mathcal{F}_\varepsilon$ (without changes in the proof).

6.3 Estimate from below of the Γ-limit for $N = 2$

Lemma 6.3.1 *Let $u_\varepsilon \in l^1(\varepsilon\mathbb{Z}^2)$ be such that $u_\varepsilon \to u$ in $L^1(\mathbb{R}^2)$. For y, $\xi \in \mathbb{Z}^2$, let $v^y_{\varepsilon,\xi} \in l^1(\varepsilon(y+\mathbb{Z}^2_\xi))$ be defined as $v^y_{\varepsilon,\xi}(x) := u_\varepsilon(x)$ for every $x \in \varepsilon(y + \mathbb{Z}^2_\xi)$. Then $v^y_{\varepsilon,\xi} \to u$ in $L^1(\mathbb{R}^2)$.*

PROOF. We call Q_ξ the unit cell of the lattice $\mathbb{Z}^2_\xi$, i.e.

$$Q_\xi := C_\xi \cap \mathbb{Z}^2 = \{\tau^1, \dots, \tau^k\}, \tag{6.3.1}$$

where C_ξ is the set defined in (6.1.1). For $j = 1, ..., k$ we set $u^j_\varepsilon(x) = u_\varepsilon(x - \varepsilon\tau^j)$. Since

$$\begin{aligned}\int_{\mathbb{R}^2} |u^j_\varepsilon(x) - u(x)|\, dx &\le \int_{\mathbb{R}^2} |u_\varepsilon(x - \varepsilon\tau^j) - u(x - \varepsilon\tau^j)|\, dx + \int_{\mathbb{R}^2} |u(x - \varepsilon\tau^j) - u(x)|\, dx \\ &= \int_{\mathbb{R}^2} |u_\varepsilon(x) - u(x)|\, dx + \int_{\mathbb{R}^2} |u(x - \varepsilon\tau^j) - u(x)|\, dx,\end{aligned}$$

we have that $u^j_\varepsilon \to u$ in $L^1(I)$ as $\varepsilon \to 0^+$, for every $j \in \{1, ..., k\}$; therefore, up to passing to a subsequence, we can suppose that

- there exists $N \subset \mathbb{R}^2$ with $\mathcal{L}^2(N) = 0$ such that $u^j_\varepsilon \to u$ pointwise in $\mathbb{R}^2 \setminus N$ for $j = 1, ..., k$;
- $|u^j_\varepsilon| \le g^j$ almost everywhere where g^j is a L^1 function, for $j = 1, ..., k$.

Since for every $x \in \mathbb{R}^2 \setminus N$ there exists $j \in \{1, ..., k\}$ such that $v^y_{\varepsilon,\xi}(x) = u^j_\varepsilon(x)$, we get $v^y_{\varepsilon,\xi} \to u$ pointwise in $\mathbb{R}^2 \setminus N$; moreover $|v^y_{\varepsilon,\xi}| \le g_1 + ... + g_k$ and therefore, by the Dominated Convergence Theorem, $v^y_{\varepsilon,\xi} \to u$ in L^1. As the same argument can be repeated for every subsequence, the lemma is proved. □

We will need also the following lemma.

Lemma 6.3.2 *Let Q_ξ the unit cell of the lattice $\mathbb{Z}^2_\xi$ as defined in* (6.3.1). *Then $\mathcal{H}^0(Q_\xi) = |\xi|^2$.*

PROOF. We refer to Figure 6.2. We associate every point $x \in Q_\xi$ with the square $x + [0,1] \times [0,1]$. The area of the shaded region, which is the union of such squares, coincides with the cardinality of Q_ξ and it is clear from the Figure that it is equal to the area of the set C_ξ (defined in (6.1.1)). □

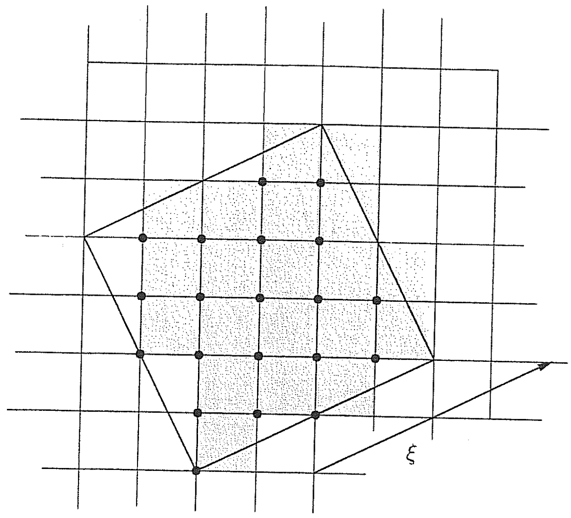

Figure 6.2: The set Q_ξ.

