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A general splitting principle on the non-smooth setting and applications

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Introduction

History and motivation of RCD spaces

In the last decades there were several attempts to generalise Riemannian manifolds with curvature bounds: the main reason is to find a set of structures which contains Riemannian manifolds with curvature bounded from one side and is compact with respect to a reasonable convergence, the Gromov-Hausdorff one, which is induced by a distance between metric spaces which measures "how far two spaces are from being isomorphic".

The concept of metric space was enough to give a synthetic concept of space with sectional curvature bounded from one side via Toponogov comparison theorem (see [BBI] and its references for an introduction about the subject).

Theorem 0.0.1 (Toponogov)**.** *Let $k \in \mathbb{R}$ and let M be a complete, connected Riemannian manifold. Then the following are equivalent:*

i) the manifold has sectional curvature always greater than or equal to k;

ii) indicating with $\mathbb{M}^k$ the simply connected model space with constant sectional curvature k, for every $p \in M$ and $\gamma : [0,1] \to \mathrm{M}$ geodesic the following holds: let $\tilde{p}, \tilde{\gamma}_0, \tilde{\gamma}_1 \in \mathbb{M}^k$ be such that the three edges of the triangle $\tilde{p}\tilde{\gamma}_0\tilde{\gamma}_1$ are equal to the ones of $p\gamma_0\gamma_1$. Then $\mathsf{d}_{\mathrm{M}}(p, \gamma_t) \geq \mathsf{d}_{\mathbb{M}^k}(\tilde{p}, \tilde{\gamma}_t)$ for every $t \in [0,1]$.

Condition (ii) is purely metric and can be written in geodesic metric spaces. We can define then "metric spaces with sectional curvature bounded from below by k" (also known as Alexandrov spaces and denoted by $\mathsf{CBB}(k)$) as the spaces for which condition (ii) holds.

In trying to approach in a similar way manifolds with a lower bound on the Ricci curvature with essential stability properties we note that working with metric spaces is not enough: taking the limit (in the Gromov-Hausdorff sense) of a sequence of manifolds with Ricci curvature greater than K seen as metric spaces might fail to have Ricci curvature greater than K.

This first problem is solved taking metric measure spaces and considering a variant of the Gromov-Hausdorff convergence (measured Gromov-Hausdorff convergence) which takes into account the measures too.

The first generalization of manifolds with Ricci curvature bounded from below is given by J. Cheeger and T. Colding in [CC96] (see also [CC97], [CC00a], [CC00b] and [CN]), where they studied the theory of Ricci limits, that are the measured Gromov-Hausdorff limits of sequences of Rimannian manifolds with Ricci curvature uniformly bounded from below. This theory has good stability properties, but it is based on an extrinsic argument: Ricci limits can only be characterized (and studied) as limits of manifolds.

In searching an intrinsic approach, as the one used for spaces with bounds on sectional curvature, ten years later, indipendently in the works [S06a] and [S06b] by K.-T. Sturm and in [LV] by J. Lott and C. Villani, it was introduced the concept of Curvature-Dimension condition $\mathsf{CD}(K, N)$, where $K \in \mathbb{R}$ is a synthetic lower bound on the Ricci curvature and $N \in [1, +\infty]$ is an upper bound on the dimension.

In the class of $\mathsf{CD}(K, N)$ spaces several properties of manifolds with Ricci curvature bounded from below hold, for instance the Bishop-Gromov volume comparison ([S06b]), the Brunn-Minkowski inequality ([S06b], [BaSt]), the Bonnet-Myers theorem ([K]), the Laplacian comparison ([G15], [CaMo20]), the Lévy-Gromov isoperimetric inequality ([CaMo17], [CMM]) and a weak local (1,1)-Poincaré inequality ([R12b]), but it also contains non-Riemannian Finsler manifolds. The definition of $\mathsf{RCD}(K, N)$ spaces, which stands for Riemannian Curvature-Dimension condition, is given in [AGS14b] (for $N = +\infty$) and in [G15] (for $N \in [1, +\infty)$) in order to exclude Finsler manifolds and keep only $\mathsf{CD}(K, N)$ spaces with a "Riemannian structure": in addition to the $\mathsf{CD}(K, N)$ condition, the $\mathsf{RCD}(K, N)$ condition requires that a sort of tangent bundle (or equivalently a generalization of the Sobolev space $W^{1,2}$ on metric measure spaces) is a Hilbert space.

The definition of $\mathsf{RCD}(K, N)$ spaces allows to introduce second order differential calculus and to prove other important results such as a weak formulation of the Bochner inequality and some rigidity theorems such as the Cheeger-Gromoll Spltting Theorem and the "volume cone implies metric cone" principle (see [G13a], [G13b] and [DPG]).

The aim of this work is to study splitting-type theorems in more generality: more precisely, we prove that if there exists a function with "good" Laplacian and Hessian then the space is isomorphic to the product of the real line $\mathbb{R}$ with another space. We talk in detail about these theorems in the next sections, where we describe, as first thing, the known results of this type, and then the new general splitting principle we obtained in [GMa] and a couple of applications.

Splitting-type theorems

In this section we see some splitting-type theorems proved in the non-smooth setting.

The first one is the Cheeger-Gromoll Splitting Theorem.

Its first version on 2-dimensional surfaces with non-negative curvature was proved by S. Cohn-Vossen in [C-V], and about twenty years later Toponogov proved a version of the Splitting Theorem for manifolds with positive sectional curvature in [T]. Short after that A. D. Milka proved it in Alexandrov spaces (see [Mil]).

In [CG] J. Cheeger and D. Gromoll proved it for manifolds with positive Ricci curvature, then J. Cheeger and T. Colding proved in [CC96] an Almost Splitting Theorem, which leads directly to the Splitting Theorem for Ricci limit spaces.

The $\mathsf{CD}(0,N)$ class of metric measure spaces is too big, indeed it contains every space of the form $(\mathbb{R}^d, \mathsf{d}_{\|\cdot\|}, \mathcal{L}^d)$ where $\mathsf{d}_{\|\cdot\|}$ is a distance induced by a norm $\|\cdot\|$. It is easy to see that if $d=2$ and the norm $\|\cdot\|$ does not come from a scalar product then the Splitting Theorem cannot hold.

Its proof in the RCD setting is due to N. Gigli and can be found in [G13a] (see also [G13b]). We recall here its statement in $\mathsf{RCD}(0,N)$ spaces.

Theorem 0.0.2 (Splitting Theorem for RCD spaces)**.** *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathsf{RCD}(0,N)$ *space containing a line, i.e. such that there exists a map* $\bar\gamma : \mathbb{R} \to \mathrm{X}$ *such that*

$$\mathsf{d}(\bar\gamma_s, \bar\gamma_t) = |t-s|$$

for every $t, s \in \mathbb{R}$*.*

Then $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *is isomorphic to the product of the Euclidean line with the Lebesgue measure* $(\mathbb{R}, \mathsf{d}_{\mathrm{Eucl}}, \mathcal{L}^1)$ *and another space* $(\mathrm{X}', \mathsf{d}', \mathfrak{m}')$ *where the product distance* $\mathsf{d}_{\mathrm{Eucl}} \times \mathsf{d}'$ *is defined by*

$$\mathsf{d}_{\mathrm{Eucl}} \times \mathsf{d}'((t,x'),(s,y'))^2 := \mathsf{d}'(x',y')^2 + |t-s|^2$$

for every $x', y' \in \mathrm{X}'$ *and for every* $t, s \in \mathbb{R}$*.*

Moreover if $N \geq 2$ *then* $(\mathrm{X}', \mathsf{d}', \mathfrak{m}')$ *is an* $\mathsf{RCD}(0, N-1)$ *space; otherwise, if* $N \in [1,2)$*, the space* X' *is just a point.*

The idea of the proof is, as in the smooth setting, to take the Busemann function $\mathrm{b} : \mathrm{X} \to \mathbb{R}$ defined as

$$\mathrm{b}(x) := \lim_{t \to +\infty} t - \mathsf{d}(x, \bar\gamma_t),$$

where $\bar{\gamma}$ is a fixed line in X. In a certain sense, the modulus of the differential of b is $\mathfrak{m}$-almost everywhere equal to 1, moreover Δb and Hess(b) are both 0.

Using these properties, it is proved that, denoting with $\mathsf{Fl}_\cdot$ the gradient flow of b, the map $(x',t) \mapsto \mathsf{Fl}_t(x')$ is a measure preserving isometry between $(\mathrm{X},\mathsf{d},\mathfrak{m})$ and $(\mathbb{R}\times\mathrm{X}',\mathsf{d}_{\mathrm{Eucl}}\times\mathsf{d}',\mathcal{L}^1\times\mathfrak{m}')$, where X' is a level set of b, d' is the distance induced on X' and $\mathfrak{m}'$ is the measure on the level set X' obtained disintegrating $\mathfrak{m}$. The proof of the isomorphism is reached by proving that the Sobolev spaces of X and the product space are isometric and passing to the metric isomorphism thanks to the Sobolev-to-Lipschitz property.

The second rigidity theorem we recall is the "volume cone to metric cone" principle: its proof in Ricci limit spaces is due to J. Cheeger and T. Colding (see [CC96], they proved an "almost rigidity" result for this too, which we do not recall here), while in the RCD setting it has been proved by G. De Philippis and N. Gigli in [DPG].

Theorem 0.0.3 (From volume cone to metric cone)**.** *Let $N \in [1,+\infty)$ and let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be an $\mathsf{RCD}(0,N)$ space with $O \in \mathrm{X}$ and $R > r > 0$ such that*

$$\mathfrak{m}\left(B_R(O)\right) = \left(\frac{R}{r}\right)^N \mathfrak{m}\left(B_r(O)\right).$$

Then exactly one of the following holds:

i) the sphere $S_{\frac{R}{2}}(O)$ contains only one point: in this case (X,d) is isometric to $[0,\mathrm{diam}(\mathrm{X})]$ ($[0,+\infty)$ if X is unbounded) with an isometry which sends O in 0 and the measure $\mathfrak{m}|_{B_R(O)}$ to $N\mathfrak{m}(B_R(O))x^{N-1}\mathrm{d}x$;

ii) the sphere $S_{\frac{R}{2}}(O)$ contains exactly two points: in this case (X,d) is a 1-dimensional Riemannian manifold, possibly with boundary, moreover there is a bijective local isometry (in the sense of distance-preserving maps) from $B_R(O)$ to $(-R,R)$ sending O to 0 and the measure $\mathfrak{m}|_{B_R(O)}$ to the measure $\frac{1}{2}N\mathfrak{m}(B_R(O))|x|^{N-1}\mathrm{d}x$, and such local isometry is an isometry when restricted to $\bar{B}_{\frac{R}{2}}(O)$;

iii) the sphere $S_{\frac{R}{2}}(O)$ contains more than two points: in this case $N \geq 2$ and there exists an $\mathsf{RCD}(N-2,N-1)$ space $(\mathrm{X}',\mathsf{d}',\mathfrak{m}')$ with $\mathrm{diam}(\mathrm{X}') \leq \pi$ such that the ball $B_R(O)$ is locally isometric to the ball $B_R(O_{\mathrm{Y}})$ of the cone Y built over X', and such local isometry is an isometry when restricted to $\bar{B}_{\frac{R}{2}}(O)$.

The strategy to prove this theorem is similar to the one of the Splitting Theorem, but it requires a more sophisticated calculus with a more precise concept of Hessian.

The Busemann function in this case is $\mathrm{b}(x) := \frac{1}{2}\mathsf{d}(O,x)^2$. The modulus of the differential of b is $\sqrt{2\mathrm{b}}$, and its Laplacian is constantly equal to N on $B_R(O)$. Moreover its Hessian is equal to the identity.

A variant of this result has been proved by N. Gigli and I. Y. Violo in [GV].

Theorem 0.0.4 (From outer functional cone to outer metric cone)**.** *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be an* $\mathsf{RCD}(0,N)$ *space with* $N \geq 2$ *and let* U *be an open subset of* X *with* ∂U *bounded. Suppose that there exists a positive* $\mathrm{b} \in D(\Delta, U) \cap C(\bar{U})$ *such that*

- *-)* $\Delta \mathrm{b} = N$ *holds* $\mathfrak{m}$*-a.e. in* U*;*
- *-)* $|D\sqrt{2\mathrm{b}}|^2 = 1$ *holds* $\mathfrak{m}$*-a.e. in* U*;*
- *-)* $\mathrm{b}_0 := \limsup_{x\to\partial U} \mathrm{b}(x) < +\infty$*;*
- *-)* $\{\mathrm{b} > \mathrm{b}_0\} \neq \emptyset$*.*

Then the following holds:

i) there exists an $\mathsf{RCD}(N-2, N-1)$ *space* $(\mathrm{X}',\mathsf{d}',\mathfrak{m}')$ *with* $\mathrm{diam}(\mathrm{X}') \leq \pi$ *such that, denoting with* $(\mathrm{Y},\mathsf{d}_\mathrm{Y},\mathfrak{m}_\mathrm{Y})$ *the* N*-cone built over* X' *with vertex in* O_Y*, there exists a bijective measure preserving local isometry* $S : \{\mathrm{b} > \mathrm{b}_0\} \to Y \setminus \overline{B_r}(O_\mathrm{Y})$ *with* $r := \sqrt{2\mathrm{b}_0}$*;*

ii) if $\mathrm{diam}(\mathrm{X}') = \pi$ *then* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *is isomorphic to* $(\mathrm{Y},\mathsf{d}_\mathrm{Y},\mathfrak{m}_\mathrm{Y})$*, while if* $\mathrm{diam}(\mathrm{X}') < \pi$ *then the local isometry* S *is an isometry between* $\mathrm{Y} \setminus \overline{B_{r'}}(O_\mathrm{Y})$ *and* $\left\{b > \frac{1}{2}r_Z^2\right\}$*, where* $r' := \sqrt{2\mathrm{b}_0}\left(1 - \sin\left(\frac{(\mathrm{diam}(\mathrm{X}')}{2}\right)\right)^{-1}$*;*

iii) the function b *has the following explicit form:*

$$\mathrm{b}(x) = \frac{1}{2}\mathsf{d}_\mathrm{Y}(O_\mathrm{Y}, S(x))^2 = \frac{1}{2}(\mathsf{d}(x, \partial\{\mathrm{b} > \mathrm{b}_0\}) + \sqrt{2\mathrm{b}_0})^2$$

for every $x \in \{\mathrm{b} > \mathrm{b}_0\}$*. In particular for every* $t > \mathrm{b}_0$ *the level set* $\left\{\mathrm{b} > \frac{t^2}{2}\right\}$ *is Lipschitz-path connected and isometric, with its induced intrinsic metric, to* $(\mathrm{X}',\mathsf{d}')$*.*

The main difference between the proof of Theorem 0.0.3 and Theorem 0.0.4 is the use of the theory of Regular Lagrangian Flows to build the flow of the "gradient" of the Busemann function b.

A similar approach is used by C. Connell, X. Dai, J. Nuñéz-Zimbrón, R. Perales, P. Suaréz-Serrato and G. Wei in [CD+] to prove the following theorem.

Theorem 0.0.5. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathsf{RCD}(-(N-1), N)$ *space for a certain* $N \in (1, +\infty)$*. Assume that there exists a function* $\mathrm{b} \in D(\Delta_{loc})$ *such that* $|D\mathrm{b}| = 1$ $\mathfrak{m}$*-a.e. and* $\Delta \mathrm{b} = N - 1$*, then* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *is isomorphic to a warped product space* $\mathbb{R} \times_w \mathrm{X}'$ *with* $w_{\mathsf{d}} = \exp$ *and* $w_{\mathfrak{m}} = \exp^{N-1}$*, where* $(\mathrm{X}', \mathsf{d}', \mathfrak{m}')$ *is an* $\mathsf{RCD}(0, N)$ *space.*

Functions w_{d} and $w_{\mathfrak{m}}$ affect respectively the distance and the measure we take on the product space $\mathbb{R} \times \mathrm{X}'$. In the next section and in Section 1.7 we see precisely their role in defining warped product spaces.

General splitting principle and applications

We state now the main result of this work (see [GMa]) and a couple of applications to spaces with positive spectrum of the Laplacian.

On a smooth manifold M, almost by definitions, the structure of warped product emerges when one can find a function $\mathrm{b} : \mathrm{M} \to \mathbb{R}$ satisfying

$$\begin{aligned} |D\mathrm{b}| &\equiv 1, \\ \Delta \mathrm{b} &= \psi_{\mathfrak{m}} \circ \mathrm{b}, \\ \mathrm{Hess}\, \mathrm{b} &= \psi_{\mathsf{d}} \circ \mathrm{b}(\mathrm{Id} - \mathrm{e}_1 \otimes \mathrm{e}_1) \qquad \text{where } \mathrm{e}_1 := \tfrac{\nabla \mathrm{b}}{|\nabla \mathrm{b}|}, \end{aligned} \tag{0.0.1}$$

for suitable functions $\psi_{\mathfrak{m}}, \psi_{\mathsf{d}} : \mathrm{M} \to \mathbb{R}$. In this case it is easy to see that $\mathrm{M} \sim \mathbb{R} \times_w \mathrm{N}$ where:

-) the smooth manifold N is given by $\mathrm{N} := \mathrm{b}^{-1}(\{0\})$;

-) denoting with $\mathrm{Pr}(x)$ the point along the gradient flow trajectory of b defined by $\gamma_t' = \nabla \mathrm{b}(\gamma_t)$, $\gamma_0 = x$ such that $\mathrm{Pr}(x) \in \mathrm{N}$, the metric-measure isomorphism sends $x \in \mathrm{M}$ to $(\mathrm{b}(x), \mathrm{Pr}(x)) \in \mathbb{R} \times \mathrm{N}$;

-) the metric tensor on $\mathbb{R} \times_w \mathrm{N}$ is given by $\mathrm{d}t^2 + w_{\mathsf{d}}(t)(\mathrm{d}x')^2$, where $(\mathrm{d}x')^2$ is the metric on N and

$$w_{\mathsf{d}}(t) := \exp\Big(\int_0^t \psi_{\mathsf{d}}(s)\, \mathrm{d}s \Big);$$

-) the measure on $\mathbb{R} \times_w \mathrm{N}$ is given by $\mathrm{d}\mathcal{L}^1(t) \otimes w_{\mathfrak{m}}(t) \mathrm{dvol}_{\mathrm{N}}(x')$, where $\mathrm{vol}_{\mathrm{N}}$ is the volume measure on N and

$$w_{\mathfrak{m}}(t) := \exp\Big(\int_0^t \psi_{\mathfrak{m}}(s)\, \mathrm{d}s \Big).$$

As shown in the previous section, the strategies used to prove Theorems 0.0.2, 0.0.3, 0.0.4 and 0.0.5 are based on the same idea (with several non-trivial technicalities):

-) in Theorem 0.0.2 the function b satisfies (0.0.1) with $\psi_{\mathsf{m}} = \psi_{\mathsf{d}} = 0$;

-) in Theorem 0.0.3 and Theorem 0.0.4 the map $\sqrt{2\mathrm{b}}$ satisfies (0.0.1) with $\psi_{\mathsf{d}} = \frac{1}{t}$ and $\psi_{\mathsf{m}} = (N-1)\frac{1}{t}$;

-) in Theorem 0.0.5 the function b satisfies (0.0.1) with $\psi_{\mathsf{m}} = N-1$ and $\psi_{\mathsf{d}} = 1$.

Our aim is then to prove that if there exists a function b which satisfies (0.0.1) for suitable ψ_{d} and ψ_{m} then the space splits, i.e. X is isomorphic as a metric measure space to the warped product $\mathbb{R} \times_w \mathrm{X}'$ with warping functions w_{d} and w_{m}, and the measure preserving isometry is given by $x \mapsto (\mathrm{b}(x), \mathsf{Fl}_{-\mathrm{b}(x)}(x))$, where Fl is the (Regular Lagrangian) flow of $\nabla \mathrm{b}$. In this sense, our splitting principle can be seen as a general tool that allows to translate the analytic information encoded in (0.0.1) into a geometric information. The theorem can be roughly stated as follows.

Theorem 0.0.6. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathsf{RCD}(K, N)$ *space with* $K \in \mathbb{R}$*,* $N \in [1, +\infty)$ *and* $\mathrm{supp}(\mathfrak{m}) = \mathrm{X}$*. Assume that there exists a function* $\mathrm{b} : \mathrm{X} \to \mathbb{R}$ *in* $H^{2,2}_{loc}(\mathrm{X}) \cap D(\Delta_{loc})$ *which satisfies* (0.0.1) *for some* $\psi_{\mathsf{d}}, \psi_{\mathsf{m}} : \mathbb{R} \to \mathbb{R}$ *locally Lipschitz.*

Then X *is isomorphic as a metric measure space to a warped product space* $\mathbb{R} \times_w \mathrm{X}'$ *for suitable warping functions* $w_{\mathsf{d}}, w_{\mathsf{m}}$*.*

A main difference between our work and the previous similar results we mentioned is that in studying how the gradient flow of b acts on vector fields we have to (more) carefully distinguish between the component parallel to $\nabla \mathrm{b}$ and the one orthogonal to it, since in the previous studies the specific geometry of the problem provided some simplifications that are not present here, see in particular Proposition 2.4.3.

In principle, one would like to obtain sharp informations about the RCD property satisfied by the quotient space X'. This, however, seems tricky to do in our generality and our results in this direction are sub-optimal: we prove, under quite general assumptions on the warping functions, that if the original space is $\mathsf{RCD}(K, N)$, then so is the quotient one. In particular, and in line with [CD+], we do not see an improving in the upper dimension bound. It seems that to obtain this a more careful analysis of the Bochner inequality - akin to that in [K] - on warped spaces is needed, but this is outside the scope of this work.

We conclude with two applications that in the smooth setting are due to P. Li and J. Wang (see [LW01] and [LW02]): in those works they studied complete manifolds with positive spectrum of the Laplacian, reaching two splitting-type theorems depending on the number (and volume) of ends of the space.

Roughly speaking, an end is a minimal unbounded connected component of $\mathrm{X} \setminus \mathrm{K}$ for a certain compact K: for instance, $\mathbb{R}^N$ with $N \geq 2$ has only one end, the line $\mathbb{R}$ has two ends, the space of lines on $\mathbb{R}^2$ passing through the origin with rational angular coefficient has infinite ends.

While the volume of an end obviously depends on the compact K, its finiteness does not: if an end has infinite volume with respect to the compact K then it has infinite volume with respect to every compact $\mathrm{K}' \supset \mathrm{K}$.

The first theorem, whose smooth version can be found in [LW01], is the following. It states that if the first eigenvalue of the Laplacian is strictly positive (more precisely greater or equal to $N-2$ with $N \geq 3$) and the space has more than one end with infinite volume then the space splits.

Theorem 0.0.7. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathsf{RCD}(-(N-1), N)$ *space with* $N \geq 3$ *and* $\mathrm{supp}(\mathfrak{m}) = \mathrm{X}$*, and assume that the first eigenvalue of the Laplacian* λ_1 *is* $\geq N-2$*. Then one of the following holds:*

i) X *has only one end with infinite volume;*

ii) X *is isomorphic as metric measure space to a warped product space* $\mathbb{R} \times_w \mathrm{X}'$*, where* X' *is a compact* $\mathsf{RCD}(-(N-1), N)$ *space and the warping functions are*

$$w_{\mathsf{d}}(t) := \cosh(t) \qquad \textit{and} \qquad w_{\mathfrak{m}}(t) := \cosh^{N-1}(t).$$

Moreover, in this case $\lambda_1 = N-2$*.*

The idea of the proof of this theorem is similar to its smooth version, but, as usually happens, translating the proof in a more general case produces some technical difficulties. The main differences between the two cases are the following:

-) to construct a non-constant bounded harmonic function on X (see Section 3.3) we need the Gauss-Green formula in the non-smooth setting (proved in [BPS]), and this requires a cautious work on sets with finite perimeter (see Theorem 3.3.6);

-) as in the smooth case the rigidity comes from an equality in the Bochner inequality: in proving this Li and Wang use the fact that the Laplacian is the trace of the Hessian, and this is not true in a generic $\mathsf{RCD}(K,N)$ space (as proved in [Han], this holds if and only if the dimension of the space is equal to N); this problem is solved using a more general Kato-type inequality and an improved version of the Bochner inequality;

-) another key point is that, given $\mathrm{d}|\mathrm{d}u| \parallel \mathrm{d}u$, we have that locally $|\mathrm{d}u| = \varphi \circ u$: in the smooth case this is pretty simple to see, in the non-smooth setting the proof is much more involved;

-) in proving that the quotient space X' is $\mathsf{RCD}(K',N')$ our argument does not provide the best constants K', N'.

Let M be an N-dimensional Riemannian manifold with Ricci curvature bounded from below by $K < 0$. Then Cheng's Theorem states that the first eigenvalue of the Laplacian λ_1 satisfies $\lambda_1 \leq \frac{-(N-1)K}{4}$. The same estimate holds in $\mathsf{RCD}(K,N)$ spaces, so the second theorem we are going to prove (see [LW02] for its smooth version) studies the case in which λ_1 is the maximum admissible. Assuming that $N > 3$, from Theorem 0.0.7 it follows that X has only one end with infinite volume; the theorem states that if it has also (at least) an end with finite volume then the space splits.

Theorem 0.0.8. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathsf{RCD}(-(N-1), N)$ *space with* $N > 3$ *and* $\mathrm{supp}(\mathfrak{m}) = \mathrm{X}$*, and assume that the first eigenvalue of the Laplacian* λ_1 *is equal to* $\frac{(N-1)^2}{4}$. *Then one of the following holds:*

i) X *has only one end;*

ii) X *is isomorphic as metric measure space to a warped product space* $\mathbb{R} \times_w \mathrm{X}'$, *where* X' *is a compact* $\mathsf{RCD}(0, N)$ *space and the warping functions are*

$$w_{\mathsf{d}}(t) := e^t \qquad \text{and} \qquad w_{\mathfrak{m}}(t) := e^{(N-1)t}.$$

Its proof is pretty similar to the proof of Theorem 0.0.7. We note that in this case we can, arguing as in [CD+], improve the bound on the curvature of the quotient space (but still not improving the bound on the dimension).

Plan of the work

In Chapter 1 we recall the definitions of all the instruments we need in order to prove the general splitting principle (Theorem 0.0.6) and their properties.

We start with some basic notations on metric measure spaces in Section 1.1. In Section 1.2 we see how to define Sobolev spaces in metric measure spaces via test plans and the modulus of the distributional differential of Sobolev functions, the so called minimal weak upper gradient. The aim of Section 1.3 is to introduce a first order differential calculus on metric measure spaces; to do that we rely on the concept of normed modules, using it to define the cotangent module (equipped with a differential operator) and its dual, the tangent module (equipped with the gradient). We conclude this section defining the divergence operator as the adjoint of the differential and the Laplacian as the divergence of the gradient. In Section 1.4 we introduce the concept of pullback of 1-forms and of speed of test plans.

In Section 1.5 we give various definitions of RCD condition on metric measure spaces: one based on the differential calculus built in Sections 1.2 and 1.3, and others based on optimal transport (different concepts of CD condition), then we see how all these are related to each other. In Section 1.6 we see how the RCD condition allows us to define a second order calculus on metric measure spaces, i.e. the space $W^{2,2}$, the Hessian and the covariant derivative.

Given that theory of differential calculus, we are ready to recall the two main instruments that we are going to use in the proof of Theorem 0.0.6: warped product spaces and Regular Lagrangian Flows. In Section 1.7 we recall the definition of warped product spaces and see how the Sobolev space of a product space behave with respect to the Sobolev spaces of the quotient spaces. Then we see when the Sobolev-to-Lipschitz property passes to the product. Section 1.8 contains the definition and the main properties of Regular Lagrangian Flows.

In Chapter 2 we prove Theorem 0.0.6 and a result about the RCD condition of the quotient space X'.

In Section 2.1 we see how to construct a Busemann function b (a function which satisfies (0.0.1)) starting with a regular enough function with a "good" Hessian. This will be useful in the applications, since usually an equality in the Bochner inequality gives a function with a "good" Hessian in that way. In Section 2.2 we fix the assumptions and the notations that we are going to use in the course of the proof. In Section 2.3 we study how the measure changes if pushed forward with the flow of $\nabla \mathrm{b}$ and use it to deduce how the disintegrated measure on the level sets of b behaves. In Section 2.4 we see how the flow of $\nabla \mathrm{b}$ changes the distances. From this follows a relation between the Sobolev space of X and the one of X'. Using the results of the last two sections, in Section 2.5 we prove that the map $\mathsf{T}(x) := (\mathrm{b}(x), \mathsf{Fl}_{-b(x)}(x))$ is a measure preserving isometry, and we conclude with a (less general and sub-optimal) result about the RCD condition of X'.

Chapter 3 contains two applications of the general splitting principle:

Theorems 0.0.7 and 0.0.8.

We start with two more sections of preliminary facts: in Section 3.1 we define harmonic functions and a more general (but coherent with respect to the one defined before) concept of Laplacian, and use it to state an improved version of the Bochner inequality, while in Section 3.2 we recall briefly the theory of quasi-continuous functions and quasi-continuous vector fields in order to state a Gauss-Green formula in the non-smooth setting.

In Section 3.3 we define ends and see how the volume of an end is related with the existence of non-constant bounded harmonic functions on it. Using this result we construct the two non-constant harmonic functions ("bounded" assuming the hypotheses of Theorem 0.0.7 and "positive" assuming the hypotheses of Theorem 0.0.8) that will lead to the proofs of the two main theorems.

In Section 3.4 we prove Theorem 0.0.7 starting, in Section 3.4.1 proving that the bounded harmonic function u constructed in Section 3.3 satisfies (0.0.1), then, in Section 3.4.2 we prove that its minimal weak upper gradient is a composite function of u, and in Section 3.4.3 we conclude the proof using Theorem 0.0.6.

Section 3.5 contains the proof of Theorem 0.0.8, which is pretty similar to the one of Theorem 0.0.7, but in this case our result about the RCD condition of the quotient space does not apply. However this precise case has been studied in [CD+], so we recall briefly their method in Section 3.5.2.

Chapter 1

Preliminaries

In this chapter we recall the definitions, notations and known results we will use along all the work, starting with differential calculus in metric measure spaces and the definition of RCD space and concluding with some properties of warped product spaces and Regular Lagrangian Flows.

1.1 Notation on metric measure spaces

As we will always work with metric measure spaces, the aim of this section is to fix some basic notation on them.

We start considering a metric space (X, d).

A curve on X is an element of $C([0,1], \mathrm{X})$, and for a curve $\gamma \in C([0,1], \mathrm{X})$ we will usually indicate with γ_t the point $\gamma(t)$. The evaluation map e_t on the set of curves is defined as follows.

Definition 1.1.1 (Evaluation map)**.** *Let* (X, d) *be a metric space. For every* $t \in [0,1]$*, we define the map* $e_t : C([0,1], \mathrm{X}) \to \mathrm{X}$ *as* $e_t(\gamma) := \gamma_t$.

We recall now the definitions of metric speed of a curve and absolutely continuous curve.

Definition 1.1.2 (Metric speed)**.** *Let* (X, d) *be a metric space. For every* $\gamma \in C([0,1], \mathrm{X})$ *and for every* $t \in [0,1]$ *we define the metric speed of* γ *at the time* t *as*

$$|\dot{\gamma}_t| := \begin{cases} \lim\limits_{h \to 0} \dfrac{\mathsf{d}(\gamma_{t+h}, \gamma_t)}{h} & \text{if such limit exists;} \\ +\infty & \text{otherwise.} \end{cases}$$

Definition 1.1.3 (Absolutely continuous curve)**.** *Let* (X, d) *be a metric space. We say that a curve* γ *is asbolutely continuous (AC) if there exists*

$f \in L^1([0,1])$ such that

$$\mathsf{d}(\gamma_t,\gamma_s) \le \int_t^s f(r)\,\mathrm{d}r \tag{1.1.1}$$

for every $t,s \in [0,1]$ with $t < s$.

It is possible to see that if γ is absolutely continuous then its metric speed is finite for a.e. $t \in [0,1]$ and it is an admissible function for (1.1.1). Moreover, for every other f satisfying (1.1.1) it holds $|\dot\gamma_t| \le f(t)$ for a.e. $t \in [0,1]$.

Other key instruments are the kinetic energy of a curve and the concept of geodesic curve.

Definition 1.1.4 (Kinetic energy). *We define the operator kinetic energy* $\mathrm{KE} : C([0,1],\mathrm{X}) \to [0,+\infty]$ *as*

$$\mathrm{KE}(\gamma) := \begin{cases} \dfrac{1}{2}\displaystyle\int_0^1 |\dot\gamma_t|^2\,\mathrm{d}t & \textit{if } \gamma \textit{ is absolutely continuous;} \\ +\infty & \textit{otherwise.} \end{cases}$$

Definition 1.1.5 (Geodesic curve). *A geodesic is a curve γ which satisfies*

$$\frac{1}{2}\mathsf{d}(\gamma_0,\gamma_1)^2 = \mathrm{KE}(\gamma),$$

or, equivalently,

$$\mathsf{d}(\gamma_t,\gamma_s) = |s-t|\mathsf{d}(\gamma_0,\gamma_1)$$

for every $t,s \in [0,1]$.

We indicate with $\mathrm{Geo}(\mathrm{X})$ *the set of geodesics on* X.

Let $(\mathrm{X},\mathsf{d}_\mathrm{X})$ and $(\mathrm{Y},\mathsf{d}_\mathrm{Y})$ be metric spaces, we will indicate with $\mathrm{LIP}(\mathrm{X},\mathrm{Y})$ the set of Lipschitz continuous functions (we omit Y if the codomain is $\mathbb{R}$), and for $f \in \mathrm{LIP}(\mathrm{X},\mathrm{Y})$ we will denote with $\mathrm{Lip}(f)$ its Lipschitz constant and with $\mathrm{lip}(f)(x)$ its local Lipschitz constant at x, given by the following definition.

Definition 1.1.6 (Local Lipschitz constant). *Let $(\mathrm{X},\mathsf{d}_\mathrm{X})$ and $(\mathrm{Y},\mathsf{d}_\mathrm{Y})$ be metric spaces and let $f \in \mathrm{LIP}(\mathrm{X},\mathrm{Y})$. The local Lipschitz constant of f is the function* $\mathrm{lip}(f) : \mathrm{X} \to [0,+\infty)$ *defined as*

$$\mathrm{lip}(f)(x) := \begin{cases} 0 & \textit{if } x \textit{ is an isolated point,} \\ \displaystyle\limsup_{y\to x} \frac{\mathsf{d}_\mathrm{Y}(f(x),f(y))}{\mathsf{d}_\mathrm{X}(x,y)} & \textit{otherwise.} \end{cases}$$

Moreover we define the asymptotic Lipschitz constant $\mathrm{lip}_a(f) : \mathrm{X} \to [0, +\infty)$ *as*

$$\mathrm{lip}_a(f)(x) := \begin{cases} 0 & \textit{if } x \textit{ is an isolated point,} \\ \limsup_{y,z \to x} \dfrac{\mathsf{d}_{\mathrm{Y}}(f(z), f(y))}{\mathsf{d}_{\mathrm{X}}(z, y)} & \textit{otherwise.} \end{cases}$$

We will indicate with $\mathrm{LIP}_{loc}(\mathrm{X}, \mathrm{Y})$ the set of locally Lipschitz functions and with $\mathrm{LIP}_{bs}(\mathrm{X}, \mathrm{Y})$ the set of Lipschitz functions with bounded support.

Let now $(\mathrm{X}, \mathfrak{m})$ be a measure space.

We indicate with $L^0(\mathfrak{m})$ the space of measurable functions on X modulo $\mathfrak{m}$-a.e. equality.

Definition 1.1.7 (Pushforward measure)**.** *Let* $(\mathrm{X}, \mathcal{A}_{\mathrm{X}})$ *and* $(\mathrm{Y}, \mathcal{A}_{\mathrm{Y}})$ *be two measurable spaces and let* $\varphi : \mathrm{X} \to \mathrm{Y}$ *be a measurable function. Given any measure* μ *on* $(\mathrm{X}, \mathcal{A}_{\mathrm{X}})$*, we define the pushforword measure* $\varphi_* \mu$ *on* $(\mathrm{Y}, \mathcal{A}_{\mathrm{Y}})$ *as*

$$(\varphi_* \mu)(B) := \mu\left(\varphi^{-1}(B)\right)$$

for every $B \in \mathcal{A}_{\mathrm{Y}}$*.*

Remark 1.1.8. For every $f \in L^1(\varphi_* \mu)$ it holds the change of variable formula

$$\int f \, \mathrm{d}(\varphi_* \mu) = \int f \circ \varphi \, \mathrm{d}\mu. \tag{1.1.2}$$

■

As we said, the "right" objects to generalize manifolds with Ricci curvature bounded from below are metric measure spaces, which are complete and measurable metric spaces equipped with a non-negative Borel measure finite on balls.

Definition 1.1.9 (Metric measure space)**.** *A triple* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *is a metric measure space if*

-) (X, d) *is a complete and separable metric space;*

-) $\mathfrak{m}$ *is a non-negative Borel measure on* X *which is finite on balls.*

1.2 Sobolev space on metric measure spaces

From the end of the last century several notions of Sobolev space on metric measure spaces has been introduced.

The first approach is due to P. Hajłasz in [Haj], who defined the Sobolev space $W^{1,p}(\mathrm{X})$ in the following way: $f \in W^{1,p}(\mathrm{X})$ if $f \in L^p(\mathfrak{m})$ and there exists $g \in L^p(\mathfrak{m})$ such that for $\mathfrak{m}$-a.e. x, y it holds

$$|f(x) - f(y)| \leq \mathsf{d}(x,y)(g(x) + g(y)).$$

The "module of the differential of f" is defined as the unique g with minimal L^p-norm that satisfies the previous inequality. This definition is consistent with the classical definition of Sobolev spaces in $\mathbb{R}^n$, but that notion of differential does not satisfy a locality property, i.e. it may depend not only on the local behaviour of the function.

A different approach is given by J. Cheeger in [C], who defined Sobolev functions using the concept of upper gradients introduced in [HeKo]: a Borel function g is an upper gradient of f provided that for every absolutely continuous curve $\gamma : [0,1] \to \mathrm{X}$ the curve $f \circ \gamma$ is absolutely continuous and satisfies $|(f \circ \gamma)'_t| \leq g(\gamma_t)|\dot{\gamma}_t|$.

Cheeger's strategy is then to define an energy functional $\mathsf{E}_{\mathsf{Ch}} : L^2 \to [0, +\infty]$ as

$$\mathsf{E}_{\mathsf{Ch}}(f) := \inf \liminf_{n \to +\infty} \frac{1}{2} \int g_n^2 \, \mathrm{d}\mathfrak{m},$$

where the infimum is taken among all the sequences $(g_n)_{n \in \mathbb{N}}$ such that there exists a sequence $f_n \to f$ in $L^2(\mathfrak{m})$ for which g_n is an upper gradient of f_n for every $n \in \mathbb{N}$. The Sobolev space $W^{1,2}$ is defined as the space of L^2 functions with finite E_{Ch} energy.

Inspired by Cheeger's approach, in [AGS14a] is proposed a definition via an "approximation with Lipschitz functions" argument: noticing that for a Lipschitz function f the maps $\mathrm{lip}(f)$ and $\mathrm{lip}_a(f)$ are upper gradients, one can define the energy functional $\mathsf{E}_* : L^2(\mathfrak{m}) \to [0, +\infty]$ as

$$\mathsf{E}_*(f) := \inf \liminf_{n \to +\infty} \frac{1}{2} \int \mathrm{lip}^2(f) \, \mathrm{d}\mathfrak{m},$$

where the infimum is taken among all the sequences $(f_n)_{n \in \mathbb{N}} \subset \mathrm{LIP}(\mathrm{X})$ such that $f_n \xrightarrow{L^2(\mathfrak{m})} f$, then the Sobolev space $W^{1,2}(\mathrm{X})$ as the space of $L^2(\mathfrak{m})$ functions with finite energy E_* (one can argue similarly with the asymptotic Lipschitz constant lip_a instead of lip in the definition of E_*).

In [Sha] N. Shanmugalingam gave a notion of Sobolev space based on how functions behave on (a good class of) curves. This approach relies on the concept of 2-modulus of a family of curves.

A similar approach, based on the concept of test plan instead of 2-moduli, is given by L. Ambrosio, N. Gigli and G. Savaré in [AGS14a].

The approaches given by Cheeger, Shanmugalingam, Ambrosio, Gigli and Savaré are consistent with the classical definition of Sobolev space in $\mathbb{R}^n$, and their definition of "modulus of the differential" are "local" (Proposition 1.2.11). Moreover, a consequence of a deep approximation result contained in [AGS13] (which we recall at the end of this section, Theorem 1.2.16) is the equivalence between these approaches.

All our work relies on the definition of Sobolev space via test plan, so the aim of this section is to recall the basic definitions and first properties of Sobolev functions in this sense.

We start recalling the definition of test plans, which are probability measures on the space of absolutely continuous curves that, in a certain sense, does not overlap too much.

Definition 1.2.1 (Test plan). *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space. A probability measure* $\boldsymbol{\pi} \in \mathscr{P}(C([0,1], \mathrm{X}))$ *is said to be a test plan on* X *provided the following two properties are satisfied:*

i) there exists a constant $C > 0$ *such that* $(e_t)_*\boldsymbol{\pi} \leq C\mathfrak{m}$ *for every* $t \in [0,1]$*;*

ii) it holds that $\int_0^1 \int |\dot{\gamma}_t|^2 \mathrm{d}\boldsymbol{\pi}(\gamma)\,\mathrm{d}t < +\infty$.

We are now ready to give the definition of weak upper gradient of a function f via test plans, and with it the definition of Sobolev class $S^2(\mathrm{X})$, which in the smooth setting is the class of functions with L^2 weak derivative.

Definition 1.2.2 (Weak upper gradient). *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space and let* $f : \mathrm{X} \to \mathbb{R}$ *be a Borel function. Then we say that a non-negative function* $G \in L^2(\mathfrak{m})$ *is a weak upper gradient of* f *if for every test plan* $\boldsymbol{\pi}$ *on* X *it holds*

$$\int |f(\gamma_1) - f(\gamma_0)|\,\mathrm{d}\boldsymbol{\pi}(\gamma) \leq \int_0^1 \int G(\gamma_t)|\dot{\gamma}_t|\,\mathrm{d}\boldsymbol{\pi}(\gamma)\mathrm{d}t.$$

Definition 1.2.3 (Sobolev class). *The Sobolev class* $S^2(\mathrm{X})$ *is defined as the space of all Borel functions* $f : \mathrm{X} \to \mathbb{R}$ *that admit a weak upper gradient.*

An easy consequence of this definition and Fatou's Lemma is the lower semicontinuity of weak upper gradients.

Proposition 1.2.4. *Let $(f_n)_n \subset S^2(\mathrm{X})$ and $f : \mathrm{X} \to \mathbb{R}$ Borel such that $f_n(x) \to f(x)$ for $\mathfrak{m}$-a.e. $x \in \mathrm{X}$. For every $n \in \mathbb{N}$ let G_n be a weak upper gradient of f_n and assume that $G_n \rightharpoonup G$ weakly in $L^2(\mathfrak{m})$ for some $G \in L^2(\mathfrak{m})$. Then $f \in S^2(\mathrm{X})$ and G is a weak upper gradient of f.*

A consequence of Proposition 1.2.4 is that the set of weak upper gradients of a function $f \in S^2(\mathrm{X})$ is closed and convex in $L^2(\mathfrak{m})$, then there exists a unique weak upper gradient with minimal $L^2(\mathfrak{m})$-norm.

Definition 1.2.5 (Minimal weak upper gradient)**.** *Let $f \in S^2(\mathrm{X})$. Then the unique weak upper gradient with minimal $L^2(\mathfrak{m})$-norm is called minimal weak upper gradient of f and it is denoted by $|Df| \in L^2(\mathfrak{m})$.*

The minimal weak upper gradient of an S^2 map f is the map that behaves like the modulus of the distributional differential of f. Note that the structures we have at this moment are not enough to give a definition of differential of f, in order to do so we need the theory of normed modules that we will introduce in the next section, but the modulus of the differential of functions is enough to define the Sobolev space $W^{1,2}(\mathrm{X})$.

Definition 1.2.6 (Sobolev space)**.** *Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be a metric measure space. The Sobolev space $W^{1,2}(\mathrm{X})$ is the set of $L^2(\mathfrak{m})$ functions that admit a weak upper gradient, i.e. $W^{1,2}(\mathrm{X}) := L^2(\mathfrak{m}) \cap S^2(\mathrm{X})$.*

For $f \in W^{1,2}(\mathrm{X})$ we define its $W^{1,2}(\mathrm{X})$-norm as

$$\|f\|_{W^{1,2}(\mathrm{X})} := \sqrt{\|f\|^2_{L^2(\mathfrak{m})} + \|Df\|^2_{L^2(\mathfrak{m})}}.$$

Remark 1.2.7. For every $f \in \mathrm{LIP}_{bs}(\mathrm{X})$ it holds $f \in S^2(\mathrm{X})$ and

$$|Df| \leq \mathrm{lip}(f) \leq \mathrm{lip}_a(f) \leq \mathrm{Lip}(f) \qquad \mathfrak{m}\text{-a.e. in X.} \tag{1.2.1}$$

■

Remark 1.2.8. The Sobolev space $W^{1,2}(\mathrm{X})$ with the norm $\|\cdot\|_{W^{1,2}(\mathrm{X})}$ is a Banach space, but in general it is not a Hilbert space. For instance if X is a smooth Finsler manifold then $W^{1,2}(\mathrm{X})$ is Hilbert if and only if X is a Riemannian manifold. ■

The previous Remark motivates the definition of infinitesimally Hilbertian metric measure space.

