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Macroscopic Gravity, the Renormalization Group and Modeling of the Riemannian Geometry in Cosmology

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Cap. LXXVII, Donde se cuentan mil zarandajas, tan impertinentes, como necesarias al verdadero entendimiento de esta grande Historia.

Cervantes, Don Quixote.

Abstract

The central problem studied in this thesis is, broadly speaking, the issue of coarse-graining in GR approximations, and *the effect of averaging on the field equations.* The important observation made is that there are some *smoothing* procedures implicit in the standard, homogeneous and isotropic Friedman-Lemaître-Robertson-Walker cosmological models. The point is that if such effects are not allowed for, we may actually be using the wrong field equations in cosmology. There has been recently an increased effort in this direction with some interesting results, as for example that the coarse-graining effects could be non-negligible in the context of affecting the age of the universe.

An interesting approach to tackle this averaging problem is the Macroscopic Gravity theory (MG) (see section 2.5). It offers an axiomatic treatment of a space-time averaging of tensors and of the pseudo-Riemannian geometry and the Einstein equations. Research presented in this thesis produced results as follows: In section 2.6 it is shown how to apply this theory to averaging of the Lagrangians, it is demonstrated further that this kind of averaging preserves the asymptotic properties of space-time and macroscopic field equations can be obtained from a variational principle. Moreover in section 2.6.5 it is pointed out how one can obtain various limiting cases of physical interest in MG, showing therefore how links can be established between MG and some previous approaches for averaging. In particular it is demonstrated how it is possible to obtain the MacCallum-Taub limit showing thus its validity in MG.

The other idea explored at length in this thesis (see chapter three) is the possibility of applying the Renormalization Group concepts in gravitation to tackle the averaging problem. In section 3.6 an explicit smoothing-out procedure for inhomogeneous cosmologies is introduced. The main result obtained is a re-interpretation of the Ricci-Hamilton flow in terms of the RG flow, thereby providing the Ricci-Hamilton flow with a physical meaning and showing how the averaging problem is rooted in the geometry. Another challenge of making use of the RG approach, in close analogy with the lattice models in Statistical Mechanics, calls for the introduction of the partition function for a gravitational system (see section 3.5). This can be achieved using the standard functional integral approach extended to constrained hamiltonian systems. This allows us to draw some conclusions

about the occurrence of phase transitions and the existence of critical exponents (see also section 3.4.1.3).

Finally in chapter four the evolution of perturbations is studied in the dust-radiation FLRW universe model. This is done in the framework of dynamical systems theory which seems well suited to this purpose. The evolution of density perturbations is presented in the form of phase diagrams. Some scales are discovered that are over-damped.

A number of ideas of relevance for future research are summarized in chapter five.

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1 Introduction

1.1 Theoretical foundations of Cosmology

The aim of cosmology is an investigation of the structure of space-time in the largest possible scale. One can think of two different directions of doing this [141]:

– "top-down" way, whereby one assumes *a priori* some postulates about the large scale of the universe and tries to deduce local physics (Milne, Bondi and Gold); and

– "bottom-up" way, whereby one extrapolates the accepted local physics as far as possible, in order to guess the global structure of space-time. This is indeed the present paradigm and it also involves some more or less aprioristic assumptions about the Universe as a whole, or cosmology itself.

In order that cosmology can be concerned with the global structure of space-time one has to make some non-empirical assumptions. In fact, it turns out that non-local assumptions are involved in every empirical prediction of standard physics. From the point of view of field theory, local extrapolations into the future can be valid only under the so-called no-interference assumption (see e.g. [98]), that there are no signals generated by a distant event, which at the next moment could affect the system under investigation. If one wants to interpret modern cosmological observations a theory of space-time is needed. The conclusions one arrives at about the properties of the universe, essentially depend on an integration of some differential equations over a "large scale" domain of the space-time manifold. Usually, as a rule it is the simplest cosmological models that are implicitly used in practice (highly symmetrical), which obscure the above fact.

General Relativity (GR henceforth) is the best classical theory of gravitation we have.

The field equations of GR are correct to a high order of accuracy for the solar system (see e.g. [250]) and relatively small binary systems (PSR 1913+16, 4U1820−30 and PSR 0655+64) [145, 258, 17, 71], i.e. to the distance scales less than 50 AU. Nevertheless GR is applied to cosmic structures such as galaxies, clusters of galaxies, superclusters and ultimately the whole universe, which are typically $10^6 \div 10^{15}$ times larger than the distances over which the theory has been tested with a high accuracy. In the theoretical studies of gravitational lensing [35] GR is assumed to be the correct theory of gravity, but doing so GR is not being tested explicitly. In fact, some authors questioned the validity of GR over large distances in the context of dark matter problem [221][1]. As far as cosmology is concerned, the most significant result obtained with the help of classical tests is the support given to the standard model against the steady-state cosmology.

The whole subject of experimental gravitation is rather subtle. This is so finally due to the fact that GR is a general covariant theory, and due to this the very concept of observables is quite involved (see e.g. [27] for still excellent review). The strategy that has to be employed in all the measurements is to use concrete physical objects as clocks and spatial references, and these objects cannot be taken as independent from the dynamics of the system. The importance of the Lorentz metric is not to be underestimated as it is its existence that makes it possible to perform space and time measurements in different reference frames. The geometry of space-time determines the laws of measurement. The Lorentz metric further allows us to construct relativistic mechanics and relativistic electrodynamics.

In GR the space-time manifold formally determined by the matter content and boundary conditions with its various geometric structures has a very special status, as a "primitive element" of the theory. It has to be a differentiable, at least class C^4 [70], Hausdorff and connected manifold, if it is to model a physical world. Locally the manifold is (pseudo-)Euclidean - concept not directly related to the curvature - in the sense that every point of it lies in a neighbourhood which can be coordinated. Moreover different points of it have coordinates (four real numbers) related in a continuous way. At every point of the

[1] One can have doubts about the validity of GR when spatial separations are of the order of l_{Planck}, but that would be relevant to the very early stages of the standard model, which we are not concerned with in the present thesis.

manifold there is a rich algebraic structure (vectors, tensors, etc.) representing various physical fields, which otherwise is introduced quite arbitrarily. This structure is built using the local differentiable properties of the manifold. Apart from the differential one there is also a geometric structure. The pseudo-Riemannian manifold is in effect taken to model the space-time in the relativistic theory of gravitation. It is only through the Einstein equations that the physical fields acquire a non-local character.

The space-time evolves during the evolution of the universe and the mathematical structure of GR space-time at different times should reflect the changes. Possibly we should even allow for a breakdown of the smoothness of the manifold. In the early universe close to the initial singularity, each physical property implied global consequences, the distinction between local and non-local ceases to be clear-cut. The singularity itself (initial) is related to the space-time being geodesically incomplete. The singularity theorems [139] show that geodesic incompleteness of the considered space-time cannot coexist with some of its properties, like e.g. compactness, non-existence of closed time-like curves, existence of Cauchy hypersurface, which are non-local ones. Every physically acceptable model of the universe can and should contain singularities [139] (e.g. the Schwarzschild solution), they had better constitute a set of measure zero! (for a comprehensive treatment of the mathematical structure of space-time, see e.g. [102]). There are different kinds of singularities and some are "more global" then others.

The principle of equivalence identifies a Riemannian space-time with the gravitational field. Einstein's equations provide a relation between the sources of the gravitational field and the field itself. They are the prescription of how matter and any energy-carrying field determines geometry of the space-time and, *a fortiori* the gravitational field. The gravitational field is represented by the space-time metric $g_{\alpha\beta}$. In this sense the gravitational field is contained in the geometry, as knowledge of the metric is sufficient (for the pseudo-Riemannian manifold) for the determination of all properties of our manifold. The information about the sources of the gravitational field is contained in the energy-momentum tensor $T_{\alpha\beta}$. In order to specify it we should know in advance the space-time structure of the manifold itself, which we cannot, until we have solved the Einstein equations[2]

[2]Natural units are chosen unless evident otherwise; the signature $(-+++)$ is adopted throughout.

$$E_{\alpha\beta} \equiv R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = T_{\alpha\beta}. \tag{1.1}$$

GR contains in fact no general recipe of how the energy-momentum tensor should be constructed, given a particular physical situation. Indeed, it can only be "constructed" on the basis of pre-relativistic physics, with no account consequently of the geometrization view. The generally adopted procedure in GR is to resort to field theories, covariant in the sense of restricted relativity (special). A covariant form of the tensor is then brought up by taking ordinary derivatives into covariant derivatives. Notwithstanding the requirement of covariance alone is not enough to construct $T_{\alpha\beta}$.

The energy-momentum tensor is postulated as a functional of the metric tensor and other relevant state variables. For example, in the case of a perfect-fluid, the energy-momentum functional has the form

$$T_{\alpha\beta} = (\mu + p) u_\alpha u_\beta + p g_{\alpha\beta}, \tag{1.2}$$

where, p is the scalar pressure, μ the (proper) energy density and the velocity field u_α of the fluid is normalized, $u_\alpha u^\alpha = -1$. Usually the barotropic equation of state $f(\mu, p) = 0$ is taken to hold.

By solving the Einstein equations we come up explicitly with the energy-momentum tensor and obviously, the components of the metric tensor. The mathematical problem is complete if we take into account the constitutive equations from outside the theory itself, e.g. the equation of state mentioned above. However, with the Bianchi identities (contracted)

$$T^{\alpha\beta}{}_{;\beta} = 0, \tag{1.3}$$

we can reduce the number of independent equations from ten to six. The remaining four degrees of freedom correspond to the freedom we have in selecting a coordinate system in advance of solving Einstein's equations. (In some special situations the number of coordinate conditions may be more.) The components of the energy-momentum tensor are not independent of this choice, and it is these components that are identified with physically measurable quantities. Here we meet again very important problem of measurability. For example, Eddington [82] has emphasized the distinction, very important in cosmology, between the invariant and relative mass, whose (or rather its associated mass density) value determines in the GR models whether the universe is closed or open. Therefore, when we talk about mass density we have to be careful whether we mean an abstract invariant or a measurable but coordinate dependent, relative density.

Any theory of gravitation must of course be germane to cosmology and the models of the universe provided by GR have to be consistent (in some sense) with the observational data. Now, one of the biggest difficulties of cosmology is that most of the observational data are theory-dependent, i.e., their meaning can be interpreted by assuming certain theoretical explanations. We interpret what we see in terms of the laws we know, extrapolating them far into the universe - this cannot be avoided. We encounter here a subtle matter of verifiability of the field equations. As far as a theory (any) is concerned, it is clear that evidence whose meaningful interpretation involves its assumptions cannot be used for its verification. Therefore, cosmological solutions should not be considered as large scale verifications of the theory of gravitation, due to the number of other assumptions invoked from outside the theory itself. This also can be looked at as one of the needs calling for a fitting (discussed at some depth later on) between the GR–models and the observations.

Most of the current cosmologies employ GR[3] and as a rule the relativistic cosmologies[4] rely on some form of Cosmological Principle[5], which is usually a smoothing-out hypothesis imposed *a priori* on the distribution of matter in the universe. A well known example is provided by the Friedmann-Lemaître-Robertson-Walker (FLRW henceforth) metric, which is usually assumed to describe the real universe. Specifying a restricted family of geometries (e.g. FLRW) and physical behaviour serves to "reduce" the fully general equations of GR, too difficult to handle, into say, a manageable set of differential equations. In this view, Cosmological Principle is just a working hypothesis leading to the simplest models

[3]We are not interested in any other theory of gravitation than GR in this thesis, neither models other than the standard one, which by itself assumes the validity of GR.

[4]With a possible exception of the De Sitter model, which does not describe the real universe anyway.

[5]A cosmological principle can also be looked at as the criterion of choice by which a certain solution to the field equations is asserted to be a model of the universe.

that are yet acceptable and at the same time operationally functioning. But one can ask questions about the limits of validity and meaning of FLRW models, since the universe around us is not any ideal, in the sense of being homogeneous and isotropic. *This question was our main objective for research pursued in this thesis.*

The above problem can be posed in terms of the so-called *fitting problem*, namely, the question of how best to fit a smooth FLRW universe to a lumpy reality. This way we recognize that the FLRW description can be valid only in some *averaged* sense. In fact the problem of construction of a physically reliable stress-energy tensor is closely related to the fitting problem. Looked at this way, there is a feedback from the observational cosmology side as our observations are all more accurate and further reaching. In effect the whole procedure of confronting the theory and observations, and fitting in particular, could proceed in iterative steps. But the point is still more complicated. Considering a particular solution with a non-vanishing energy-momentum tensor, we can always ask whether the energy-momentum tensor has been correctly constructed or not. It seems that this question on its own is extremely difficult to answer experimentally, if not in principle impossible. This is because, due to the principle of equivalence, GR is a theory of gravitational field and has been empirically confirmed only in this rôle (and as far as solutions of the empty field equations are concerned). Consequently it is not possible to circumscribe the domain of validity of GR from within, because of the non-geometrization of anything other than gravitation. The ideal solution would be to calculate $T_{\alpha\beta}$ once the field equations are solved, instead of postulating it before attempting to solve the equations. In other words, the Einstein field equations only determine the gravitational field corresponding to a given energy distribution and by doing so, they do not provide us with a theoretical description of non-gravitational fields. Certainly, even if the resulting geometrical structure were observed, the question whether the equations have been correctly applied or not, would still be an open question. This disadvantage was already recognised by Einstein himself [87]. He emphasized the non-relativistic nature of any assumed form of $T_{\alpha\beta}$ rather than the possible non-verifiability of the resulting theory. This was one of the reasons that in fact prompted him to work on his unified theory.

With these remarks in mind, it is a matter of great importance to make every effort in order that the foundations from which a cosmological model is obtained are as sound as possible and free from assumptions that may not be warranted. Let us stress at this point that we are not going to be concerned with what might be regarded as a completion of GR in this respect, in the sense of meaningful incorporation of fields into a continuous picture of reality[6], but we will rather try to follow a way enabling to find some more realistic approximations to $T_{\alpha\beta}$ in the cosmological setting from within the theory itself, and then see what the field equations can tell us about the evolution of the universe and the domain of applicability of GR. (Cf. section 1.3. These remarks are particularly relevant in light of section 2.5.)

Having chosen the model, we have in practice at our disposal a procedure for testing the cosmological models. Let us reiterate again the facts that must be kept in mind when we are discussing the universe at large. Cosmological information is obtained along the null geodesics of a pseudo-Riemannian space-time, which represent light rays and the very understanding of what is happening is always to a certain degree theory-dependent, in the sense that the interpretation of cosmological observations is impossible without assuming a working model of space-time. The choice for this model (or a class of models) is eventually made on the basis of postulates, principles and even philosophical tastes.

The comparison is then made between the relations amongst observables in the model and these same relations implied by observations. This line of approach was paved by a beautiful paper of Kristian and Sachs [166]. Worth mentioning is Kristian's [165] attempt to measure the distortion of images of clusters, due to conformal curvature of the universe. This effect is very important, since it is in principle capable of detecting departures from the FLRW geometry in the real universe (it is zero in conformally flat models). The idea is that in a galaxy cluster, the angular distribution of galaxies of each shape should be random, and the anisotropy in their observed images would be determined by the magnetic part of the Weyl tensor.

Further, the problem of the observational basis of cosmology was treated in a series of papers by G. Ellis and collaborators (see [92] and references therein), who discussed in de-

[6]In any case, a description of the fields related to the weak and strong interactions would have to be left aside within this approach, since their current description involves ideas which cannot be meaningfully incorporated into the continuous picture.

tails such problems as: what quantities can be estimated and how from observations, how could observations imply that our universe is FLRW, the practical limitations of observations. The lack of well-defined criteria for acceptance or rejection of the FLRW models was emphasized. In particular, the near-isotropy of the cosmic microwave background (CMB) radiation does not prove the near-isotropy of the matter distribution nor that of the space-time. The general conclusion of Ellis' programme is, that the ideal observations on our past light cone are not sufficient to uniquely construct its space-time geometry. In other words, without dynamical equations (field equations) we can only reach a conclusion of consistency of many different cosmological models with the observations. If we take into account the field equations the situation gets much more involved. However, it is possible to ascertain that the maximal observable data set $D(w_o, z^*)$, is at the same time, the minimal one needed to uniquely determine the geometry of our light cone up to z^* (the redshift of the last scattering surface) [92]. In reality however observations can never be made precise and this aggravates even more the whole problem of fitting.

Clearly the standard cosmological model, the FLRW one, used to describe the real inhomogeneous universe is a very special one and very unlikely at the same time. Interestingly enough, though not surprising, is the fact that the FLRW solutions make up in fact only a set of measure zero in the space of solutions of the Einstein field equations. To ask, how well and how bad [208] describe they the real universe, is a sound attitude and should be a matter of temperate and balanced evaluation, and in particular, the question whether the FLRW line element is a good representation of the geometry of the universe should be asked and embarked to answer quantitatively. Only recently, as new observational data is coming to the fore, the importance of this issue has begun being realised. New more precise data are expected to make the issue of fitting more conspicuous and urgent to address in any cosmological considerations. Current observational information does not exclusively mandate the standard theory.

Interestingly, many cosmologists and physicists appreciated that the FLRW models are an oversimplification of Nature. Below, we quote several of them, but a great merit, it should be said, of the standard model is the absence of any *ad hoc* modification of prevalent theoretical ideas [209].

...the grounds on which homogeneity is generally assumed appearing to be those of convenience rather than generality. (...) We must categorically dissent from the extreme idea (...) that homogeneity is included in the definition of the universe (...) We hold that the assumption of spatial homogeneity is (...) a working hypothesis, valid so long as it does not conflict with observation or with theoretical probability, and justifiable during that time as a restriction on arbitrary speculation. (Dingle, 1933)

The foregoing results demonstrate the lack of existence of any general kind of gravitational action which would necessarily lead to the disappearance of inhomogeneities in cosmological models. (...) it is at least evident from the results obtained, that we must proceed with caution in applying to the actual universe any *wide* extrapolations - either spatial or temporal - of results deduced from strictly homogeneous models. (Tolman, 1934)

It is often claimed that the universe in the large must be isotropic or homogeneous. Certainly this view has immense aesthetic and philosophical appeal, but is it strongly supported by current observations? Unfortunately, it is not (...) observations neither confirm nor deny the "cosmological principle" that the universe is isotropic and homogeneous, or even homogeneous, and (...) measurements at the present time cannot prove, but can only disprove, that particular models represent the actual structure of the universe. (...) global theoretical models that are inhomogeneous should be looked for. (Kristian and Sachs, 1966)

But this approximation[7] is a very crude one. The Einstein field equations are not linear, so that the disturbances in the field produced by the various stars cannot be just added, or averaged in any way. (Dirac, 1981)

Yet, it[8] seems based on absurdly simple assumptions. The universe is assumed to be spatially homogeneous and isotropic while at first sight it is

[7] "This approximation" means replacing the real inhomogeneous distribution of matter in the universe by a smoothed-out one.

[8] I.e. the standard model.

remarkably non-homogeneous and anisotropic. True, one talks of a "large scale" in this connection but that large scale remains beautifully vague and undefined. (Raychaudhuri, 1988)

A more detailed discussion of the question of modeling of the universe will be given in the following sections – in fact, the large part of the burden of this thesis is devoted to it.

The question of choice of an appropriate cosmological model is an important one, but so is the problem of the global understanding of Einstein's equations which can be gained by studying the space of their solutions, the so-called *ensemble* of universes[9]. The field equations are defined on and at the same time determine the space-time manifold. This is what the construction of cosmological models refers to, and this is the point of view of observational cosmology. One simply assumes that each solution to the Einstein field equations describes a universe, and cosmology should but specify boundary or initial conditions relevant to the universe resulting from the astronomical observations.

The observations are never precise, consequently, a cosmological model can describe the real universe only within some limits of accuracy. It is usually assumed that there exists a one-to-one correspondence between the observations of a real system and a mathematical model. Assuming Einstein's field equations to be an appropriate mathematical model one is allowed to predict the evolution of the system. We are already aware that this picture is too idealistic. Even more, the model is usually assumed to have the property of *structural stability* [242], in the sense that the single exact solutions upon which the whole modeling rests are assumed to be in some sense representative, implying that our inherent inability to specify the model precisely would not have significant effects on the qualitative dynamical outcome. But as shown in [242] such a framework cannot be assumed *a priori*[10] and in fact it might be that the appropriate theoretical framework turns out to be that of structural *fragility*.

[9]E.g., the Cauchy problem in a cosmological context naturally leads to the ensemble of possible universes: the set of all admissible kinds of initial data (on a space-like hypersurface) is equivalent to the set of universes evolving from it.

[10]This argument is essentially based on a number of theorems in dynamical systems theory, which show that structurally stable systems are not dense in the space of all systems.

Given these caveats the progress in cosmology can be considered amazing.

This thesis is organized as follows:

In chapter one, apart from rewieving the standard cosmological model, we introduce the problem of fitting and of averaging. A discussion of them follows in a cosmological as well as more general physical contexts. In particular, the motivation for a macroscopic theory of gravity is presented.

In chapter two, a classification and discussion of approaches attempting the averaging of the small-scale (microscopic) Einstein equations to obtain the large scale cosmological field equations is given. In section 2.4.1 the smoothing of cosmological space-times making use of the Ricci-Hamilton flow is presented. In section 2.5 the Macroscopic Gravity Theory (MG) is reviewed. It is an axiomatic approach to a space-time averaging out of the pseudo-Riemannian geometry and of the Einstein equations by using a covariant space-time averaging scheme. In section 2.6 original work is presented (done jointly with R. Zalaletdinov) in the Lagrangian formulation of Macroscopic Gravity [269]. The next section 2.6.5 contains a re-derivation of the MacCallum-Taub limit in MG, showing that MacCallum-Taub results are valid in MG and serve as one of the limiting cases for MG. This part belongs also to our original contributions [206].

Chapter three is concerned with the Renormalization Group (RG) approach in gravity. The main ideas of the RG theory and its application for studying critical phenomena in statistical mechanical models are discussed. Recently discovered numerically critical phenomena in GR gravitational collapse are reviewed. In section 3.4.1.3 some rather radical conclusions are offered pertaining to critical phenomena in gravitational collapse and the significance of the critical exponent close to 0.37. For example, in [65] a point is made of the fact that $0.37 \approx \frac{1}{e}$ while in our opinion it has no significance. Section 3.5 contains original material (in collaboration with J. Pérez-Mercader). The partition function for a gravitational system is introduced formally by using the standard functional integral approach, extended to constrained hamiltonian systems [205]. This treatment is motivated by the RG and is in principle capable of giving the relevant critical exponents. It turns out that the partition function of a homogeneous and isotropic universe model is identical to the partition function of a continuous parameter Ising model (see [205]). This allows to draw some conclusions about the occurrence of phase transitions and the existence of critical exponents (cf. section 3.4.1.3). Section 3.6 contains original material (work done in collaboration with M. Carfora), dealing explicitly with a smoothing procedure for inhomogeneous cosmologies based on the RG approach [61].

Chapter four contains original material (work done jointly with M. Bruni), a description of the evolution of perturbations in form of phase diagrams in the dust-radiation FLRW universe from the viewpoint of dynamical systems theory [52]. (As such this contribution is not directly related to the rest of the thesis.)

Finally, chapter five contains conclusions and some indications for future research.

The bibliography at the end is arranged in the alphabetical order.

The original material presented in this thesis has been or is going to be published. Moreover, an extended version of the three chapters is being prepared for a review paper on the averaging problem and constructing realistic cosmological models via averaging out inhomogeneities of geometry and matter, and currently some physical applications of MG are under study.

1.2 Meaning of cosmological models

1.2.1 Standard cosmological solution

In this section a standard material is overwied and references may not be given explicitly each time; for pertinent references see e.g. [194] or [89].

The assumptions of the standard model are the validity of GR and the Cosmological Principle. The classical term "cosmological model" usually means a geometrical description of the space-time structure and the smoothed-out matter and radiation content of an expanding universe, upon adopting GR as a fundamental theory of gravity. In cosmology, in order to handle the Einstein equations, one introduces simplifications, better or less founded, dealing necessarily with models, which by virtue of their building tend to eschew detailed realism.

To make the above assumptions more explicit we will spell out hypotheses that have been introduced, namely, the *isotropy hypothesis* - motivated by the fact that from our observation point space appears to be nearly isotropic with no indication of noticeable anisotropy. The counts of galaxies and radio-sources give near the same results for all parts of the sky; the cosmological expansion seems to follow the same pattern in various directions and the cosmic microwave background (CMB) radiation is very nearly isotropic. In this respect, (taken on its face value) there exist direct observational evidence favouring the isotropy of the universe, namely the COBE data [231].

As a result, we can write ds^2 in a particular form with a spherically symmetric spatial form and with an orthogonal time

$$ds^2 = -dt^2 + e^{f(r,t)}(dr^2 + r^2 d\Omega^2), \tag{1.4}$$

where, r is a radial coordinate measured from the Earth and $d\Omega^2$ is the angular element corresponding to the angular coordinates θ and ϕ (i.e. $d\Omega^2 = d\theta^2 + sin^2\theta d\phi^2$).

The *second hypothesis* is to suppose that the universe is filled with a *perfect fluid.* It is justified assuming that one considers galaxies as the molecules of a gas that fills the space. Likewise, at the epoch when galaxies would not have existed and the universe would have been filled with a photon gas, it would behaved like a perfect fluid, as well. This particular form of the energy-momentum tensor is given by (1.2).

With these premises along, under the assumption about the symmetry of the space-like hypersurfaces, it is possible to arrive at a metric element known as the FLRW metric. In other words, a system of coordinates can always be found in which the line element can be written in one of its standard FLRW form

$$\begin{aligned} ds^2 &= -dt^2 + R^2(t)\frac{dr^2 + r^2 d\Omega^2}{(1 + kr^2/4)^2}, \quad or \\ ds^2 &= -dt^2 + R^2(t)\frac{dx^2 + dy^2 + dz^2}{(1 + \frac{1}{4}kr^2)^2}, \end{aligned} \tag{1.5}$$

where, $r^2 = x^2 + y^2 + z^2$; or

$$\begin{aligned} ds^2 &= -dt^2 + R^2(t)[\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2] \\ ds^2 &= -dt^2 + R^2(t)[\frac{dr^2}{1 - kr^2} + \frac{r^2(dx^2 + dy^2)}{1 + \frac{1}{4}(x^2 + y^2)}], \end{aligned} \tag{1.6}$$

where, the function $R(t)$ is unknown before solving the Einstein equations entirely; k is an arbitrary constant.

FLRW models have exact spherical symmetry about every point, i.e. space-time is spatially homogeneous and admits a 6-parameter group of isometries, whose orbits are space-like 3-surfaces of constant curvature (Minkowski and De Sitter space are examples of FLRW space-times with additional higher symmetry). The groups are direct products $B_3 \otimes O(3)$, where B_3 is one of the Bianchi groups [126].

Upon assuming the metric in the form of (1.6) it defines automatically a perfect fluid energy momentum tensor. $R(t)$ can be constrained only by an equation of state.

The necessary and sufficient conditions for a space-time to be FLRW are the following:

(1) the metric obeys Einstein equations with a perfect fluid source;

(2) the source velocity field has zero vorticity, shear and acceleration.

The necessity of these conditions can be ascertained by a direct computation. The sufficiency follows from the evolution equations for vorticity $\omega_{\alpha\beta}$, shear $\sigma_{\alpha\beta}$ and acceleration $\dot{u}^\alpha$ [89], which imply that a perfect fluid solution with $\sigma = \omega = 0 = \dot{u}^\alpha$ must be conformally flat. All such solutions were found by Stephani, in general they have $\dot{u}^\alpha \neq 0$, by requiring $\dot{u}^\alpha = 0$ we get FLRW models (e.g. [163]).

Another invariant definition of the FLRW space-times which makes no use of the field equations is the following:

(1′) the space-time admits a foliation into spacelike hypersurfaces of constant curvature;

(2′) the congruence orthogonal to the leaves of foliation are shear-free geodesics;

(3′) the expansion scalar of the geodesic congruence has its gradient tangent to the geodesics.

Another often met representation of FLRW metric is the following

$$ds^2 = -dt^2 + R^2(t)(dr^2 + f^2(r)d\Omega^2), \tag{1.7}$$

where,

$$f(r) = \begin{cases} \sin r & \text{for } k > 0 \\ r & \text{for } k = 0 \\ \sinh r & \text{for } k < 0 \end{cases}$$

All three cases are covered by

$$ds^2 = -dt^2 + R^2(t)(dr^2 + k^{-1}\sin^2(k^{1/2}r)d\Omega^2). \tag{1.8}$$

The range of r is finite or infinite, depending on the sign of k and coordinates used.

A few points are worth recalling, namely.

1. FLRW metric (or element) is a very particular solution of the Einstein equations.

2. For the time variable t fixed ($dt = 0$), the ds^2 is that of a 3-space with constant Riemannian curvature at every point of the space and therefore the space is spherical, Euclidean or hyperbolic, for $k = 1, 0$ or -1, respectively. When $k = 0$, the group B_3 can be of Bianchi type I or VII_o; for $k < 0$ (open FLRW models) its B_3 can be of Bianchi type V or VII_h and for $k > 0$ (closed FLRW models) its B_3 is of type IX.

3. The FLRW metric leads to homogeneous model universes, through the assumption of isotropy with respect to the Earth, which leads to a space of constant curvature and, consequently, to a space that is isotropic with respect to all other points. One should not forget however that the near-isotropy with respect to the Earth is valid within certain limits: redshifts up to 0.5 for galaxies, 5 for radio-sources, 7 for the CMB, and in any case all observations of this kind are limited by the cosmological horizon.

4. We are dealing with uniform model universes, that is to say for t fixed, the density μ and pressure p each have the same values at every point in the space.

5. The worldlines $r = const, \theta = const, \phi = const$ are geodesics of the space-time. What this means is that the worldlines in question are possible solutions of the equations of GR and they therefore represent possible motions, e.g. of field galaxies (after correcting for random motions) or galaxy clusters, which in this picture have fixed coordinates r, θ, ϕ, termed as comoving coordinates.

6. $R(t)$ appears as a scaling parameter for the universe, if it is an increasing function of t we are then led to an expansion of the universe. It gives the rate at which two points at fixed comoving coordinates increase their mutual physical distance as $R(t)$ increases.

The time t (cosmic time) is the proper time at every point fixed in the comoving coordinates, in particular, it would be shown by clocks in various galaxies synchronized by exchanging light signals.

The comoving coordinates r, θ, ϕ owe their properties to the fact that effectively the distribution of mass in the universe may be described by masses having fixed comoving coordinates[11].

[11] The comoving distance is not an observable, unlike e.g. the luminosity distance.

To completely solve the cosmological problem, the two Einstein equations together with the equation of state for the perfect fluid $p = p(\mu)$, we have to determine the three functions: $R(t), \mu(t)$ and $p(t)$. If we consider only the case in which the cosmological constant $\Lambda = 0$, we obtain the simplest family of relativistic models, the FLRW models. These models are necessarily an approximation to the real universe, due to various simplifying assumptions employed in their derivation.

In the case of FLRW model the Einstein equations reduce to a system of two equations

$$\left(\frac{\dot{R}}{R}\right)^2 \equiv H^2 = \frac{8\pi G}{3}\mu - \frac{k}{R^2}, \tag{1.9}$$

$$-\frac{\ddot{R}}{R} = \frac{4\pi G}{3}(\mu + 3p). \tag{1.10}$$

Heuristically, these equations can be seen as the equivalent of the energy conservation and the second dynamics law of classical (non-relativistic) mechanics. We see that two points at a distance $d \equiv R(t)r$ move apart with velocity $v = Hd$. The Hubble constant determinations at the present epoch give $H = 100\, h\, km\, s^{-1}\, Mpc^{-1}$ with $0.4 \leq h \leq 1$. The large uncertainty is mainly due to the discrepancies between different distance estimators.

A standard model - one of the homogeneous, isotropic FLRW model (and the present epoch (t_o)) can be determined by the measurement of two dimensionless numbers: the deceleration parameter $q_o \equiv -\ddot{R}_o/(R_o H^2)$, the density parameter $\Omega_o \equiv \mu_o/\mu_{crit}$, and a scale constant H_o ($\Lambda = 0$ is equivalent to $2q_o = \Omega_o$). The critical density $\mu_{crit} = 3H_o^2/(8\pi G) = 1.9 \times 10^{-29}\, h^{-2}\, g\, cm^{-3}$ and present density values of μ, above, below or equal to μ_{crit} correspond to closed, open or flat geometries, respectively. Current limits on Ω_o are, $0.2 < \Omega_o \leq 1$. The limit of $\Omega_o = 1$ is usually preferred on the grounds of the standard inflationary universe scenario. Standard cosmology puts forward the equation $4\pi G\mu/3 = qH^2$. If and when the precise value of μ_o, q_o and H_o are available, the decision about the standard model will be easy to reach.

