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Bose-glass, superfluid, and rung-Mott phases of hard-core bosons in disordered two-leg ladders

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By means of Monte Carlo techniques, we study the role of disorder on a system of hard-core bosons in a two-leg ladder with both intra-chain (t) and inter-chain (t') hoppings. We find that the phase diagram as a function of the boson density, disorder strength, and t'/t is far from being trivial. This contrasts the case of spin-less fermions where standard localization arguments apply and an Anderson-localized phase pervades the whole phase diagram. A compressible Bose-glass phase always intrudes between the Mott insulator with zero (or one) bosons per site and the superfluid that is stabilized for weak disorder. At half filling, there is a direct transition between a (gapped) rung-Mott insulator and a Bose glass, which is driven by exponentially rare regions where disorder is suppressed. Finally, by doping the rung-Mott insulator, a direct transition to the superfluid is possible only in the clean system, whereas the Mott phase is always surrounded by the a Bose glass when disorder is present. The phase diagram based on our numerical evidence is finally reported.

I. INTRODUCTION

Interacting bosons in one-dimensional (1D) or quasi-1D lattices are of interest in many physical contexts, ranging from Josephson-junction arrays[1] to more recent experiments on ultracold bosons loaded in 1D optical traps.[2] Especially the latter ones offer the unique opportunity to fine tune the experimental parameters and realize in laboratory a wide variety of bosonic models where kinetic energy, inter-particle interaction and disorder can be varied at will. In particular, it is possible to tune interaction to such an extend that atoms essentially behave as hard-core bosons confined along 1D tubes with[3] and without[4] a superimposed optical lattice. In this condition, the huge repulsion prevents bosons from occupying the same position in space and induces a sort of Pauli exclusion principle. It is well known that hard-core bosons can be mapped onto spin-less fermions by the so-called Jordan-Wigner transformation.[5] However, since the Jordan-Wigner fermions are non local in terms of the original bosonic operators, the local bosonic Hamiltonian is transformed into a very complicated non-local interacting fermion model. This approach simplifies in 1D tight-binding models with only nearest-neighbor hopping: here, non-locality is absent and the bosonic model maps onto non-interacting fermions, easily solvable. Whenever the lattice is not rigorously 1D, i.e., when more chains are coupled together or longer-range hoppings are considered, the corresponding fermionic problem contains complicated interaction terms that may become highly non-local when the number of chains increases.

The difference between hard-core bosons and spin-less fermions is even more pronounced in the presence of disorder. Indeed, non-interacting spin-less fermions in a quasi-1D system are always Anderson localized for any disorder strength. On the contrary, hard-core bosons can become superfluid in the presence of disorder as soon as they can exchange among each other, a property that occurs already in the simplest case of a two-leg ladder.[6,7] From this point of view, ladders of hard-core bosons represent an ideal case study to uncover the role of Bose statistics versus Fermi statistics in the presence of disorder.

From the purely theoretical side, we think this is an interesting issue. Single-particle wave functions are always localized in a quasi-1D disordered lattice. It follows that any Slater determinant built with such wave functions is localized too, so any many-body wave function for non-interacting fermions. On the contrary, hard-core bosons, which like spin-less fermions cannot occupy the same site but whose wave function is symmetric under exchanging two particles, can cooperatively act and give rise to a delocalized superfluid phase.

Also from the experimental side ladder systems are of interest, as they can be realized with optical lattices.[8,9] In realistic experimental setups, a two-leg ladder can be realized through a double-well potential along a direction (say, y) like in Ref. 10, and a potential creating a cigar geometry in the x-axis. Further superimposing a periodic potential along x, one could finally realize a two-leg Bose-Hubbard model with tunable hopping rates among and between the legs. Disorder can be introduced by superimposing a disordering lattice or introducing a speckle potential. We further mention that bosonic ladder systems are realized also in magnetic materials.[11] For example, the disorder-free compound IPA-CuCl$_3$ has been found to be a prototypical $S = 1/2$ antiferromagnetic spin ladder material, which can be thought as a system of interacting hard-core bosons.[12] Here disorder is introduced

by means of random chemical substitution, i.e., IPA-Cu(Cl$_{1-x}$Br$_x$)$_3$. Neutron scattering experiments have shown convincing evidences of the spin-analogous of a Bose-glass.[13]

Given its both theoretical and experimental interest, we decided to study the phase diagram of a simple model of hard-core bosons hopping on a two-leg ladder with bounded on-site disorder by means of Green's function Monte Carlo.[14] In short, we find that the phase diagram as function of the density and the ratio between inter- and intra-chain hopping includes three phases: a localized Bose-glass, a superfluid and, at half-filling, a so-called rung-Mott insulator that seems to be always surrounded by the glass.