Definition 1.2.9 (Infinitesimal Hilbertianity)**.** *The metric measure space $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ is said to be infinitesimally Hilbertian if $\big(W^{1,2}(\mathrm{X}), \|\cdot\|_{W^{1,2}(\mathrm{X})}\big)$ is a Hilbert space.*

Remark 1.2.10. In all this work we heavily rely on the infinitesimal Hilbertianity of the space since, as we said in the Introduction, it is the hypothesis that allows to exclude Finsler manifolds and allows us to define a second order differential calculus on metric measure spaces. ■

The minimal weak upper gradient obeys to some natural calculus rules such as a locality property, the chain rule and the Leibniz rule.

Proposition 1.2.11 (Locality for the minimal weak upper gradient)**.** *Let* $f, g \in S^2(\mathrm{X})$. *Then* $|Df| = |Dg|$ *holds* $\mathfrak{m}$*-a.e. on* $\{f = g\}$.

Proposition 1.2.12 (Chain rule for the minimal weak upper gradient)**.** *Let* $f \in S^2(\mathrm{X})$. *If a Borel set* $N \subset \mathbb{R}$ *is* $\mathcal{L}^1$*-negligible then* $|Df| = 0$ *holds* $\mathfrak{m}$*-a.e. on* $f^{-1}(N)$.

Moreover, if $\varphi \in \mathrm{LIP}(\mathbb{R})$, *then* $\varphi \circ f \in S^2(\mathrm{X})$ *and*

$$|D(\varphi \circ f)| = |\varphi'| \circ f |Df| \tag{1.2.2}$$

holds $\mathfrak{m}$*-a.e..*

Proposition 1.2.13 (Leibniz rule for the minimal weak upper gradient)**.** *Let* $f, g \in S^2(\mathrm{X}) \cap L^\infty(\mathfrak{m})$. *Then* $fg \in S^2(\mathrm{X}) \cap L^\infty(\mathfrak{m})$ *and the inequality*

$$|D(fg)| \leq |f||Dg| + |g||Df|$$

holds $\mathfrak{m}$*-a.e. on* X.

Thanks to Proposition 1.2.11, we can define the local Sobolev class $S^2_{loc}(\mathrm{X})$ and, with that, the local Sobolev space $W^{1,2}_{loc}(\mathrm{X})$.

Definition 1.2.14 (Local Sobolev class)**.** *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space and let* $f : \mathrm{X} \to \mathbb{R}$ *be a Borel function. We say that* $f \in S^2_{loc}(\mathrm{X})$ *provided that for every bounded Borel set* $B \subset \mathrm{X}$ *there exists a function* $f_B \in S^2(\mathrm{X})$ *such that* $f_B = f$ *holds* $\mathfrak{m}$*-a.e. in* B.

For every $f \in S^2_{loc}(\mathrm{X})$ *we define its minimal weak upper gradient as the function (unique up to* $\mathfrak{m}$*-negligible sets)* $|Df|$ *such that for every bounded Borel set* B *and every function* $f_B \in S^2(\mathrm{X})$ *which coincides with* f *on* B *it holds* $|Df| = |Df_B|$.

Definition 1.2.15 (Local Sobolev space)**.** *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space. The local Sobolev space* $W^{1,2}_{loc}(\mathrm{X})$ *is defined as* $S^2_{loc}(\mathrm{X}) \cap L^2_{loc}(\mathfrak{m})$.

Moreover, given $U \subset \mathrm{X}$ *open, we define* $W^{1,2}_0(U)$ *as the closure in* $W^{1,2}(\mathrm{X})$ *of* $\mathrm{LIP}_{bs}(U)$.

We conclude this section with the following approximation theorem: its proof can be found in [AGS13] (see also [AGS14a]).

Theorem 1.2.16. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space. Then Lipschitz functions on* X *are dense in energy in* $W^{1,2}(\mathrm{X})$*, namely for every* $f \in W^{1,2}(\mathrm{X})$ *there exists a sequence* $(f_n)_n \subset \mathrm{LIP}(\mathrm{X}) \cap L^2(\mathfrak{m})$ *such that*

$$f_n \xrightarrow{L^2} f, \qquad \textit{and} \qquad \mathrm{lip}_a(f_n) \xrightarrow{L^2} |Df|. \tag{1.2.3}$$

An easy consequence of (1.2.1) and (1.2.3) (and the minimality of the minimal weak upper gradient) is that, given f and $(f_n)_n$ as in Theorem 1.2.16, also the sequences $(\mathrm{lip}(f_n))_n$ and $(|Df_n|)_n$ converge in $L^2(\mathfrak{m})$ to $|Df|$.

1.3 Normed modules and first order differential calculus

In the previous section we recalled the definition of Sobolev space on a metric measure space defining what the "modulus of the differential" of a function is. In order to develop a tensor calculus on metric measure spaces we recall now a definition of "tangent/cotangent bundle" in the non-smooth setting given by N. Gigli in [G18b].

More precisely, it generalizes the concept of "space of L^2 sections of a normed vector bundle", and the idea, inspired by [W], is to consider moduli over the commutative ring $L^\infty(\mathrm{X})$ with a pointwise norm operator.

Definition 1.3.1 (L^2-normed L^∞-module)**.** *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space. An* $L^2(\mathfrak{m})$*-normed* $L^\infty(\mathfrak{m})$*-module is a quadruple* $(\mathscr{M}, \|\cdot\|_{\mathscr{M}}, \cdot, |\cdot|)$ *with the following properties:*

i) $(\mathscr{M}, \|\cdot\|_{\mathscr{M}})$ *is a Banach space;*

ii) the multiplication by $L^\infty(\mathfrak{m})$ *functions* $\cdot : L^\infty(\mathfrak{m}) \times \mathscr{M} \to \mathscr{M}$ *is a bilinear map satisfying*

$$f \cdot (g \cdot v) = (fg) \cdot v \qquad \textit{for every } f, g \in L^\infty(\mathfrak{m}) \textit{ and } v \in \mathscr{M},$$
$$\hat{1} \cdot v = v \qquad \textit{for every } v \in \mathscr{M},$$

where $\hat{1}$ *is the function identically equal to 1 on* X*;*

iii) the pointwise norm $|\cdot| : \mathscr{M} \to L^2(\mathfrak{m})$ *satisfies*

$$|v| \geq 0 \qquad \textit{holds } \mathfrak{m}\textit{-a.e. for every } v \in \mathscr{M},$$
$$|f \cdot v| = |f||v| \qquad \textit{holds } \mathfrak{m}\textit{-a.e. for every } f \in L^\infty(\mathfrak{m}) \textit{ and } v \in \mathscr{M},$$
$$\|v\|_{\mathscr{M}} = \||v|\|_{L^2(\mathfrak{m})} \qquad \textit{for every } v \in \mathscr{M}.$$

A variant of this definition which does not rely on integrability hypotheses is the following.

Definition 1.3.2 (L^0-normed L^0-module)**.** *An $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-module is a quadruple $(\mathscr{M}^0, \tau, \cdot, |\cdot|)$ with the following properties:*

i) *$(\mathscr{M}^0, \tau)$ is a topological vector space;*

ii) *the multiplication by L^0 functions $\cdot : L^0(\mathfrak{m}) \times \mathscr{M}^0 \to \mathscr{M}^0$ is a bilinear map satisfying*

$$f \cdot (g \cdot v) = (fg) \cdot v, \text{ and } \hat{1} \cdot v = v$$

for every $f, g \in L^0(\mathfrak{m})$ and $v \in \mathscr{M}^0$, where $\hat{1}$ is the functions identically equal to 1;

iii) *the pointwise norm $|\cdot| : \mathscr{M}^0 \to L^0(\mathfrak{m})$ satisfies*

$$\begin{aligned} |v| &\geq 0 && \mathfrak{m}\text{-a.e. for every } v \in \mathscr{M}^0, \\ |f \cdot v| &= |f||v| && \mathfrak{m}\text{-a.e. for every } f \in L^0(\mathfrak{m}) \text{ and } v \in \mathscr{M}^0; \end{aligned}$$

iv) *the topology τ is induced by the map $\mathsf{d}_{\mathscr{M}^0} : \mathscr{M}^0 \times \mathscr{M}^0 \to [0, +\infty)$ defined by*

$$\mathsf{d}_{\mathscr{M}^0}(v, w) := \int |v - w| \wedge 1 \, \mathrm{d}\mathfrak{m}'$$

for some $\mathfrak{m}' \in \mathscr{P}(\mathrm{X})$ with $\mathfrak{m} \ll \mathfrak{m}' \ll \mathfrak{m}$.

The concepts of L^2-normed L^∞-module and L^0-normed L^0-module are strictly related, indeed they can be seen one as the restriction/completion of the other thanks to the following propositions.

Proposition 1.3.3 (L^2-restriction)**.** *Let $\mathscr{M}^0$ be an L^0-normed L^0-module and let $\mathscr{M}$ be defined as*

$$\mathscr{M} := \left\{ v \in \mathscr{M}^0 : |v| \in L^2(\mathfrak{m}) \right\}.$$

Then $\mathscr{M}$ with the norm $\|\cdot\|_{\mathscr{M}} := \||\cdot|\|_{L^2(\mathfrak{m})}$ and the product and pointwise norm induced by the ones of $\mathscr{M}^0$ is an L^2-normed L^∞-module.

Proposition 1.3.4 (L^0-completion)**.** *Let $\mathscr{M}$ be an L^2-normed L^∞-module. Then there exists a unique couple $(\mathscr{M}^0, \iota)$ where $\mathscr{M}^0$ is an L^0-normed L^0-module and the map $\iota : \mathscr{M} \to \mathscr{M}^0$ is a linear operator with dense image that preserves the pointwise norm.*

Uniqueness is intended up to unique isomorphism .

In a very natural way we define Hilbert modules and dual normed modules.

Definition 1.3.5 (Hilbert module). *Let $\mathscr{M}$ be an L^2-normed L^∞-module. We say that $\mathscr{M}$ is a Hilbert module if $(\mathscr{M}, \|\cdot\|_{\mathscr{M}},)$ is a Hilbert space, or, equivalently, if the pointwise norm satisfies the parallelogram rule, i.e.*

$$|v+w|^2 + |v-w|^2 = 2(|v|^2 + |w|^2)$$

holds $\mathfrak{m}$-a.e. for every $v, w \in \mathscr{M}$.

Definition 1.3.6 (Dual normed module). *Let $\mathscr{M}$ be an L^2-normed L^∞-module. Its dual normed module $\mathscr{M}^*$ is defined as the set of linear and continuous operators $L : \mathscr{M} \to L^1(\mathfrak{m})$ such that $L(fv) = fL(v)$ for every $v \in \mathscr{M}$ and $f \in L^\infty(\mathfrak{m})$ endowed with the operator norm*

$$\|L\|_* := \sup_{\|v\|_{\mathscr{M}} \leq 1} \|L(v)\|_{L^1(\mathfrak{m})}$$

and the pointwise norm

$$|L|_* := \underset{v \in \mathscr{M}, |v| \leq 1 \ \mathfrak{m}\text{-}a.e.}{\text{ess-sup}} L(v).$$

It can be proved that the dual normed module just defined is an L^2-normed L^∞-module.

Tangent and cotangent modules, that we will define in the next section, will be the dual one of the other. Moreover their Hilbertianity will be related to the infinitesimal Hilbertianity of the space.

1.3.1 Cotangent and tangent modules

Thanks to the next existence and uniqueness theorem we define the cotangent module and the differential operator on a generic metric measure space.

Theorem 1.3.7 (Cotangent Module). *Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be a metric measure space. Then there exists a unique couple $(L^2(T^*\mathrm{X}), \mathrm{d})$, where $L^2(T^*\mathrm{X})$ is an L^2-normed L^∞-module and $\mathrm{d} : S^2(\mathrm{X}) \to L^2(T^*\mathrm{X})$ is a linear operator, such that the following conditions hold:*

i) $|\mathrm{d}f| = |Df|$ *holds $\mathfrak{m}$-a.e. for every $f \in S^2(\mathrm{X})$;*

ii) $L^2(T^*\mathrm{X})$ *is generated by $\{\mathrm{d}f : f \in S^2(\mathrm{X})\}$.*

We shall refer to $L^2(T^*\mathrm{X})$ as cotangent module and to d as differential.

We will indicate with $L^0(T^*\mathrm{X})$ the L^0-completion of $L^2(T^*\mathrm{X})$.

The differential just defined inherits from the weak upper gradient similar calculus rules.

Proposition 1.3.8 (Locality for the differential). *Let* $f, g \in S^2(\mathrm{X})$. *Then* $\mathrm{d}f = \mathrm{d}g$ *holds* $\mathfrak{m}$*-a.e. on* $\{f = g\}$.

Proposition 1.3.9 (Chain rule for the differential). *Let* $f \in S^2(\mathrm{X})$. *If a Borel set* $N \subset \mathbb{R}$ *is* $\mathcal{L}^1$*-negligible then* $\mathrm{d}f = 0$ *holds* $\mathfrak{m}$*-a.e. on* $f^{-1}(N)$.

Moreover, if $\varphi \in \mathrm{LIP}(\mathbb{R})$, *then* $\varphi \circ f \in S^2(\mathrm{X})$ *and* $\mathrm{d}(\varphi \circ f) = \varphi' \circ f \mathrm{d}f$.

Proposition 1.3.10 (Leibniz rule for the differential). *Let* $f, g \in S^2(\mathrm{X}) \cap L^\infty(\mathfrak{m})$. *Then* $fg \in S^2(\mathrm{X}) \cap L^\infty(\mathfrak{m})$ *and* $\mathrm{d}(fg) = f\mathrm{d}g + g\mathrm{d}f$.

We define now the tangent module, which will be the space of "vector fields", as the dual of the cotangent module (as in Definition 1.3.6).

Definition 1.3.11 (Tangent module). *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space. The tangent module* $L^2(T\mathrm{X})$ *of* X *is defined as the dual normed module of* $L^2(T^*\mathrm{X})$. *Its elements are called vector fields.*

Similarly to the definition of $L^0(T^*\mathrm{X})$, we indicate with $L^0(T\mathrm{X})$ the L^0-completion of $L^2(T\mathrm{X})$.

In order to define the gradient of a Sobolev function we need to assume that the space is inifinitesimally Hilbertian and combine the two following results: the first is a Riesz representation theorem for normed modules, the second is the anticipated relation between Hilbert modules and infinitesimal Hilbertianity.

Theorem 1.3.12 (Riesz). *Let* $\mathscr{M}$ *be a Hilbert* L^2*-normed* L^∞*-module. Then the map* $\mathscr{M} \to \mathscr{M}^*$ *which sends* v *to* $\langle \cdot, v \rangle$ *is an isomorphism of modules.*

Proposition 1.3.13 (Infinitesimal Hilbertianity). *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space. Then* X *is infinitesimally Hilbertian if and only if* $L^2(T\mathrm{X})$ *and* $L^2(T^*\mathrm{X})$ *are Hilbert modules.*

The gradient can then be defined, in infinitesimally Hilbertian spaces, as the dual of the differential.

Definition 1.3.14 (Gradient). *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an infinitesimally Hilbertian metric measure space. Then we denote by* $\nabla f \in L^2(T\mathrm{X})$ *the element corresponding to* $\mathrm{d}f \in L^2(T^*\mathrm{X})$ *via the Riesz isomorphism. We call* ∇f *the gradient of* f.

Remark 1.3.15. Infinitesimal Hilbertianity is not really necessary to define the gradient of a function, indeed given $f \in S^2(\mathrm{X})$ the gradient of f can be defined as the set of vector fields v such that $\mathrm{d}f(v) = |\mathrm{d}f|^2 = |v|^2$ holds $\mathfrak{m}$-a.e.. Gradients defined like that obey to good calculus rules, but in general they are not a singleton, and even if they are they may not depend linearly on f. It can be proved that the space is infinitesimally Hilbertian if and only if for every $f \in S^2(\mathrm{X})$ the gradient of f is a singleton (whose element is denoted by ∇f) and for every $f, g \in W^{1,2}(\mathrm{X})$ the formula $\nabla(f+g) = \nabla f + \nabla g$ holds $\mathfrak{m}$-a.e. in X. In this case this gradient is equivalent to the one defined in Definition 1.3.14. ■

From the calculus rules for the differential immediately follow the ones for the gradient.

Proposition 1.3.16 (Chain rule for the gradient)**.** *Let* X *be an infinitesimally Hilbertian metric measure space and let* $f \in S^2_{loc}(\mathrm{X})$ *and* $\varphi \in \mathrm{LIP}_{loc}(\mathbb{R})$*. Then* $\nabla(\varphi \circ f) = \varphi' \circ f \nabla f$*.*

Proposition 1.3.17 (Leibniz rule for the gradient)**.** *Let* X *be an infinitesimally Hilbertian metric measure space and let* $f, g \in S^2_{loc}(\mathrm{X}) \cap L^\infty_{loc}(\mathfrak{m})$*. Then* $\nabla(fg) = f\nabla g + g\nabla f$*.*

1.3.2 Divergence and Laplace operators

We define the divergence operator as the adjoint of the differential.

Definition 1.3.18 (Divergence)**.** *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space,* $U \subset \mathrm{X}$ *open and* $v \in L^0(T\mathrm{X})$*. We say that* $v \in D(\mathrm{div}_{loc}, U)$ *provided there is* $f \in L^2_{loc}(U)$ *such that*

$$\int \mathrm{d}g(v)\,\mathrm{d}\mathfrak{m} = -\int fg\,\mathrm{d}\mathfrak{m}$$

for every $g \in W^{1,2}_0(U)$*.*

In this case the function f*, that is easily seen to be uniquely determined, is denoted by* $\mathrm{div}(v)$*. In the case* $U = \mathrm{X}$ *we simply write* $v \in D(\mathrm{div}_{loc})$*.*

The divergence operator just defined satisfies a locality property and a Leibniz rule.

Proposition 1.3.19 (Locality for the divergence)**.** *Let* $v, w \in D(\mathrm{div}_{loc}, U)$ *and assume that* $v = w$ *holds* $\mathfrak{m}$*-a.e. on an open set* $\Omega \subset U$*. Then* $\mathrm{div}(v) = \mathrm{div}(w)$ *holds* $\mathfrak{m}$*-a.e. on* Ω*.*

Proposition 1.3.20 (Leibniz rule for the divergence)**.** *Let* $v \in D(\mathrm{div}_{loc}, U)$ *and* $f \in \mathrm{LIP}(U) \cap L^\infty(U)$*. Then* $fv \in D(\mathrm{div}_{loc}, U)$ *and*

$$\mathrm{div}(fv) = \mathrm{d}f(v) + f\mathrm{div}(v)$$

holds $\mathfrak{m}$*-a.e. in* U*.*

The Laplacian operator can be defined as the divergence of the gradient. We note that, in order to define the Laplace operator, infinitesimal Hilbertianity is a necessary condition.

Definition 1.3.21 (Laplacian)**.** *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be infinitesimally Hilbertian,* $U \subset \mathrm{X}$ *open and* $f \in W^{1,2}_{loc}(\mathrm{X})$*. We say that* $f \in D(\Delta_{loc}, U)$ *provided* $\nabla f \in D(\mathrm{div}_{loc}, U)$*, i.e. if there is* $h \in L^2_{loc}(\mathrm{X})$ *such that*

$$\int \langle \nabla f, \nabla \varphi \rangle \,\mathrm{d}\mathfrak{m} = -\int \varphi h \,\mathrm{d}\mathfrak{m}$$

for every $\varphi \in W^{1,2}_0(U)$*.*

In this case the function h *is denoted by* Δf*. In the case* $U = \mathrm{X}$ *we simply write* $f \in D(\Delta_{loc})$*.*

The Laplacian operator obviously inherits the following calculus rules from the ones of gradient and divergence.

Proposition 1.3.22 (Chain rule for the Laplacian)**.** *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be infinitesimally Hilbertian and* $U \subset \mathrm{X}$ *open. For every* $f \in D(\Delta_{loc}, U) \cap \mathrm{LIP}(U)$ *and* $\varphi \in C^2(\mathbb{R})$ *with* $\varphi'' \in L^\infty(\mathbb{R})$ *it holds* $\varphi \circ f \in D(\Delta_{loc}, U)$ *and*

$$\Delta(\varphi \circ f) = \varphi' \circ f \Delta f + \varphi'' \circ f |\nabla f|^2. \tag{1.3.1}$$

Proposition 1.3.23 (Leibniz rule for the Laplacian)**.** *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be infinitesimally Hilbertian and* $U \subset \mathrm{X}$ *open. Let* $f, g \in D(\Delta_{loc}, U) \cap \mathrm{LIP}(U) \cap L^\infty(U)$*. Then* $fg \in D(\Delta_{loc}, U)$ *and*

$$\Delta(fg) = f\Delta g + g\Delta f + 2\langle \nabla f, \nabla g \rangle. \tag{1.3.2}$$

Remark 1.3.24. In general the set $D(\Delta_{loc}) \cap \mathrm{LIP}(\mathrm{X})$ could contain only constant functions. However this does not happen in the RCD setting thanks to the Bakry-Émery inequality and the regularizing properties of the heat flow. For the details see for instance [AGS14b]. ■

1.4 Pullback of 1-forms and speed of a test plan

The aim of this section is to give a definition of speed of a test plan and a chain rule for it.

All the proofs of this section (except where indicated otherwise) can be found in [G18b] and [G18a].

We start defining maps of bounded compression, i.e. maps for which the push-forward measure (Definition 1.1.7) does not concentrate the mass too much, and pullback modules with respect to maps of bounded compression.

Definition 1.4.1 (Maps of bounded compression). *Let* $(\mathrm{X}, \mathfrak{m}_{\mathrm{X}})$ *and* $(\mathrm{Y}, \mathfrak{m}_{\mathrm{Y}})$ *be measured spaces. We say that* $\varphi : \mathrm{Y} \to \mathrm{X}$ *is a map of bounded compression if there exists* $C > 0$ *such that* $\varphi_* \mathfrak{m}_{\mathrm{Y}} \leq C \mathfrak{m}_{\mathrm{X}}$.

The least such constant C *is called compression constant and is denoted by* $\mathrm{Comp}(\varphi)$.

The pullback module is defined via the following existence and uniqueness theorem.

Theorem 1.4.2 (Pullback module). *Let* $(\mathrm{X}, \mathsf{d}_{\mathrm{X}}, \mathfrak{m}_{\mathrm{X}})$ *and* $(\mathrm{Y}, \mathsf{d}_{\mathrm{Y}}, \mathfrak{m}_{\mathrm{Y}})$ *be metric measure spaces. Let* $\mathscr{M}$ *be an* $L^2(\mathfrak{m}_{\mathrm{X}})$*-normed* $L^\infty(\mathfrak{m}_{\mathrm{X}})$*-module and let* $\varphi : \mathrm{Y} \to \mathrm{X}$ *be a map of bounded compression. Then there exists a unique couple* $(\varphi^*\mathscr{M}, [\varphi^* \cdot])$ *where* $\varphi^*\mathscr{M}$ *is an* $L^2(\mathfrak{m}_{\mathrm{Y}})$*-normed* $L^\infty(\mathfrak{m}_{\mathrm{Y}})$*-module and* $[\varphi^*\cdot] : \mathscr{M} \to \varphi^*\mathscr{M}$ *is a linear and continuous operator such that*

i) $|[\varphi^* v]| = |v| \circ \varphi$ *holds* $\mathfrak{m}_Y$*-a.e. for every* $v \in \mathscr{M}$*;*

ii) the set $\{[\varphi^* v] : v \in \mathscr{M}\}$ *generates* $\varphi^*\mathscr{M}$*.*

We note that for every test plan $\boldsymbol{\pi}$ the evaluation map e_t has bounded compression (by item (i) of Definition 1.2.1), then Theorem 1.4.2 guarantees the existence and uniqueness of the pullback module $L^2(T\mathrm{X})$. We define the speed of a test plan as follows.

Theorem 1.4.3 (Speed of a test plan). *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an infinitesimally Hilbertian metric measure space and let* $\boldsymbol{\pi}$ *be a test plan on* X*. Then for a.e.* $t \in [0,1]$ *there exists a unique* $\boldsymbol{\pi}'_t \in e_t^* L^2(T\mathrm{X})$ *such that for every* $f \in W^{1,2}(\mathrm{X})$ *it holds*

$$L^1(\boldsymbol{\pi}) - \lim_{h \to 0} \frac{f \circ e_{t+h} - f \circ e_t}{h} = [e_t^* \mathrm{d} f](\boldsymbol{\pi}'_t).$$

Moreover $|\boldsymbol{\pi}'_t|(\gamma) = |\dot\gamma_t|$ *for* $(\boldsymbol{\pi} \times \mathcal{L}^1)$*-a.e.* (γ, t)*.*

We shall refer to $\boldsymbol{\pi}'_t$ *as speed of* $\boldsymbol{\pi}$ *at time* t*.*

In order to write a sort of chain rule for the speed of test plans we need to restrict to Lipschitz maps of bounded compression and say what their differential is.

Definition 1.4.4 (Maps of bounded deformation)**.** *Given two metric measure spaces* $(\mathrm{X}, \mathsf{d}_\mathrm{X}, \mathfrak{m}_\mathrm{X})$ *and* $(\mathrm{Y}, \mathsf{d}_\mathrm{Y}, \mathfrak{m}_\mathrm{Y})$*, we say that* $\varphi : \mathrm{Y} \to \mathrm{X}$ *is a map of bounded deformation if it is both Lipschitz continuous and of bounded compression.*

Theorem 1.4.5 (Pullback of 1-forms)**.** *Let* $(\mathrm{X}, \mathsf{d}_\mathrm{X}, \mathfrak{m}_\mathrm{X})$ *and* $(\mathrm{Y}, \mathsf{d}_\mathrm{Y}, \mathfrak{m}_\mathrm{Y})$ *be metric measure spaces and* $\varphi : \mathrm{Y} \to \mathrm{X}$ *of bounded deformation. Then there exists a unique linear and continuous operator* $\varphi^* : L^2(T^*\mathrm{X}) \to L^2(T^*\mathrm{Y})$ *such that*

i) $\varphi^* \mathrm{d}f = \mathrm{d}(f \circ \varphi)$ *for every* $f \in S^2(\mathrm{X})$*;*

ii) $\varphi^*(g\omega) = g \circ \varphi \, \varphi^* \omega$ *for every* $g \in L^\infty(\mathfrak{m}_\mathrm{X})$ *and* $\omega \in L^2(T^*\mathrm{X})$*.*

Moreover

$$|\varphi^* \omega| \leq \mathrm{Lip}(\varphi) |\omega| \circ \varphi$$

holds $\mathfrak{m}_\mathrm{Y}$*-a.e. for every* $\omega \in L^2(T^*\mathrm{X})$*.*

Theorem 1.4.6 (Differential of maps of bounded deformation)**.** *Let* X *and* Y *be metric measure spaces with* X *infinitesimally Hilbertian, and* $\varphi : \mathrm{Y} \to \mathrm{X}$ *of bounded deformation. Then there exists a unique* $L^\infty(\mathfrak{m}_\mathrm{Y})$*-linear and continuous map* $\mathrm{d}\varphi : L^2(T\mathrm{Y}) \to \varphi^* L^2(T\mathrm{X})$ *such that*

$$[\varphi^* \omega](\mathrm{d}\varphi(v)) = \varphi^* \omega(v)$$

for every $v \in L^2(T\mathrm{Y})$ *and* $\omega \in L^2(T^*\mathrm{X})$*.*

Moreover, for every $v \in L^2(T\mathrm{Y})$ *it holds* $\mathfrak{m}_\mathrm{Y}$*-a.e. that*

$$|\mathrm{d}\varphi(v)| \leq \mathrm{Lip}(\varphi) |v|.$$

In case φ is invertible with inverse of bounded compression we can express the differential of φ as a function from $L^2(T\mathrm{Y})$ to $L^2(T\mathrm{X})$ denoting with $\mathrm{d}\varphi(v)$ the map $\omega \mapsto (\varphi^* \omega(v)) \circ \varphi^{-1}$. The precise statement is the following.

Proposition 1.4.7. *Let* $(\mathrm{X}, \mathsf{d}_\mathrm{X}, \mathfrak{m}_\mathrm{X})$ *and* $(\mathrm{Y}, \mathsf{d}_\mathrm{Y}, \mathfrak{m}_\mathrm{Y})$ *be two metric measure spaces and let* $\varphi : \mathrm{Y} \to \mathrm{X}$ *be an invertible map of bounded deformation and assume that its inverse has bounded compression. Then there exists a unique linear continuous operator* $\mathrm{d}\varphi : L^2(T\mathrm{Y}) \to L^2(T\mathrm{X})$ *such that*

$$\omega(\mathrm{d}\varphi(v)) = (\varphi^* \omega(v)) \circ \varphi^{-1} \qquad \textit{holds } \mathfrak{m}_\mathrm{X}\textit{-a.e.}$$

for every $v \in L^2(T\mathrm{Y})$ *and* $\omega \in L^2(T^*\mathrm{X})$.

Moreover

$$|\mathrm{d}\varphi(v)| \leq \mathrm{Lip}(\varphi)|v| \circ \varphi^{-1} \qquad \textit{holds } \mathfrak{m}_{\mathrm{X}}\textit{-a.e.}$$

for every $v \in L^2(T\mathrm{Y})$.

We conclude this section recalling the chain rule for speeds of test plans (for its proof see [DPG, Proposition 3.28]).

With a slight abuse of notation we still denote with $F : C([0,1],\mathrm{Y}) \to C([0,1],\mathrm{X})$ the map such that $(F(\gamma))_t = F(\gamma_t)$ for every $t \in [0,1]$ and every $\gamma \in C([0,1],\mathrm{Y})$. Given a test plan $\boldsymbol{\pi}$ on Y, assuming that F is of bounded deformation, it is easy to see that $F_*\boldsymbol{\pi}$ is a test plan on X.

Let $F : \mathrm{Y} \to \mathrm{X}$ be of bounded deformation, invertible and with inverse of deformation and let $\boldsymbol{\pi}$ be a test plan on Y. For every $t \in [0,1]$ the differential $\mathrm{d}F : L^2(T\mathrm{Y}) \to L^2(T\mathrm{X})$ defined in Proposition 1.4.7 naturally induces a map $e_t^*L^2(T\mathrm{Y}) \to e_t^*L^2(T\mathrm{X})$, i.e. the unique linear and continuos map, still denoted by $\mathrm{d}F$, such that

$$\begin{aligned} &\mathrm{d}F(e_t^*v) = e_t^*(\mathrm{d}F(v)) && \text{for every } v \in L^2(T\mathrm{Y}), \\ &\mathrm{d}F(gV) = g \circ F\mathrm{d}F(V) && \text{for every } V \in e_t^*L^2(T\mathrm{Y}),\ g \in L^\infty(\boldsymbol{\pi}). \end{aligned}$$

Proposition 1.4.8 (Chain rule for speeds). *Let* $F : \mathrm{X} \to \mathrm{Y}$ *be a map of bounded deformation, invertible and with inverse with bounded deformation. Then, for every test plan* $\boldsymbol{\pi}$ *on* X *we have*

$$(F_*\boldsymbol{\pi})'_t = (\mathrm{d}F)(\boldsymbol{\pi}'_t), \qquad \textit{for a.e. } t \in [0,1].$$

1.5 RCD spaces

In this section we show the two main intrinsic approaches to define the Riemannian Curvature-Dimension condition:

-) via optimal transport and convexity of the entropy;

-) via differential calculus and Bochner inequality.

For the second approach, due to L. Ambrosio, N. Gigli and G. Savaré (see [AGS15]) we already recalled all the needed notions in the previous sections.

Definition 1.5.1 (RCD space via Bochner inequality). *Let* $N \in [1,+\infty)$ *and* $K \in \mathbb{R}$. *We say that the metric measure space* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *is an* $\mathsf{RCD}(K,N)$ *space if the following properties are satisfied:*

-) *the space* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *is infinitesimally Hilbertian;*

-) *there exists a constant* $C > 0$ *such that* $\mathfrak{m}(B_r(x)) \leq e^{Cr^2}$ *for every* $r > 0$ *and for every* $x \in \mathrm{X}$*;*

-) *the Sobolev-to-Lipschitz property holds, i.e. every* $f \in W^{1,2}(\mathrm{X})$ *with* $|Df| \leq C$ *admits a* C*-Lipschitz representative;*

-) *for every* $f \in D(\Delta)$ *such that* $\Delta f \in W^{1,2}(\mathrm{X})$ *and every* $g \in D(\Delta) \cap L^\infty(\mathfrak{m})$ *such that* $g \geq 0$ *and* $\Delta g \in L^\infty(\mathfrak{m})$ *the weak Bochner inequality holds, i.e.*

$$\frac{1}{2}\int |Df|^2 \Delta g \, \mathrm{d}\mathfrak{m} \geq \int \left[\frac{(\Delta f)^2}{N} + \langle \nabla f, \nabla \Delta f \rangle + K|Df|^2\right] g \, \mathrm{d}\mathfrak{m}. \tag{1.5.1}$$

The same definition can be extended to the case $N = +\infty$ simply deleting the term $\frac{(\Delta f)^2}{N}$ in the right-hand-side of (1.5.1).

We talk about the first approach in the following section.

1.5.1 Wasserstein distance and CD conditions

In this section we recall some basic tools of optimal transport, use them to give three definitions of curvature-dimension condition ($\mathsf{CD}/\mathsf{CD}^*/\mathsf{CD}^e$) and see how they are related. They were introduced by K.-T. Sturm, J. Lott and C. Villani (CD) in [S06a], [S06b], [LV], by K. Bacher and K.-T. Sturm (CD^*) in [BaSt] and by M. Erbar, K. Kuwada and K.-T. Sturm (CD^e) in [EKS]. The "R" to obtain RCD spaces were added by L. Ambrosio, N. Gigli and G. Savaré in [AGS14b] in the case $N = +\infty$ and by N. Gigli in [G15] for the finite dimensional case, and it adds the requirement that the space is infinitesimally Hilbertian.

We start defining the space of measures with finite second moment and the Wasserstein distance on it.

Definition 1.5.2 (Measure with finite second moment)**.** *Let* (X, d) *be a complete and separable metric space. We indicate with* $\mathscr{P}_2(\mathrm{X})$ *the set of probability measures* $\mu \in \mathscr{P}(\mathrm{X})$ *such that*

$$\int_{\mathrm{X}} \mathsf{d}^2(x, y) \, \mathrm{d}\mu(y) < +\infty$$

for some $x \in \mathrm{X}$ *(it is easy to see that if it holds for a point* $x \in \mathrm{X}$ *then it holds for every point). A measure* $\mu \in \mathscr{P}_2(\mathrm{X})$ *is said to have finite second moment.*

Definition 1.5.3 (Wasserstein distance)**.** *Let $\mu, \nu \in \mathscr{P}_2(\mathrm{X})$. Denoting with π^1 and π^2 the projections on the first and the second coordinate respectively, $\pi^1(x,y) := x$ and $\pi^2(x,y) := y$, we define*

$$W_2^2(\mu,\nu) := \inf \int \mathsf{d}^2(x,y)\,\mathrm{d}\boldsymbol{\pi}(x,y), \tag{1.5.2}$$

where the infimum is taken among all $\boldsymbol{\pi} \in \mathscr{P}(\mathrm{X}^2)$ such that $\pi^1_*\boldsymbol{\pi} = \mu$ and $\pi^2_*\boldsymbol{\pi} = \nu$.

It turns out that the infimum in (1.5.2) is always attained, i.e. it is actually a minimum.

Proposition 1.5.4 (Wasserstein space)**.** *The map $W_2 : \mathscr{P}_2(\mathrm{X}) \times \mathscr{P}_2(\mathrm{X}) \to \mathbb{R}$ is a distance and $(\mathscr{P}_2(\mathrm{X}), W_2)$, called Wasserstein space, is complete and separable. Moreover, if (X, d) is geodesic then the Wasserstein space is geodesic too.*

The Ricci curvature bound from below by K comes from a convexity property (depending on the bound K) of the Boltzmann-Shannon entropy along Wasserstein geodesics.

Definition 1.5.5 (Boltzmann-Shannon relative entropy)**.** *Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be a metric measure space such that there exist $c_1, c_2 > 0$ and $x \in \mathrm{X}$ for which*

$$\mathfrak{m}(B_r(x)) \leq c_1 e^{c_2 r} \qquad \text{for every } r > 0. \tag{1.5.3}$$

We define the Boltzmann-Shannon entropy for every $\mu \in \mathscr{P}_2(\mathrm{X})$ as

$$\mathrm{Ent}_{\mathfrak{m}}(\mu) := \begin{cases} \int_{\mathrm{X}} \rho \log(\rho)\,\mathrm{d}\mathfrak{m} & \text{if } \mu = \rho\mathfrak{m}; \\ +\infty & \text{otherwise.} \end{cases} \tag{1.5.4}$$

Remark 1.5.6. Condition (1.5.3) is satisfied in every CD space, we introduce this as further assumption here in order to give a good definition of entropy for spaces with infinite mass too. ■

Definition 1.5.7 (K-convexity)**.** *Let (X, d) be a metric space. We say that a functional $E : \mathrm{X} \to \mathbb{R} \cup \{+\infty\}$ is K-convex if for every $x, y \in E^{-1}(\mathbb{R})$ there exists a geodesic γ connecting them such that*

$$E(\gamma_t) \leq (1-t)E(x) + tE(y) - \frac{1}{2}Kt(1-t)\mathsf{d}^2(x,y)$$

for every $t \in [0,1]$.

In the smooth setting, as proved in [SvR] (see also [OV] and [CEMS] for preliminary results), K-convexity of the relative entropy is equivalent to the lower bound on the Ricci curvature.

Theorem 1.5.8. *Let* M *be a smooth, connected and complete Riemannian manifold and let* $K \in \mathbb{R}$*. Then the following are equivalent:*

i) for every $x \in \mathrm{M}$ *and* $v \in T_x\mathrm{M}$ *it holds* $\mathsf{Ric}_x(v,v) \geq K|v|^2$*;*

ii) the relative entropy $\mathrm{Ent}_{\mathrm{vol}}$ *is* K*-convex on* $(\mathscr{P}_2(\mathrm{X}), W_2)$*.*

Property (ii) can be formulated in any metric measure space, and this motivates the following definition.

Definition 1.5.9 ($\mathsf{CD}(K,\infty)$ condition)**.** *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space and let* $K \in \mathbb{R}$*. We say that* X *is a* $\mathsf{CD}(K,\infty)$ *space if the Boltzmann-Shannon relative entropy defined in* (1.5.4) *is* K*-convex on* $(\mathscr{P}_2(\mathrm{X}), W_2)$*.*

The definition of $\mathsf{CD}(K,N)$ spaces with finite dimension bound N requires more work. We recall three different definitions of it and see the relations between them.

To begin we define the volume distortion coefficients τ and the coefficients σ.

For every $\kappa \in \mathbb{R}$ and $\vartheta \geq 0$ we denote by $\mathfrak{s}_\kappa$ the solution of the Cauchy problem

$$f'' + \kappa f = 0, \qquad f(0) = 0 \text{ and } f'(0) = 1,$$

and can be written explicitly as

$$\mathfrak{s}_\kappa(\vartheta) := \begin{cases} \frac{1}{\sqrt{\kappa}} \sin(\sqrt{\kappa}\vartheta) & \text{if } \kappa > 0, \\ \vartheta & \text{if } \kappa = 0, \\ \frac{1}{\sqrt{-\kappa}} \sinh(\sqrt{-\kappa}\vartheta) & \text{if } \kappa < 0. \end{cases}$$

Definition 1.5.10 (Volume distortion coefficients)**.** *For* $t \in [0,1]$*,* $K \in \mathbb{R}$ *and* $N \in (0, +\infty)$ *we define*

$$\sigma^{(t)}_{K,N}(\vartheta) := \begin{cases} \frac{\mathfrak{s}_{K/N}(t\vartheta)}{\mathfrak{s}_{K/N}(\vartheta)} & \textit{if } \frac{K}{N}\vartheta^2 \neq 0 \textit{ and } \frac{K}{N}\vartheta^2 < \pi^2, \\ t & \textit{if } \frac{K}{N}\vartheta^2 = 0, \\ +\infty & \textit{if } \frac{K}{N}\vartheta^2 \geq \pi^2 \end{cases} \tag{1.5.5}$$

and for $t \in [0,1]$*,* $K \in \mathbb{R}$ *and* $N \in (1, +\infty)$

$$\tau^{(t)}_{K,N}(\vartheta) := t^{\frac{1}{N}} \left(\sigma^{(t)}_{K,N-1}(\vartheta)\right)^{\frac{N-1}{N}}. \tag{1.5.6}$$

The $\mathsf{CD}(K,N)$ condition is a sort of convexity for the Rényi entropy functional defined by

$$S_N(\mu|\mathfrak{m}) := -\int_{\mathrm{X}} \rho^{-\frac{1}{N}}\,\mathrm{d}\mu,$$

where ρ is the density of the absolutely continuous part of μ with respect to $\mathfrak{m}$.

For the definition of $\mathsf{CD}(K,N)$ space we will focus on optimal geodesic plans, which are geodesics connecting μ and ν with respect to the Wasserstein distance.

Definition 1.5.11 (Optimal geodesic plan). *Let $\mu,\nu \in \mathscr{P}_2(X)$ be such that $W_2(\mu,\nu) < +\infty$. We say that $\boldsymbol{\pi} \in \mathscr{P}(\mathrm{Geo}(\mathrm{X}))$ is an optimal geodesic plan between μ and ν if $(e_0)_*\boldsymbol{\pi} = \mu$, $(e_1)_*\boldsymbol{\pi} = \nu$ and*

$$\int \mathsf{d}^2(\gamma_0,\gamma_1)\,\mathrm{d}\boldsymbol{\pi} = \int\int_0^1 |\dot\gamma_s|^2\,\mathrm{d}s\,\mathrm{d}\boldsymbol{\pi}(\gamma) = W_2^2(\mu,\nu).$$

We indicate with $\mathrm{OptGeo}(\mu,\nu)$ *the set of optimal geodesic plans connecting μ and ν.*

Definition 1.5.12 (Curvature-Dimension condition). *Let $K \in \mathbb{R}$ and $N \in [1,\infty)$. We say that a metric measure space $(\mathrm{X},\mathsf{d},\mathfrak{m})$ is $\mathsf{CD}(K,N)$ if for every $\mu_0,\mu_1 \ll \mathfrak{m}$ with bounded support there exists $\boldsymbol{\pi}$ optimal geodesic plan between μ_0 and μ_1 such that, indicating with $\rho_t\mathfrak{m}$ the absolutely continuous part with respect to $\mathfrak{m}$ of $(e_t)_*\boldsymbol{\pi}$, it holds*

$$\begin{aligned} -\int \rho_t^{1-\frac{1}{N'}}\,\mathrm{d}\mathfrak{m} \le -\int & \tau_{K,N'}^{(1-t)}(\mathsf{d}(\gamma_0,\gamma_1))\rho_0^{-\frac{1}{N'}}(\gamma_0) \\ &+ \tau_{K,N'}^{(t)}(\mathsf{d}(\gamma_0,\gamma_1))\rho_1^{-\frac{1}{N'}}(\gamma_1)\,\mathrm{d}\boldsymbol{\pi}(\gamma) \end{aligned} \tag{1.5.7}$$

for every $t \in [0,1]$ and $N' \ge N$.

The following consistency result can be found in [S06b].

Theorem 1.5.13. *Let* M *be a complete connected Riemannian manifold with Riemannian distance d and Riemannian volume $\mathfrak{m}$, and let $K \in \mathbb{R}$, $N \in [1,+\infty)$. Then the following are equivalent:*

i) the metric measure space $(\mathrm{M},\mathsf{d},\mathfrak{m})$ satisfies the $\mathsf{CD}(K,N)$ condition;

ii) for every $x \in \mathrm{M}$ and $v \in T_x\mathrm{M}$ it holds $\mathsf{Ric}_x(v,v) \ge K|v|^2$ and M *has dimension at most N.*

In order to study a local-to-global property for the $\mathsf{CD}(K,N)$ condition, K. Bacher and K.-T. Sturm in [BaSt] introduced a weaker notion of curvature-dimension condition, the so called reduced curvature-dimension condition, indicated with $\mathsf{CD}^*(K,N)$.