At present the cosmological observables include also: the age of the universe, composition of the universe, CMB radiation and other cosmological background radiation, the abundance of light elements, the baryon number of the universe (quantified as the baryon-to-photon ratio), and the distribution of galaxies and larger structures. Observations of the universe on scales similar to the size of a galaxy $\sim 10\,kpc$ display significant inhomogeneities, but the current interpretation is that below such scales non-gravitational forces are dynamically dominant. On the other hand, scales $\gg 10\,kpc$ are considered relevant to the large scale structure. In any case, non-negligible anisotropies were never observed at scales comparable to the horizon scale $d_H \sim (cH_o)^{-1} = 3000\,h^{-1}\,Mpc$. Therefore the cosmological framework of the hot Big Bang in a spatially homogeneous and isotropic universe - the Standard Model - is taken to be experimentally vindicated, through the successful prediction of primeval element abundance and the observations of relic radiation in the form of CMB radiation [250][12].

However, from observational point of view the standard models have currently some problems. Some of them were listed in [26][13].

1. Dark matter not visible, but revealed through non-Keplerian motions of galaxies and stars in galaxies.

2. Streaming motions of galaxies with velocities $v \leq 5000\,km/s$ towards invisible attractors.

3. Honeycomb-like large scale structure and possibly spatial periodicity in the distribution of galaxies.

There seems to exist observational evidence in favour of larger and larger structures (e.g. [67] recently), nevertheless, [41] indicate that there is a tendency to homogeneity at large scales, although it is difficult to point out the scale at which homogeneity is reached, due to the small size of present redshift surveys. The striking feature of the (luminous) matter distribution is the existence of voids (in the scales probed so far, up to a few hundred Mpc) surrounded by sheet-like structures containing galaxies (e.g. [118]), with a typical size of voids $50 - 60\,h^{-1}\,Mpc$. Also, larger underdense regions of $\sim 130\,h^{-1}\,Mpc$ probably exist [46]. Interestingly, dynamical estimates of FLRW density parameter Ω_o give different results on different scales. Observations of galactic haloes on scales $10 - 30\,Mpc$ give $\Omega \simeq 0.2 \pm 0.1$ [220], whereas smoothing the observations over scales $\sim 100\,Mpc$ indicates the existence of a less clustered component with a contribution probably as high

[12] Doubts still persist, e.g. regarding the exact primordial abundance of helium.

[13] Several objections can be accommodated within either the Lemaître-Tolman [168, 246] or Szekeres [241] models with a non-simultaneous Big Bang.

as 0.8 ± 0.2 [83]. Hopes have been raised that new inputs like the inflationary scenario and cosmic strings may solve the problem.

4. According to the present evidence quasars must contain large magnitude density concentrations (especially if they accrete black holes), but matter distribution at the last scattering should be nearly isotropic, as confirmed by very high degree of isotropy of the CMB radiation. The question therefore is, how such density contrasts might have evolved from such a homogeneous initial state in the time implied by the standard model.

Generally, the problem of structure formation in spatially homogeneous models is an important problem of principle. *"Statistical fluctuations in FRW models cannot collapse fast enough to form the observed galaxies. This suggests that there must be real inhomogeneities at all stages in the universe. Moreover, some perturbations of FRW models are decaying modes which would have been more important in the past."* (MacCallum, 1979). (Cf. point **3.**)

5. Astronomical observations contradicting the Hubble law [13].

The linear velocity distance relation was questioned in [199]. Their conclusion about the quadratic relation between velocity and distance did not gain an acceptance[14].

McCrea in 1939 [192] on the basis of Milne's result, that any cloud of particles with different velocities and initially being confined to a finite space volume will disperse and eventually obey the Hubble law in the first approximation, concluded that in order to infer matter distribution in the universe from the observed motion of galaxies, a higher approximation is needed than the Hubble law.

6. The quadrupole anisotropy in the CMB radiation on large angular scales $\simeq 80^o$ [106].

The quadrupole anisotropy could serve as a test of the anisotropic expansion of the universe at the last scattering epoch since it is due to the non-vanishing vorticity of the universe; its matter is rather subtle[15] and still disputable.

7. Anomalous redshifts and "quantisation" of redshifts.

The so-called anomalous redshifts refer to objects which are presumably close together

[14] It was shown later that even such a relation may be accommodated in the standard model [210].

[15] It might be seen as contradicting Mach's Principle; attempts are therefore advanced to attribute it rather to local causes, like the non-homogeneity in the mass distributions of galactic clusters.

and show significantly different redshifts. There is also the curious quantization of redshifts, with Δz multiples of Δz_o corresponding to a velocity of $72\, km\, s^{-1}$ [14]. Both of these cast doubt on the idea of redshift being *only* due to universal expansion.

8. Still controversial interpretations on the observations of radio sources, like the angular size – flux density relations.

All the above facts cause problems with fitting, however not only the data but also the logic of the standard approach make the fitting problem particularly urgent (cf. next section).

1.2.2 Modeling of the real universe

A standard way to analyze a real system is to make a mathematical model, which can then be studied analytically or numerically. The relation between the real system and its model should be properly assessed, though quite often it remains obscure.

On local scales like the scale of planetary systems, we can safely assume GR to hold, but there the space-time geometry determined by Einstein's equations should be very complex. In fact, a mathematical model of the matter distribution going down to these small scales is almost impossible to obtain, as there is no observational data on which it could be based. Therefore, when considering the kinematics and dynamics of the universe as a whole, one usually ignores the fine-graining due to the local inhomogeneities and deals with the simpler structure of space-time geometry, which is more illuminating from the point of view of cosmology.

As we have already seen, the standard FLRW universe models are perfectly homogeneous and isotropic. Therefore one can question their applicability for modeling the universe accurately, as it is manifestly not a FLRW universe (on at least some scales). But, these standard models are usually taken to represent the real universe in some vaguely "average sense", or on some averaged scale. This implied averaging procedure should be of great importance in cosmology, especially in terms of interpreting the meaning of FLRW models. However, it hardly ever receives a due attention, even though it underlies the geometric and physical applications of FLRW metrics to describing the real universe, and even though it has been posed by a number of authors [228, 226, 90]. Shirokov and Fisher

[228] seem to have been the first to consider it in 1962. Recently, this problem has been brought to general attention by G.Ellis.

Problems arise when we start to relate the realistic inhomogeneous universe models to these idealised, smoothed-out models. The very relation between them is not clear, in particular, it is not quite obvious how the galaxies or clusters of galaxies are related to the comoving coordinates of the averaged idealised models, nor how particular light rays correspond to the idealised geodesics of these models, etc. The standard approach in this respect is a theory of the perturbed FLRW models (see e.g. [21]) and their relation to observations of galaxies and background radiation [157, 218], which is still a matter of investigation (see e.g. [237, 51, 215]).

In observational cosmology it is standard to follow this scheme: firstly - to observe the distribution, masses and velocities relative to us of neighbouring galaxies; secondly - to calculate the averaged quantities assuming the Hubble law, i.e., the isotropy (on average) of the relative velocity field, and homogeneity of the distribution of galaxies; thirdly - to compare these mean properties with those of FLRW models, having the same density as that of the total mass of the galaxies uniformly distributed in the observed region. This procedure basically means that the FLRW models are assumed instead of deduced from observations. But in principle, what one could hope to get in this way, are the best-fit parameters of the FLRW models. Unfortunately, the discrepancy between the observational data and the properties of FLRW models is such that it is usually necessary to introduce an additional matter contents besides that estimated from the visible matter (dark matter), or even a cosmological constant (see e.g. [158] and [223]). However in effect, even without the benefit of a sound theory as to how to obtain the best-fit of idealised model universes to reality, the observers are already "fitting" the observed velocity patterns to a hypothetical FLRW velocity field (e.g. in the efforts to determine the velocity of the local supercluster relative to a FLRW background).

An attempt to translate the mathematical prescriptions into practical observational procedure, unfortunately seems to be a very difficult issue with various problems, concerning in particular, the determination of the average density and velocity of matter at a given distance down the past null cone [88, 90]. Other alternative analyses of homogeneity based on "almost Killing vectors" [191] or "observational homogeneity" [39], do not yet seem able to resolve these issues as well.

But a reliable description of the inhomogeneities in the expanding universe is wanting above all. Clumpiness obviously affects the analysis of observations in an inhomogeneous universe model, by affecting the dynamics of photons in the universe altering the shear and the convergence of null rays. This has its effect on the cosmological area-distance relation and in consequence on apparent sizes and luminosities (see e.g. [29]).

It is possible to take this effect into account, but the form of the focusing depends sensitively on how strongly the matter is clumped. A useful idealization is the Dyer-Roeder formula (see e.g. [224]) valid for universes "not too much filled with clumps", it represents the largest possible angular diameters distance (for a given redshift) for light bundles which have not yet passed through a caustic.

The effect of clumpiness on geodesics paths can change the relations adopted in standard model, but that strongly depends on the clustering of matter in the universe. The same applies to the issue of Ω_o. The underlying assumption is that there is a good limit to Ω_o if we take large enough averaging volumes, but it is not so if we live in a hierarchical universe [249], where there is no non-zero limit to Ω_o as we use larger and larger averaging volumes[16].

The transition from an inhomogeneous model to an averaged (or smoothed) standard model is of fundamental importance, also in the context of structure formation and the interpretation of distance measurements.

In practical terms the heuristical justification for using FLRW models asserts that for them to hold the matter inhomogeneities have to be averaged (or smoothed-out) and redistributed homogeneously (e.g. in the form of a perfect fluid). A series of related mathematical problems arise. First of all, let us notice that we are using continuous functions in modeling the universe (matter density, pressure or kinematical scalars of the velocity field), assuming that they represent "volume averages" of the corresponding fine-scale

[16]Suggestions that the universe might have a fractal structure [40] might support this argument.

quantities. The results of such averaging in an inhomogeneous medium depend on the scale, but this scale (of averaging) was never explicitly agreed upon. Additional problem is that a volume average for tensors is a non-covariant quantity (unlike for scalars), so a more sophisticated definition is required. The basic and tacit assumption which underlies this whole procedure is that a smoothed-out universe and the actual, inhomogeneous one, behave identically under their own gravitation. Or, to be more precise, almost identically on some scales of interest, e.g. the ones that are much greater than a characteristic scale of the local inhomogeneities and much smaller than a characteristic length of the universe model under study. This assumption is usually taken for granted, but does by no means need to be true. Indeed, the non-commutativity of averaging of the metric and calculating the Einstein tensor (highly non-linear in the metric) is a severe problem (this will be discussed in more detail in section 1.3).

1.2.2.1 The "fitting" problem in cosmology

The basic idea here is that we do not *a priori* assume that the universe is necessarily well described at all times by the FLRW model, but nevertheless decide to use such a model for, say, pragmatic reasons [99].

The problem is then how to determine a *best-fit* between a clumpy cosmological model $\mathcal{U}$, which is supposed to give a realistic representation of the universe including all inhomogeneities down to some specified length scale, and a smoothed-out, idealised FLRW model $\mathcal{U}'$. The focus in this approach is on the relation between the idealised model and more realistic descriptions of reality. Therefore, one should also be able to specify details of that fitting, including the issue of how good the fit is. Basically, one could aim at the repeated use of the smoothing procedure, i.e., to consider a best fit between any lumpy universe model and a model $\mathcal{U}''$, which gives an even better description of the real universe than $\mathcal{U}'$, by describing the inhomogeneities at an even more detailed level.

In principle, this process would allow one to determine the best description at any prescribed level of detail.

The above task can be approached in many ways, based on:

(i) the space of space-times,
(ii) initial data for space-times,
(iii) the "gauges" adopted in perturbation studies,
(iv) near equivalence,
(v) average behaviour,
(vi) null data,
(vii) normal coordinates.

The (i) approach gives a useful concept of the fitting idea, but it does not take into account the dynamics of GR. In this approach, the lumpy model $\mathcal{U}$ and the ideal one (FLRW) $\mathcal{U}'$ are represented by points P, P' in the space of space-times S, and the point P' is constrained to lie in the hypersurface of FLRW space-times. If a suitable distance function on S is known; then given $\mathcal{U}, \mathcal{U}'$ is chosen in such a way as to minimize the distance between P and P'. Unfortunately, no natural positive definite metric on S exists. Also to be able to distinguish whether two different points in S represent the same space-time or not, one would need to factor out the coordinate freedom (the diffeomorphism group) which seems also to be problematic, unless one could find the best fit in some specific and operationally defined coordinate system rather than the general one. Moreover, matter distributions would need to be fitted as well. Most likely this approach cannot be easily related to an observational procedure on the past null cone.

Instead, in the (ii) approach we consider the space S^* of initial data for space-times (the phase space of a cosmological model in GR), with initial data given on a spacelike hypersurface Σ : $(g_{ab}, K^{ab}, \mu, q^a)$, where, g_{ab} is the 3-metric on Σ, K^{ab} its second fundamental form, μ the matter energy density, q^a a 3-momentum relative to Σ. Given this, each point Q in S^* will correspond to a specification of all of these quantities at each point on the initial hypersurface Σ, chosen so as to satisfy the Einstein constraints on Σ.

The problem then is stated similarly as in the (i) approach. The difficulties of (i) remain. One has to take into account here as well, the fact that different data can represent the same cosmology $\mathcal{U}$ (e.g., choose two different spacelike slicing Σ of $\mathcal{U}$ to get two different 3-space metrics g_{ab}). However, involved here is rather the hamiltonian structure of GR, and roughly speaking, the symplectic form of that structure gives a way of comparing metrics on two different spacelike hypersurfaces, to see if they represent slicings of the same space-time. Therefore in principle, once the fit of 3-metrics is determined, the 4-dimensional fit would be as well, even if not explicitly known [120].

Clearly, the (ii) approach originates from the *ensemble* viewpoint. To understand the equivalence of models and the "distance" between them calls for the introduction of the appropriate topology and metric structure on the *ensemble*, which is a difficult and up to now an open problem.

Usually, one starts with $Lor(\mathcal{M})$, the space of all Lorentz metrics on the manifold $\mathcal{M}$. This space admits infinitely many topologies, none of which is "natural". However, what one is really interested in is a subset of $Lor(\mathcal{M})$, namely, those metrics that are the solutions of Einstein's equations. Fortunately, this set usually is a smooth manifold, with a local representation in the space of four functions of three variables, and Einstein's equations acting as a hamiltonian system. Further, the space of linearized solutions of the Einstein equations is tangent to it (this is so only, if our set of metrics is a smooth manifold in the neighborhood of a given solution) and in this region Einstein's equations are stable with respect to the linearization[17]. This approach can be helpful for the global analysis of the solutions to Einstein's equations.

In fact the averaging procedure advocated in section 3.6 takes on just this viewpoint.

The point of view of approach (iii) is quite useful since when describing perturbations of FLRW universe we need to choose a gauge, which is essentially a question of fitting a FLRW universe $\mathcal{U}'$ to a lumpy one $\mathcal{U}$ [21, 235]. Choice of a gauge in a perturbed universe model $\mathcal{U}$ is equivalent to choosing a point identification between the FLRW model $\mathcal{U}'$ and $\mathcal{U}$. Given a choice of local coordinates in $\mathcal{U}'$ and in $\mathcal{U}$, the above correspondence can be expressed in terms of a relation between these coordinates. But instead this is usually taken to be the identity, i.e. the coordinates of the corresponding points are taken to be the same.

A specific gauge can be characterized in terms of a choice of "hypersurfaces of simultaneity in the physical space-time" [21], which is actually the choice of a mapping of the FLRW surfaces ($t = const$) in $\mathcal{U}'$ into the lumpy model $\mathcal{U}$. Once a point correspondence

[17] In general, this is not true for space-times with Killing vectors.

between space-times is set up one can then look for the parameters H_o and q_o giving the best fit. Seen this way the problem of fitting is the problem of choice of gauge in disguise.

The classic equivalence problem of GR (see [84] for a review) is much of relevance for the fitting problem - (iv) approach. A direct approach by comparison of curvature invariants of two space-times is problematic because the metric tensors are indefinite. However, to prove an equivalence of two space-times it is sufficient to have the equivalence of the curvature tensors and their covariant derivatives, evaluated in orthonormal frames [155]. Using a procedure of this kind would allow to determine if two cosmologies are almost equivalent. There is also here the question of how to choose which pairs of points p, p' to try to identify in the two cosmologies. Secondly, the idea of almost equivalence is more complex in the context of specific choices of tetrad canonically related to particular Petrov types. An extended algebraic classification of the curvature tensor and its derivatives should make possible determination of near-equivalence of space-times by extending the methods used to examine exact equivalence. Moreover, the problem would be simplified in the cosmological context. Thus this is a promising approach though its relation to possible observational procedure seems obscure.

The (v) approach, based on the concept that the smoothed-out model $\mathcal{U}'$ should accurately represent the average behaviour of the more realistic model $\mathcal{U}$ is going to be discussed at length in the next section. The important issue to address here is what is precisely meant by the "average" model, which is what most of this thesis is devoted to. This approach to fitting is also made use of in section 3.6.

The null data approach (vi) [99] is a specific observational prescription to best-fit null data to obtain optimal, i.e. best fit parameters describing FLRW universe, given an optimal correspondence between any FLRW model and lumpy universe. This approach is in practice similar to what is done at present by observers (as it extends the approach of Kristian and Sachs [166]), but it can further be related to the averaging approach and suggests moreover the nature of possible criteria of acceptable fit.

Finally the (vii) approach, closely related to the nature of local physics in a lumpy universe, is in fact a local almost equivalence approach. This is so since in the analytic case, one can examine the curvature near any space-time point by using normal coordinates about that point. Different space-times can be locally compared by writing each of them in such coordinate systems centered on points p, p' and directly comparing the metric components up to some required order of accuracy. As before the choice of correspondence of points p, p' to make in the two space-times is a problem. This approach can nevertheless be related to the question of a best fit to astronomical observations by an appropriate adaptation of [166], whereas carrying out the analysis in a non-local way ends up in (vi) approach [99].

The implications of fitting can be analysed in terms of Traschen integral constraints [247]. In an almost homogeneous universe model with inhomogeneities due to local physical processes, the local energy and momentum conservation imply the existence of conserved quantities expressing the conservation of monopole and dipole terms. In such models, the Sachs–Wolfe effect is reduced with respect to the ones where the effect is ignored [247][18].

According to [248] this argument would not apply to matter perturbations created by quantum fluctuations in the inflationary epoch because of their non-local size, in terms of today's scales. Nevertheless the constraints can be thought of as the fitting conditions, required to be satisfied if the chosen FLRW background model has the right monopole and dipole terms to correspond correctly to the real universe [96].

For example, the very definition of Ω_o refers always to an idealised background model, which cannot be determined without simultaneously solving for its perturbations.

We can start with a uniform universe model A and model high density regions in it by adding some over-densities here and there, resulting in a non-uniform model B. Obviously, the average density Ω_B in B is greater than the background density Ω_A of A, thus using model B means also changing the background model to, now A', which has the density $\Omega_{A'} = \Omega_B$ and different dynamics from the initial model considered. If we use the same background density value (Ω_A) in the lumpy universe model, model B has to be replaced by model C, say, in which the high density regions are surrounded by void regions, in order that the average density $\Omega_C = \Omega_A$. This is basically Traschen's condition [96].

[18] Other dynamic relations are affected as well.

1.3 The "averaging" problem in cosmology

1.3.1 Statement of the problem

One important way of thinking of the use of a smoothed-out model is that it represents the average properties of an inhomogeneous model. If $\mathcal{U}^*$ is the smoothed-out model universe, obtained form a clumpy one $\mathcal{U}$ by an averaging procedure, then it represents the nature of $\mathcal{U}$, when described over some averaging length scale L. The best-fit FLRW model $\mathcal{U}'$ should be the same as the averaged model $\mathcal{U}^*$, if one can indeed describe the large scale nature of $\mathcal{U}$ by a FLRW space-time [99][19].

In standard cosmology to describe the discrete matter distribution in the universe, we use a continuously distributed stress tensor, most often the so-called perfect fluid form of it. But strictly speaking using the Einstein equations in this situation is not well-founded. What it means is that an effective averaging of real inhomogeneities has been carried out, while at the same time the unchanged left hand side of Einstein's equations is tacitly assumed to describe the "averaged" gravitational field. However we should bear in mind that the Einstein equations are highly non-linear, which is why any averaging process is far from trivial in general[20]. In other words, the averaging process may change their structure and consequently the geometric and physical meaning of the very gravitational field would be changed.

In particular, Ellis conjectured [90] that upon smoothing-out the space-time geometry, there would appear geometric correction terms in the sources to Einstein equations. They may have influence on the dynamics of the universe. In general, there would always be a non-zero backreaction of inhomogeneities on the dynamic behaviour at the smoothed-out scales affecting the expansion rate and the estimate of age for the universe.

To put it differently, if we calculate the Einstein tensor $\tilde{E}_{\mu\nu}$ from an "averaged" (whatever this means) metric $\bar{g}_{\mu\nu}$, it will not be equal to the Einstein tensor $\bar{E}_{\mu\nu}$ which was first

[19] An intriguing possibility one can think of is that one could construct clumpy Small Universe [91] models which look like the perturbed FLRW models, but for which the smoothed-out (large scale) version is not a FLRW universe. They would appear approximately homogeneous, but their topology would be incompatible with the symmetry of exactly homogeneous universes.

[20] Unlike in electrodynamics, where the macroscopic Maxwell equations can be derived by averaging out the microscopic Maxwell-Lorentz equations over 4-regions in Minkowski space-time [128].

calculated from the fine-scale metric $g_{\mu\nu}$ and then averaged. As a consequence, the Einstein equations seem not to hold on scales where averaging is required if they hold on say, planetary scale. Most probably, the Einstein tensor $\tilde{E}_{\mu\nu}$ determined by the smoothed-out metric $\bar{g}_{\mu\nu}$ will be related to $E_{\mu\nu}$ by a map S'' distinct from the smoothing operator S' acting on the matter tensor $T_{\mu\nu}$.

However in cosmology the following is assumed: one calculates the Einstein tensor from a metric that is supposed to be already averaged and equates it to an energy momentum tensor – already averaged as well. But let us now introduce a tensor $\Pi_{\mu\nu}$ to represent the difference between the Einstein tensor $\tilde{E}_{\mu\nu}$ obtained from the smoothed-out metric $\bar{g}_{\mu\nu}$ and the smoothed-out matter tensor $\bar{T}_{\mu\nu}$ [90]. The point is that the Einstein equations should in fact be "corrected", so that the difference $\Pi_{\mu\nu} \equiv \tilde{E}_{\mu\nu} - \bar{E}_{\mu\nu}$ is compensated, namely,

$$\tilde{E}_{\mu\nu} - \Pi_{\mu\nu} = \kappa \bar{T}_{\mu\nu}. \tag{1.11}$$

Writing the term $\Pi_{\mu\nu}$ on the right hand side of (1.11), we can interpret it as a correction to the source resulting from averaging out the small scale inhomogeneities of the gravitational field. This brings back the equations again to their familiar form but with the effective source term,

$$\tilde{E}_{\mu\nu} = \kappa \bar{T}_{\mu\nu} + \Pi_{\mu\nu}. \tag{1.12}$$

This correction term is to be added to the field equations at other scales, than the scale at which they are verified and which is therefore the scale on which they are supposed to hold.

If we assume that the averaged metric is the FLRW metric and that $\bar{T}_{\mu\nu}$ has the perfect fluid form (homogeneous and isotropic), the correction $\Pi_{\mu\nu}$ will perturb the energy density and pressure of the source, invalidating the FLRW relation between the sign of spatial curvature on the one hand and the size and lifetime of the universe on the other[21].

A correct and consistent treatment of this problem would require one to average the geometry *and* matter present, i.e. a microscopic matter distribution and the Einstein field operator, to determine both sides of the averaged Einstein field equations. Considering the above kind of averaging, one has to determine a relation between the manifold structures

[21] It holds anyway only for dust without a cosmological constant.

and corresponding points in the two models that we deal with at each step of averaging. Finally, this would allow us to explicitly determine the averaged field equations responsible for the large-scale, average dynamics and study the observational properties of the average universes and the relation between more detailed and the averaged behaviour in them.

In the weak-field or almost FLRW cases, one can use direct methods to define the averaging (see chapter two for a review of various approaches); in the full theory the situation is more complicated.

It is probably worth stressing at this point that speaking of averaging of the Einstein equations requires an utmost care and one has to clearly state what this means, since by themselves the Einstein equations (as any differential equations) *do not contain any in-built fundamental length*, so that they can be used *a priori* to describe cosmoses of any size. It should be emphasized that while doing this, these different metric and matter tensors used are intended to describe *the same* physical system, i.e. the same space-time, but at different scales of description enabling one to resolve less or more details[22].

Let us stress again that the motivation for a setting of the problem within GR came from cosmology, where one is usually solving the Einstein equations with a smooth (continuously distributed) stress tensor which implies that a space-time (or *ensemble*) averaging of a discrete matter distribution in forms of stars, galaxies, etc, has been carried out.

1.3.2 Micro-, macro-physics and the issue of observables

The averaging problem, as discussed above, if looked at from a more general perspective can be formulated as the problem of a *macroscopic description of the gravitational field* [228, 226, 212, 265]. Some authors (see e.g. [273]) advocate a multiple averaging (done over various e.g. increasing scales) whereas, in fact, it is enough to average out the relevant equations only *once* to arrive at, following again the electrodynamics' example, the *macroscopic* field equations and then use them for any specific situation one can think of. This indeed seems to be the most general and correct statement of the problem. The first

[22] Posing the problem this way turns out to be very useful from the Renormalization Group approach viewpoint, which we are going to advocate in chapter three.

fundamental question concerning this issue is the question about the character of Einstein's gravity. The overwhelming majority of authors dealing with an averaging in GR consider it together with the Einstein field equations as a theory of microscopic gravity[23]. This is strictly speaking an assumption, but it is well grounded from the physical point of view. The support for such an identification lies in the fact that the Einstein equations are the ones that provide us with an exact, physically meaningful solution for the gravitational field of an isolated point-like mass (Schwarzschild solution and its generalizations). Another support in favour of this identification can be claimed in superstring theories which in their low energy limit give precisely the Einstein equations.

To conclude, unlike in electrodynamics where the macroscopic equations had been discovered by Maxwell and only afterwards the corresponding microscopic ones were established by Lorentz, Einstein's GR can be considered as a theory of *microscopic gravity* with Einstein field equations as the *microscopic field equations.*

The second fundamental question to consider is how to formulate the macroscopic theory of gravity. Generally adopted way which had been evoked by Lorentz follows closely the electrodynamics case. In this sense the idea of macroscopic gravity can be considered as an extension of Lorentz' idea in electrodynamics, about two-level description of classical physical phenomena, namely, microscopic and macroscopic [175]. The process of constructing the macroscopic field equations should include then, as it was in electrodynamics' case, two steps: 1) a space-time averaging of the Einstein field operator, and 2) the same kind of averaging for the matter distribution[24]. To carry out this program within GR requires one to face problems of geometrical and statistical nature, unlike in electrodynamics where the linear field operator of Maxwell's equations is known and easily averaged out, and the problem consists basically in constructing an electromagnetic medium model [175, 128]. Apart from the problem of the macroscopic description of gravitating matter, the space-time averaging of the Einstein field operator is not a trivial task due to a non-flat geometry underlying GR.

The interpretation of a macroscopic theory and its meaning is of great physical impor-

[23]Exceptions are [6] and [183].

[24]A space-time averaging is assumed in most cases to be equivalent to *ensemble* averaging due to the property of ergodicity.

tance. The observation of the state of a physical system involves, as a rule an averaging of the instantaneous values of the pertinent variables, either over time or over a volume[25]. Field quantities, the solutions of the basic macroscopic equations, are by definition meaningful and measurable. A macroscopic theory can thus provide the answers as to which objects can be observed in classical measurements of fields and matter. The distinction between macrophysics and microphysics seems to lie at a pretty fundamental level of the concept of what constitutes an observation and an observable. Macrophysics as a rule can be defined as that aspect of reality in which the interference of an observer can always be neglected.

Likewise in the framework of GR, the space-time averaging gives meaning to a measurement procedure in a finite measurement time with a measuring system of definite (finite) space size. Such an averaging of the objects of a microscopic physical theory, i.e. a theory taking into account the structure of matter, is carried out over space regions and time intervals physically small compared with the *macroscopic* distances and times characteristic for the process under consideration and physically large with respect to the microscopic regions and intervals [175]. It describes measurements of the processes. A macroscopic theory has no doubt a direct observational status and tells us which objects can in principle be observed in classical measurements of fields and characteristics of matter (e.g. induction in electrodynamics)[26].

To accomplish this program the averaging procedure should rather be a linear one. The non-linear procedure would not have a clear physical meaning. This is dictated by the fact that the essence of any measurement lies in the process of integrating and thereby it is intrinsically linear, even for a non-linear theory.

In the case GR, there is a consensus of opinions concerning the rôle of the averaged microscopic metric tensor, $G_{\alpha\beta} =< g_{\alpha\beta} >$, $G_{\alpha\beta}$ being the macroscopic metric tensor. Since the metric tensor $g_{\alpha\beta}$ is an observable and measurable object in GR, the averaged metric $G_{\alpha\beta}$ acquires directly a physical meaning and is a measurable object describing

[25]The reader is advised to consult an excellent paper [36] (and one of the classic) which deals with the question of the measurability of electromagnetic field quantities on a deep level.

[26]Note that after quantization of the microscopic theory, a quantum theory will describe the classical measurements of quantum processes through the quantum averaging.

now the macroscopic gravitational field as usual.

1.3.3 Setting of the problem of the macroscopic description of gravity

All approaches tackling the averaging problem followed in fact the generally adopted way of Lorentz for electrodynamics, i.e. of averaging Einstein's equations to arrive at the averaged field equations, and tried to understand thereby a structure of the averaged gravity. Such an approach however failed to give and cannot in fact give a satisfactory solution to the problem, since no proposal has been made about the correlation functions. They should inevitably emerge in averaging out a non-linear theory (see [263, 266] for a discussion). The very averaging of the Einstein equations cannot provide us with a definition of the correlation functions which should be defined in addition to the averaged field equations together with the differential equations to find them. Indeed upon averaging out the Einstein equations $r_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} g^{\mu\nu} r_{\mu\nu} = -\kappa t^{(micro)}_{\alpha\beta}$, we obtain

$$< r_{\alpha\beta} > -\frac{1}{2} < g_{\alpha\beta} g^{\mu\nu} r_{\mu\nu} >= -\kappa < t^{(micro)}_{\alpha\beta} >, \tag{1.13}$$

which can be re-written as

$$\bar{r}_{\alpha\beta} - \frac{1}{2} \bar{g}_{\alpha\beta} \bar{g}^{\mu\nu} \bar{r}_{\mu\nu} + C_{\alpha\beta} = -\kappa < t^{(micro)}_{\alpha\beta} >, \tag{1.14}$$

where $C_{\alpha\beta}$ stands for a correlation function. The Einstein equations now become only a definition of the correlation function. In order to bring back their status of the field equations one should make a proposal regarding $C_{\alpha\beta}$ from outside. (Look for example at equations (1.11) which are now only a definition of $\Pi_{\mu\nu}$.)

A new setting of the problem to formulate a theory of macroscopic gravity has been recently put forward by Zalaletdinov (see section 2.5 for a review). Following [263] one should study, first of all, the problem of how *to average out a (pseudo-)Riemannian space-time itself*, i.e. Cartan's structure equations describing the structure of Riemannian geometry. Meanwhile it is necessary to understand which averaged geometrical objects - metric, connection, or curvature - can characterize an averaged space-time. Another necessary ingredient of such a theory is a splitting of the average of products of relevant objects encountered in averaging out Cartan's equations. *This is the problem of introducing the* *correlation functions.* Upon deriving the structure equations for the averaged manifold can the Einstein equations, which from a geometrical point of view are known to be the additional conditions to Cartan's equations, be successfully averaged out. Such an approach to formulate a macroscopic theory of gravity is essentially *non-perturbative* and provides us with the geometry underlying the macroscopic gravitational phenomena. *This theory possesses the correlation length as the fundamental length scale.*

Let us add that from the point of view of further possible generalizations this is not a final theory however, since Macroscopic Gravity theory is proposed as the theory of equilibrium gravitating systems where the correlation functions are functions of a space-time point, whereas we would expect them generally to be functions of many points (critical phenomena could then possibly emerge).