The paper is organized as follows: in Sec. II, we introduce the model and the numerical methods; in Sec. III, we briefly discuss the clean system; in Sec. IV, we present our results for the disordered model; in Sec. V, we finally draw our conclusions.

II. THE MODEL

We shall consider a system of disordered hard-core bosons on a $L = 2 \times L_x$ lattice. The Hamiltonian reads

$$\mathcal{H} = - t \sum_{i, \eta=1,2} \left(b_{i,\eta}^\dagger b_{i+1,\eta} + h.c. \right) - t' \sum_i \left(b_{i,1}^\dagger b_{i,2} + h.c. \right) + \sum_{i,\eta} \epsilon_{i,\eta} n_{i,\eta}, \quad (1)$$

where $b_{i,\eta}^\dagger$ ($b_{i,\eta}$) creates (destroys) a boson at rung i on the chain $\eta = 1, 2$. The matrix elements t and t' are the hopping amplitudes along legs and rungs, respectively. The disordered potential couples to the density operator $n_{i,\eta} = b_{i,\eta}^\dagger b_{i,\eta}$ and it is described by random variables $\epsilon_{i,\eta}$ that are uniformly distributed in $[-\Delta, \Delta]$. Finally, the hard-core constraint of $n_{i,\eta} \leq 1$ is implied.

We study the Hamiltonian Eq. (1) by Green's function Monte Carlo with a fixed number M of bosons on L sites,[15] $n = M/L$ being the average density. The Green's function Monte Carlo approach is based on a stochastic implementation of the power method technique that allows, in principle, to extract the actual ground state $|\Psi_{GS}\rangle$ of a given Hamiltonian $\mathcal{H}$, from any starting trial (e.g. variational) wave function $|\Psi_V\rangle$, provided that $\langle \Psi_V | \Psi_{GS} \rangle \neq 0$. In order to improve the numerical efficiency, it is important to consider a good starting wave function, for which we use one- and two-body Jastrow factors applied to a state where all bosons are condensed at momentum $q = 0$. The one-body Jastrow factor makes it possible to vary the local density of the bosons. On the contrary, the two-body Jastrow is taken to be translationally invariant.[16]
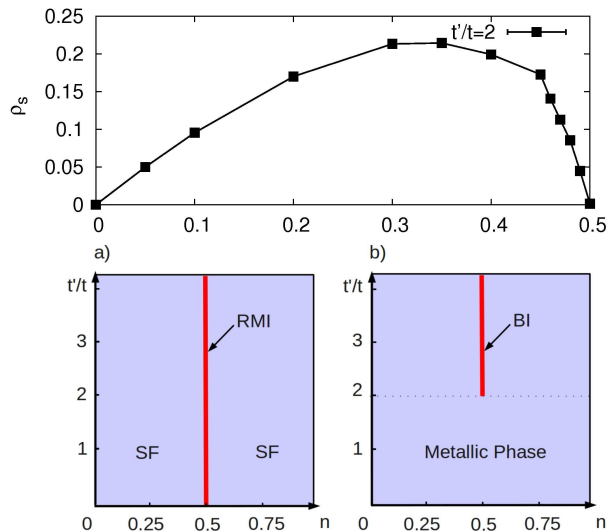

FIG. 1: (Color on-line) Upper panel: Superfluid stiffness as function of the density n at fixed $t'/t = 2$ on a lattice with $L = 2 \times 50$. Lower panel: clean phase diagram of hard-core bosons (a) and spin-less fermions (b) on the two-leg ladder as function of t'/t and density n. The rung-Mott insulator is denoted by RMI, the superfluid by SF and the band insulator by BI.