Definition 1.5.14 (Reduced Curvature-Dimension condition)**.** *Let $K \in \mathbb{R}$ and $N \in (0,\infty)$. We say that a metric measure space $(\mathrm{X},\mathsf{d},\mathfrak{m})$ is $\mathsf{CD}^*(K,N)$ if for every $\mu_0,\mu_1 \ll \mathfrak{m}$ with bounded support there exists $\boldsymbol{\pi}$ optimal geodesic plan between μ_0 and μ_1 such that, indicating with $\rho_t\mathfrak{m}$ the absolutely continuous part with respect to $\mathfrak{m}$ of $(e_t)_*\boldsymbol{\pi}$, it holds*

$$
\begin{aligned}
-\int \rho_t^{1-\frac{1}{N'}}\,\mathrm{d}\mathfrak{m} \leq -\int \sigma_{K,N'}^{(1-t)}(\mathsf{d}(\gamma_0,\gamma_1))\rho_0^{-\frac{1}{N'}}(\gamma_0)& \\
+\,\sigma_{K,N'}^{(t)}(\mathsf{d}(\gamma_0,\gamma_1))\rho_1^{-\frac{1}{N'}}(\gamma_1)\,\mathrm{d}\boldsymbol{\pi}(\gamma)&
\end{aligned}
\tag{1.5.8}
$$

for every $t \in [0,1]$ and $N' \geq N$.

The only difference between the definition of CD spaces and CD^* spaces is that in (1.5.7) we used the coefficients $\tau_{K,N}$ as weights and in (1.5.8) we changed them with the coefficients $\sigma_{K,N}$.

From Definitions 1.5.12, 1.5.14 and 1.5.10 it follows easily that $\mathsf{CD}(0,N)$ condition and $\mathsf{CD}^*(0,N)$ condition are equivalent, indeed

$$\tau_{0,N}^{(t)}(\vartheta) = \sigma_{0,N}^{(t)}(\vartheta) = t.$$

From an easy computation we conclude that, for every $K,\vartheta \in \mathbb{R}$ and $N \in (1,+\infty)$ it holds

$$\sigma_{K,N}^{(t)}(\vartheta) \leq \tau_{K,N}^{(t)}(\vartheta), \tag{1.5.9}$$

and if $K > 0$, defining $K^* := K\frac{N-1}{N}$, it holds

$$\tau_{K^*,N}^{(t)}(\vartheta) \leq \sigma_{K,N}^{(t)}(\vartheta). \tag{1.5.10}$$

By Definitions 1.5.12, 1.5.14 and inequality (1.5.9) we conclude easily that

$$\mathsf{CD}(K,N) \Rightarrow \mathsf{CD}^*(K,N),$$

and for $K > 0$, using (1.5.10), that

$$\mathsf{CD}^*(K,N) \Rightarrow \mathsf{CD}(K^*,N).$$

We recall the definition of essentially non-branching space given in [RS].

Definition 1.5.15 (Essentially non-branching space). *A metric measure space* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *is said essentially non-branching if for every* $\mu, \nu \in \mathscr{P}_2(\mathrm{X})$ *absolutely continuous with respect to the reference measure* $\mathfrak{m}$ *any element of* $\mathrm{OptGeo}(\mu, \nu)$ *is concentrated on a set of non-branching geodesics, i.e. given* $\boldsymbol{\pi} \in \mathrm{OptGeo}(\mu, \nu)$ *there exists* $\Gamma \subset C([0,1], \mathrm{X})$ *Borel such that* $\boldsymbol{\pi}(\Gamma) = 1$ *and if* $\gamma, \eta \in \Gamma$ *coincide on the interval* $[0, t]$ *for some* $t \in (0,1]$ *then* $\gamma \equiv \eta$.

In [CaMi] F. Cavalletti and E. Milman proved that the $\mathsf{CD}(K, N)$ condition is equivalent to the $\mathsf{CD}^*(K, N)$ condition for normalized essentially non-branching spaces. A proof for spaces with infinite mass is still missing.

The third notion of curvature-dimension condition we recall here is due to M. Erbar, K. Kuwada and K.-T. Sturm (see [EKS]).

We indicate with $\mathscr{P}_2^*(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ the set of probability measures on X with finite entropy, and with $U_N : \mathscr{P}_2(\mathrm{X}) \to [0, +\infty]$ the map

$$U_N(\mu) := \exp\left(-\frac{1}{N}\,\mathrm{Ent}_{\mathfrak{m}}(\mu)\right).$$

We define $\mathsf{CD}^e(K, N)$ spaces as the spaces for which the entropy satisfies a $(K, N)-$convexity condition in the following way.

Definition 1.5.16 (Entropic Curvature Dimension condition). *Given* $K \in \mathbb{R}$ *and* $N \in (0, +\infty)$ *we say that a metric measure space* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *satisfies the entropic curvature-dimension condition* $\mathsf{CD}^e(K, N)$ *if and only if for each pair* $\mu_0, \mu_1 \in \mathcal{P}_2^*(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *there exists a constant speed geodesic* $(\mu_t)_{t\in[0,1]} \subset \mathcal{P}_2^*(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *connecting* μ_0 *to* μ_1 *such that for all* $t \in [0,1]$ *it holds*

$$U_N(\mu_t) \geq \sigma_{K/N}^{(1-t)}(W_2(\mu_0, \mu_1))U_N(\mu_0) + \sigma_{K/N}^{(t)}(W_2(\mu_0, \mu_1))U_N(\mu_1).$$

In [EKS] it is proven that for essentially non-branching metric measure spaces the $\mathsf{CD}^*(K, N)$ condition and the $\mathsf{CD}^e(K, N)$ condition are equivalent, moreover, a major result from the same paper, is the equivalence between the CD^e condition and the weak formulation of the Bochner inequality (1.5.1).

Theorem 1.5.17. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space. The following are equivalent:*

i) the space $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *is* $\mathsf{RCD}(K, N)$*, i.e. it satisfies the conditions of Definition 1.5.1;*

ii) the space $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *is infinitesimally Hilbertian and satisfies the entropic curvature-dimension condition* $\mathsf{CD}^e(K, N)$*.*

In [RS] T. Rajala and K.-T. Sturm proved that every infinitesimally Hilbertian $\mathsf{CD}(K,\infty)$ space is essentially non-branching. From this we conclude that, at least for normalized spaces, if we define $\mathsf{RCD}/\mathsf{RCD}^*/\mathsf{RCD}^e$ spaces as "$\mathsf{CD}/\mathsf{CD}^*/\mathsf{CD}^e$+infinitesimal Hilbertianity" the three definitions are equivalent. Given that, slightly abusing the notation, we will not distinguish between RCD, RCD^* and RCD^e conditions.

Remark 1.5.18. In [D] Q. Deng proved that $\mathsf{RCD}(K,N)$ spaces are non-branching, i.e. if $\gamma,\eta \in \mathrm{Geo}(\mathrm{X})$ coincide on the interval $[0,t]$ for some $t \in (0,1)$ then $\gamma \equiv \eta$. ■

As anticipated in the Introduction, an important result holding true in every $\mathsf{CD}(K,N)$ spaces is the Bishop-Gromov inequality. Its proof can be found in [S06b].

Theorem 1.5.19 (Bishop-Gromov Inequality)**.** *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be a* $\mathsf{CD}(K,N)$ *metric measure space with* $K\in\mathbb{R}$ *and* $N\in(1,+\infty)$*. Fix* $p\in\mathrm{X}$ *and let* $\mu := \mathsf{d}(\cdot,p)_*\mathfrak{m}$*. Then* $\mu = s\mathcal{L}^1$ *for some function* $s:[0,\mathrm{diam}(\mathrm{X})]\to\mathbb{R}^+$ *such that*

$$r \mapsto s(r)\big(\mathfrak{s}_{\frac{K}{N-1}}(r)\big)^{1-N} \qquad \textit{is not increasing on } [0,\mathrm{diam}(\mathrm{X})]. \tag{1.5.11}$$

Also, the following integrated version of the monotonicity holds:

$$R \mapsto \mathfrak{m}(B_R(p))\Big(\int_0^R \big(\mathfrak{s}_{\frac{K}{N-1}}(r)\big)^{N-1}\,\mathrm{d}r\Big)^{-1} \textit{ is not increasing on } [0,\mathrm{diam}(\mathrm{X})]. \tag{1.5.12}$$

The Bishop-Gromov inequality for volumes (1.5.12) estimates the ratio of the volume of balls, then an easy consequence is that every $\mathsf{CD}(K,N)$ space is uniformly locally doubling.

Definition 1.5.20 (locally doubling space)**.** *We say that a metric measure space* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *is locally doubling if for every* $x\in\mathrm{X}$ *there are an open set* $U\ni x$ *and constants* $C,R>0$ *such that*

$$\mathfrak{m}(B_{2r}(y)) \le C\mathfrak{m}(B_r(y))$$

for every $y\in U$ *and* $r\in(0,R)$*.*

A space is said uniformly locally doubling if constants C *and* R *do not depend on* x*.*

1.5.2 Convergence of (pointed) metric measure spaces and stability

An important property of the class of $\mathsf{CD}(K,N)$ spaces is its stability. In literature several different convergences of metric measure spaces (and pointed metric measure spaces) have been studied in this direction, for instance:

-) in [S06a] and [S06b] Sturm defined a distance between normalized metric measure spaces and proved that curvature-dimension bounds are stable with respect to the convergence induced by that distance;

-) in [LV] Lott and Villani worked with proper pointed metric measure spaces and pointed measured Gromov Hausdorff (pmGH) convergence for $\mathsf{CD}(K,N)$ spaces (note that for $N<+\infty$ a $\mathsf{CD}(K,N)$ space is always proper, but this is not true for $N=+\infty$) (see also [V]);

-) in [GMS] Gigli, Mondino and Savaré introduced the so called pointed measured Gromov (pmG) convergence to prove the stability in case the space has infinite mass (and is not necessarily proper), so to cover also the case $\mathsf{CD}(K,\infty)$. Then they proved that for uniformely doubling spaces pmG and pmGH convergences coincide.

We recall the needed definitions and the compactness result for pmGH convergence of $\mathsf{CD}(K,N)$ spaces (see [GMS] for the proof of the stability theorem and for more details about the convergence of metric measure spaces).

Definition 1.5.21 (Pointed metric measure space)**.** *A pointed metric measure space* $(\mathrm{X},\mathsf{d},\mathfrak{m},x)$ *is a quadruple where* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *is a metric measure space (as in Definition 1.1.9) and* x *is a fixed point of* $\mathrm{supp}(\mathfrak{m})$.

Definition 1.5.22 (Pointed measured Gromov Hausdorff convergence)**.** *Let* $((\mathrm{X}_n,\mathsf{d}_n,\mathfrak{m}_n,x_n))_{n\in\mathbb{N}}$ *be a sequence of pointed metric measure spaces. We say that* $((\mathrm{X}_n,\mathsf{d}_n,\mathfrak{m}_n,x_n))_{n\in\mathbb{N}}$ *converges in the pointed measured Gromov Hausdorff sense to* $(\mathrm{X}_\infty,\mathsf{d}_\infty,\mathfrak{m}_\infty,x_\infty)$ *provided that there exist two sequences* $R_n\uparrow+\infty$ *and* $\varepsilon_n\downarrow 0$ *and Borel maps* $f_n:\mathrm{X}_n\to\mathrm{X}_\infty$ *such that the following hold:*

-) $f_n(x_n)=x_\infty$;

-) $\sup_{x,y\in B_{R_n}(x_n)}|\mathsf{d}_n(x,y)-\mathsf{d}_\infty(f_n(x),f_n(y))|\leq\varepsilon_n$;

-) the ε_n*-neighborhood of* $f_n(B_{R_n}(x_n))$ *contains* $B_{R_n-\varepsilon_n}(x_\infty)$;

-) for every $\varphi\in C_{bs}(\mathrm{X}_\infty)$ *it holds*

$$\lim_{n\to\infty}\int\varphi\circ f_n\,\mathrm{d}\mathfrak{m}_n=\int\varphi\,\mathrm{d}\mathfrak{m}_\infty.$$

We denote this convergence with $(\mathrm{X}_n, \mathsf{d}_n, \mathfrak{m}_n, x_n) \xrightarrow{pmGH} (\mathrm{X}_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, x_\infty)$.

Theorem 1.5.23. *Let* $((\mathrm{X}_n, \mathsf{d}_n, \mathfrak{m}_n, x_n))_{n\in\mathbb{N}}$ *be a sequence of pointed metric measure spaces such that for every* $n \in \mathbb{N}$ *the space* $(X_n, \mathsf{d}_n, \mathfrak{m}_n)$ *is* $\mathsf{CD}(K_n, N_n)$ *(resp.* $\mathsf{RCD}(K_n, N_n)$*) for certain* $K_n \in \mathbb{R}$ *and* $N_n \in [1, +\infty)$. *Moreover assume that* $K_n \to K \in \mathbb{R}$, $N_n \to N \in [1, +\infty)$ *and*

$$0 < \liminf_{n\to+\infty} \mathfrak{m}_n(B_1(x_n)) \le \limsup_{n\to+\infty} \mathfrak{m}_n(B_1(x_n)) < +\infty.$$

Then $((\mathrm{X}_n, \mathsf{d}_n, \mathfrak{m}_n, x_n))_{n\in\mathbb{N}}$ *has a subsequence that converges to a* $\mathsf{CD}(K, N)$ *(resp.* $\mathsf{RCD}(K, N)$*) space* $(X_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, x_\infty)$.

1.6 Second order differential calculus

The goal of this section is to give the definition of Hessian and covariant derivative in RCD spaces and see their calculus rules.

We start giving the definition of tensor product of Hilbert modules.

If $\mathscr{H}_1^0$ and $\mathscr{H}_2^0$ are the $L^0(\mathfrak{m})$-completions of two Hilbert modules $\mathscr{H}_1$ and $\mathscr{H}_2$, we indicate with $\mathscr{H}_1^0 \otimes_{Alg} \mathscr{H}_2^0$ the space of formal finite sums of objects of the form $v \otimes w$ with $v \in \mathscr{H}_1^0$ and $w \in \mathscr{H}_2^0$, with $(v, w) \mapsto v \otimes w$ being bilinear. On $\mathscr{H}_1^0 \otimes_{Alg} \mathscr{H}_2^0$ we define a "scalar product" in the following way: for every $v_1, v_2 \in \mathscr{H}_1^0$ and $w_1, w_2 \in \mathscr{H}_2^0$ we define

$$\langle v_1 \otimes w_1, v_2 \otimes w_2\rangle := \langle v_1, v_2\rangle\langle w_1, w_2\rangle \in L^0(\mathfrak{m}),$$

then we extend it by linearity.

The scalar product just defined is symmetric and the following hold:

$$\begin{aligned} \langle A, A\rangle &\ge 0 && \mathfrak{m}\text{-a.e. for every } A \in \mathscr{H}_1^0 \otimes_{Alg} \mathscr{H}_2^0;\\ \langle A, A\rangle &= 0 && \mathfrak{m}\text{-a.e. on } E \text{ if and only if } \chi_E A = 0;\\ f\langle A, B\rangle &= \langle fA, B\rangle && \mathfrak{m}\text{-a.e. for every } A, B \in \mathscr{H}_1^0 \otimes_{Alg} \mathscr{H}_2^0 \text{ and } f \in L^0(\mathfrak{m}). \end{aligned}$$

Definition 1.6.1 (Hilbert-Schmidt norm)**.** *On* $\mathscr{H}_1^0 \otimes_{Alg} \mathscr{H}_2^0$ *we define the Hilbert-Schmidt norm as*

$$|A|_{\mathsf{HS}} := \sqrt{\langle A, A\rangle}.$$

Definition 1.6.2. *We define the space* $\mathscr{H}_1^0 \otimes \mathscr{H}_2^0$ *as the completion of* $\mathscr{H}_1^0 \otimes_{Alg} \mathscr{H}_2^0$ *with respect to the distance*

$$\mathsf{d}_\otimes(A, B) := \sum_{i\in\mathbb{N}} \frac{1}{2^i \mathfrak{m}(E_i)} \int (|A - B|_{\mathsf{HS}} \wedge 1)\, \mathrm{d}\mathfrak{m},$$

where $(E_i)_{i\in\mathbb{N}}$ *is a Borel partition of* X *in sets of positive measure.*

Definition 1.6.3 (Tensor product of Hilbert modules). *The tensor product* $\mathscr{H}_1 \otimes \mathscr{H}_2$ *is defined as the subset of* $\mathscr{H}_1^0 \otimes \mathscr{H}_2^0$ *with* $L^2(\mathfrak{m})$ *Hilbert-Schmidt norm.*

We observe that the space $\mathscr{H}_1 \otimes \mathscr{H}_2$ endowed with the norm

$$\|A\| := \sqrt{\int |A|_{\mathsf{HS}}^2 \, d\mathfrak{m}}$$

is a Hilbert space. It easily follows that $\mathscr{H}_1 \otimes \mathscr{H}_2$ is a Hilbert module.

We introduce now the following notation.

Definition 1.6.4. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathsf{RCD}(K, \infty)$ *space. Then we define*

$$L^2((T^*)^{\otimes 2}\mathrm{X}) := L^2(T^*\mathrm{X}) \otimes L^2(T^*\mathrm{X}).$$

Given $A \in L^2((T^*)^{\otimes 2}\mathrm{X})$ *we indicate with* $A(v, w)$ *the function* $A(v \otimes w) \in L^0(\mathfrak{m})$ *for every* $v, w \in L^2(T\mathrm{X})$.

Similarly, we define $L^2(T^{\otimes 2}\mathrm{X}) := L^2(T\mathrm{X}) \otimes L^2(T\mathrm{X})$.

We observe that $L^2((T^*)^{\otimes 2}\mathrm{X})$ can be seen as the dual of $L^2(T^{\otimes 2}\mathrm{X})$ via the duality map

$$(\omega \otimes \eta)(v \otimes w) := \omega(v)\eta(w)$$

for $v, w \in L^2(T\mathrm{X})$ and $\omega, \eta \in L^2(T^*\mathrm{X})$.

In order to define the space $W^{2,2}_{loc}(\mathrm{X})$ and the Hessian of a function we recall the definition of test functions, which are the most regular functions we can consider in the non-smooth setting, and they will be used instead of the C_c^∞ functions.

Definition 1.6.5 (Local test functions). *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathsf{RCD}(K, \infty)$ *space with* $K \in \mathbb{R}$. *We define the set of local test functions as*

$$\mathrm{Test}_{loc}(\mathrm{X}) := \left\{ f \in \mathrm{LIP}_{loc}(\mathrm{X}) \cap L^\infty_{loc}(\mathrm{X}) \cap D(\Delta_{loc}) : \Delta f \in W^{1,2}_{loc}(\mathrm{X}) \right\}.$$

From the weak Bochner inequality (1.5.1) follows that for every function $f \in \mathrm{Test}_{loc}(\mathrm{X})$ it holds $|Df| \in W^{1,2}_{loc}(\mathrm{X})$. A consequence of this fact is that the set of local test functions is an algebra and it is dense in $W^{1,2}_{loc}(\mathrm{X})$.

The following result provides the existence of good cut-off functions (for the construction see [MN], for similar results [AMS] or [GP20]).

Proposition 1.6.6 (Good cut-off functions). *Let* $K \in \mathbb{R}$, $N \in (1, +\infty)$ *and let* X *be an* $\mathsf{RCD}(K, N)$ *space. Then for every* $0 < r < R < +\infty$, *every compact* $\mathrm{K} \subset \mathrm{X}$ *and every open* $U \subset \mathrm{X}$ *such that* $\mathrm{diam}(U) \leq R$ *and* $\mathsf{d}(P, U^c) > r$ *there exists a function* $\eta \in \mathrm{Test}_{loc}(\mathrm{X})$ *such that* *i)* $0 \le \eta \le 1$ *on* X, $\mathrm{supp}(\eta) \subset U$ *and* $\eta \equiv 1$ *in* P,

ii) $r|D\eta| + r^2|\Delta\eta| \le C$, *where* C *depends only on* R, N, K.

We give now the definition of $W^{2,2}_{loc}(\mathrm{X})$ and Hessian of a function $f \in W^{2,2}_{loc}(\mathrm{X})$.

Definition 1.6.7 ($W^{2,2}_{loc}(\mathrm{X})$ space and Hessian). *Let* $K \in \mathbb{R}$ *and let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathsf{RCD}(K,\infty)$ *space. For* $f \in W^{1,2}_{loc}(\mathrm{X})$ *we say that* $f \in W^{2,2}_{loc}(\mathrm{X})$ *if there exists* $A \in L\left((T^*)^{\otimes 2}\mathrm{X}\right)$ *such that*

$$2\int \varphi A(\nabla\psi_1, \nabla\psi_2)\,\mathrm{d}\mathfrak{m} = -\int \langle\nabla f, \nabla\psi_1\rangle\,\mathrm{div}(\varphi\nabla\psi_2) + \langle\nabla f, \nabla\psi_2\rangle\,\mathrm{div}(\varphi\nabla\psi_1) + \varphi\langle\nabla f, \nabla\langle\nabla\psi_1, \nabla\psi_2\rangle\rangle\,\mathrm{d}\mathfrak{m}$$

for every $\psi_1, \psi_2 \in \mathrm{Test}_{loc}(\mathrm{X})$ *and* $\varphi \in \mathrm{LIP}_{bs}(\mathrm{X})$.

Such tensor A *is uniquely determined and it will be denoted by* $\mathrm{Hess}(f)$.

Remark 1.6.8. This definition is justified by the following formula on Riemannian manifolds: let M be a Riemannian manifold, then for every functions $f, \psi_1, \psi_2 \in C^\infty(\mathrm{M})$ it holds

$$2\,\mathrm{Hess}(f)(\psi_1, \psi_2) = \langle\nabla\langle\nabla f, \nabla\psi_1\rangle, \nabla\psi_2\rangle + \langle\nabla\langle\nabla f, \nabla\psi_2\rangle, \nabla\psi_1\rangle - \langle\nabla\langle\nabla\psi_1, \nabla\psi_2\rangle, \nabla f\rangle.$$

■

The existence of many $W^{2,2}_{loc}$ functions is given by the following inclusion result proved in [G18b].

Proposition 1.6.9. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathsf{RCD}(K,\infty)$ *space with* $K \in \mathbb{R}$. *Then* $\mathrm{Test}_{loc}(\mathrm{X}) \subset D(\Delta_{loc}) \subset W^{2,2}_{loc}(\mathrm{X})$.

The set of local test functions is not dense in $W^{2,2}_{loc}(\mathrm{X})$, then we define the space $H^{2,2}_{loc}(\mathrm{X})$ as the closure of $\mathrm{Test}_{loc}(\mathrm{X})$ in $W^{2,2}_{loc}(\mathrm{X})$.

Definition 1.6.10 (The space $H^{2,2}_{loc}(\mathrm{X})$). *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathsf{RCD}(K,\infty)$ *space with* $K \in \mathbb{R}$. *We define the space* $H^{2,2}_{loc}(\mathrm{X}) \subset W^{2,2}_{loc}(\mathrm{X})$ *as the collection of all functions* $f \in W^{2,2}_{loc}(\mathrm{X})$ *such that the following holds: for any* $U \subset \mathrm{X}$ *open bounded there is a sequence* $(f_n) \subset \mathrm{Test}_{loc}(\mathrm{X})$ *such that*

$$\|f_n - f\|_{L^2(U)} + \||\mathrm{d}f_n - \mathrm{d}f|\|_{L^2(U)} + \||\mathrm{Hess}\,f_n - \mathrm{Hess}\,f|_{\mathsf{HS}}\|_{L^2(U)} \to 0$$

as $n \to \infty$, *where here and in what follows we denote by* $|\cdot|_{\mathsf{HS}}$ *the pointwise norm in* $L^2_{loc}(T^{\otimes 2}\mathrm{X})$.

We see now some calculus rules for the Hessian.

Proposition 1.6.11 (Leibniz rule for the Hessian). *Let* $f, g \in W^{2,2}_{loc}(\mathrm{X}) \cap \mathrm{LIP}_{loc}(\mathrm{X}) \cap L^\infty_{loc}(\mathfrak{m})$. *Then* $fg \in W^{2,2}_{loc}(\mathrm{X})$ *and*

$$\mathrm{Hess}(fg) = f\,\mathrm{Hess}(g) + g\,\mathrm{Hess}(f) + \mathrm{d}f \otimes \mathrm{d}g + \mathrm{d}g \otimes \mathrm{d}f$$

holds $\mathfrak{m}$*-a.e. in* X.

Proposition 1.6.12 (Chain rule for the Hessian). *Let* $f, g \in W^{2,2}_{loc}(\mathrm{X}) \cap \mathrm{LIP}_{loc}(\mathrm{X})$ *and let* $\varphi \in C^2(\mathbb{R})$ *such that* φ' *and* φ'' *are bounded and* $\varphi(0) = 0$. *Then* $\varphi \circ f \in W^{2,2}_{loc}(\mathrm{X})$ *and the following formula holds* $\mathfrak{m}$*-a.e.:*

$$\mathrm{Hess}(\varphi \circ f) = \varphi' \circ f\,\mathrm{Hess}(f) + \varphi'' \circ f \mathrm{d}f \otimes \mathrm{d}f. \tag{1.6.1}$$

Proposition 1.6.13 (Product rule for gradients). *Let* $f, g \in H^{2,2}_{loc}(\mathrm{X}) \cap \mathrm{LIP}_{loc}(\mathrm{X})$. *Then* $\langle \nabla f, \nabla g\rangle \in W^{1,2}_{loc}(\mathrm{X})$ *and*

$$\mathrm{d}\langle \nabla f, \nabla g\rangle = \mathrm{Hess}(f)(\nabla g, \cdot) + \mathrm{Hess}(g)(\nabla f, \cdot) \tag{1.6.2}$$

holds $\mathfrak{m}$*-a.e. in* X.

As we saw for the Hessian, thanks to an identity on Riemannian manifolds we can define the covariant derivative on RCD spaces. On M Riemannian manifold, for every v vector field and $\varphi, \psi \in C^\infty(\mathrm{M})$ it holds

$$\langle \nabla_{\nabla\varphi} v, \nabla\psi\rangle = \langle \nabla\langle v, \nabla\psi\rangle, \nabla\varphi\rangle - \mathrm{Hess}(\psi)(v, \nabla\varphi).$$

We indicate with : the scalar product between two elements of $L^2(T^{\otimes 2}\mathrm{X})$.

Definition 1.6.14 (Covariant derivative). *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathsf{RCD}(K, \infty)$ *space for a certain* $K \in \mathbb{R}$. *The space* $W^{1,2}_C(T\mathrm{X})$ *is the set of vector fields* $v \in L^2(T\mathrm{X})$ *for which there exists* $T \in L^2(T^{\otimes 2}\mathrm{X})$ *such that for every* $\varphi, \psi, \xi \in \mathrm{Test}(\mathrm{X})$ *it holds*

$$\int \xi T : (\nabla\varphi \otimes \nabla\psi)\,\mathrm{d}\mathfrak{m} = -\int \langle v, \nabla\psi\rangle\,\mathrm{div}(\xi\nabla\varphi) + \xi\,\mathrm{Hess}(\psi)(v, \nabla\varphi)\,\mathrm{d}\mathfrak{m}.$$

Such T*, which is uniquely determined, is called covariant derivative of* v *and denoted by* ∇v.

Remark 1.6.15. The space $W^{1,2}_C(T\mathrm{X})$ with the norm

$$\|v\|_{W^{1,2}_C(T\mathrm{X})} := \left(\|v\|^2_{L^2(T\mathrm{X})} + \|\nabla v\|^2_{L^2(T^{\otimes 2}\mathrm{X})}\right)^{\frac{1}{2}}$$

is a separable Hilbert space. Moreover, denoting with $\sharp$ the Riesz isomorphism between $L^2((T^*)^{\otimes 2}\mathrm{X})$ and $L^2(T^{\otimes 2}\mathrm{X})$, for every $f \in H^{2,2}(\mathrm{X}) \cap \mathrm{LIP}(\mathrm{X})$ it holds $\nabla f \in W^{1,2}(T\mathrm{X})$ and $\nabla(\nabla f) = (\mathrm{Hess}(f))^\sharp$. ■

The covariant derivative satisfies the following Leibniz rule.

Proposition 1.6.16 (Leibniz rule for the covariant derivative). *Let* X *be an* $\mathsf{RCD}(K,\infty)$ *space. Let* $v \in W^{1,2}_C(T\mathrm{X}) \cap L^\infty(T\mathrm{X})$ *and* $f \in W^{1,2}(\mathrm{X}) \cap L^\infty(\mathfrak{m})$. *Then* $fv \in W^{1,2}_C(T\mathrm{X})$ *and*

$$\nabla(fv) = \nabla f \otimes v + f\nabla v.$$

We conclude this section recalling the definition of test vector field.

Definition 1.6.17 (Test vector field). *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathsf{RCD}(K,\infty)$ *space with* $K \in \mathbb{R}$. *We define the set of test vector fields as*

$$\mathrm{TestV}(\mathrm{X}) := \left\{ \sum_{i=1}^n g_i \nabla f_i : n \in \mathbb{N} \textit{ and } f_i, g_i \in \mathrm{Test}(\mathrm{X}) \textit{ for } i = 1, ..., n \right\}.$$

Similarly to test functions and $W^{2,2}(\mathrm{X})$, it holds $\mathrm{TestV}(\mathrm{X})$ is contained $W^{1,2}_C(T\mathrm{X})$, but it is not dense in it. Then we indicate with $H^{1,2}_C(T\mathrm{X})$ the closure of $\mathrm{TestV}(\mathrm{X})$ in $W^{1,2}_C(T\mathrm{X})$.

1.7 Warped product spaces

As we saw in the Introduction, all rigidity theorems state that, under certain hypotheses, the space is isomorphic to a warped product space $\mathbb{R} \times_w \mathrm{X}'$. In this section we define warped products of metric measure spaces and see how the Sobolev spaces behave on them. All these results can be found in [GH].

Definition 1.7.1 (Warped kinetic energy of curves). *Let* $(\mathrm{X}, \mathsf{d}_\mathrm{X})$ *and* $(\mathrm{Y}, \mathsf{d}_\mathrm{Y})$ *be length spaces and let* $w_\mathsf{d} : \mathrm{Y} \to [0, +\infty)$ *be a continuous function. For every curve* $\gamma := (\gamma^\mathrm{Y}, \gamma^\mathrm{X})$ *with* $\gamma^\mathrm{Y} \in AC(\mathrm{Y})$ *and* $\gamma^\mathrm{X} \in AC(\mathrm{X})$ *we define its* w_d*-kinetic energy as*

$$\mathrm{KE}_w[\gamma] := \int_0^1 |\dot\gamma_t^\mathrm{Y}|^2 + w_\mathsf{d}^2(\gamma_t^\mathrm{Y})|\dot\gamma_t^\mathrm{X}|^2 \, \mathrm{d}t.$$

In a natural way we define the warped distance and the warped product of metric spaces.

Definition 1.7.2 (Warped product of metric spaces). *Taken two length spaces* $(\mathrm{X}, \mathsf{d}_\mathrm{X})$ *and* $(\mathrm{Y}, \mathsf{d}_\mathrm{Y})$, *let* $w_\mathsf{d} : \mathrm{Y} \to [0, +\infty)$ *be a continuous function. For every points* $p, q \in \mathrm{X} \times \mathrm{Y}$ *we define the pseudo-metric* $\mathsf{d}_w(p,q)$ *as*

$$\mathsf{d}_w(p,q)^2 := \inf \{\mathrm{KE}_w[\gamma]\},$$

where the infimum is taken among the curve $\gamma = (\gamma^{\mathrm{Y}}, \gamma^{\mathrm{X}})$ *such that* $\gamma^{\mathrm{Y}} \in AC(\mathrm{Y})$, $\gamma^{\mathrm{X}} \in AC(\mathrm{X})$, $\gamma_0 = p$ *and* $\gamma_1 = q$.

We denote with $(\mathrm{Y} \times_w \mathrm{X}, \mathsf{d}_w)$ *the completion of the quotient of* $\mathrm{Y} \times \mathrm{X}$ *with respect to the equivalence relation induced by the pseudo-metric* d_w.

We indicate with $\pi : \mathrm{Y} \times \mathrm{X} \to \mathrm{Y} \times_w \mathrm{X}$ the quotient map induced by d_w.

Definition 1.7.3 (Warped product of metric measure spaces)**.** *Assume that* $(\mathrm{X}, \mathsf{d}_{\mathrm{X}}, \mathfrak{m}_{\mathrm{X}})$ *and* $(\mathrm{Y}, \mathsf{d}_{\mathrm{Y}}, \mathfrak{m}_{\mathrm{Y}})$ *are two complete, separable and length metric spaces equipped with non-negative Radon measures, and let* $w_{\mathfrak{m}}, w_{\mathsf{d}} : \mathrm{Y} \to [0, \infty)$ *be continuous functions. Moreover, assume that at least one of the following holds:*

-) $\mathfrak{m}_{\mathrm{X}}$ *is a finite measure;*

-) w_{d} *is always positive.*

The warped product space $(\mathrm{Y} \times_w \mathrm{X}, \mathsf{d}_w, \mathfrak{m}_w)$ *is defined as the warped product of metric spaces* $(\mathrm{Y} \times_w \mathrm{X}, \mathsf{d}_w)$ *equipped with the Radon measure* $\mathfrak{m}_w$ *defined as*

$$\mathfrak{m}_w := \pi_* \left((w_{\mathfrak{m}} \mathfrak{m}_{\mathrm{Y}}) \times \mathfrak{m}_{\mathrm{X}} \right).$$

Remark 1.7.4. The alternative conditions in Definition 1.7.3 ($\mathfrak{m}_{\mathrm{X}}$ finite or w_{d} never 0) are needed to ensure that the measure $\mathfrak{m}_w$ is Radon. ■

In all the work we indicate with (y, x) the elements of $\mathrm{Y} \times_w \mathrm{X}$, omitting the quotient map.

We introduce some notations and assumptions that we are going to use along this section.

Assumption 1.7.5. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space and* $I \subset \mathbb{R}$ *be a closed and possibly unbounded interval. Given* $w_{\mathsf{d}}, w_{\mathfrak{m}} : I \to [0, +\infty)$ *continuous functions, assume that at least one of the alternative conditions in Definition 1.7.3 is satisfied. Then we indicate with* $(\mathrm{X}_w, \mathsf{d}_w, \mathfrak{m}_w)$ *the warped product space* $I \times_w \mathrm{X}$ *with warping functions* w_{d} *and* $w_{\mathfrak{m}}$.

The first result we recall from [GH] is the characterization of the Sobolev space on warped product spaces. It relates the Sobolev space of X_w with the ones of the quotient spaces and tells us who the minimal weak upper gradient of a Sobolev function in X_w is.

Theorem 1.7.6. *Given Assumption 1.7.5,* $f \in W^{1,2}(\mathrm{X}_w)$ *if and only if the following conditions hold:*

i) for $\mathfrak{m}$*-a.e.* $x \in \mathrm{X}$ *the map* $f^{(x)} := f(\cdot, x)$ *is in* $W^{1,2}(\mathbb{R}, \mathsf{d}_{\mathrm{Eucl}}, w_{\mathfrak{m}} \mathcal{L}^1)$*;*

ii) for $\mathcal{L}^1$*-a.e.* $r \in \mathbb{R}$ *the map* $f^{(r)} := f(r,\cdot)$ *is in* $W^{1,2}(\mathrm{X},\mathsf{d},\mathfrak{m})$*;*

iii) the identity

$$|Df|^2_{\mathrm{X}_w}(r,x) = |Df^{(x)}|^2_{\mathbb{R}}(r) + \frac{1}{w^2_{\mathsf{d}}(r)}|Df^{(r)}|^2_{\mathrm{X}}(x) \tag{1.7.1}$$

holds for $\mathfrak{m}_w$*-a.e.* (r,x)*.*

The proof of points (i) and (ii) is an easy consequence of Theorem 1.2.16 and Fubini's Theorem, while the proof of point (iii) follows from a nontrivial comparison argument with the Cartesian product case ($w_{\mathfrak{m}} = w_{\mathsf{d}} = 1$).

The second result we recall from [GH] studies the Sobolev-to-Lipschitz property for warped product spaces.

In general it is not true that if X has the Sobolev-to-Lipschitz property then the warped product space $I \times_w \mathrm{X}$ has it too. For instance it is trivial to observe that if $w_{\mathfrak{m}}$ is 0 on some interval $J \subset I$ which disconnects I then $\mathfrak{m}_w$ has disconnected support, and this violates the Sobolev-to-Lipschitz property. Even assuming $w_{\mathfrak{m}}$ strictly positive it is not clear if the Sobolev-to-Lipschitz property passes to the warped product space or not. In order to prove the Sobolev-to-Lipschitz property for the warped product space then Gigli and Han introduced a variant of the length property which takes into account the measure.

Definition 1.7.7 (Measured-length space)**.** *We say that a metric measure space* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *is a measured-length space if there exists a Borel set* $A \subset \mathrm{X}$ *whose complement is* $\mathfrak{m}$*-negligible with the following property: for every* $x_0, x_1 \in A$ *there exists* $\varepsilon > 0$ *such that for every* $\varepsilon_0, \varepsilon_1 \in (0,\varepsilon]$ *there is a test plan* $\boldsymbol{\pi}^{\varepsilon_0,\varepsilon_1}$ *such that*

-) the map $(0,\varepsilon]^2 \ni (\varepsilon_0,\varepsilon_1) \mapsto \boldsymbol{\pi}^{\varepsilon_0,\varepsilon_1}$ *is weakly Borel in the sense that for any* $\varphi \in C_b(C([0,1],\mathrm{X}))$ *the map*

$$(0,\varepsilon]^2 \ni (\varepsilon_0,\varepsilon_1) \mapsto \int \varphi \,\mathrm{d}\boldsymbol{\pi}^{\varepsilon_0,\varepsilon_1}$$

is Borel;

-) for every $\varepsilon_0, \varepsilon_1 \in (0,\varepsilon]$ *we have*

$$(e_0)_*\boldsymbol{\pi}^{\varepsilon_0,\varepsilon_1} = \frac{1}{\mathfrak{m}(B_{\varepsilon_0}(x_0))}\mathfrak{m}|_{B_{\varepsilon_0}(x_0)}, \quad (e_1)_*\boldsymbol{\pi}^{\varepsilon_0,\varepsilon_1} = \frac{1}{\mathfrak{m}(B_{\varepsilon_1}(x_1))}\mathfrak{m}|_{B_{\varepsilon_1}(x_1)};$$

-) we have

$$\limsup_{\varepsilon_0,\varepsilon_1\downarrow 0}\int\int_0^1 |\dot\gamma_t|^2\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}^{\varepsilon_0,\varepsilon_1}(\gamma)\le \mathsf{d}^2(x_0,x_1).$$

The measured-length property and the a.e. locally doubling property (a variant of Definition 1.5.20 requiring that the property described in Definition 1.5.20 holds for every $x\in B$ with $B\subset \mathrm{X}$ with $\mathfrak{m}$-negligible complement instead that for every $x\in\mathrm{X}$) both passes to the warped product space. Notice also that in case w_{d} and $w_{\mathfrak{m}}$ strictly positive (as we have in Chapters 2 and 3) the locally doubling property pass to the warped product space.

Proposition 1.7.8. *Consider assumptions and notations as in Assumption 1.7.5, moreover assume that $w_{\mathfrak{m}}$ is striclty positive in the interior of I and that X is a measured-length space. Then the warped product space $I\times_w \mathrm{X}$ is a measured-length space.*

Proposition 1.7.9. *Consider assumptions and notations as in Assumption 1.7.5, and assume that X is a.e. locally doubling. Then the warped product space $I\times_w\mathrm{X}$ is a.e. locally doubling.*

We conclude this section seeing how the last two definitions are related to the Sobolev-to-Lipschitz property.

Proposition 1.7.10. *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be an a.e. locally doubling measured-length space. Then it has the Sobolev-to-Lipschitz property.*

In particular, the following theorem holds.

Theorem 1.7.11. *Consider assumptions and notations as in Assumption 1.7.5, moreover assume that X is an a.e. locally doubling measured-length space and that $w_{\mathfrak{m}}$ is strictly positive in the interior of I. Then the warped product space $I\times_w X$ is an a.e. locally doubling measured-length space. In particular, it has the Sobolev-to-Lipschitz property.*

1.8 Regular Lagrangian Flows

As we stated in the Introduction, the strategy to prove rigidity theorems is based on taking a "good" function and studying the flow of its gradient. In Section 1.3 we defined the gradient of a function and the space of vector fields thanks to the theory of normed modules. In this section we give the notion of Regular Lagrangian Flow and some useful properties such as existence, uniqueness and regularity. This theory is due to R. J. DiPerna and P.-L. Lions (see [DPL]) in the Euclidean space (see also [A]) and generalized to the metric setting by L. Ambrosio and D. Trevisan in [AT14] (see also [AT15]). We refer to [GR18] for a reformulation of the results in [AT14] using differential calculus in RCD spaces.

As in the smooth setting, there are two approaches to this problem: Lagrangian and Eulerian.

The former studies the trajectories of particles: given v_t time dependent vector field, for $x \in \mathrm{X}$ we want to find a curve $\gamma : [0,T] \to \mathrm{X}$ that in some sense satisfies

$$\begin{cases} \gamma_t' = v_t(\gamma_t), \\ \gamma_0 = x. \end{cases}$$

The Eulerian approach studies how the mass changes in time in a fixed point: as before, let v_t be a time dependent vector field, then given a starting measure $\bar{\mu}$, the push-forward measure with respect to the flow $\mu_s := (\mathsf{Fl}_s^{(v_t)})_* \bar{\mu}$ must satisfy, in a weak sense, the continuity equation

$$\begin{cases} \frac{d}{dt}\mu_t + \mathrm{div}(v_t \mu_t) = 0, \\ \mu_0 = \bar{\mu}. \end{cases}$$

In this section we recall the definitions of Regular Lagrangian Flow and weak solution of the continuity equation in the non-smooth setting, we see how the two approaches are related and we state an existence and uniqueness theorem for the flow of a regular enough vector field. We conclude recalling a regularity theorem (from [BS20b]), which states that if a vector field has bounded (symmetric) covariant derivative then its Regular Lagrangian Flow admits a Lipschitz representative, and some properties of flows of vector fields that do not depend on time.

In this section we assume that $(\mathrm{X}, \mathrm{d}, \mathfrak{m})$ is an $\mathsf{RCD}(K, \infty)$ space for some $K \in \mathbb{R}$.

Definition 1.8.1 (Solution of the continuity equation). *Let $\mu : [0,T] \to \mathscr{P}(\mathrm{X})$ and $v : [0,T] \to L^0(T\mathrm{X})$ be Borel maps. We say that they solve the continuity equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu_t + \mathrm{div}(v_t \mu_t) = 0 \tag{1.8.1}$$

if the following conditions hold:

i) there exists $C > 0$ such that $\mu_t \le C\mathfrak{m}$ for every $t \in [0,T]$;

ii) it holds

$$\int_0^T \int |v_t|^2 \, \mathrm{d}\mu_t \, \mathrm{d}t < +\infty;$$

iii) for every $f \in W^{1,2}(\mathrm{X})$ the map $t \mapsto \int f \,\mathrm{d}\mu_t$ is absolutely continuous and for a.e. $t \in [0,T]$ it holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\int f \,\mathrm{d}\mu_t = \int \mathrm{d}f(v_t)\,\mathrm{d}\mu_t.$$

Existence and uniqueness of solutions of the continuity equation in the sense of Definition 1.8.1 can be established under suitable regularity assumption on the vector field v_t. Among other things, a control on the covariant derivative ∇v_t is required.

Theorem 1.8.2 (Uniqueness of solutions of the continuity equation)**.** *Let $(v_t) : [0,T] \to L^2(T\mathrm{X})$ be Borel and such that $v_t \in D(\mathrm{div})$ for every $t \in [0,T]$. Moreover, assume that $\||v_t|\|_{L^2(\mathfrak{m})} \in L^1(0,T)$, $\|\mathrm{div}(v_t)\|_{L^2(\mathfrak{m})} \in L^1(0,T)$, $\|\mathrm{div}(v_t)^-\|_{L^\infty} \in L^\infty(0,T)$, and $\|\nabla v_t\|_{L^2(T^{\otimes 2}\mathrm{X})} \in L^1(0,T)$. Moreover, let $\bar\mu \in \mathscr{P}(\mathrm{X})$ be such that $\bar\mu \le C\mathfrak{m}$ for a constant $C > 0$.*

Then there exists a unique $\mu : [0,T] \to \mathscr{P}(X)$ such that the couple (μ, v) is a solution of the continuity equation (1.8.1) with $\mu_0 = \bar\mu$.

Passing now to the Lagrangian approach, we recall the definition of Regular Lagrangian Flow.

Definition 1.8.3 (Regular Lagrangian Flow)**.** *Let $(v_t) : [0,T] \to L^2(T\mathrm{X})$ be Borel. We say that a map $\mathsf{Fl} : [0,T] \times \mathrm{X} \to \mathrm{X}$ is a Regular Lagrangian Flow associated to (v_t) if the following are satisfied:*

i) there exists $C > 0$ such that

$$(\mathsf{Fl}_t)_*\mathfrak{m} \le C\mathfrak{m} \qquad \text{for every } t \in [0,T]; \tag{1.8.2}$$

ii) for $\mathfrak{m}$-a.e. $x \in \mathrm{X}$ the function $[0,T] \ni t \to \mathsf{Fl}_t(x)$ is continuous and satisfies $\mathsf{Fl}_0(x) = x$;

iii) for every $f \in \mathrm{Test}(\mathrm{X})$ it holds that for $\mathfrak{m}$-a.e. $x \in \mathrm{X}$ the function $(0,T) \ni t \to f \circ \mathsf{Fl}_t(x)$ is absolutely continuous and

$$\frac{\mathrm{d}}{\mathrm{d}t} f \circ \mathsf{Fl}_t(x) = \langle \nabla f, v_t \rangle \circ \mathsf{Fl}_t(x) \tag{1.8.3}$$

for a.e. $t \in (0,T)$.