2 A survey of approaches for averaging

In this chapter we shall review various approaches for constructing the relativistic cosmological models via averaging out inhomogeneities of geometry and matter. A helpful reference on the subject is [164] and [267], listing many of the relevant papers.

We do not adopt a unique notational convention in this chapter. This was mainly dictated by the fact that there are various conventions adopted by various authors and for the reader's convenience (who might wish to consult the original papers) we decided to maintain the notation of each reviewed paper unchanged.

2.1 Averaging within approximation schemes

The definition of averaging can be based on or coupled with approximation schemes.

In the first attempt of this kind [240] (not meant to be applied to cosmology) Szekeres showed that linearized Einstein equations, i.e. when the metric is assumed to be a small correction superposed on the Minkowski background, are formally similar to the Maxwell equations. As a consequence of this, a macroscopic gravitation theory could be formulated in analogy with the Lorentz theory in electrodynamics. In particular, estimates of the terms Π in equations (1.12) have been given

$$\Pi_{ab} = {Q'}_{ab}{}^{cd}{}_{,cd}, \tag{2.1}$$

with Π_{ab} determined as the double divergence of an effective quadrupole gravitational polarization tensor Q'_{abcd} with suitable symmetries $Q'_{abcd} = Q'_{[ab]cd} = Q'_{ab[cd]} = Q'_{cdab}$

(Q_{abcd} incorporates also any dipole polarization effects that may occur). Such models might cover the larger-scale transitions.

This approach does not invoke any specific kind of averaging. It was applied to the propagation of gravitational waves through a medium whose molecules were supposed to be harmonic oscillators. The result obtained was that gravitational waves slow down in such kind of a medium.

Another approach is the one due to Sibgatullin [230]. He does not give a definition of averaging either. The Einstein equations are averaged after the metric was decomposed into a "background" and a "fluctuation". No criterion to separate the metric into the background and fluctuation was provided. With the assumption that the characteristic scale of correlations between matter and geometry is small, with respect to the scale of variation of the smoothed-out geometry, the result was calculated approximately that fluctuations in matter do not influence the equations of zero-th and first order in the small parameter.

Independent considerations on the subject was presented by Bialko [31]. In this approach the metric was developed into a FLRW background and a high frequency perturbation. The Einstein equations were averaged over spatial volumes, under the assumption that the characteristic wavelength of the perturbation is small as compared to the curvature radius. He further obtained that the equations governing the evolution of the averaged perturbation differ from those for linearized perturbations by a logarithmically varying factor.

Another approach is due to Noonan [200]. He showed within the weak field slow motion approximation that when the Einstein equations are averaged by volume, the energy momentum tensor splits into three parts. The interpretation of the first part is kinematical, due to averaged microscopic motions. The second contribution is mechanical, due to averaged microscopic stresses, and the third one - gravitational, due to averaged small scale variations in the gravitational field. In [201] the author showed in addition, that the time-space components of the above macroscopic energy momentum tensor, can be interpreted as the flux of gravitational energy of the microscopic field.

In the recent approach developed by Futamase [112, 113], the averaging is performed in the perturbation framework (see section 2.1.2 for details). The components of tensors are averaged over the spatial volume. In [115] the author considered inhomogeneous space-times with preferred slicings. Assuming that in the limit of zero perturbation, the preferred slicings go over into the homogeneous spaces of the FLRW models, the effect on the Friedmann equation of averaging by 3−dimensional volumes within the preferred slices was calculated. In [33] in the same approximation, the backreaction of inhomogeneities on the evolution was calculated, the result being that inhomogeneities slow down the expansion as compared to the standard Friedmann equation. Therefore the age of the Universe calculated from the Hubble law should be underestimated.

Another option mentioned in [114] could be statistical kind of averaging. Suppose that we have a statistical ensemble containing all possible density and velocity distribution of fluid elements (representing galaxies) with some constraints, which characterize the universe we wish to approximate. Choosing the particular ensemble in which the density and velocity distribution satisfy the condition $< \delta\mu >=< v^i >= 0$, and the averaging of any quantity with spatial derivative vanishes, then the averaging procedure obtained could be also appropriate to treat the situation where there are singularities, unlike within the spatial averaging concept [114].

2.1.1 Perturbation approaches

There is the "usual" perturbation approach, by which we mean here that one first introduces a fixed, i.e., unaffected by the perturbations, background space-time, e.g. the FLRW metric, and assumes that the perturbation variables in the given background are small. With this assumption, one can expand these variables to higher orders, keeping only the zero and first order terms. This allows to tackle only weakly non-linear situations. The important fact is ignored in this approach, namely that the material distribution itself determines the geometry and in the presence of inhomogeneities one cannot specify the background metric independently from the inhomogeneities - the backreaction problem (this is taken into account by the approaches to be discussed below).

The study of perturbations of the Einstein equations in the cosmological context started with the pioneering work of Lifshitz [170]. Of particular interest, are the scalar perturbations since they are directly related to density fluctuations in the early universe and are thus relevant to the structure formation. Lifshitz's theory is however not easy to interpret due to its gauge ambiguity. This ambiguity is eliminated in the theory of gauge invariant perturbations due to Bardeen [21]. Both approaches are in reality valid only in the linear regime.

In principle, there exists a general method of determining the equations for any order of perturbations, but in practice the generalization of these schemes to the non-linear situations is not straightforward.

Recently, a new gauge invariant version of perturbation theory has been given [93, 95] (henceforth we call it EBH scheme) and we review it below (see also [49]).

The standard $\delta\mu/\mu$ approach compares two evolutions (the actual one and a fictitious comparison one) along a world line. The covariant and gauge-invariant EBH scheme compares evolutions along neighbouring worldlines in the actual universe, reflecting thus the spatial density variation in the fluid.

The advantages of this scheme are, firstly that it does not necessarily assume the background geometry *a priori*, since exact equations are given governing the evolution of density inhomogeneities in arbitrary space-time without any reference to a background space-time. Secondly, it deals with exact quantities, like e.g. the comoving fractional gradient of the energy density orthogonal to the fluid flow (spatial projection of the energy density gradient). These are both directly observable and gauge invariant in the case of linear perturbations about FLRW universe. It has been shown that the linearized EBH equations are equivalent to the Bardeen gauge invariant equations [146]. The EBH equations are not however restricted only to the linear case[1].

We consider the exact covariant fluid equations for a general fluid flow in a curved space-time [89]. The 4-velocity vector tangent to the flow lines (the world-lines of fundamental observers in the universe, which are at rest w.r.t. our volume element of fluid) is $u^a = dx^a/d\tau$ $(u^a u_a = -1)$, where τ is the proper time along the fluid flow lines. The

[1] An extension of EBH scheme, combined with the spatial averaging to tackle non-linear case can be also developed (see [116] for a sketch of the scheme).

projection tensor into the tangent 3-spaces orthogonal to u^a (into the local rest frame of a comoving observer) is

$$h_{ab} \equiv g_{ab} + u_a u_b \tag{2.2}$$

and $h_b{}^a h_c{}^b = h_c{}^a, h_a{}^b u_b = 0$.

The time derivative of any tensor $T^{ab}{}_{cd}$ along the fluid flow lines is $\dot{T}^{ab}{}_{cd} \equiv T^{ab}{}_{cd;e}u^e$, the covariant derivative along u^a (the rate of change of $T^{ab}{}_{cd}$ as measured by a fundamental observer).

The first covariant derivative of the 4-velocity vector is

$$u_{a;b} \;=\; \omega_{ab} + \sigma_{ab} + \frac{1}{3}\Theta h_{ab} - \dot{u}_a u_b \tag{2.3}$$

where $\Theta \equiv u^a{}_{;a}$ is the expansion, $\omega_{ab} = \omega_{[ab]}$ is the vorticity tensor ($\omega_{ab}u^b = 0$), and $\sigma_{ab} = \sigma_{(ab)}$ is the shear tensor ($\sigma_{ab}u^b = 0,\ \sigma^a{}_a = 0$). A representative length scale S along the flow lines is defined by

$$\frac{\dot{S}}{S} = \frac{1}{3}\Theta. \tag{2.4}$$

The vorticity and shear magnitudes are defined by $\omega^2 = \frac{1}{2}\omega_{ab}\omega^{ab}$, $\sigma^2 = \frac{1}{2}\sigma_{ab}\sigma^{ab}$.

As we restrict our attention to the case of a perfect fluid, the matter stress tensor takes the form

$$T_{ab} = \mu u_a u_b + p h_{ab} \tag{2.5}$$

where, μ is the energy density, $\mu = T_{ab}u^a u^b$ and the pressure $p = \frac{1}{3}h^{ab}T_{ab}$ (in the local rest frame of a comoving observer). In general, μ and p will be related through an equation of state.

In a FLRW universe model the shear, vorticity, acceleration, and Weyl tensor vanish, and the energy density μ, the pressure p and expansion Θ are functions of the cosmic time t only. Three simple gauge invariant quantities give us the information we need to discuss the time evolution of density fluctuations.

The first is the *comoving fractional density gradient*

$$\mathcal{D}_a \equiv S h_a{}^b \frac{\mu_{,b}}{\mu} \tag{2.6}$$

which is gauge-invariant and dimensionless, and represents the spatial density variation over a fixed comoving scale [93]. Note that S, and so $\mathcal{D}_a$ is defined only up to a constant along each world-line by equation (2.6); this allows it to represent the density variation between any neighbouring world-lines. The time variation of this quantity precisely reflects the relative growth of density in neighbouring fluid comoving volumes.

The second is the *pressure gradient*

$$\mathcal{Y}_a \equiv h_a{}^b p_{,b}\,. \tag{2.7}$$

The third is the *comoving expansion gradient*

$$\mathcal{Z}_a \equiv S h_a{}^b \Theta_{,b}\,. \tag{2.8}$$

We can determine exact propagation equations along the fluid flow lines for these quantities, and then linearize these to the almost-FLRW case. The linearized equations are those given in [138] (see equations (13) to (19) there) plus the linearized propagation equations for the gauge-invariant spatial gradients defined above (see [93, 95] and [49]).

The basic equations are: the energy and momentum-conservation equations (the time and space components of the 4-dimensional equation $T^{ab}{}_{;b} = 0$)

$$\dot{\mu} + (\mu + p)\Theta = 0, \tag{2.9}$$

$$(\mu + p)\dot{u}_a + \mathcal{Y}_a = 0; \tag{2.10}$$

the Raychaudhuri equation

$$\dot{\Theta} + \frac{1}{3}\Theta^2 + 2(\sigma^2 - \omega^2) + \frac{1}{2}\kappa(\mu + 3p) - \dot{u}^a{}_{;a} = 0 \tag{2.11}$$

where, $\dot{u}^a{}_{;a}$ is the acceleration divergence; and the propagation equations for the gauge-invariant variables $\mathcal{D}_a$, $\mathcal{Z}_a$:

$$h_c{}^a(\mathcal{D}_a)^{\cdot} = -\mathcal{D}_a(\omega^a{}_c + \sigma^a{}_c) + \frac{p}{\mu}\Theta\mathcal{D}_c - (1 + \frac{p}{\mu})\mathcal{Z}_c, \tag{2.12}$$

$$\begin{aligned} h_c{}^a(\mathcal{Z}_a)^{\cdot} \;=\; & -\Theta\mathcal{Z}_c + h_c{}^a(\frac{1}{3}\Theta\mathcal{Z}_a - \frac{1}{2}\mu\kappa\mathcal{D}_a - 2S(\sigma^2)_{,a} + 2S(\omega^2)_{,a} + \\ & S\dot{u}^b{}_{;ba}) - \mathcal{Z}_b(\sigma^b{}_c + \omega^b{}_c) + \dot{u}_c S\mathcal{R}, \end{aligned} \tag{2.13}$$

where, $\mathcal{R} \equiv \frac{1}{3}\Theta^2 - 2\sigma^2 + 2\omega^2 + \kappa\mu + \dot{u}^a{}_{;a}$.

In the above, $\dot{u}^b{}_{;ba}$ stands for the gradient of the acceleration divergence, and $\kappa = 8\pi G$.

Once the equation of state of the fluid is known, the evolution of $\mathcal{Y}_a$ will follow from that for $\mathcal{D}_a$.

For completeness, we give also the propagation equation for the acceleration $a_c \equiv \dot{u}_c$

$$h_a{}^c(a_c)^{\cdot} = a_a\Theta(\frac{dp}{d\mu} - \frac{1}{3}) + h_a{}^b\left(\frac{dp}{d\mu}\Theta\right)_{,b} - a_c(\omega^c{}_a + \sigma^c{}_a) \tag{2.14}$$

($\frac{dp}{d\mu}$ is taken along the fluid flow lines).

Further equations can be used as the basis of various systematic approximation schemes. The major point to notice is that in using equations (2.12) and (2.13) in an approximation scheme to determine propagation of density inhomogeneities to the nth. order, we only need solve the other equations of the model to the $(n-1)$th. order [116]. This gives the behaviour of the coefficients in these equations (and the Christoffel terms implied in the covariant derivatives on the left) to that order; then they directly determine the behaviour of inhomogeneities at the nth. order.

As equations (2.12) and (2.13) are gauge-invariant as well as covariant, we can use any coordinates and any convenient choice of background FLRW model in their further investigation. However equations (2.9) and (2.11) are not gauge-invariant; we can deal with this by using an averaging procedure to determine a background model (see footnote[1] in this chapter).

2.1.2 Approaches of Futamase and Kasai

Futamase's approach

In a series of papers [112, 113, 114] Futamase developed an approximation scheme for describing an inhomogeneous universe, valid in non-linear case, basically with an arbitrary density contrast. This is a perturbative approach and the averaging introduced, gives a clean separation between the global and local quantities.

The aim is to construct the approximate, reliable metric representing the real, clumpy universe in General Relativity. Obviously, the averaged, smooth metric coincides nowhere with the real inhomogeneous metric, but we know that the FLRW description is valid only in some averaged sense (if so). It seems then natural, to suppose that the space-time is close to a FLRW space-time, i.e., the inhomogeneous space-time is in a sense a small deviation away from the averaged smooth space-time, which is not *a priori* given.

The crucial observation is the fact that the size of the metric perturbation and that of the density contrast are independent of each other in the exact theory, as well as in post-Newtonian approximations. For example, in the Solar System the metric coefficients in nearly orthonormal coordinates deviate from their special relativistic values, by no more than $\sim 2GM_{\odot}/c^2R_{\odot} \sim 10^{-6}$, whereas the density contrast between the interior of the Sun, planets and interplanetary space is $> 10^{20}$.

The *ansatz* for the metric is taken as:

$$g_{\mu\nu} = a^2(\eta)(\tilde{g}^{(b)}_{\mu\nu} + h_{\mu\nu}), \tag{2.15}$$

where, $h_{\mu\nu}$ are generated by local matter inhomogeneities and the gravitational waves (we neglect the latter), assumed to be small; this does not imply the smallness of density contrast. The scale factor $a(\eta)$ describes as usual, the global FLRW expansion (averaged). In other words, $g_{\mu\nu}$ is the standard FLRW metric when $h_{\mu\nu} = 0$, with $\tilde{g}^{(b)}_{\mu\nu} = -d\eta^2 + d\Omega_3^2(k)$, where $d\Omega_3^2(k)$ is the standard metric on S^3 if $k = 1$ and on $\mathbb{R}^3$ if $k = 0, -1$.

The *ansatz* for the metric is such that the deviations from the FLRW models are small, but this does not imply that the zero-th order space-time is the FLRW one. It depends on the approximation chosen, e.g., within the linear approximation the zero-th order space-time is indeed taken to be the FLRW space-time. The approximation chosen depends on the kind of physical situation that one deals with, for the case at hand, what we have in mind is the matter clumps of various scales interacting gravitationally with each other and the density contrast between them and the mean density is $\gg 1$.

Above all, we have to restrict our space-times to those in which there is a well-defined meaning of the spatial average. The spatial averaging is therefore defined in a family of geometrically preferred slices, i.e., such that the metric deviation away from the FLRW metric is small everywhere on them.

The scheme is worked out in harmonic gauge:

$$\overline{h}^{\mu\nu}_{|\nu} = 0, \tag{2.16}$$

where, "|" stands for covariant derivation with respect to $\tilde{g}^{(b)}$, and $\overline{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\tilde{g}^{(b)}_{\mu\nu}h$, with $h = \tilde{g}^{(b)\mu\nu}h_{\mu\nu}$ is the so-called trace reversed perturbation.

The equations are derived under the assumption that h is small and that the scale on which h varies is small compared to that of a and $\tilde{g}^{(b)}$.

The above *ansatz* is used to calculate the Einstein equations as follows (a prime stands for the derivative w.r.t. the conformal time η)

$$\begin{aligned}(\tfrac{a'}{a})^2(4\tilde{g}^{(b)\mu\eta}\tilde{g}^{(b)\nu\eta} - \tilde{g}^{(b)\eta\eta}\tilde{g}^{(b)\mu\eta}) - 2\tfrac{a''}{a}(\tilde{g}^{(b)\mu\eta}\tilde{g}^{(b)\nu\eta} - \tilde{g}^{(b)\eta\eta}\tilde{g}^{(b)\mu\nu}) + A^{\mu\nu} + \\ \tfrac{a'}{a}(2\overline{h}^{\eta(\mu|\nu)} - \overline{h}^{\mu\nu|\eta} - \tilde{g}^{(b)\eta(\mu}\overline{h}^{|\nu)} + \tfrac{1}{2}\tilde{g}^{(b)\mu\nu}\overline{h}^{|\eta}) - \tfrac{1}{2}\overline{h}^{\mu\nu|\rho}{}_{|\rho} = 8\pi G\tau^{\mu\nu},\end{aligned} \tag{2.17}$$

where, $A^{\mu\nu}$ is the background spatial curvature term, given by $A^{\eta\eta} = -3k\tilde{g}^{(b)\eta\eta} = 3k$, $A^{ij} = -k\tilde{g}^{(b)ij}$, and $A^{\eta i} = 0$. $\tau^{\mu\nu} = a^4T^{\mu\nu} + t^{\mu\nu}$ may be regarded as the effective (total) stress-energy tensor, i.e., material stress-energy tensor $T^{\mu\nu}$ plus gravitational stress-energy pseudotensor $t^{\mu\nu}$, consisting of terms quadratic in $\overline{h}$.

Effectively, this means that the Einstein equations are expanded in terms of two small parameters, ϵ and κ, whose meaning is the following:

ϵ is the size (amplitude) of the metric perturbation h and it is assumed that h, $h_{\mu\nu} \simeq \mathcal{O}(\epsilon^2)$, $h_{\mu\nu,\rho} \simeq \mathcal{O}(\epsilon^2/l)$; The ϵ defined this way is also an amplitude of the peculiar gravitational potential Φ, the gravitational potential generated by the inhomogeneous distribution of matter, where $\Phi \simeq \mathcal{O}(\epsilon^2)$, as well.

$\kappa = \frac{l}{L}$ is the ratio between the scale of the variation of h, and that of a and $\tilde{g}^{(b)}$, where for L we can take the present horizon size $\sim 10^4\,Mpc$, and for l the typical scale of inhomogeneities.

Note that $\kappa \in [0, 1]$, with small κ indicating condensed density contrast, large κ - diffused density contrast.

The relative size of κ and ϵ depends on the physical system considered. It is straightforward to see that the density contrast is of the order of ϵ^2/κ^2, and consequently the linear stage is characterised by $\kappa \gg \epsilon$, and in the non-linear stage we have $\epsilon \gg \kappa$. For galaxies we typically have $\epsilon > \kappa$, and the ratio ϵ/κ increases, when we consider smaller regions. Basically, if the typical size of inhomogeneities is larger than a galactic scale, the approximation is valid in the parameter range $\epsilon^2 \ll \kappa^2$.

In deriving (2.17), terms like $\frac{a''}{a}\overline{h} \simeq \mathcal{O}(\epsilon^2/L^2)$, $\frac{a'}{a}\overline{h}_{|\rho}\overline{h} \simeq \mathcal{O}(\epsilon^4/lL)$, $\overline{h}_{|\rho}\overline{h}_{|\sigma}\overline{h} \simeq \mathcal{O}(\epsilon^6/l^2)$ and of higher orders, were neglected, whereby it was taken that $\frac{a'}{a} \simeq \mathcal{O}(1/L)$ and $\frac{a''}{a} \simeq \mathcal{O}(1/L^2)$. Physically, it means that the effect of the self-gravity of clumps on their dynamics is more important than the expansion of the universe. The neglected terms are negligible as far as $\epsilon, \kappa \ll 1$ and $\kappa \gg \epsilon^2$.

A perfect fluid is taken for the material source, which is characterized by the density field ρ, its peculiar velocity, $\vec{v}$, and peculiar gravitational potential Φ;

$$T^{\mu\nu} = [\rho + p(\rho)]u^\mu u^\nu + p(\rho)g^{\mu\nu}. \tag{2.18}$$

One works with conformally rescaled variables: $\tilde{u}^\mu = au^\mu$, $\tilde{g}^{\mu\nu} = a^2 g^{\mu\nu}$. Then $\tau^{\mu\nu} = a^2\tilde{T}^{\mu\nu} + t^{\mu\nu}$, where, $\tilde{T}^{\mu\nu} = [\rho + p]\tilde{u}^\mu\tilde{u}^\nu + p\tilde{g}^{\mu\nu} = a^2 T^{\mu\nu}$.

On very large scales the universe is assumed to be homogeneous in space. In the next step the spatial averaging is applied to the truncated Einstein field equations (2.17), assuming spatial periodicity of the material initial data[3] (as well as of the free data for the gravitational field) and no coherent motion over the volume to be averaged[4].

The spatial average over a volume V is defined as:

$$< Q >= \frac{1}{V}\int_V Q\sqrt{\tilde{g}^{(b)}}d^3x, \tag{2.19}$$

where, $\tilde{g}^{(b)}$ is the determinant of the spatial part of the background metric $\tilde{g}^{(b)}_{\mu\nu}$ and $\sqrt{\tilde{g}^{(b)}}d^3x$ is the invariant volume element in the background space, and for the density we have $< \rho >= \rho_b$ (background density). The only property used in the calculation is $< Q_{|i} >= 0$, which is implied by the spatial periodicity. Also, $< \tau^{\eta i} >= 0$, which just means no coherent motion over the averaging volume.

[2] In [114] an approximate metric is constructed in the situation with strong gravity and/or smaller regions of inhomogeneity, where $\epsilon^2 \gg \kappa$, but we will not discuss this case here.

[3] Space-time averaging is another possibility; in [147] temporal periodicity for the inhomogeneities had to be assumed, in order to get a separation between global and local evolution.

[4] This is almost always safe, i.e. with large enough averaging volumes and randomly distributed perturbations.

The spatial average of (2.17) under the above requirements, implies $<\overline{h}^{\eta i}>=0$, and of (2.16) $<\overline{h}_{\eta\eta}>= const.$, which under a redefinition of the time variable and scale factor can be put to zero. Also, $<\overline{h}^{k}{}_{k}>$ can be put to zero, since it expresses an additional isotropic expansion, which can be absorbed into the scale factor upon its redefinition.

The averaged sources are used to calculate the global expansion and the following averaged Einstein equations are obtained from (2.17), to the first non-trivial order (by non-trivial order, we mean the order at which the first backreaction effect due to inhomogeneities on the expansion of the universe appears)

$$\begin{aligned}(\tfrac{a'}{a})^2 &= \tfrac{8\pi G}{3} < \tau^{\eta\eta} > -k \\ \tfrac{a''}{a} &= \tfrac{4\pi G}{3} < \tau^{\eta\eta} - \tau^{k}{}_{k} > -k \\ \tfrac{1}{a^2}(a^2 < \overline{h}^{ij} >_{|\eta})_{|\eta} &= 16\pi G < \hat{\tau}^{ij} >,\end{aligned} \tag{2.20}$$

where, $\hat{\tau}^{ij} = \tau^{ij} - \frac{1}{3}\bar{g}^{(b)ij}\tau^{k}{}_{k}$ is the trace free part of τ^{ij}.

The averaged line element

$$< ds^2 >= a^2(\eta)(-d\eta^2 + (\delta_{ij}+ < \overline{h}_{ij} >)dx^i dx^j), \tag{2.21}$$

tells us that the averaged space-time expands anisotropically, except when $<\overline{h}_{ij}>$ vanishes identically, since $<\overline{h}_{ij}>$ expresses the deviation from the isotropic expansion, due to the inhomogeneities $<\hat{\tau}^{ij}>$. Note that the first two equations of (3.19) are the Friedmann equations with source terms replaced by the effective stress-energy tensor, therefore the effect of the local inhomogeneities on the global expansion can be partly expressed by the effective density and pressure $\rho_{eff} = a^2 < \tau^{\eta\eta} >$, $p_{eff} = \frac{1}{3}a^2 < \tau^{k}{}_{k} >$.

One can integrate the last equation of the system (3.19) to obtain the expression for $<\overline{h}^{ij}>(\eta)$, and see that a sufficient condition for global isotropic expansion is given by $<\overline{h}^{ij}>_{,\eta}(\eta_o) \equiv 0$ and $<\hat{\tau}^{ij}>\equiv 0$. The equations determining the evolution of the local inhomogeneities are derived by substituting the above equations into (2.17), additionally we have also the equations of motion (derived from the conservation of the stress-energy tensor).

Now, we can employ a particular approximation scheme. The evolution of the density perturbations in our picture is sufficiently well described by means of a post-Newtonian approximation. The post-Newtonian approximation is characterized by small parameter ϵ, of the order of a typical peculiar velocity divided by the speed of light. ϵ is introduced by a coordinate transformation $\eta_N = \epsilon\eta$, η_N is the Newtonian time, which means physically that a typical time scale gets longer as ϵ^{-1} as the velocity goes to zero. This parameter is identified with the already introduced ϵ.

The orders for material variables are assumed to be $\rho_N = \epsilon^{(-2)}\rho$, $v^i_N = \epsilon^{(-1)}v^i$, $p_N = \epsilon^{(-4)}p$.

The other small parameter, κ is associated with global cosmic expansion (when $\kappa \to 0$, the expansion of the universe slows down). ϵ and κ are our order parameters, in the sense, that they parameterize a sequence of space-times and we study the Newtonian limit on that sequence.

The evolution equations for the local inhomogeneities are solved perturbatively up to the first non-trivial order.

An interesting outcome of the application of this approximation scheme is that the backreactions lead to an underestimation of the age of the universe as inferred from a measurement of today's Hubble constant [33] (also see [32]). For a simple model within the framework of pancake theory for structure formation on a flat expanding background, it is shown in [33] that the age problem (severe in view of the recent determinations of globular cluster ages) may be solved by taking into account the backreaction of inhomogeneities in an averaged sense.

This scheme can also be used for the correct interpretation of observations of gravitational lenses.

Kasai's approach

Kasai's scheme [156] to construct inhomogeneous relativistic universes which are homogeneous and isotropic on average, goes further than Futamase's approach. It is not assumed here that the deviations from a FLRW model are small to acquire FLRW-like behaviour on average.

Here also, spatial averaging is introduced, but the description is based on the deformation tensor and can give yet another possibility to obtain solutions to describe more realistic situations, and on the other hand to formulate a relativistic version of the Zel'dovich approximation, used to handle the evolution of the large scale structure in Newtonian cosmology.

In [156] the inhomogeneous irrotational dust universe models are constructed in the framework of General Relativity, with the property of being homogeneous an isotropic on average.

The averaging is introduced for matter only, on the hypersurfaces Σ_t orthogonal to dust motion, and the mean ("background") density for the inhomogeneous universe model is written as

$$\rho_b = <\rho> \equiv \lim_{V\to\Sigma_t} \frac{1}{\int_V [\det(g_{ij})]^{1/2} d^3x} \int_V \rho [\det(g_{ij})]^{1/2} d^3x, \tag{2.22}$$

for $V \subset \Sigma_t$ (assuming that this limit exists).

The "scale factor" (averaged) is then defined by

$$\dot{\rho}_b + 3(\frac{\dot{a}}{a})\rho_b = 0. \tag{2.23}$$

The peculiar deformation tensor is introduced as

$$V^i_j \equiv u^i_j - \frac{\dot{a}}{a}\delta^i_j. \tag{2.24}$$

This quantity describes the deviation from a uniform Hubble expansion. As usual, u^μ is a 4-velocity and comoving coordinates are used.

The deformation tensor $u^i_j \equiv u^i_{;j}$, describing the change of the relative position X^i between the world lines of neighbouring "particles" (galaxies), $\dot{X}^i = u^i_j X^j$ is at the same time, the extrinsic curvature of the $t = const.$ hypersurfaces Σ_t.

Another quantity introduced is a density contrast[5]

$$\Delta \equiv \frac{\rho - \rho_b}{\rho}. \tag{2.25}$$

[5] Note that it differs form the conventionally adopted $\delta \equiv \frac{\rho-\rho_b}{\rho_b}$.

Given this, the Einstein equations are the following:

$$\dot{\Delta} + (1-\Delta)V^i_i = 0 \tag{2.26}$$

$$(\frac{\dot{a}}{a})^2 = \frac{8\pi G}{3}\frac{\rho_b}{1-\Delta} - \frac{1}{6}{}^{(3)}R - \frac{2}{3}\frac{\dot{a}}{a}V^i_i - \frac{1}{6}[(V^i_i)^2 - V^i_j V^j_i] \tag{2.27}$$

$$\ddot{\Delta} + 2(\frac{\dot{a}}{a})\dot{\Delta} - 4\pi G\rho_b\Delta = -(1-\Delta)[(V^i_i)^2 - V^i_j V^j_i] \tag{2.28}$$

where, $\dot{} \equiv \frac{\partial}{\partial t}$, ${}^{(3)}R = {}^{(3)}R^i_i$ and $i, j = 1, 2, 3$.

FLRW model (with ρ_b and a) is the background model. Note that (2.27) reduces to the Friedmann equation $(\frac{\dot{a}}{a})^2 = \frac{8\pi G}{3}\rho_b - \frac{k}{a^2}$, when there are no inhomogeneities , if and only if the condition

$$\frac{8\pi G}{3}\rho_b\frac{\Delta}{1-\Delta} - \frac{1}{6}{}^{(3)}R + \frac{k}{a^2} - \frac{2}{3}\frac{\dot{a}}{a}V^i_i - \frac{1}{6}[(V^i_i)^2 - V^i_j V^j_i] = 0 \tag{2.29}$$

holds.

On the other hand, when the left hand side of (2.28) is zero, one gets the evolution equation for $\delta \equiv \frac{\rho-\rho_b}{\rho_b}$ in the linear perturbation theory.

The nice property of this approach is the fact that equations similar to (2.26) and (2.28) appear in Newtonian cosmology in the context of extending the Zel'dovich-type approximations, with V^i_j spatial gradient of peculiar velocity $v^i_{,j}$.

In particular, when $(V^i_i)^2 - V^i_j V^j_i = 2(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1) = 0$, where λ_i are eigenvalues of V^i_j, Δ obeys the same equation as δ and the solutions can be extrapolated from the result of the linear perturbation theory.

Therefore the present construction can represent "relativistic pancake solutions" analogous to those in Newtonian cosmology.

2.2 Exact non-covariant averaging

Despite the fact that quite often the averaging procedure is applied on top of a perturbative scheme, we will review in this chapter approaches in which the averaging (by volume in all cases) is defined (non-covariantly) without any recourse to approximations. There is no covariant definition of spatial averaging available in a general space-time.

The earliest attempt is due to Shirokov and Fisher [228], where they proposed to define the components of the averaged (macroscopic) metric as the volume averages of the corresponding components of the small scale (microscopic) metric. This procedure is explicitly non-covariant (volume integrals of tensor components do not constitute a tensor!) and has no geometrical interpretation, it is therefore not clear what is actually represented by a volume average of a metric component. Nevertheless, this definition was applied to metrics which are small perturbations of the FLRW models and from them the Einstein tensor was calculated, as well as the averages of all its components. They were then equated to the averages of the appropriate components of the energy momentum tensor. Terms non-linear in the small quantities were neglected. The result they obtained was a generalization of the FLRW solutions, with a repulsive term preventing the singularity for all three curvatures. At a maximal crunch, each particle fills the interior of a sphere of a radius equal to that of the particle's own gravitational radius.