III. THE CLEAN SYSTEM

Before considering the disordered case, it is useful to briefly discuss the clean system, where $\epsilon_{i,\eta} = 0$ (see also Appendix). First, we consider the limit $t' = 0$, i.e., two uncoupled chains. In this situation, the ground state is a superfluid with quasi-long-range order for any density $0 < n < 1$. At densities $n = 0$ and 1, there is a (trivial) "frozen" Mott insulator due to the infinite on-site repulsion, which completely suppresses charge fluctuations. Let us now analyze the the opposite limit $t'/t \gg 1$. Exactly at half filling, i.e., $n = 0.5$, there is one boson per rung and the wave function can be approximately written as a independent product of single-particle rung states as

$$|\Psi_{GS}\rangle \simeq \prod_i^{L_x} (b_{i,1}^\dagger + b_{i,2}^\dagger)|0\rangle. \quad (2)$$

The system is in the so-called rung-Mott insulator with a unique ground state and a gap to all excitations.[17] At half filling, the transition between the rung-Mott insulator and the superfluid takes place exactly at $t' = 0$, since the inter-chain hopping represents a relevant perturbation that immediately opens a gap in the excitation spectrum, see Appendix. This transition is of the Berezinskii-Kosterlitz-Thouless type, which makes it difficult to observe by numerical simulations on finite clusters.[18]

Now, if few bosons are added or removed to the insulating state, a superfluid is stabilized. Therefore, at any other filling $0 < n < 0.5$ and $0.5 < n < 1$ the system will
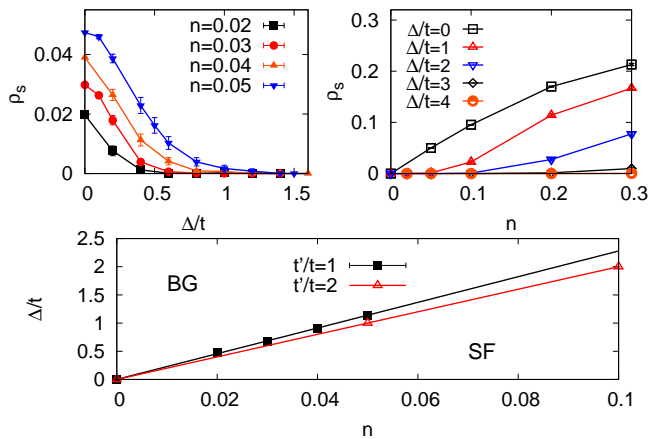
FIG. 2: (Color on-line) Upper left panel: superfluid stiffness ρ_s as a function of the disorder strength Δ/t for different n and $t'/t = 1$. Upper right panel: superfluid stiffness ρ_s as a function of the density for different Δ/t and fixed $t'/t = 2$. Lower panel: low-density phase diagram of the hard-core bosonic model for $t'/t = 1$ and $t'/t = 2$. Calculations have been done on a $L = 2 \times 50$ system.
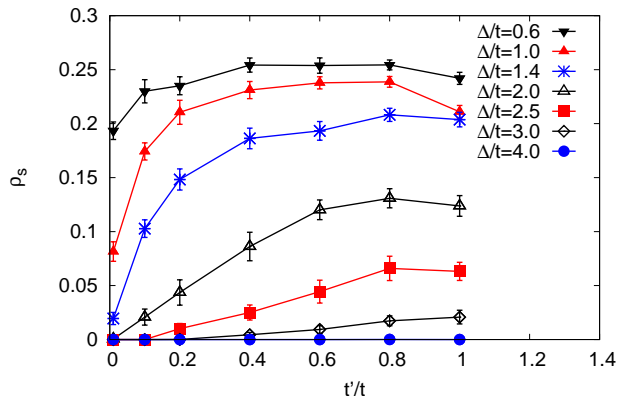

FIG. 3: (color on-line) Superfluid stiffness ρ_s as a function of t'/t for different Δ/t on a two-leg ladder with $L = 2 \times 50$ sites. The density has been fixed to $n = 0.4$.

be always superfluid, as can be seen in Fig. 1 where the superfluid stiffness as function of the density of particles n is shown for fixed $t'/t = 2$. The superfluid stiffness starts to grow at low density, it reaches a maximum, and finally vanishes at $n = 0.5$ at the rung-Mott insulator.

We conclude this section by mentioning that, for spin-less fermions at half filling, there is a transition from a metallic to a band insulator at $t'/t = 2$, where a gap opens up. For all the other densities $n \neq 0$ and 1, the ground state is metallic. The clean phase diagrams for hard-core bosons and spin-less fermions on a two-leg ladder are sketched in Fig. 1 for comparison.