Before starting the proof of the Γ-liminf inequality it is convenient to rewrite the functional F_ε in a suitable way. After observing that

$$\bigcup_{y\in Q_\xi}(\mathbb{Z}^2_\xi + y) = \mathbb{Z}^2,$$

we can write, for every $u \in l^1(\varepsilon\mathbb{Z}^2 \cap \Omega)$,

$$F_\varepsilon(u) = \varepsilon^2 \sum_{x\in\Omega\cap\varepsilon\mathbb{Z}^2} \sum_{\substack{\xi\in\mathbb{Z}^2 \\ x+\varepsilon\xi\in\Omega}} \frac{1}{a_\varepsilon|\xi|}\log\left(1 + a_\varepsilon|\xi|\frac{|u(x+\varepsilon\xi)-u(x)|^2}{\varepsilon^2|\xi|^2}\right)\rho(\xi) = \sum_{\xi\in\mathbb{Z}^2}\rho(\xi)\sum_{y\in Q_\xi} G^y_{\varepsilon,\xi}(u), \tag{6.3.2}$$

where

$$G^y_{\varepsilon,\xi}(u) := \varepsilon^2 \sum_{\substack{x\in\varepsilon(y+\mathbb{Z}^2_\xi)\cap\Omega \\ x+\varepsilon\xi\in\Omega}} \frac{1}{a_\varepsilon|\xi|}\log\left(1 + a_\varepsilon|\xi|\frac{|u(x+\varepsilon\xi)-u(x)|^2}{\varepsilon^2|\xi|^2}\right).$$

Let $u_\varepsilon \to u$ such that $\sup_\varepsilon F_\varepsilon(u_\varepsilon) < +\infty$. Taking u_ε and u equal to zero outside $(\varepsilon\mathbb{Z}^2 \cap \Omega)$ and Ω respectively, we can suppose that $u_\varepsilon \in l^1(\varepsilon\mathbb{Z}^2)$, $u \in L^1(\mathbb{R}^2)$, and $u_\varepsilon \to u$ in $L^1(\mathbb{R}^2)$. If we are able to prove that

$$u \in GSBV(\Omega) \quad \text{and} \quad \liminf_{\varepsilon\to 0+} G^y_{\varepsilon,\xi}(u_\varepsilon) \geq \frac{1}{|\xi|^2}\left(\int_\Omega |\nabla u\cdot\hat{\xi}|^2\,dx + \int_{S_u}|\nu_u\cdot\hat{\xi}|\,d\mathcal{H}^1\right), \tag{6.3.3}$$

for every $\xi \in \mathbb{Z}^2$ and every $y \in Q_\xi$, then, by (6.3.2), Lemma 6.3.2, (6.1.3) and (6.1.4), we have

$$\begin{aligned}
\liminf_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon) &\geq \sum_{\xi\in\mathbb{Z}^2} \rho(\xi) \sum_{y\in Q_\xi} \liminf_{\varepsilon\to 0^+} G^y_{\varepsilon,\xi}(u_\varepsilon) \\
&\geq \sum_{\xi\in\mathbb{Z}^2} \rho(\xi) \sum_{y\in Q_\xi} \frac{1}{|\xi|^2}\left(\int_\Omega |\nabla u\cdot\hat{\xi}|^2\,dx + \int_{S_u} |\nu_u\cdot\hat{\xi}|\,d\mathcal{H}^1\right) \\
&= \sum_{\xi\in\mathbb{Z}^2} \rho(\xi)\left(\int_\Omega |\nabla u\cdot\hat{\xi}|^2\,dx + \int_{S_u} |\nu_u\cdot\hat{\xi}|\,d\mathcal{H}^1\right) \\
&= \int_\Omega \frac{1}{2}\sum_{\xi\in\mathbb{Z}^2} \rho(\xi)(|\nabla u\cdot\hat{\xi}|^2 + |\nabla u\cdot\hat{\xi^\perp}|^2)\,dx + \int_{S_u}\sum_{\xi\in\mathbb{Z}^2}\rho(\xi)|\nu_u\cdot\hat{\xi}|\,d\mathcal{H}^1 \\
&= c_\rho\int_\Omega |\nabla u|^2\,dx + \int_{S_u}\Phi(\nu_u)\,d\mathcal{H}^1\,. \qquad (6.3.4)
\end{aligned}$$