Further in [229] (which is really only a short conference report on work in progress), the author considers the following deformation of the FLRW metrics:

$$ds^2 = dt^2 - (1 + \frac{1}{4}kr^2)^{-2}G^2(y)f(t,x,y,z)(dx^2 + dy^2 + dz^2), \tag{2.30}$$

where, $G(y)$ and $f(t,x,y,z)$ are unknown functions. The consequences of averaging of the Einstein tensor (calculated from an "averaged" metric) with the assumption that it obeys the cosmological principle, are calculated, exactly without approximations. The term due to averaging was interpreted as a negative contribution to pressure capable of preventing the Big Bang.

In a series of papers, Saar [217] considered the influence of averaged rapid fluctuations on a slowly changing background metric. The metric (without considering any explicit form of it) was split into the background and a fluctuation times a small parameter. Einstein equations were then developed to the second order in the small parameter and averaged by volume. As a result, the contribution of averaged fluctuations can be interpreted, as a negative pressure in the background and cosmological expansion can proceed slower, with more time available for structure formation.

The result of Nelson [198] on the other hand, seems to contradict the others. The metric was split into smooth background and small perturbations describing lumps, and then averaged by integrating metric components over volume. The background was obtained to be approximately equal to the average of the whole metric, with the average obeying *"the usual set of cosmological equations"*. The corrections to the average obey equations *"equivalent to instability equations. The large scale development of such a Universe is therefore shown to be almost independent of the formation of condensations provided the average of the energy-stress tensor is unaffected by the condensations"*. The relevant assumptions were that the averaging volume contained many condensations, the number of condensations was big, they were evenly distributed in space and their radii large with respect to the Schwarzschild radius.

Interesting approach was put forward in [184]. A series of following papers by Marochnik and co-workers, initiated in fact properly, a direction of research taking into account the small scale inhomogeneities.

The metric, the density and the velocity field were taken to be sums of an average (background) and a correction (called turbulence) whose average is zero. The following equalities were assumed to hold

$$\lambda_T^2 < L^2 \ll 1/\bar{R}, \tag{2.31}$$

where, λ_T is the characteristic scale of the turbulence, L the averaging scale, and $\bar{R}$ the background curvature. The effect of averaging in terms of volume integrals on the Einstein equations was examined. They claimed that corrections to energy and pressure (due to averaging out small scale inhomogeneities) do not need to be positive and may be interpreted formally as antigravitation. This can further lead, for example to a situation, where perturbations of a FLRW background can yield a non-expanding, static universe; or the corrections due to averaging might prevent the Big Bang singularity.

Further on, in [185] the results of averaging on linearized perturbations of the FLRW models were calculated and it turns out that:

1. [6] For $p = \rho/3$ (valid in the FLRW background)

[6] It was noted in [185] that this result is in contradiction to [31], where the main reason for deviation from the Friedmann equation was attributed to the gravitational waves.

- if the potential turbulence exceeds the vortical one [171], then the perturbations decelerate the expansion and the initial singularity can be avoided. (The authors notice however that before the minimal radius is approached the linear approximation is invalidated.)
- if the vortical turbulence dominates, the expansion is accelerated (the singularity remains).

2. For others equations of state, the turbulence could modify the evolution of the universe when $p \geq \rho/3$. In this regime ($p > \rho/3$) however, the linear approximation does not hold any more.

In [186] the influence of long-wave turbulence on the background expansion was examined, taking into account only those modes of turbulence that remain finite when close to singularity. It turned out that they influence the expansion in the same way as in the perturbative solution of Lifshitz and Khalatnikov [172].

In [187] the equations of [185] were written up to the second approximation in the turbulent perturbation and solved in [188], where also the effects of the perturbations on the background were studied. No pronounced qualitative effects were found though.

In [189] the changes in the most important cosmological parameters due to averaged out small scale inhomogeneities were calculated and the author found that this affects the transition moment between the hadron and lepton eras (by a factor of up to 1.4); the temperature in the transition moment (by a factor of 0.88); and the helium abundance[7].

One of the most recent papers considering the effect of averaging over spatial volumes on the Einstein equations is the one due to Zotov and Stoeger [273]. They simply compare an exact FLRW model with one where galaxies, represented by a Schwarzschild metric, are superposed on the FLRW background with a constant number density. They find that upon averaging the metric components, the background FLRW model with the scale factor $R(t)$, changes to another FLRW model with the scale factor $S(t)$, and

$$S^2(t) = R^2(t)(1 - K), \tag{2.32}$$

[7]It may remain unchanged if the energy density of short wave fluctuations is smaller than 1.5 $\bar{\rho}$ (where, $\bar{\rho}$ is the large scale average energy density).

where, $K = NV_1/V_2$, N is the number density of galaxies, V_1 the averaging volume, V_2 the volume per one galaxy in the space. The effect of matter, as calculated through the average values of metric components is therefore to squeeze the space volume. If calculated through substituting the new average density in the Einstein equations, the effect is the same, but now the dependence of $R(t)$ on the density parameter σ_o is given in the form of parametric equations

$$\begin{aligned} R_h(t) &= H_o\sigma_o(1-2\sigma_o)^{-3/2}(\cosh 2\psi - 1), \\ ct &= H_o\sigma_o(1-2\sigma_o)^{-3/2}(\sinh 2\psi - 2\psi). \end{aligned}$$

The authors do not calculate nor discuss in their paper the terms due to averaging in the Einstein equations.

2.3 Exact covariant averaging

Isaacson was the first to consider the problem of covariant averaging involved in "coarse-grain" viewpoint. In [147] he considered the vacuum Einstein equations in the short-wave approximation, assuming that the metric of a gravitational wave space-time can be split into a low frequency background and high frequency wave, namely, $g_{\mu\nu} = g^{(B)}_{mu\nu} + h_{\mu\nu}$. High frequency means small wavelength with respect to the curvature radius of the background and further, the smallness of the waves' amplitude was assumed. The Einstein equations were then linearized with respect to the high frequency correction and the waves were shown to obey a covariant generalization of the equation of massless spin 2 fields in flat background. They were shown to travel on null geodesics of the background, their amplitude, frequency and polarization modified by the background curvature.

Further in [148] the same considerations were carried out to a higher order of approximation. The author showed that the corrections of the first non-linear order to the vacuum Einstein equations provide a term ("stress-energy" tensor) that can be interpreted as the effective energy of the gravitational waves. It was then used to define the total energy and momentum carried off to infinity by the waves. The stress-energy tensor of gravitational waves was shown to be well defined only in a smeared out sense (cf. also [194]).

The energy momentum tensor of the gravitational waves arises once the metric is averaged. The average is defined by parallel-transporting the tensors from the point x' to the representative point x, along the geodesic between x' and x, and then integrating the resulting object with respect to x' with a weighting function $f(x,x')$ (this function was not defined explicitly):

$$< T_{\mu\nu}(x) >= \int_{all\ space} G_{\mu}^{\alpha'}(x,x') G_{\nu}^{\beta'}(x,x') T_{\alpha'\beta'}(x') f(x,x') d^4x', \tag{2.33}$$

where, $G_{\beta}^{\alpha'}$ are the propagators of parallel displacement. From the weighting function it was demanded that:

1. $f(x,x') \to 0$, when the distance $d(x,x')$ obeys $\lambda \ll d \ll L$, where λ is the wavelength of (high frequency) waves and L is the wavelength of the low frequency background.

2. $\int_{all\ space} f(x,x') d^4x' = 1$.

This definition was applied within a perturbative scheme, but it is perfectly covariant.

The evaluation of the effective stress-energy tensor for the gravitational waves requires averaging of various quantities over several wavelengths. From (2.33) one can derive Isaacson's averaging rules (he names the averaging scheme used as "Brill-Hartle averaging") which are the following:

- covariant derivatives commute; e.g. $< h\, h_{\mu\nu|\alpha\beta} >=< h\, h_{\mu\nu|\beta\alpha} >$, where $|$ stands for the covariant derivative with respect to $g_{\mu\nu}^{(B)}$.

- gradients average out to zero; e.g. $< (h_{|\alpha} h_{\mu\nu})_{|\beta} >= 0$.

- one can integrate by parts, flipping derivatives from one h to the other; $< h\, h_{\mu\nu|\alpha\beta} >=< -h_{|\beta} h_{\mu\nu|\alpha} >$.

In harmonic gauge $\bar{h}_{\mu}{}^{\alpha}{}_{|\alpha} = \bar{h} = 0$ the stress-energy for the gravitational waves was shown to have the following form

$$T_{\mu\nu}^{GW} = \frac{1}{32\pi} < \bar{h}_{\alpha\beta|\mu} \bar{h}^{\alpha\beta}{}_{|\nu} >, \tag{2.34}$$

where, $\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} h g_{\mu\nu}^{(B)}$.

Further improvement was due to Matzner. In [190] he does not need the assumption that the wave is of high frequency in order to identify the background. Moreover, he gives another definition of averaging the geometry, namely, a metric is Lie-dragged along a specific vector field, defined by

$$\xi^{(\alpha;\beta)}{}_{;\beta} + \lambda\xi^{\alpha} = 0, \tag{2.35}$$

to a chosen point, averaged there over all points, and then Lie-dragged back. Equation (2.35) is in fact a generalization of the Killing equation. The above definition was applied to the $t = const$ sections of the Taub-NUT space (space of finite volume). It turned out, that the averaged metric has the FLRW algebraic form, however it is not in any simple way related to the original metric (e.g. the volume of the averaged space is different from the initial volume).

In [191] a measure of symmetry in a Riemannian manifold with a positive-definite metric was put forward. Namely, it was defined as the minimum value of the functional:

$$\lambda[\xi] = (\int_{all\ space} \xi^{(\alpha;\beta)}\xi_{(\alpha;\beta)}dv)/(\xi^{\mu}\xi_{\mu}dv), \tag{2.36}$$

where, ξ^{α} is a vector field, $dv = \sqrt{g}d^n x$. For compact space or when the integral over the boundary at infinity is zero, the minima of $\lambda[\xi]$ obey (2.35); on a positive-definite manifold, $\lambda = 0$ in (2.35) implies Killing equation. The above definition was applied to some space-times and the parameters λ defined by (2.35) could be interpreted as averaged energy-density and averaged stresses of gravitational waves.

In [212] Rosen pointed out that for a stochastic stress-energy tensor associated with cosmic turbulence the Einstein equations imply fluctuations in the space-time metric tensor (of a purely classical-statistical character). He then showed that averaging the metric produces corrections to the energy momentum tensor and calculated them explicitly, assuming that: (1) the perturbed metric is conformally equivalent to the background metric, and (2) the averaged energy-momentum tensor has the algebraic form of a perfect fluid. The conclusion reached, was that fluctuations in the metric always accelerate the expansion in a FLRW background (i.e. increase $\ddot{R}/R$). However, no definition of averaging was proposed and the author called his averages *"with respect to the statistical ensemble"*; the whole treatment was rather axiomatic. This paper nevertheless appears to be one of the first tackling the problem from a geometric viewpoint.

Further, there is the approach of Carfora and Marzuoli [55] (which will be reviewed in section 2.4.1). We will only mention here that they were the first to confirm exactly and in a covariant manner the result predicted by Shirokov and Fisher [228].

In [159] Khiet influenced by Shirokov and Fisher paper, studied the results of averaging microscopic gravitational equations and without any definition of an averaging procedure, simply guessed the correction term $C_{\alpha\beta}(g)$ in the macroscopic equations $G_{\alpha\beta} + C_{\alpha\beta} = \kappa T_{\alpha\beta}$ on the basis of covariance requirements. He obtained new, non-linear in curvature, field equations and studied their exact solutions for a general FLRW metric. These were however field equations of a new metric gravitation theory, instead of averaged Einstein equations.

In [8] a space-time averaging procedure was introduced which appears to be a special case of that in [10]. It is a bilocal approach where the averaging bivectors (cf. section 2.5.1) were taken to be the tetrad components. No averaging of the metric could be achieved this way since always $< g_{\alpha\beta}(x) >= g_{\alpha\beta}(x)$. The averaged curvature tensor was therefore taken as a characteristic of the averaged space-time. A space-time, with the curvature tensor $R^{\alpha}{}_{\beta\gamma\delta}$ satisfying $R_{\mu\nu} = 0$ was shown to be a (pseudo-)Riemannian one, if and only if $R^{\alpha}{}_{\beta\gamma\delta} = 0$ or $R^{\alpha}{}_{\beta\gamma\delta} = r^{\alpha}{}_{\beta\gamma\delta}$, where $< r^{\alpha}{}_{\beta\gamma\delta} >= R^{\alpha}{}_{\beta\gamma\delta}$. The average curvature tensor taken as the curvature tensor is thus non-Riemannian. This averaging scheme is not viable from a physical point of view.

There is yet another approach due to Arifov and Shayn [7], which is similar in spirit to Khiet's approach. The averaging scheme is stated to be the same as in [9] (see section 2.5), but this scheme was not however explicitly used at all in this approach, in which it is in fact enough to introduce averages formally.

The average curvature $\bar{r}_{\alpha\beta\gamma\delta}$ and metric tensor $\bar{g}_{\alpha\beta}$ are no longer the curvature and metric tensors of the same Riemannian manifold. It was proved that when dim $n \leq 4$, there exists the Riemannian manifold with the metric tensor $G_{\alpha\beta}$ of the same signature as $g_{\alpha\beta}$ and curvature tensor $R_{\alpha\beta\gamma\delta} \equiv \bar{r}_{\alpha\beta\gamma\delta}$, which is not Riemannian with $\bar{g}_{\alpha\beta}$. Effectively, this is a bi-metric gravitation theory. The macroscopic Einstein equations were obtained upon averaging out of the left hand side of Einstein's equations and extracting the Einstein tensor (made up of the averaged curvature tensor), namely

$$R^{\alpha}_{\beta} - \frac{1}{2}\delta^{\alpha}_{\beta}R = -T^{\alpha}_{\beta} + D^{\alpha}_{\beta}, \tag{2.37}$$

where D^{α}_{β} is defined as

$$D^{\alpha}_{\beta} \equiv D'^{\alpha}_{\beta} + \delta^{\alpha}_{\beta}(H^{\mu\nu}R_{\mu\nu} + \frac{1}{2}H^{\rho\sigma}H^{\mu\nu}R_{\rho\mu\nu\sigma}) - H^{\mu\nu}R^{\alpha}_{\mu\nu\beta} - H^{\alpha\rho}R_{\rho\beta} - H^{\alpha\rho}H^{\mu\nu}R_{\rho\mu\nu\beta}, \tag{2.38}$$

with further definitions following simply

$$D'^{\alpha}_{\beta} \equiv \frac{1}{2}\delta^{\alpha}_{\beta}(\bar{r} - \bar{g}^{\rho\sigma}\bar{g}^{\mu\nu}\bar{r}_{\rho\mu\nu\sigma}) - (\bar{r}^{\alpha}_{\beta} - \bar{g}^{\alpha\rho}\bar{g}^{\mu\nu}\bar{r}_{\rho\mu\nu\beta}) \tag{2.39}$$

for the correlation tensor, and $H^{\alpha\beta} \equiv \bar{g}^{\alpha\beta} - G^{\alpha\beta}$, $\bar{g}^{\alpha\beta}\bar{g}_{\gamma\beta} = \delta^{\alpha}_{\gamma}$, $G^{\alpha\beta}G_{\gamma\beta} = \delta^{\alpha}_{\gamma}$.

It can be argued whether any real averaging has been achieved this way, in any way no geometrical considerations were offered. Consequently, the correlation tensor could only be modeled phenomenologically (unlike in [263, 264]). For example, in the case of gravitational field of the spherically symmetric source $D^{\alpha}_{\beta} = \sigma C^{\alpha\mu\nu\sigma}C_{\beta\mu\nu\sigma}$, where $C_{\alpha\mu\nu\sigma}$ is the conformal curvature tensor and $\sigma = const.$ The solution found in this case was argued to yield a macroscopic metric that has neither an event horizon nor a singularity for certain values of the parameter σ (contrary to the usual Schwarzschild solution).

Recently, a promising approach has been proposed by Zalaletdinov, who introduced an axiomatic description of covariant averaging of tensors [263, 264] (see also [266] and [265]). So doing he carried out the averaging according to Lorentz' ideas of the (pseudo-)Riemannian geometry. By applying this scheme to GR he obtained a new "averaged" theory called by him Macroscopic Gravity (MG henceforth; reviewed in chapter (2.5)). The cosmological consequences of the theory are not clear at the moment.

The problem of averaging is very much related to the issue of approximating a fine-scale cosmological model with a large-scale model. In [232] Spero and Baierlein proposed an independent approach, namely, to define an approximate symmetry of an inhomogeneous model by "best-fitting" to it a Bianchi-type model. The "best-fit" was defined in terms of a minimum of a functional, with respect to variations of a triad of orthonormal vectors in the given space-time and variations of the structure constants, to be found. As shown, the resulting Bianchi type is not always unique, unless it is one of the generic types: IX, $VIII$, VII_h or VI_h. The classification depends on the slicing and is not necessarily preserved in time. Moreover, the approximating type of space-time is not guaranteed to obey Einstein equations.

The second paper of the same authors [233] is basically an application of the earlier ideas to two slicings of the Gowdy solutions [124], providing approximants of Bianchi type I and VI_o, and to Kantowski-Sachs metric with the approximant Bianchi type I.

Another interesting paper is that by Stoeger, Ellis and Hellaby [236], where they proposed a criterion of continuous homogeneity of the universe, namely, if the mean mass density[8] in a sphere of volume V_L centered at the point $\vec{r}$ is given by

$$\bar{\rho}_L(\vec{r}) = \frac{1}{V_L}\int_{V_L} \rho(\vec{r'})\, d^3\vec{r'}, \tag{2.40}$$

where, V_L is assumed to be small enough so that the curvature inside V_L does not need to be taken into account. Then we say that the density distribution is spatially homogeneous on average at the level ϵ, on scales larger than L_c, if and only if, $\exists\ \epsilon \ll 1$ and L_c such that:

$$|\bar{\rho}_{L_1}(\vec{r_1}) - \bar{\rho}_{L_2}(\vec{r_2})| < \epsilon\bar{\rho}_{L_1}(\vec{r_1}) \tag{2.41}$$

for all $\vec{r_1}$, $\vec{r_2}$ and all L_1, $L_2 \geq L_c$. In principle, this criterion is falsifiable by observations. As an example, they considered galaxies randomly distributed in space according to Poisson distribution, and showed that without further assumptions such a distribution is not in agreement with (2.41) and so cannot be described by a FLRW in this sense. In the observed Universe one probably has $L_c \approx 200\, Mpc$ and $\epsilon \leq 0.01$.

2.4 Smoothing of cosmological spacetimes

The aim we are interested in can be rephrased as setting up a program for approximating the evolution of cosmological space-time solutions of Einstein's equations via the development of a procedure for "smoothing" sets of initial data for such space-times.

[8]This criterion applies to any scalar.

Looked at this way, smoothing is equivalent to a physical approximation scheme for particular space-times. The idea is the following. Given an initial data set: the spatial metric g, the extrinsic curvature K, and matter fields ψ, one would like to build a new smooth (i.e. spatially homogeneous) initial data set $(\overline{g}, \overline{K}, \overline{\psi})$, so that the new initial data is more easily evolved than the old one, and at the same time the evolution of new initial data models certain aspects of the evolution of the original initial data.

On the other hand, one can think of smoothing as a mathematical method for making general statements about a collection of space-times. The smoothing procedure could then be used as a map from general space-times to the spatially homogeneous ones, in order to study the space of space-times, in particular, the collection of space-times whose large scale dynamics are closely represented by the dynamics of spatially homogeneous space-times [149].

The flows of the metric are an important part of the smoothing we have in mind, namely, the Ricci-Hamilton flow (facts concerning it are given in Appendix B) and the Renormalization Group flow (see section 3.6).

General mathematical preliminaries can be found in Appendix A.

2.4.1 Smoothing-out spatially closed cosmologies

In [55] (see also [57]) a specific smoothing-out procedure was put forward, deforming a family of locally inhomogeneous and anisotropic spatially closed space-times into closed FLRW universes. These space-times are associated with gravitational configurations which can be considered near to the standard ones generating closed FLRW cosmological models. This class is large, it contains solutions to the Einstein field equations that are not just perturbations of closed FLRW space-times.

The smoothing-out procedure is employed in the full theory, and a precise content to the averaging hypothesis, by providing explicitly the correction terms to the physical sources induced upon smoothing-out the space-time geometry, can thus be given.

The idea is the following. We pick up an appropriate initial data set, on a closed spacelike hypersurface, which upon the Cauchy evolution is going to be the space-time to be averaged out. Such data set is then smoothly deformed into a FLRW initial data set, by the action of parabolic-type operators. This deformation is constructed in such a way as to make the deformed data satisfy the four constraints associated with Einstein's equations. It follows then that the flow of deformed initial data generates a one parameter family of solutions to the field equations, which interpolates between the original space-time and a closed FLRW space-time, considered to be the smoothed-out counterpart of the given universe model.

To make the above precise, let $({}^{(4)}V \overset{\phi}{\simeq} \mathcal{M} \times I, {}^{(4)}g)$ be the space-time manifold, the Cauchy evolution of a regular initial data set $(\mathcal{M}, g, K)$, where ϕ is a diffeomorphism mapping ${}^{(4)}V$ to $\mathcal{M} \times I$ ($I \subset \mathbb{R}$), with $\mathcal{M}$ the (closed) 3-manifold carrier of the initial data (i.e., a space-like 3-hypersurface in the space-time manifold) and $g, K \in S^2\mathcal{M}$, representing in the final space-time the induced Riemannian 3-metric on $\mathcal{M}$, and the second fundamental form of the embedding $\mathcal{M} \to ({}^{(4)}V, {}^{(4)}g)$, respectively.

We assume that $\mathcal{M}$ is topologically a 3-sphere S^3 and that the class of initial data supported by $\mathcal{M}$ is such that $Ric(g)$ is a positive definite bilinear form for them ($Ric(g)$ is the Ricci tensor associated with g).

Due to the results of R. Hamilton (see Appendix B), $({}^{(4)}V, {}^{(4)}g)$ resulting from the time evolution of data from the above class, can be taken as modeling a locally anisotropic and inhomogeneous universe, not too far from a closed FLRW space-time. A smoothing-out mapping associates with the given initial data set a one parameter family $(\mathcal{M}, g(\beta), K(\beta))$ with $0 \le \beta < \infty$, $g(0) = g, K(0) = K$, approximating closer and closer, the standard initial data set for a closed FLRW model and reaching it uniformly as $\beta \to \infty$ (β is the parameter labelling the family (flow) of 3−metrics).

According to Hamilton's theorem, we can deform the metric g into the constant-curvature metric $\overline{g}$ on S^3, by the flow of metrics $g(\beta), 0 \le \beta < \infty$, solution to the non-linear, weakly parabolic, initial value problem

$$\frac{\partial}{\partial \beta} g_{ab}(\beta) = \frac{2}{3} < R(\beta) >_\beta g_{ab}(\beta) - 2R_{ab}(\beta), \tag{2.42}$$

with $g_{ab}(0) = g_{ab}$, $(a, b = 1, 2, 3)$, where, $< R(\beta) >_\beta$ is the average scalar curvature over $(\mathcal{M}, g(\beta))$, and $R_{ab}(\beta)$, $R(\beta)$ are the components of the Ricci tensor and the scalar curvature associated with $g(\beta)$, respectively. We will recall here that the family (2.42) has the following properties (for more details see Appendix B):

(1) the volume of $\mathcal{M}(\beta)$ is independent of β;

(2) any symmetries of $g_{ab}(\beta_o)$ are inherited by all $g_{ab}(\beta)$ with $\beta \geq \beta_o$, and

(3) the limiting smoothed metric $\bar{g}_{ab} = \lim_{\beta\to\infty} g_{ab}(\beta)$ has positive constant curvature.

Now, g_{ab} is the inhomogeneous metric to be smoothed-out. Equation (2.42) (with the initial condition) defines a smooth family of deformations of the initial manifold, deforming it into a 3−space of constant curvature and of the same volume as the initial manifold.

In order to smooth-out the whole data set, we need to average out the second fundamental form K, as well.

Obviously, we have for the smoothed metric $\overline{g}_{ab} = \lim_{\beta\to\infty} g_{ab}(\beta)$ (presuming that the flow converges), where $g_{ab}(\beta)$ satisfies the Ricci flow equation (2.42). Given (g, K), let us then define a nearby flow $\tilde{g}_{ab}(\beta; \epsilon)$, with initial condition

$$\tilde{g}_{ab}(\beta = 0; \epsilon) \equiv g_{ab}(\beta = 0) + \epsilon K_{ab}(\beta = 0). \tag{2.43}$$

These flows evolve with β yielding as "connecting vector" the bilinear form

$$K_{ab}(\beta) = \lim_{\epsilon\to 0} \frac{1}{\epsilon}(\tilde{g}_{ab}(\beta; \epsilon) - g_{ab}(\beta)), \tag{2.44}$$

so we can define

$$\overline{K}_{ab} = \lim_{\beta\to\infty} K_{ab}(\beta). \tag{2.45}$$

The β evolution of $K(\beta)$ is found by linearizing (2.42). Formally we define it as

$$\frac{\partial}{\partial\beta} K_{ab}(\beta) = \frac{2}{3} g_{ab}(\beta)(\frac{1}{2} < R(\beta) K^c_c(\beta) >_\beta - \frac{1}{2} < R(\beta) >_\beta < K^c_c(\beta) >_\beta - \\ < R_{ab}(\beta) K^{ab}(\beta) >_\beta) + \frac{2}{3} < R(\beta) >_\beta K_{ab}(\beta) - \Delta_\beta K_{ab}(\beta) - L_Y g_{ab}(\beta) \tag{2.46}$$

with $K_{ab}(0) = K_{ab}$, and where Δ_β denotes the DeRham-Lichnerowicz Laplacian associated with $g(\beta)$, $\Delta_\beta K_{ab}(\beta) = -\nabla^c \nabla_c K_{ab} + R_{ac} K^c_b + R_{bc} K^c_a - R^d_{abc} K^c_d$, and L_Y is the Lie derivative along the vector field Y.

It can be shown that the flow $K(\beta)$, solution of (2.46), is such that $\frac{\partial}{\partial\beta}(< K^a_a(\beta) >_\beta) = 0$, i.e., the average over $(\mathcal{M}, g(\beta))$ of the trace of K is constant during the deformation.

Also, as (2.46) is the formal linearization of (2.42), we have $\lim_{\beta\to\infty} K_{ab}(\beta) = \frac{1}{3} < K^a_a >_o \overline{g}_{ab}$; $< \cdots >_o$ stands for the space average of the original physical quantity. Thus, the flow $K(\beta)$ deforms the given K by eliminating its shear: $K_{ab} - \frac{1}{3}K^c_c g_{ab}$, and replacing the original (position-dependent) rate of volume expansion K^a_a with its average value.

For each β the space average of matter density $< \rho(\beta) >_\beta$ can be defined analogously, as the average of scalar curvature, namely

$$< \rho(\beta) >_\beta = (V(\beta))^{-1} \int_{\mathcal{M}(\beta)} \rho(\beta) dV(\beta), \tag{2.47}$$

and then $\bar{\rho} = \lim_{\beta\to\infty} < \rho(\beta) >_\beta$ is the matter density in the limiting FLRW model.

The smoothing flow of regular initial data sets has to be such that for each value of β, the four constraints of the Einstein equations:

$$R(\beta) - K_{ab}(\beta)K^{ab}(\beta) + (K^a_a(\beta))^2 = 2\rho(\beta) \tag{2.48}$$

$$\nabla_a K^{ab}(\beta) - \nabla^b K^a_a(\beta) = J^b(\beta), \tag{2.49}$$

have to be satisfied; where $\rho(\beta), J(\beta)$ are the mass and momentum density, respectively (of the external sources as described by a system of observers instantaneously at rest on $\mathcal{M}$) referred to the β dependent measure associated with $g(\beta)$.

For $\beta = 0$, (2.48) and (2.49) hold true, since $\rho(\beta = 0) = \rho$ and $J(\beta = 0) = J$ are the physical densities of sources of a given gravitational configuration $(\mathcal{M}, g, K)$. The averaging flows $\rho(\beta)$ and $J(\beta)$ cannot be defined independently, once g and K are given and deformed according to (2.42) and (2.46), in order for the constraints (2.48) and (2.49) to remain valid. In other words, to properly average the sources one has to take into account the backreaction of the geometry, determined by the constraints.

We therefore interpret the constraints as actually defining $\rho(\beta)$ and $J(\beta)$. Indeed, then $\overline{\rho} \equiv \lim_{\beta\to\infty} < \rho(\beta) >_\beta$ and $\lim_{\beta\to\infty} < J(\beta) >_\beta \equiv \overline{J} = 0$ (from (2.48) and (2.49) and the properties of the Ricci-Hamilton flow). One can show explicitly that

$$\overline{\rho} = [< \rho >_o + \frac{1}{2} < (K_{ab} - \frac{1}{3}K^c_c g_{ab})(K^{ab} - \frac{1}{3}K^c_c g^{ab}) >_o + \frac{1}{2}\overline{R}(\eta + \sigma^2)]/(1 + \sigma^2) \tag{2.50}$$

where, $\sigma \equiv (< (K^a_a)^2 >_o - < (K^a_a) >^2_o)/ < K^a_a >_o$, $\eta \equiv (\overline{R} - < R >_o)/\overline{R}$, i.e., σ is the standard deviation describing the fluctuations of the original (position dependent) value of K^a_a with respect to its average (conserved) value $< K^a_a >_o$; η, $0 \leq \eta < 1$ denotes the relative function of the physical scalar curvature with respect to the averaged one $\overline{R} = \lim_{\beta \to \infty} < R(\beta) >_\beta$. ($K_{ab} - \frac{1}{3}K^c{}_c g_{ab}$ is the traceless part of K_{ab} on the initial manifold $\mathcal{M}(0)$).

Now, we can build an effective stress tensor modeling the dynamical effects of deviations from a spatially homogeneous geometry (also those which are too big to be handled by perturbation techniques) that have been smoothed-out.

If $(\mathcal{M}, g_t)$ defines a normal geodesic slicing of $({}^{(4)}V, {}^{(4)}g)$ (for sufficiently small t) the stress tensor enters only into the evolution part of Einstein's equations,

$$\frac{\partial}{\partial t}K_{ab} = R_{ab} + K^c_c K_{ab} - 2K_{ac}K^c_b - (T_{ab} - \frac{1}{2}T^c_c g_{ab}) - \frac{1}{2}\rho g_{ab}. \tag{2.51}$$

The smoothing flow $T(\beta)$ of the spatial stress tensor is defined by requiring that for each t for which the evolution of the data $(g(\beta), K(\beta))$ is defined, the flows $(g_t(\beta), K_t(\beta))$ (resulting from the evolution equations) are Ricci-Hamilton flows, with initial conditions $g_t(0) = g_t$, $K_t(0) = K_t$, respectively.

The physical meaning of the presented results enables us to state precisely what is meant by the requirement that the original physical model universe and its FLRW smoothed-out ideal should behave as close as possible under their own gravitation. Namely, for $\beta \to \infty$, the volume $V(S^3, \overline{g}_t) = V(\mathcal{M}, g_t)$; this shows how the dynamics of the closed FLRW model is related to the dynamics of the original space-time. The fact that $V(\mathcal{M}, g_t) = V(\mathcal{M}, g_t(\beta))$ implies that $\frac{\partial}{\partial t}(< K^a_a >_o) = \frac{\partial}{\partial t}(< K^a_a(\beta) >_\beta)$ along the flow $(g_t(\beta), K_t(\beta))$.

The smoothed-out pressure $\overline{p} \equiv \lim_{\beta \to \infty} < p(\beta) >_\beta$ (in the final FLRW model on the surface of homogeneity $t = 0$; $\overline{p}_t$ (as well as $\overline{\rho}_t$, if the equation of state is known) can be determined by the evolution equation) is shown to be (taking into account (2.51));

$$\begin{aligned} \overline{p} &= \frac{1}{3} < T^a_a >_o + \frac{2}{3} < (K_{ab} - \frac{1}{3}K^c_c g_{ab})(K^{ab} - \frac{1}{3}K^c_c g^{ab}) >_o - \frac{4}{9}\sigma^2 < K^a_a >^2_o \\ &\quad + \frac{1}{3}(< \rho >_o - \overline{\rho}) \end{aligned} \tag{2.52}$$

with $\overline{p}$ given by (2.50). We see from (2.50) and (2.52) that $\overline{p} > 0$ and $|\overline{p}| \leq \overline{\rho}$, i.e., the dominant energy condition is satisfied.