IV. THE DISORDERED SYSTEM

A. Low-density phase diagram

Now we turn to the disordered case. Let us start by the low-density regime of the phase diagram at fixed inter-chain hopping as function of the density and disorder. In Fig. 2, we report our results of the superfluid stiffness as a function of disorder strength (for $t'/t = 1$) and density (for $t'/t = 2$); the low-density phase diagram is reported as well. We find that, for any finite disorder Δ/t, the low-density phase is a Bose glass that turns superfluid above a critical density. The trivial Mott insulator with zero (or one) bosons per site is therefore always separated from the superfluid by the Bose-glass phase. This is a remark-able result since, in a single chain with nearest-neighbor hopping only, hard-core bosons are equivalent to spin-less fermions, which Anderson localize for any density. Hence, in a two-leg ladder, hard-core bosons behave dif-ferently from spin-less fermions; while the latter ones re-main always localized, the former ones show a superfluid phase stabilized by the inter-chain hopping, as it was predicted using bosonization and renormalization-group techniques.[7] The idea is that, in a strictly 1D geometry with only nearest-neighbor hopping, the statistics of the particles does not matter. However, whenever particles may be interchanged (by non-strictly 1D paths) bosons can form a superfluid, even in presence of disorder. We just mention that the same behavior holds also on a sin-gle chain with longer-range hopping. This scenario can be understood in very simple terms as follows. At very low fillings, the statistics of the particles does not matter so much and hard-core bosons, as free fermions, local-ize due to disorder, giving rise to the Bose glass. This is because the length over which the single-particle wave function extends is short enough that the wave functions of two particles never overlap. As soon as the filling is increased, the particles get closer to each other and the single-particle wave functions begin to overlap. At this point the statistics of the particles starts to play a role. If the particles are fermions they will still be localized (in $D \leq 2$), whereas bosons may stabilize a superfluid, as confirmed by our numerical simulations.

In the range of values of inter-chain hopping that we have studied, the effect of a larger t'/t in the low-density phase diagram is to slightly reduce the superfluid re-sponse of the system, as can be also seen in Fig. 2. Al-though the actual thermodynamic value of the transition between the Bose glass and the superfluid may be rather different from the one obtained by our calculations, be-cause of strong size effects, the present results give a qual-itative correct insight into the phase diagram. Finally, we would like to mention that the exact behavior of the transition line between Bose glass and superfluid phases $\Delta_c(n) \propto n^\alpha$ is hard to be found by numerical calcula-tions. Although an almost linear fit is found, i.e., $\alpha = 1$, a different power-law cannot be excluded, as implied by the arguments of Ref. 19.
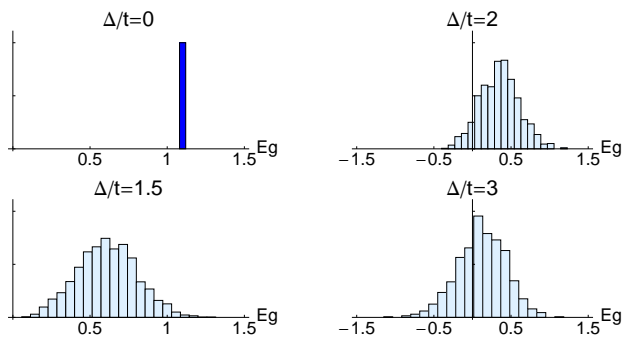
FIG. 4: (Color on-line) Distribution $P(E_g)$ of the gap as function of disorder strength and fixed density $n = 0.5$ and $t'/t = 2$ on a two-leg ladder with $L = 2 \times 50$ sites. The clean gap is shown for comparison in the upper-left box as a blue bar.

B. The effect of the inter-chain hopping

Here, we want to investigate the effect of the coupling t' on the otherwise insulating (Anderson localized) decoupled chains. Generally speaking, for any value of disorder, a certain finite ratio t'/t is necessary to drive the system into a superfluid phase, hence the system remains in the Bose-glass phase for small t'/t. However, for small disorder, the localization length of the Bose glass is expected to be very large. This means that on clusters that are accessible to numerical simulations it may be very hard to see the Bose glass region. This fact is indeed confirmed by our results on the superfluid stiffness as a function of the inter-chain hopping, see Fig. 3. Rapidly, as a small t'/t is introduced, a large superfluid response is found for small disorder (e.g., $\Delta/t = 0.6$ and 1.0 in the figure). It is also observed that, by a further increase of the inter-chain hopping the superfluid stiffness reaches a maximum and then eventually decays, since for $t/t' \to 0$ the system decouples in a collection of decoupled rungs (which are obviously not superfluid). Moreover, as the disorder is increased, the superfluid response is suppressed, until the system cannot attain superfluidity any longer and remains localized for any value of t'.