Regualar Lagrangian Flows can be characterized via the following proposition.

Proposition 1.8.4. *Let $(v_t) : [0,T] \to L^2(T\mathrm{X})$ be Borel and such that $|v_t| \in L^\infty([0,T], L^\infty(\mathrm{X}))$. Moreover, let $\mathsf{Fl} : [0,T]\times\mathrm{X} \to \mathrm{X}$ be a Borel map satisfying (i) and (ii) of Definition 1.8.3. Then the following are equivalent:*

i) Fl *is a Regular Lagrangian Flow associated to (v_t);*

ii) for every $f \in W^{1,2}(\mathrm{X})$ the map $[0,T] \ni t \mapsto f \circ \mathsf{Fl}_t \in L^2(\mathrm{X})$ is Lipschitz and

$$\lim_{h\to 0} \frac{f \circ \mathsf{Fl}_{t+h} - f \circ \mathsf{Fl}_t}{h} = \mathrm{d}f(v_t) \circ \mathsf{Fl}_t \qquad \text{for a.e. } t \in [0,T], \tag{1.8.4}$$

the limit being intended in $L^2(\mathfrak{m})$;

iii) for every $f \in W^{1,2}_{loc}(\mathrm{X})$ the map $[0,T] \ni t \mapsto f\circ\mathsf{Fl}_t \in L^2_{loc}(\mathrm{X})$ is Lipschitz and for a.e. $t \in [0,T]$ formula (1.8.4) *holds with the limit being intended in $L^2_{loc}(\mathfrak{m})$.*

As mentioned before, existence and uniqueness of Regular Lagrangian Flows are related to existence and uniqueness of solutions of the continuity equation.

Theorem 1.8.5 (Existence and uniqueness of Regular Lagrangian Flows)**.** *Let (v_t) be as in Theorem 1.8.2. Then there exists a unique Regular Lagrangian Flow Fl_t associated to v_t.*

Uniqueness is intended as: if both Fl and $\tilde{\mathsf{Fl}}$ are two such flows, then for $\mathfrak{m}$-a.e. x we have $\mathsf{Fl}_t(x) = \tilde{\mathsf{Fl}}_t(x)$ for any $t \in [0,T]$.

Moreover, the following hold:

-) for $\mathfrak{m}$-a.e. $x \in \mathrm{X}$ the curve $t \mapsto \mathsf{Fl}_t(x)$ is absolutely continuous and

$$\mathsf{ms}(\mathsf{Fl}_\cdot(x), t) = |v_t| \circ F_t(x) \qquad \text{a.e. } t \in [0,1]; \tag{1.8.5}$$

-) indicating with μ_t the unique solution of the continuity equation given by Theorem 1.8.2, μ_t satisfies

$$\mu_t = (\mathsf{Fl}^{(v)}_t)_*\bar{\mu}.$$

We conclude recalling a regularity theorem about Regular Lagrangian Flows. Its proof is due to E. Bruè and D. Semola and can be found in [BS20b].

They introduced the notion of symmetric covariant derivative defined as follows.

Definition 1.8.6 (Symmetric covariant derivative). *The space $W^{1,2}_{C,s}(T\mathrm{X})$ is defined as the space of all vector fields $v \in L^2(T\mathrm{X})$ for which there exists a tensor $S \in L^2(T^{\otimes 2}\mathrm{X})$ such that for every $\varphi, \psi_1, \psi_2 \in \mathrm{Test}(\mathrm{X})$ it holds*

$$2\int \varphi S(\nabla\psi_1, \nabla\psi_2)\,\mathrm{d}\mathfrak{m} = \int -\langle v, \nabla\psi_1\rangle \,\mathrm{div}(\varphi\nabla\psi_2) - \langle v, \nabla\psi_2\rangle \,\mathrm{div}(\varphi\nabla\psi_1) + \mathrm{div}(\varphi v)\langle\nabla\psi_1, \nabla\psi_2\rangle\rangle \,\mathrm{d}\mathfrak{m}.$$

The tensor S is called symmetric covariant derivative of v and we denote it with $\nabla_{sym}v$. We endow the space $W^{1,2}_{C,s}(T\mathrm{X})$ with the norm

$$\|v\|^2_{W^{1,2}_{C,s}(T\mathrm{X})} := \|v\|^2_{L^2(T\mathrm{X})} + \|\nabla_{sym}v\|^2_{L^2(T^{\otimes 2}\mathrm{X})}.$$

As noted in [BS20b], $W^{1,2}_C(T\mathrm{X}) \subset W^{1,2}_{C,s}(T\mathrm{X})$ and the symmetric covariant derivative of a vector field is the symmetric part of its covariant derivative whenever the latter exists.

Let (v_t) be a time dependent vector field such that for every $t \in [0, T]$ it admits a symmetric covariant derivative. We define

$$L(v) := \sup_{t\in[0,T]} \|\nabla_{sym}v_t\|_{L^\infty}.$$

Theorem 1.8.7. *Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be an $\mathsf{RCD}(K, N)$ space. Assume that (v_t) satisfies the hypotheses of Theorem 1.8.5 and assume that $L(v) < +\infty$. Then the Regular Lagrangian Flow Fl of (v_t) admits a Lipschitz representative and for every $x, y \in \mathrm{X}$ it holds*

$$\mathsf{d}(\mathsf{Fl}_t(x), \mathsf{Fl}_t(y)) \le e^{Lt}\mathsf{d}(x, y).$$

Remark 1.8.8. In [BS20b] the authors assumed the space to be compact in order to deduce Lipschitz regularity of the flow. This was needed as they were using the main result in [GT18] that, at that time, provided a necessary second-order differentiation formula on finite-dimensional and compact RCD spaces. A subsequent improvement of this paper [GT21] established the same second-order differentiation formula in the non-compact setting, thus allowing Bruè-Semola's result to be extended to the non-compact setting. ■

We consider now an autonomous vector field (v_t), i.e. such that there exists $v \in L^2(T\mathrm{X})$ which satisfies $v_t \equiv v$ for every $t \in [0, T]$.

Let v be as in Theorem 1.8.5. From Theorem 1.8.5 follows easily that Fl can be extended uniquely to a map $[0, +\infty) \times \mathrm{X} \to \mathrm{X}$ such that for every $t, s \in [0, +\infty)$ it holds the group property

$$\mathsf{Fl}_t \circ \mathsf{Fl}_s = \mathsf{Fl}_{t+s} \qquad \mathfrak{m}\text{-a.e.}, \tag{1.8.6}$$

where, as in Theorem 1.8.5, uniqueness is to be intended up to a negligible set of trajectories.

Assuming that $\mathrm{div}(v) \in L^\infty(\mathfrak{m})$ (notice that before we only assumed that its negative part was in $L^\infty(\mathfrak{m})$) and denoting with Fl^{-v} the Regular Lagrangian Flow of $-v$, it holds (see for instance [GR18, Lemma 3.18]) that $\mathsf{Fl}_t^{-v} \circ \mathsf{Fl}_t = \mathrm{Id}$ for every $t \in [0, +\infty)$, then, defining $\mathsf{Fl}_{-t} := \mathsf{Fl}_t^{-v}$ for every $t \in [0, +\infty)$ we can extend the flow Fl on $\mathbb{R} \times \mathrm{X}$ and in this case the group property (1.8.6) holds for every $t, s \in \mathbb{R}$.

Moreover the following holds:

in (ii) and (iii) of Proposition 1.8.4 we can replace "Lipschitz" with "C^1" and formula 1.8.4 holds for every t. (1.8.7)

Chapter 2

A general splitting principle on RCD spaces

In this chapter we present the main result of [GMa]: the general splitting principle Theorem 0.0.6.

2.1 Producing a coordinate function

In the practice of studying rigidity properties of spaces with lower Ricci bounds, one often finds out a function u with somehow controlled gradient and Laplacian and for which equality holds in the Bochner inequality, and this in turn gives information about the structure of the Hessian of u. The typical informations of u are the identities

$$\begin{aligned} |\mathrm{d}u| &= \varphi \circ u, \\ \Delta u &= \xi \circ u, \\ \mathrm{Hess}(u) &= \zeta \circ u \, |\mathrm{d}u| \, \mathrm{Id} + \tilde{\zeta} \circ u \, |\mathrm{d}u| \, \mathrm{e}_1 \otimes \mathrm{e}_1, \end{aligned}$$

where $\mathrm{e}_1 = \frac{\nabla u}{|\nabla u|}$ on $|\nabla u| > 0$, for suitable functions $\varphi, \zeta, \tilde{\zeta}, \xi : \mathbb{R} \to \mathbb{R}$.

As we shall see, when this occurs the space splits as warped product $\mathbb{R} \times_w \mathrm{X}'$ and the '$\mathbb{R}$-coordinate' of the isomorphism is the post-composition of u with a suitable function $\eta : \mathbb{R} \to \mathbb{R}$.

To have a better understanding of the warped product we will ultimately obtain, it is convenient to identify right now who is such suitable post-composition. This is the scope of the following lemma, whose proof only relies in handling chain rules.

Lemma 2.1.1 (Producing a coordinate function)**.** *Let $K \in \mathbb{R}$, $N \in [1, +\infty)$ and let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be a $\mathsf{RCD}(K, N)$ space. Assume that $u \in H^{2,2}_{loc}(\mathrm{X})$ satisfies*

$$|\mathrm{d}u| = \varphi \circ u \qquad \mathfrak{m}\text{-a.e.} \tag{2.1.1}$$

for some $\varphi : u(\mathrm{X}) \to (0, \infty)$ in $C^{1,1}_{loc}$. Put $\mathrm{e}_1 := \frac{\nabla u}{|\nabla u|}$ (this is $\mathfrak{m}$-a.e. well defined as $|\nabla u| > 0$ $\mathfrak{m}$-a.e. as a consequence of our assumption on φ) and assume that for some locally Lipschitz functions $\zeta, \tilde{\zeta} : u(\mathrm{X}) \to \mathbb{R}$ we have

$$\operatorname{Hess} u = \zeta \circ u |\mathrm{d}u| \mathrm{Id} + \tilde{\zeta} \circ u |\mathrm{d}u| \mathrm{e}_1 \otimes \mathrm{e}_1. \tag{2.1.2}$$

Then

$$\varphi' = \zeta + \tilde{\zeta}, \tag{2.1.3}$$

any function $\eta : u(\mathrm{X}) \to (0, \infty)$ in $C^{1,1}_{loc}$ such that $\eta' = \frac{1}{\varphi}$ is invertible and the function $\mathrm{b} := \eta \circ u$ is in $H^{2,2}_{loc}(\mathrm{X})$ with

$$|\mathrm{d}\mathrm{b}| = 1, \tag{2.1.4a}$$

$$\operatorname{Hess} \mathrm{b} = \zeta \circ \eta^{-1} \circ \mathrm{b}\big(\mathrm{Id} - \mathrm{e}_1 \otimes \mathrm{e}_1\big). \tag{2.1.4b}$$

If moreover we have $u \in D(\Delta_{loc})$ with $\Delta u = \xi \circ u$ for some $\xi : u(\mathrm{X}) \to \mathbb{R}$ Borel locally bounded, then $\mathrm{b} \in D(\Delta_{loc})$ with

$$\Delta \mathrm{b} = \big(\tfrac{\xi}{\varphi} - \varphi'\big) \circ \eta^{-1} \circ \mathrm{b}. \tag{2.1.5}$$

Proof. Since $u \in H^{2,2}_{loc}(\mathrm{X})$ is with $|\mathrm{d}u| \in L^\infty_{loc}(\mathrm{X})$ (by (2.1.1)) we have that $|\mathrm{d}u| \in W^{1,2}_{loc}(\mathrm{X})$ (recall (1.6.2)), thus we can write

$$\varphi' \circ u \, \mathrm{d}u = \mathrm{d}|\mathrm{d}u| = \operatorname{Hess} u\big(\tfrac{\mathrm{d}u}{|\mathrm{d}u|}\big) = (\zeta \circ u + \tilde{\zeta} \circ u) \, \mathrm{d}u$$

and since $|\mathrm{d}u| > 0$ $\mathfrak{m}$-a.e., property (2.1.3) follows.

The fact that η is invertible follows directly from $\eta' = \frac{1}{\varphi} > 0$ and the fact that $\mathrm{b} \in H^{2,2}_{loc}(\mathrm{X})$ from the chain rule noticing that:

-) if $u \in \mathrm{Test}_{loc}(\mathrm{X})$ and $\varphi \in C^\infty_{loc}(u(\mathrm{X}))$, then $\varphi \circ u \in \mathrm{Test}_{loc}(\mathrm{X})$ (by direct computation);

-) formula (1.6.1) shows that if u is also locally Lipschitz, then the Hessian of $\varphi \circ u$ is locally in L^2.

Then (2.1.4a) follows from $|\mathrm{d}\mathrm{b}| = \eta' \circ u |\mathrm{d}u| = (\eta'\varphi) \circ u$ and the choice of η. For (2.1.4b) we use the chain rule for the Hessian (1.6.1) to compute

$$\begin{aligned} \operatorname{Hess} \mathrm{b} &= \eta' \circ u \operatorname{Hess} u + \eta'' \circ u \mathrm{d}u \otimes \mathrm{d}u \\ \text{(by (2.1.1),(2.1.2))} \quad &= \zeta \circ u \, \mathrm{Id} + \tilde{\zeta} \circ u \mathrm{e}_1 \otimes \mathrm{e}_1 - \varphi' \circ u \mathrm{e}_1 \otimes \mathrm{e}_1 \\ \text{(by (2.1.3))} \quad &= \zeta \circ u \, \big(\mathrm{Id} - \mathrm{e}_1 \otimes \mathrm{e}_1\big), \end{aligned}$$

which is (2.1.4b). The last claim is also a direct consequence of the assumptions and of the chain rule for the Laplacian (1.3.1). □

2.2 Set up and statement of the splitting result

From now on we shall assume the following:

a) $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ is an $\mathsf{RCD}(K, N)$ space, $K \in \mathbb{R}$, $N < \infty$, with $\mathrm{supp}(\mathfrak{m}) = \mathrm{X}$.

b) $\mathrm{b} : \mathrm{X} \to \mathbb{R}$ is a function in $H^{2,2}_{loc} \cap D(\Delta_{loc})$ such that

$$|\mathrm{db}| = 1, \qquad \mathfrak{m}\text{-a.e.} \tag{2.2.1a}$$

$$\Delta \mathrm{b} = \psi_{\mathfrak{m}} \circ \mathrm{b}, \tag{2.2.1b}$$

$$\mathrm{Hess}\,\mathrm{b} = \psi_{\mathsf{d}} \circ \mathrm{b}\,(\mathrm{Id} - \mathrm{e}_1 \otimes \mathrm{e}_1) \qquad \text{where } \mathrm{e}_1 := \nabla \mathrm{b} = \tfrac{\nabla \mathrm{b}}{|\nabla \mathrm{b}|}, \tag{2.2.1c}$$

for some $\psi_{\mathfrak{m}}, \psi_{\mathsf{d}} : \mathbb{R} \to \mathbb{R}$ locally Lipschitz. By (2.2.1a) and the Sobolev-to-Lipschitz property, b has a 1-Lipschitz representative and we shall always identify it with such representative.

These assumptions have the following rather direct consequences:

-) The Regular Lagrangian Flow $\mathsf{Fl} : \mathbb{R} \times \mathrm{X} \to \mathrm{X}$ of $\nabla \mathrm{b}$ is well defined. Indeed, for $\eta \in \mathrm{LIP}_{bs}(\mathrm{X})$ the vector field $\eta \nabla \mathrm{b}$ admits a Regular Lagrangian Flow thanks to the properties (2.2.1). Then taking into account the finite speed of propagation of this flow (from (2.2.1a) and (1.8.5)) it is easy to see that taking η equal 1 on a ball of radius R and then letting $R \uparrow \infty$ and using the fact that $\mathrm{div}(\nabla \mathrm{b}) = \Delta \mathrm{b}$ is bounded on the level sets of b we can find the desired flow (Fl_t).

-) $\mathsf{Fl} : \mathbb{R} \times \mathrm{X} \to \mathrm{X}$ has a continuous representative, still denoted by Fl, and its Lipschitz constant on $[-T, T] \times \mathrm{X}$ is bounded for every $T > 0$. Such Lipschitz regularity follows from the fact that the covariant derivative of $\nabla \mathrm{b}$ is bounded on $\mathrm{b}^{-1}([-T, T])$, because of (2.2.1c), and Theorem 1.8.7.

-) The formula

$$\mathrm{b}(\mathsf{Fl}_t(x)) = \mathrm{b}(x) + t \tag{2.2.2}$$

holds for every $t \in \mathbb{R}$, $x \in \mathrm{X}$. Indeed, we know from (1.8.3) that for $\mathfrak{m}$-a.e. x the function $t \mapsto \mathrm{b}(\mathsf{Fl}_t(x))$ is in $W^{1,1}_{loc}(\mathbb{R})$ (in fact locally absolutely continuous, as b is Lipschitz) with derivative equal to $|\mathrm{db}|^2(\mathsf{Fl}_t(x))$. By integration and using (1.8.2) we see from (2.2.1a) that for given $t \in \mathbb{R}$ the formula (2.2.2) holds for $\mathfrak{m}$-a.e. x. The claim follows by the continuity of Fl and b.

-) For any two $x, y \in \mathrm{X}$ with $\mathrm{b}(x) = \mathrm{b}(y)$ there is a Lipschitz curve joining them lying entirely on the same level set. Indeed, if γ is any Lipschitz path connecting x any y, say a geodesic, then $t \mapsto \mathsf{Fl}_{\mathrm{b}(x)-\mathrm{b}(\gamma_t)}(\gamma_t)$ is still Lipschitz (by the above Lipschitz regularity) joins x and y and lies on the same level set of x, y (by (2.2.2)).

We can now define the metric measure space $(\mathrm{X}', \mathsf{d}', \mathfrak{m}')$ by putting $\mathrm{X}' := \mathrm{b}^{-1}(0)$ and then defining

$$\mathsf{d}'(x, y)^2 := \inf \int_0^1 |\dot\gamma_t|^2 \,\mathrm{d}t, \tag{2.2.3}$$

the inf being taken among all continuous curves $\gamma : [0, 1] \to \mathrm{X}' \subset \mathrm{X}$ joining x and y and

$$\mathfrak{m}' := c^{-1} \,\mathrm{Pr}_*(\mathfrak{m}|_{\mathrm{b}^{-1}([0,1])}), \qquad \text{for} \qquad c := \int_0^1 w_{\mathfrak{m}} \,\mathrm{d}\mathcal{L}^1, \tag{2.2.4}$$

where $\mathrm{Pr} : \mathrm{X} \to \mathrm{X}'$ is defined as $\mathrm{Pr}(x) := \mathsf{Fl}_{-\mathrm{b}(x)}(x)$, and $w_{\mathfrak{m}}$ is a positive function that we will define soon (see (2.2.9a) and (2.2.10), for now it is enough that c is a positive constant). It is clear that d' is a (finite) distance on X' and, since clearly from the above Pr is locally Lipschitz, that it is complete and induces the same topology on X' as the one induced by d. Then obviously $\mathfrak{m}'$ is a Borel measure and from (2.2.6) below it easily follows that it is finite on bounded sets (and thus Radon).

We also define

$$\begin{array}{rccc} \mathsf{T} : & \mathrm{X} & \to & \mathbb{R} \times \mathrm{X}', \\ & x & \mapsto & (\mathrm{b}(x), \mathrm{Pr}(x)) \end{array} \tag{2.2.5}$$

and notice that

$$\mathsf{T} \text{ is locally Lipschitz, invertible, with locally Lipschitz inverse,} \tag{2.2.6}$$

where on $\mathbb{R} \times \mathrm{X}'$ we are, for the moment, putting the distance

$$\mathsf{d}_+((t, x), (s, y)) := \mathsf{d}'(x, y) + |t - s|.$$

Indeed, the local Lipschitz regularity of T is obvious. For the other inequality notice that

$$\mathsf{d}(x, y) \le \mathsf{d}(\mathsf{Fl}_{\mathrm{b}(x)}(\mathrm{Pr}(x)), \mathsf{Fl}_{\mathrm{b}(x)}(\mathrm{Pr}(y))) + \mathsf{d}(\mathsf{Fl}_{\mathrm{b}(x)-\mathrm{b}(y)}(y), y) \tag{2.2.7}$$

for every $x, y \in \mathrm{X}$, and use that $t \mapsto \mathsf{Fl}_t(y)$ is 1-Lipschitz (by (2.2.1a) and (1.8.5)), that $\mathsf{Fl}_z : \mathrm{X}' \to \mathrm{X}$ is Lipschitz uniformly on $z \in [-T, T]$ for any

$T > 0$ (by Theorem 1.8.7 applied to the vector field $\eta \circ \mathrm{b}\nabla\mathrm{b}$ with η smooth cut-off function with bounded support) and that $\mathsf{d} \le \mathsf{d}'$ on X' (obviously from the definition). Observe also that

$$\text{the inverse of } \mathsf{T} \text{ is the map } (t, x') \mapsto \mathsf{Fl}_t(x') \tag{2.2.8}$$

as it is easily seen from the definitions recalling also (2.2.2).

With this said, we now want to equip the set $\mathbb{R}\times\mathrm{X}'$ with a warped product structure. To this aim, let us introduce the locally absolutely continuous functions $w_{\mathsf{m}}, w_{\mathsf{d}} : \mathbb{R} \to (0, +\infty)$ defined by

$$(\log(w_{\mathfrak{m}}))' = \psi_{\mathfrak{m}}, \tag{2.2.9a}$$
$$(\log(w_{\mathsf{d}}))' = \psi_{\mathsf{d}}, \tag{2.2.9b}$$

normalized in such a way that

$$w_{\mathfrak{m}}(0) = w_{\mathsf{d}}(0) = 1. \tag{2.2.10}$$

Then on $\mathbb{R} \times \mathrm{X}'$ we define the (Radon) warped product measure $\mathfrak{m}_w$ by the formula

$$\int f(t,x)\,\mathrm{d}\mathfrak{m}_w := \int \Big(\int f(t,x)\,\mathrm{d}\mathfrak{m}'(x) \Big) w_{\mathfrak{m}}(t)\,\mathrm{d}\mathcal{L}^1(t) \tag{2.2.11}$$

and the warped product distance d_w as

$$\mathsf{d}_w\big((t,x),(s,y)\big)^2 = \inf \int_0^1 |\dot\eta_r|^2 + w_{\mathsf{d}}^2(\eta_r)|\dot\gamma_r|^2\,\mathrm{d}r, \tag{2.2.12}$$

the inf being taken among all absolutely continuous curves $\gamma : [0,1] \to \mathrm{X}'$ and $\eta : [0,1] \to \mathbb{R}$ joining x to y and t to s respectively. Since w_{d} is continuous and strictly positive, it is clear that d_w is locally equivalent to (i.e. up to multiplicative constants controls and is controlled by) the distance d_+ used above. In particular, it induces the product topology.

We can now state the main result of this chapter (which is, stated slightly differently, Theorem 0.0.6).

Theorem 2.2.1. *Under the assumptions (a, b) stated above, the following holds.*

The map T defined in (2.2.5) is a measure preserving isometry from X to $\mathbb{R} \times_w \mathrm{X}'$, the latter being the space $\mathbb{R}\times\mathrm{X}'$ equipped with the distance d_w and the measure $\mathfrak{m}_w$.

The proof of this result will come as a result of the analysis in the following sections, the conclusion being in Section 2.5.

2.3 Behaviour of the measure under the flow

Let us define the locally Lipschitz map $\mathbb{R}^2 \ni (t,z) \mapsto \Psi_{\mathfrak{m},t}(z) \in \mathbb{R}$ as

$$\Psi_{\mathfrak{m},t}(z) := \tfrac{w_{\mathfrak{m}}(z-t)}{w_{\mathfrak{m}}(z)}. \tag{2.3.1}$$

Then from (2.2.9a) we get

$$\partial_t \Psi_{\mathfrak{m},t}(z) = -\psi_{\mathfrak{m}}(z-t)\Psi_{\mathfrak{m},t}(z), \qquad \text{and} \qquad \Psi_{\mathfrak{m},0} \equiv 1 \tag{2.3.2}$$

and

$$\partial_t \Psi_{\mathfrak{m}} + \partial_z \Psi_{\mathfrak{m}} + \psi_{\mathfrak{m}}\Psi_{\mathfrak{m}} = 0. \tag{2.3.3}$$

Lemma 2.3.1. *With the same assumptions and notation of Section 2.2 and for $\Psi_{\mathfrak{m}}$ defined as in* (2.3.1) *we have*

$$(\mathsf{Fl}_t)_*\mathfrak{m} = \Psi_{\mathfrak{m},t} \circ \mathrm{b}\, \mathfrak{m} \qquad \forall t \in \mathbb{R}. \tag{2.3.4}$$

Proof. Recalling the simple implication "$T_*\mu = \nu$ implies $T_*(\rho\mu) = \rho \circ T^{-1}\nu$", and by the finite speed of propagation of the flow, to conclude it is sufficient to prove that for any Lipschitz probability density ρ with bounded support we have $(\mathsf{Fl}_t)_*(\rho\mathfrak{m}) = \rho_t\mathfrak{m}$, where $\rho_t = (\rho \circ \mathsf{Fl}_{-t})\,(\Psi_{\mathfrak{m},t} \circ \mathrm{b})$.

By the uniqueness result for the continuity equation (that is central for the theory of Regular Lagrangian Flows, see Section 1.8), this latter claim will follow if we prove that (ρ_t) solves

$$\partial_t \rho_t + \mathrm{div}(\rho_t \nabla \mathrm{b}) = 0 \qquad a.e.\ t.$$

Notice that $(t,x) \mapsto \rho_t(x)$ is Lipschitz, thus the computations that we are going to perform are justified. Now observe that letting $h \to 0$ in $\frac{\rho\circ\mathsf{Fl}_{-t}\circ\mathsf{Fl}_{-h} - \rho\circ\mathsf{Fl}_{-t}}{h}$ we see that $\partial_t(\rho \circ \mathsf{Fl}_{-t}) = -\langle \nabla(\rho \circ \mathsf{Fl}_{-t}), \nabla \mathrm{b}\rangle$, thus

$$\partial_t \rho_t = -\langle \nabla(\rho \circ \mathsf{Fl}_{-t}), \nabla \mathrm{b}\rangle\ \Psi_{\mathfrak{m},t} \circ \mathrm{b} + (\rho \circ \mathsf{Fl}_{-t})(\partial_t \Psi_{\mathfrak{m},t}) \circ \mathrm{b}.$$

On the other hand

$$\begin{aligned}\mathrm{div}(\rho_t \nabla \mathrm{b}) &= \langle \nabla \rho_t, \nabla \mathrm{b}\rangle + \rho_t \Delta \mathrm{b} \\ &= \langle \nabla(\rho \circ \mathsf{Fl}_{-t}), \nabla \mathrm{b}\rangle\, \Psi_{\mathfrak{m},t} \circ \mathrm{b} + (\rho \circ \mathsf{Fl}_{-t})(\partial_z \Psi_{\mathfrak{m},t}) \circ \mathrm{b}\, |\mathrm{db}|^2 + \rho_t \Delta \mathrm{b},\end{aligned}$$

thus recalling (2.2.1) and adding up we conclude that

$$\partial_t \rho_t + \mathrm{div}(\rho_t \nabla \mathrm{b}) = (\rho \circ \mathsf{Fl}_{-t})\big(\partial_t \Psi_{\mathfrak{m},t} + \partial_z \Psi_{\mathfrak{m},t} + \psi_{\mathfrak{m}}\Psi_{\mathfrak{m},t}\big) \circ \mathrm{b} \overset{(2.3.3)}{=} 0,$$

as desired. □

For $B \subset \mathrm{X}'$ Borel we define $\hat{B} \subset \mathrm{X}$ as

$$\hat{B} := \mathsf{Fl}_{\cdot}^{-1}(B) = \bigcup_{t\in\mathbb{R}} \mathsf{Fl}_t^{-1}(B) = \{\mathsf{Fl}_t(B) : t \in \mathbb{R}\}. \tag{2.3.5}$$

Clearly $\hat{B}$ is Borel. Then we have the following.

Lemma 2.3.2. *With the same assumptions and notation of Section 2.2 (recall in particular the definition* (2.2.9a) *and the normalization* (2.2.10)*) the following holds.*

Let $B \subset \mathrm{X}'$ be Borel and put $\mu := \mathrm{b}_*(\mathfrak{m}|_{\hat{B}})$. Then

$$(\mathrm{tr}_t)_*\mu = \Psi_{\mathfrak{m},t}\,\mu, \qquad \text{for every } t \in \mathbb{R}, \tag{2.3.6}$$

where $\mathrm{tr}_t : \mathbb{R} \to \mathbb{R}$ is the translation map sending z to $z + t$ and moreover

$$\mu = \mathfrak{m}'(B)\, w_{\mathfrak{m}}\, \mathcal{L}^1. \tag{2.3.7}$$

Proof. If $\mathfrak{m}'(B) = \infty$ the conclusion follows from the definition of $\mathfrak{m}'$ and (2.3.4), thus we assume $\mathfrak{m}'(B) < \infty$. By (2.3.4) and the invariance of $\hat{B}$ under the flow it follows immediately that $(\mathsf{Fl}_t)_*(\mathfrak{m}|_{\hat{B}}) = \Psi_{\mathfrak{m},t} \circ \mathrm{b}\, \mathfrak{m}|_{\hat{B}}$. Then (2.3.6) follows from

$$\begin{aligned}(\mathrm{tr}_t)_*\mu = (\mathrm{tr}_t)_*\mathrm{b}_*(\mathfrak{m}|_{\hat{B}}) &\overset{(2.2.2)}{=} \mathrm{b}_*(\mathsf{Fl}_t)_*(\mathfrak{m}|_{\hat{B}})\\ &= \mathrm{b}_*(\Psi_{\mathfrak{m},t} \circ \mathrm{b}\, \mathfrak{m}|_{\hat{B}}) \overset{*}{=} \Psi_{\mathfrak{m},t}\, \mathrm{b}_*(\mathfrak{m}|_{\hat{B}}) = \Psi_{\mathfrak{m},t}\mu,\end{aligned}$$

where the starred equality is justified by

$$\int \varphi \,\mathrm{d}\mathrm{b}_*(\Psi_{\mathfrak{m},t} \circ \mathrm{b}\, \mathfrak{m}|_{\hat{B}}) = \int (\varphi \Psi_{\mathfrak{m},t}) \circ \mathrm{b}\, \mathrm{d}\mathfrak{m}|_{\hat{B}} = \int \varphi \Psi_{\mathfrak{m},t}\, \mathrm{d}\mathrm{b}_*(\mathfrak{m}|_{\hat{B}}).$$

Averaging (2.3.6) in t we see that the left hand side becomes absolutely continuous with respect to $\mathcal{L}^1$, thus showing that $\mu \ll \mathcal{L}^1$, say $\mu = \rho\mathcal{L}^1$. Then (2.3.6) becomes: for any $t \in \mathbb{R}$ we have $\rho(z - t) = \Psi_{\mathfrak{m},t}(z)\rho(z)$ for $\mathcal{L}^1$-a.e. z. From this identity and $\partial_t \Psi_{\mathfrak{m},t}|_{t=0} = -\psi_{\mathfrak{m}}$ (that comes from (2.3.2)) it easily follows that the distributional derivative ρ' of ρ satisfies $\rho' = \psi_{\mathfrak{m}}\rho$, thus ensuring that ρ is a multiple of $w_{\mathfrak{m}}$, say $\rho = cw_{\mathfrak{m}}$.

To find the value of c notice that

$$\mathrm{b}_*(\mathfrak{m}|_{\hat{B}})([0, 1]) = \mathfrak{m}(\hat{B} \cap \mathrm{b}^{-1}[0, 1])) = \mathfrak{m}'(B) \int_0^1 w_{\mathfrak{m}}\, \mathrm{d}\mathcal{L}^1$$

by definition of $\mathfrak{m}'$, and also that

$$\mathrm{b}_*(\mathfrak{m}|_{\hat{B}})([0, 1]) = \mu([0, 1]) = \int_0^1 \rho \,\mathrm{d}\mathcal{L}^1 = c \int_0^1 w_{\mathfrak{m}}\, \mathrm{d}\mathcal{L}^1.$$

□

Let now $B \subset \mathrm{X}'$ be Borel as before and with $\mathfrak{m}'(B) < \infty$. Define $\hat{B}$ as in (2.3.5) and notice that since $\mu := \mathrm{b}_*(\mathfrak{m}|_{\hat{B}})$ is σ-finite (in fact Radon), we can disintegrate $\mathfrak{m}|_{\hat{B}}$ along the map b, as the standard proof (see e.g. [B, Section 10.6]) naturally carries over. Thus we obtain a weakly measurable family $z \mapsto \nu_z$ of probability measures on X such that ν_z is concentrated on $\mathrm{b}^{-1}(z)$ for μ-a.e. z (equivalently by Lemma 2.3.2 above: for $\mathcal{L}^1$-a.e. z) and so that

$$\int \varphi \,\mathrm{d}\mathfrak{m}|_{\hat{B}} = \iint \varphi \,\mathrm{d}\nu_z \,\mathrm{d}\mu(z) \overset{(2.3.7)}{=} \mathfrak{m}'(B) \int \Big(\int \varphi \,\mathrm{d}\nu_z \Big) w_{\mathfrak{m}}(z) \,\mathrm{d}\mathcal{L}^1(z) \tag{2.3.8}$$

holds for every $\varphi : \mathrm{X} \to \mathbb{R}^+$ Borel. Then we have:

Lemma 2.3.3. *With the same assumptions and notation of Section 2.2 the following holds.*

Let $B \subset \mathrm{X}'$ be Borel with $\mathfrak{m}'(B) < \infty$ and $\hat{B}$ as in (2.3.5). Then the disintegration $\{\nu_z\}_{z\in\mathbb{R}}$ of $\mathfrak{m}|_{\hat{B}}$ with respect to b satisfies: for any $t \in \mathbb{R}$ it holds

$$(\mathsf{Fl}_t)_*\nu_z = \nu_{z+t} \qquad \mu\text{-a.e. } z. \tag{2.3.9}$$

Proof. Fix $t \in \mathbb{R}$. We shall prove that the family $\{(\mathsf{Fl}_t)_*\nu_{z-t}\}$ is an admissible disintegration of $\mathfrak{m}|_{\hat{B}}$ with respect to b: this is (equivalent to) the claim. For μ-a.e. z we know that ν_z is concentrated on $\mathrm{b}^{-1}(z)$, thus ν_{z-t} is concentrated on $\mathrm{b}^{-1}(z-t)$ and therefore, by (2.2.2), $(\mathsf{Fl}_t)_*\nu_{z-t}$ is concentrated on $\mathrm{b}^{-1}(z)$. To conclude, with a density argument based also on (2.2.6) it is therefore sufficient to prove that for any Borel functions $g : \mathrm{X}' \to \mathbb{R}^+$ and $h : \mathbb{R} \to \mathbb{R}^+$ we have

$$\int h \circ \mathrm{b}\, g \circ \mathrm{Pr} \,\mathrm{d}\mathfrak{m}|_{\hat{B}} = \int \int h \circ \mathrm{b}\, g \circ \mathrm{Pr} \,\mathrm{d}(\mathsf{Fl}_t)_*\nu_{z-t} \,\mathrm{d}\mu(z). \tag{2.3.10}$$

We start claiming that it holds

$$\int h(z)\Psi_{\mathfrak{m},t}(z)\Big(\int g\circ\mathrm{Pr}\,\mathrm{d}\nu_{z-t}\Big)\,\mathrm{d}\mu(z) = \int h(z)\Psi_{\mathfrak{m},t}(z)\Big(\int g\circ\mathrm{Pr}\,\mathrm{d}\nu_z\Big)\,\mathrm{d}\mu(z). \tag{2.3.11}$$

Since $\mathrm{Pr} \circ \mathsf{Fl}_t = \mathrm{Pr}$, we have on one hand

$$\begin{aligned}\int_{\hat{B}} h \circ \mathrm{b}\, g \circ \mathrm{Pr} \,\mathrm{d}(\mathsf{Fl}_t)_*\mathfrak{m} &= \int_{\hat{B}} (\Psi_{\mathfrak{m},t} h) \circ \mathrm{b}\, g \circ \mathrm{Pr} \,\mathrm{d}\mathfrak{m} \\ &= \int \Psi_{\mathfrak{m},t}(z) h(z) \Big(\int g \circ \mathrm{Pr} \,\mathrm{d}\nu_z \Big) \,\mathrm{d}\mu(z)\end{aligned}$$

and on the other

$$\begin{aligned}\int_{\hat B} h\circ \mathrm{b}\, g\circ \mathrm{Pr}\,\mathrm{d}(\mathsf{Fl}_t)_*\mathfrak{m} &= \int_{\hat B} h\circ \mathrm{b}\circ \mathsf{Fl}_t\, g\circ \mathrm{Pr}\,\mathrm{d}\mathfrak{m}\\
\text{(by (2.2.2))}\qquad &= \int h(z+t)\Big(\int g\circ \mathrm{Pr}\,\mathrm{d}\nu_z\Big)\,\mathrm{d}\mu(z)\\
&= \int h(z)\Big(\int g\circ \mathrm{Pr}\,\mathrm{d}\nu_{z-t}\Big)\,\mathrm{d}(\mathrm{tr}_t)_*\mu(z)\\
\text{(by (2.3.6))}\qquad &= \int \Psi_{\mathfrak{m},t}(z)h(z)\Big(\int g\circ \mathrm{Pr}\,\mathrm{d}\nu_{z-t}\Big)\,\mathrm{d}\mu(z),\end{aligned}$$

thus proving our claim (2.3.11). Now notice that replacing h with $h\Psi_{\mathfrak{m},t}$ (recall that $\Psi_{\mathfrak{m},t}>0$ everywhere) and using again that $\mathrm{Pr}\circ\mathsf{Fl}_t=\mathrm{Pr}$, from (2.3.11) we get

$$\int h(z)\Big(\int g\circ \mathrm{Pr}\,\mathrm{d}((\mathsf{Fl}_t)_*\nu_{z-t})\Big)\,\mathrm{d}\mu(z) = \int h(z)\Big(\int g\circ \mathrm{Pr}\,\mathrm{d}\nu_z\Big)\,\mathrm{d}\mu(z).$$

To conclude we use again the fact that $(\mathsf{Fl}_t)_*\nu_{z-t}$ and ν_z are both concentrated on $\mathrm{b}^{-1}(z)$ to deduce from the above that

$$\begin{aligned}\iint h\circ \mathrm{b}\, g\circ \mathrm{Pr}\,\mathrm{d}((\mathsf{Fl}_t)_*\nu_{z-t})\,\mathrm{d}\mu(z) &= \iint h\circ \mathrm{b}\, g\circ \mathrm{Pr}\,\mathrm{d}\nu_z\,\mathrm{d}\mu(z)\\
&= \int h\circ \mathrm{b}\, g\circ \mathrm{Pr}\,\mathrm{d}\mathfrak{m}|_{\hat B},\end{aligned}$$

that is the desired (2.3.10). □

Proposition 2.3.4. *With the same assumptions and notation of Section 2.2 the following holds.*

The map $\mathsf{T}:\mathrm{X}\to\mathrm{X}'\times_w\mathbb{R}$ *(recall* (2.2.5)*) is measure preserving. In other words and recalling* (2.2.8)*, for any* $\varphi:\mathrm{X}\to\mathbb{R}^+$ *Borel we have*

$$\int\varphi\,\mathrm{d}\mathfrak{m} = \int\Big(\int\varphi\,\mathrm{d}(\mathsf{Fl}_t)_*\mathfrak{m}'\Big)w_{\mathfrak{m}}(t)\,\mathrm{d}t$$

Proof. Letting $B\subset\mathrm{X}'$ be arbitrary Borel with $\mathfrak{m}'(B)<\infty$ and letting φ be 0 outside $\hat B$, we see that to conclude it suffices to prove that we can choose $\nu_z:=\mathfrak{m}'(B)^{-1}(\mathsf{Fl}_z)_*(\mathfrak{m}'|_B)$ in the disintegration formula (2.3.8).

To see this, observe that from Lemma 2.3.3 above and Fubini's theorem we see that for a.e. z we have $(\mathsf{Fl}_t)_*\nu_z=\nu_{z+t}$ for a.e. t. Fix $\bar z$ for which this holds and for which $\nu_{\bar z}$ is concentrated on $\mathrm{b}^{-1}(\bar z)$ and then define $\bar\nu_z:=(\mathsf{Fl}_{z-\bar z})_*\nu_{\bar z}$ for any z. Then $\bar\nu_z=\nu_z$ for a.e. z, and thus the $\bar\nu_z$'s are admissible in formula (2.3.8).

To conclude it is therefore enough to show that $\bar{\nu}_0 = \mathfrak{m}'(B)^{-1}\mathfrak{m}'|_B$. To see this, let $C \subset B$ be Borel, define $\hat{C}$ as in (2.3.5) and recall the definition (2.2.4) of $\mathfrak{m}'$ to get

$$\mathfrak{m}'(C) = \Big(\int_0^1 w_{\mathfrak{m}}\,\mathrm{d}\mathcal{L}^1\Big)^{-1}\mathfrak{m}(\hat{C}\cap \mathrm{b}^{-1}([0,1]))$$
$$\overset{(2.3.8)}{=} \Big(\int_0^1 w_{\mathfrak{m}}\,\mathrm{d}\mathcal{L}^1\Big)^{-1}\mathfrak{m}'(B)\int_0^1 \bar{\nu}_t(\hat{C})w_{\mathfrak{m}}(t)\,\mathrm{d}\mathcal{L}^1(t) = \mathfrak{m}'(B)\bar{\nu}_0(C),$$

having also used the definition of $\bar{\nu}_t$ in the last equality. The conclusion follows by the arbitrariness of $C \subset B$. □

2.4 Behaviour of the distance under the flow

We shall work under the same assumptions and notation as in Section 2.2.

We start decomposing the tangent module $L^0(T\mathrm{X})$ into the submodules $V^{\|}, V^{\perp}$ of vector fields that are pointwise parallel/orthogonal to $\nabla \mathrm{b}$. Thus we put

$$V^{\|} := \{v \in L^0(T\mathrm{X}) \ : \ v = f\nabla \mathrm{b} \text{ for some } f : \mathrm{X} \to \mathbb{R}\},$$
$$V^{\perp} := \{v \in L^0(T\mathrm{X}) \ : \ \langle v, \nabla \mathrm{b}\rangle \equiv 0\}.$$

Given an arbitrary vector field $v \in L^0(T\mathrm{X})$ it components $v^{\|}, v^{\perp}$ in $V^{\|}, V^{\perp}$ respectively are defined as (recall that $|\mathrm{db}| \equiv 1$):

$$v^{\|} := \langle v, \nabla \mathrm{b}\rangle\,\nabla \mathrm{b} \qquad \text{and} \qquad v^{\perp} := v - v^{\|}.$$

To better understand the structure of $V^{\|}, V^{\perp}$, it is convenient to introduce the following two classes of functions:

$$\mathcal{G} := \{g \circ \mathrm{Pr} \ : \ g \in L^\infty \cap W^{1,2}(\mathrm{X}') \text{ with bounded support}\},$$
$$\mathcal{H} := \{h \circ \mathrm{b} \ : \ h \in L^\infty \cap W^{1,2}(\mathbb{R}) \text{ with bounded support}\}.$$

Notice that both $\mathcal{G}$ and $\mathcal{H}$ are algebra of functions. We shall typically use letters g, h for functions on $\mathrm{X}', \mathbb{R}$ respectively and $\hat{g}, \hat{h}$ for $g\circ \mathrm{Pr}, h \circ \mathrm{b}$ respectively. It is clear that $\nabla \hat{g} \in V^{\perp}$ and $\nabla \hat{h} \in V^{\|}$ and thus that

$$\begin{aligned} V^{\perp} &\supset \text{ sub-module of the tangent module generated by } \nabla\hat{g} \text{ with } \hat{g}\in\mathcal{G}, \\ V^{\|} &\supset \text{ sub-module of the tangent module generated by } \nabla\hat{h} \text{ with } \hat{h}\in\mathcal{H}. \end{aligned} \tag{2.4.1}$$

We shall prove in a moment that these inclusion are actually identities. To see this, it is convenient to introduce the algebra $\mathcal{A}$ of functions on X as

$$\mathcal{A} := \{\text{algebra of functions on X of the form } \sum_{i=1}^{n} \hat{g}_i \hat{h}_i, \text{ with } n \in \mathbb{N}\}. \quad (2.4.2)$$

We then have the following result:

Lemma 2.4.1. *With the same assumptions and notation of Section 2.2 and with the definitions just given, the following holds.*

The algebra $\mathcal{A}$ is densely contained in $W^{1,2}(\mathrm{X})$. Similarly, the algebra of functions $f : \mathrm{X}' \times \mathbb{R} \to \mathbb{R}$ such that $f \circ \mathsf{T} \in \mathcal{A}$ is densely contained in $W^{1,2}(\mathrm{X}' \times_w \mathbb{R})$.