In case when $\sigma^2 \approx 0$ (homogeneous expansion) and $\eta \ll 1$ (fluctuations of the physical curvature w.r.t. FLRW background curvature small, on average) the closed FLRW universe is the proper model, only if we add to the physical sources $< \rho >_o$ and $< T^a_a >_o$ the term: $< (K_{ab} - \frac{1}{3}K^c_c g_{ab})(K^{ab} - \frac{1}{3}K^c_c g^{ab}) >_o$, taking this way into account the contribution of cosmological gravitational radiation. This term can influence the dynamics of the universe, and there is no evidence by now that the relative magnitude of this term with respect to $< \rho >_o$ is $\ll 1$.

Let us note that the smoothed stress tensor is defined by requiring that the smoothing commutes with the Einstein evolution, i.e., at each stage of this smoothing we in fact, appeal to the standard form of Einstein's equations, while the effect of smoothing on those equations was something to be investigated. Therefore in this scheme one cannot say anything about what the effect of smoothing is on the form of the equations.

Let us also stress that this smoothing program has in general a few unresolved issues, like e.g. dependence on the spacelike slice chosen, making an identification between hypersurfaces of the original space-time and those of the smoothed space-time[9]. Secondly, the issue is how to define the smoothed stress tensor T_{ab} if one uses a more complicated form for $T_{\mu\nu}$ than a perfect fluid.

One possible application of the averaging procedure could be to examine the conjecture that all cosmological models with S^3 spatial topology have a time of maximum expansion.

Hemmerich [142] criticized rather strongly this approach by raising, first of all, the slice dependence problem, namely that the space-time metric in the limit $\beta \to \infty$ is non-unique as it depends not only on $g_{ab}(\infty)$, but also on the foliation, i.e. the lapse and shift functions in the ADM formalism. In particular, the limiting 4−metric may not be FLRW at all.

It was shown in [142] that the lapse and shift can be limited, so that $g^{(4)}_{\mu\nu}(\infty)$ is indeed the FLRW geometry.

[9] We will offer some comments on this point in section 3.6.2.1.

The second objection was that the equations of Carfora–Marzuoli approach, among them (2.50), may be obtained directly from ADM formalism without employing the Ricci-Hamilton flow (2.42). To answer this, we can point out that equation (2.42) provides not only a relation between the average values of scalars on the initial manifold $\mathcal{M}(0)$ and the smoothed-out one $\mathcal{M}(\infty)$, but also a mapping of points of $\mathcal{M}(0)$ into points of $\mathcal{M}(\infty)$, which is lacking in other approaches. Moreover, with (2.42) one can consider any intermediate scale smoothing, from $\beta = 0$ to $\beta = \beta_o < \infty$, in addition to global smoothing from $\beta = 0$ to ∞.

The only real limitation of this scheme seems to be the assumption, that all $\mathcal{M}(\beta)$ are closed with their Ricci tensors positive definite.

2.5 Macroscopic Gravity

We have discussed the physical motivation for such a theory in some detail in chapter one (cf. in particular section 1.3.3), therefore now we will concentrate on the mathematical structure of the theory[10].

The notation adopted in this section follows that of [263] and [264], namely: the signature of the metric tensor $g_{\alpha\beta}$ is $(-+++)$, the Riemann tensor is $r^{\alpha}_{\beta\gamma\delta} = 2\Gamma^{\varepsilon}_{\beta[\delta}\Gamma^{\alpha}_{\underline{\varepsilon}\gamma]}$ (no (anti-)symmetrization in the underlined indices), the Ricci tensor $r_{\alpha\beta} = r^{\gamma}_{\alpha\beta\gamma}$, the scalar curvature $r = g^{\alpha\beta}r_{\alpha\beta}$, the Einstein equations:

$$r^{\varepsilon}_{\gamma} - \frac{1}{2}\delta^{\varepsilon}_{\gamma} r = -\kappa t^{\varepsilon\,(micro)}_{\gamma}, \tag{2.53}$$

where, $t^{\varepsilon\,(micro)}_{\gamma}$ is the energy momentum tensor of a microscopic matter distribution[11] and $\kappa = \frac{8\pi G}{c^4}$.

Tensors are denoted by italic letters and their bilocal extensions are in bold face, whereas (tensor-valued) forms are given in normal letters with their bilocal extensions in script.

[10] For more detailed presentation the reader is referred to the original papers [263, 264].

[11] The re-writing of the field equations in the mixed form has been done on purpose since then, the only splitting rule needed for averaging them out is the one for $< \mathbf{g}^{\alpha\beta}\mathbf{r}_{\beta\gamma} > -\bar{g}^{\alpha\beta}\bar{r}_{\beta\gamma}$.

2.5.1 The averaging scheme

We will describe here a space-time averaging procedure put forward in [10] for the case of Riemannian manifold, which was further generalized in [263, 264] for the case of a differentiable manifold.

In an open region $\Sigma \subset \mathcal{M}$ of a differentiable manifold $\mathcal{M}$, choose a (supporting) point $x \in \Sigma$. Let there be a tensor field $p^{\alpha}_{\beta}(x)$, $x \in \mathcal{M}$. The average value of the tensor field p^{α}_{β} over a region Σ at the supporting point $x \in \Sigma$ is defined as

$$\bar{p}^{\alpha}_{\beta}(x) = \frac{1}{V_{\Sigma}} \int_{\Sigma} p^{\mu'}_{\nu'}(x') \mathcal{A}^{\alpha}_{\mu'}(x, x') \mathcal{A}^{\nu'}_{\beta}(x, x') d\Omega', \tag{2.54}$$

where,

$$V_{\Sigma} = \int_{\Sigma} d\Omega = \int_{\Sigma} (-g)^{1/2} d^4x. \tag{2.55}$$

The object $\mathcal{A}^{\alpha'}_{\beta}(x, x')$ is a bilocal one, a vector at x' and 1−form at x. To begin with, we suppose that for a given Σ, $\mathcal{A}$ exists for all points $x, x' \in \tilde{\Sigma} \subset \mathcal{M}$, $\Sigma \subseteq \tilde{\Sigma}$. The averaging bivector $\mathcal{A}^{\alpha'}_{\beta}$ is defined through its postulated properties:

1. coincidence limit: $\lim_{x \to x'} \mathcal{A}^{\alpha'}_{\beta}(x, x') = \delta^{\alpha}_{\beta}$ (ensures a correct limit of (2.54) as $V_{\Sigma} \to 0$).
2. existence of an inverse operator $\mathcal{A}^{-1}{}^{\beta}_{\gamma'} \equiv \mathcal{A}^{\beta}_{\gamma'}$ (implied by 1.): $\mathcal{A}^{\alpha'}_{\beta} \mathcal{A}^{\beta}_{\gamma'} = \delta^{\alpha'}_{\gamma'}$, $\mathcal{A}^{\alpha'}_{\beta} \mathcal{A}^{\gamma}_{\alpha'} = \delta^{\gamma}_{\beta}$.
3. associativity: $\mathcal{A}^{\alpha'}_{\beta} \mathcal{A}^{\beta}_{\gamma''} = \mathcal{A}^{\alpha'}_{\gamma''}$.
4. flat space limit, $r^{\alpha}_{\beta\gamma\delta} = 0$: $\mathcal{A}^{\alpha'}_{\beta} = \delta^{\alpha'}_{\beta}$ (ensures a correct limit of (2.54) as $r^{\alpha}_{\beta\gamma\delta} \to 0$, giving the standard definition of the average value in a flat space).

The averaging operator defines a mapping from a point x to a point x', which is an equivalence relation. For notational convenience, we denote the integrand of (2.54), i.e. $p^{\mu'}_{\nu'}(x') \mathcal{A}^{\alpha}_{\mu'} \mathcal{A}^{\nu'}_{\beta}$, as $\mathrm{p}^{\alpha}_{\beta}$ and write the averages either as $\bar{p}^{\alpha}_{\beta}$, or $< \mathrm{p}^{\alpha}_{\beta} >$.

A concept of $\mathcal{A}$−constancy generalizing that of a covariant constancy can be introduced, namely, q^{α}_{β} is $\mathcal{A}$−constant if

$$q^{\alpha}_{\beta}(x) = \mathcal{A}^{\alpha}_{\mu'}(x, x') \mathcal{A}^{\nu'}_{\beta}(x, x') q^{\mu'}_{\nu'}(x'). \tag{2.56}$$

As seen from the definition (2.54), the average of a sum of tensors is equal to the sum of the averages. For an $\mathcal{A}-$constant tensor q^{α}_{β}, $< \mathbf{q}^{\alpha}_{\beta}\mathbf{p}^{\mu}_{\nu} >= q^{\alpha}_{\beta}\bar{p}^{\mu}_{\nu}$. Moreover it can be shown that $\bar{\bar{p}}^{\alpha}_{\beta} = \bar{p}^{\alpha}_{\beta}$.

To derive the commutation formulae for the averaging and the derivation, we define first a directional derivative of an average along a vector field $\xi = d/d\lambda$,

$$\frac{d}{d\lambda}\bar{p}^{\alpha}_{\beta}(x) = \lim_{\Delta\lambda\to 0}\frac{1}{\Delta\lambda}[\bar{p}^{\alpha}_{\beta}(x+\Delta x) - \bar{p}^{\alpha}_{\beta}(x)]. \tag{2.57}$$

In order to compare two neighbouring regions and define, as a result, commutation formulae for partial (covariant) derivative and averaging, let us define the shift field for every point $x' \in \Sigma$ as

$$S^{\alpha'}(x,x') = \mathcal{W}^{\alpha'}_{\beta}(x,x')\xi^{\beta}. \tag{2.58}$$

This means that the averaging region Σ is Lie-dragged along the integral lines of a given field $S^{\alpha'}$ on parametric length $\Delta\lambda$ (for all $x' \in \Sigma$). Here, $\mathcal{W}^{\alpha'}_{\beta}(x,x')$ is another bilocal operator, namely, the averaging region coordination bivector. It is assumed to satisfy the properties from 1. to 4. above.

The bivector $\mathcal{W}^{\alpha'}_{\beta}$ enables us to calculate the shift vector $S^{\alpha'}$ for any averaging region in accordance with a given vector ξ^{α} at the supporting point for that region. By choosing four such linearly independent vector fields ξ^{α} (for a $n = 4$ dim manifold), and shifting averaging regions along them one can build a covering of the Riemannian manifold with the averaging regions for all points $x \in \mathcal{M}$ as the supporting points.

The following formula holds as a result of applying (2.58)

$$\bar{p}^{\alpha}_{\beta;\lambda} =< \mathcal{A}^{\alpha}_{\mu'}\mathcal{A}^{\nu'}_{\beta}p^{\mu'}_{\nu';\epsilon'}\mathcal{W}^{\epsilon'}_{\lambda} > + < \mathbf{p}^{\alpha}_{\beta}\mathcal{W}^{\epsilon'}_{\lambda;\epsilon'} > -\bar{p}^{\alpha}_{\beta} < \mathcal{W}^{\epsilon'}_{\lambda;\epsilon'} > - < S^{\alpha}_{\sigma\lambda}\mathbf{p}^{\sigma}_{\beta} > + < S^{\sigma}_{\beta\lambda}\mathbf{p}^{\alpha}_{\sigma} > . \tag{2.59}$$

Here, $\mathcal{S}$ are the so called structural functions of the Riemannian manifold, namely, $S^{\alpha}_{\beta\gamma} = \mathcal{A}^{\alpha}_{\epsilon'}(\mathcal{A}^{\epsilon'}_{\beta;\gamma} + \mathcal{A}^{\epsilon'}_{\beta;\sigma'}\mathcal{W}^{\sigma'}_{\gamma})$. The same formulae remain valid for the case of partial differentiation, with the covariant differentiation symbol replaced by the partial one at point x. The change of the volume V_{Σ} of the averaging region Σ along the vector field ξ is given by $\frac{d}{d\lambda}V_{\Sigma} = \int_{\Sigma}\operatorname{div}\xi d\Omega$. Shifting the regions by $\mathcal{W}^{\alpha'}_{\beta}$ we get $V_{\Sigma,\beta} =< \mathcal{W}^{\alpha'}_{\beta;\alpha'} > V_{\Sigma}$.

The problem we have to face now is whether it is possible to define in this way a unique covering of the manifold $\mathcal{M}$. In other words, whether it is possible to determine an average tensor field as a local one of the supporting point and unique, for a covering of the manifold under study by the averaging regions of fixed volume. More importantly, upon averaging a tensor field we want to obtain an average tensor field again. Therefore, we require first of all,

$$\mathcal{W}^{\alpha'}{}_{\beta;\alpha'} = 0, \tag{2.60}$$

to hold, in order that volumes of the regions are kept constant while being shifted (coordinated). Further, in the averaging region coordination by $\mathcal{W}^{\alpha'}_{\beta}$ it is necessary and sufficient to require

$$\mathcal{W}^{\alpha'}{}_{[\beta,\gamma]} + \mathcal{W}^{\alpha'}{}_{[\rho,\delta'}\mathcal{W}^{\delta'}{}_{\gamma]} = 0, \tag{2.61}$$

in order that the average tensor field $\bar{p}^{\alpha}_{\beta}(x)$ is a single valued local function on $\mathcal{M}$: $\bar{p}^{\alpha}{}_{\beta,[\mu\nu]} = 0$. Geometrically, (2.61) means that an averaging region transported along an infinitesimal parallelogram due to (2.58) coincides with the original region.

Thus if (2.61) is satisfied an average tensor field is unique and local function of the supporting point x, i.e $\bar{p}^{\alpha}_{\beta} = p^{\alpha}_{\beta}(x, V)$. Requiring in addition (2.60), the tensor field does not depend explicitly on the value of the region's volume, i.e. $\bar{p}^{\alpha}_{\beta} = \bar{p}^{\alpha}_{\beta}(x)$. The volume's value itself is a free parameter of the theory. The average tensor fields can therefore be treated in the framework of standard analysis. In addition, a convenient choice is

$$\mathcal{W}^{\alpha'}{}_{\beta} = \mathcal{A}^{\alpha'}{}_{\beta}\,. \tag{2.62}$$

The formula (2.59) now takes a remarkably simple form [264]

$$\mathrm{D}\bar{\mathbf{p}}^{\alpha}_{\beta} = < \mathrm{Đ}\mathbf{p}^{\alpha}_{\beta} > . \tag{2.63}$$

In tensor notation it reads

$$\bar{p}^{\alpha}{}_{\beta;\gamma} = < \mathbf{p}^{\alpha}{}_{\beta;\gamma} + \mathbf{p}^{\alpha}{}_{\beta;\alpha'}\mathcal{A}^{\alpha'}{}_{\gamma} > . \tag{2.64}$$

Obviously it holds also for the case of partial differentiation (with the covariant differentiation symbol replaced by the partial one on the left hand side of it and in the first term on its right hand side).

As a consequence of (2.60), (2.61) and (2.62), all the conditions for $\mathcal{A}^{\alpha'}{}_{\beta}$ and $\mathcal{W}^{\alpha'}_{\beta}$ can be written as a system of bilocal algebraic and partial differential equations. They can be shown to be necessarily integrable on an arbitrary differentiable manifold. In particular, a factorized solution can be obtained to (2.61) in the form $\mathcal{W}^{\alpha'}_{\beta} = \frac{\partial x^{\alpha'}}{\partial \phi^i}\frac{\partial \phi^i}{\partial x^{\beta}}$, which depends on four arbitrary scalar functions ϕ^i. A freedom to choose them allows to also satisfy (2.60). This being the case, the bilocal operator $\mathcal{A}^{\alpha'}_{\beta}$ determines a (proper) coordinate system $x^\alpha = \phi^i(x)$, $x \in \mathcal{M}$ in which $\mathcal{A}^{\alpha'}_{\beta} = \delta^i_j$, and averaging regions are shifted on a constant vector $S^\alpha(x) = \xi^i(x)$.

Thus the bilocal operator $\mathcal{A}^{\alpha'}_{\beta}$ has been shown to exist locally on a manifold. Its definition can be extended though for it to exist globally. In the case of $\mathbb{R}^4$ topology it is evidently determined globally. In the case of manifolds with non-trivial topology this would require defining $\mathcal{A}^{\alpha'}_{\beta}$ in the appropriate separate patches. For example, in the case of 2–sphere S^2 it should be respectively determined in the upper and lower patches.

2.5.2 The structure of macroscopic geometry

The original (microscopic) manifold is a Riemannian space-time with the metric $g_{\alpha\beta}$, connection 1–form $\omega^\alpha{}_\beta$ and the curvature 2–form $\mathrm{r}^\alpha{}_\beta$.

Cartan's structure equations are known to hold on the *metric* part of the geometry,

$$\mathrm{d}g_{\alpha\beta} - g_{\alpha\gamma}\omega^\gamma{}_\beta - g_{\gamma\beta}\omega^\gamma{}_\alpha = 0, \tag{2.65}$$

$$g_{\alpha\varepsilon}\mathrm{r}^\varepsilon{}_\beta + g_{\beta\varepsilon}\mathrm{r}^\varepsilon{}_\alpha = 0, \tag{2.66}$$

and its *affine* part, namely,

$$\omega^\alpha{}_\varepsilon \wedge \mathrm{d}x^\varepsilon = 0, \tag{2.67}$$

$$\mathrm{r}^\alpha{}_\varepsilon \wedge \mathrm{d}x^\varepsilon = 0, \tag{2.68}$$

$$\mathrm{r}^\alpha{}_\beta = \mathrm{d}\omega^\alpha{}_\beta + \omega^\alpha{}_\varepsilon \wedge \omega^\varepsilon{}_\beta, \tag{2.69}$$

$$\mathrm{d}\mathrm{r}^\alpha{}_\beta - \mathrm{r}^\alpha{}_\varepsilon \wedge \omega^\varepsilon{}_\beta + \omega^\alpha{}_\varepsilon \wedge \mathrm{r}^\varepsilon{}_\beta = 0. \tag{2.70}$$

The first pair of the equations express the covariant constancy of the metric and a symmetry of the first pair of indices. The second equation is at the same time the integrability condition for (2.65). Equation (2.67) is a condition for the torsion-free connection, equation (2.68) being its integrability condition. It is also called the cyclic Bianchi identities. Finally, the last two equations express the curvature through the connection and the differential Bianchi identities, respectively, the last equation being again the integrability condition of the previous one. This system includes moreover the condition of equi-affinity,

$$\mathrm{r}^{\epsilon}{}_{\epsilon} = 0. \tag{2.71}$$

To average out the above equations we have to first write down their bilocal extensions. Then we can average out the above system according to the rules of the preceeding section. We will concentrate first on the

<u>AFFINE PART</u>.

The averaged objects here are: the averaged connection $\bar{\Omega}^{\alpha}{}_{\beta}$ and the averaged curvature $\bar{\mathrm{r}}^{\alpha}{}_{\beta} \equiv < \mathcal{R}^{\alpha}{}_{\beta} > \equiv \mathrm{R}^{\alpha}{}_{\beta}$. The averaged connection can be shown to be an affine symmetric connection 1−form.

In particular, note that writing the structural functions $\mathcal{S}^{\alpha}{}_{\beta\gamma}$ as $\mathcal{S}^{\alpha}{}_{\beta\gamma} = \mathcal{F}^{\alpha}{}_{\beta\gamma} - \Gamma^{\alpha}{}_{\beta\gamma}$, where $\Gamma^{\alpha}{}_{\beta\gamma}$ are the Christoffel symbols of the connection 1−form $\omega^{\alpha}{}_{\beta}$, one can notice that the bilocal objects

$$\mathcal{F}^{\alpha}{}_{\beta\gamma} = \mathcal{W}^{\alpha}{}_{\epsilon'}(\mathcal{W}^{\epsilon'}{}_{\beta,\gamma} + \mathcal{W}^{\epsilon'}{}_{\beta;\sigma'}\mathcal{W}^{\sigma'}{}_{\gamma})\,, \tag{2.72}$$

behave as connection coefficients under coordinate transformations at x.

The coincidence limit of $\mathcal{F}^{\alpha}{}_{\beta\gamma}$ is[12]

$$\lim_{x'\to x} \mathcal{F}^{\alpha}{}_{\beta\gamma} = \Gamma^{\alpha}{}_{\beta\gamma}\,, \tag{2.73}$$

and consequently $\mathcal{F}^{\alpha}{}_{\beta\gamma}$ can be considered as a bilocal extension of ordinary connection coefficients $\Gamma^{\alpha}{}_{\beta\gamma}$. Upon averaging, $< \mathcal{F}^{\alpha}{}_{\beta\gamma} > \equiv \bar{\mathcal{F}}^{\alpha}{}_{\beta\gamma}$ will represent an average symmetric (due to (2.61)) connection[13]. In the language of forms $< \Omega^{\alpha}{}_{\beta} > \equiv \bar{\Omega}^{\alpha}{}_{\beta} = \bar{\mathcal{F}}^{\alpha}{}_{\beta\gamma} dx^{\gamma}$ is an affine connection 1−form of the averaged space.

Whether the averaged curvature $\mathrm{R}^{\alpha}{}_{\beta}$ is the Riemannian curvature 2−form remains to be seen later on. By directly averaging out equations (2.67), (2.68) and (2.71) we obtain

$$\bar{\Omega}^{\alpha}{}_{\epsilon} \wedge \mathrm{d}x^{\epsilon} = 0, \tag{2.74}$$

$$\mathrm{R}^{\alpha}{}_{\epsilon} \wedge \mathrm{d}x^{\epsilon} = 0, \tag{2.75}$$

---

[12] A theorem due to Christensen [66] determines similar coincidence limits.

[13] For $\mathcal{W}^{\alpha'}{}_{\beta}$ in the factorized form, there is a coordinate system $x^{\alpha} = \phi^{i}$, where $\mathcal{W}^{\alpha'}{}_{\beta} = \delta^{i}{}_{j}$ and where $\bar{\mathcal{F}}^{\alpha}{}_{\beta\gamma}$ are manifestly the averaged $\Gamma^{\alpha}{}_{\beta\gamma}$.

$$\mathrm{R}^{\alpha}{}_{\alpha} = 0. \tag{2.76}$$

The exterior derivative above is understood to be its bilocal extension. The first of the above equations means again the absence of torsion. Averaging equation (2.69) yields an important structure relation

$$\mathrm{M}^{\alpha}{}_{\beta} = \mathrm{R}^{\alpha}{}_{\beta} - \mathrm{Q}^{\alpha}{}_{\beta}, \tag{2.77}$$

where,

$$\mathrm{M}^{\alpha}{}_{\beta} = \mathrm{d}\bar{\Omega}^{\alpha}{}_{\beta} + \bar{\Omega}^{\alpha}{}_{\varepsilon} \wedge \bar{\Omega}^{\varepsilon}{}_{\beta}, \tag{2.78}$$

is the curvature 2−form due to $\bar{\Omega}$ (the algebraic identities $\mathrm{M}^{\alpha}{}_{\varepsilon} \wedge \mathrm{d}x^{\varepsilon} = 0$ and $\mathrm{M}^{\alpha}{}_{\alpha} = 0$ hold). The relation (2.77), $M^{\alpha}{}_{\beta\rho\sigma} = R^{\alpha}{}_{\beta\rho\sigma} - Q^{\alpha}{}_{\beta\rho\sigma}$[14], is of the form of a constitutive relation between the induction, $M^{\alpha}{}_{\beta\rho\sigma}$, and the average field $R^{\alpha}{}_{\beta\rho\sigma}$, with $Q^{\alpha}{}_{\beta\rho\sigma}$ standing for the polarization tensor (defined below). The origin of this *geometric relation* lies in the simple, but non-trivial, geometric fact of the non-linear definition of the affine curvature in terms of connection, which results in the curvature being determined by the average connection not equal to the average curvature.

We can now introduce the following correlation 2−form

$$\mathrm{Q}^{\alpha}{}_{\beta} = < \Omega^{\alpha}{}_{\varepsilon} \wedge \Omega^{\varepsilon}{}_{\beta} > - \bar{\Omega}^{\alpha}{}_{\varepsilon} \wedge \bar{\Omega}^{\varepsilon}{}_{\beta}, \tag{2.79}$$

with the algebraic properties $\mathrm{Q}^{\alpha}{}_{\beta} \wedge \mathrm{d}x^{\beta} = 0$, $\mathrm{Q}^{\alpha}{}_{\alpha} = 0$.

Let now $\mathrm{R}^{\alpha}{}_{\beta}$ be a curvature 2−form of an affine connection 1−form $\Pi^{\alpha}{}_{\beta}$, namely,

$$\mathrm{R}^{\alpha}{}_{\beta} = \mathrm{d}\Pi^{\alpha}{}_{\beta} + \Pi^{\alpha}{}_{\varepsilon} \wedge \Pi^{\varepsilon}{}_{\beta}. \tag{2.80}$$

This connection is symmetric as well. Due to [225] one can choose,

(a) $\mathrm{M}^{\alpha}{}_{\beta}$ to be a Riemannian curvature 2−form and $\mathrm{R}^{\alpha}{}_{\beta}$ a non-Riemannian one; or

(b) $\mathrm{R}^{\alpha}{}_{\beta}$ to be a Riemannian curvature 2−form and $\mathrm{M}^{\alpha}{}_{\beta}$ a non-Riemannian one.

We choose to proceed with (a). If so a metric tensor $G_{\alpha\beta}$ exists, such that $\mathrm{D}_{\bar{\Omega}} G_{\alpha\beta} = 0$, $\mathrm{D}_{\Pi} G_{\alpha\beta} \neq 0$, where D stands for the exterior covariant derivative. This metric tensor corresponds to the Levi-Civita connection $\bar{\mathcal{F}}^{\alpha}{}_{\beta\gamma}$ and plays the rôle of the *macroscopic metric tensor*. There is then an affine deformation 1−form defined as

$$\mathrm{A}^{\alpha}{}_{\beta} = \bar{\Omega}^{\alpha}{}_{\beta} - \Pi^{\alpha}{}_{\beta}, \tag{2.81}$$

[14] Note the change of sign in the definition of $Q^{\alpha}{}_{\beta\rho\sigma}$ as compared with the papers [263, 264, 265, 266].

which plays the rôle of the polarization potential. According to (2.77) the following equations for $A^\alpha{}_\beta$ holds

$$D_{\bar{\Omega}} A^\alpha{}_\beta - A^\alpha{}_\epsilon \wedge A^\epsilon{}_\beta = -Q^\alpha{}_\beta. \tag{2.82}$$

They are shown to be always integrable.

It should be noted that the non-Riemannian geometry in the two cases (a) and (b) is always equi-affine, since due to (2.82) (and similar equation for the case (b) with $\bar{\Omega}$ replaced by Π and $Q^\alpha{}_\beta$ on the right hand side there) $A^\alpha{}_\alpha$ can be shown to be a gradient of some scalar function.

The differential Bianchi identities hold as the result of (2.78) and (2.80)

$$dM^\alpha{}_\beta - M^\alpha{}_\epsilon \wedge \bar{\Omega}^\epsilon{}_\beta + \bar{\Omega}^\alpha{}_\epsilon \wedge M^\epsilon{}_\beta = 0, \tag{2.83}$$

$$dR^\alpha{}_\beta - R^\alpha{}_\epsilon \wedge \Pi^\epsilon{}_\beta + \Pi^\alpha{}_\epsilon \wedge R^\epsilon{}_\beta = 0. \tag{2.84}$$

Recall however that the differential Bianchi identities (2.70) so far are not averaged out.

We can introduce a correlation 2–form

$$Z^\alpha{}_\beta{}^\mu{}_\nu = < \Omega^\alpha{}_\beta \wedge \Omega^\mu{}_\nu > - \bar{\Omega}^\alpha{}_\beta \wedge \bar{\Omega}^\mu{}_\nu, \tag{2.85}$$

so that $Z^\alpha{}_\epsilon{}^\epsilon{}_\beta = Q^\alpha{}_\beta$. Equivalently there is the correlation tensor defined as

$$Z^\alpha{}_{\beta[\gamma}{}^\mu{}_{\underline{\nu}\sigma]} = < \mathcal{F}^\alpha{}_{\beta[\gamma} \mathcal{F}^\mu{}_{\underline{\nu}\sigma]} > - \bar{\mathcal{F}}^\alpha{}_{\beta[\gamma} \bar{\mathcal{F}}^\mu{}_{\underline{\nu}\sigma]}, \tag{2.86}$$

with $Q^\alpha{}_{\beta\gamma\sigma} = 2Z^\alpha{}_{\epsilon[\gamma}{}^\epsilon{}_{\underline{\beta}\sigma]}$. Taking its exterior derivative gives the structure equations for Z

$$D_{\bar{\Omega}} Z^\alpha{}_\beta{}^\mu{}_\nu = -2\mathbb{P} Y^\alpha{}_\delta{}^\delta{}_\beta{}^\mu{}_\nu + 2\mathbb{P}(< \mathcal{R}^\alpha{}_\beta \wedge \Omega^\mu{}_\nu > - R^\alpha{}_\beta \wedge \bar{\Omega}^\mu{}_\nu). \tag{2.87}$$

Here, $\mathbb{P}$ is a matrix permutation operator acting on a k–matrix-valued k–form. This equation can also be shown to be integrable. The structure equation (2.87) simultaneously is the splitting rule for $< \mathcal{R}\Omega >$ and applying it one can average out (2.70) which results in (2.83). The correlation 3–form $Y^\alpha{}_\beta{}^\mu{}_\nu{}^\theta{}_\kappa$ is defined as

$$Y^\alpha{}_\beta{}^\mu{}_\nu{}^\theta{}_\kappa = < \Omega^\alpha{}_\beta \wedge \Omega^\mu{}_\nu \wedge \Omega^\theta{}_\kappa > - 3\mathbb{P}(Z^\alpha{}_\beta{}^\mu{}_\nu \wedge \Omega^\theta{}_\kappa) - \bar{\Omega}^\alpha{}_\beta \wedge \bar{\Omega}^\mu{}_\nu \wedge \bar{\Omega}^\theta{}_\kappa. \tag{2.88}$$

The structure equation for Y can be similarly found (see [263, 264]),

$$D_{\bar{\Omega}} Y^\alpha{}_\beta{}^\mu{}_\nu{}^\theta{}_\kappa = \ldots, \tag{2.89}$$

which is, at the same time, the splitting rule for $< \mathcal{R}\Omega\Omega >$ and is again integrable. It requires the introduction of the correlation 4–form $\mathrm{X}^{\alpha}{}_{\beta}{}^{\mu}{}_{\nu}{}^{\theta}{}_{\kappa}{}^{\tau}{}_{\varphi}$ which is a function of the other correlation tensors. *It is the last correlation function for a 4-dimensional space-time.*

Given X and upon fixing the structure of correlators in (2.87) and (2.89), these equations can be used as the equations to find Z and Y. A possible choice could be

$$\mathrm{D}_{\bar{\Omega}}\mathrm{Z}^{\alpha}{}_{\beta}{}^{\mu}{}_{\nu} = 0, \qquad (2.90)$$
$$\mathrm{Y}^{\alpha}{}_{\beta}{}^{\mu}{}_{\nu}{}^{\theta}{}_{\kappa} = \mathrm{X}^{\alpha}{}_{\beta}{}^{\mu}{}_{\nu}{}^{\theta}{}_{\kappa}{}^{\tau}{}_{\varphi} = 0.$$

In view of the importance of the correlation 2–form Z it is worth writing its algebraic properties:

$$\begin{aligned} \mathrm{Z}^{\alpha}{}_{\beta}{}^{\mu}{}_{\nu} &= -\mathrm{Z}^{\mu}{}_{\nu}{}^{\alpha}{}_{\beta}, \\ \mathrm{Z}^{\alpha}{}_{\beta}{}^{\mu}{}_{\nu} \wedge dx^{\nu} &= 0, \\ \mathrm{Z}^{\alpha}{}_{\varepsilon}{}^{\varepsilon}{}_{\nu} &= \mathrm{Q}^{\alpha}{}_{\nu}, \\ \mathrm{Z}^{\varepsilon}{}_{\varepsilon}{}^{\mu}{}_{\nu} &= 0. \end{aligned}$$

The last relation is a consequence of the equi-affinity. Now one can define a Ricci-like object for tensor $Z^{\alpha}{}_{\beta[\gamma}{}^{\mu}{}_{\underline{\nu}\sigma]}$, namely, $Z^{\alpha}{}_{\beta\nu\sigma} = 2Z^{\alpha}{}_{\beta[\mu}{}^{\mu}{}_{\underline{\nu}\sigma]}$. It has the following algebraic properties:

$$\begin{aligned} Z^{\alpha}{}_{[\beta\sigma]\nu} &= \frac{1}{2}Q^{\alpha}{}_{\nu\beta\sigma}, \\ Z^{\alpha}{}_{\beta[\nu\sigma]} &= 0, \\ Z^{\varepsilon}{}_{\varepsilon\nu\sigma} &= 0, \\ Z^{\varepsilon}{}_{\beta\varepsilon\sigma} &= Q_{\beta\sigma}, \end{aligned}$$

where $Q_{\beta\sigma} = Q^{\mu}_{\beta\sigma\mu}$. This tensor will play an important rôle in the dynamics of the macroscopic gravitational field (see the next section 2.5.3).