C. The rung-Mott phase in presence of disorder

The effect of disorder on the rung-Mott phase at density $n = 0.5$ system is now discussed. From general grounds, it is expected that the presence of disorder will fill the gap with localized states, so to induce a transition to a gapless Bose-glass phase. In practice, given a ratio t'/t, the rung-Mott insulator will survive up to a certain critical value of Δ/t, where the gap will be completely filled and the system will become compressible. This situation is similar to the one of the Bose-Hubbard model at integer fillings[20], where a direct transition between the Mott insulator and the Bose-glass phase is expected by decreasing the ratio between the on-site repulsion U and

the disorder strength Δ.[20–23] Also in our case of hard-core bosons, we can make use of the argument based on the fact that, if Δ is larger than half of the energy gap of the clean insulator E_g^{clean}, then the ground state must be compressible; otherwise the system is incompressible with a reduced gap given by $E_g = E_g^{\text{clean}} - 2\Delta$. In particular, for $t'/t \gg 1$ we have that $E_g^{\text{clean}} \sim 2t'$. Therefore, the gap will vanish around a critical value of disorder $\Delta_c \sim t'$. These arguments should hold exactly only in the infinite system and large size effects are expected because this transition is of the Griffiths type, i.e., driven by exponentially rare regions which are locally ordered.[24] On the other hand, on any finite system the transition from the gapped to the compressible phase will appear at a larger Δ_c, since these exponentially rare configurations will be hardly sampled on finite clusters.

We recently proposed[16] a method to alleviate the strong size effects that consists of computing directly the distribution probability of the gap

$$P(E_g) = \sum_{\alpha\beta} \delta \left(E_g - \mu_\alpha^+ + \mu_\beta^- \right), \qquad (3)$$

where $\mu_\alpha^\pm = \pm \left(E_{M\pm1}^\alpha - E_M^\alpha \right)$ (E_M^α being the ground-state energy with M particles on the realization α of disorder). This definition of the gap distribution is introduced because in disordered systems the gap can be overcome by transferring particles between two rare regions with almost flat disorder shifting the local chemical potential upward and downward. These exponentially rare regions may be far apart in space and represent rare fluctuations (Lifshitz's tail regions), thus it is useful to imagine that a large system is made by several subsystems, each represented by a different disorder realization of our L-site cluster, and construct the gap by using the process of taking one particle from region α to region β.[16] If such processes are allowed at no energy cost, i.e., $P(0) \neq 0$, the corresponding system will be gapless. One could define an alternative estimate of the gap as

$$E_g^{\text{min}} = \min_{\alpha,\beta} | \mu_\alpha^+ - \mu_\beta^- |, \qquad (4)$$

with all the disorder realizations α and β.

In Fig. 4, we show the distribution probability at half filling as function of disorder, $t'/t = 2$ and $L = 2 \times 50$. For small values of disorder the gap survives, while for $\Delta/t = 2$ the probability to find zero gap is finite, which we interpret as signalling zero gap in the infinite system and a Bose glass phase. We note that, when considering the case of the rung-Mott insulator, this method performs a bit worse than in the case of the Bose-Hubbard model (at integer fillings). Indeed, for this value of the hopping parameters, we have that $E_g^{\text{clean}}/t \simeq 1.11$, giving rise to $\Delta_c/t \simeq 0.55$, which is much smaller than the value obtained by numerical simulations. However, we would like to mention that, even though the finite-size analysis of $P(E_g)$ overestimates the actual value of the transition, it gives a sizable improvement with respect to
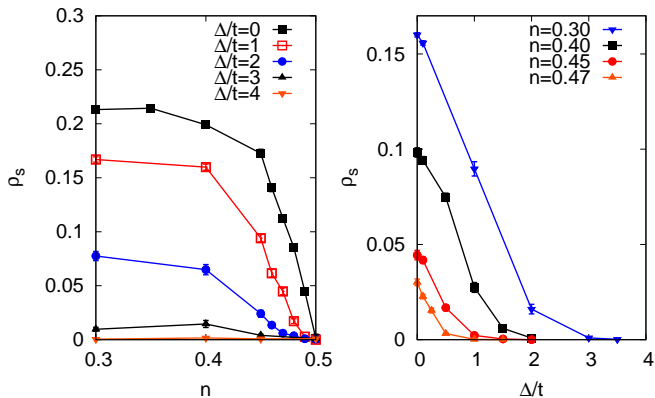
FIG. 5: (Color on-line) Left panel: Superfluid stiffness ρ_s as a function of the density and several values of Δ/t and fixed $t'/t = 2$. The stiffness of the clean system is also shown for comparison. Right panel: Superfluid stiffness ρ_s as function of the disorder bound for several densities close to half filling and fixed $L = 2 \times 50$ and $t'/t = 10$.
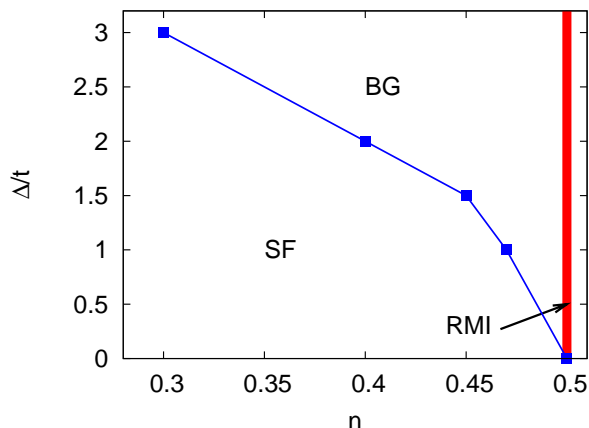