Following the notation introduced in Subsection 1.1.1, we denote the hyperplane orthogonal to ξ by Π_ξ and Ω_ξ the projection of Ω on Π_ξ. For every $w \in \Pi_\xi$ we set $\Omega^w_\xi := \{t \in \mathbb{R} : w + t\hat{\xi} \in \Omega\}$ and, given a function, we define $f^w_\xi(t) := f(w + t\hat{\xi})$. We also define $O_{\varepsilon,\xi} := \Omega_\xi \cap \varepsilon\mathbb{Z}^2$ (see Figure 6.3) and for every $x \in \mathbb{R}^2$

$$O^x_{\varepsilon,\xi} := \{y \in x + \varepsilon\mathbb{Z}\xi : y + \varepsilon\xi \in \Omega\}\,.$$

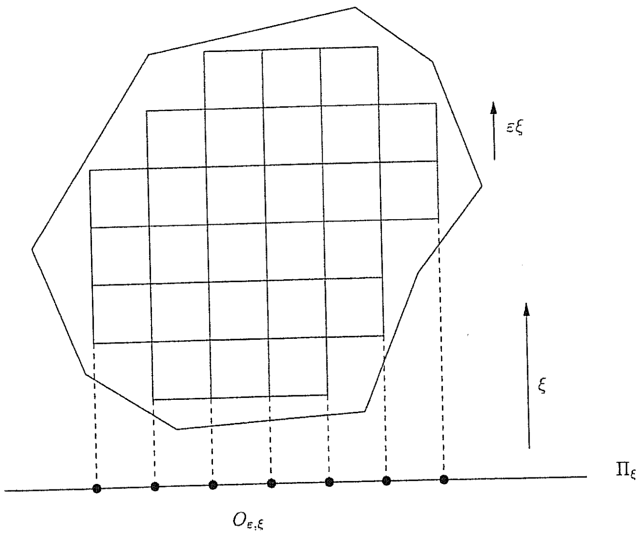

Figure 6.3: The sets Π_ξ and $O_{\varepsilon,\xi}$.

Note that we can write

$$\begin{aligned} G^y_{\varepsilon,\xi}(u_\varepsilon) &= \varepsilon^2 \sum_{w\in O_{\varepsilon,\xi}} \sum_{O^{w+\varepsilon y}_{\varepsilon,\xi}} \frac{1}{a_\varepsilon|\xi|}\log\left(1+a_\varepsilon|\xi|\frac{|u_\varepsilon(x+\varepsilon\xi)-u_\varepsilon(x)|^2}{\varepsilon^2|\xi|^2}\right) \\ &= \frac{1}{|\xi|^2}\sum_{w\in O_{\varepsilon,\xi}} \varepsilon|\xi| \sum_{O^{w+\varepsilon y}_{\varepsilon,\xi}} \frac{\varepsilon|\xi|}{a_\varepsilon|\xi|}\log\left(1+a_\varepsilon|\xi|\frac{|u_\varepsilon(x+\varepsilon\xi)-u_\varepsilon(x)|^2}{\varepsilon^2|\xi|^2}\right) \\ &= \frac{1}{|\xi|^2}\int_{\Omega_\xi}\left[\frac{\varepsilon|\xi|}{a_\varepsilon|\xi|}\sum_{O^{w+\varepsilon y}_{\varepsilon,\xi}} \log\left(1+a_\varepsilon|\xi|\frac{|v^y_{\varepsilon,\xi}(x+\varepsilon\xi)-v^y_{\varepsilon,\xi}(x)|^2}{\varepsilon^2|\xi|^2}\right)\right] d\mathcal{H}^1(w), \qquad (6.3.5) \end{aligned}$$