Proof. The second claim follows directly from [GH, Section 3]. Now recall that, directly from the definition of Sobolev spaces, if $T : (\mathrm{X}_1, \mathsf{d}_1, \mathfrak{m}_1) \to (\mathrm{X}_2, \mathsf{d}_2, \mathfrak{m}_2)$ is measure preserving and biLipschitz, then

$$L^{-1} \|f\|_{W^{1,2}(\mathrm{X}_2)} \leq \|f \circ T\|_{W^{1,2}(\mathrm{X}_1)} \leq L \|f\|_{W^{1,2}(\mathrm{X}_2)},$$

where L is the biLipschitz constant. Thus the conclusion would follow if we knew that T was globally biLipschitz, as the measure preserving property comes from Proposition 2.3.4.

In general, it might be that T is only locally biLipschitz, but this is sufficient to conclude, as with a cut-off argument we see that functions with bounded support are dense in $W^{1,2}$ and we can approximate functions in $W^{1,2}(\mathrm{X} \times_w \mathbb{R})$ with bounded support with functions as in the statement with uniformly bounded support. □

From this density result it follows that gradients of functions in $\mathcal{A}$ generate the whole tangent module. Since for $f = \sum_i \hat{g}_i \hat{h}_i$ we have $\nabla f = \sum_i \hat{h}_i \nabla \hat{g}_i + \hat{g}_i \nabla \hat{h}_i$, it follows that the sub-module generated by gradients of functions in $\mathcal{G}$ and $\mathcal{H}$ is the whole $L^0(T\mathrm{X})$. Hence (2.4.1) improves into

$$\begin{aligned} V^{\perp} &= \text{sub-module of the tangent module generated by } \nabla \hat{g} \text{ with } \hat{g} \in \mathcal{G}, \\ V^{\parallel} &= \text{sub-module of the tangent module generated by } \nabla \hat{h} \text{ with } \hat{h} \in \mathcal{H}. \end{aligned} \quad (2.4.3)$$

In particular, we obtain:

$$v \in L^0(T\mathrm{X}) \quad \text{with} \quad \langle v, \nabla \hat{g} \rangle = 0 \quad \text{for every } \hat{g} \in \mathcal{G} \qquad \Rightarrow \qquad v \in V^{\parallel} \quad (2.4.4)$$

Let us now pick $\varphi \in C^2(\mathbb{R})$ with globally bounded second derivative, put $\tilde{\mathrm{b}} := \varphi \circ \mathrm{b}$ and notice that the Regular Lagrangian Flow $(\tilde{\mathsf{Fl}}_t)$ of $\nabla\tilde{\mathrm{b}}$ is globally well defined (the uniform bound on φ'' grants, among other things, that $|\nabla\tilde{\mathrm{b}}|$ grows at most linearly, that in turn shows that $\tilde{\mathsf{Fl}}_t(x)$ does not go to infinity in finite time). We claim that for some 'reparametrization map' $\mathrm{rep} : \mathbb{R}^2 \to \mathbb{R}$ locally Lipschitz we have

$$\tilde{\mathsf{Fl}}_t(x) = \mathsf{Fl}_{\mathrm{rep}(t,\mathrm{b}(x))}(x) \qquad \text{for every } t \in \mathbb{R},\ x \in \mathrm{X}. \tag{2.4.5}$$

Since $\mathsf{T} : \mathrm{X} \to \mathrm{X}' \times \mathbb{R}$ is locally biLipschitz, this will follow if we show that

$$\mathrm{Pr} \circ \tilde{\mathsf{Fl}}_t = \mathrm{Pr}, \quad \text{on } \mathrm{X}, \qquad \text{for every } t \in \mathbb{R}, \tag{2.4.6a}$$

$$\mathrm{b}(\tilde{\mathsf{Fl}}_t(x)) = f_t(\mathrm{b}(x)), \qquad \text{for every } t \in \mathbb{R},\ x \in \mathrm{X}, \tag{2.4.6b}$$

for some locally Lipschitz function $f : \mathbb{R}^2 \to \mathbb{R}$.

As for the flow (Fl_t) of $\nabla\mathrm{b}$, we have that for any $t \in \mathbb{R}$ the map $\tilde{\mathsf{Fl}}_t : \mathrm{X} \to \mathrm{X}$ is Lipschitz, with a control on the Lipschitz constant locally uniformly bounded in t. For $\hat{g} \in \mathcal{G}$ we can compute $\partial_t(g \circ \tilde{\mathsf{Fl}}_t) = \langle \nabla\hat{g}, \nabla\tilde{\mathrm{b}}\rangle \circ \tilde{\mathsf{Fl}}_t = \varphi' \circ \mathrm{b}\langle \nabla\hat{g}, \nabla\mathrm{b}\rangle \circ \tilde{\mathsf{Fl}}_t = 0$, i.e. $\hat{g} \circ \tilde{\mathsf{Fl}}_t = \hat{g}$ $\mathfrak{m}$-a.e.. Then the arbitrariness of $\hat{g}$ and a continuity argument give (2.4.6a).

To prove (2.4.6b) we start noticing that with the same continuity arguments used to deduce (2.2.2), we see that for any $x \in \mathrm{X}$ the curve $t \mapsto \mathrm{b}(\tilde{\mathsf{Fl}}_t(x))$ is C^1 and satisfies

$$\partial_t \mathrm{b}(\tilde{\mathsf{Fl}}_t(x)) = \varphi'(\mathrm{b}(\tilde{\mathsf{Fl}}_t(x))) \qquad \text{for every } t \in \mathbb{R}. \tag{2.4.7}$$

In other words, the function $g_t(x) := \mathrm{b}(\tilde{\mathsf{Fl}}_t(x))$ solves the Cauchy problem $\partial_t g_t(x) = \varphi'(g_t(x))$ with $g_0(x) = \mathrm{b}(x)$, hence its value at time t depends only on t and $\mathrm{b}(x)$, as in (2.4.6b). The local Lipschitz regularity of f then follows by standard ODE estimates (recall that φ is C^2).

Also, $\tilde{\mathsf{Fl}}_t : \mathrm{X} \to \mathrm{X}$ has bounded compression. Since this map is invertible with inverse $\tilde{\mathsf{Fl}}_{-t}$ that is also of bounded compression, it admits a differential $\mathrm{d}\tilde{\mathsf{Fl}}_t : L^0(T\mathrm{X}) \to L^0(T\mathrm{X})$ (see Theorem 1.4.6) characterized by

$$\langle \mathrm{d}\tilde{\mathsf{Fl}}_t(v), \mathrm{d}f\rangle \circ \tilde{\mathsf{Fl}}_t = \langle v, \mathrm{d}(f \circ \tilde{\mathsf{Fl}}_t)\rangle \qquad \text{for every } v \in L^0(T\mathrm{X}),\ f \in W^{1,2}(\mathrm{X}) \tag{2.4.8}$$

Arguing verbatim as for [DPG, Proposition 3.31] we obtain the following result, whose proof we omit (see also [GV, Appendix A]).

Lemma 2.4.2. *With the same assumptions and notation of Section 2.2 and with $\tilde{\mathrm{b}}$, $(\tilde{\mathsf{Fl}}_t)$ as just defined, the following holds.*

Let $v \in L^2(T\mathrm{X})$ and put $v_s := \mathrm{d}\tilde{\mathsf{Fl}}_s(v)$. Then the map $\mathbb{R} \ni s \mapsto \frac{1}{2}|v_s|^2 \circ \tilde{\mathsf{Fl}}_s \in L^1(\mathrm{X})$ is C^1 and its derivative, intended as limit of the difference quotients both strongly in L^1 and pointwise a.e., is given by

$$\partial_s\big(\tfrac{1}{2}|v_s|^2 \circ \tilde{\mathsf{Fl}}_s\big) = \mathrm{Hess}\,\tilde{\mathrm{b}}(v_s, v_s) \circ \tilde{\mathsf{Fl}}_s \qquad \textit{for every } s \in \mathbb{R}. \tag{2.4.9}$$

The special structure of the Hessian of $\tilde{\mathrm{b}}$ allows for a more explicit computation of the above. Notice that the chain rule (1.6.1) grants that $\tilde{\mathrm{b}} \in H^{2,2}_{loc}(\mathrm{X})$ with

$$\mathrm{Hess}\,\tilde{\mathrm{b}} = (\psi_{\mathsf{d}}\varphi') \circ \mathrm{b}\,(\mathrm{Id} - \mathrm{e}_1 \otimes \mathrm{e}_1) + \varphi'' \circ \mathrm{b}\,\mathrm{e}_1 \otimes \mathrm{e}_1 \tag{2.4.10}$$

Also, let us notice that

$$v \in V^{\perp} \quad \Rightarrow \quad \mathrm{d}\tilde{\mathsf{Fl}}_t(v) \in V^{\perp} \qquad \text{for every } t \in \mathbb{R}, \tag{2.4.11a}$$

$$v \in V^{\parallel} \quad \Rightarrow \quad \mathrm{d}\tilde{\mathsf{Fl}}_t(v) \in V^{\parallel} \qquad \text{for every } t \in \mathbb{R}. \tag{2.4.11b}$$

Indeed, for the first notice that

$$\langle \mathrm{d}\tilde{\mathsf{Fl}}_t(v), \nabla \mathrm{b}\rangle \circ \tilde{\mathsf{Fl}}_t \overset{(2.4.8)}{=} \langle v, \nabla(b \circ \tilde{\mathsf{Fl}}_t)\rangle \overset{(2.4.6b)}{=} (\partial_x f_t) \circ \mathrm{b}\,\langle v, \nabla \mathrm{b}\rangle = 0$$

while for the second we pick $\hat{g} \in \mathcal{G}$ and compute

$$\langle \mathrm{d}\tilde{\mathsf{Fl}}_t(v), \nabla \hat{g}\rangle \circ \tilde{\mathsf{Fl}}_t \overset{(2.4.8)}{=} \langle v, \nabla(\hat{g} \circ \tilde{\mathsf{Fl}}_t)\rangle \overset{(2.4.6a)}{=} \langle v, \nabla \hat{g}\rangle = 0,$$

thus the arbitrariness of $\hat{g}$ and (2.4.4) give the claim.

We are now ready to state and prove the main result of this section.

Proposition 2.4.3. *With the same assumptions and notation of Section 2.2 the following holds.*

Let $\boldsymbol{\pi}$ be a test plan on X. *Then for $\boldsymbol{\pi}$-a.e. γ the curve* $\mathrm{Pr}(\gamma)$ *defined as* $t \mapsto \mathrm{Pr}(\gamma_t)$ *is absolutely continuous and*

$$\mathsf{ms}(\mathrm{Pr}(\gamma), t) \le \tfrac{1}{w_{\mathsf{d}}(\mathrm{b}(\gamma_t))}\mathsf{ms}(\gamma, t) \qquad \text{for a.e. } t, \tag{2.4.12}$$

where we are writing $\mathsf{ms}(\gamma, t)$ *for the metric speed of the curve γ at the time t.*

Moreover, equality holds in (2.4.12) *provided $t \mapsto \mathrm{b}(\gamma_t)$ is constant for $\boldsymbol{\pi}$-a.e. γ i.e. if $\boldsymbol{\pi}$ is concentrated on curves lying on level sets of* b.

Proof. We follow the line of thought used to prove [DPG, Proposition 3.23], this time paying attention to the action of the flow on the parallel and perpendicular directions.

Let $\varphi(z) := -\frac{1}{2}z^2$, $v \in L^2(T\mathrm{X})$ and let $v^{\parallel}, v^{\perp} \in L^2(T\mathrm{X})$ be its components in $V^{\parallel}, V^{\perp}$, respectively. Put $v_s := \mathrm{d}\tilde{\mathsf{Fl}}_s(v)$, $v_s^{\parallel} := \mathrm{d}\tilde{\mathsf{Fl}}_s(v^{\parallel})$ and $v_s^{\perp} := \mathrm{d}\tilde{\mathsf{Fl}}_s(v^{\perp})$. Then by (2.4.9) and (2.4.10) it follows that

$$|v_s^{\parallel}| \circ \tilde{\mathsf{Fl}}_s = |v^{\parallel}|\, e^{-s}, \tag{2.4.13a}$$

$$|v_s^{\perp}| \circ \tilde{\mathsf{Fl}}_s = |v^{\perp}| \exp\Big(\int_0^s -(\mathrm{b}\,\psi_{\mathsf{d}} \circ \mathrm{b}) \circ \tilde{\mathsf{Fl}}_r\, \mathrm{d}r\Big). \tag{2.4.13b}$$

Formula (2.4.7) gives $\mathrm{b}\circ\tilde{\mathsf{Fl}}_r = e^{-r}\mathrm{b}$, thus we have

$$\int_0^s -(\mathrm{b}\psi_{\mathsf{d}}\circ\mathrm{b})\circ\tilde{\mathsf{Fl}}_r\,\mathrm{d}r = \int_0^s -\mathrm{b}e^{-r}\psi_{\mathsf{d}}(\mathrm{b}e^{-r})\,\mathrm{d}r = \int_{\mathrm{b}e^{-s}}^{\mathrm{b}} -\psi_{\mathsf{d}}(z)\,\mathrm{d}z$$
$$= \log\left(\tfrac{w_{\mathsf{d}}(\mathrm{b}e^{-s})}{w_{\mathsf{d}}(\mathrm{b})}\right)$$

and therefore

$$|v_s|\circ\tilde{\mathsf{Fl}}_s = \sqrt{|v_s^{\|}|^2\circ\tilde{\mathsf{Fl}}_s + |v_s^{\perp}|^2\circ\tilde{\mathsf{Fl}}_s} \le |v|\big(e^{-s} + \tfrac{w_{\mathsf{d}}(\mathrm{b}e^{-s})}{w_{\mathsf{d}}(\mathrm{b})}\big) \tag{2.4.14}$$

Now the inequality (2.4.12) follows along the same lines used in [DPG, Proposition 3.33]:

-) the test plan admits a derivative $\boldsymbol{\pi}'_t$ as an element of a suitable pullback of the tangent module satisfying $|\boldsymbol{\pi}'_t|(\gamma) = \mathsf{ms}(\gamma,t)$ for $\boldsymbol{\pi}$-a.e. γ and a.e. t (see Theorem 1.4.3);

-) the classical formula $\partial_t(\tilde{\mathsf{Fl}}_s(\gamma_t)) = \mathrm{d}\tilde{\mathsf{Fl}}_{s,\gamma_t}(\gamma'_t)$ admits a natural analogue in this setting if one works with derivatives of test plans in place of derivative of single curves (see Proposition 1.4.8);

-) coupling these two informations with (2.4.14) we see that for $\boldsymbol{\pi}$-a.e. γ we have

$$\mathsf{ms}(\tilde{\mathsf{Fl}}_s(\gamma),t) \le \mathsf{ms}(\gamma,t)(e^{-s} + \tfrac{w_{\mathsf{d}}(\mathrm{b}e^{-s})}{w_{\mathsf{d}}(\mathrm{b})}) \qquad \text{for a.e. } t;$$

-) the conclusion (2.4.12) follows letting $s\to\infty$ in the above, noticing that $\tilde{\mathsf{Fl}}_s \to \mathrm{Pr}$ pointwise, recalling the normalization choice (2.2.10) and using the lower semicontinuity of the metric speed.

For the equality case we argue along the following lines:

-) If the test plan $\boldsymbol{\pi}$ is concentrated on curves lying on level sets of b, then its speed is orthogonal to $\nabla\mathrm{b}$ (as in the proof of Proposition 3.4.6);

-) formula (2.4.13b) and the link between derivatives of test plans and metric speed recalled before ensure that for $\boldsymbol{\pi}$-a.e. γ we have

$$\mathsf{ms}(\tilde{\mathsf{Fl}}_s(\gamma),t) = \mathsf{ms}(\gamma,t)\tfrac{w_{\mathsf{d}}(\mathrm{b}e^{-s})}{w_{\mathsf{d}}(\mathrm{b})} \qquad \text{for a.e. } t; \tag{2.4.15}$$

-) recall that from Theorem 1.8.7 if $L := \||\mathrm{Hess}\,\mathrm{b}|_{\mathsf{HS}}\|_{L^\infty} < \infty$, then we have the bi-Lipschitz estimate

$$e^{-L|s|}\mathsf{d}(x,y) \le \mathsf{d}(\mathsf{Fl}_s(x),\mathsf{Fl}_s(y)) \le e^{L|s|}\mathsf{d}(x,y) \tag{2.4.16}$$

valid for any $x, y \in \mathrm{X}$ and $s \in \mathbb{R}$, for the flow of $\nabla \mathrm{b}$; in our setting we don't know if $\mathrm{Hess}\,\mathrm{b}$ is bounded, but with a cut-off argument based on the fact that $|\mathrm{Hess}\,\mathrm{b}|_{\mathsf{HS}}$ is bounded by a function of b, it is not hard to see that (2.4.16) still holds for $s \in [-1,1]$ and $x, y \in \mathrm{b}^{-1}([-1,1])$ for some constant L;

-) we can assume that $\boldsymbol{\pi}$ is concentrated on curves lying on a bounded set. Thus for s sufficiently big $(\tilde{\mathsf{Fl}}_s)_*\boldsymbol{\pi}$ is concentrated on curves lying in $\mathrm{b}^{-1}([-1,1])$;

-) we use the identity $\mathrm{Pr}(x) = \mathsf{Fl}_{-e^{-s}\mathrm{b}(x)}(\tilde{\mathsf{Fl}}_s(x))$ and the previous item to deduce that for all s sufficiently big the estimate

$$\begin{aligned} \mathsf{ms}(\mathrm{Pr}(\gamma), t) &\overset{(2.4.16)}{\geq} e^{-Le^{-s\mathrm{b}(\gamma_t)}} \mathsf{ms}(\tilde{\mathsf{Fl}}_s(\gamma), t) \\ &\overset{(2.4.15)}{=} e^{-Le^{-s\mathrm{b}(\gamma_t)}} \mathsf{ms}(\gamma, t) \tfrac{w_{\mathsf{d}}(\mathrm{b}e^{-s})}{w_{\mathsf{d}}(\mathrm{b})} \qquad \text{for a.e. } t \end{aligned}$$

holds for $\boldsymbol{\pi}$-a.e. γ. Letting $s \uparrow \infty$ and recalling again the normalization (2.2.10) we get the equality in (2.4.12).

□

Corollary 2.4.4. *With the same assumptions and notation of Section 2.2 the following holds.*

Let $g \in L^2_{loc}(\mathrm{X}')$ be Borel and put, as before, $\hat{g} := g \circ \mathrm{Pr}$. Then $g \in W^{1,2}_{loc}(\mathrm{X}')$ if and only if $\hat{g} \in W^{1,2}_{loc}(\mathrm{X})$ and in this case

$$|\mathrm{d}\hat{g}| = \tfrac{1}{w_{\mathsf{d}} \circ \mathrm{b}} |\mathrm{d}g| \circ \mathrm{Pr} \qquad \mathfrak{m}\text{-}a.e.. \tag{2.4.17}$$

Proof. It is clear from Proposition 2.3.4 that $g \in L^2_{loc}(\mathrm{X}')$ if and only if $\hat{g} \in L^2_{loc}(\mathrm{X})$. Now assume that $g \in W^{1,2}_{loc}(\mathrm{X}')$ and let $\boldsymbol{\pi}$ be a test plan on X concentrated on curves lying on some bounded set. Then, since Pr is locally Lipschitz and by Proposition 2.3.4, we have that $\mathrm{Pr}_*\boldsymbol{\pi}$ is a test plan on X', where with a little abuse of notation we are denoting by Pr the map sending the curve γ to the curve $t \mapsto \mathrm{Pr}(\gamma_t)$. Since g is Sobolev we have

$$\begin{aligned} \int |\hat{g}(\gamma_1) - \hat{g}(\gamma_0)| \,\mathrm{d}\boldsymbol{\pi}(\gamma) &= \int |g(\eta_1) - g(\eta_0)| \,\mathrm{d}\mathrm{Pr}_*\boldsymbol{\pi}(\eta) \\ &\leq \iint_0^1 |\mathrm{d}g|(\eta_t)|\dot{\eta}_t| \,\mathrm{d}t \,\mathrm{d}\mathrm{Pr}_*\boldsymbol{\pi}(\eta) \\ (\text{"}\eta = \mathrm{Pr}(\gamma)\text{" } + (2.4.12)) \qquad &\leq \iint_0^1 \tfrac{1}{w_{\mathsf{d}}(\mathrm{b}(\gamma_t))} |\mathrm{d}g|(\mathrm{Pr}(\gamma_t))|\dot{\gamma}_t| \,\mathrm{d}t \,\mathrm{d}\boldsymbol{\pi}(\gamma), \end{aligned}$$

thus proving, by the arbitrariness of $\boldsymbol{\pi}$, that $\hat{g} \in W^{1,2}_{loc}(\mathrm{X})$ and that inequality $\leq$ holds in (2.4.17).

Now assume that $\hat{g} \in W^{1,2}_{loc}(\mathrm{X})$ and let $\boldsymbol{\pi}'$ be a test plan on X' concentrated on curves lying on some bounded set. Fix $T > 0$, and consider the push forward $\boldsymbol{\pi}$ of the plan $\boldsymbol{\pi}' \times (\frac{1}{2T}\mathcal{L}^1|_{[-T,T]})$ via the map $(x,t) \mapsto \mathsf{Fl}_t(x)$. Then from Proposition 2.3.4, identity (2.2.2) and the fact that $\mathsf{Fl} : \mathrm{X}' \times \mathbb{R} \to \mathrm{X}$ is locally Lipschitz we see that $\boldsymbol{\pi}$ is a test plan on X concentrated on curves lying on level sets of b, thus since $\hat{g}$ is Sobolev we have

$$\begin{aligned}\int |g(\eta_1) - g(\eta_0)|\,\mathrm{d}\boldsymbol{\pi}'(\eta) &= \int |\hat{g}(\gamma_1) - \hat{g}(\gamma_0)|\,\mathrm{d}\boldsymbol{\pi}(\gamma)\\ &\leq \iint_0^1 |\mathrm{d}\hat{g}|(\gamma_t)|\dot{\gamma}_t|\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}(\gamma)\\ \text{((2.4.12) + def. } \boldsymbol{\pi}) \qquad &= \iint_0^1 \Big(\tfrac{1}{2T}\int_{-T}^{T} w_{\mathsf{d}}(s)|\mathrm{d}\hat{g}|(\mathsf{Fl}_s(\eta_t))\,\mathrm{d}s\Big)|\dot{\eta}_t|\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}'(\eta).\end{aligned}$$

Since $\boldsymbol{\pi}'$ was arbitrary, we see that g is locally Sobolev with

$$|\mathrm{d}g|(x') \leq \tfrac{1}{2T}\int_{-T}^{T} w_{\mathsf{d}}(s)|\mathrm{d}\hat{g}|(\mathsf{Fl}_s(x'))\,\mathrm{d}s \overset{*}{\leq} |\mathrm{d}g|(x') \qquad \mathfrak{m}'\text{-a.e. } x',$$

where the starred inequality comes from the already proven inequality $\leq$ in (2.4.17). Thus the starred inequality must be an equality, and this forces the equality in (2.4.17) to hold $\mathfrak{m}$-a.e. on $\mathrm{b}^{-1}([-T,T])$. The conclusion follows by the arbitrariness of T. □

2.5 Isometry and RCD condition of the quotient space

Lemma 2.5.1. *With the same notation and assumptions as in Section 2.2 the following holds.*

The warped product space $\mathbb{R} \times_w \mathrm{X}'$ has the Sobolev-to-Lipschitz property.

Proof. The arguments used in [DPG, Theorem 3.34] carry over. One first proves that X' is locally doubling (because X is and Pr is locally Lipschitz) and has the measured length property, see Definition 1.7.7 (this follows from the fact that X has such property and the estimate (2.4.12)).

Then Theorem 1.7.11 applies. □

The isomorphism between X and $\mathbb{R} \times_w \mathrm{X}'$ is reached via duality with Sobolev norms thanks to the following result (see [G13a, Proposition 4.20] for its proof).

Proposition 2.5.2 (Isomorphisms via duality with Sobolev norms)**.** *Let* $(\mathrm{X}_1, \mathsf{d}_1, \mathfrak{m}_1)$ *and* $(\mathrm{X}_2, \mathsf{d}_2, \mathfrak{m}_2)$ *be two metric measure spaces with the Sobolev-to-Lipschitz property and let* $\mathsf{T} : \mathrm{X}_1 \to \mathrm{X}_2$ *be a Borel map. Assume that both* $\mathfrak{m}_1$ *and* $\mathfrak{m}_2$ *give finite mass to bounded sets. Then the following are equivalent:*

i) up to a modification on a $\mathfrak{m}_1$*-negligible set,* T *is an isomorphism of metric measure spaces, i.e.* $\mathsf{T}_*\mathfrak{m}_1 = \mathfrak{m}_2$ *and* $\mathsf{d}_2(\mathsf{T}(x), \mathsf{T}(y)) = \mathsf{d}_1(x, y)$ *for any* $x, y \in \mathrm{supp}(\mathfrak{m}_1)$*;*

ii) the following two are true:

ii-a) there exists a Borel $\mathfrak{m}_1$*-negligible set* $\mathcal{N} \subset \mathrm{X}_1$ *and a Borel map* $\mathsf{S} : \mathrm{X}_2 \to \mathrm{X}_1$ *such that* $\mathsf{S}(\mathsf{T}(x)) = x$ *for every* $x \in \mathrm{X}_1 \setminus \mathcal{N}$*;*

ii-b) the right composition with T *produces a bijective isometry of the Sobolev space* $W^{1,2}(\mathrm{X}_2)$ *in* $W^{1,2}(\mathrm{X}_1)$*, i.e.* $f \in W^{1,2}(\mathrm{X}_2)$ *if and only if* $f \circ \mathsf{T} \in W^{1,2}(\mathrm{X}_1)$ *and in this case* $\|f\|_{W^{1,2}(\mathrm{X}_2)} = \|f \circ \mathsf{T}\|_{W^{1,2}(\mathrm{X}_1)}$.

Theorem 2.5.3. *With the same notation and assumptions as in Section 2.2 the following holds.*

The map $\mathsf{T} : \mathrm{X} \to \mathbb{R} \times_w \mathrm{X}'$ *is a measure preserving isometry.*

Proof. Since we already know that T is measure preserving (by Proposition 2.3.4) and both X and $\mathbb{R} \times_w \mathrm{X}'$ have the Sobolev-to-Lipschitz property, according to Proposition 2.5.2 it is sufficient to prove that $f \in W^{1,2}(\mathbb{R} \times_w \mathrm{X}')$ if and only if $f \circ \mathsf{T} \in W^{1,2}(\mathrm{X})$ and in this case

$$|\mathrm{d}f|_{\mathbb{R}\times_w \mathrm{X}'} \circ \mathsf{T} = |\mathrm{d}(f \circ \mathsf{T})|_{\mathrm{X}} \qquad \mathfrak{m}\text{-a.e.}. \tag{2.5.1}$$

By a density argument based on Lemma 2.4.1, it suffices to prove the above for functions f such that $f \circ \mathsf{T} \in \mathcal{A}$.

Corollary 2.4.4 and (1.7.1) ensure that (2.5.1) holds if $f \circ \mathsf{T} \in \mathcal{G}$, while (2.2.1a) give that (2.5.1) holds for $f \circ \mathsf{T} \in \mathcal{H}$ (see also the arguments used in [G13a, Section 6.2]). The conclusion now follows exactly as in [G13a, Section 6.2] (see also [DPG, Section 3.8]) using the fact that both X and $\mathbb{R} \times_w \mathrm{X}'$ are infinitesimally Hilbertian (X' is so because of identity (2.4.17), then the property carries to warped products by Theorem 1.7.6) and that functions in $\mathcal{G}$ and $\mathcal{H}$ have orthogonal gradients. We omit the details. □

Proposition 2.5.4. *With the same notation and assumptions as in Section 2.2 the following holds.*

Assume furthermore that for some $\bar{z} \in \mathbb{R}$ *we have* $\psi_{\mathsf{d}} \le 0$ *on* $(-\infty, \bar{z}]$ *and* $\psi_{\mathsf{d}} \ge 0$ *on* $[\bar{z}, +\infty)$ *(thus in particular* $\psi_{\mathsf{d}}(\bar{z}) = 0$*). Then* $(\mathrm{X}', \mathsf{d}', \mathfrak{m}')$ *is an* $\mathsf{RCD}(\frac{1}{w_{\mathsf{d}}^2(\bar{z})}K, N)$ *space.*

Proof. As already noticed, by Corollary 2.4.4 it easily follows that X' is infinitesimally Hilbertian, so we need only to prove the $\mathsf{CD}(K,N)$ condition.

For $z\in\mathbb{R}$ let $\mathrm{X}'_z:=\mathrm{b}^{-1}(z)$ and equip it with the distance d'_z defined as

$$\mathsf{d}'_z(x,y)^2:=\inf\int_0^1|\dot\gamma_t|^2\,\mathrm{d}t,$$

the inf being taken among all absolutely continuous curves $\gamma:[0,1]\to\mathrm{X}'_z\subset\mathrm{X}$. We also equip X'_z with the measure $\mathfrak{m}'_z:=(\mathsf{Fl}_z)_*\mathfrak{m}'$. Then using the equality case in Proposition 2.4.3 and using the fact that X' has the measured length property (briefly mentioned in the proof of Lemma 2.5.1 above) it is not hard to see that $\mathsf{Fl}_z:\mathrm{X}'\to\mathrm{X}'_z$ satisfies

$$\mathsf{d}'_z(\mathsf{Fl}_z(x),\mathsf{Fl}_z(y))=w_{\mathsf{d}}(z)\mathsf{d}'(x,y)\qquad\forall x,y\in\mathrm{X}'.$$

Thus if we establish that $\mathrm{X}'_{\bar z}$ is $\mathsf{RCD}(K,N)$, taking into account how the CD condition scales with the distance (see [S06b, Proposition 1.4]) we conclude. With this said, replacing b with $\mathrm{b}+\bar z$ we can assume that $\bar z=0$ and then the goal is to prove that X' is $\mathsf{RCD}(K,N)$.

The key geometric property that allows us to conclude, and for which we shall use the assumption on ψ_{d}, is the following:

$$\begin{aligned}&\text{Let } \mu_0,\mu_1\in\mathscr{P}(\mathrm{X}) \text{ be } \ll\mathfrak{m} \text{ and } \boldsymbol{\pi} \text{ so that } W_2^2(\mu_0,\mu_1)=\iint_0^1|\dot\gamma_t|^2\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}(\gamma).\\&\text{Assume that } \mathrm{supp}(\mu_i)\subset\mathrm{b}^{-1}([-T,T]),\ i=0,1.\\&\text{Then } \boldsymbol{\pi} \text{ is concentrated on curves lying on } \mathrm{b}^{-1}([-T,T]).\end{aligned}\tag{2.5.2}$$

To see why this holds, let $\varphi:\mathbb{R}\to\mathbb{R}$ be C^2 with bounded second derivative, convex, identically 0 on $[-T,T]$ and strictly positive elsewhere. Then the Regular Lagrangian Flow $(\tilde{\mathsf{Fl}}_t)$ of $\nabla\tilde b$ with $\tilde{\mathrm{b}}=\varphi\circ\mathrm{b}$ is the identity on the strip $\mathrm{b}^{-1}([-T,T])$ and converges to the 'projection on the boundary of such strip' outside of it, namely defining $\mathrm{Pr}_T:\mathrm{X}\to\mathrm{X}$ as:

$$\mathrm{Pr}_T(x):=\begin{cases}x, & \text{if } \mathrm{b}(x)\in[-T,T],\\ \mathsf{Fl}_{T-\mathrm{b}(x)}(x), & \text{if } \mathrm{b}(x)>T,\\ \mathsf{Fl}_{-T-\mathrm{b}(x)}(x), & \text{if } \mathrm{b}(x)<-T,\end{cases}$$

we have $\tilde{\mathsf{Fl}}_s(x)\to\mathrm{Pr}_T(x)$ as $s\uparrow\infty$ for any $x\in\mathrm{X}$. Then arguing exactly as for Proposition 2.4.3 we obtain that: for every test plan $\boldsymbol{\pi}$ we have that for $\boldsymbol{\pi}$-a.e. γ it holds

$$\mathsf{ms}(\mathrm{Pr}_T(\gamma),t)\leq\tfrac{w_{\mathsf{d}}(\mathrm{b}(\mathrm{Pr}_T(\gamma_t)))}{w_{\mathsf{d}}(\mathrm{b}(\gamma_t))}\mathsf{ms}(\gamma,t)\qquad\text{for a.e. } t.\tag{2.5.3}$$

Now recall that by (2.2.9b) we have $\log(\frac{w_{\mathsf{d}}(z_2)}{w_{\mathsf{d}}(z_1)}) = \int_{z_1}^{z_2} \psi_{\mathsf{d}}\, \mathrm{d}\mathcal{L}^1$ for every $z_1, z_2 \in \mathbb{R}$, $z_1 < z_2$ and use the assumption on ψ_{d} (with $\bar{z} = 0$) to conclude from (2.5.3) that for $\boldsymbol{\pi}$-a.e. γ it holds

$$\mathsf{ms}(\mathrm{Pr}_T(\gamma), t) \leq \mathsf{ms}(\gamma, t) \qquad \text{for a.e. } t. \tag{2.5.4}$$

It directly follows from this that the total kinetic energy of the test plan $\mathsf{KE}(\boldsymbol{\pi}) := \frac{1}{2}\iint_0^1 |\dot{\gamma}_t|^2\, \mathrm{d}t\, \mathrm{d}\boldsymbol{\pi}(\gamma)$ does not increase under the action of Pr_T. In particular, this occurs for $\boldsymbol{\pi}$ as in (2.5.2), thus obtaining $\mathsf{KE}((\mathrm{Pr}_T)_*\boldsymbol{\pi}) \leq \frac{1}{2}W_2^2(\mu_0, \mu_1)$. On the other hand, we know from [GRS] that for $\mu_0, \mu_1 \ll \mathfrak{m}$ there is exactly one plan $\boldsymbol{\pi}$ for which $\mathsf{KE}(\boldsymbol{\pi}) \leq \frac{1}{2}W_2^2(\mu_0, \mu_1)$, thus we conclude that $(\mathrm{Pr}_T)_*\boldsymbol{\pi} = \boldsymbol{\pi}$, which is the claim (2.5.2).

A direct consequence of (2.5.2) and the very definition of $\mathsf{CD}(K, N)$ spaces (see Section 1.5.1) is that the space $(\mathrm{X}_T, \mathsf{d}_T, \mathfrak{m}_T)$ given by $\mathrm{X}_T := \mathrm{b}^{-1}([-T, T])$ with d_T being the restriction of the distance and $\mathfrak{m}_T := \frac{1}{\int_{-T}^{T} w_{\mathfrak{m}}}\mathfrak{m}|_{\mathrm{b}^{-1}(-T,T)}$ is $\mathsf{CD}(K, N)$ as a consequence of $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ being so (recall that a scaling of the measure does not affect Curvature-Dimension bounds). Therefore by the stability of the CD condition (see Theorem 1.5.23) to conclude it suffices to prove that for any fixed $p \in \mathrm{X}'$ the spaces $(\mathrm{X}_T, \mathsf{d}_T, \mathfrak{m}_T, p)$ converge to $(\mathrm{X}', \mathsf{d}', \mathfrak{m}', p)$ as $T \downarrow 0$ in the pointed-measured-Gromov-Hausdorff sense.

To see this, consider the map $\mathrm{Pr} : \mathrm{X}_T \to \mathrm{X}'$ and notice that $\mathrm{Pr}(p) = p$, that $\mathrm{Pr}_*\mathfrak{m}_T = \mathfrak{m}'$ (by (2.2.4)) and that

$$|\mathsf{d}(\mathrm{Pr}(x), \mathrm{Pr}(y)) - \mathsf{d}(x, y)| \leq \mathsf{d}(\mathrm{Pr}(x), x) + \mathsf{d}(\mathrm{Pr}(y), y) \leq 2T \qquad \forall x, y \in \mathrm{X}_T.$$

This suffices to prove that $(\mathrm{X}_T, \mathsf{d}_T, \mathfrak{m}_T, p)$ pmGH-converge to $(\mathrm{X}', \mathsf{d}, \mathfrak{m}', p)$, which therefore is a $\mathsf{CD}(K, N)$ space, so we are left to prove that $\mathsf{d} = \mathsf{d}'$ on X'. To see this, notice that since $(\mathrm{X}', \mathsf{d}, \mathfrak{m}')$ is $\mathsf{CD}(K, N)$ and $\mathrm{supp}(\mathfrak{m}') = \mathrm{X}'$ (as direct consequence of the assumption $\mathrm{supp}(\mathfrak{m}) = \mathrm{X}$), we have that $(\mathrm{X}', \mathsf{d})$ is a geodesic space, i.e. given $x, y \in \mathrm{X}'$ there is a curve $\gamma : [0, 1] \to \mathrm{X}'$ with $\int_0^1 |\dot{\gamma}_t|^2\, \mathrm{d}t = \mathsf{d}^2(x, y)$. It follows by the very definition of d' that $\mathsf{d}' \leq \mathsf{d}$, and since the other inequality is trivially true, the proof is complete. □

Chapter 3

Applications to spaces with positive spectrum

In this chapter, using the general strategy saw in Chapter 2, we prove Theorems 0.0.7 and 0.0.8.

All we have to do is to construct functions with the properties required in Lemma 2.1.1, use them to find the Busemann functions and conclude with Theorem 2.2.1.

In order to find the required functions we will need other preliminary definitions and results that we show in the next two sections.

3.1 Preliminaries: harmonic functions and Bochner inequality

The construction of the Busemann functions we need is based on the strong maximum principle for harmonic functions. In this section we recall the definition of harmonic function and the statement of the maximum principles in the non-smooth setting. Moreover we use the notion of measure valued Laplacian to state an improved version of the Bochner inequality.

We start recalling the definition of measure valued Laplacian, a generalization of the notion of Laplacian that we have seen in Definition 1.3.21 given by Gigli in [G15].

Definition 3.1.1 (Measure valued Laplacian). *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an infinitesimally Hilbertian space,* $U \subset \mathrm{X}$ *open and* $f \in W^{1,2}_{loc}(\mathrm{X})$*. We shall say that* $f \in D(\mathbf{\Delta}_{loc}, U)$ *provided there exists a Radon functional* μ *such that*

$$\int \langle \nabla f, \nabla \varphi \rangle \,\mathrm{d}\mathfrak{m} = -\int \varphi \,\mathrm{d}\mu$$

for every $\varphi \in \mathrm{LIP}(\mathrm{X})$ *with support bounded and contained in* U.

In this case the measure μ *is denoted by* $\mathbf{\Delta} f$. *In case* $U = \mathrm{X}$ *we simply write* $f \in D(\mathbf{\Delta}_{loc})$.

Recall that a *Radon functional* is a linear functional L from the space $C_{bs}(\mathrm{X})$ of continuous functions on X with bounded support to $\mathbb{R}$ such that for every $K \subset \mathrm{X}$ there is $C_K \geq 0$ such that

$$|L(f)| \leq C_K \sup |f| \qquad \text{for every } f \in C_{bs}(\mathrm{X}) \text{ with support in } K.$$

Thus Radon functionals should be thought of (and we shall do so) as signed measures that have finite total variation on compact sets, but that in principle might have both positive and negative parts of infinite mass, see also the discussion in [CaMo20], where it was observed that Radon functionals are the correct object to use in place of the Radon measures used in [G15]. Notice also that in the field of non-smooth analysis some authors use the term Radon measure for what we are calling here Radon functionals, see for instance [AB, Remark 2.12].

There are natural compatibility conditions between this notion of Laplacian and the one in Definition 1.3.21, in particular the following proposition holds (see [G15, Chapter 4]).

Proposition 3.1.2. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an infinitesimally Hilbertian metric measure space. Assume that* $f \in D(\mathbf{\Delta}_{loc})$ *and* $\mathbf{\Delta} f = g\mathfrak{m}$ *for a certain* $g \in L^2_{loc}(\mathrm{X})$. *Then* $f \in D(\Delta_{loc})$ *with* $\Delta f = g$.

Moreover, as the Laplacian Δ, the measure valued Laplacian satisfies natural calculus rules.

Proposition 3.1.3 (Leibniz rule for the measure valued Laplacian)**.** *Let* $f \in D(\mathbf{\Delta}_{loc}, U)$ *and let* $g \in \mathrm{LIP}_{loc}(\mathrm{X}) \cap D(\mathbf{\Delta}_{loc}, U)$. *Then* $fg \in D(\mathbf{\Delta}_{loc}, U)$ *and it holds the following formula:*

$$\mathbf{\Delta}(fg) = f\mathbf{\Delta} g + g\mathbf{\Delta} f + \langle \nabla f, \nabla g \rangle \mathfrak{m}_{|_U}.$$

Proposition 3.1.4 (Chain rule for the measure valued Laplacian)**.** *Let* $f \in D(\mathbf{\Delta}_{loc}, U) \cap \mathrm{LIP}_{loc}(\mathrm{X})$ *and let* $\varphi \in C^2(\mathbb{R})$. *Then* $\varphi \circ f \in D(\mathbf{\Delta}_{loc}, U)$ *and it holds the following formula:*

$$\mathbf{\Delta}(\varphi \circ f) = \varphi' \circ f \mathbf{\Delta} f + \varphi'' \circ f |\nabla f|^2 \mathfrak{m}_{|_U}. \tag{3.1.1}$$

These concepts of Laplacian are linked to energy minimizers via Theorems 3.1.5 and 3.1.7 (see [G15], [GMo] and [GR19]).

Theorem 3.1.5. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a proper infinitesimally Hilbertian metric measure space,* $U \subset \mathrm{X}$ *open and* $f \in L^2_{loc}(U)$*. Then the following are equivalent:*

-) $f \in D(\Delta_{loc}, U)$ *and* $\Delta f = 0$*;*

-) $f \in D(\boldsymbol{\Delta}_{loc}, U)$ *and* $\boldsymbol{\Delta} f = 0$*;*

-) *for any* $g \in W^{1,2}_0(U)$ *we have*

$$\int_U |\mathrm{d}f|^2 \,\mathrm{d}\mathfrak{m} \leq \int_U |\mathrm{d}(f+g)|^2 \,\mathrm{d}\mathfrak{m}. \tag{3.1.2}$$

Definition 3.1.6 (Harmonic function)**.** *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a proper infinitesimally Hilbertian space,* $U \subset \mathrm{X}$ *be open and* $f \in L^2_{loc}(U)$*. We say that* f *is harmonic if it satisfies any of the three equivalent conditions of Theorem 3.1.5.*

In a similar way we can define sub/super-harmonic functions. Indeed, similarly to Theorem 3.1.5, the following holds.

Theorem 3.1.7. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a proper infinitesimally Hilbertian metric measure space,* $U \subset \mathrm{X}$ *open and* $f \in L^2_{loc}(U)$*. Then the following are equivalent:*

-) $f \in D(\boldsymbol{\Delta}_{loc}, U)$ *and* $\boldsymbol{\Delta} f \geq 0$ *(resp.* ≤ 0*);*

-) (3.1.2) *holds for every* g *negative (resp. positive)* $W^{1,2}_0(U)$ *function.*

Definition 3.1.8 (Sub/Super-harmonic function)**.** *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a proper infinitesimally Hilbertian,* $U \subset \mathrm{X}$ *be open and* $f \in L^2_{loc}(U)$*. We say that* f *is subharmonic (resp. superharmonic) if it satisfies any of the two equivalent conditions of Theorem 3.1.7.*

We state now the weak and strong maximum principle in the non-smooth setting. For their proof see [BB] (see also [GR19] for a different proof).

Theorem 3.1.9 (Weak maximum principle)**.** *Let* $K \in \mathbb{R}$ *and let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathsf{RCD}(K, \infty)$ *space. Let* U *be an open bounded subset of* X *and let* $f \in W^{1,2}_{loc}(U) \cap C(\bar{U})$ *be subharmonic. Then*

$$\sup_U f \leq \sup_{\partial U} f.$$

Theorem 3.1.10 (Strong maximum principle)**.** *Let $K \in \mathbb{R}$ and $N \in [1, +\infty)$, and let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be an $\mathsf{RCD}(K, N)$ space. Let U be an open and connected subset of X and let $f \in W^{1,2}_{loc}(U) \cap C(\bar{U})$ be subharmonic and such that for some $\bar{x} \in U$ it holds*

$$f(\bar{x}) = \max_{x \in \bar{U}} f(x).$$

Then f is constant.

In order to study the properties of specific harmonic functions we will rely on the following regularity result for harmonic functions, that extend to the RCD setting a classical gradient estimate due to Cheng and Yau (see [CY]) in the smooth setting. Its proof in the RCD setting can be found in [HKX].