FIG. 6: (Color on-line) Results of the 2×50 lattice for the phase diagram in the vicinity of the rung-Mott phase for $t'/t = 10$.

the simple calculation of the average gap value in presence of disorder. The precise determination of the critical point is well beyond any numerical calculations and, therefore, we take advantage of the fact that its estimation can be done by using the criterion of Ref. 20–23, namely $\Delta_c = E_g^{\text{clean}}/2$.

D. Transition from the superfluid to the gapped phase

In connection with the rung-Mott phase at half filling, we investigate the phase diagram in the vicinity of such a phase and address the question of whether it is possible to have a direct transition from the superfluid phase

to the rung-Mott insulator as the density $n \to 0.5$, or there is always an intruding Bose glass phase. In this regard, whenever the gapped state is doped with a few particles or holes such that their typical spacing will be large, those few carriers on top of the rung-Mott phase will effectively see a disordered background. Therefore, standard single-particle Anderson localization arguments apply and the system remains insulating by localizing those few carriers on the Lifshitz's tails that are in the Mott gap. By further increasing the density of particles (or holes), a superfluid is eventually formed. This simple single-particle argument implies the presence of an intervening Bose glass between the rung-Mott phase and the superfluid. We proceed to test this argument quantitatively. In Fig. 5, we report our numerical results for the superfluid stiffness at densities close to $n = 0.5$ and $t'/t = 2$. Our data is consistent with a transition driven by density from the superfluid phase through the Bose glass to finally end up with the rung-Mott insulator. For example, for $\Delta/t = 1$, the superfluid stiffness appears to vanish just before $n = 0.5$, however the region in which the Bose glass takes place is very small. For $\Delta/t = 2$ the rung-Mott insulator has already been wiped out by the effect of disorder, as observed in Fig. 4, and, therefore, this issue cannot be addressed.

By considering a larger value of t'/t we have two advantages: first, the Mott gap is larger, such that the gapped phase is more robust, second, the localization due to disorder is expected to be enhanced, thus, enlarging the Bose glass region at $n < 0.5$ (or $n > 0.5$). These facts enable us to provide further evidence in favor of an intervening Bose glass in between the rung-Mott and the superfluid as follows. We have performed simulations with a rather large $t'/t = 10$. For such a value of the hoppings, we have that $E_g^{\text{clean}} \simeq 17.31$, such that the transition from the gapped to the compressible (Bose-glass) phase is argued to occur at a $\Delta_c \simeq 8.65$. Therefore, the system is expected to be gapped for $\Delta \lesssim 8.65$. In Fig. 5, we present our results for the superfluid stiffness as function of the disorder strength for several densities close to half filling. From these calculations we can easily see that, as the density approaches $n = 0.5$, the critical point where the stiffness vanishes gets smaller, leaving room for a large Bose-glass phase in between the superfluid and the rung-Mott insulator. Given our results, we can draw the phase diagram for densities close to the rung-Mott phase, see Fig. 6. Notice that the large value of t'/t (and therefore the clean Mott gap) ensures the existence of a truly gapped state at half filling. All together, we can make the safe statement that the transition between the superfluid and the rung-Mott phases is not direct, but through an intervening Bose-glass state.