where $v^y_{\varepsilon,\xi}$ is the sequence defined in Lemma 6.3.1. Set $\eta := \varepsilon|\xi|$, $w^y_{\eta,\xi} := v^y_{\varepsilon,\xi}$, $z := y/|\xi|$ and observe that

$$\lim_{\varepsilon\to 0^+}\frac{a_\eta}{a_\varepsilon|\xi|} = 1\,. \qquad (6.3.6)$$

Fix $\delta\in(0,1)$; by (6.3.5), by Fatou's Lemma, and by (6.3.6), we obtain

$$\begin{aligned} \liminf_{\varepsilon\to 0^+} G^y_{\varepsilon,\xi}(u_\varepsilon) &\geq \frac{1}{|\xi|^2}\int_{\Omega_\xi}\liminf_{\varepsilon\to 0^+}\left[\frac{\varepsilon|\xi|}{a_\varepsilon|\xi|}\sum_{O^{w+\varepsilon y}_{\varepsilon,\xi}}\log\left(1+a_\varepsilon|\xi|\frac{|v^y_{\varepsilon,\xi}(x+\varepsilon\xi)-v^y_{\varepsilon,\xi}(x)|^2}{\varepsilon^2|\xi|^2}\right)\right] d\mathcal{H}^1(w) \\ &\geq \frac{1}{|\xi|^2}\int_{\Omega_\xi}\liminf_{\eta\to 0^+}\left[\frac{\eta}{a_\eta}\sum_{O^{w+\eta z}_{\eta,\hat\xi}}\log\left(1+\delta a_\eta\frac{|w^y_{\eta,\xi}(x+\eta\hat\xi)-w^y_{\eta,\xi}(x)|^2}{\eta^2}\right)\right] d\mathcal{H}^1(w) \\ &= \frac{1}{|\xi|^2}\int_{\Omega_\xi}\liminf_{\eta\to 0^+}\mathcal{F}^t_\eta\Big((\sqrt{\delta}w^y_{\eta,\xi})^w_\xi,\Omega^w_\xi\Big)\,d\mathcal{H}^1(w)\,, \end{aligned}$$

where $t := z\cdot\hat\xi$ and $\mathcal{F}^t_\eta$ is the functional defined in Remark 6.2.5. Since $(w^y_{\eta,\xi})^w_\xi \to u^w_\xi$ for $\mathcal{H}^1$-a.e. $w\in\Pi_\xi$, as $\eta\to 0$ (thanks to Lemma 6.3.1), by Proposition 6.2.1 and Remark 6.2.5 we deduce

$$\liminf_{\varepsilon\to 0^+} G^y_{\varepsilon,\xi}(u_\varepsilon) \geq \frac{1}{|\xi|^2}\int_{\Omega_\xi}\left(\int_{\Omega^w_\xi}\delta|(u^w_\xi)'|^2\,dt + \mathcal{H}^0(S_{u^y_\xi})\right) d\mathcal{H}^1(w),$$

from which (6.3.3) follows by letting $\delta\uparrow 1$ and by applying Theorem 1.1.1.

6.4 Estimate from above of the Γ-limit

Thanks to a standard approximation argument based on the use of Theorem 1.1.5, it will be enough to prove the Γ-limsup inequality for a function $u\in\mathcal{W}(\Omega)$ whose discontinuity set consists of the union of a finite family $\{S_1,..,S_k\}$ of disjoint segments compactly contained in Ω (see Remark 1.1.6). Let $\varepsilon_n\to 0$ and set, for every $u\in L^1(\Omega)$, $F''(u) := \Gamma\text{-}\limsup_{n\to\infty} F_{\varepsilon_n}(u)$; we aim to prove that

$$F''(u)\leq F(u). \qquad (6.4.1)$$

We begin by assuming that

$$S_i\cap\varepsilon_n\mathbb{Z}^2=\emptyset \qquad \forall n\in\mathbb{N},\ \forall i\in\{1,..,k\}. \qquad (6.4.2)$$