Coming now to the

METRIC PART,

we have the following averaged objects $\bar{g}_{\alpha\beta}$, $\bar{g}^{\alpha\beta}$, and in general $\bar{g}_{\alpha\beta}\bar{g}^{\beta\gamma} \neq \delta^{\gamma}{}_{\alpha}$. We will have to investigate what relation or condition can be put between these tensors and the macroscopic metric tensor.

In order to average out (2.65) let us assume the following *splitting rule* for the objects which have a large-scale mode (e.g. covariantly constant objects, Killing vectors and tensors)

$$< \mathbf{g}_{\alpha\beta}\Omega^{\mu}{}_{\nu} >= \bar{g}_{\alpha\beta}\bar{\Omega}^{\mu}{}_{\nu}, \tag{2.91}$$

$$< \mathbf{g}^{\alpha\beta}\Omega^{\mu}{}_{\nu} >= \bar{g}^{\alpha\beta}\bar{\Omega}^{\mu}{}_{\nu}. \tag{2.92}$$

This means that upon averaging, as is natural to expect, the covariantly constant objects remain so, Killing vectors remain Killing vectors, etc. We therefore obtain as a result

$$\mathrm{D}_{\bar{\Omega}}\bar{g}_{\alpha\beta} = 0, \tag{2.93}$$

$$\mathrm{D}_{\bar{\Omega}}\bar{g}^{\alpha\beta} = 0. \tag{2.94}$$

The integrability conditions of these equations are

$$\bar{g}_{\alpha\beta}\mathrm{M}^{\beta}{}_{\gamma} + \bar{g}_{\gamma\beta}\mathrm{M}^{\beta}{}_{\alpha} = 0, \tag{2.95}$$

$$\bar{g}^{\alpha\beta}\mathrm{M}^{\gamma}{}_{\beta} + \bar{g}^{\gamma\beta}\mathrm{M}^{\alpha}{}_{\beta} = 0. \tag{2.96}$$

This being the case we can identify $\bar{g}_{\alpha\beta} = G_{\alpha\beta}$ (but $\bar{g}^{\alpha\beta} \neq G^{\alpha\beta}$). So we have

$$G_{\alpha\beta}\mathrm{M}^{\beta}{}_{\gamma} + G_{\gamma\beta}\mathrm{M}^{\beta}{}_{\alpha} = 0. \tag{2.97}$$

The same relation above we have directly as the integrability condition of $\mathrm{D}_{\bar{\Omega}}G_{\alpha\beta} = 0$.

To derive the splitting rule for $< \mathbf{gr} >$ and consequently average out (2.65), to obtain exactly (2.95) and (2.96), we should differentiate (2.91) and (2.92) [263, 264]. Upon assuming now the following splitting rules (which are in agreement with (2.91) and (2.92))

$$< \mathbf{g}_{\rho\sigma}\Omega^{\alpha}{}_{\beta} \wedge \Omega^{\mu}{}_{\nu} >= \bar{g}_{\rho\sigma} < \Omega^{\alpha}{}_{\beta} \wedge \Omega^{\mu}{}_{\nu} >, \tag{2.98}$$

$$< \mathbf{g}^{\rho\sigma}\Omega^{\alpha}{}_{\beta} \wedge \Omega^{\mu}{}_{\nu} >= \bar{g}^{\rho\sigma} < \Omega^{\alpha}{}_{\beta} \wedge \Omega^{\mu}{}_{\nu} >, \tag{2.99}$$

we can include the correlation tensor Z into that rule which reads now

$$< \mathbf{r}^{\alpha}{}_{\beta\gamma\lambda}\mathbf{g}^{\varepsilon\rho} > -R^{\alpha}{}_{\beta\gamma\lambda}\bar{g}^{\varepsilon\rho} = -2Z^{\alpha}{}_{\beta[\gamma}{}^{\varepsilon}{}_{\underline{\delta}\lambda]}\bar{g}^{\delta\rho} - 2Z^{\alpha}{}_{\beta[\gamma}{}^{\rho}{}_{\underline{\delta}\lambda]}\bar{g}^{\varepsilon\delta}. \tag{2.100}$$

This rule is of physical importance since it is the one that is needed for the averaging out of the Einstein equations.

Let us also write down for completeness the splitting rule for the Ricci tensor $r_{\alpha\beta}$

$$< \mathbf{g}^{\rho\varepsilon}\mathbf{r}_{\varepsilon\gamma} > -\bar{g}^{\rho\varepsilon}M_{\varepsilon\gamma} = -\bar{g}^{\varepsilon\delta}Z^{\rho}{}_{\delta\varepsilon\gamma}, \tag{2.101}$$

and for the scalar curvature

$$< \mathbf{g}^{\rho\varepsilon}\mathbf{r}_{\rho\varepsilon} > -\bar{g}^{\rho\varepsilon}M_{\rho\varepsilon} = -\bar{g}^{\varepsilon\delta}Q_{\varepsilon\delta}. \tag{2.102}$$

By differentiating further (2.100) it is possible to arrive at a splitting rule for (see [263, 264])

$$< \mathbf{g}^{\mu\nu}\mathcal{R}^{\alpha}{}_{\beta} \wedge \Omega^{\mu}{}_{\nu} > -\bar{g}^{\mu\nu}\mathrm{R}^{\alpha}{}_{\beta} \wedge \bar{\Omega}^{\mu}{}_{\nu} = \ldots . \tag{2.103}$$

2.5.3 Averaging out the Einstein equations

To average out the metric structure equations and the Einstein equations one should apply now the rule (2.100) (there is a similar one for the covariant metric tensor) for splitting of *the average of the metric times the curvature.* This rule is important from a physical point of view for it is the only rule needed for the Einstein equations to be averaged out. Using the contracted version of (2.100), namely, (2.101) and (2.102), enables us to carry this out. The result is the *macroscopic field equations*

$$G^{\alpha\varepsilon}M_{\varepsilon\beta} - \frac{1}{2}\delta^{\alpha}_{\beta}G^{\mu\nu}M_{\mu\nu} = -\kappa T^{\alpha(macro)}_{\beta}, \tag{2.104}$$

where, the macroscopic stress tensor $T^{\alpha(macro)}_{\beta}$ has the following form

$$\kappa T^{\alpha(macro)}_{\beta} = \kappa < \mathbf{t}^{\alpha(micro)}_{\beta} > -(Z^{\alpha}{}_{\mu\nu\beta} - \frac{1}{2}\delta^{\alpha}_{\beta}Q_{\mu\nu})\bar{g}^{\mu\nu} + U^{\alpha\varepsilon}M_{\varepsilon\beta} - \frac{1}{2}\delta^{\alpha}_{\beta}U^{\mu\nu}M_{\mu\nu}. \tag{2.105}$$

This stress tensor is conserved

$$T^{\alpha(macro)}{}_{\beta|\alpha} = 0, \tag{2.106}$$

where, the vertical | denotes the covariant derivative with respect to the connection coefficients $\bar{\mathcal{F}}^{\alpha}{}_{\beta\gamma}$, i.e. $\mathrm{D}_{\bar{\Omega}}$. $Z^{\alpha}{}_{\mu\nu\beta}$ defined in the previous section is a Ricci-tensor-like object; $< \mathbf{t}^{\alpha(micro)}_{\beta} >$ is the averaged energy-momentum tensor. The algebraic structure of the covariantly constant tensor $U^{\alpha\beta} = \bar{g}^{\alpha\beta} - G^{\alpha\beta}$, $(U^{\alpha\beta}{}_{|\gamma} = 0)$ can be further determined [264] (see section 2.5.4).

Though somewhat unexpected, the result is natural: the space-time averaging of the Einstein equations yields the field equations which *can be written in the form of the Einstein equations again for the induction Ricci tensor defined through the macroscopic metric.*

The macroscopic stress tensor includes, in addition to the averaged matter, the correlation tensor terms for the geometric correction of the averaged matter. However, in their geometrical meaning equations (2.104) are not Riemannian ones, as is the case in GR. This fact can be thought of as reflecting a different geometry which underlies the macroscopic gravitation.

The *macrovacuum* equations of the theory

$$M_{\alpha\beta} = -Q_{\alpha\beta} \ , \quad Z^{\alpha}{}_{\mu\nu\beta}\bar{g}^{\mu\nu} = -\bar{g}^{\alpha\varepsilon}Q_{\beta\varepsilon}, \tag{2.107}$$

($\bar{g}^{\alpha\beta}Q_{\alpha\beta} = 0$) state the *Ricci non-flat* character of the macroscopic gravitation in the absence of the averaged matter. This is a non-trivial geometrical and physical fact.

In the case when all the correlation function vanish and in addition $U^{\alpha\beta} = 0$ (no metric correlations), equations (2.104) (and (2.107) as well) become the usual Einstein equations for the macroscopic metric $G_{\alpha\beta}$ (for the macrovacuum $M_{\alpha\beta}(G) = 0$), with the tensor $< t^{\alpha(micro)}_{\beta} >$ on their right hand side. It is this tensor that is usually taken as the perfect fluid tensor in cosmological applications of Einstein's GR. These remarks clarify the range of validity of Einstein's equations in cosmology.

It is remarkable that the macroscopic Einstein equations can be put in the form of Hooke's law [266]

$$(M^{\alpha}{}_{\mu\nu\beta} - Z^{\alpha}{}_{\mu\nu\beta})\bar{g}^{\mu\nu} = -\kappa(< \mathrm{t}^{\alpha(micro)}_{\beta} > -\frac{1}{2}\delta^{\alpha}_{\beta} < \mathrm{t}^{\varepsilon(micro)}_{\varepsilon} >), \tag{2.108}$$

with $M^{\alpha}{}_{\mu\nu\beta} - Z^{\alpha}{}_{\mu\nu\beta}$ and $\bar{g}^{\alpha\beta}$ playing the rôle of the tensor of elastic moduli and the deformation tensor of the macroscopic space-time, respectively.

2.5.4 Classification of macroscopic space-times

The analysis of (2.96) and the integrability conditions of $D_{\bar{\Omega}}G^{\alpha\beta} = 0$,

$$G^{\alpha\beta}\mathrm{M}^{\gamma}{}_{\beta} + G^{\gamma\beta}\mathrm{M}^{\alpha}{}_{\beta} = 0, \tag{2.109}$$

by means of the invariant algebraic classification [163] of the symmetric tensor $\bar{g}^{\alpha\beta}$ through the macroscopic metric tensor $G^{\alpha\beta}$ yields a theorem which makes of the contents of this section:

Due to the relations (2.96) and (2.109) there is the following correspondence between the algebraic structure of the averaged tensor $\overline{g}^{\alpha\beta}$ and the Riemannian curvature tensor $M^{\alpha}{}_{\beta\rho\sigma}$:

A All four algebraic types A_1, A_2, A_3 and B without generations lead to the existence of four linearly independent covariantly constant vectors, so that $M^{\alpha}{}_{\beta\rho\sigma} = 0$.

B For all cases with generations, except $A_1[(111,1)]$ and $A_3[(11,2)]$, the non-trivial components of $M^{\alpha}{}_{\beta\rho\sigma}$ can be expressed through the Ricci tensor $M_{\alpha\beta}$ and all the space-times are algebraically special and reducible; different types of them depend on the algebraic types of $\overline{g}^{\alpha\beta}$.

C For the case $A_3[(11,2)]$, when

$$\overline{g}^{\alpha\beta} = \mathrm{const} \cdot G^{\alpha\beta} - 2\xi^{\alpha}\xi^{\beta}, \tag{2.110}$$

the space-time has a covariantly constant null vector ξ^{α} ($\xi^{\alpha}{}_{|\beta} = 0$), i.e. $G^{\alpha\beta}$ is a pp-wave metric [163].

For the case $A_1[(111,1)]$, when

$$\overline{g}^{\alpha\beta} = \mathrm{const} \cdot G^{\alpha\beta}, \tag{2.111}$$

there are no conditions on the curvature tensor and the space-time structure.

The above correspondence scheme provides us with a classification of possible macroscopic space-times. The algebraic structure of $M^{\alpha}{}_{\beta\rho\sigma}$, the tensor which plays a rôle of the gravitational induction, depends on the algebraic structure of the averaged microscopic metric, i.e. the structure of the microscopic space-time. As a rule, the more symmetric the original manifold the more symmetric the macroscopic manifold.

In the case [A], the microscopic space-time has no symmetries and the induction vanishes duly. In the cases [B] and [C], the symmetries of the microscopic space-time translate into a non-trivial structure of the macroscopic space-time being reducible as a result of the averaging.

2.5.5 Perturbation theory in MG

Let us apply the high frequency approximation, due to Isaacson [147, 148], to the theory of MG. This will also enable us to study the self-consistency of the theory. In this approach we assume that there exists a family of geometries for which the microscopic metric can be written as

$$g_{\alpha\beta} = G_{\alpha\beta} + h_{\alpha\beta}, \tag{2.112}$$

where, $G_{\alpha\beta}$ is the macroscopic metric (or, the background metric [147]) which changes slowly over a characteristic macroscopic length L, and $h_{\alpha\beta}$ is the high frequency oscillating part over a microscopic length λ, $\lambda \ll d \ll L$, d being a typical diameter of the averaging region.

In [147, 148] the Brill-Hartle (BH) [45] space-time averaging was carried out over the background regions with $[-\det(G_{\alpha\beta})]^{\frac{1}{2}}$ the integration measure and $\mathcal{A}^{\alpha'}_{\beta} = g^{\alpha'}_{\beta}$. A coordinate system (2.112) can be chosen, without loss of generality, as the system $\mathcal{A} = e_{\alpha'}dx^{\alpha}$ and the averaging scheme applied.

The averaging over the microscopic space-time regions is done here with the measure $[-\det(g_{\alpha\beta})]^{\frac{1}{2}}$, and $\mathcal{A}^{\alpha'}_{\beta} = \mathcal{W}^{\alpha'}_{\beta}$.

The space-time averages of $h_{\alpha\beta}$ and its derivatives vanish

$$< h_{\alpha\beta} >_{BH} = 0, \quad < \partial_\gamma h_{\alpha\beta} >_{BH} = 0, \ldots, \tag{2.113}$$

$$< h_{\alpha\beta} > = 0, \quad < \partial_\gamma h_{\alpha\beta} > = 0, \ldots, \tag{2.114}$$

but averaging out the microscopic metric (2.112) gives

$$< g_{\alpha\beta} >_{BH} = G_{\alpha\beta} - \frac{1}{2} < h_{\alpha\beta} h^{\mu}_{\mu} >_{BH} - \frac{1}{8} G_{\alpha\beta} < h^{\nu}_{[\nu} h^{\mu}_{\mu]} >_{BH} + \ldots, \tag{2.115}$$

$$< g_{\alpha\beta} > \equiv \overline{g}_{\alpha\beta} = G_{\alpha\beta}. \tag{2.116}$$

The extra terms in the BH average above as compared to $< g_{\alpha\beta} >$, appear due to

$$\det(g_{\alpha\beta}) = \det(G_{\alpha\beta})[1 + h^{\alpha}_{\alpha} + \frac{1}{2} h^{\alpha}_{[\alpha} h^{\beta}_{\beta]} + \frac{1}{3!} h^{\alpha}_{[\alpha} h^{\beta}_{\beta} h^{\delta}_{\delta]} + \det(h_{\alpha\beta})].$$

Averaging the inverse microscopic metric

$$g^{\alpha\beta} = G^{\alpha\beta} - G^{\alpha\rho} G^{\beta\sigma} h_{\rho\sigma} + G^{\alpha\rho} G^{\varepsilon\sigma} G^{\beta\kappa} h_{\rho\sigma} h_{\kappa\varepsilon} - \ldots$$

gives

$$< g^{\alpha\beta} >_{BH}= \quad G^{\alpha\beta} + \frac{1}{2} < h^{\alpha\beta} h^{\mu}_{\mu} >_{BH} - \frac{1}{8} G^{\alpha\beta} < h^{\nu}_{[\nu} h^{\mu}_{\mu]} >_{BH} + \ldots, \qquad (2.117)$$

$$< g^{\alpha\beta} >\equiv \quad \overline{g}^{\alpha\beta} = G^{\alpha\beta} + < h^{\alpha\epsilon} h^{\beta}_{\epsilon} > - \ldots. \qquad (2.118)$$

Formulae (2.116) and (2.117), (2.118) clear up the origin of $U^{\alpha\beta}$.

Now, let us consider the second order corrected metric

$$g_{\alpha\beta} = G_{\alpha\beta} + h_{\alpha\beta} + j_{\alpha\beta}, \qquad (2.119)$$

where, the amplitudes of $g_{\alpha\beta}$, $h_{\alpha\beta}$ and $j_{\alpha\beta}$ are taken as $g_{\alpha\beta} \sim \mathcal{O}(1)$, $h_{\alpha\beta} \sim \mathcal{O}(a)$, $j_{\alpha\beta} \sim \mathcal{O}(a^2)$, with $a \ll 1$. Expanding all microscopic objects in powers of a up to a^4, and keeping only dominant terms for each object yield

$$g = \underset{\mathcal{O}(1)}{G} + \underset{a}{h} + \underset{a^2}{j}\,, \qquad (2.120)$$

$$g^{-1} = \underset{\mathcal{O}(1)}{G^{-1}} + \underset{a}{G^{-2}h} + \underset{a^2}{(g^{-1})^{(2)}(h,^2 j)} + \underset{a^3}{(g^{-1})^{(3)}(h^3, hj)} + \underset{a^4}{(g^{-1})^{(4)}(h^4, h^2 j, j^2)}\,, \qquad (2.121)$$

$$\gamma = \underset{L^{-1}}{\overline{\mathcal{F}}} + \underset{a\lambda^{-1}}{\gamma^{(1)}(h)} + \underset{a^2\lambda^{-1}}{\gamma^{(2)}(h^2, j)} + \underset{a^3\lambda^{-1}}{\gamma^{(3)}(h^3, hj)} + \underset{a^4\lambda^{-1}}{\gamma^{(4)}(h^4, h^2 j, j^2)}\,, \qquad (2.122)$$

$$r = \underset{L^{-2}}{M} + \underset{a\lambda^{-2}}{r^{(1)}(h)} + \underset{a^2\lambda^{-2}}{r^{(2)}(h^2, j)} + \underset{a^3\lambda^{-2}}{r^{(3)}(h^3, hj)} + \underset{a^4\lambda^{-2}}{r^{(4)}(h^4, h^2 j, j^2)}\,. \qquad (2.123)$$

Symbolically we have used here $g^{-1}g = 1$, $\gamma = g^{-1}\partial g$ for Christoffel symbols, $r = g^{-1}\partial^2 g + g^{-2}(\partial g)^2$ for the Riemann tensor, $\overline{\mathcal{F}} = G^{-1}\partial G$ for the connection coefficients and $M = G^{-1}\partial^2 G + G^{-2}(\partial G)^2$ for the curvature of the background.

Now averaging the above expansions (having in mind (2.119), with h of frequency ω and j of ω and 2ω [148]) one obtains

$$< g >= \underset{\mathcal{O}(1)}{G} \qquad (2.124)$$

$$< g^{-1} >= \underset{\mathcal{O}(1)}{G^{-1}} + \underset{a^2}{< (g^{-1})^{(2)}(h^2) >} + \underset{a^4}{< (g^{-1})^{(4)}(h^2, h^2 j, j^2) >}\,, \qquad (2.125)$$

$$< \gamma >= \underset{L^{-1}}{\overline{\mathcal{F}}} + \underset{a^2 L^{-1}}{< \gamma^{(2)}(h^2) >} + \underset{a^3\lambda^{-1}}{< \gamma^{(3)}(hj) >} + \underset{a^4 L^{-1}}{< \gamma^{(4)}(h^4, h^2 j, j^2) >}\,, \qquad (2.126)$$

$$< r >= \underset{L^{-2}}{M} + \underset{a^2\lambda^{-2}}{< r^{(2)}(h^2) >} + \underset{a^3\lambda^{-1}L^{-1}}{< r^{(3)}(hj) >} + \underset{a^4\lambda^{-2}}{< r^{(4)}(h^4, h^2j, j^2) >} . \tag{2.127}$$

Throughout, the properties $< h^2 > \neq 0$, $< h\partial h >= 0$, $< (\partial h)^2 > \neq 0$, $< j >= 0$, $< \partial j >= 0$, $< jh >= 0$, $< j\partial h > \neq 0$, $< h\partial j > \neq 0$, etc., have been used. Note specially, the leading terms for $< \gamma^{(2)} >$, $< \gamma^{(4)} >$ and $< r^{(3)} >$ above, as compared with (2.122) and (2.123). The formula (2.127) clears up the origin of (2.77) and (2.79). Further using (2.120)-(2.123) and (2.124)-(2.127) one can now estimate the order of magnitude in the structure equations for the correlation forms (2.87) and (2.89)

$$\underset{a^2\lambda^{-2}L^{-1}+a^3\lambda^{-1}L^{-2}+a^4\lambda^{-2}L^{-1}}{\partial Z} = \underset{a^4\lambda^{-2}L^{-1}}{-2Y} + \underset{a^3\lambda^{-3}+a^4\lambda^{-2}L^{-1}}{2 < \gamma r > -2 < \gamma >< r >} \tag{2.128}$$

$$\begin{aligned} \underset{a^4\lambda^{-2}L^{-2}}{\partial Y} = & \underset{a^4\lambda^{-4}}{-3X + 6Z \wedge Z} \\ & \underset{a^4\lambda^{-4}}{-3 < \gamma\gamma r > +3 < \gamma\gamma >< r > -6 < \gamma >< \gamma r >} \\ & \underset{a^4\lambda^{-4}}{+6 < \gamma >< \gamma >< r >} \end{aligned} \tag{2.129}$$

Note that Y has no terms in the power a^3.

The similar estimation for the splitting rules gives:

$$< \gamma g > - < \gamma >< g > \simeq a^2L^{-1} + a^3\lambda^{-1} + a^4L^{-1}, \tag{2.130}$$

$$< \gamma\gamma g > - < \gamma\gamma >< g > \simeq a^2L^{-2} + a^3\lambda^{-1}L^{-1} + a^4\lambda^{-2}, \tag{2.131}$$

$$\underset{a^2\lambda^{-2}+a^3\lambda^{-1}L^{-1}+a^4\lambda^{-2}}{< rg > - < r >< g >} = \underset{a^2\lambda^{-2}+a^3\lambda^{-1}L^{-1}+a^4\lambda^{-2}}{2Z < g >} \tag{2.132}$$

As can be seen, the leading terms in powers of a are $\frac{a^2}{\lambda^2}$, $\frac{a^3}{\lambda^3}$ and $\frac{a^4}{\lambda^4}$. Due to (2.130) and (2.131) the splitting rules (2.91) (and (2.92)) and (2.98) (and (2.99)) take place always in powers of a^2, a^3 and a^4, and their error is always less that of the averaging, λL^{-1}. The expressions (2.128), (2.129) and (2.130) manifest the self-consistency of the structure equations for Z and Y and the splitting rule for $< rg >$.

The high-energy frequency analysis can be shown to be valid also for the higher order corrections to (2.119).

2.5.6 The correspondence principle

In the macroscopic field equations the macrovacuum source is defined in terms of the correlation tensor Z, i.e. $Q_{\alpha\beta} = 2Z^{\delta}{}_{\alpha[\epsilon}{}^{\epsilon}{}_{\delta\beta]}$. In order to discover the physical meaning of $Q_{\alpha\beta}$, let us take the high frequency limit of macroscopic equations (2.107) up to a^2 for the metric (2.112), making use of the results of the previous section. To be explicit let us assume that the macroscopic metric is a high frequency perturbation superposed on the microscopic metric.

The tensor $Q_{\alpha\beta}$ is found in the following form

$$Q^{HF}_{\alpha\beta} =< \delta\gamma^{\epsilon}{}_{\alpha\sigma}\delta\gamma^{\sigma}{}_{\epsilon\beta} - \delta\gamma^{\epsilon}{}_{\alpha\beta}\delta\gamma^{\sigma}{}_{\epsilon\sigma} >, \tag{2.133}$$

where

$$\delta\gamma^{\alpha}{}_{\beta\gamma} \equiv \gamma^{\alpha}{}_{\beta\gamma} - \overline{\mathcal{F}}^{\alpha}{}_{\beta\gamma} = \frac{1}{2}G^{\alpha\delta}(h_{\beta\delta|\gamma} + h_{\gamma\delta|\beta} + h_{\beta\gamma|\delta}). \tag{2.134}$$

After the calculation of $Q^{HF}_{\alpha\beta}$ in Lorentz gauge $h^{\mu\nu}{}_{|\nu} = 0$, $h^{\alpha}_{\alpha} = 0$, we get the following result using the averaging rules of [147, 148] (up to a^2 order the Brill-Hartle averaging is equivalent to the one adopted here)

$$Q^{HF}_{\alpha\beta} = \frac{1}{4} < h^{\mu\nu}{}_{|\alpha}h_{\mu\nu|\beta} >= \kappa T^{GW}_{\alpha\beta}, \tag{2.135}$$

where, $T^{GW}_{\alpha\beta}$ is Isaacson's stress-energy tensor for gravitational waves. The gravitational stress-energy tensor for the macrovacuum is constructed from the correlation tensor $Z^{\alpha}{}_{\beta[\gamma}{}^{\mu}{}_{\underline{\nu}\sigma]}$. As was already shown, the Ricci tensor $M_{\alpha\beta}$ describes the background and within the accuracy of the approximation (cf. (2.127)) we have $Q_{\alpha\beta}G^{\alpha\beta} = 0$ and $M_{\alpha\beta}G^{\alpha\beta} = 0$. Therefore the macrovacuum equations in the high frequency limit become

$$M_{\alpha\beta} = -\kappa T^{GW}_{\alpha\beta}, \tag{2.136}$$

which is the same as the equations obtained by Isaacson (see equation (3.1 b) of [147]).

The conservation of $T^{GW}_{\alpha\beta}$ is ensured by the averaged contracted Bianchi identities and reads (again within this approximation scheme)

$$G^{\alpha\epsilon}M_{\alpha\beta|\epsilon} = 0. \tag{2.137}$$

The fact that the macrovacuum equations $M_{\alpha\beta} = -Q_{\alpha\beta}$ become Isaacson's in the high frequency limit acts as the correspondence principle for the theory of Macroscopic Gravity, similarly as the Newtonian limit for GR. This demonstrates that the macroscopic field equations have the correct high frequency limit (2.136) for the case of the macrovacuum. As a result, the correlation tensor $Q_{\mu\nu}$ which serves as the macrovacuum source and is equal to $\kappa T^{(GW)}_{\alpha\beta}$ in the high-frequency limit, has been established to describe the energy momentum of the macrovacuum gravitational field. Thus, in the macroscopic theory of gravity there is *the tensor object for the energy momentum of macroscopic gravitation*

$$-\kappa T^{\alpha(grav)}_{\beta} = (Z^{\alpha}{}_{\mu\nu\beta} - \frac{1}{2}\delta^{\alpha}_{\beta} Q_{\mu\nu})\bar{g}^{\mu\nu}. \tag{2.138}$$

Due to the conservation of the macroscopic stress tensor $T^{\alpha(macro)}_{\beta}$ the following equation of motion for the averaged matter holds

$$\kappa < \mathbf{t}^{\alpha(micro)}_{\beta} >_{|\alpha} = (Z^{\varepsilon}{}_{\mu\nu\beta|\varepsilon} - \frac{1}{2} Q_{\mu\nu|\beta})\bar{g}^{\mu\nu} \equiv -\kappa T^{\alpha(grav)}_{\beta|\alpha}. \tag{2.139}$$

Indeed, a simple consideration shows that after being summed up in averaging out over a space-time region the gravitational field energy becomes localizable and can be treated, as a consequence, by a tensor object. Clearly this fact has first been established in [148] within the high frequency approximation for GR. The full macroscopic theory provides a general solution to this problem.

2.6 Averaged Lagrangians

Averaging of Lagrangians (i.e. averaging a scalar density) can be thought of as another approach for averaging (see [180, 63, 181, 252]).

A problem which we wish to study now is whether it is possible to obtain the field equations of MG from a variational principle. In order to carry this project out we have to formulate MG in the Lagrangian framework. The field equations of MG above were obtained by averaging out directly the field equations of GR. To obtain them from the variational principle, we will follow the way which is to average out the (microscopic) action. This will provide us with the averaged (macroscopic) action which upon the variation of the macroscopic field variables, namely, the macroscopic metric should yield the field equations of MG.

We will denote $\mathcal{M}$ for the original (microscopic) manifold and $\bar{\mathcal{M}}$ for the averaged manifold on which all averaged objects live (see section 2.5.2). Both manifold have the same topology and differentiable structure so $\partial\mathcal{M} = \partial\bar{\mathcal{M}}$.

2.6.1 Splitting rules and averaging of the measure

First, we give the necessary splitting rules in order to be able to carry out the averaging of Lagrangians. We need two splitting rules. The first is the contracted version of the splitting rule for the average of the metric times the curvature $< \mathbf{rg} >$ which follows from the rule (2.100) (see also (2.101) and (2.102)),

$$< \mathbf{g}^{\alpha\beta}\mathbf{r}_{\alpha\beta} >= \bar{g}^{\alpha\beta}M_{\alpha\beta} - \bar{g}^{\alpha\beta}Q_{\alpha\beta}. \tag{2.140}$$

The second splitting rule is that for the product of the metric and two connections $< \mathbf{g}\mathcal{F}\mathcal{F} >$ (see equation (2.99)), contracted again, namely,

$$< \mathbf{g}^{\beta\gamma}\mathcal{F}^{\alpha}{}_{\epsilon[\gamma}\mathcal{F}^{\epsilon}{}_{\underline{\beta}\alpha]} >= \bar{g}^{\beta\gamma} < \mathcal{F}^{\alpha}{}_{\epsilon[\gamma}\mathcal{F}^{\epsilon}{}_{\underline{\beta}\alpha]} > \tag{2.141}$$

together with

$$< \mathcal{F}^{\alpha}{}_{\epsilon[\gamma}\mathcal{F}^{\epsilon}{}_{\underline{\beta}\alpha]} >= \frac{1}{2}Q_{\beta\gamma} + \bar{\mathcal{F}}^{\alpha}{}_{\epsilon[\gamma}\bar{\mathcal{F}}^{\epsilon}{}_{\underline{\beta}\alpha]}, \tag{2.142}$$

which is only the definition of $Q_{\alpha\beta}$ (see section 2.5.2).

Now we need to consider the effect of averaging on the measure of integration. This is a question of its own value as concerning the measure of integration of average tensor fields within the space-time averaging procedure applied (we follow here [268]). And here it is the question to be necessarily addressed when we come to averaging out of the action itself.

From (2.76) and $\mathrm{Q}^{\alpha}{}_{\alpha} = 0$ it follows that $\mathrm{M}^{\alpha}{}_{\alpha} = 0$ (equi-affinity of the geometry with curvature tensor $M^{\alpha}{}_{\beta\gamma\delta}$) and consequently

$$\bar{\Omega}^{\alpha}{}_{\alpha} = \mathrm{d}\ln\sqrt{-G}. \tag{2.143}$$

We should proceed further to derive the 1−form $\bar{\Omega}^{\alpha}{}_{\alpha}$ in terms of the averaged $f = \sqrt{-g}$, to clear up the relation between $\bar{f}$ and $\sqrt{-G}$.