Theorem 3.1.11 (Cheng-Yau type gradient estimate)**.** *Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be an $\mathsf{RCD}(K, N)$ metric measure space with $K \leq 0$ and $N \in [1, \infty)$. Then there exists a constant C depending only on N such that every positive harmonic function u on any geodesic ball $B_{2R} \subset \mathrm{X}$ satisfies*

$$\frac{|\mathrm{d}u|}{u} \leq C \frac{1 + \sqrt{-K}R}{R} \qquad \text{in } B_R. \tag{3.1.3}$$

We conclude this introductory section recalling (a suitable version of) the Bochner inequality. In order to state it, we need to recall the concept of *essential dimension* $\dim(\mathrm{X})$ of a finite dimensional RCD space.

Theorem 3.1.12. *Let X be an $\mathsf{RCD}(K, N)$ space with $K \in \mathbb{R}$ and $N \in [1, +\infty)$. Then there exists an integer $\dim(\mathrm{X}) \in [1, N]$ such that the tangent module $L^0(T\mathrm{X})$ has constant dimension equal to $\dim(\mathrm{X})$.*

The proof of this result is highly non-trivial, and ultimately coming from [BS20a] (but see also [G13a], [MN], [DPR], [DPMR], [KM], [GP22], [GP21]).

We can now state the desired inequality.

Theorem 3.1.13 (Improved Bochner Inequality)**.** *Let X be an $\mathsf{RCD}(K, N)$ space with $K \in \mathbb{R}$ and $N \in [1, +\infty)$. Then for any $f \in \mathrm{Test}_{loc}(\mathrm{X})$ we have $|\mathrm{d}f|^2 \in D(\mathbf{\Delta}_{loc})$ and*

$$\mathbf{\Delta}\left(\tfrac{|\mathrm{d}f|^2}{2}\right) \geq \left(|\operatorname{Hess}(f)|^2_{\mathsf{HS}} + K|\mathrm{d}f|^2 + \langle \mathrm{d}f, \mathrm{d}\Delta f\rangle + \tfrac{(\Delta f - \operatorname{tr}\operatorname{Hess}(f))^2}{N - \dim(\mathrm{X})}\right)\mathfrak{m}, \tag{3.1.4}$$

where $\frac{(\Delta f - \operatorname{tr}\operatorname{Hess}(f))^2}{N-n}$ is taken to be 0 in the case $\dim(\mathrm{X}) = N$.

This result was proved in [Han] (strongly based on the earlier [GKO], [AGS14b], [EKS], [G18b]).

3.2 Preliminaries: sets of finite perimeter and Gauss-Green formula

In this section we recall the statement of the Gauss-Green formula for RCD spaces proved by E. Bruè, E. Pasqualetto and D. Semola in [BPS]. More precisely, we give the definitions and results in [DGP] and [BPS] that allow us to generalize in the non-smooth setting the formula

$$\int_E \operatorname{div}(v)\,\mathrm{d}x = \int_{\partial E} v \cdot \nu \,\mathrm{d}\mu_{\partial E}, \tag{3.2.1}$$

where E is a regular bounded subset of $\mathbb{R}^n$, v is a C^1 vector field on a neighborhood of $\bar{E}$, $\mu_{\partial E}$ is the $(n-1)$-dimensional measure on ∂E and ν is the outward normal vector field defined on ∂E.

With the structure we presented in the previous sections we are already able to write the left hand side of (3.2.1).

To define the measure on ∂E we recall the notion of perimeter of a set.

Definition 3.2.1 (Perimeter and sets of finite perimeter). *Given a Borel set $E \subset \mathrm{X}$ and an open set $A \subset \mathrm{X}$ we define the perimeter* $\operatorname{Per}(E, A)$ *as*

$$\operatorname{Per}(E, A) := \inf \left\{ \liminf_{n\to+\infty} \int_A |\mathrm{d}u_n| \,\mathrm{d}\mathfrak{m} : u_n \in \operatorname{LIP}_{loc}(A), u_n \xrightarrow{L^1_{loc}(A,\mathfrak{m})} \chi_E \right\}.$$

We say that E has finite perimeter if $\operatorname{Per}(E) < +\infty$. *In this case it can be proved that the set function $A \mapsto \operatorname{Per}(E, A)$ is the restriction to open sets of a finite and positive Borel measure* $\operatorname{Per}(E, \cdot)$ *on* X *defined by*

$$\operatorname{Per}(E, B) := \inf \left\{ \operatorname{Per}(E, A) : A \text{ open and } A \supset B \right\}.$$

The measure $\operatorname{Per}(E, \cdot)$ is usually denoted with $|D\chi_E|$: this notation comes from the fact that a set has finite perimeter if and only if its characteristic function χ_E has bounded variation, and in this case its total variation coincides with the perimeter measure.

An important result about sets of finite perimeter is the coarea formula that is well known to be valid even in the metric setting (see [BPS], [BCM] and references therein, starting from the original [Mir], where the first instance of the coarea formula in the metric setting has been obtained).

Theorem 3.2.2 (Coarea formula). *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathsf{RCD}(K, N)$ *metric measure space with $N < +\infty$ and let $U \subset \mathrm{X}$ open. Let $u \in \operatorname{LIP}_{loc}(U)$ be positive and such that $u^{-1}([a, b])$ is compact in U for every $[a, b] \subset (0, 1)$.*

Then $\{u<t\}$ *has finite perimeter for a.e.* $t\in(0,1)$*, and for any* $f:U\to[-\infty,+\infty]$ *Borel and in* $L^1_{loc}(U,|\nabla u|\mathfrak{m}|_U)$ *it holds that*

$$\int_U \varphi(u)f|\mathrm{d}u|\,\mathrm{d}\mathfrak{m}=\int_U \varphi(t)\int f\mathrm{d}\,\mathrm{Per}(\{u<t\},\cdot)\,\mathrm{d}t$$

for every $\varphi:[0,1]\to\mathbb{R}$ *Borel with* $\mathrm{supp}(\varphi)\subset(0,1)$.

The other instrument we need is the outward normal ν_E to a set E with finite perimeter. We note that the theory of L^0-modules is not enough for this, since vector fields as we introduced them are defined only $\mathfrak{m}$-a.e..

3.2.1 Capacity and quasi-continuous functions

In this section we introduce the concept of capacity of a set, and thanks to that we define quasi-continuous functions. The results in this section are classical (see for instance [EG]).

Definition 3.2.3 (Capacity)**.** *Let* X *be a metric measure space and fix* $E\subset\mathrm{X}$*. We indicate with* $\mathscr{F}_E$ *the set of* $W^{1,2}(X)$ *functions* f *which satisfy* $f_{|_U}\geq 1$ *for some* U *open neighborhood of* E*.*

The capacity of E*,* $\mathrm{Cap}(E)\in[0,+\infty]$ *is defined as*

$$\mathrm{Cap}(E):=\begin{cases}\inf\limits_{f\in\mathscr{F}_E}\|f\|^2_{W^{1,2}(X)} & \text{if } \mathscr{F}_E\neq\emptyset;\\ +\infty & \text{if } \mathscr{F}_E=\emptyset\end{cases}$$

Proposition 3.2.4. *The capacity* Cap *is a submodular (for every* $E,F\subset\mathrm{X}$ *it holds* $\mathrm{Cap}(E\cup F)+\mathrm{Cap}(E\cap F)\leq\mathrm{Cap}(E)+\mathrm{Cap}(F)$*) outer measure on* X*. Moreover it is bounded on bounded sets and for every Borel set* $E\subset\mathrm{X}$ *it holds* $\mathfrak{m}(E)\leq\mathrm{Cap}(E)$*.*

Remark 3.2.5. Via Cavalieri's formula we can define the integral with respect to outer measures. In our case, for every function $f:X\to[0,+\infty]$ we define

$$\int f\,\mathrm{dCap}:=\int_0^{+\infty}\mu(\{f>t\})\,\mathrm{d}t. \tag{3.2.2}$$

Since Cap is submodular then the integral defined in (3.2.2) is subadditive, and this is used to prove that the function $\mathsf{d}_{\mathrm{Cap}}$ below is a distance. ■

Definition 3.2.6 (The space $L^0(\mathrm{Cap})$)**.** *We say that two Borel functions* $f,g:\mathrm{X}\to\mathbb{R}$ *are equal* Cap*-a.e. if it holds* $\mathrm{Cap}(\{f\neq g\})=0$*, and we* *indicate with* $L^0(\mathrm{Cap})$ *the set of all equivalence classes of Borel functions up to* Cap-*a.e. equality.*

Moreover, let $(A_k)_k$ *be an increasing sequence of open subsets of* X *with finite capacity such that for every bounded* $B \subset \mathrm{X}$ *there exists a* $k \in \mathbb{N}$ *such that* $B \subset A_k$*. We define the following distance on* $L^0(\mathrm{Cap})$*:*

$$\mathsf{d}_{\mathrm{Cap}}(f,g) := \sum_{k\in\mathbb{N}} \frac{1}{2^k(\mathrm{Cap}(A_k)\vee 1)} \int_{A_k} |f-g| \wedge 1 \,\mathrm{dCap}.$$

Proposition 3.2.7. *The metric space* $(L^0(\mathrm{Cap}), \mathsf{d}_{\mathrm{Cap}})$ *is complete.*

Definition 3.2.8 (Quasi-continuous function)**.** *A function* $f : \mathrm{X} \to \mathbb{R}$ *is said to be quasi-continuous if for every* $\varepsilon > 0$ *there exists a Borel set* $E \subset \mathrm{X}$ *with* $\mathrm{Cap}(E) < \varepsilon$ *such that the function* $f_{|\mathrm{X}\setminus E} : \mathrm{X} \setminus E \to \mathbb{R}$ *is continuous.*

Remark 3.2.9. By definitions easily follows that the set of quasi-continuous functions (up to equality Cap-a.e.) is contained in $L^0(\mathrm{Cap})$. ■

Theorem 3.2.10 (Quasi-continuous representative of Sobolev functions)**.** *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an infinitesimally Hilbertian metric measure space. Then there exists a unique continuous map* $\mathsf{QCR} : W^{1,2}(\mathrm{X}) \to L^0(\mathrm{Cap})$ *such that for every* $f \in W^{1,2}(\mathrm{X})$ *the function* $\mathsf{QCR}(f)$ *is quasi-continuous and* $\mathfrak{m}$*-a.e. it holds* $\mathsf{QCR}(f) = f$*.*

3.2.2 $L^0(\mathrm{Cap})$-modules and quasi-continuous vector fields

In this section we define the tangent $L^0(\mathrm{Cap})$-module and see, similarly to quasi-continuous functions, that Sobolev vector fields have a quasi-continuous representative. The proofs of the results in this section can be found in [DGP].

Similarly to what we saw in Definition 1.3.2, we define $L^0(\mathrm{Cap})$-normed $L^0(\mathrm{Cap})$-modules.

Definition 3.2.11 ($L^0(\mathrm{Cap})$-normed $L^0(\mathrm{Cap})$-module)**.** *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space. We say that a quadruple* $(\mathscr{M}, \tau, \cdot, |\cdot|)$ *is a* $L^0(\mathrm{Cap})$*-normed* $L^0(\mathrm{Cap})$*-module over* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *if the following hold:*

i) $(\mathscr{M}, \tau)$ *is a topological vector space;*

ii) the bilinear map $\cdot : L^0(\mathrm{Cap}) \times \mathscr{M} \to \mathscr{M}$ *satisfies*

$$f\cdot(g\cdot v) = (fg)\cdot v \qquad \text{for every } f,g \in L^0(\mathrm{Cap}) \text{ and } v \in \mathscr{M},$$
$$\hat{1}\cdot v = v \qquad \text{for every } v \in \mathscr{M},$$

where $\hat{1}$ *is the function identically equal to 1 on* X*;*

iii) the pointwise norm $|\cdot|:\mathscr{M}\to L^0(\mathrm{Cap})$ *satisfies*

$$\begin{aligned} |v| &\geq 0 && \text{for every } v\in\mathscr{M},\text{ equality if and only if } v=0,\\ |v+w| &\leq |v|+|w| && \text{for every } v,w\in\mathscr{M},\\ |f\cdot v| &= |f||v| && \text{for every } f\in L^0(\mathrm{Cap}) \text{ and } v\in\mathscr{M}, \end{aligned}$$

where all equalities and inequalities are intended in the Cap*-a.e. sense;*

iv) taking $(A_k)_k$ *as in Definition 3.2.6, the distance* $\mathsf{d}_{\mathscr{M}}$ *on* $\mathscr{M}$ *defined as*

$$\mathsf{d}_{\mathscr{M}}(v,w):=\sum_{k\in\mathbb{N}}\frac{1}{2^k(\mathrm{Cap}(A_k)\vee 1)}\int_{A_k}|v-w|\wedge 1\,\mathrm{dCap}$$

for every $v,w\in\mathscr{M}$ *is complete and induces the topology* τ*.*

Definition 3.2.12 (Hilbert $L^0(\mathrm{Cap})$-module)**.** *Let* $\mathscr{M}$ *be a* $L^0(\mathrm{Cap})$*-normed* $L^0(\mathrm{Cap})$*-module. We say that it is a Hilbert module if the parallelogram rule*

$$|v+w|^2+|v-w|^2=2(|v|^2+|w|^2)$$

holds Cap*-a.e. in* X *for every* $v,w\in\mathscr{M}$*.*

With the following theorem we define the tangent $L^0(\mathrm{Cap})$-module.

Theorem 3.2.13 (Tangent $L^0(\mathrm{Cap})$-module)**.** *Let* $K\in\mathbb{R}$ *and let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be an* $\mathsf{RCD}(K,\infty)$ *space. Then there exists a unique couple* $(L^0_{\mathrm{Cap}}(T\mathrm{X}),\bar\nabla)$ *where* $L^0_{\mathrm{Cap}}(T\mathrm{X})$ *is an* $L^0(\mathrm{Cap})$*-module over* X *and* $\bar\nabla:\mathrm{Test}(\mathrm{X})\to L^0_{\mathrm{Cap}}(T\mathrm{X})$ *is a linear operator satisfying the following properties:*

i) for every $f\in\mathrm{Test}(\mathrm{X})$ *the equality* $|\bar\nabla f|=|\,\mathsf{QCR}(|Df|)|$ *holds* Cap*-a.e. on* X*;*

ii) the space

$$\left\{\sum_{n\in\mathbb{N}}\chi_{E_n}\bar\nabla f_n:(f_n)_{n\in\mathbb{N}}\subset\mathrm{Test}(\mathrm{X})\text{ and }(E_n)_{n\in\mathbb{N}}\text{ Borel partition of }\mathrm{X}\right\}$$

is dense in $L^0_{\mathrm{Cap}}(T\mathrm{X})$*.*

Uniqueness is intended up to unique isomorphism.

The space $L^0_{\mathrm{Cap}}(T\mathrm{X})$ *is called tangent* $L^0(\mathrm{Cap})$*-module, its elements are the* Cap*-vector fields, and the operator* $\bar\nabla$ *is called gradient.*

Proposition 3.2.14. *The tangent* $L^0(\mathrm{Cap})$*-module* $L^0_{\mathrm{Cap}}(T\mathrm{X})$ *is a Hilbert module.*

We define, similarly to what we did for test vector fields and $H^{1,2}_C$ in Section 1.6, test Cap-vector fields and the set of quasi-continuous vector fields.

Definition 3.2.15 (Test Cap-vector fields). *We define the set of test* Cap-*vector fields* $\mathrm{Test}\bar{\mathrm{V}} \subset L^0_{\mathrm{Cap}}(T\mathrm{X})$ *as*

$$\mathrm{Test}\bar{\mathrm{V}}(\mathrm{X}) := \left\{ \sum_{i=0}^{n} \mathsf{QCR}(g_i)\bar{\nabla} f_i : n \in \mathbb{N},\ (f_i)_{i=0}^n, (g_i)_{i=0}^n \subset \mathrm{Test}(\mathrm{X}) \right\}.$$

Definition 3.2.16 (Quasi-continuous Cap-vector field). *The set of quasi-continuous vector fields $\mathcal{QC}(T\mathrm{X})$ is defined as the closure of* $\mathrm{Test}\bar{\mathrm{V}}(\mathrm{X})$ *in* $L^0_{\mathrm{Cap}}(T\mathrm{X})$.

We conclude this section recalling the last result of [DGP], which prove the existence of a (unique) quasi-continuous representative of $H^{1,2}_C(\mathrm{X})$ vector fields.

We indicate with $\bar{\mathrm{Pr}}$ the natural projection from $L^0_{\mathrm{Cap}}(T\mathrm{X})$ to $L^0(T\mathrm{X})$, i.e. the linear continuous operator such that

i) $\bar{\mathrm{Pr}}(\bar{\nabla} f) = \nabla f$ for every $f \in \mathrm{Test}(\mathrm{X})$;

ii) $\bar{\mathrm{Pr}}(gv) = \mathrm{Pr}(g)\bar{\mathrm{Pr}}(v)$ for every $g \in L^0(\mathrm{Cap})$ and $v \in L^0_{\mathrm{Cap}}(T\mathrm{X})$, where Pr is the natural projection from $L^0(\mathrm{Cap})$ to $L^0(\mathfrak{m})$.

Theorem 3.2.17 (Quasi-continuous representative of Sobolev vector fields). *Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be an $\mathsf{RCD}(K,\infty)$ space for some $K \in \mathbb{R}$. Then there exists a unique map $\bar{\mathsf{QCR}} : H^{1,2}_C(T\mathrm{X}) \to \mathcal{QC}(T\mathrm{X})$ such that $\bar{\mathrm{Pr}} \circ \bar{\mathsf{QCR}} : H^{1,2}_C(T\mathrm{X}) \to L^0(T\mathrm{X})$ coincides with the inclusion $H^{1,2}_C(T\mathrm{X}) \subset L^0(T\mathrm{X})$.*

Moreover $\bar{\mathsf{QCR}}$ is linear and for every $v \in H^{1,2}_C(T\mathrm{X})$ it holds $|\bar{\mathsf{QCR}}(v)| = \mathsf{QCR}(|v|)$.

3.2.3 Tangent module over ∂E and Gauss-Green formula

In this section we recall the main results of [BPS, Chapter 2]: the existence and uniqueness of a tangent module over the boundary of a set with finite perimeter and the Gauss-Green formula for RCD spaces.

In order to do that we observe that the following proposition holds.

Proposition 3.2.18. *Let* X *be an $\mathsf{RCD}(K,N)$ metric measure space and let $E \subset \mathrm{X}$ be of finite perimeter. Then $|D\chi_E| \ll \mathrm{Cap}$.*

We will indicate with $\pi_{|D\chi_E|}$ the projection of $L^0(\mathrm{Cap})$ on $L^0(|D\chi_E|)$.

Definition 3.2.19 (Trace operator over ∂E)**.** *We define the trace operator over the boundary of E as the function $\mathrm{tr}_E : W^{1,2}(\mathrm{X}) \to L^0(|D\chi_E|)$ given by*

$$\mathrm{tr}_E := \pi_{|D\chi_E|} \circ \mathsf{QCR}\,.$$

Theorem 3.2.20 (Tangent module over ∂E)**.** *Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be an $\mathsf{RCD}(K,N)$ space and let $E \subset \mathrm{X}$ be a set of finite perimeter. Then there exists a unique couple $(L^2_E(T\mathrm{X}), \tilde{\nabla})$ where $L^2_E(T\mathrm{X})$ is an $L^2(|D\chi_E|)$-normed $L^\infty(|D\chi_E|)$-module and $\tilde{\nabla} : \mathrm{Test}(\mathrm{X}) \to L^2_E(T\mathrm{X})$ is a linear operator such that*

i) the equality $|\tilde{\nabla} f| = \mathrm{tr}_E(|\nabla f|)$ holds $|D\chi_E|$-a.e. for every $f \in \mathrm{Test}(\mathrm{X})$;

ii) the set

$$\left\{ \sum_{i=1}^n \chi_{E_i} \tilde{\nabla} f_i : (E_i)_{i=1}^n \text{ is a Borel partition of } \mathrm{X},\ (f_i)_{i=1}^n \subset \mathrm{Test}(\mathrm{X}) \right\}$$

is dense in $L^2_E(T\mathrm{X})$.

Uniqueness is intended up to unique isomorphism.

The space $L^2_E(T\mathrm{X})$ is called tangent module over the boundary of the set E and the operator $\tilde{\nabla}$ is called gradient.

Similarly to how we defined the trace operator of functions over ∂E, we define the trace operator of vector fields. To do that we indicate with $\bar{\pi}_{|D\chi_E|}$ the projection of $L^0_{\mathrm{Cap}}(T\mathrm{X})$ on $L^0_E(T\mathrm{X})$, the L^0-completion of $L^2_E(T\mathrm{X})$.

Definition 3.2.21 (Trace operator of vector fields)**.** *We define the trace operator of vector fields over ∂E as the function $\bar{\mathrm{tr}}_E : H^{1,2}_C \cap L^\infty(T\mathrm{X}) \to L^2_E(T\mathrm{X})$ defined by*

$$\bar{\mathrm{tr}}_E := \bar{\pi}_{|D\chi_E|} \circ \mathsf{Q\bar{C}R}\,.$$

Theorem 3.2.22 (Gauss-Green formula on RCD spaces)**.** *Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be an $\mathsf{RCD}(K,N)$ space and let $E \subset \mathrm{X}$ be a set of finite perimeter such that $\mathfrak{m}(E) < +\infty$. Then there exists a unique vector field $\nu_E \in L^2_E(T\mathrm{X})$ such that $|\nu_E| = 1$ holds $\mathrm{Per}(E, \cdot)$-a.e. and*

$$\int_E \mathrm{div}(v)\,\mathrm{d}\mathfrak{m} = -\int \langle \bar{\mathrm{tr}}_E(v), \nu_E \rangle\,\mathrm{d}\,\mathrm{Per}(E,\cdot) \tag{3.2.3}$$

for every $v \in H^{1,2}_C(T\mathrm{X}) \cap D(\mathrm{div})$ with $|v| \in L^\infty(\mathfrak{m})$.

A similar result for $\mathsf{RCD}(K,\infty)$ space can be found in [BCM].

3.3 Volume of the ends and harmonic functions

The concept of 'end' borns in the smooth category of Riemannian manifolds, but it can easily be adapted to metric spaces.

Definition 3.3.1. *Let* (X, d) *be a metric space and* $K \subset \mathrm{X}$ *compact. A set* $E \subset \mathrm{X}$ *is called end of* X *with respect to* K *provided:*

-) *E is an unbounded connected component of* $\mathrm{X} \setminus K$*;*

-) *for any $K' \supset K$ compact the set $E \setminus K'$ has only one unbounded connected component.*

Suppose that (X, d) *is equipped with a Radon measure* $\mathfrak{m} \geq 0$*. Then we say that an end E has infinite volume if* $\mathfrak{m}(E) = +\infty$*.*

Notice that if E is an end of X with respect to K and $K' \supset K$ is compact, then the only unbounded connected component E' of $E \setminus K'$ is an end with respect to K'. Also, in this case E has infinite volume if and only if E' does.

Let E be an end of X with respect to K and let $p \in K$. We indicate with $E(R) := E \cap B_R(p)$ for every $R > \mathrm{dist}(E, p)$. Moreover we define $\partial E := \partial K \cap \bar{E}$ and $\partial E(R) := \partial B_R(p) \cap E$.

We indicate with $V_E(R)$ the volume of $E(R)$ and with $V_E(\infty)$ the volume of the end E.

We conclude defining the first eigenvalue of the Laplacian λ_1.

Definition 3.3.2 (First eigenvalue of the Laplacian)**.** *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space. We define*

$$\lambda_1 := \inf \left\{ \frac{\int_{\mathrm{X}} |\mathrm{d}f|^2 \, \mathrm{d}\mathfrak{m}}{\int_{\mathrm{X}} |f|^2 \, \mathrm{d}\mathfrak{m}} : f \in W^{1,2}(\mathrm{X}),\ \int f^2 \, \mathrm{d}\mathfrak{m} \neq 0 \right\}.$$

Notice that the definition makes sense on arbitrary metric measure spaces, regardless of the linearity of the Laplacian (but we shall only work on infinitesimally Hilbertian spaces).

In this section, following the steps used by Li and Wang, we prove that, assuming $\lambda_1 > 0$, an end has infinite volume if and only if there exists a non-constant bounded harmonic function on it. In order to do this we begin studying some decay estimates for a class of harmonic functions.

In [GV, Section B.1] the following result has been established, the point being the continuity at $\partial \mathrm{K}^r$ (see also [BB]). Below for $\mathrm{K} \subset \mathrm{X}$ and $r > 0$ we put

$$\mathrm{K}^r := \{x : \mathsf{d}(x, \mathrm{K}) < r\}.$$

We consider such enlargements to gain sufficient regularity of the boundary to be sure that the harmonic function given by the statement below is continuous up to the boundary.

Theorem 3.3.3. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathsf{RCD}(K, N)$ *space with* $K \in \mathbb{R}$ *and* $N \in [1, +\infty)$*, and fix* $\mathrm{K} \subset \mathrm{X}$ *a bounded subset and* $r > 0$*. Let also* $B \subset \mathrm{X}$ *be a ball containing* K^r*. Then there is* $f \in W_0^{1,2}(B) \cap C(B)$ *with* $0 \leq f \leq 1$ *on* B *that is harmonic on* $B \setminus \overline{\mathrm{K}^r}$ *and equal to 1 in* $\overline{\mathrm{K}^r}$*.*

In particular, let E be an end of X, say with respect to a compact set K. Then

$$E \text{ is also an end with respect to the compact set } \mathrm{K}' := \overline{\mathrm{K}^1}. \tag{3.3.1}$$

This means that for $p \in \mathrm{K}'$ and $R > 0$ large enough we can apply the above with $B = B_R(p)$ with $R > 0$ big enough to find $f_R \in W_0^{1,2}(B) \cap C(B)$ with $0 \leq f_R \leq 1$ on B that is harmonic on $B \setminus \mathrm{K}'$ and equal to 1 in K'.

Thanks to the maximum principle (see Theorems 3.1.9 and 3.1.10) the functions f_R generated above are pointwise increasing in R, thus we can define:

Definition 3.3.4. *The function* $f_E : E \backslash \mathrm{K}' \to [0, 1]$ *is defined as the pointwise limit of the* f_R*'s described above.*

It is easy to verify that f_E depends only on E and K', and not on the particular $p \in \mathrm{K}'$ chosen, and that it is harmonic. The dependence of this function on K' is omitted from the notation for brevity. Also, in what follows we relabel E to be $E \setminus \mathrm{K}'$.

Our first goal is to prove some key decay estimates for the function f_E.

In all the following for a fixed point $p \in \mathrm{X}$ we indicate with $\mathsf{d}_p : \mathrm{X} \to \mathbb{R}$ the function $\mathsf{d}(\cdot, p)$.

Lemma 3.3.5. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathsf{RCD}(K, N)$ *metric measure space and* E *an end of it. Assume that* $\lambda_1 > 0$*. Then for the harmonic function* f_E *given by Definition 3.3.4 there exists a constant* $C = C(E, \mathrm{K}, \lambda_1)$ *such that for every* R *large enough the following estimates hold:*

$$\int_{E(R+1)\backslash E(R)} f_E^2 \, \mathrm{d}\mathfrak{m} \leq C e^{-2\sqrt{\lambda_1} R}, \tag{3.3.2}$$

$$\int_{E(R+1)\backslash E(R)} |\nabla f_E|^2 \, \mathrm{d}\mathfrak{m} \leq C e^{-2\sqrt{\lambda_1} R}, \tag{3.3.3}$$

$$\int_{E(R)} e^{2\sqrt{\lambda_1}\mathsf{d}_p} |\nabla f_E|^2 \, \mathrm{d}\mathfrak{m} \leq C R. \tag{3.3.4}$$

Proof. Let $\mathrm{K} \subset \mathrm{X}$ be compact, put $\mathrm{K}' := \overline{\mathrm{K}^1}$ and, as before, say that E is an end with respect to K'. Also, let $p \in \mathrm{X}$ and $R_0 > 0$ fixed so that $\mathrm{K}' \subset B_{R_0}(p)$. Recall that we put $E(R) := E \cap B_R(p)$ and that the function $f_R \in W_0^{1,2} \cap C(B_R(p))$ has been defined after Theorem 3.3.3.

Step 0. We claim that for every $g \in \mathrm{LIP}_{loc}(\mathrm{X})$ be identically 0 on K' and for every R large enough it holds

$$\int_{E(R)} |\nabla(g f_R)|^2 \, \mathrm{d}\mathfrak{m} = \int_{E(R)} |\nabla g|^2 f_R^2 \, \mathrm{d}\mathfrak{m}. \tag{3.3.5}$$

Using the Leibniz and chain rules we compute

$$\begin{aligned}\int_{E(R)} |\nabla(g f_R)|^2 \, \mathrm{d}\mathfrak{m} &= \int_{E(R)} |\nabla g|^2 f_R^2 \, \mathrm{d}\mathfrak{m} + \int_{E(R)} g^2 |\nabla f_R|^2 \, \mathrm{d}\mathfrak{m} \\ &\quad + \int_{E(R)} f_R \left\langle \nabla(g^2), \nabla f_R \right\rangle \, \mathrm{d}\mathfrak{m}.\end{aligned}$$

Now we notice that the integration by parts in

$$\begin{aligned}\int_{E(R)} f_R \left\langle \nabla(g^2), \nabla f_R \right\rangle \, \mathrm{d}\mathfrak{m} &= -\int_{E(R)} g^2 \operatorname{div}(f_R \nabla f_R) \, \mathrm{d}\mathfrak{m} \\ &= -\int_{E(R)} g^2 |\nabla f_R|^2 \, \mathrm{d}\mathfrak{m}\end{aligned}$$

is justified by the fact that $\partial(E(R)) \subset (\partial \mathrm{K}') \cup (\partial B_R(p))$ and the assumption $g \equiv 0$ on K' and $f_R \in W_0^{1,2}(B_R(p))$. The claim (3.3.5) follows.

The same line of thought proves that

$$\int_E |\nabla(g f_E)|^2 \, \mathrm{d}\mathfrak{m} = \int_E |\nabla g|^2 f_E^2 \, \mathrm{d}\mathfrak{m} \tag{3.3.6}$$

if g is as above and moreover in $W^{1,2}(\mathrm{X})$.

Step 1. Let $\xi := \exp(\sqrt{\lambda_1} \mathsf{d}_p)$. We claim that for some $C = C(E, \mathrm{K}, \lambda_1)$ we have

$$\int_E \xi^{2\delta} f_E^2 \, \mathrm{d}\mathfrak{m} \le \frac{C}{(1-\delta)^2} \qquad \text{for every } \delta \in (0,1). \tag{3.3.7}$$

Let $\varphi \in \mathrm{LIP} \cap C_b(\mathrm{X})$ be defined as $\varphi := \hat{\varphi} \circ \mathsf{d}_p$, where

$$\hat{\varphi}(z) := \begin{cases} 0 & \text{if } z \le R_0, \\ \frac{z - R_0}{R_0} & \text{if } z \in [R_0, 2R_0], \\ 1 & \text{if } z \ge 2R_0, \end{cases}$$

and let $R > 2R_0$. By (3.3.5) and the Cauchy-Schwarz inequality we have that

$$\begin{aligned}\int_{E(R)} \left|\nabla(\varphi\xi^\delta f_R)\right|^2 \mathrm{d}\mathfrak{m} &= \int_{E(R)} |\nabla(\varphi\xi^\delta)|^2 f_R^2 \,\mathrm{d}\mathfrak{m} \\ &\le (1+\varepsilon)\int_{E(R)} \varphi^2|\nabla\xi^\delta|^2 f_R^2\,\mathrm{d}\mathfrak{m} \\ &\quad + \left(1+\tfrac{1}{\varepsilon}\right)\int_{E(R)} \xi^{2\delta}\left|\nabla\varphi\right|^2 f_R^2\,\mathrm{d}\mathfrak{m} \\ &\le (1+\varepsilon)\delta^2\lambda_1\int_{E(R)} \varphi^2\xi^{2\delta} f_R^2\,\mathrm{d}\mathfrak{m} \\ &\quad + \left(1+\tfrac{1}{\varepsilon}\right)\tfrac{1}{R_0^2}\int_{E(2R_0)\setminus E(R_0)} \xi^{2\delta} f_R^2\,\mathrm{d}\mathfrak{m}.\end{aligned}$$

By definition of λ_1 we have $\lambda_1\int_{E(R)}\varphi^2\xi^{2\delta}f_R^2\,\mathrm{d}\mathfrak{m} \le \int_{E(R)}|\nabla(\varphi\xi^\delta f_R)|^2\,\mathrm{d}\mathfrak{m}$ and thus

$$\lambda_1\left(1-(1+\varepsilon)\delta^2\right)\int_{E(R)}\varphi^2\xi^{2\delta}f_R^2\,\mathrm{d}\mathfrak{m} \le (1+\tfrac{1}{\varepsilon})\tfrac{1}{R_0^2}\int_{E(2R_0)\setminus E(R_0)}\xi^{2\delta}f_R^2\,\mathrm{d}\mathfrak{m},$$

and choosing $\varepsilon = \frac{1-\delta}{\delta}$, recalling that $0 \le f_R \le 1$ we obtain

$$\lambda_1(1-\delta)^2\int_{E(R)}\varphi^2\xi^{2\delta}f_R^2\,\mathrm{d}\mathfrak{m} \le \frac{1}{R_0^2}\int_{E(2R_0)\setminus E(R_0)}\xi^{2\delta}f_R^2\,\mathrm{d}\mathfrak{m} \le C. \tag{3.3.8}$$

Since φ is positive and identically equal to 1 on $E(R)\setminus E(2R_0)$, we have that

$$\begin{aligned}\int_{E(R)}\xi^{2\delta}f_R^2\,\mathrm{d}\mathfrak{m} &= \int_{E(R)\setminus E(2R_0)}\xi^{2\delta}f_R^2\,\mathrm{d}\mathfrak{m} + \int_{E(2R_0)}\xi^{2\delta}f_R^2\,\mathrm{d}\mathfrak{m} \\ &\le \int_{E(R)}\varphi^2\xi^{2\delta}f_R^2\,\mathrm{d}\mathfrak{m} + C \overset{(3.3.8)}{\le} \frac{C}{(1-\delta)^2} + C = \frac{C}{(1-\delta)^2}\end{aligned}$$

and letting $R\to\infty$ we get the claim (3.3.7).

Step 2. Put for simplicity

$$F(R) := \int_{E(R)}\xi^2 f_E^2\,\mathrm{d}\mathfrak{m} \qquad \text{for every } R \ge R_0. \tag{3.3.9}$$

Let $R_0 < R_1 < R$. We claim that for any $t \in (0, R-R_1)$ it holds

$$\begin{aligned}\tfrac{2\sqrt{\lambda_1}t}{(R-R_1)^2}(F(R-t)-F(R_1)) \le{}& \tfrac{2\sqrt{\lambda_1}(R_1-R_0)+1}{(R_1-R_0)^2}(F(R_1)-F(R_0)) \\ &+ \tfrac{1}{(R-R_1)^2}(F(R)-F(R_1)).\end{aligned} \tag{3.3.10}$$

To see this let $\psi \in \mathrm{LIP}(\mathrm{X})$ be defined as $\psi := \hat{\psi} \circ \mathsf{d}_p$, where

$$\hat{\psi}(z) := \begin{cases} \frac{z-R_0}{R_1-R_0} & \text{if } z \in [R_0, R_1], \\ \frac{R-z}{R-R_1} & \text{if } z \in [R_1, R], \\ 0 & \text{otherwise.} \end{cases}$$

By definition of λ_1, (3.3.6) and the identity $\nabla \xi = \sqrt{\lambda_1} \xi \nabla \mathsf{d}_p$ we get

$$\begin{aligned} \lambda_1 \int_E \psi^2 \xi^2 f_E^2 \, \mathrm{d}\mathfrak{m} &\le \int_E |\nabla (\psi \xi f_E)|^2 \, \mathrm{d}\mathfrak{m} = \int_E |\nabla (\psi \xi)|^2 f_E^2 \, \mathrm{d}\mathfrak{m} \\ &= \int_E |\nabla \psi|^2 \xi^2 f_E^2 \, \mathrm{d}\mathfrak{m} + \lambda_1 \int_E \psi^2 \xi^2 f_E^2 \, \mathrm{d}\mathfrak{m} \\ &\quad + 2\sqrt{\lambda_1} \int_E \psi \xi^2 \langle \nabla \psi, \nabla \mathsf{d}_p \rangle f_E^2 \, \mathrm{d}\mathfrak{m}, \end{aligned}$$

that can be rewritten as

$$-2\sqrt{\lambda_1} \int_E \psi \xi^2 \langle \nabla \psi, \nabla \mathsf{d}_p \rangle f_E^2 \, \mathrm{d}\mathfrak{m} \le \int_E |\nabla \psi|^2 \xi^2 f_E^2 \, \mathrm{d}\mathfrak{m}.$$

Using the explicit expression of ψ, this can be further rewritten as

$$\begin{aligned} &\frac{2\sqrt{\lambda_1}}{(R-R_1)^2} \int_{E(R) \setminus E(R_1)} (R - \mathsf{d}_p) \xi^2 f_E^2 \, \mathrm{d}\mathfrak{m} \\ &\le \frac{2\sqrt{\lambda_1}}{(R_1-R_0)^2} \int_{E(R_1) \setminus E(R_0)} \underbrace{(\mathsf{d}_p - R_0)}_{(\le R_1 - R_0 \text{ on } E(R_1) \setminus E(R_0))} \xi^2 f_E^2 \, \mathrm{d}\mathfrak{m} \\ &\quad + \frac{1}{(R_1-R_0)^2} \int_{E(R_1) \setminus E(R_0)} \xi^2 f_E^2 \, \mathrm{d}\mathfrak{m} + \frac{1}{(R-R_1)^2} \int_{E(R) \setminus E(R_1)} \xi^2 f_E^2 \, \mathrm{d}\mathfrak{m} \\ &\le \text{Right Hand Side of (3.3.10).} \end{aligned}$$

Then to get (3.3.10) notice that for $t \in (0, R - R_1)$ we have $t \le R - \mathsf{d}_p$ on $E(R-t) \setminus E(R_1)$.

Step 3. We claim that there exists a constant $C = C(E, \mathrm{K}, \lambda_1) > 0$ such that

$$F(R) \le CR \qquad \text{for all } R \text{ large enough and } F \text{ as in (3.3.9).} \tag{3.3.11}$$

To see this pick $t = 1$, $R_1 = R_0 + 1$ and replace R with $R + 1$ in (3.3.10) to get, after an easy manipulation, that

$$F(R) - F(R_0 + 1) \le CR^2 + \frac{1}{2\sqrt{\lambda_1}} (F(R+1) - F(R_0 + 1))$$

(observe that R_0 depends only on E and K). By iteration we get

$$\begin{aligned} F(R)-F(R_0+1) &\leq C\sum_{i=1}^{k}\frac{(R+i)^2}{2^{i-1}}+(2\sqrt{\lambda_1})^{-k}\big(F(R+k)-F(R_0+1)\big)\\ &\leq CR^2+(2\sqrt{\lambda_1})^{-k}\big(F(R+k)-F(R_0+1)\big) \end{aligned} \tag{3.3.12}$$

for every $k\in\mathbb{N}$, $k>0$.

Now notice that for any $\delta\in(0,1)$ we have

$$\begin{aligned} F(R+k)-F(R_0+1) &\leq e^{2\sqrt{\lambda_1}(R+k)(1-\delta)}\int_{E(R+k)\setminus E(R_0+1)}\xi^{2\delta}f_E^2\,\mathrm{d}\mathfrak{m}\\ &\overset{(3.3.7)}{\leq}\frac{Ce^{2\sqrt{\lambda_1}(R+k)(1-\delta)}}{(1-\delta)^2}, \end{aligned}$$

thus picking δ so that $2\sqrt{\lambda_1}(1-\delta)<\log(2\sqrt{\lambda_1})$, by letting $k\to\infty$ in (3.3.12) we conclude that

$$F(R)\leq CR^2. \tag{3.3.13}$$

To improve this, pick $t=\frac{R}{2}$ and $R_1=R_0+1$ in (3.3.10) to obtain, after little manipulation, that

$$\tfrac{1}{R}\big(F(\tfrac{R}{2})-F(R_0+1)\big)\leq C\big(1+\tfrac{1}{R^2}(F(R)-F(R_0+1))\big)\overset{(3.3.13)}{\leq}C$$

for every R large enough. This is (equivalent to) our claim (3.3.11).

Step 4. We claim that (3.3.2) holds. To see this, in (3.3.10) pick $t=\frac{2}{\sqrt{\lambda_1}}$, $R+t$ in place of R and $R-t$ in place of R_1: with little manipulation we deduce that

$$\begin{aligned} \frac{1}{t^2}\big(F(R)-F(R-t)\big) &\leq \underbrace{\frac{C}{R}\big(F(R-t)-F(R_0)\big)}_{\leq C\ \text{by (3.3.11)}}\\ &\quad+\frac{1}{4t^2}\underbrace{\big(F(R+t)-F(R-t)\big)}_{=(F(R+t)-F(R))+(F(R)-F(R-t))} \end{aligned}$$

and thus that $F(R)-F(R-t)\leq C+\frac{1}{3}(F(R+t)-F(R))$. Iterating we get

$$F(R)-F(R-t)\leq C\sum_{i=0}^{k-1}\frac{1}{3^i}+\frac{1}{3^k}(F(R+kt)-F(R+(k-1)t))$$

for every $k\in\mathbb{N}$, $k>0$.

Recalling (3.3.11) and letting $k \to \infty$ we conclude that $F(R) - F(R-t) \leq C$ for all R large enough (recall that here $t = \frac{2}{\sqrt{\lambda_1}}$). The claim (3.3.2) easily follows.

Step 5. We claim that (3.3.3) and (3.3.4) hold.

Let $R > R_0 + 1$ and $\zeta \in \mathrm{LIP}_{bs}(\mathrm{X})$ be defined as $\zeta := \hat{\zeta} \circ \mathsf{d}_p$, where

$$\hat{\zeta}(z) := \begin{cases} z - (R-1) & \text{if } z \in [R-1, R], \\ 1 & \text{if } z \in [R, R+1], \\ R+2-z & \text{if } z \in [R+1, R+2], \\ 0 & \text{otherwise.} \end{cases}$$

Then using that f_E is harmonic we get the standard estimate

$$\begin{aligned} \int_E \zeta^2 |\nabla f_E|^2 \, \mathrm{d}\mathfrak{m} &= -2 \int_E \zeta f_E \langle \nabla \zeta, \nabla f_E \rangle \, \mathrm{d}\mathfrak{m} \\ &\leq \frac{1}{2} \int_E \zeta^2 |\nabla f_E|^2 \, \mathrm{d}\mathfrak{m} + 2 \int_E |\nabla \zeta|^2 f_E^2 \, \mathrm{d}\mathfrak{m} \end{aligned}$$

that gives $\int_E \zeta^2 |\nabla f_E|^2 \, \mathrm{d}\mathfrak{m} \leq 4 \int_E |\nabla \zeta|^2 f_E^2 \, \mathrm{d}\mathfrak{m} \leq 4 \int_{E(R+2) \setminus E(R-1)} f_E^2 \, \mathrm{d}\mathfrak{m}$. Thus

$$\begin{aligned} \int_{E(R+1) \setminus E(R)} |\nabla f_E|^2 \, \mathrm{d}\mathfrak{m} &\leq \int_E \zeta^2 |\nabla f_E|^2 , \mathrm{d}\mathfrak{m} \\ &\leq 4 \int_{E(R+2) \setminus E(R-1)} f_E^2 \, \mathrm{d}\mathfrak{m} \overset{(3.3.2)}{\leq} C e^{-2\sqrt{\lambda_1} R}, \end{aligned}$$

as desired. Then (3.3.4) is a direct consequence of (3.3.3). □

These estimates allow us to deduce the following important dichotomy result. In proving it we shall use the main results shown in Section 3.2: the Gauss-Green formula (Theorem 3.2.22) and the coarea formula (Theorem 3.2.2).

Theorem 3.3.6. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathsf{RCD}(K, N)$ *metric measure space and let* E *be an end of* X. *Assume that* $\lambda_1 > 0$. *Then exactly one of the following holds:*

i) there exists a bounded non-constant harmonic function on E *and* E *has exponential volume growth, more precisely*

$$V_E(R) \geq C \exp\left(2\sqrt{\lambda_1} R\right) \tag{3.3.14}$$

for every R *large enough and some constant* $C > 0$*;*

ii) every bounded harmonic function on E is constant and E has exponential volume decay, more precisely

$$V_E(\infty) - V_E(R) \le C \exp\left(-2\sqrt{\lambda_1} R\right) \tag{3.3.15}$$

for every R large enough and some constant $C > 0$.