As for the proof of the Γ-liminf inequality, the thesis is achieved once we have shown that for a suitable sequence (u_n) converging to u, we have

$$\limsup_{n\to\infty} G^y_{\varepsilon_n,\xi}(u_n) \le \frac{1}{|\xi|^2}\left(\int_\Omega |\nabla u\cdot\hat{\xi}|^2\,dx + \int_{S_u}|\nu_u\cdot\hat{\xi}|\,d\mathcal{H}^1\right) \quad \forall\xi\in\mathbb{Z}^2,\ y\in Q_\xi. \tag{6.4.3}$$

To simplify the notation we will prove (6.4.3) only for $y=0$. In the sequel, given x_1 and x_2 in $\mathbb{R}^2$, we denote by $[x_1,x_2]$ the segment joining the two points. Let us define the following sets:

$$A_n := \{x\in\varepsilon_n\mathbb{Z}^2_\xi\cap\Omega:\ x+\varepsilon_n\xi\in\Omega,\ [x,x+\varepsilon_n\xi]\cap S_j=\emptyset \text{ for } j=1,\dots,k\},$$

and

$$B^j_n := \{x\in\varepsilon_n\mathbb{Z}^2_\xi:\ [x,x+\varepsilon_n\xi]\cap S_j\neq\emptyset\} \qquad j=1,\dots,k.$$

Clearly for n large enough, $B^j_n\cap B^i_n=\emptyset$ if $i\neq j$. Note now that we can write

$$\begin{aligned}
G^y_{\varepsilon_n,\xi}(u_n) &= \varepsilon_n^2\sum_{A_n}\frac{1}{a_{\varepsilon_n}|\xi|}\log\left(1+a_{\varepsilon_n}|\xi|\frac{|u_n(x+\varepsilon_n\xi)-u_n(x)|^2}{\varepsilon_n^2|\xi|^2}\right)\\
&\quad+\sum_{j=1}^k\varepsilon_n^2\sum_{B^j_n}\frac{1}{a_{\varepsilon_n}|\xi|}\log\left(1+a_{\varepsilon_n}|\xi|\frac{|u_n(x+\varepsilon_n\xi)-u_n(x)|^2}{\varepsilon_n^2|\xi|^2}\right)\\
&= \underbrace{\frac{1}{|\xi|^2}\int_\Omega\frac{1}{a_{\varepsilon_n}|\xi|}\log\left(1+a_{\varepsilon_n}|\xi|\frac{|v^0_{n,\xi}(x+\varepsilon_n\xi)-v^0_{n,\xi}(x)|^2}{\varepsilon_n^2|\xi|^2}\right)\chi_{(A_n+\varepsilon_nC_\xi)}}_{=\,I_{n,1}}\\
&\quad+\underbrace{\frac{1}{|\xi|^2}\sum_{j=1}^k\varepsilon_n^2|\xi|^2\sum_{B^j_n}\frac{1}{a_{\varepsilon_n}|\xi|}\log\left(1+a_{\varepsilon_n}|\xi|\frac{|u_n(x+\varepsilon_n\xi)-u_n(x)|^2}{\varepsilon_n^2|\xi|^2}\right)}_{=\,I_{n,2}},
\end{aligned}$$

where $v^0_{n,\xi}$ is the sequence defined in Lemma 6.3.1, while C_ξ is the set defined in (6.1.1). It is immediate to see that

$$\chi_{(A_n+\varepsilon_nC_\xi)}\to\chi_{\Omega\setminus S_u}. \tag{6.4.4}$$

Take $x\in\Omega\setminus S_u$ and let $y_n\in\varepsilon_n\mathbb{Z}^2_\xi$ be such that $x\in y_n+\varepsilon_nC_\xi$; by Lagrange's Theorem it turns out that