Let us re-write the equi-affinity condition,

$$\epsilon_{\alpha\beta\gamma\delta;\varepsilon} = 0, \tag{2.144}$$

where, ϵ is the volume 4−form, $\epsilon_{\alpha\beta\gamma\delta} = f e_{\alpha\beta\gamma\delta}$ ($e_{\alpha\beta\gamma\delta}$ is the Levi-Civita symbol), with $f = \sqrt{-g}$ for the case of Riemannian geometry, as

$$\mathrm{d}f = f\omega^{\alpha}{}_{\alpha}. \tag{2.145}$$

The bilocal extension of this equation is

$$\text{đ}\mathbf{f} = \mathbf{f}\Omega^{\alpha}{}_{\alpha}. \tag{2.146}$$

Here đ is an exterior bilocal derivative [264] and the bilocal extension $\mathbf{f}$ of $f = \sqrt{-g}$ is given by

$$\mathbf{f} = f' \det(\mathcal{A}^{\mu'}{}_{\alpha}), \tag{2.147}$$

where, $\det(\mathcal{A}^{\mu'}{}_{\alpha})$ is a determinant of $\mathcal{A}^{\mu'}{}_{\alpha}$ and it is a bidensity of weight -1 at x' and $+1$ at x. The transformation law of the determinant is $\det(\tilde{\mathcal{A}}^{\mu'}{}_{\alpha}) = (J')^{-1}\det(\mathcal{A}^{\mu'}{}_{\alpha})J$, with Jacobian of a coordinate transformation $x \to \tilde{x}'(x)$, $J = \det(\partial x^{\alpha}/\partial \tilde{x}^{\beta})$. So, $\mathbf{f}$ is a scalar at x' and a density of weight $+1$ at x. Now, upon averaging out this condition (2.146) we obtain directly

$$\mathrm{d}\bar{f} = \bar{f}\bar{\Omega}^{\alpha}{}_{\alpha} + \mathrm{K}, \tag{2.148}$$

where, K is a correlation 1−form density of weight +1, namely,

$$< \mathbf{f}\Omega^{\alpha}{}_{\alpha} > = \bar{f}\bar{\Omega}^{\alpha}{}_{\alpha} + \mathrm{K}. \tag{2.149}$$

The last equation can be conveniently written in the following form

$$\bar{\Omega}^{\alpha}{}_{\alpha} = \mathrm{d}\ln\bar{f} - \mathrm{K}/\bar{f}, \tag{2.150}$$

where, $\mathrm{K}/\bar{f}$ is now a 1−form. The equation for $\mathrm{K}/\bar{f}$ follows directly from (2.143) and (2.150)

$$\mathrm{d}(\mathrm{K}/\bar{f}) = 0, \tag{2.151}$$

which yields locally

$$\mathrm{K}/\bar{f} = \mathrm{d}b \quad \text{i.e.,} \quad \mathrm{K} = \bar{f}\mathrm{d}b. \tag{2.152}$$

The correlation 1−form K does vanish if db = 0 in some (particular) coordinate system since $\bar{f} \neq 0$. To show it we use the proper coordinate system (see section 2.5.1), $x^\alpha = \phi^i$, in which

$$\mathcal{A}^{\alpha'}{}_{\beta} = \frac{\partial x^\alpha}{\partial \phi^i}\frac{\partial \phi^i}{\partial x^\beta} = \delta^i{}_j, \tag{2.153}$$

$\mathcal{W}^{\gamma'}{}_{\sigma;\gamma'} = 0$, $\mathcal{W}^{\alpha}{}_{\beta';\alpha} = 0$ and

$$\Omega^\alpha{}_\alpha = \omega^\alpha{}_\alpha = \mathrm{d}\ln\sqrt{-g} = 0 \tag{2.154}$$

(see [10]). Taking into account now the definition of K (2.149) one can show easily that b = const, and $\bar{\Omega}^\alpha{}_\alpha$ has consequently the form

$$\bar{\Omega}^\alpha{}_\alpha = \mathrm{d}\ln\bar{f}. \tag{2.155}$$

Thus the solutions of (2.143) and (2.155) imply that

$$\bar{f} = \mathrm{const}\cdot\sqrt{-G}. \tag{2.156}$$

This result means that upon averaging, the integration measure becomes

$$\bar{f}d^4x = \mathrm{const}\cdot\sqrt{-G}d^4x. \tag{2.157}$$

Now we will show that the average of the product of any geometric object p^α_β by the measure function f can always be split out without correlation. Consider the splitting rule

$$<\mathbf{p}^\alpha_\beta\mathbf{f}> = \bar{p}^\alpha_\beta\bar{f} + C^\alpha_\beta, \tag{2.158}$$

where C^α_β is the correlation object. This rule has a tensorial character as it can be shown, by a definition of the object $C^\alpha_\beta/\bar{f}$ that is a tensor. Considering now again the proper coordinate system (2.153), in which f = const due to f = const and $\det(\mathcal{A}^{\mu'}{}_\alpha) = 1$, one arrives at $C^\alpha_\beta = 0$ and the rule (2.158) reads

$$<\mathbf{p}^\alpha_\beta\mathbf{f}> = \bar{p}^\alpha_\beta\bar{f}. \tag{2.159}$$

2.6.2 Variation and averaging

Considering a variation of an averaged quantity a natural question arises about commutativity, or not, of taking variation and averaging. This is a question of great importance for the setting of the variational principle for the averaged Lagrangians.

Consider a geometric object field ϕ^A to be variated and evaluate now the variation of the average object $\delta\bar{\phi}^A$ due to the variation of the metric $\delta g_{\mu\nu}$,

$$\delta\bar{\phi}^A =< \delta\Phi^A > -\frac{1}{2} < g^{\mu\nu}\delta g_{\mu\nu} > \bar{\phi}^A + \frac{1}{2} < \Phi^A g^{\mu\nu}\delta g_{\mu\nu} >, \tag{2.160}$$

where we used the well known expression for variation of the measure function $\sqrt{-g}$,

$$\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}g^{\mu\nu}\delta g_{\mu\nu}, \tag{2.161}$$

and the variation δV_Σ in the second term is given by

$$\delta V_\Sigma = \int_\Sigma g^{\mu\nu}\delta g_{\mu\nu}\sqrt{-g}d^4x. \tag{2.162}$$

These two terms in (2.160) reflect *the change in the volume value* due to the metric variation, since the volume value is defined in terms of the metric measure function $\sqrt{-g}$. The formula (2.160) shows thereby that a variation in the average value of the geometric object $\bar{\phi}^A$ has two parts: one is due to the variation of the object itself and the second one is due to the change of the value of the averaging region(s). The space-time procedure formulated in section 2.5.1 includes the special prescription for a unique covering (coordination) of the manifold to be averaged out by averaging regions, which provides us with a unique local average tensor field with a given constant value of the averaging regions' volume. So it is natural to keep this prescription valid and require the variation of an average geometric object not to include the terms responsible for the change of volume and, consequently, for change in the value of the averaged tensor field. This requirement is a splitting rule in fact

$$< \Phi^A g^{\mu\nu}\delta g_{\mu\nu} >= \bar{\phi}^A < g^{\mu\nu}\delta g_{\mu\nu} >, \tag{2.163}$$

which means that the variation $\delta\sqrt{-g}$ in the metric measure function is a small arbitary function (infinitesimal variation) which can be taken out of the averaging sign. With the rule (2.163) we have from (2.160) the remarkable formula for commutation of variation and averaging

$$\delta\bar{\phi}^A = < \delta\Phi^A > . \tag{2.164}$$

This formula states that the ordinary variational analysis is valid for the variational problems posed in the averaged field theories, in particular, in macroscopic gravity.

The formula (2.164) results also in the preservation of the asymptotic conditions for the variation of field ϕ^A at spatial infinity

$$\delta\phi^A{}_{|\partial\mathcal{M}} = 0. \tag{2.165}$$

With (2.165) valid it is easy to show from (2.164) that

$$\delta\bar{\phi}^A{}_{|\partial\bar{\mathcal{M}}} = 0. \tag{2.166}$$

Indeed, for the supporting point $x \in \partial\mathcal{M}$ let us choose an averaging region Σ as $\Sigma = \Delta t \cdot \Delta S$ with $\Delta S \subset \partial\mathcal{M}$. Taking into account (2.165), the evaluation of $< \delta\Phi^A > = \delta\bar{\phi}^A(x), x \in \partial\bar{\mathcal{M}}$ for the region gives zero for the average value and results in (2.166).

In addition to (2.163) one can write a generalized rule for taking the metric variation out of the averaging sign. For an arbitrary function of field variables $f(\Phi^A)$

$$< f(\Phi^A)\delta g_{\alpha\beta} > = < f(\Phi^A) > \delta\bar{g}_{\alpha\beta} \ , \ < f(\Phi^A)\delta g^{\alpha\beta} > = < f(\Phi^A) > \delta\bar{g}^{\alpha\beta} \tag{2.167}$$

which are valid, again, due to the arbitrariness and infinitesimal character of the metric variation. The variations in a microscopic metric $\delta g_{\alpha\beta}$, $\delta g^{\alpha\beta}$, macroscopic metric $\delta G_{\alpha\beta}$, $\delta G^{\alpha\beta}$ and the averaged metric $\delta\bar{g}_{\alpha\beta}$, $\delta\bar{g}^{\alpha\beta}$ are the same in order of magnitude (see section 2.5.5)

$$\delta g_{\alpha\beta} \simeq \delta G_{\alpha\beta} \simeq \delta\bar{g}_{\alpha\beta}, \tag{2.168}$$

the same identities holding for contravariant metrics.

On the basis of (2.167) let us now derive a useful relation between the variations in the metric and in the averaged metric. Consider the average of the definition of the inverse metric

$$< g^{\alpha\beta} g_{\sigma\beta} > = \delta^\alpha_\sigma, \tag{2.169}$$

and take its variation, which gives

$$< \delta g^{\alpha\beta} g_{\sigma\beta} > + < g^{\alpha\beta} \delta g_{\sigma\beta} >= 0. \tag{2.170}$$

Taking into account the rule (2.167) one arrives immediately at the relation

$$\delta \bar{g}^{\alpha\beta} = -G^{\alpha\epsilon} \bar{g}^{\beta\rho} \delta G_{\epsilon\rho}, \tag{2.171}$$

and the inverse one

$$\delta G_{\alpha\beta} = -G_{\alpha\epsilon} \bar{g}^{-1}_{\beta\rho} \delta \bar{g}^{\epsilon\rho}. \tag{2.172}$$

2.6.3 Averaging out the Lagrangians

We are going to average out Hilbert's form of the action as well as the Einstein one, which read respectively

$$I_H = -\frac{1}{2\kappa c} \int_{\mathcal{M}} g^{\alpha\beta} r_{\alpha\beta} \sqrt{-g} d^4x, \tag{2.173}$$

$$I_E = \frac{1}{2\kappa c} \int_{\mathcal{M}} 2 g^{\alpha\beta} \Gamma^{\mu}{}_{\sigma[\alpha} \Gamma^{\sigma}{}_{\underline{\beta}\mu]} \sqrt{-g} d^4x. \tag{2.174}$$

The integration in the above formulae is over *all space* $\mathcal{M}$.

An action is a scalar and a Lagrangian density is used to derive the field equations from a variational principle. In order to meaningfully speak of a possibility to average out an action we have to define their bilocal extensions. This can be done analogously as already discussed in this section (see section 2.5.1). Namely, the object to be now averaged out according to the space-time averaging procedure applied, should be expressed as a bilocal extension of the Lagrangian scalar. We will further consider in detail the space-time averaging of Hilbert's action. The bilocal extension of the above action (2.173) has now the following form

$$\mathcal{I}_H(x) = -\frac{1}{2\kappa c} \int_{\mathcal{M}} \mathbf{g}^{\alpha\beta}(x,x') \mathbf{r}_{\alpha\beta}(x,x') \mathbf{f}(x,x') d^4x'. \tag{2.175}$$

The average value of (2.175) is

$$\bar{I}_H = \frac{1}{V_\Sigma} \int_\Sigma \mathcal{I}_H(x) \sqrt{-g} d^4x = -\frac{1}{2\kappa c} \frac{1}{V_\Sigma} \int_\Sigma \sqrt{-g} d^4x \int_{\mathcal{M}} \mathbf{g}^{\alpha\beta} \mathbf{r}_{\alpha\beta} \mathbf{f} d^4x'. \tag{2.176}$$

Now taking into account that $V_\Sigma = \text{const}$ we can change the order of integration

$$\bar{I}_H = -\frac{1}{2\kappa c}\int_{\mathcal{M}} d^4x' \frac{1}{V_\Sigma}\int_\Sigma \mathbf{g}^{\alpha\beta}\mathbf{r}_{\alpha\beta}\mathbf{f}\sqrt{-g}d^4x. \tag{2.177}$$

The second factor in the formula above is the average value of the scalar curvature

$$\frac{1}{V_\Sigma}\int_\Sigma \mathbf{g}^{\alpha\beta}\mathbf{r}_{\alpha\beta}\mathbf{f}d^4x = <\mathbf{g}^{\alpha\beta}\mathbf{r}_{\alpha\beta}\mathbf{f}>, \tag{2.178}$$

and taking into account (2.156), (2.157) and (2.159) we obtain the following result

$$\bar{I}_H = -\frac{1}{2\kappa c}\int_{\bar{\mathcal{M}}} <\mathbf{g}^{\alpha\beta}\mathbf{r}_{\alpha\beta}> \sqrt{-G}d^4x. \tag{2.179}$$

The integration in the above formula is over *all space* $\bar{\mathcal{M}}$.

Now using the splitting rule (2.140) we get the following form of the averaged Hilbert action

$$\bar{I}_H(x) = -\frac{1}{2\kappa c}\int_{\bar{\mathcal{M}}} \bar{g}^{\alpha\beta}(M_{\alpha\beta} - Q_{\alpha\beta})\sqrt{-G}d^4x. \tag{2.180}$$

In a similar manner we can obtain the averaged form of the Einstein form of the action (2.174)

$$\bar{I}_E = \frac{1}{2\kappa c}\int_{\bar{\mathcal{M}}} 2 <\mathbf{g}^{\alpha\beta}\mathcal{F}^{\mu}{}_{\sigma[\alpha}\mathcal{F}^{\sigma}{}_{\underline{\beta}\mu]}> \sqrt{-G}d^4x, \tag{2.181}$$

which upon applying the rule (2.141) acquires the form below

$$\bar{I}_E = \frac{1}{2\kappa c}\int_{\bar{\mathcal{M}}} \bar{g}^{\alpha\beta}(2\bar{\mathcal{F}}^{\mu}{}_{\sigma[\alpha}\bar{\mathcal{F}}^{\sigma}{}_{\underline{\beta}\mu]} + Q_{\alpha\beta})\sqrt{-G}d^4x. \tag{2.182}$$

Now we show that these actions, (2.180) and (2.182), coincide up to a perfect divergence term, as it was in the microscopic case of GR. It is convenient to use again the proper coordinate system (2.153) in which $\mathcal{F}^{\alpha}{}_{\beta\gamma} = \Gamma^{i}{}_{jk}$. Writing down the difference between (2.180) and (2.182) we obtain

$$\bar{I}_H = \bar{I}_E + \frac{1}{2\kappa c}\int_{\bar{\mathcal{M}}} \bar{g}^{\alpha\beta}(-M_{\alpha\beta} - 2\bar{\Gamma}^{\mu}{}_{\sigma[\alpha}\bar{\Gamma}^{\sigma}{}_{\underline{\beta}\mu]})\sqrt{-G}d^4x. \tag{2.183}$$

Since $M^{\alpha}{}_{\beta\gamma\delta}$ is the Riemannian curvature tensor, we have

$$\bar{g}^{\alpha\beta}M_{\alpha\beta} = \bar{g}^{\alpha\beta}\bar{\Gamma}^{\gamma}{}_{\alpha\gamma,\beta} - \bar{g}^{\alpha\beta}\bar{\Gamma}^{\gamma}{}_{\alpha\beta,\gamma} + 2\bar{g}^{\alpha\beta}\bar{\Gamma}^{\mu}{}_{\sigma[\alpha}\bar{\Gamma}^{\sigma}{}_{\underline{\beta}\mu]}. \tag{2.184}$$

Upon substitution into (2.183) we are left with

$$\bar{I}_H = \bar{I}_E + \frac{1}{2\kappa c}\int_{\bar{\mathcal{M}}} \bar{g}^{\alpha\beta}(\bar{\Gamma}^{\gamma}{}_{\alpha\beta,\gamma} - \bar{\Gamma}^{\gamma}{}_{\alpha\gamma,\beta} - 4\bar{\Gamma}^{\mu}{}_{\sigma[\alpha}\bar{\Gamma}^{\sigma}{}_{\underline{\beta}\mu]})\sqrt{-G}d^4x. \qquad (2.185)$$

Rewriting the two first terms in the integrands of the above formula as

$$\sqrt{-G}\,\bar{g}^{\alpha\beta}\bar{\Gamma}^{\gamma}{}_{\alpha\gamma,\beta} = (\sqrt{-G}\,\bar{g}^{\alpha\beta}\bar{\Gamma}^{\gamma}{}_{\alpha\gamma})_{,\beta} - \bar{\Gamma}^{\gamma}{}_{\alpha\gamma}(\sqrt{-G}\,\bar{g}^{\alpha\beta})_{,\beta}, \qquad (2.186)$$

$$\sqrt{-G}\,\bar{g}^{\alpha\beta}\bar{\Gamma}^{\gamma}{}_{\alpha\beta,\gamma} = (\sqrt{-G}\,\bar{g}^{\alpha\beta}\bar{\Gamma}^{\gamma}{}_{\alpha\beta})_{,\gamma} - \bar{\Gamma}^{\gamma}{}_{\alpha\beta}(\sqrt{-G}\,\bar{g}^{\alpha\beta})_{,\gamma}, \qquad (2.187)$$

and substituting now to (2.185), we find that the only surviving terms are in the form of a total divergence, like the first terms on the right hand side of the above equations. According to Gauss' theorem such term can be written in the form of a surface integral. We can thus write

$$\bar{I}_H = \bar{I}_E + \int_{\partial\bar{\mathcal{M}}} J^{\mu}(\bar{g},\bar{\Gamma})dS_{\mu}. \qquad (2.188)$$

The total divergence term does not contribute to the variation $\delta\bar{I}_H$ due to the vanishing of variations in macroscopic metric and the average connection at spatial infinity

$$\delta G_{\alpha\beta|\partial\bar{\mathcal{M}}} = 0 \text{ and } \delta\bar{\Gamma}^{\alpha}{}_{\beta\gamma|\partial\bar{\mathcal{M}}} = 0, \qquad (2.189)$$

as resulting from (2.166), and vanishing of the variations in the corresponding microscopic quantities at spatial infinity

$$\delta g_{\alpha\beta|\partial\mathcal{M}} = 0 \text{ and } \delta\Gamma^{\alpha}{}_{\beta\gamma|\partial\mathcal{M}} = 0. \qquad (2.190)$$

2.6.4 Variational Principle

Now we will consider the variation of the action $\bar{I}_H$ (2.180), and show how one can derive from the variational principle

$$\delta\bar{I}_H = -\frac{1}{2\kappa c}\int_{\bar{\mathcal{M}}} \frac{\delta\bar{\mathcal{L}}}{\delta\Psi^A}\delta\Psi^A d^4x = 0, \qquad (2.191)$$

the macroscopic field equations. Ψ^A represent all the fields to be variated and $\bar{\mathcal{L}} \equiv \bar{L}\sqrt{-G}$ stands for the averaged Lagrangian density. For the purpose of this section we write the field equations as (this form is of course equivalent to (2.104) and (2.105))

$$\bar{g}^{\beta\varepsilon}M_{\beta\gamma} - \frac{1}{2}\delta^{\varepsilon}_{\gamma}\bar{g}^{\mu\nu}M_{\mu\nu} = -\kappa < t^{\varepsilon(micro)}_{\gamma} > +(Z^{\varepsilon}{}_{\mu\nu\gamma} - \frac{1}{2}\delta^{\varepsilon}_{\gamma}Q_{\mu\nu})\bar{g}^{\mu\nu}, \qquad (2.192)$$

since they were obtained in this form upon averaging directly (2.53), i.e.

$$< \mathbf{g}^{\beta\epsilon}\mathbf{r}_{\beta\gamma} > -\frac{1}{2}\delta^{\epsilon}_{\gamma} < \mathbf{g}^{\mu\nu}\mathbf{r}_{\mu\nu} >= -\kappa < t^{\epsilon(micro)}_{\gamma} > .$$

In the following we consider the macrovacuum case and assume $t^{\epsilon(micro)}_{\gamma} = 0$.

Under the independent variations in (2.191), the variation in the Lagrangian density $\bar{\mathcal{L}}$ reads

$$\begin{aligned} \delta\bar{\mathcal{L}} &= -\frac{1}{2\kappa c}[(M_{\alpha\beta} - Q_{\alpha\beta})\sqrt{-G}\delta\bar{g}^{\alpha\beta} + \bar{g}^{\alpha\beta}\sqrt{-G}\delta M_{\alpha\beta} - \\ &\quad \bar{g}^{\alpha\beta}\sqrt{-G}\delta Q_{\alpha\beta} + \bar{g}^{\alpha\beta}(M_{\alpha\beta} - Q_{\alpha\beta})\delta\sqrt{-G}]. \end{aligned} \quad (2.193)$$

In calculation of variations of some terms in (2.193) we should estimate the variation of $\bar{\Gamma}$. Let us derive the relevant formulae. For the variation of Christoffel symbols we have according to [250]

$$\delta\Gamma^{\lambda}_{\mu\nu} = -g^{\lambda\rho}\delta g_{\rho\sigma}\Gamma^{\sigma}_{\mu\nu} + \frac{1}{2}g^{\lambda\rho}(\delta g_{\mu\rho,\nu} + \delta g_{\nu\rho,\mu} - \delta g_{\mu\nu,\rho}). \quad (2.194)$$

Variation of $\bar{\Gamma}$ is given by the same formula since $\bar{\Gamma}$ corresponds to Riemannian geometry of $M^{\alpha}_{\beta\gamma\delta}$, i.e.

$$\delta\bar{\Gamma}^{\lambda}_{\mu\nu} = -G^{\lambda\rho}\delta G_{\rho\sigma}\bar{\Gamma}^{\sigma}_{\mu\nu} + \frac{1}{2}G^{\lambda\rho}(\delta G_{\mu\rho,\nu} + \delta G_{\nu\rho,\mu} - \delta G_{\mu\nu,\rho}). \quad (2.195)$$

On the other hand using formula (2.164) we can show that

$$\delta\bar{\Gamma}^{\lambda}_{\mu\nu} =< \delta\Gamma^{\lambda}_{\mu\nu} > . \quad (2.196)$$

Indeed, after averaging we have

$$< \delta\Gamma^{\lambda}_{\mu\nu} >=< -g^{\lambda\rho}\delta g_{\rho\sigma}\Gamma^{\sigma}_{\mu\nu} > +\frac{1}{2} < g^{\lambda\rho}(\delta g_{\mu\rho,\nu} + \delta g_{\nu\rho,\mu} - \delta g_{\mu\nu,\rho}) > . \quad (2.197)$$

Using the splitting rules (2.167) and (2.91) and (2.92) and noticing that due to (2.168), terms like $\bar{g}^{\alpha\beta}\delta\bar{g}_{\alpha\beta} = \bar{g}^{\alpha\beta}\delta G_{\alpha\beta}$ can be written as

$$\bar{g}^{\alpha\beta}\delta\bar{g}_{\alpha\beta} = G^{\alpha\beta}\delta G_{\alpha\beta} + \mathcal{O}(\delta^2), \quad (2.198)$$

so we have within the accuracy the relation (2.196), which is in agreement with (2.164).

Now we will consider each term in (2.193) separately. First let us concentrate on the fourth term. Due to the variation $\delta\sqrt{-G}$, which is

$$\delta\sqrt{-G} = \frac{1}{2}\sqrt{-G}G^{\mu\nu}\delta G_{\mu\nu}, \tag{2.199}$$

we can re-write it in the following form

$$\frac{1}{2}(\bar{g}^{\alpha\beta}M_{\alpha\beta} - \bar{g}^{\alpha\beta}Q_{\alpha\beta})G^{\mu\nu}\delta G_{\mu\nu}\sqrt{-G}. \tag{2.200}$$

In the second term we variate $\delta M_{\alpha\beta}$ (we recall that $M_{\alpha\beta}$ is the Ricci tensor for Riemannian curvature $M^{\alpha}{}_{\beta\gamma\delta}$). Using (2.195) and (2.196) and recalling the well-known expression for the variation of Ricci tensor in Riemannian geometry, we have

$$\delta M_{\alpha\beta} = \delta\bar{\Gamma}^{\varepsilon}_{\alpha\varepsilon|\beta} - \delta\bar{\Gamma}^{\varepsilon}_{\alpha\beta|\varepsilon}. \tag{2.201}$$

Thus

$$\begin{aligned} \bar{g}^{\alpha\beta}\delta M_{\alpha\beta} &= (\bar{g}^{\alpha\beta}\delta\bar{\Gamma}^{\varepsilon}{}_{\alpha\varepsilon})_{|\beta} - (\bar{g}^{\alpha\beta}\delta\bar{\Gamma}^{\varepsilon}_{\alpha\beta})_{|\varepsilon} \\ &= (\bar{g}^{\alpha\beta}\delta\bar{\Gamma}^{\varepsilon}{}_{\alpha\varepsilon})_{,\beta} - (\bar{g}^{\alpha\beta}\delta\bar{\Gamma}^{\varepsilon}_{\alpha\beta})_{,\varepsilon} \end{aligned} \tag{2.202}$$

which does not contribute upon integration over the surface "encompassing" all space (the boundary of $\bar{\mathcal{M}}$) due to (2.166).

The formula (2.171) in section 2.6.2 for the variation $\delta\bar{g}^{\alpha\beta}$ helps us to re-write the first term in (2.193) in the following form

$$-G^{\alpha\mu}\bar{g}^{\beta\nu}M_{\alpha\beta}\delta G_{\mu\nu}. \tag{2.203}$$

Now we will consider variations of the remaining terms

$$-\delta\bar{g}^{\alpha\beta}Q_{\alpha\beta} - \bar{g}^{\alpha\beta}\delta Q_{\alpha\beta}.$$

We should prove that

$$\delta\bar{g}^{\alpha\beta}Q_{\alpha\beta} + \bar{g}^{\alpha\beta}\delta Q_{\alpha\beta} = -G^{\alpha\mu}\bar{g}^{\varepsilon\sigma}Z^{\nu}{}_{\varepsilon\sigma\alpha}\delta G_{\mu\nu}. \tag{2.204}$$

Let us consider the variation $\delta Q_{\alpha\beta}$, namely,

$$\delta Q_{\alpha\beta} = 2 < \delta\Gamma^{\varepsilon}_{\alpha[\mu}\Gamma^{\mu}_{\underline{\varepsilon}\beta]} + \Gamma^{\varepsilon}_{\alpha[\mu}\delta\Gamma^{\mu}_{\underline{\varepsilon}\beta]} > -2\delta\bar{\Gamma}^{\varepsilon}_{\alpha[\mu}\bar{\Gamma}^{\mu}_{\underline{\varepsilon}\beta]} - 2\bar{\Gamma}^{\varepsilon}_{\alpha[\mu}\delta\bar{\Gamma}^{\mu}_{\underline{\varepsilon}\beta]}. \tag{2.205}$$

We will analyze in detail only the first and third terms in (2.205), the remaining pair of terms can be treated in a similar manner. Using the splitting rules (2.167) together with (2.91) and (2.92) of section 2.5.2 and (2.198), we have for the first term

$$\begin{aligned} &< \delta\Gamma^{\varepsilon}_{\alpha[\mu}\Gamma^{\mu}_{\underline{\varepsilon}\beta]} > = < -g^{\varepsilon\sigma}\delta g_{\sigma\rho}\Gamma^{\rho}_{\alpha[\mu}\Gamma^{\mu}_{\underline{\varepsilon}\beta]} > + < \frac{1}{2}g^{\varepsilon\sigma}(\delta g_{\alpha\sigma,[\mu} + \delta g_{[\mu\underline{\alpha},\underline{\sigma}} - \delta g_{\alpha[\mu,\underline{\sigma}})\Gamma^{\mu}_{\underline{\varepsilon}\beta]} > \\ &= -G^{\varepsilon\sigma}\delta G_{\sigma\rho} < \Gamma^{\rho}_{\alpha[\mu}\Gamma^{\mu}_{\underline{\varepsilon}\beta]} > + \frac{1}{2}G^{\varepsilon\sigma}(\delta G_{\alpha\sigma,[\mu} + \delta G_{[\mu\underline{\alpha},\underline{\sigma}} - \delta G_{\alpha[\mu,\underline{\sigma}})\bar{\Gamma}^{\mu}_{\underline{\varepsilon}\beta]}. \end{aligned} \tag{2.206}$$

Now taking into account (2.195) we obtain for the considered pair

$$\begin{aligned} &2 < \delta\Gamma^{\varepsilon}_{\alpha[\mu}\Gamma^{\mu}_{\underline{\varepsilon}\beta]} > -2\delta\bar{\Gamma}^{\varepsilon}_{\alpha[\mu}\bar{\Gamma}^{\mu}_{\underline{\varepsilon}\beta]} = -2G^{\varepsilon\sigma}\delta G_{\sigma\rho} < \Gamma^{\rho}_{\alpha[\mu}\Gamma^{\mu}_{\underline{\varepsilon}\beta]} > -2G^{\varepsilon\sigma}\delta G_{\rho\sigma}\bar{\Gamma}^{\rho}_{\alpha[\mu}\bar{\Gamma}^{\mu}_{\underline{\varepsilon}\beta]} \\ &= -2G^{\varepsilon\sigma}\delta G_{\sigma\rho}Z^{\rho}{}_{\alpha[\mu}{}^{\mu}{}_{\underline{\varepsilon}\beta]} = -G^{\varepsilon\sigma}\delta G_{\sigma\rho}Z^{\rho}{}_{\alpha\varepsilon\beta}, \end{aligned} \tag{2.207}$$

with the tensor $Z^{\rho}{}_{\alpha\varepsilon\beta}$ defined in section 2.5.2.