Proof.
Step 1. Assume that every bounded harmonic function on E is constant. Then this is the case for the function f_E given by Definition 3.3.4. We have then that f_E is equal to 1 a.e. on E, and by (3.3.2) follows that

$$V_E(R+1) - V_E(R) \le C \exp\left(-2\sqrt{\lambda_1} R\right).$$

Summing up we obtain

$$V_E(\infty) - V_E(R) \le C \sum_{i=0}^{+\infty} \exp\left(-2\sqrt{\lambda_1}(R+i)\right) \le C \exp\left(-2\sqrt{\lambda_1} R\right).$$

Step 2. Assume now that there exists a bounded non-constant harmonic function u on E. By translation and rescaling we can assume that $u \ge 0$ on X, $u \ge 1$ on K$'$ and $u < 1$ somewhere in E (recall (3.3.1)). Then by the maximum principle and the construction of f_E it follows that f_E is not constant as well.

Now notice that by the spherical version of the Bishop-Gromov inequality we have that $B_R(p)$ has finite perimeter for any $R > 0$. It follows that for any $R > R_0$ the set $E^R := E \setminus E(R)$ has finite perimeter with $\mathrm{Per}(E^R, \cdot)$ being the restriction of $\mathrm{Per}(B_R(p), \cdot)$ to E.

We claim that there is $c > 0$ such that

$$\int |\nabla f_E| \, \mathrm{d}\,\mathrm{Per}(E^r, \cdot) \ge c \qquad \text{for every } r > R_0, \tag{3.3.16}$$

where here, with a slight abuse of notation, we are denoting by $\nabla f_E \in L^0(T\mathrm{X}, \mathrm{Per}(E^r(\cdot)))$ the trace of $\nabla f_E \in W^{1,2}_{C,loc}(E)$ in the sense of Definition 3.2.21.

Let us show how from (3.3.16) we can conclude. Denoting with $P(r)$ the perimeter $\mathrm{Per}(E^r, \mathrm{X})$, by (3.3.16) and the Cauchy-Schwarz inequality we immediately have $\frac{1}{P(r)} \le \frac{1}{c} \int |\nabla f_E|^2 \, \mathrm{d}\,\mathrm{Per}(E^r, \cdot)$ and therefore

$$\begin{aligned}
1 &\le \int_R^{R+1} P(r)\,\mathrm{d}r \int_R^{R+1} \frac{1}{P(r)}\,\mathrm{d}r \\
&\overset{*}{\le} \frac{1}{c}\big(V_E(R+1) - V_E(R)\big) \int_{E(R+1)\setminus E(R)} |\nabla f_E|^2 \,\mathrm{d}\mathfrak{m} \qquad (3.3.17) \\
\text{(by (3.3.3))} \quad &\le C\big(V_E(R+1) - V_E(R)\big) e^{-2R\sqrt{\lambda_1}},
\end{aligned}$$

where in the starred inequality we used the coarea formula to deduce

$$\begin{aligned} V_E(R+1) - V_E(R) &= \mathfrak{m}(E(R+1) \setminus E(R)) \\ &= \int_{E(R+1)\setminus E(R)} |\nabla \mathsf{d}_p| \, \mathrm{d}\mathfrak{m} = \int_R^{R+1} P(r) \, \mathrm{d}r \end{aligned}$$

and the fact that, if we call $\tilde{\nabla} f_E \in L^0_{\mathrm{Cap}}(T\mathrm{X})$ the quasi-continuous representative of $\nabla f_E \in W^{1,2}_{C,loc}(E)$ (set to 0 outside E, say) and $\mathrm{tr}_r \nabla f_E$ the trace of ∇f_E in $L^0(T\mathrm{X}, \mathrm{Per}(E^r, \cdot))$, then, by definition, $\mathrm{tr}_r \nabla f_E$ is the equivalence class of $\tilde{\nabla} f_E$ in in $L^0(T\mathrm{X}, \mathrm{Per}(E^r, \cdot))$ and thus we have $|\,\mathrm{tr}_r \nabla f_E| = |\tilde{\nabla} f_E|$ $\mathrm{Per}(E^r, \cdot)$-a.e.. This and the trivial identity $|\tilde{\nabla} f_E| = |\nabla f_E|$ $\mathfrak{m}$-a.e. justifies the computation above.

Since (3.3.17) is equivalent to the claim, we are left to prove (3.3.16). Since f_E is harmonic, putting $F(r) := \int \langle \nabla f_E, \nu_{E(r)} \rangle \, \mathrm{d}\,\mathrm{Per}(E^r, \cdot)$ for brevity, by the Gauss-Green formula we have that, for every $r_2 > r_1 > R_0$ it holds

$$\begin{aligned} F(r_2) - F(r_1) &= \int \langle \nabla f_E, \nu_{E(r_2)\setminus E(r_1)} \rangle \, \mathrm{d}\,\mathrm{Per}(E(r_2) \setminus E(r_1), \cdot) \\ &= -\int_{E(r_2)\setminus E(r_1)} \Delta f_E \, \mathrm{d}\mathfrak{m} = 0, \end{aligned}$$

thus $r \mapsto F(r)$ is constant on $(R_0, +\infty)$ and since clearly

$$|F(r)| \le \int |\nabla f_E| \, \mathrm{d}\,\mathrm{Per}(E^r, \cdot),$$

to get (3.3.16) it suffices to prove that $F(r) \neq 0$ for some $r > R_0$.

To see this, fix $\bar{R} > R_0$ and notice that by the strong maximum principle and the fact that f_E is not constant, there must be $a > 0$ such that $\sup_{\partial E^{\bar{R}}} f_E \le 1 - a$. In particular, taking into account the continuity of f_E and the Sobolev-to-Lipschitz property it easily follows that there is $b \in (0, \frac{a}{2})$ such that

$$\int_{E \cap B_{\bar{R}}(p) \cap \{1-2b < f_E < 1-b\}} |\nabla f_E|^2 \, \mathrm{d}\mathfrak{m} > 0. \tag{3.3.18}$$

Put $\rho := \varphi \circ f_E$ where

$$\varphi(z) := \begin{cases} 1, & \text{if } z \le 1 - 2b, \\ 0, & \text{if } z \ge 1 - b, \\ \text{affine and continuous on } [1-2b, 1-b]. \end{cases}$$

and notice that we have

$$\begin{aligned} \int_{E(\bar{R})} \mathrm{div}(\rho \nabla f_E) \, \mathrm{d}\mathfrak{m} &= \int_{E(\bar{R})} \langle \nabla \rho, \nabla f_E \rangle \, \mathrm{d}\mathfrak{m} \\ &= -\frac{1}{b} \int_{E \cap B_{\bar{R}}(p) \cap \{1-2b < f_E < 1-b\}} |\nabla f_E|^2 \, \mathrm{d}\mathfrak{m} \overset{(3.3.18)}{<} 0. \end{aligned}$$

We now claim that there is a finite perimeter set $\tilde{E}$ such that $\tilde{E}\Delta E(\bar{R}) \subset f_E^{-1}((1-b,1))$. Assuming we have such $\tilde{E}$, we conclude with

$$\begin{aligned} 0 > \int_{E(\bar{R})} \operatorname{div}(\rho \nabla f_E)\, \mathrm{d}\mathfrak{m} &= \int_{\tilde{E}} \operatorname{div}(\rho \nabla f_E)\, \mathrm{d}\mathfrak{m} \\ &= \int \rho \langle \nabla f_E, \nu_{\tilde{E}} \rangle \, \mathrm{d}\operatorname{Per}(\tilde{E}, \cdot) = -F(\bar{R}), \end{aligned}$$

where in the last step we used the fact that $\rho \equiv 1$ on $\partial E^{\bar{R}} = \partial\tilde{E} \cap f_E^{-1}((0,1-b])$ and $\rho \equiv 0$ on $\partial\tilde{E} \setminus f_E^{-1}((0,1-b])$. It thus remains to show the existence of such $\tilde{E}$. To see this, recall that balls have finite perimeter (by the spherical version of the Bishop-Gromov inequality (1.5.11)) and that ∂E is compact. Then using the continuity of f_E we can find $r > 0$ such that for any $x \in \partial E$ we have $B_r(x) \subset \{f_E \geq 1-b\}$ and by compactness a finite number of points $x_1, \ldots, x_n$ such that $\partial E \subset \cup_i B_r(x_i)$. It is then clear that the set $\tilde{E} := E \setminus \cup_i B_r(x_i)$ does the job. □

As a direct consequence of the above result we obtain the following corollaries.

Corollary 3.3.7. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathsf{RCD}(K,N)$ *space with* $\lambda_1 > 0$ *and with at least two ends with infinite volume.*

Then there exists a bounded non-constant harmonic function on X.

Proof. Let E_1, E_2 be the two given ends with infinite volume and consider the associated functions f_{E_1}, f_{E_2} as in Definition 3.3.4 (possibly enlarging R_0 we can take $\mathrm{K}' := B_{R_0}(p)$), which by Theorem 3.3.6 are bounded non-constant and harmonic functions. Recall that $f_{E_i} \equiv 1$ on ∂E_i and $\operatorname{ess\,inf} f_{E_i} = 0$ for $i = 1, 2$.

Then fix $R > R_0 + 1$ and let $C_i := \{x \in B_R(p) : \mathsf{d}(x, E_i \cap \partial B_{R+1}(p)) \geq 1\}$. Then a simple variant of Theorem 3.3.3 gives the existence of a function $f_R \in (W_0^{1,2} \cap C)(B_{R+2}(p))$ with values in $[0,1]$ that is harmonic in $B_{R+2}(p) \setminus (C_1 \cup C_2) \supset B_R(p)$, equal to 1 on C_1 and equal to 0 on C_2.

We claim that on $B_R \cap E_2$ we have $f_R \leq f_{E_2}$. To see this notice that the boundary $\partial(B_R \cap E_2)$ is contained in the disjoint union of ∂E_2 and C_2 and by construction we have $f_{E_2} = 1$ on ∂E_2 and $f_R = 0$ on C_2. Since both functions have values in $[0,1]$, are harmonic in $B_R \cap E_2$ and continuous up to the boundary of such set, by the maximum principle our claim follows.

Analogously we can prove that $f_R \geq 1 - f_{E_1}$ on $B_R \cap E_1$.

We now want to send $R \uparrow +\infty$ and find, possibly after passing to a subsequence, a limit harmonic function on the whole X. This is possibile thanks to the Lipschitz estimates (3.1.3), that grants that for some $C(R)$ the

Lipschitz constant of $f_{R'}$ on $B_{R/2}(p)$ is bounded from above by $C(R)$ for any $R' > R$. By Arzelá-Ascoli's theorem, this suffices to find a limit function f and it is then easy to see that this function is harmonic and bounded (in fact with values in $[0,1]$).

Also, a direct consequence of the construction and of the previous claims is that $f \le f_{E_2}$ on E_2 and $f \ge 1 - f_{E_1}$ on E_1. If we knew that $\inf_{E_i} f_{E_i} = 0$ for $i = 1, 2$ the conclusion would directly follow, as $\inf f \le \inf_{E_2} f_{E_2} = 0$ and $\sup f \ge \sup_{E_1} 1 - f_{E_1} = 1 - \inf_{E_1} f_{E_1} = 1$, proving that f is not constant.

Thus it remains to show that $\inf_{E_1} f_{E_1} = 0$ (the argument for E_2, f_{E_2} being analogous). Say not. Then by construction $a := \inf_{E_1} f_{E_1} > 0$. Define the new function $\tilde{f}_{E_1} := 1 - \frac{1-f_{E_1}}{1-a}$ and notice that it is still harmonic and continuous on E_1 with boundary value equal to 1 and that it is still positive. Thus by the maximum principle the function $\tilde{f}_{E_1}$ bounds from above each of the harmonic functions defined on $B_R(p) \cap E_1$ that are used in the definition of f_{E_1}. It follows by the definition of f_{E_1} that it would hold $f_{E_1} \le \tilde{f}_{E_1}$. This, however, contradicts the definition of $\tilde{f}_{E_1}$, as this ensures that $\tilde{f}_{E_1} < f_{E_1}$ on E_1. □

A similar argument allows us to find a non-constant positive harmonic function on X in the case of one end with infinite volume and one with finite volume.

Corollary 3.3.8. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathsf{RCD}(K, N)$ *space with* $\lambda_1 > 0$ *and with at least one end with infinite volume and one end with finite volume.*

Then there exists a non-constant positive harmonic function on X.

Proof. Let E_1 be an end with infinite volume and E_2 be an end with finite volume both with respect to a compact $\mathrm{K} \subset \mathrm{X}$, and fix $p \in \mathrm{K}$. Taking $R_0 > 0$ large enough we can assume, as before, $\mathrm{K}' := \overline{B_{R_0}}(p) \supset \overline{\mathrm{K}^1}$.

Let f_{E_1} and f_R be defined as in the proof of Corollary 3.3.7 (we note that the definition of f_R does not depend on the volume of the ends, so the same argument holds in this case too): f_{E_1} is defined as in Definition 3.3.4, and for R large enough $f_R \in (W_0^{1,2} \cap C)(\overline{B_R}(p))$ is harmonic in $B_R(p)$ identically equal to 1 on $\partial E_2(R)$ and identically equal to 0 on $\partial E_1(R)$ and on every other eventual end $\partial E_i(R)$ with $i > 2$.

To be precise, we should modify $B_R(p)$ near its boundary as in the proof of Corollary 3.3.7 to achieve the regularity needed to apply Theorem 3.3.3. We omit these technical details here referring instead to the proof of Corollary 3.3.7.

We fix $\varepsilon > 0$ and take $(\partial E_2)^\varepsilon := \{p \in \mathrm{X} : \mathsf{d}(p, \partial E_2) \le \varepsilon\}$. Thanks to the strong maximum principle, the compactness of $(\partial E_2)^\varepsilon$ and the continuity of f_R we have that $c_R := \max_{(\partial E_2)^\varepsilon} f_R > 0$ for every R large enough. Moreover, by

(3.1.3), the functions $\log(f_R/c_R)$ are locally equi-Lipschitz, and since they are equibounded on $(\partial E_2)^\varepsilon$ (by the definition of c_R) we conclude by Arzelà-Ascoli's theorem that there exists a subsequence of $(f_R/c_R)_R$ that converges uniformely on compact sets to a function $u : \mathrm{X} \to \mathbb{R}$.

The function u is clearly positive and harmonic. We observe that it is also non-constant. Indeed, as in the proof of Corollary 3.3.7 it is easy to see that $u \leq f_{E_1}$ on E_1, then $\inf_{\mathrm{X}} u = 0$, moreover, by the definition of c_R we conclude that $u \geq 1$ on $(\partial E)^\varepsilon$.

We also note that on every other end E_i with $i > 2$, thanks to the maximum principle, $f_R < c_R$, then $u \leq 1$ on every end E_i with $i > 2$ (if the space has more than two ends). □

Corollaries 3.3.7 and 3.3.8 should be coupled with the following simple and general result, stating that if $\lambda_1 > 0$ (an assumption that is present in Theorems 0.0.7 and 0.0.8), then the space has at least one end of infinite volume.

Proposition 3.3.9. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space with* $\lambda_1 > 0$. *Then* X *has at least one end with infinite volume.*

Proof. By the definition of λ_1 it follows immediately that if $\mathfrak{m}(\mathrm{X}) < \infty$ then $\lambda_1 = 0$, this means that $\mathfrak{m}(\mathrm{X}) = \infty$.

Assume by contradiction that X has no ends with infinite volume. Then there exists a compact K which has positive volume and cuts the space in ends with finite volume. Let $\{E_i\}_{i\in\mathbb{N}}$ be the set of the ends with respect to K. Since $\mathfrak{m}(E_i) < \infty$ for every $i \in \mathbb{N}$ then for every $\varepsilon > 0$ there exist a couple of radii $r_{i,\varepsilon}$, $R_{i,\varepsilon} > r_{i,\varepsilon} + 2$ such that $V_{E_i}(R_{i,\varepsilon}) - V_{E_i}(r_{i,\varepsilon}) \leq \frac{\varepsilon}{2^i}$. Taking now the function

$$f_\varepsilon(x) := \begin{cases} 1 & \text{for } x \in \mathrm{K} \cup \left(\bigcup (E_i(r_{i,\varepsilon}))\right), \\ \frac{\mathsf{d}(x,p) - r_{i,\varepsilon}}{R_{i,\varepsilon} - r_{i,\varepsilon}} & \text{for } x \in E_i(R_{i,\varepsilon}) \setminus E_i(r_{i,\varepsilon}), \\ 0 & \text{for } x \in E_i \setminus E_i(R_{i,\varepsilon}), \end{cases}$$

we conclude that

$$\frac{\int_{\mathrm{X}} |\nabla f_\varepsilon|^2 \, d\mathfrak{m}}{\int_{\mathrm{X}} |f_\varepsilon|^2 \, d\mathfrak{m}} \leq \frac{\varepsilon}{\mathfrak{m}(\mathrm{K})} \xrightarrow[\varepsilon \to 0]{} 0.$$

□

3.4 Rigidity theorem for spaces with positive spectrum

We focus now on Theorem 0.0.7, which states that if a space with strictly positive λ_1 has two ends with infinite volume then it splits (see Definitions 3.3.2 and 3.3.1 for the definitions of λ_1 and ends in metric measure spaces). We recall here its precise statement.

Theorem 3.4.1. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathsf{RCD}(-(N-1), N)$ *space with* $N \geq 3$ *and* $\mathrm{supp}(\mathfrak{m}) = \mathrm{X}$*, and assume that the first eigenvalue of the Laplacian* λ_1 *is* $\geq N-2$*. Then one of the following holds:*

i) X *has only one end with infinite volume;*

ii) X *is isomorphic as metric measure space to a warped product space* $\mathbb{R} \times_w \mathrm{X}'$*, where* X' *is a compact* $\mathsf{RCD}(-(N-1), N)$ *space and the warping functions are*

$$w_{\mathsf{d}}(t) := \cosh(t) \qquad \text{and} \qquad w_{\mathfrak{m}}(t) := \cosh^{N-1}(t).$$

Moreover, in this case $\lambda_1 = N-2$*.*

Given the statement of Theorem 3.4.1 and remembering the results of Section 3.3, let us fix the following notations and assumptions that we will use along all the proof of this result.

Assumption 3.4.2. $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *is an* $\mathsf{RCD}(-(N-1), N)$ *space with* $N \geq 3$ *and* $\lambda_1 \geq N-2$*. We also assume that* $\mathrm{supp}(\mathfrak{m}) = \mathrm{X}$ *and that* X *has two ends with infinite volume.*

Finally, we shall denote by u *a fixed bounded and non-constant harmonic function on* X*. The existence of such* u *is granted by Corollary 3.3.7.*

3.4.1 Properties of the bounded harmonic function

We start with the following simple regularity statement.

Lemma 3.4.3. *With the same assumptions and notation as in Assumption 3.4.2 the following holds.*

The function u *is in* $\mathrm{Test}_{loc}(\mathrm{X})$ *and globally Lipschitz.*

Proof. By the Cheng-Yau gradient estimate (3.1.3) and the fact that u is bounded it follows that it is globally Lipschitz. This information together with the fact that $\Delta u \equiv 0$ suffices to ensure that $u \in \mathrm{Test}_{loc}(\mathrm{X})$. □

The rigidity result we are going to prove in this section is ultimately a consequence, as customary, of the equality in the improved Bochner inequality (see inequality (3.1.4)). Before coming to that it is useful to recall the following result, that in the RCD setting has been proved in [GV]. We provide anyway the complete proof because we shall be interested in the equality case, that was not explicitly studied in [GV].

Lemma 3.4.4 (Generalized refined Kato inequality). *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathsf{RCD}(K, N)$ *space with* $K \in \mathbb{R}$ *and* $N \in [1, +\infty)$*. Then for any* $u \in H^{2,2}_{loc}(\mathrm{X})$ *it holds*

$$\frac{t + \dim(\mathrm{X})}{t + \dim(\mathrm{X}) - 1} |\mathrm{d}|\mathrm{d}u||^2 \leq |\operatorname{Hess}(u)|^2_{\mathsf{HS}} + \frac{(\operatorname{tr}\operatorname{Hess}(u))^2}{t} \quad \mathfrak{m}\text{-}a.e. \tag{3.4.1}$$

for every $t > 0$*, where* $\dim(\mathrm{X}) \in \mathbb{N} \cap [1, N]$ *is the (constant) dimension of* $L^0(T\mathrm{X})$ *(recall Theorem 3.1.12). Equality holds* $\mathfrak{m}$*-a.e. for a given* t *if and only if*

$$\operatorname{Hess}(u) = \alpha\big(\mathrm{Id} - (t + \dim(\mathrm{X}))\, \mathrm{e}_1 \otimes \mathrm{e}_1\big), \tag{3.4.2}$$

for some function $\alpha \in L^2_{loc}(\mathrm{X})$*, where* $\mathrm{e}_1 \in L^0(T\mathrm{X})$ *is a pointwise unitary vector that is equal to* $\frac{\nabla u}{|\nabla u|}$ *on* $|\nabla u| > 0$*.*

Proof. We indicate with n the dimension $\dim(\mathrm{X})$. Let A be a symmetric $n \times n$ real matrix. We claim that for every $t > 0$ and every $v \in \mathbb{R}^n$ we have:

$$\frac{t + n}{t + n - 1} |A \cdot v|^2 \leq |v|^2 |A|^2_{\mathsf{HS}} + \frac{|v|^2 (\operatorname{tr} A)^2}{t}. \tag{3.4.3}$$

Indeed, by the spectral theorem there is an orthonormal base $\mathrm{e}_1, \dots, \mathrm{e}_n$ of $\mathbb{R}^n$ so that A is diagonalwith respect to such base, with diagonal entries $\alpha_1, ..., \alpha_n$. We can also assume that $|\alpha_1| \geq |\alpha_i|$ for $i = 2, ..., n$. Then, applying twice the Cauchy-Schwarz inequality we obtain

$$\begin{aligned}
&|v|^2 \left(\frac{(\alpha_1 + \alpha_2 + ... + \alpha_n)^2}{t} + \alpha_1^2 + \alpha_2^2 + ... + \alpha_n^2 \right) \\
&\overset{(1)}{\geq} |v|^2 \left(\frac{(\alpha_1 + \alpha_2 + ... + \alpha_n)^2}{t} + \frac{(\alpha_2 + \alpha_3 + ... + \alpha_n)^2}{n-1} + \alpha_1^2 \right) \\
&\overset{(2)}{\geq} |v|^2 \left(\frac{\alpha_1^2}{t + n - 1} + \alpha_1^2 \right) \\
&\overset{(3)}{\geq} \frac{t + n}{t + n - 1} |A \cdot v|^2,
\end{aligned}$$

which is (3.4.3). Equality holds in (3.4.3) if and only if the inequalities above are all equality. Equality in (1) holds if and only if $\alpha_2 = \dots = \alpha_n =: \alpha$, equality in (2) if and only if $\alpha = \alpha_1(1-(t+n))$. Finally, equality in (3) holds if and only if v is a multiple of e_1. In other words, equality in (3.4.3) holds if and only if

$$A = \alpha(\mathrm{Id} - (t+n)\mathrm{e}_1 \otimes \mathrm{e}_1).$$

We come to (3.4.1) and its equality case and start noticing that $\mathrm{d}|\mathrm{d}u| = \chi_{\{|\nabla u|>0\}} \operatorname{Hess} u(\frac{\nabla u}{|\nabla u|})$ (for $u \in \mathrm{Test}_{loc}(\mathrm{X})$ this follows from Proposition 1.6.13 and the chain rule - see also [DGP, Lemma 2.5], then the general case comes by approximation). Now the conclusion follows picking a suitable pointwise orthonormal base $\mathrm{e}_1, \dots, \mathrm{e}_{\dim(\mathrm{X})} \in L^0(T\mathrm{X})$ of $L^0(T\mathrm{X})$ that diagonalizes $\operatorname{Hess} u$. □

Proposition 3.4.5. *With the same assumptions and notation as in Assumption 3.4.2 the following hold:*

i) $\lambda_1 = N-2$;

ii) $|\mathrm{d}u|$ *is locally Lipschitz (i.e. it has a locally Lipschitz representative) and* strictly positive;

iii) Putting $\mathrm{e}_1 := \frac{\nabla u}{|\nabla u|}$ *(this is well defined by item (ii)) we have*

$$\operatorname{Hess}(u) = \alpha(\mathrm{Id} - N\mathrm{e}_1 \otimes \mathrm{e}_1), \tag{3.4.4}$$

for some $\alpha \in L^2_{loc}(\mathrm{X})$;

iv) we have $|\mathrm{d}u|^{\frac{N-2}{N-1}} \in D(\Delta_{loc})$ *with*

$$\Delta(|\mathrm{d}u|^{\frac{N-2}{N-1}}) = -(N-2)|\mathrm{d}u|^{\frac{N-2}{N-1}}; \tag{3.4.5}$$

v) u *is an open map, i.e.* $u(U) \subset \mathbb{R}$ *is open for any* $U \subset \mathrm{X}$ *open.*

Proof. The required rigidity will follow by closely inspecting the proof of [GV, Theorem 3.4]. Start noticing that since we know from Lemma 3.4.3 above that $u \in \mathrm{Test}_{loc}(\mathrm{X})$, we can apply the improved Bochner inequality 3.1.4 (say $N > \dim(\mathrm{X})$; the case $N = \dim(\mathrm{X})$ follows along similar lines recalling that in this case $\operatorname{tr}\operatorname{Hess} u = \Delta u = 0$ and that the last addend in the Bochner inequality below is 0 - see [Han]) recalling that $\Delta u = 0$ to get

$$\mathbf{\Delta}\frac{|\mathrm{d}u|^2}{2} \geq \Big(|\operatorname{Hess} u|^2_{\mathsf{HS}} - (N-1)|\mathrm{d}u|^2 + \frac{(\operatorname{tr}\operatorname{Hess} u)^2}{N-\dim(\mathrm{X})}\Big)\mathfrak{m}.$$

Kato's inequality as in Lemma 3.4.1 with $t = N - \dim(\mathrm{X})$ can be written as

$$|\operatorname{Hess} u|^2_{\mathsf{HS}} = \frac{N}{N-1}|\mathrm{d}|\mathrm{d}u||^2 - \frac{(\operatorname{tr}\operatorname{Hess} u)^2}{N-\dim(\mathrm{X})} + F, \qquad \text{for some } F \geq 0\ \mathfrak{m}\text{-a.e.}$$

and the equality case reads as

$$F = 0\ \mathfrak{m}\text{-a.e.} \qquad \Rightarrow \qquad \text{identity (3.4.4) holds.} \tag{3.4.6}$$

We thus have

$$\mathbf{\Delta}\frac{|\mathrm{d}u|^2}{2} \geq \left(\tfrac{N}{N-1}|\mathrm{d}|\mathrm{d}u||^2 - (N-1)|\mathrm{d}u|^2 + F\right)\mathfrak{m} \tag{3.4.7}$$

Now let us fix $\beta, \varepsilon > 0$ and put $\varphi(z) := \varphi_{\beta,\varepsilon}(z) := (z+\varepsilon)^\beta$. Then (3.4.7) and basic calculus rules (see also the proof of [GV, Theorem 3.4]) give

$$\begin{aligned}\mathbf{\Delta}(\varphi(|\mathrm{d}u|^2)) \geq \Big(&\beta(|\mathrm{d}u|^2+\varepsilon)^{\beta-1}(2F - 2(N-1)|\mathrm{d}u|^2) \\ &+ 2\beta|\mathrm{d}|\mathrm{d}u||^2\Big(\tfrac{N}{N-1} + \tfrac{2(\beta-1)|\mathrm{d}u|^2}{|\mathrm{d}u|^2+\varepsilon}\Big)\Big)\mathfrak{m}.\end{aligned}$$

Now pick $\beta := \frac{N-2}{2(N-1)}$ and let $\varepsilon \downarrow 0$: the rightmost addend goes to 0 and putting for brevity $f := |\nabla u|^{\frac{N-2}{N-1}}$ (and summing $\lambda_1 f\mathfrak{m}$ to both sides) we are left with

$$\mathbf{\Delta}f + \lambda_1 f\mathfrak{m} \geq \Big((\lambda_1 - (N-2))f + \tfrac{N-2}{2(N-1)}F|\mathrm{d}u|^{-\frac{N}{N-1}}\Big)\mathfrak{m} \geq 0 \tag{3.4.8}$$

(notice that the passage to the limit is justified also by the fact that $f \in W^{1,2}_{loc}(\mathrm{X})$, as established in [GV, Theorem 3.4] - the proof of this follows from the computations we are repeating here).

To deduce the desired rigidity from the above, start observing that for any $\varphi \in \mathrm{LIP}_{bs}(\mathrm{X})$ we have

$$\int \varphi^2 f\,\mathrm{d}\mathbf{\Delta}f = \int -\varphi^2|\mathrm{d}f|^2 - 2f\varphi\langle\mathrm{d}\varphi,\mathrm{d}f\rangle\,\mathrm{d}\mathfrak{m} = \int -|\mathrm{d}(\varphi f)|^2 + f^2|\mathrm{d}\varphi|^2\,\mathrm{d}\mathfrak{m},$$

and thus using that $\lambda_1\int f^2\varphi^2\,\mathrm{d}\mathfrak{m} \leq \int|\mathrm{d}(\varphi f)|^2\,\mathrm{d}\mathfrak{m}$ we get

$$\int \varphi^2 f\,\mathrm{d}(\mathbf{\Delta}f + \lambda_1 f\mathfrak{m}) \leq \int f^2|\mathrm{d}\varphi|^2\,\mathrm{d}\mathfrak{m}. \tag{3.4.9}$$

Notice that (3.4.8) and the assumption $\lambda_1 \geq N-2$ tell in particular that $\mathbf{\Delta}f + \lambda_1 f\mathfrak{m} \geq 0$, then fix $R > 0$, let $\varphi := (1 - R^{-1}\mathsf{d}(\cdot, B_R(p)))^+$ in the above then let $R \uparrow \infty$ to get

$$\int f\,\mathrm{d}(\mathbf{\Delta}f + \lambda_1 f\mathfrak{m}) \leq \liminf_{R\to+\infty}\frac{1}{R^2}\int_{B_{2R}(p)\setminus B_R(p)} f^2\,\mathrm{d}\mathfrak{m}. \tag{3.4.10}$$

Let us prove that the right hand side is zero. Putting $A_R := B_{2R}(p) \setminus B_R(p)$, from Holder's inequality we get

$$\begin{aligned} \int_{A_R} f^2 \, \mathrm{d}\mathfrak{m} &\leq \Big(\int_{A_R} |\mathrm{d}u|^2 e^{2\sqrt{\lambda_1}\mathsf{d}_p} \, \mathrm{d}\mathfrak{m} \Big)^{\frac{N-2}{N-1}} \\ &\quad \cdot \Big(\int_{A_R} e^{-2(N-2)\sqrt{\lambda_1}\mathsf{d}_p} \, \mathrm{d}\mathfrak{m} \Big)^{\frac{1}{N-1}} \\ \text{(by (3.3.4))} \qquad &\leq C R^{\frac{N-2}{N-1}} \Big(\int_R^{2R} e^{-2(N-2)\sqrt{\lambda_1}\mathsf{d}_p} s(\mathsf{d}_p) \, \mathrm{d}\mathfrak{m} \Big)^{\frac{1}{N-1}} \\ \text{(by (1.5.11))} \qquad &\leq C R^{\frac{N-2}{N-1}} \Big(\int_R^{2R} e^{(-2(N-2)\sqrt{\lambda_1}+N-1)\mathsf{d}_p} \, \mathrm{d}\mathfrak{m} \Big)^{\frac{1}{N-1}} \leq CR, \end{aligned}$$

where in the last inequality we used the fact that $-2(N-2)\sqrt{\lambda_1}+N-1 \leq 0$, that in turn follows from $\lambda_1 \geq N-2$ and $N \geq 3$.

Therefore from (3.4.10) we see that $\int f \, \mathrm{d}(\boldsymbol{\Delta} f + \lambda_1 f \mathfrak{m}) \leq 0$ and thus from (3.4.8) we conclude that

$$(\lambda_1 - (N-2)) \int f^2 \, \mathrm{d}\mathfrak{m} + \int |\mathrm{d}u|^{-\frac{2}{N-1}} F \, \mathrm{d}\mathfrak{m} \leq 0.$$

Since $\lambda_1 \geq N-2$ and f is not identically 0 (as u is not constant), we deduce that $\lambda_1 = N-2$, i.e. item (i) holds. Also, since $|\mathrm{d}u|^{-\frac{2}{N-1}} \geq \mathrm{LIP}(u)^{-\frac{2}{N-1}} > 0$, we see that $F = 0$ $\mathfrak{m}$-a.e., and thus by (3.4.6) item (iii) holds as well.

Now notice that we now know that

$$\boldsymbol{\Delta} f = -(N-2) f \mathfrak{m} \tag{3.4.11}$$

and by the compatibility of the concepts of measure-valued and L^2-valued Laplacian (see Proposition 3.1.2), item (iv) follows. Moreover, we also deduce that $\boldsymbol{\Delta} f \leq 0$, hence f is superharmonic on any bounded open subset of X. Since X is locally doubling and supports a local weak Poincaré inequality, point (ii) follows from the weak Harnack inequality (see e.g. [BB, Chapter 8]) recalling, once again, that f is not identically 0. Even more, knowing (3.4.11) and the Bochner inequality it is not hard to see that

$$\boldsymbol{\Delta} \tfrac{|\mathrm{d}f|^2}{2} \geq \langle \mathrm{d}f, \mathrm{d}\Delta f \rangle - (N-1)|\mathrm{d}f|^2 = -(2N-3)|\mathrm{d}f|^2,$$

thus the same arguments just expressed tell that $|\mathrm{d}f|$ is locally bounded from above, i.e. that f is locally Lipschitz (by the Sobolev-to-Lipschitz property). Hence the same holds for $|\mathrm{d}u| = f^{\frac{N-1}{N-2}}$.

It remains to prove that u is open. Fix $\bar{x} \in \mathrm{X}$, let $\xi : \mathrm{X} \to [0,1]$ be a Lipschitz cut-off function with compact support identically 1 on $B_2(\bar{x})$.

Let (Fl_t) be the Regular Lagrangian Flow of $\xi\nabla u$, that is easily seen to exists as $|\xi\nabla u| \in L^\infty$, $\mathrm{div}(\xi\nabla u) = \langle\nabla\xi, \nabla u\rangle \in L^\infty$ and $\xi\nabla u \in W^{1,2}_C(T\mathrm{X})$ by Proposition 1.6.16. By the defining property (1.8.3) and the finite speed of propagation that follows from (1.8.5), we see that there is $T > 0$ such that $\partial_t u(\mathsf{Fl}_t(x)) = |\mathrm{d}u|^2(\mathsf{Fl}_t(x))$ holds for $\mathfrak{m}$-a.e. $x \in B_1(\bar{x})$ and $t \in [-T, T]$. For these x, t, recalling also item (ii), we see that $\partial_t u(\mathsf{Fl}_t(x)) \geq c$ for $c := \inf_{B_2(\bar{x})} |\mathrm{d}u|r > 0$. Fix x for which this holds for a.e. $t \in [-T, T]$ and use the continuity of u to deduce that the image under u of $\{\mathsf{Fl}_t(x) : t \in [-T, T]\}$ contains $[u(x) - Tc, u(x) + Tc]$. Picking x sufficiently close to $\bar{x}$ we conclude that $u(B_1(\bar{x}))$ contains a neighbourhood of $u(x)$ and repeating the argument with $B_r(\bar{x})$, $r \ll 1$, in place of $B_1(\bar{x})$ we conclude. □

3.4.2 $|\mathrm{d}u|$ as a function of u

In this section we prove that the minimal weak upper gradient of u is of the form $\varphi \circ u$ for a suitable smooth function φ. Notice that from (3.4.4) it follows that

$$\mathrm{d}|\mathrm{d}u| = \mathrm{Hess}\, u\big(\tfrac{\nabla u}{|\mathrm{d}u|}\big) = \alpha(1 - N)\tfrac{\mathrm{d}u}{|\mathrm{d}u|},$$

i.e. $\mathrm{d}|\mathrm{d}u| = h\mathrm{d}u$ for some function h. In the smooth category, this suffices to conclude that, locally, $|\mathrm{d}u|$ is a function of u, a natural line of thought being: let U be a small open set such that level sets of u are smooth-path-connected in U (recall that $|\mathrm{d}u| > 0$), then pick $x, y \in U$ with $u(x) = u(y)$ and find a smooth curve γ joining them, with values in U and with $t \mapsto u(\gamma_t)$ constant. Then $0 = \partial_t u(\gamma_t) = \mathrm{d}u(\gamma'_t)$ and thus $\partial_t(|\mathrm{d}u|(\gamma_t)) = \mathrm{d}|\mathrm{d}u|(\gamma'_t) = h(\gamma_t)\mathrm{d}u(\gamma'_t) = 0$, proving that $|\mathrm{d}u|$ is also constant along γ and thus that, on U, the value of $|\mathrm{d}u|$ depends solely on that of u.

We are going to prove an analogous statement in our setting: roughly said, the underlying idea is the same just exposed, but the technicalities are much more involved.

Proposition 3.4.6. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathsf{RCD}(K, N)$ *space and fix two functions* $f, g \in \mathrm{LIP}_{loc}(\mathrm{X})$ *such that* $\mathrm{d}g = h\mathrm{d}f$ *for some* $h : \mathrm{X} \to \mathbb{R}$.

Also, assume that $f \in D(\Delta_{loc}) \cap W^{2,2}_{loc}(\mathrm{X})$ *is with* $\frac{1}{|\mathrm{d}f|}, |\Delta f|, |\mathrm{Hess}\, f|_{\mathsf{HS}} \in L^\infty_{loc}(\mathrm{X})$.

Then for every $x \in \mathrm{X}$ *there are a neighbourhood* U *and a Lipschitz function* $\varphi : \mathbb{R} \to \mathbb{R}$ *such that* $g = \varphi \circ f$ *on* U.

Proof. Fix $\bar{x} \in \mathrm{X}$ and let $\eta : \mathrm{X} \to [0, 1]$ be Lipschitz, with bounded support and identically 1 on $B_3(\bar{x})$. Then by direct computation we see that for the vector field $v := \eta\frac{\nabla f}{|\nabla f|^2}$ we have

$$\mathrm{div}(v) = \eta\frac{\Delta f}{|\nabla f|^2} - 2\eta\frac{\langle\nabla|\nabla f|, \nabla f\rangle}{|\nabla f|^3} + \frac{\langle\nabla\eta, \nabla f\rangle}{|\nabla f|^2}$$
$$\nabla v = \eta\frac{\mathrm{Hess}(f)}{|\nabla f|^2} - 2\eta\frac{\nabla|\nabla f| \otimes \nabla f}{|\nabla f|^3} + \frac{\nabla\eta \otimes \nabla f}{|\nabla f|^2}$$

so that our assumptions on f and η grant that v has bounded divergence and bounded covariant derivative. It follows from Theorem 1.8.7 that the Regular Lagrangian Flow $\mathsf{Fl} : \mathbb{R} \times \mathrm{X} \to \mathrm{X}$ of v is Lipschitz in space and time. By definition of Regular Lagrangian Flow and of v the equation $\partial_t(f(\mathsf{Fl}_t(x))) = \eta(\mathsf{Fl}_t(x))$ holds for $\mathfrak{m}$-a.e. x and a.e. t, but thanks to the continuity of η, Fl and f it is easy to see that in fact for any $x \in \mathrm{X}$ the map $t \mapsto f(\mathsf{Fl}_t(x))$ is C^1 with derivative equal to $\eta(\mathsf{Fl}_t(x))$.

Now consider the map $H : \mathrm{X} \times \mathbb{R} \times \mathbb{R} \to \mathrm{X}$ defined as $H(x, t, s) := \mathsf{Fl}_{t+s-f(x)}(x)$. Then this is clearly Lipschitz. Also, let $B \subset \mathrm{X}$ be bounded and $I \subset \mathbb{R}$ be a bounded interval. We claim that for some $C > 0$ we have

$$H_*(\mathfrak{m}|_B \times \delta_t \times \mathcal{L}^1|_I) \leq C\mathfrak{m} \qquad \text{for every } t \in I. \tag{3.4.12}$$

Indeed, since f is Lipschitz there is $T > 0$ such that $|t + s - f(x)| \leq T$ holds for any $(x, t, s) \in B \times I \times I$. It follows that for any $\varphi \in C_b(\mathrm{X})$ non-negative it holds

$$\begin{aligned}\int \varphi \,\mathrm{d}H_*(\mathfrak{m}|_B \times \delta_t \times \mathcal{L}^1|_I) &= \int_B \int_I \varphi(\mathsf{Fl}_{t+s-f(x)}) \,\mathrm{d}s \,\mathrm{d}\mathfrak{m}(x) \\ &\leq \int_B \int_{-T}^{T} \varphi(\mathsf{Fl}_r) \,\mathrm{d}r \,\mathrm{d}\mathfrak{m}(x) \leq 2T\mathsf{Comp}(\mathsf{Fl}) \int \varphi \,\mathrm{d}\mathfrak{m},\end{aligned}$$

thus proving our claim (3.4.12). Notice also that since $|v| \in L^\infty$, the estimate (1.8.5) trivially ensures that

$$\exists \bar{t} > 0 \text{ so that for any } x \in B_2(\bar{x}) \text{ we have } \mathsf{Fl}_t(x) \in B_3(\bar{x}) \text{ for any } t \in [-\bar{t}, \bar{t}]. \tag{3.4.13}$$

Put $\bar{r} := \min\{\frac{1}{2}, \frac{\bar{t}}{8}(\mathrm{LIP}(f|_{B_3(\bar{x})}))^{-1}\}$. We claim that

$$\text{there is } \varphi : \mathbb{R} \to \mathbb{R} \text{ such that } g = \varphi \circ f \text{ on } B_{\bar{r}}(\bar{x}) \tag{3.4.14}$$

and argue by contradiction. If not, there are $x_0, x_1 \in B_{\bar{r}}(\bar{x})$ with $f(x_0) = f(x_1)$ and $g(x_0) < g(x_1)$. By continuity we can find neighbourhoods $U_0, U_1 \subset B_{\bar{r}}(\bar{x})$ of x_0, x_1 respectively such that $\inf_{U_1} g > \sup_{U_0} g$. Then we can find other neighbourhoods $V_i \subset U_i$ of x_i, $i = 0, 1$, and $\bar{s} \in (0, \frac{\bar{t}}{2})$ such that $\mathsf{Fl}_s(x) \in U_i$ for every $x \in V_i$, $s \in [-\bar{s}, \bar{s}]$, $i = 0, 1$.

Let $\mu_i := \mathfrak{m}(V_i)^{-1}\mathfrak{m}|_{V_i}$, $i = 0, 1$ and let $\boldsymbol{\pi}$ be the only (by [GRS]) optimal geodesic plan joining μ_0 to μ_1. Then [R12a] ensures that $\boldsymbol{\pi}$ is a test plan and clearly it is concentrated on geodesics taking values in $B_{2\bar{r}}(\bar{x}) \subset B_1(\bar{x})$. Also, let $G : C([0,1], \mathrm{X}) \times \mathbb{R} \to C([0,1], \mathrm{X})$ be given by $G(\gamma, s)_t := H(\gamma_t, f(x_0), s)$ and put $\hat{\boldsymbol{\pi}} := G_*(\boldsymbol{\pi} \times (\frac{1}{2\bar{s}}\mathcal{L}^1|_{[-\bar{s},\bar{s}]}))$. From the fact that H is Lipschitz and (3.4.12) it directly follows that $\hat{\boldsymbol{\pi}}$ is a test plan as well and the construction also ensures that $(\mathrm{e}_i)_*\hat{\boldsymbol{\pi}}$ is concentrated on V_i, $i = 0, 1$.

We make now the intermediate claim:

$$\hat{\boldsymbol{\pi}} \text{ is concentrated on curves along which } f \text{ is constant.} \tag{3.4.15}$$

To see this, notice that $\boldsymbol{\pi}$ is concentrated on geodesics having endpoints in $B_{\bar{r}}(\bar{x})$ and thus that it is sufficient to prove that for any such geodesic γ, any $t \in [0,1]$ and $s \in [-\bar{s}, \bar{s}]$ we have

$$f(\mathsf{Fl}_{f(x_0)+s-f(\gamma_0)}(\gamma_t)) = f(x_0) + s.$$

Since $r \mapsto f(\mathsf{Fl}_r(\gamma_t))$ is C^1 with derivative $\eta(\mathsf{Fl}_r(\gamma_t))$, the claim follows if we show that $\mathsf{Fl}_r(\gamma_t) \in B_3(\bar{x}) \subset \{\eta = 1\}$ for any $|r| \le |f(x_0) + s - f(\gamma_0)|$. To see this, notice that γ takes values in $B_{2\bar{r}}(\bar{x}) \subset B_2(\bar{x})$ and thus $|f(\gamma_0) + s - f(\gamma_t)| \le \bar{s} + \mathrm{LIP}(f|_{B_3(\bar{x})})\mathsf{d}(\gamma_0, \gamma_t) \le \bar{t}$ for any $t \in [0,1]$ and $s \in [-\bar{s}, \bar{s}]$. The claim then follows from (3.4.13).