$$\begin{aligned}
\frac{1}{a_{\varepsilon_n}|\xi|}\log\left(1+a_{\varepsilon_n}|\xi|\frac{|v^0_{n,\xi}(x+\varepsilon_n\xi)-v^0_{n,\xi}(x)|^2}{\varepsilon_n^2|\xi|^2}\right) &= \frac{1}{a_{\varepsilon_n}|\xi|}\log\left(1+a_{\varepsilon_n}|\xi|\frac{|u(y_n+\varepsilon_n\xi)-u(y_n)|^2}{\varepsilon_n^2|\xi|^2}\right)\\
&= \frac{1}{a_{\varepsilon_n}|\xi|}\log\left(1+a_{\varepsilon_n}|\xi||\nabla u(\xi_n)\cdot\hat{\xi}|^2\right)\\
&\le |\nabla u(\xi_n)\cdot\hat{\xi}|^2,
\end{aligned}$$

where $\xi_n\in[y_n,y_n+\varepsilon_n\xi]$ and therefore $\xi_n\to x$. Taking into account the continuity of ∇u and recalling (6.4.4), we deduce that

$$\limsup_{n\to\infty} I_{n,1}\le\frac{1}{|\xi|^2}\int_\Omega|\nabla u(x)\cdot\hat{\xi}|^2\,dx. \tag{6.4.5}$$

Moreover, for every $x \in B_n^j$, we have

$$\frac{\varepsilon_n|\xi|}{a_{\varepsilon_n}|\xi|}\log\left(1+a_{\varepsilon_n}|\xi|\frac{|u_n(x+\varepsilon_n\xi)-u_n(x)|^2}{\varepsilon_n^2|\xi|^2}\right) \le \frac{\varepsilon_n}{a_{\varepsilon_n}}\log\left(1+a_{\varepsilon_n}\frac{4\|u\|_\infty^2}{\varepsilon_n^2}\right) \to 1, \tag{6.4.6}$$

where the last limit follows from the definition of a_{ε_n}. Denote by $l_\xi(S_j)$ the length of the projection of S_j on Π_ξ; using the fact that $l_\xi(S_j) = \int_{S_j} \nu_u \cdot \hat{\xi}\, d\mathcal{H}^1$, we easily obtain (see Figure 6.4 below)

$$\mathcal{H}^0(B_n^j) \le \frac{l_\xi(S_j)}{\varepsilon_n|\xi|} + 1 \le \frac{1}{\varepsilon_n|\xi|}\int_{S_j} \nu_u \cdot \hat{\xi}\, d\mathcal{H}^1 + 1; \tag{6.4.7}$$

therefore from (6.4.6) and (6.4.7) we get

$$\limsup_{n\to\infty} I_{2,n} \le \frac{1}{|\xi|^2}\limsup_{n\to\infty}\sum_{j=1}^k \varepsilon_n|\xi|\mathcal{H}^0(B_n^j) \le \frac{1}{|\xi|^2}\sum_{j=1}^k\int_{S_j}\nu_u\cdot\hat{\xi}\,d\mathcal{H}^1 = \frac{1}{|\xi|^2}\int_{S_u}\nu_u\cdot\hat{\xi}\,d\mathcal{H}^1,$$

which, combined with (6.4.5), gives (6.4.3) and therefore (6.4.1).

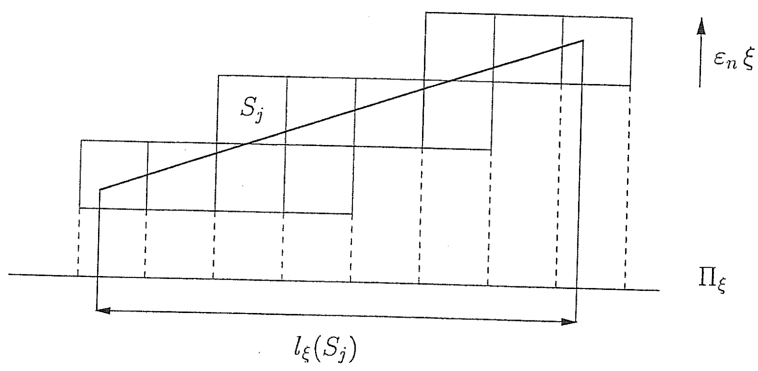

Figure 6.4: The projection of S_j on Π_ξ.