V. CONCLUSIONS

We have studied hard-core bosons on disordered two-leg ladders by using both numerical techniques and an-
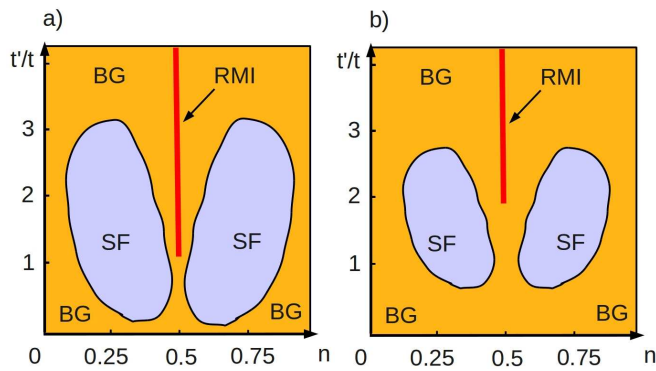
FIG. 7: (Color on-line) a) Tentative zero-temperature phase diagram of hard-core bosons on the two-leg ladder system. b) Same phase diagram in a) but for a larger value of disorder Δ/t.

alytical arguments borrowed from similar problems in Bose-Hubbard models. We have shown that the zero-temperature phase diagram is rather rich and contains different phases; apart from the trivial Mott insulators at $n = 0$ and 1, that are totally frozen due to the hard-core constraint, we found superfluid, Bose-glass, and rung-Mott phases. This contrasts the case of spin-less fermions, where no metallic phases are possible and Anderson localization takes place for any density $n \neq 0.5$ at finite Δ. A final sketched phase diagram, based upon our results, is reported in Fig. 7. In the case of no disorder, i.e., $\Delta = 0$, the superfluid phase pervades the phase diagram for all densities $0 < n < 0.5$ and $0.5 < n < 1$ and all $t'/t \neq \infty$. When considering a finite disorder strength, the superfluid shrinks and a Bose-glass phase appears. Most importantly, the transition between the Mott and the rung-Mott phases and the superfluid ones is never direct, like in the Bose-Hubbard model.[20–23]

Acknowledgments

Appendix A: Basic bosonization formulas

There are several papers that discuss at length the harmonic-fluid representation of bosonic lattice Hamiltonians following the seminal work by Haldane.[25] Nevertheless, we believe that it is worth listing some useful formulas, referring the interested readers to existing literature for further details.[25–27]

In the long-wavelength limit, the boson density and creation operator on each chain $\alpha = 1, 2$ can be written as

$$
\rho_\alpha(x) = \left(\rho_0 + \frac{1}{\pi} \nabla \phi_\alpha(x) \right) \sum_{m=-\infty}^{\infty} e^{i2m\left(\phi_\alpha(x) + \pi \rho_0 x \right)},
$$

$$
\psi_\alpha^\dagger(x) = \sqrt{\rho_\alpha(x)}\, e^{i\theta_\alpha(x)}
$$

$$
\sim e^{i\theta_\alpha(x)} \sum_{m=-\infty}^{\infty} e^{i2m\left(\phi_\alpha(x) + \pi \rho_0 x \right)},
$$

where ρ_0 is the average density and the two fields $\phi(x)$ and $\theta(x)$ satisfy

$$
\left[\phi_\alpha(x), \nabla \theta_\beta(y) \right] = i\pi \delta_{\alpha\beta}\, \delta(x - y). \tag{A1}
$$

In the case of hard-core bosons, it could be useful to define $\phi(x)$ and $\theta(x)$ in terms of right (R) and left (L) chiral fields:[28]

$$
\phi_\alpha(x) = \frac{1}{2}\left(\phi_{\alpha R}(x) + \phi_{\alpha L}(x) \right), \tag{A2}
$$

$$
\theta_\alpha(x) = \frac{1}{2}\left(\phi_{\alpha L}(x) - \phi_{\alpha R}(x) \right). \tag{A3}
$$

The dimension d of each operator $\Delta(x)$, defined through $\langle \Delta(x)\, \Delta(0) \rangle \sim x^{-2d}$, can be easily evaluated by recalling that

$$
\langle e^{i\gamma \phi_R(x)} e^{-i\gamma \phi_R(0)} \rangle \sim \langle e^{i\gamma \phi_L(x)} e^{-i\gamma \phi_L(0)} \rangle \sim \left(\frac{1}{x} \right)^{\gamma^2}.
$$