From (3.4.15) and the definition of speed of a test plan (see Theorem 1.4.3) we have that

$$0 = f \circ \mathrm{e}_s - f \circ \mathrm{e}_t = \int_t^s [\mathrm{e}_r^* \mathrm{d}f](\hat{\boldsymbol{\pi}}'_r)\,\mathrm{d}r \qquad \hat{\boldsymbol{\pi}}\text{-a.e.}$$

for every $t, s \in [0,1]$, $t < s$. By Fubini's theorem this implies that for a.e. $t \in [0,1]$ the identity $[\mathrm{e}_t^* \mathrm{d}f](\hat{\boldsymbol{\pi}}'_t) = 0$ holds $\hat{\boldsymbol{\pi}}$-a.e.. Now we can use our assumption $\mathrm{d}g = h\mathrm{d}f$ (noticing that $|h| \in L^\infty_{loc}(\mathrm{X})$ as a consequence of $|\mathrm{d}g|, \frac{1}{|\mathrm{d}f|} \in L^\infty_{loc}(\mathrm{X})$) to deduce that

$$g \circ \mathrm{e}_1 - g \circ \mathrm{e}_0 = \int_0^1 [\mathrm{e}_r^* \mathrm{d}g](\hat{\boldsymbol{\pi}}'_r)\,\mathrm{d}r = \int_0^1 h \circ \mathrm{e}_r [\mathrm{e}_r^* \mathrm{d}f](\hat{\boldsymbol{\pi}}'_r)\,\mathrm{d}r = 0 \qquad \hat{\boldsymbol{\pi}}\text{-a.e..}$$

This latter identity, however, is in contradiction with the fact that

$$\int g \circ \mathrm{e}_0\,\mathrm{d}\hat{\boldsymbol{\pi}} = \int g\,\mathrm{d}(\mathrm{e}_0)_*\hat{\boldsymbol{\pi}} \le \sup_{V_0} g < \inf_{V_1} g \le \int g\,\mathrm{d}(\mathrm{e}_1)_*\hat{\boldsymbol{\pi}} = \int g \circ \mathrm{e}_1\,\mathrm{d}\hat{\boldsymbol{\pi}},$$

thus proving the claim (3.4.14).

Property (3.4.14) defines the real valued function φ on the connected - being the continuous image of a connected - set $I := f(B_{\bar r}(\bar x)) \subset \mathbb{R}$. To conclude the proof it is therefore enough to show that $\varphi : I \to \mathbb{R}$ is locally Lipschitz, with a control on the local Lipschitz constant independent on the chosen neighbourhood.

Thus let $x \in B_{\bar r}(\bar x)$, put $\alpha := f(x)$ and notice that for $|t| \ll 1$ we have $\mathsf{Fl}_t(x) \in B_{\bar r}(\bar x)$ as well with - by the above discussion - $f(\mathsf{Fl}_t(x)) = \alpha + t$. Conclude noticing that

$$\begin{aligned}|\varphi(\alpha+t) - \varphi(\alpha)| &= |\varphi(f(\mathsf{Fl}_t(x))) - \varphi(f(x))|\\ &= |g(\mathsf{Fl}_t(x)) - g(x)| \leq \mathrm{LIP}(g|_{B_{\bar r}(\bar x)})\|v\|_{L^\infty}|t|,\end{aligned}$$

where in the last inequality we used the fact that the speed of the curve $s \mapsto \mathsf{Fl}_s(x)$ is uniformly bounded by $\|v\|_{L^\infty}$. □

In general this last result cannot be globabilized. In our case, however, this is possible thanks to the global properties of the function u we proved in Section 3.4.1.

Proposition 3.4.7. *With the same notation and assumptions as in Assumption 3.4.2 the following holds.*

There exists a function $\varphi \in C^\infty_{loc}(I)$ *such that* $|\mathrm{d}u| = \varphi \circ u$, *where* $I := u(\mathrm{X})$, *and it satisfies*

$$\tfrac{1}{1-N}\varphi\varphi'' = 1 - \tfrac{1}{(1-N)^2}(\varphi')^2. \tag{3.4.16}$$

Proof. For $x \in \mathrm{X}$ let U_x and φ_x be the neighbourhood and the function given by Proposition 3.4.6 above. By item (ii) in Proposition 3.4.5 we know that φ_x is strictly positive. Now suppose for a moment that we already know that φ_x is $C^{1,1}_{loc}$ for every $x \in \mathrm{X}$.

We know that $|\mathrm{d}u| = \varphi_x \circ u$ holds on U_x, thus the regularity of φ_x and its positivity justify the chain rules

$$\begin{aligned}\Delta(|\mathrm{d}u|^{\frac{N-2}{N-1}}) &= \mathrm{div}(\tfrac{N-2}{N-1}(\varphi_x^{-\frac{1}{N-1}}\varphi_x') \circ u\nabla u)\\ &= \Big(- \tfrac{N-2}{(N-1)^2}\,\varphi_x^{\frac{N-2}{N-1}}(\varphi_x')^2 + \tfrac{N-2}{N-1}\,\varphi_x^{1+\frac{N-2}{N-1}}\varphi_x''\Big) \circ u\end{aligned}$$

having used also the fact that u is harmonic.

Since (3.4.5) can now be written as $\Delta(|\mathrm{d}u|^{\frac{N-2}{N-1}}) = -(N-2)\varphi_x^{\frac{N-2}{N-1}} \circ u$ we conclude that φ_x satisfies (3.4.16). It then follows by standard bootstrapping that φ_x is C^∞_{loc}, as in the statement.

The ODE (3.4.16) also gives, by ODE uniqueness, the desired rigidity, as it is clear that any two solutions $\varphi_x, \varphi_{x'}$ defined on some intervals $I_x, I_{x'} \subset$

$(0,1)$ that coincide in some non-trivial interval $I \subset I_x \cap I_{x'}$ must coincide on the whole $I_x \cap I_{x'}$ and that the natural 'glued' function defined on $I_x \cup I_{x'}$ is still a solution. Thus fix $\bar{x} \in \mathrm{X}$, let φ be the maximal solution of (3.4.16) extending $\varphi_{\bar{x}}$ and let $A \subset \mathrm{X}$ the set of those x's such that φ_x is the restriction of φ to some subinterval of its domain of definition. Since U_x is a neighbourhood of x and φ_x is defined on the open set $u(U_x)$ (by item (v) in Proposition 3.4.5), it is clear that A is both open and closed. Since it is not empty and X is connected (being geodesic) we conclude that $A = \mathrm{X}$.

We now prove that φ_x is $C^{1,1}_{loc}$ and to this aim we start claiming that for $\eta \in W^{1,2}(\mathrm{X})$ with support compact and contained in U_x the measure $\mu := u_*(\eta\mathfrak{m})$ is absolutely continuous with respect to $\mathcal{L}^1$ with density that is also absolutely continuous. Indeed, let ζ be a Lipschitz cut-off function with compact support and identically 1 on a neighbourhood of $\mathrm{supp}(\eta)$ and let (Fl_t) be the Regular Lagrangian Flow of $\zeta\nabla u$ (whose existence and uniqueness follow from $|\zeta\nabla u| \in L^\infty$, $\mathrm{div}(\zeta\nabla u) = \langle\nabla\zeta, \nabla u\rangle \in L^\infty$ and $\zeta\nabla u \in W^{1,2}_C(T\mathrm{X})$ by Proposition 1.6.16). Then by Proposition 1.8.4 we see that $\frac{u\circ\mathsf{Fl}_t - u}{t} \to |\mathrm{d}u|^2$ in $L^2(\mathrm{supp}(\eta), \mathfrak{m})$ as $t \to 0$. Notice also that item (ii) in Proposition 3.4.5 yields that $\frac{1}{|\mathrm{d}u|^2} \in W^{1,2}_{loc} \cap L^\infty_{loc}(\mathrm{X})$ (for Sobolev regularity recall that $u \in H^{2,2}_{loc}(\mathrm{X})$ and [G18b, Proposition 3.3.22]), thus the following computation is justified for any $\xi \in C^\infty_c(\mathbb{R})$:

$$
\begin{aligned}
-\int \xi' \,\mathrm{d}\mu = -\int \xi' \circ u\, \eta \,\mathrm{d}\mathfrak{m} &= -\lim_{t\to 0}\int \frac{\xi\circ u\circ\mathsf{Fl}_t - \xi\circ u}{t}\,\frac{\eta}{|\mathrm{d}u|^2}\,\mathrm{d}\mathfrak{m}\\
&= \int \xi\circ u\,\langle \mathrm{d}\Big(\frac{\eta}{|\mathrm{d}u|^2}\Big), \mathrm{d}u\rangle\,\mathrm{d}\mathfrak{m} = \int \xi\,\mathrm{d}\nu,
\end{aligned}
$$

where $\nu := u_*\Big(\langle \mathrm{d}\Big(\frac{\eta}{|\mathrm{d}u|^2}\Big), \mathrm{d}u\rangle\,\mathfrak{m}\Big)$.

This proves that the distributional derivative of μ is a Radon measure, and thus that $\mu \ll \mathcal{L}^1$ with BV density. To prove that the density is absolutely continuous it suffices to prove that $\nu \ll \mathcal{L}^1$. But this is obvious, as a direct consequence of what just proved is that $u_*\mathfrak{m} \ll \mathcal{L}^1$.

We are now ready to show that $\varphi = \varphi_x$ is $C^{1,1}_{loc}(u(U_x))$. We know from Proposition 3.4.6 that it is locally Lipschitz and that in U_x we have $|\mathrm{d}u| = \varphi\circ u$. Since we already recalled that $|\mathrm{d}u| \in W^{1,2}_{loc}$ we see that $\mathrm{d}|\mathrm{d}u| = \varphi'\circ u\,\mathrm{d}u$. Also, we know from (3.4.5) and the chain rule for the Laplacian that in U_x we have $\Delta(|\mathrm{d}u|) = \psi\circ u$ for some locally bounded function ψ, that we can rewrite as $\mathrm{div}(\varphi'\circ u\,\mathrm{d}u) = \psi\circ u$. Now we take $\eta \in \mathrm{Test}(\mathrm{X})$ with support in U_x and $\xi \in C^\infty(\mathbb{R})$ and observe that

$$
\int \langle \mathrm{d}(\xi\circ u\,\eta), \varphi'\circ u\,\mathrm{d}u\rangle\,\mathrm{d}\mathfrak{m} = -\int (\psi\,\xi)\circ u\,\eta\,\mathrm{d}\mathfrak{m} = -\int \psi\,\xi\rho\,\mathrm{d}\mathcal{L}^1,
$$

where ρ is the density of $u_*(\eta\mathfrak{m})$. On the other hand we also have

$$\int \langle \mathrm{d}(\xi\circ u\,\eta), \varphi'\circ u\,\mathrm{d}u\rangle\,\mathrm{d}\mathfrak{m} = \int (\xi'\varphi')\circ u\,|\mathrm{d}u|^2\eta + (\varphi'\xi)\circ u\langle \mathrm{d}\eta, \mathrm{d}u\rangle\,\mathrm{d}\mathfrak{m}$$
$$= \int \xi'\varphi'\rho_1 + \xi\varphi'\rho_2\,\mathrm{d}\mathcal{L}^1,$$

where ρ_1, ρ_2 are the densities of $u_*(|\mathrm{d}u|^2\eta\mathfrak{m})$ and $u_*(\langle \mathrm{d}\eta, \mathrm{d}u\rangle\,\mathfrak{m})$, respectively. By the arbitrariness of ξ we proved that the distributional derivative of $\varphi'\rho_1$ is equal to $\varphi'\rho_2 + \psi\rho \in L^1_{loc}$, and thus that $\varphi'\rho_1$ is (more precisely: has a representative that is) absolutely continuous. Since $|\mathrm{d}u|^2\eta$ is in $W^{1,2}(\mathrm{X})$ and has support in U_x, what previously proved shows that ρ_1 is also absolutely continuous.

We thus deduce that φ' is locally absolutely continuous on $\{\rho_1 > 0\} = u(\{\eta > 0\})$. Taking $\eta = \eta_n$ for (η_n) such that of $\cup_n\{\eta_n > 0\} = U_x$ we conclude. □

3.4.3 Proof of Theorem 3.4.1

We are now ready to prove Theorem 3.4.1 combining the results of Chapter 2 and Sections 3.3, 3.4.1 and 3.4.2.

Proof of Theorem 3.4.1. Since $\lambda_1 \geq N-2 > 0$, we know from Proposition 3.3.9 that X has at least one end of infinite measure. Assume it has at least two of these. Then Corollary 3.3.7 gives the existence of a non-constant bounded harmonic function u on the whole X and then Proposition 3.4.7 that

$$|\mathrm{d}u| = \varphi\circ u \tag{3.4.17}$$

for a positive smooth function φ on $u(\mathrm{X})$ that satisfies

$$\tfrac{1}{1-N}\varphi\varphi'' = 1 - \tfrac{1}{(1-N)^2}(\varphi')^2. \tag{3.4.18}$$

Then Proposition 3.4.5 tells that $|\mathrm{d}u| \neq 0$ a.e., and that putting $\mathrm{e}_1 := \frac{\nabla u}{|\nabla u|}$ we have

$$\operatorname{Hess} u = \zeta\circ u|\mathrm{d}u|\mathrm{Id} - N\zeta\circ u|\mathrm{d}u|\mathrm{e}_1\otimes\mathrm{e}_1 \qquad \text{for} \qquad \zeta = \tfrac{\varphi'}{1-N}. \tag{3.4.19}$$

We can therefore apply Lemma 2.1.1: let η be so that $\eta' = \frac{1}{\varphi}$ (η is defined up to an additive constant: the value of such constant will be chosen in a moment) and define the function $\mathrm{b} := \eta \circ u$. Then $\mathrm{b} \in H^{2,2}_{loc}(\mathrm{X})$ with

$$|\mathrm{db}| \equiv 1, \tag{3.4.20}$$

$$\mathrm{Hess}\,\mathrm{b} = \psi_{\mathsf{d}} \circ \mathrm{b}\big(\mathrm{Id} - \mathrm{e}_1 \otimes \mathrm{e}_1\big), \quad \text{for} \quad \psi_{\mathsf{d}} := \zeta \circ \eta^{-1} \tag{3.4.21}$$

$$\Delta \mathrm{b} = \psi_{\mathfrak{m}} \circ \mathrm{b}, \quad \text{for} \quad \psi_{\mathfrak{m}} := (N-1)\zeta \circ \eta^{-1} = (N-1)\psi_{\mathsf{d}}. \tag{3.4.22}$$

To find explicitly ψ_{d} and $\psi_{\mathfrak{m}}$ notice that

$$\zeta'\varphi \overset{(3.4.19)}{=} \frac{\varphi''\varphi}{1-N} \overset{(3.4.18)}{=} 1 - \frac{(\varphi')^1}{(N-1)^2} \overset{(3.4.19)}{=} 1 - \zeta^2 \tag{3.4.23}$$

and thus

$$\psi_{\mathsf{d}}' = (\zeta \circ \eta^{-1})' = (\zeta' \tfrac{1}{\eta'}) \circ \eta^{-1} = (\zeta'\varphi) \circ \eta^{-1} \overset{(3.4.23)}{=} 1 - \zeta^2 \circ \eta^{-1} = 1 - \psi_{\mathsf{d}}^2$$

proving that $\psi_{\mathsf{d}}(z) = \tanh(z + c)$ for some $c \in \mathbb{R}$. Since $\psi_{\mathsf{d}} \circ \eta$ is equal to the given function ζ, we see that replacing η with $\tilde{\eta} := \eta + \alpha$ for $\alpha \in \mathbb{R}$ means replacing ψ_{d} with $\tilde{\psi}_{\mathsf{d}} = \psi_{\mathsf{d}}(\cdot - \alpha)$. Thus we can, and will, choose η so that $\psi_{\mathsf{d}} = \tanh$.

It follows by the defining properties (2.2.9b), (2.2.10) that $w_{\mathsf{d}} = \cosh$. By (3.4.22) we also have $\psi_{\mathfrak{m}} = (N-1)\tanh$ and thus $w_{\mathfrak{m}} = \cosh^{N-1}$.

Then Theorem 2.5.3 gives the required warped product structure, where the fiber is the space $\mathrm{X}' := \mathrm{b}^{-1}(0)$ equipped with the distance d' and the measure $\mathfrak{m}'$ defined in (2.2.3) and (2.2.4) respectively. Moreover, we can apply Proposition 2.5.4 with $\bar{z} = 0$ to deduce that $(\mathrm{X}', \mathsf{d}', \mathfrak{m}')$ is $\mathsf{RCD}(-(N-1), N)$.

It remains to prove that X' is compact. Say not. We are going to prove that in this case X has at most one end, thus contradicting the assumption made at the beginning of the proof.

Let $\mathrm{K} \subset \mathrm{X}$ be compact and let $\mathrm{K}' := \mathrm{Pr}^{-1}(\mathrm{Pr}(\mathrm{K})) \cap \mathrm{b}^{-1}(\mathrm{b}(\mathrm{K}))$. Since $\mathsf{T}(\mathrm{K}') = \mathrm{b}(\mathrm{K}) \times \mathrm{Pr}(\mathrm{K}) \subset \mathbb{R} \times_w \mathrm{X}'$ we see that the 'rectangle' K' is also compact. Let $x_0, x_1 \in \mathrm{X} \setminus \mathrm{K}'$. Then for $i = 0, 1$ either $\mathrm{Pr}(x_i) \notin \mathrm{Pr}(\mathrm{K}')$ or $\mathrm{b}(x_i) \notin \mathrm{b}(\mathrm{K}')$ (or both). Say $\mathrm{b}(x_i) \notin \mathrm{b}(\mathrm{K}')$ for $i = 0, 1$ and use the assumption that X' is not compact to find $z \in \mathrm{X}' \setminus \mathrm{Pr}(\mathrm{K}')$. Then the curve $t \mapsto \mathsf{Fl}_t(z)$ (here Fl is the flow of $\nabla \mathrm{b}$) does not meet K' and, moreover, there are $t_0, t_1 \in \mathbb{R}$ with $\mathrm{b}(\mathsf{Fl}_{t_i}(z)) = \mathrm{b}(x_i)$, $i = 0, 1$. Recalling that the level sets of b are path connected, we can find curves joining x_i and $\mathsf{Fl}_{t_i}(z)$ lying entirely on level sets. The assumption $\mathrm{b}(x_i) \notin \mathrm{b}(\mathrm{K}')$ ensures that these curves do not meet K', thus the path obtained by gluing these curves and $t \mapsto \mathsf{Fl}_t(z)$ produces a curve from x_0 to x_1 that does not meet K', showing that x_0, x_1 belong to the same connected component of $\mathrm{X} \setminus \mathrm{K}'$. Since an analogous construction can be made if one, or both, of the x_i's are with $\mathrm{Pr}(x_i) \notin \mathrm{Pr}(\mathrm{K}')$, we see that $\mathrm{X} \setminus \mathrm{K}'$ has only one connected component, providing the desired contradiction. □

3.5 Rigidity theorem for spaces with maximal spectrum

By Cheng's Theorem the first eigenvalue of the Laplacian of an n-dimensional Riemannian manifold with Ricci curvature bounded from below by $-(n-1)$ is bounded from above by $\frac{(n-1)^2}{4}$. The same result holds in the non-smooth setting: it is a consequence of the following result proved in metric measure space by Sturm (see [S94, Theorem 5]).

Theorem 3.5.1. *Let* $v_*(r) := \inf_{x\in \mathrm{X}} \frac{\mathfrak{m}(B_r(x))}{B_1(x)}$. *If there exists* $k \geq 0$ *such that*

$$\liminf_{r\to+\infty} \frac{1}{r} \log(v_*(r)) \leq k,$$

then $\lambda_1 \in \left[0, \frac{k^2}{4}\right]$.

Theorem 3.5.2. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathsf{RCD}(K, N)$ *with* $K < 0$ *and* $N \in (1, +\infty)$. *Then*

$$\lambda_1 \leq \frac{-(N-1)K}{4}.$$

Proof. By the Bishop-Gromov inequality (1.5.12) we can estimate $v_*(r)$ from above with $Ce^{r\sqrt{-(N-1)K}}$. Theorem 3.5.1 concludes the proof. □

Its proof easily follows combining the Bishop-Gromov inequality (1.5.12) with the following theorem (see [S94, Theorem 5]).

Theorem 3.5.2 motivates the study of the case $\lambda_1 = \frac{(N-1)^2}{4}$, i.e. the spaces in which λ_1 is the maximum admissible.

3.5.1 Isomorphism with the warped product

The proof of Theorem 0.0.8 is very similar to the one of Theorem 3.4.1 saw in the previous section. We recall the precise statement of the theorem.

Theorem 3.5.3. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathsf{RCD}(-(N-1), N)$ *space with* $N > 3$ *and* $\mathrm{supp}(\mathfrak{m}) = \mathrm{X}$, *and assume that the first eigenvalue of the Laplacian* λ_1 *is equal to* $\frac{(N-1)^2}{4}$. *Then one of the following holds:*

i) X *has only one end;*

ii) X *is isomorphic as metric measure space to a warped product space* $\mathbb{R}\times_w \mathrm{X}'$, *where* X' *is a compact* $\mathsf{RCD}(0, N)$ *space and the warping functions are*

$$w_{\mathsf{d}}(t) := e^t \qquad \text{and} \qquad w_{\mathfrak{m}}(t) := e^{Nt}.$$

To begin we observe that $\lambda_1(\mathrm{X}) > N-2$, then, by Theorem 3.4.1, the space has only one end with infinite volume. We fix then the following assumptions.

Assumption 3.5.4. $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *is an* $\mathsf{RCD}(-(N-1), N)$ *space with* $N > 3$ *and* $\lambda_1 = \frac{(N-1)^2}{4}$. *We also assume that* $\mathrm{supp}(\mathfrak{m}) = \mathrm{X}$ *and that* X *has at least two ends.*

Finally, we shall denote by u *the positive and non-constant harmonic function on* X *given by Corollary 3.3.8.*

As we did in Proposition 3.4.5, we prove that the positive non-constant harmonic function u satisfies the equality in the Bochner+Kato inequality (3.4.7) with $F = 0$, and from this we deduce its Hessian.

As in the proof of Proposition 3.4.5, an inequality (the equivalent of (3.4.8)) comes from the Bochner inequality, and the other follows by the definition of λ_1 and the estimates (3.3.14) and (3.3.15) (as in (3.4.10)). The first inequality in this case has been studied by H.-C. Zhang and X.-P. Zhu in [ZZ], where they proved a sharp estimate for the gradient of harmonic functions.

Theorem 3.5.5 (Li-Yau gradient estimate)**.** *Let* $K \leq 0$ *and* $N \in (1, +\infty)$, *and let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathsf{RCD}(K, N)$ *metric measure space. Taking* u *positive harmonic function on* X *it holds*

$$|\mathrm{d}u| \leq \sqrt{-K(N-1)}u.$$

Proposition 3.5.6 (Properties of the Busemann function)**.** *With the same assumptions and notation as in Assumption 3.5.4 the following hold:*

i) $|\mathrm{d}u| = (N-1)u$ *holds* $\mathfrak{m}$*-a.e., in particular* $|\mathrm{d}u|$ *is locally Lipschitz and strictly positive;*

ii) putting $\mathrm{e}_1 := \frac{\nabla u}{|\nabla u|}$, *which is well defined by the above, we have*

$$\mathrm{Hess}(u) = -(N-1)u(\mathrm{Id} - N\mathrm{e}_1 \otimes \mathrm{e}_1). \tag{3.5.1}$$

Proof. Also in this case it is easy to see, as in Lemma 3.4.3, that $u \in \mathrm{Test}_{loc}(\mathrm{X})$ (since here u is not globally bounded we do not have the global Lipschitzianity in this case).

Starting with item (i), the inequality $|\mathrm{d}u| \leq (N-1)u$ is given by Theorem 3.5.5, so we only have to prove that the opposite inequality holds.

Let $f := \sqrt{u}$. Then, by the chain rule for the Laplacian (3.1.1) it holds

$$\mathbf{\Delta} f = -\frac{1}{4}u^{-\frac{3}{2}}|\mathrm{d}u|^2\mathfrak{m} \geq -\frac{(N-1)^2}{4}f\mathfrak{m}, \tag{3.5.2}$$

which means that, as in the proof of Proposition 3.4.5, $\boldsymbol{\Delta} f + \lambda_1 f \mathfrak{m} \geq 0$.

Moreover, repeating the same computations as in (3.4.9) we conclude that

$$\int \varphi^2 f \,\mathrm{d}(\boldsymbol{\Delta} f + \lambda_1 f \mathfrak{m}) \leq \int f^2 |\mathrm{d}\varphi|^2 \,\mathrm{d}\mathfrak{m}. \tag{3.5.3}$$

As in the proof of Proposition 3.4.5 the aim is to choose a sequence of functions $(\varphi_R) \to \hat{1}$ that sends the right hand side of (3.5.3) to 0. We start noticing that we can ignore every end with finite volume on which $u \leq 1$ (so every end with finite volume except for E_2), since we can choose φ_R similarly to what we did in the proof of Proposition 3.3.9 and see that their contribution in the right hand side of (3.5.3) is negligible.

In the following, for brevity and simplicity of the notation, we assume that the space has only the two ends E_1 and E_2 (noting that we can define φ_R independently on every end, so the argument above is enough to conclude in presence of other ends with finite volume).

Take φ_R defined as $\varphi_R := (1 - R^{-1}\mathsf{d}(\cdot, B_R(p)))^+$. Letting $R \uparrow +\infty$ we get

$$\int f \,\mathrm{d}(\boldsymbol{\Delta} f + \lambda_1 f \mathfrak{m}) \leq \liminf_{R \to +\infty} \frac{1}{R^2} \int_{B_{2R}(p) \setminus B_R(p)} f^2 \,\mathrm{d}\mathfrak{m}. \tag{3.5.4}$$

We prove that the right hand side is zero. More precisely, we claim that there exists a constant $C > 0$ such that for every $R > 0$ large enough it holds

$$\int_{B_R(p)} f^2 \,\mathrm{d}\mathfrak{m} = \int_{B_R(p)} u \,\mathrm{d}\mathfrak{m} \leq CR. \tag{3.5.5}$$

On the end with infinite volume E_1 we have that

$$\begin{aligned} \int_{E_1(R+1)\setminus E_1(R)} u \,\mathrm{d}\mathfrak{m} &\leq \left(\int_{E_1(R+1)\setminus E_1(R)} u^2 \,\mathrm{d}\mathfrak{m} \right)^{\frac{1}{2}} \sqrt{V_{E_1}(R+1)} \\ (3.3.2) + (3.3.14) \qquad &\leq C e^{-\frac{N-1}{2}R} e^{\frac{N-1}{2}R} = C, \end{aligned}$$

and summing up on R

$$\int_{E_1(R)} f^2 \,\mathrm{d}\mathfrak{m} \leq CR. \tag{3.5.6}$$

Focusing now on E_2, we note that, by Theorem 3.5.5 and the Sobolev-to-Lipschitz property it holds

$$u \leq C + e^{(N-1)\mathsf{d}_p} \qquad \mathfrak{m}\text{-a.e.},$$

then for every R large enough

$$\begin{aligned}\int_{E_2(R+1)\setminus E_2(R)} u\,\mathrm{d}\mathfrak{m} &\le \int_{E_2(R+1)\setminus E_2(R)} C + e^{(N-1)\mathsf{d}_p}\,\mathrm{d}\mathfrak{m}\\ &\le (C + e^{(N-1)(R+1)})(V_{E_2}(R+1) - V_{E_2}(R))\\ &\le (C + e^{(N-1)(R+1)})(V_{E_2}(\infty) - V_{E_2}(R))\\ &\le C(C + e^{(N-1)(R+1)})e^{-(N-1)R} = C. \qquad (3.3.15)\end{aligned}$$

Summing up on R we obtain that

$$\int_{E_2(R)} f^2\,\mathrm{d}\mathfrak{m} \le CR, \tag{3.5.7}$$

and this, together with (3.5.6) concludes the proof of (3.5.5).

We conclude that $\boldsymbol{\Delta} f = -\lambda_1 f\mathfrak{m}$, then, since the equality holds in (3.5.2), we have that

$$|\mathrm{d}u| = (N-1)u. \tag{3.5.8}$$

From an easy computation, we see that u satisfies an equality in the Bochner+Kato inequality, i.e. (3.4.7) holds with $F = 0$: indeed, using the chain rule for the Laplacian (3.1.1), $\boldsymbol{\Delta} u \equiv 0$ and (3.5.8) we have

$$\boldsymbol{\Delta}\tfrac{|\mathrm{d}u|^2}{2} = \frac{(N-1)^2}{2}\boldsymbol{\Delta}(u^2) = (N-1)^2|\mathrm{d}u|^2\mathfrak{m}$$

and

$$\begin{aligned}(\tfrac{N}{N-1}|\mathrm{d}|\mathrm{d}u||^2 - (N-1)|\mathrm{d}u|^2)\mathfrak{m} &= (N(N-1)|\mathrm{d}u|^2 - (N-1)|\mathrm{d}u|^2)\mathfrak{m}\\ &= (N-1)^2|\mathrm{d}u|^2\mathfrak{m}.\end{aligned}$$

Arguing as in the proof of Proposition 3.4.5 we conclude that the Hessian of u is of the form $\alpha(\mathrm{Id} - Ne_1 \otimes e_1)$ for $e_1 := \frac{\nabla u}{|\nabla u|}$ and

$$\alpha = \frac{\langle \mathrm{d}|\mathrm{d}u|, \mathrm{d}u\rangle}{(1-N)|\mathrm{d}u|} = -(N-1)u.$$

□

We observe that in this case we do not need to repeat the argument of Section 3.4.2, since we already have that $|\mathrm{d}u| = (N-1)u$, then we can conclude the proof of the "splitting" in Theorem 3.5.3 arguing as in Section 3.4.3.

Proposition 3.5.7. *With the same assumptions and notation as in Assumption 3.5.4 the following holds: the space* X *is isomorphic as metric measure space to a warped product space* $\mathbb{R}\times_w \mathrm{X}'$*, where* X' *is compact and the warping functions are*

$$w_{\mathsf{d}}(t) := e^t, \qquad \text{and} \qquad w_{\mathsf{m}}(t) := e^{(N-1)t}.$$

Proof. The proof is identical to the one of Theorem 3.4.1 except for the RCD condition for the quotient space X' (we will talk about that in the next section).

Fixing $\mathrm{b} := -\frac{\log(u)}{N-1}$, from Proposition 3.5.6 follows that

$$|\mathrm{db}| \equiv 1, \tag{3.5.9}$$
$$\mathrm{Hess}\,\mathrm{b} = \mathrm{Id} - \mathrm{e}_1 \otimes \mathrm{e}_1, \tag{3.5.10}$$
$$\Delta \mathrm{b} = N-1. \tag{3.5.11}$$

Defining w_{d} and w_{m} as in (2.2.9) and (2.2.10), we have that $w_{\mathsf{d}}(t) = e^t$ and $w_{\mathsf{m}} = e^{(N-1)t}$.

From Theorem 2.2.1 follows that X is isomorphic to $\mathbb{R}\times_w \mathrm{X}'$ with $\mathrm{X}' := \mathrm{b}^{-1}(0)$ and warping functions w_{d} and w_{m}. Moreover X' is compact since, if it was not then the space $\mathbb{R}\times_w \mathrm{X}'$ would have only one end (the argument is identical to the one in the proof of Theorem 3.4.1). □

3.5.2 RCD condition of the quotient space

In this case Theorem 2.5.4 cannot be applied, so we have to argue differently to prove the RCD condition of the quotient space X'.

This case has been studied in [CD+], we recall briefly their argument since it is completely different from ours in Theorem 2.5.4.

As seen in Section 2.5, the first three properties of Definition 1.5.1 are satisfied in the space X'. We prove that the Bochner inequality (1.5.1) holds too.

The first step to prove it is to study how the Laplacian of the warped product space $\mathbb{R}\times_w \mathrm{X}'$ behaves with respect to the Laplacian in X'. We denote by Δ the Laplacian in $\mathbb{R}\times_w \mathrm{X}'$ and by $\underline{\Delta}$ the Laplacian in X'.

Lemma 3.5.8. *Let* $\rho \in C^\infty_c(\mathbb{R})$ *and* $f \in D(\underline{\Delta})$*. Moreover assume that* $w_{\mathsf{m}} \in C^1(\mathbb{R})$*. Indicating with* $\overline{f}, \overline{\rho} : \mathbb{R}\times_w \mathrm{X}' \to \mathbb{R}$ *the functions* $\overline{f}(t,x') := f(x')$ *and* $\overline{\rho}(t,x') := \rho(t)$*, it holds* $\overline{f}\overline{\rho} \in D(\Delta)$ *and* $\Delta(\overline{f}\overline{\rho}) \in W^{1,2}(\mathbb{R}\times_w \mathrm{X}')$*.*

Moreover

$$\Delta(\overline{f}\overline{\rho}) = \rho w_{\mathsf{d}}^{-2}\underline{\Delta} f + (\rho' w_{\mathsf{m}})' w_{\mathsf{m}}^{-1} f. \tag{3.5.12}$$

Remark 3.5.9. In (3.5.12) with a little abuse of notation we omitted the right compositions with the projections $\pi_1(t,x') := t$ and $\pi_2(t,x') := x'$. ■

Proof. We indicate with X_w the warped product $\mathbb{R} \times_w \mathrm{X}'$. By Theorem 1.7.6 follows easily that function $\overline{f\rho}$ is in $W^{1,2}(\mathrm{X}_w)$ for every f, ρ as in the assumptions.

Recalling nations (2.2.5) and (2.4.2), let $\tilde{\mathcal{A}}$ be the set of functions $f : \mathrm{X}_w \to \mathbb{R}$ such that $f \circ \mathsf{T} \in \mathcal{A}$, and similarly we define $\tilde{\mathcal{G}}$ and $\tilde{\mathcal{H}}$. By Lemma 2.4.1 we have that $\tilde{\mathcal{A}}$ is dense in $W^{1,2}(\mathrm{X}_w)$.

Let $\varphi \in \mathrm{Test}(\mathrm{X}_w) \cap \tilde{\mathcal{A}}$ given by $\varphi = \sum_{i=1}^n a_i \overline{h}_i \overline{g}_i$, where $a_i \in \mathbb{R}$, $\overline{h}_i \in \tilde{\mathcal{H}}$ and $\overline{g}_i \in \tilde{\mathcal{G}}$.

Using the Leibniz rule and remembering the argument in Section 2.4, we have that

$$\int \langle \nabla(\overline{f}\overline{\rho}), \nabla\varphi\rangle \,\mathrm{d}\mathfrak{m}_w = \sum_{i=1}^n a_i \int \Big(\overline{f}\overline{g}_i \langle \nabla\overline{\rho}, \nabla\overline{h}_i\rangle + \overline{\rho}\overline{h}_i\langle \nabla\overline{f}, \nabla\overline{g}_i\rangle\Big)\,\mathrm{d}\mathfrak{m}_w.$$

Noting that, by Theorem 1.7.6 follows that

$$\langle \nabla\overline{f}, \nabla\overline{g}_i\rangle_{\mathrm{X}_w} = w_{\mathsf{d}}^{-2}\langle \nabla f, \nabla g_i\rangle_{\mathrm{X}'} \quad \text{and} \quad \langle \nabla\overline{\rho}, \nabla\overline{h}_i\rangle_{\mathrm{X}_w} = \rho' h_i',$$

we conclude that

$$\begin{aligned}\sum_{i=1}^n a_i \int \overline{\rho}\overline{h}_i \langle \nabla\overline{f}, \nabla\overline{g}_i\rangle \,\mathrm{d}\mathfrak{m}_w &= \sum_{i=1}^n a_i \int_{\mathbb{R}} f g_i w_{\mathsf{d}}^{-2} w_{\mathfrak{m}} \int_{\mathrm{X}'} \langle \nabla f, \nabla g_i\rangle \,\mathrm{d}\mathfrak{m}' \,\mathrm{d}\mathcal{L}^1 \\ &= -\sum_{i=1}^n a_i \int_{\mathbb{R}} \rho h_i w_{\mathsf{d}}^{-2} w_{\mathfrak{m}} \,\mathrm{d}\mathcal{L}^1 \int_{\mathrm{X}'} g_i \underline{\Delta} f, \mathrm{d}\mathfrak{m}'\end{aligned}$$

and

$$\begin{aligned}\sum_{i=1}^n a_i \int \overline{f}\overline{g}_i \langle \nabla\overline{\rho}, \nabla\overline{h}_i\rangle \,\mathrm{d}\mathfrak{m}_w &= \sum_{i=1}^n a_i \int_{\mathbb{R}} \rho' h_i' w_{\mathfrak{m}} \,\mathrm{d}\mathcal{L}^1 \int_{\mathrm{X}'} f g_i \,\mathrm{d}\mathfrak{m}' \\ &= -\sum_{i=1}^n a_i \int_{\mathbb{R}} (\rho' w_{\mathfrak{m}})' h_i \,\mathrm{d}\mathcal{L}^1 \int_{\mathrm{X}'} f g_i, \mathrm{d}\mathfrak{m}'.\end{aligned}$$

Then for every $\varphi \in \mathrm{Test}(\mathrm{X}_w) \cap \tilde{A}$ it holds

$$\int \langle \nabla(\overline{f}\overline{\rho}), \nabla\varphi\rangle \,\mathrm{d}\mathfrak{m}_w = -\int \varphi\Big(\rho w_{\mathsf{d}}^{-2} \underline{\Delta} f + f(w_{\mathfrak{m}}^{-1}(\rho' w_{\mathfrak{m}})'\Big)\,\mathrm{d}\mathfrak{m}_w, \qquad (3.5.13)$$

where, as in the previous remark, we omitted the right composition with the projections.

Since $\mathrm{Test}(\mathrm{X}_w) \cap \tilde{A}$ is dense in $W^{1,2}(\mathrm{X}_w)$, we have that (3.5.13) holds for every $\varphi \in W^{1,2}(\mathrm{X}_w)$, then we conclude that $\overline{f\rho} \in D(\Delta)$ and

$$\Delta(\overline{f\rho}) = \rho w_{\mathsf{d}}^{-2} \underline{\Delta} f + (\rho' w_{\mathsf{m}})' w_{\mathsf{m}}^{-1} f.$$

□

Starting from the weak Bochner inequality (1.5.1) on the space $\mathbb{R} \times_w \mathrm{X}'$ and using (3.5.12) we can see that, in our case, the Bochner inequality holds on X'.

Proposition 3.5.10. *Assume that* X' *and* $\mathbb{R} \times_w \mathrm{X}'$ *are as in Proposition 3.5.7. Then for every* $f \in D(\underline{\Delta})$ *such that* $\underline{\Delta} f \in W^{1,2}(\mathrm{X}')$ *and all non-negative* $g \in D(\underline{\Delta}) \cap L^\infty(\mathrm{X}')$ *such that* $\underline{\Delta} g \in L^\infty(\mathrm{X}')$ *the following is satisfied:*

$$\frac{1}{2} \int \underline{\Delta} g |\mathrm{d} f|^2_{\mathrm{X}'} \,\mathrm{d}\mathfrak{m}' \geq \frac{1}{N} \int g (\underline{\Delta} f)^2 \,\mathrm{d}\mathfrak{m}' + \int g \langle \nabla \underline{\Delta} f, \nabla f \rangle_{\mathrm{X}'} \,\mathrm{d}\mathfrak{m}'. \tag{3.5.14}$$

Sketch of the proof. Let $f \in D(\underline{\Delta})$ such that $\underline{\Delta} f \in W^{1,2}(\mathrm{X}')$, and $g \in D(\underline{\Delta}) \cap L^\infty(\mathrm{X}')$ such that $g \geq 0$ and $\underline{\Delta} g \in L^\infty(\mathrm{X}')$.

Take $\rho \in C_c^\infty(\mathbb{R})$. Since f and g satisfy the hypotheses of Lemma 3.5.8, we have that the following weak Bochner inequality holds for $\overline{f\rho}$ and $\overline{g\rho}$:

$$\begin{aligned} \frac{1}{2} \int \Delta(\overline{g\rho}) |\mathrm{d}(\overline{f\rho})|^2 \,\mathrm{d}\mathfrak{m}_w \geq \frac{1}{N} & \int \overline{g\rho} (\Delta(\overline{f\rho}))^2 \,\mathrm{d}\mathfrak{m}_w \\ & - (N-1) \int \overline{g\rho} |\mathrm{d}(\overline{f\rho})|^2 \,\mathrm{d}\mathfrak{m}_w \\ & + \int \overline{g\rho} \langle \nabla(\Delta(\overline{f\rho})), \nabla(\overline{f\rho}) \rangle \,\mathrm{d}\mathfrak{m}_w. \end{aligned} \tag{3.5.15}$$

Using (3.5.12), (1.7.1) and the Leibniz rule all the terms of (3.5.15) can be split in products of integrals on $\mathbb{R}$ and on X'. For instance:

$$\begin{aligned} \int \Delta(\overline{g\rho}) |\mathrm{d}(\overline{f\rho})|^2 \,\mathrm{d}\mathfrak{m}_w = & \int (\rho w_{\mathsf{d}}^{-2} \underline{\Delta} g + (\rho' w_{\mathsf{m}})' w_{\mathsf{m}}^{-1} g) \\ & (\rho^2 w_{\mathsf{d}}^{-2} |\mathrm{d} f|^2_{\mathrm{X}'} + f^2 (\rho')^2) \,\mathrm{d}\mathfrak{m}_w \\ = & \int_{\mathbb{R}} \rho^3 w_{\mathsf{d}}^{-4} w_{\mathsf{m}} \,\mathrm{d}\mathcal{L}^1 \int_{\mathrm{X}'} |\mathrm{d} f|^2_{\mathrm{X}'} \underline{\Delta} g \,\mathrm{d}\mathfrak{m}' \\ & + \int_{\mathbb{R}} \rho (\rho')^2 w_{\mathsf{d}}^{-2} w_{\mathsf{m}} \,\mathrm{d}\mathcal{L}^1 \int_{\mathrm{X}'} f^2 \underline{\Delta} g \,\mathrm{d}\mathfrak{m}' \\ & + \int_{\mathbb{R}} \rho^2 w_{\mathsf{d}}^{-2} (\rho' w_{\mathsf{m}})' \,\mathrm{d}\mathcal{L}^1 \int_{\mathrm{X}'} g |\mathrm{d} f|^2_{\mathrm{X}'} \,\mathrm{d}\mathfrak{m}' \\ & + \int_{\mathbb{R}} (\rho')^2 (\rho' w_{\mathsf{m}})' \,\mathrm{d}\mathcal{L}^1 \int_{\mathrm{X}'} g f^2 \,\mathrm{d}\mathfrak{m}', \end{aligned} \tag{3.5.16}$$

and similarly for the other three terms of (3.5.15).

The idea to conclude is then to find a sequence of functions (ρ_n) such that all the "unwanted terms" go to 0 as $n \to +\infty$ leaving only (3.5.14).

Let $\rho_0 \in C_c^\infty(\mathbb{R})$ be a cut-off function which is identically 1 on $[-1,1]$ and 0 outside $[-2,2]$ and let $\rho_n(t) := \rho_0(t+n)$. Remembering that $w_{\mathsf{d}}(t) = e^t$ and $w_{\mathfrak{m}}(t) = e^{(N-1)t}$, we can observe that all the "real integrals" in (3.5.16) and in the similar terms derived from (3.5.15) goes to 0 faster than $\int \rho^3 w_{\mathsf{d}}^{-4} w_{\mathfrak{m}} \,\mathrm{d}\mathcal{L}^1$:

$$\int \rho^3 w_{\mathsf{d}}^{-4} w_{\mathfrak{m}} \,\mathrm{d}\mathcal{L}^1 \geq \int_{-(n+1)}^{-(n-1)} e^{(N-5)s} \,\mathrm{d}s = Ce^{-(N-5)n},$$

$$\int_{\mathbb{R}} \rho(\rho')^2 w_{\mathsf{d}}^{-2} w_{\mathfrak{m}} \,\mathrm{d}\mathcal{L}^1 \leq C \int_{-(n+2)}^{-(n-2)} e^{(N-3)s} \,\mathrm{d}s = Ce^{-(N-3)n},$$

$$\int_{\mathbb{R}} \rho^2 w_{\mathsf{d}}^{-2} (\rho' w_{\mathfrak{m}})' \,\mathrm{d}\mathcal{L}^1 \leq C \int_{-(n+2)}^{-(n-2)} e^{(N-3)s} \,\mathrm{d}s = Ce^{-(N-3)n},$$

$$\int_{\mathbb{R}} (\rho')^2 (\rho' w_{\mathfrak{m}})' \,\mathrm{d}\mathcal{L}^1 \leq C \int_{-(n+2)}^{-(n-2)} e^{(N-1)s} \,\mathrm{d}s = Ce^{-(N-1)n},$$

and so on for the other terms. Dividing (3.5.15) by $\int \rho^3 w_{\mathsf{d}}^{-4} w_{\mathfrak{m}} \,\mathrm{d}\mathcal{L}^1$ and sending n to $+\infty$ we obtain (3.5.14). □

Thanks to Proposition 3.5.10 we conclude that the space $(\mathrm{X}', \mathsf{d}', \mathfrak{m}')$ is $\mathsf{RCD}(0, N)$, and this concludes the proof of Theorem 3.5.3.

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