If (6.4.2) is not true we can argue in the following way. We first observe that it is possible to find a sequence $(\tau_k) \subset \mathbb{R}^2$ such that $\tau_k \to 0$ and $S_u + \tau_k$ satisfy (6.4.2) for every k. Let $u_k(x) := u(x - \tau_k)$, then $u_k \to u$, $S_{u_k} = S_u + \tau_k$ satisfies (6.4.2), and $F(u_k) \to F(u)$; using he previous step and the semicontinuity of F'', we have

$$F''(u) \le \liminf_{k\to\infty} F''(u_k) \le \lim_{k\to\infty} F(u_k) = F(u),$$

which concludes the proof.

□

6.5 Compactness

In this section we prove the equicoerciveness of the approximating functionals F_ε. We will use the L^1-precompactness criterion by slicing introduced by Alberti, Bouchitté & Seppecher (see Lemma 1.1.2).

Proposition 6.5.1 *Let (u_ε) be a sequence of equibounded functions such that $\sup_\varepsilon F_\varepsilon(u_\varepsilon) < M < +\infty$; then (u_ε) is strongly precompact in $L^p(\Omega)$, for every $p \geq 1$.*

Proof. Clearly it is enough to prove the precompactness in L^1. Let $\{e_1, e_2\}$ be the canonical basis in $\mathbb{R}^2$. Since for $\xi = e_i$ (for $i = 1, 2$) the function $v^y_{\varepsilon,\xi}$ defined in Lemma 6.3.1 coincides with u_ε, from (6.3.5) we have

$$M > \sup_\varepsilon F_\varepsilon(u_\varepsilon) \geq \sup_\varepsilon G^0_{\varepsilon,e_i} \geq \sup_\varepsilon \int_{\Omega_{e_i}} \mathcal{F}_\varepsilon\Big((u_\varepsilon)^w_{e_i}, \Omega^w_{e_i}\Big)\, d\mathcal{H}^1(w) = \sup_\varepsilon \int_{\Omega_{e_i}} f_\varepsilon(w)\, d\mathcal{H}^1(w), \tag{6.5.1}$$

where

$$f_\varepsilon(w) := \mathcal{F}_\varepsilon\Big((u_\varepsilon)^w_{e_i}, \Omega^w_{e_i}\Big).$$

Fix $\delta \in (0,1)$ and choose $k > 0$ so large that

$$M \frac{\sup_\varepsilon \|u_\varepsilon\|_\infty}{k} \operatorname{diam}(\Omega) < \delta; \tag{6.5.2}$$

setting $A^k_{\varepsilon,i} := \{w \in \Omega_{e_i} : f_\varepsilon(w) > k\}$, by Chebychev Inequality and (6.5.1), we can estimate

$$|A^k_{\varepsilon,i}| \leq \frac{1}{k} \sup_\varepsilon \int_{\Omega_{e_i}} f_\varepsilon(w)\, d\mathcal{H}^1(w) \leq \frac{M}{k}. \tag{6.5.3}$$

Let $z_{\varepsilon,\delta}$ be such that $z_{\varepsilon,\delta}(x) = 0$ if the projection of x on Ω_{e_i} belongs to $A^k_{\varepsilon,i}$ and $z_{\varepsilon,\delta}(x) = u_\varepsilon(x)$ otherwise. We clearly have

$$\|z_{\varepsilon,\delta} - u_\varepsilon\|_{L^1} \leq \sup_\varepsilon \|u_\varepsilon\|_\infty |A^k_{\varepsilon,i}| \operatorname{diam}(\Omega) \leq \delta,$$

where the last inequality follows from (6.5.3) and (6.5.2). Moreover $\mathcal{F}_\varepsilon\Big((z_{\varepsilon,\delta})^w_{e_i}, \Omega^w_{e_i}\Big) \leq f_\varepsilon(w)(1 - \chi_{A^k_{\varepsilon,i}}) \leq k$ for every $w \in \Omega_{e_i}$ and therefore $((z_{\varepsilon,\delta})^w_{e_i})$, by the one dimensional result, is precompact in $L^1(\Omega^w_{e_i})$ for every $w \in \Omega_{e_i}$. Thus we have constructed a sequence which is δ-closed to (u_ε) and such that the one-dimensional sections in the e_i-direction are precompact, for $i = 1, 2$. The thesis follows from Lemma 1.1.2. □

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