In the two-leg ladder it is convenient to introduce the symmetric (s) and anti-symmetric (a) combinations $\phi_s = \left(\phi_1 + \phi_2 \right)/\sqrt{2}$ and $\phi_a = \left(\phi_1 - \phi_2 \right)/\sqrt{2}$, respectively (and seemingly for θ_s and θ_a). It follows that the inter-chain hopping becomes

$$
\psi_1^\dagger(x)\, \psi_2(x) \sim e^{-i\sqrt{2}\,\theta_a(x)} \sum_{m,n} e^{i2\pi\rho_0(m+n)x} \times
$$

$$
\times\; e^{i\sqrt{2}\,(m+n)\,\phi_s(x)}\, e^{i\sqrt{2}\,(m-n)\,\phi_a(x)}.
$$

The first term in the right hand side is the most relevant one and opens a gap in the anti-symmetric sector such that θ_a acquires a finite average value while ϕ_a has exponentially decaying correlations. It follows that the leading operator generated by the inter-chain hopping is

$$
\psi_1^\dagger(x)\, \psi_2(x) \sim e^{-i\sqrt{2}\,\theta_a(x)} \left[1 + 2\cos\left(\sqrt{8}\,\phi_s(x) + 4\pi\rho_0 x \right) \right]. \tag{A4}
$$

In other words, t' not only gaps the anti-symmetric sector but also generates an umklapp scattering in the symmetric channel that becomes marginally relevant at half filling, where $4\pi\rho_0 = 2\pi$. It is just this umklapp that is responsible for the appearance of the rung-Mott insulator

at any finite $t' \ll t$. In addition, being marginally relevant, it opens a gap in a Berezinskii-Kosterlitz-Thouless fashion, which is hard to detect numerically. Following Ref. 26, one finds that disorder gives rise to a fluctuating umklapp

$$\mathcal{H}_{\mathrm{umklapp}} = \int dx\, \xi(x)\, \cos\left(\sqrt{8}\, \phi_s(x)\right), \qquad \text{(A5)}$$

with $\overline{\xi(x)\xi(y)} = u^2\, \delta(x-y)$ in case of a Gaussian noise. Therefore, at half filling the coupling constant of the umklapp has a finite value plus a fluctuating one $\xi(x)$; therefore, if the latter one is small, the gap is on average finite (the Mott phase), while, for u above a certain threshold, the gap is washed out by disorder, leading to the Bose glass.

Away from half filling, we should keep into account a renormalization of the symmetric sector that can be parameterized by a Luttinger liquid parameter K_s through[25,26,28]

$$\phi_s \to \sqrt{K_s}\, \phi_s, \qquad \theta_s \to \sqrt{\frac{1}{K_s}}\, \theta_s.$$

Since the full density

$$\rho(x) = \rho_1(x) + \rho_2(x) = \frac{\sqrt{2}}{\pi}\, \nabla\phi_s(x) \to \frac{\sqrt{2K_s}}{\pi}\, \nabla\phi_s(x),$$

K_s can be easily extracted by the static structure factor in momentum space

$$\langle \rho(q)\, \rho(-q) \rangle = 2K_s\, \frac{qL_x}{2\pi},$$

where L_x is the number of sites per chain. When $\rho_0 \neq 1/2$, the umklapp scattering in Eq. (A4) ceases to play a role and what survives is just the disorder-generated term (A5). Conventional scaling arguments predict that such a term is relevant when $K_s < 3/4$.[26] This would suggest that for $K_s > 3/4$ a superfluid phase is stable, otherwise disorder is relevant and the Bose glass occurs. According to the theory of the commensurate-incommensurate transition in 1D, we expect in the clean case that, as the density ρ_0 approaches half filling $K_s \to 1/2$, which would imply that disorder becomes relevant already before the Mott transition is approached in density. However, even if the density $\rho_0 \to 0$ we should expect $K_s \to 1/2$. Therefore, whatever is the behavior close to half filling, it must be qualitatively the same also close to zero filling. We know that, at very low density, bosons localize in the Lifshitz's tails and superfluidity can arise only when the localization length becomes of the order of the interparticle distance. This also suggests that a harmonic-fluid representation is likely inadequate at low density, hence that the scaling criterium $K_s > 3/4$ for the appearance of superfluidity may not work. Seemingly, the same argument must apply close to half filling, so that it must not be surprising that the phase boundary between the superfluid and the Bose glass goes smoothly and almost linearly to zero, see Fig. 5.

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