In an analogous manner we obtain for the second pair of terms

$$2 < \Gamma^{\varepsilon}_{\alpha[\mu}\delta\Gamma^{\mu}_{\underline{\varepsilon}\beta]} > -2\bar{\Gamma}^{\varepsilon}_{\alpha[\mu}\delta\bar{\Gamma}^{\mu}_{\underline{\varepsilon}\beta]} = -G^{\mu\sigma}\delta G_{\sigma\rho}Q^{\rho}{}_{\alpha\beta\mu}. \tag{2.208}$$

Therefore

$$\delta Q_{\alpha\beta} = -G^{\varepsilon\sigma}\delta G_{\sigma\rho}Z^{\rho}{}_{\alpha\varepsilon\beta} - G^{\mu\sigma}\delta G_{\sigma\rho}Q^{\rho}{}_{\alpha\beta\mu}, \tag{2.209}$$

and

$$\bar{g}^{\alpha\beta}\delta Q_{\alpha\beta} = -G^{\varepsilon\sigma}\bar{g}^{\alpha\beta}Z^{\rho}{}_{\alpha\beta\varepsilon}\delta G_{\sigma\rho} - \delta\bar{g}^{\alpha\beta}Q_{\alpha\beta}, \tag{2.210}$$

where we have used the symmetry property $Z^{\rho}{}_{\alpha\varepsilon\beta} = Z^{\rho}{}_{\alpha\beta\varepsilon}$ and the relation

$$\bar{g}^{\alpha\beta}G^{\mu\sigma}\delta G_{\rho\sigma}Q^{\rho}{}_{\alpha\beta\mu} = \delta\bar{g}^{\alpha\beta}Q_{\alpha\beta}. \tag{2.211}$$

Thus, the variation principle (2.191) has provided us with the following variation of the Lagrangian density

$$\delta\bar{\mathcal{L}} = (G^{\alpha\mu}\bar{g}^{\beta\nu}M_{\alpha\beta} - G^{\alpha\mu}\bar{g}^{\rho\sigma}Z^{\nu}{}_{\rho\sigma\alpha} - \frac{1}{2}G^{\mu\nu}(\bar{g}^{\rho\sigma}M_{\rho\sigma} - \bar{g}^{\rho\sigma}Q_{\rho\sigma})\delta G_{\mu\nu}. \tag{2.212}$$

Considering macrovacuum, we have the macroscopic field equations

$$\bar{g}^{\beta\nu}M_{\alpha\beta} - \frac{1}{2}\delta^{\nu}_{\alpha}\bar{g}^{\rho\sigma}M_{\sigma\rho} = \bar{g}^{\rho\sigma}Z^{\nu}{}_{\rho\sigma\alpha} - \frac{1}{2}\delta^{\nu}_{\alpha}\bar{g}^{\rho\sigma}Q_{\rho\sigma}. \tag{2.213}$$

Let us take into account the geometric relation

$$\bar{g}^{\rho\sigma} Q_{\sigma\varepsilon} = -\bar{g}^{\mu\nu} Z^{\rho}{}_{\mu\nu\varepsilon}, \tag{2.214}$$

which follows from (2.101)

$$\langle \mathbf{r}_{\varepsilon\gamma} \mathbf{g}^{\rho\varepsilon} \rangle - R_{\varepsilon\gamma} \bar{g}^{\rho\varepsilon} = -\bar{g}^{\rho\sigma} Q_{\sigma\gamma} - \bar{g}^{\mu\nu} Z^{\rho}{}_{\mu\nu\gamma},$$

under the microvaccum condition and

$$\bar{g}^{\rho\sigma} Q_{\rho\sigma} = 0,$$

which follows from (2.214) upon contraction with using $Z^{\rho}{}_{\mu\nu\rho} = Q_{\mu\nu}$ (see section 2.5.2). The last relation and (2.214) with

$$\bar{g}^{\alpha\rho} M_{\alpha\rho} = \bar{g}^{\alpha\rho} Q_{\alpha\rho}, \tag{2.215}$$

as a result of the splitting rule (2.102) for microvacuum

$$\langle \mathbf{r}^{\alpha\beta} \mathbf{g}_{\alpha\beta} \rangle = \bar{g}^{\mu\nu} M_{\mu\nu} - \bar{g}^{\mu\nu} Q_{\mu\nu},$$

give us finally

$$\bar{g}^{\beta\nu} M_{\alpha\beta} = -\bar{g}^{\beta\nu} Q_{\alpha\beta}. \tag{2.216}$$

Multiplying by $\bar{g}^{-1}_{\beta\varepsilon}$, the last equation becomes

$$M_{\alpha\beta} = -Q_{\alpha\beta}. \tag{2.217}$$

This way the macrovacuum equations here obtained from the variational principle (2.191) with the action (2.180) are

$$M_{\alpha\beta} = -Q_{\alpha\beta} \quad , \quad \bar{g}^{\rho\sigma} Q_{\sigma\varepsilon} = -\bar{g}^{\mu\nu} Z^{\rho}{}_{\mu\nu\varepsilon},$$

with

$$\bar{g}^{\rho\sigma} Q_{\rho\sigma} = 0.$$

They are the same as the ones obtained earlier by direct averaging out of the field equations of GR.

2.6.5 Physical limits of Macroscopic Gravity

2.6.5.1 Setting the generalized perturbation theory in MG

We will discuss here a general scheme to consider some limiting cases in MG as the generalization of results presented section 2.5.5. In particular, our aim is to consider the MacCallum-Taub [180] limit in MG. In [180] it was claimed that one can re-derive the Isaacson limit [147] for the gravitational waves propagating in the vacuum within the "averaged - Lagrangian" method of Whitham [252] combined with the two-timing method [63]. It can be shown that if special choices and assumptions are made this indeed can be arranged. But in fact Isaacson's approach, as far as its philosophy is concerned, is of broader applicability and its scheme will enable us to formulate the generalized perturbation theory in MG (as a generalization of the perturbation theory put forward in [264], see also section 2.5.5 here) and to derive as a result various limiting cases in MG, MacCallum-Taub one including (see below for a list).

Let us describe briefly foundations of the generalized perturbation theory following Isaacson [147]. Assume, as is usually done to discuss waves, that the metric can be written as

$$g_{\mu\nu}(x^\alpha) = G_{\mu\nu}(x^\alpha) + eh_{\mu\nu}(x^\alpha) + e^2 j_{\mu\nu}(x^\alpha) + \dots, \qquad (2.218)$$

where, $G_{\mu\nu}$, $h_{\mu\nu}$, $j_{\mu\nu}, \dots$ are of the same magnitude $\mathcal{O}(1)$ and e is some parameter measuring the value of h. The background metric $G_{\mu\nu}$ is a slowly varying function of space-time, whereas $h_{\mu\nu}$ is the high-frequency ripple and the structure of perturbations is constrained by the field equations. We introduce estimates of how fast the metric components vary by saying that their derivatives are of order

$$\partial G \sim 1/L, \;\; \partial h \sim h/\lambda, \qquad (2.219)$$

where, L and λ are characteristic lengths over which the background and ripple change significantly, and $L \gg \lambda$. Another parameter (being always small) is determined by the ratio $\lambda/L = \epsilon \ll 1$. Within this approach we do not require any special relation between e and ϵ. Further relations for ϵ and e and between them should be defined in formulation of the approximation scheme.

Further an expansion of Ricci curvature tensor is determined in terms of (2.218) and vacuum field equations have been shown to be split into the "wave-part" $\propto e$ and "coarse-grain part" $\propto e^2$ smooth on scale λ plus corrections due to fluctuations $\propto e^2$ and ripply on scale λ [147]. The wave part is the propagation equation for $h_{\mu\nu}$. To extract a part smooth on scale λ, averaging is carried out over several wavelengths. The smooth part expresses how the stress-energy in the waves creates the background curvature. This is the main and fundamental result of Isaacson derived within the high frequency approximation, when $e \leq \epsilon$.

In order to find a solution $h_{\mu\nu}(x)$ and $j_{\mu\nu}(x)$ to the above system of equations we have to solve them with a given background $G_{\mu\nu}$. A solution, usually attempted within W.K.B. approximation is of the form

$$\Psi_{\mu\nu} = A_{\mu\nu} \exp(\imath\phi), \tag{2.220}$$

where,

$$\Psi \equiv h_{\mu\nu} - \frac{1}{2} G_{\mu\nu} h, \tag{2.221}$$

and $h = G^{\alpha\beta} h_{\alpha\beta}$. $A_{\mu\nu}$ is a slowly changing real function of position, and ϕ is a real function with a large first derivative. (Actually only the real part of (2.220) is to be used.) Wave-vectors can be introduced (normals to surfaces of constant phase ϕ), namely,

$$l_\alpha \equiv \phi_{,\alpha}. \tag{2.222}$$

In MG the background metric is exactly the macroscopic metric, L is a macroscopic length and λ a microscopic length, and due to

$$< h_{\mu\nu} >=< \partial^n h_{\mu\nu} >= 0,$$

the propagation equation vanishes. Only terms of order e^2 and higher are meaningful in MG.

2.6.5.2 Limiting cases

We take into account, from now on, the split of the metric (2.218) up to e^2 terms for the metric. The macrovacuum equations

$$M_{\alpha\beta} = -Q_{\alpha\beta}, \tag{2.223}$$

tell us how the macroscopic (background) metric is created by the correlations of gravitational field through $Q_{\alpha\beta}$. In the perturbation theory [264] (see section 2.5.5)

$$Q^{HF}_{\alpha\beta} = < \delta\gamma^{\varepsilon}{}_{\alpha\sigma}\delta\gamma^{\sigma}{}_{\varepsilon\beta} - \delta\gamma^{\varepsilon}{}_{\alpha\beta}\delta\gamma^{\sigma}{}_{\varepsilon\sigma} > .$$

It is expressed through the perturbation, in the following way

$$\begin{aligned} Q_{\beta\rho} &= \frac{e^2}{4} < G^{\alpha\varepsilon}(h_{\delta\varepsilon|\rho} + h_{\rho\varepsilon|\delta} - h_{\delta\rho|\varepsilon})G^{\delta\sigma}(h_{\beta\sigma|\alpha} + h_{\alpha\sigma|\beta} - h_{\beta\alpha|\sigma}) \\ &\quad - G^{\alpha\varepsilon}h_{\alpha\varepsilon|\delta}G^{\delta\sigma}(h_{\beta\sigma|\rho} + h_{\rho\sigma|\beta} - h_{\beta\rho|\sigma}) > \qquad (2.224) \\ &= \frac{e^2}{4} < 2h^{\alpha}{}_{\rho|\delta}h^{\delta}_{\beta|\alpha} - 2h_{\delta\rho}{}^{|\alpha}h_{\beta}{}^{\delta}{}_{|\alpha} + h^{\alpha}{}_{\delta|\rho}h^{\delta}{}_{\alpha|\beta} - h_{|\delta}h^{\delta}{}_{\beta|\rho} - h_{|\delta}h^{\delta}{}_{\rho|\beta} + h_{|\delta}h_{\beta\rho}{}^{|\delta} > . \end{aligned}$$

This is a general expression in so far as no restrictions have been placed on the gauge, and independent on the type of perturbation, whether $h_{\mu\nu}$ are generated by the gravitational waves or local matter inhomogeneities. It should be pointed out that no assumption was made about the smallness of the density contrast even if $h_{\mu\nu}$ are assumed to be small. Recall (section 2.1.2 on the approach of Futamase) that the size of the metric perturbation is independent from the density contrast in exact theory. Thus the expression for $Q_{\alpha\beta}$ (2.224) will enable us to derive the averaged stress-energy tensor in various physical situations when the split of the microscopic metric is assumed. In particular, taking account of the gauges and rules for dealing with the perturbation terms within the averaging brackets in (2.224) which are used by various authors, we can compile the following list:

1. Isaacson's limit – high-frequency limit ($e = \epsilon$; see section 2.5.6 and equation (2.135)), [264].

2. Futamase's limit – i.e. a covariant analog of it (see section 2.1.2); notice that the parameters e and ϵ used in this section correspond respectively to ϵ and κ of Futamase.

3. Madore's limit – assumes $\epsilon^2 \gg e$ [181].

4. MacCallum-Taub's limit – see the next section.

2.6.5.3 MacCallum-Taub limit

The treatment in [180] enables us to cover any perturbation of the form,

$$h_{\mu\nu} = F(X^\mu, \theta) \tag{2.225}$$

periodic in the rapid variable θ. In the formula above

$$X^\mu = \epsilon x^\mu \text{ and } \theta = \epsilon^{-1}\Theta(X^\mu), \tag{2.226}$$

and it is assumed that the derivatives of $h_{\mu\nu}$ with respect to X^μ and θ are of equal magnitude (which we may take as of order unity). The small parameter ϵ measures the ratio of the fast length scale (microscopic) to the slow one (macroscopic). Moreover, the rapid variations are assumed to propagate in the direction of vector (2.222)

$$l_\mu = \frac{\partial\theta}{\partial x^\mu} = \frac{\partial\Theta}{\partial X^\mu} = \Theta_{,\mu} \sim \mathcal{O}(\epsilon). \tag{2.227}$$

Note that (2.225), being the main *ansatz* in two-timing method, can be considered as a generalization of W.K.B. approximation (2.220), with $\Theta \sim \phi$. In particular, the above treatment (2.225) can be extended to sums such as

$$h_{\mu\nu} = \Sigma_n(\alpha_{\mu\nu}^{(n)} \exp(\imath\Theta_n/\epsilon) + \text{c.c.}) \tag{2.228}$$

where, $\alpha_{\mu\nu}^{(n)}$, Θ_n have derivatives of order unity with respect to X^ρ and c.c. stands for complex conjugate. In fact, MacCallum and Taub have therefore restricted themselves to the W.K.B. *ansatz* for $h_{\mu\nu}$, periodic in θ,

$$k_{\mu\nu} = a_{\mu\nu}(X^\rho)\exp(\imath\Theta(X^\rho)\epsilon^{-1}) + \text{c.c.}, \tag{2.229}$$

where, $k_{\mu\nu} = h_{\mu\nu} + \frac{k}{2}G_{\mu\nu}$, $k = -G^{\mu\nu}h_{\mu\nu}$, with X^μ, θ as above in (2.226) to help keep track of the orders of the different terms.

High frequency approximation is expressed by $\epsilon \ll 1$ and the derivatives of $\alpha_{\mu\nu}$ and Θ with respect to X^ρ are of order unity. Moreover, for the high frequency approximation to remain valid we must have $e \leq \epsilon$; the amplitude of the wave and the ratio of its wavelength to the curvature radius of the background have to be small.

Isaacson in his approach assumes $e = \epsilon$ and it is therefore more restrictive than the MacCallum-Taub approximation. But Isaacson does not assume that rapid variations propagate in one direction, namely that determined by l_μ, which is the case for MacCallum-Taub's approach. From this point of view Isaacson's limit is less restrictive.

Taking into account (2.224) and (2.229) yields the stress-energy tensor due to the perturbation which is of the form [206] (cf. formula (3.15) in [180])

$$\begin{aligned} Q^{\mu\nu} &= e^2[-a^{\gamma(\nu}\bar{a}^{\mu)\tau}l_\gamma l_\tau + a^{\gamma(\mu}\bar{a}^{\nu)}{}_\gamma(l^\sigma l_\sigma) + \frac{1}{2}(a^{\alpha\beta}\bar{a}_{\alpha\beta} - \frac{1}{2}a\bar{a})l^\mu l^\nu \\ &\quad -([\bar{a}_\gamma{}^\tau a^{\gamma(\mu}l^{\nu)}l_\tau + \frac{1}{4}\bar{a}^{\mu\nu}a(l^\sigma l_\sigma) + \text{c.c.}]) \\ &\quad +\frac{1}{4}G^{\mu\nu}[2a^{\sigma\alpha}\bar{a}_\alpha{}^\tau l_\sigma l_\tau - (a^{\alpha\beta}\bar{a}_{\alpha\beta} - \frac{1}{2}a\bar{a})l^\sigma l_\sigma]. \end{aligned} \tag{2.230}$$

In [180] it has been arrived at by varying the averaged action in the sense of Whitham, where averaging is performed over the fast variable, taking into account (2.229), whereas Isaacson [147] used the Brill-Hartle technique (cf.section 2.3). Our results agree, wherever relevant, with that of [180] and [147].

2.7 Specific exact solutions

An alternative in a situation when there are spherically symmetric inhomogeneities in the lumpy universe model is to use some exact solutions, e.g. in the form of Swiss Cheese models. These are made by combining matched sections of the FLRW expanding universe with spherically symmetric (Schwarzschild) vacuum or Lemaître-Bondi-Tolman [168, 37, 246] dust solutions[15], to yield an exact inhomogeneous universe model representing growth of inhomogeneities with a spatially homogeneous and isotropic background.

This kind of treatment, in a certain sense achieves the goals of averaging without averaging *per se*. It is in fact a conceptually different approach to the fitting problem.

The first model is due to Bonnor [38]. He examined the Einstein-Strauss vacuole as representing a bound cluster of galaxies embedded in a standard pressure free cosmological model, using the Darmois junction conditions. The key point of his analysis is that the

[15]This solution though first proposed by Lemaître is often called the Tolman-Bondi or just Tolman model.

average energy density of the whole cluster is not the same as the density of the smoothed-out model. Therefore the field equations, with the energy density being the average density over a cluster of galaxies, are not satisfied.

The case when the lumpy universe model is an exact Tolman spherically symmetric dust universe was considered by Hellaby [140]. The approach taken was that of volume matching. The idea is that the behaviour of a certain region in space can be found to yield a particular FLRW model. One can then compare the density and equation of state of the resulting FLRW model with the matter content of the original volume. The results do not need to correspond to the mean density and pressure of the region derived by a simple averaging procedure. And if they do not, then we cannot count on this averaging of data to provide the parameters of a suitable homogeneous equivalent of our universe. Hellaby showed that there is a family of parabolic Tolman models in which the volume-averaged behaviour of the energy density and pressure is identical to that of a $k = 0$ pressure free FLRW universe (however local matching over infinitesimal volumes does not satisfy this condition). But in hyperbolic and elliptic cases the effective (macroscopic) equation of state is not the same as the average of the Tolman values, e.g. although the pressure is zero in the inhomogeneous Tolman models, the effective pressure in the FLRW (averaged) model is non-zero.

Recently, Moffat and Tatarski [195] studied a local void embedded in the globally FLRW model and discussed observational properties of such a model. The inhomogeneity was described using the Lemaître-Bondi-Tolman solution (the same as in [140]) with the spherically symmetric matter distribution taken to be dust, based on the faint galaxies number counts. The model has a property of being very similar to the FLRW one at the beginning of the expansion but becomes observationally different at later times. The authors studied its effects on the cosmological time scale, the measurement of the Hubble constant and the redshift-luminosity distance relation, which were shown to be fully compatible with cosmological observations. However if we happened to live in such a void and insisted on interpreting cosmological observations through the FLRW model, then e.g. the Hubble constant measurement could give results depending on the separation of the source and the observer and quasars could be younger than we think and also less distant (consequently less energetic).

Worth mentioning here, for its own sake, is Lindquist and Wheeler approach [174] where the idea of Schwarzschild–cell method was elaborated. (The method is similar in spirit to that of Wigner and Seitz in the solid state problem.) A number of mass concentrations is considered, such that the zone of influence of each can be approximated by a sphere. Inside each cell the actual gravitational potential is replaced by the Schwarzschild expression. Its important feature is that its normal derivative at the boundary of each lattice cell is non-zero and moreover, does not go to zero at finite distance. Due to this fact, the mass concentrations on either side of the cell boundary accelerate towards that boundary, at such a rate as to nullify the discontinuity in matching of the normal derivative of the gravitational potentials that would otherwise occur. This feature expresses the equation of motion of the mass at the center of a cell as a dynamic condition on the boundary of the cell. When applied to the problem of the expanding universe, this idea enables one to derive the whole of the dynamics of the expansion and subsequent contraction from the elementary static Schwarzschild solution.

2.8 The Green Function approach

This is a perturbative approach, which does not rely on any averaging procedure.

With the aim of studying the effects of a given matter distribution on the metric, and hence on the radiation, Jacobs and colleagues put forward a new scheme of determining the realistic metric of our universe [152, 151].

The idea is of solving the field equations through the use of scalar harmonics as spatial basis functions, while avoiding the use of any averaging procedure for the metric perturbations. Small metric perturbations are assumed (again, this does not restrict the size of perturbations to the matter variables), and the global expansion rate is that of FLRW model. Also assumed is the matter distribution and its evolution (known from observations and/or theory). The results do not assume a particular model for the formation of structure in the matter distribution, and are valid everywhere in our universe outside of strong field regions.

In the presence of inhomogeneities we have

$$ds^2 = a^2(\eta)[\gamma_{\mu\nu}(\vec{x}) + h_{\mu\nu}(\eta, \vec{x})]dx^\mu dx^\nu, \tag{2.231}$$

where, $h_{\mu\nu}$ describes the metric perturbations. It is assumed that $h_{\mu\nu} \equiv \mathcal{O}(\epsilon^2) \ll \gamma_{\mu\nu} \equiv \mathcal{O}(1)$ (background terms are of order 1), then $\Delta_\delta h_{\mu\nu} \simeq \mathcal{O}(\epsilon^2/\kappa)$. [16] Also, $\epsilon^2 \ll 1$ and $\epsilon^2 \ll \kappa$. The latter means that the matter inhomogeneities move non-relativistically and the effective stress-energy of metric perturbations is small.

Since the background is homogeneous and isotropic one can perform a separation of space and time dependencies in the field equations, enabling perturbations to be written not as functions of $h_{\mu\nu}$, but as harmonic decomposition. The spatial dependence of perturbations is then expanded as eigenfunctions (normal modes) of the covariant Laplacian ${}^{(3)}\nabla^2$ on the 3-dimensional static background γ_{ij}. The field equations are reduced this way to the equations for the time dependent amplitudes of the modes.

Only scalar harmonics (scalar modes Q) are considered, in terms of solutions of

$${}^{(3)}\nabla^2 Q(\vec{x}, \vec{q}) = -q^2 Q(\vec{x}, \vec{q}) \tag{2.232}$$

and obviously, $Q \simeq \mathcal{O}(1)$ and $\nabla_i Q = \mathcal{O}(q) \simeq \mathcal{O}(\kappa^{-1})$.

The metric perturbations (h_{oo}, h_{oi}, h_{ij}) are then expanded in terms of scalar harmonics Q [21]. The longitudinal gauge is assumed, and the Einstein tensor is written, including terms linear in $h_{\mu\nu}$ and its derivatives. Non-linear terms of $\mathcal{O}(\epsilon^4)$, $\mathcal{O}(\epsilon^4/\kappa)$, $\mathcal{O}(\epsilon^4/\kappa^2)$, or smaller are neglected. With this, we retain non-linear interactions of energy density of perturbations and their backreaction on the FLRW component. The stress-energy tensor is constructed in the usual way, taking perfect fluid as the background model, and perturbations (scalar) to the energy density μ, pressure p, and velocity v_i. The components of $T_{\mu\nu}$ are then written to the first order in the velocity.

By exploiting the harmonic decomposition of the field equations one can solve them by taking their spatial projections against different scalar modes. The important result is then

$$ds^2 = a^2[-(1+2\phi)d\eta^2 + (1-2\phi)\gamma_{ij}dx^i dx^j], \tag{2.233}$$

[16] The parameters ϵ and κ are the same as already described in Futamase's approach.

where, $\phi(\eta,\vec{x}) = -\frac{1}{2}h_{oo} = -\int d\mu(\vec{q})Q(\vec{x},\vec{q})H(\eta,\vec{q}) + \mathcal{O}(\epsilon^4)$, and $H \simeq \mathcal{O}(\epsilon^2)$ is the amplitude ($d\mu$ is the measure associated with the eigenvalue spectrum).

$\phi(x^\mu)$ is the effective quasi-Newtonian potential of the inhomogeneities, characterizing metric perturbations.

In terms of the matter variables the equation for ϕ can be obtained from the following equation for H:

$$3(\frac{a'}{a})H' + (q^2 + 8\pi a^2\mu - 6k)H = 4\pi a^2\mu\Delta, \tag{2.234}$$

where, $\Delta(\eta,\vec{q})$ is a suitable density fluctuation variable describing perturbations to the energy density μ, and *a priori* $|\Delta| > 1$; as usual $' = \frac{d}{d\eta}$.

To this level of approximation the Friedmann equation holds as well, and there are two additional equations relating the metric perturbations to the pressure and velocity perturbations.

The estimation of the orders of magnitude of matter variables perturbations allows us to conclude that in any allowed regime (linear $\epsilon/\kappa \ll 1$, non-linear $\epsilon/\kappa \gg 1$) the pressure and velocity perturbations are much weaker than the density fluctuations. In other words, the metric perturbation $H(\eta,\vec{q})$ are determined primarily by $\Delta(\eta,\vec{q})$, i.e., hydrodynamically the density fluctuations can be treated as the source.

To any order of magnitude arguments we have to consider effects on the scale factor a, since it makes an implicit contribution. On physical grounds:

$$a(\eta) = a_{FLRW}[1 + \mathcal{O}(< \epsilon^4/\kappa^2 >)], \tag{2.235}$$

and clearly, $\mathcal{O}(< \epsilon^4/\kappa^2 >) \ll \epsilon^4/\kappa^2 \ll 1$, so using the background scale factor does not alter the arguments about the matter variables perturbations.

Solving equation (2.234) for $H(\eta,\vec{q})$ we can obtain the pseudo-Newtonian potential:

$$\begin{aligned}\phi(\eta,\vec{q}) &= \int dV(\vec{y})G(\eta_o,\eta,\vec{x},\vec{y})\phi(\eta_o,\vec{y}) - \frac{4\pi}{3}\int_{\eta_o}^{\eta} du\frac{a^3\mu}{a'}\int dV(\vec{y}) \\ &\quad G(u,\eta,\vec{x},\vec{y})\Delta(u,\vec{y}) + \mathcal{O}(\epsilon^4),\end{aligned} \tag{2.236}$$

where, dV is a coordinate volume element, and $G(u,\eta,\vec{x},\vec{y})$ is a Green function for metric perturbations due to the scalar density fluctuations in a FLRW background.

In case of flat spatial sections ($k = 0$) it can be proved that

$$G_{k=0}(u, \eta, \vec{x}, \vec{y}) = \frac{a(u)}{a(\eta)} \frac{1}{[4\pi C(u,\eta)]^{\frac{3}{2}}} \exp[-\frac{|\vec{y} - \vec{x}|^2}{4C(u,\eta)}]. \tag{2.237}$$

The latter gives an interesting analogue with the diffusion [151].

The Green function expression for the potential can be reduced to a Newtonian form $\phi_{Newt} \simeq -\int dV(\frac{3}{8\pi}(\frac{a'}{a})^2 \frac{\Delta}{|\vec{y}-\vec{x}|}) \simeq -\int a^3 dV \frac{\mu\Delta}{a|\vec{y}-\vec{x}|}$ under appropriate conditions. But more interesting are situations where the time evolution of density fluctuations makes a significant contribution to the metric, e.g. post-Newtonian ones.

Formula (2.236) offers a relativistically correct way of calculating the metric perturbations, taking into account the cosmological expansion, non-linear density evolution and also, e.g. deviations from the thin-lens approximation.

3 The Renormalization Group approach in Gravitation

3.1 Introduction

Many phenomena in Nature are so complicated that they do not succumb to reductionist approaches. For systems with many (infinitely many) degrees of freedom new types of collective behaviour emerge and their large-scale behaviour cannot be predicted from their microscopic origin. The phenomena we have in mind are complex precisely because they contain events and information over a wide range of length- and time-scales [255].

Luckily, physics is accustomed to ignoring inconvenient details and making use of large simplifications to get to the heart of issues. For example, in the standard models of magnetism many details of multi-spin and long range interactions, magneto-elastic coupling, are omitted, as a rule, and yet the theory captures much of the essence of magnetic phenomena. Usually, a successful theory enables us to isolate some limited range of length-scales or select a not too big set of variables, to render the problem tractable and at the same time preserve its essence. In many circumstances, fortunately, it is not necessary to resolve the details associated with each scale, since generally phenomena at each scale can be treated independently. For example, in hydrodynamics there is no need to specify the motion of each water molecule in order to describe waves as a disturbance of a continuous fluid. However, in the complex phenomena, where each length-scale's contribution is of equal importance one would need to take into account the entire spectrum of length scales, dealing with fluctuations of practically any wavelengths and consequently many coupled degrees of freedom. The problem would become therefore intractable. Examples of this class of phenomena are critical phenomena, turbulent flow, the internal structure of elementary particles and confinement in QCD.

There exist however a general class of theories (and methods) known as the Renormalization Group (RG) approach, which enabled to make progress in understanding the dynamics of complex phenomena. The value and significance of RG ideas that have pervaded much of today's Statistical Mechanics and Quantum Field Theory, should not be underestimated, since not only is the RG approach a method (unlike, e.g. High-Temperature Series Expansion method to calculate the values of critical exponents), but also a theory with essential physical ideas behind, which can explain phenomena like scaling and universality, observed in various facets in Nature. From an even more open point of view, it is also the RG *philosophy*, interpreted broadly to include various kinds of "multi-length-scale" and "coarse-graining" arguments.

What is the RG then? Usually, when we speak of a group we are thinking of symmetry operations, i.e. transformations that leave the physics invariant. In particular, this means that the RG procedure (whatever it means at the moment) can be iterated. Actually, the RG should be properly called a semi-group, because the inverse of the transformation is not defined. The RG approach can be very loosely described as follows: our aim is to study some properties of a certain function H and we perform a change of variables transforming the initial problem into an identical one in terms of a new function, now H', such that $H' = \mathcal{R}H$. The transformation $\mathcal{R}$ has to be chosen in such a way that after a few iterations (or at least in the limit $n \to \infty$) $H^{(n)} = \mathcal{R}^{(n)}H$ becomes tractable by some other techniques. In the successive changes of coordinates there is some loss of information, due to the fact that the changes of coordinates will not be one to one or everywhere defined - as a rule, since otherwise the problem could not really become simpler in the new coordinates.

Generally, the subject of the RG is then the modification of the fundamental laws of physics with the change of the observational length scale, and one may probe the dependence of the effective (RG-improved) couplings on the characteristic length through the RG flow equation(s). The otherwise complicated flow pattern becomes particularly simple in the vicinity of the fixed points, where the linearized RG flow and scaling holds.

3.2 A short history of RG

The RG equations were first introduced into particle physics. It was observed that the conformal invariance, a space-time symmetry of the classical electro-magnetic action[1] can be violated by quantum effects in the energy regime $E \gg 0.5MeV$, where the electron mass can be ignored. In the seminal papers of [239] and [119], it was shown that quantum effects induce a scale dependence in the electro-magnetic charge and that the derivative of the electro-magnetic coupling with respect to the scale, is an analytic function of the coupling itself, the so-called β function.

Further, there are physical amplitudes in QFT, which depend on the couplings and also on points in space, labeled by the coordinates x_i (e.g. Cartesian in flat space). In [53] it was shown that the variation in the couplings under a change of scale (at which they are defined) can be always compensated for by a rescaling of the coordinates x_i, so that the vacuum amplitudes remain invariant. This results in the RG equation - an inhomogeneous partial differential equation for the amplitudes, further extended in [251, 143] to a homogeneous equation.

The issue of course is that QED or any renormalizable field theory is plagued by "infinities" which can be renormalized. This requires an introduction of "bare" couplings $g_{bare}(g, \epsilon)$, which are analytic functions of the renormalized couplings, g_i, and a regularization parameter or parameters, ϵ, e.g. for a cut-off, Λ, $\epsilon = \kappa/\Lambda$, where κ is a renormalization point and for dimensional regularization $\epsilon = 4 - d$, where d is the dimension of space or space-time. One can prove then that to any given order in perturbation theory, it is possible to choose the couplings in a cut-off dependent way so as to make physics at momenta much smaller than the cut-off, independent of it. As eventually the cut-off is sent to infinity, the above means physics at any finite momentum. The change in the cut-off accompanied by a suitable change in the couplings is an invariance of the theory.

These transformations form a group, with the law that changing the cut-off by a factor s_1 and then by s_2, should be equivalent to a change by a factor $s_1 s_2$. Writing

$$s = e^{-t} \tag{3.1}$$

[1] This symmetry is only angles preserving, not lengths.

i.e., $\Lambda(t) = \Lambda_o e^{-t}$ (Λ_o is some fixed number), the above composition rule just means that when two transformations are implemented in a row, the parameters t add. The β-function is defined as

$$\beta(g) = \frac{dg}{dt}, \tag{3.2}$$

where, g stands for the coupling(s).[2] In the case of Yang-Mills theory we have

$$\frac{dg}{dt} = cg^3 + \text{higher orders}, \quad c > 0. \tag{3.3}$$

Integrating this from $t = 0$ to $t = t$ (i.e. from Λ_o to $\Lambda_o e^{-t}$) we find

$$g^2(t) = \frac{g^2(0)}{1 - 2g^2(0)ct}. \tag{3.4}$$

What this means is that as we send the cut-off to infinity ($t \to -\infty$) we have to reduce the coupling to zero logarithmically $g^2(t) \simeq 1/|t|$.

In statistical physics the RG equation was raised to central importance through the work of Kadanoff [154] and Wilson [256]. There one contemplates changing the cut-off (and the couplings) even in a problem where nature provides a natural cut-off such as the inverse lattice spacing, $\Lambda \simeq 1/a$, and there are no UV infinities. In this view-point, the cut-off is not to be viewed as an artifact to be sent to infinity, but as the dividing line between the modes we are interested in and the ones we are not interested in. We may change the cut-off and the couplings without affecting the slow mode physics (for a clear exposition, see e.g. the introduction to [227]).

The RG in the study of phase transitions provides a mathematical scaffold for understanding scaling and universality.

More recently, the idea inspired by O'Connor and Stephens [202] that the RG equations should be viewed in terms of a coordinate transformation on the space of couplings, was further pursued by Dolan [78], who showed that the RG equations for vacuum amplitudes can be interpreted as a Lie derivative on the space of couplings.

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[2]There are different conventions - in condensed matter physics increasing t decreases Λ, opposite to the field theory convention.

3.3 Real-space RG and critical phenomena

Critical points (second-order phase transitions) occur in liquid-gas transitions, ferromagnetic transitions, binary alloys transitions, etc. There are close analogies relating all these critical points, which is one of the fascinations of the subject.

The critical temperature (Curie temperature) T_c of a ferromagnet marks the onset of spontaneous magnetization in the absence of an external field. Above the critical temperature T_c, it is zero (the ferromagnet is in a paramagnetic state), below (near) T_c, varies as $(T_c - T)^\beta$. The exponent β is an example of a critical exponent. Theory of critical phenomena should be able to predict the values of the exponents. Other critical exponents characterize other power laws near the critical point. The critical point of the ferromagnet is at zero applied magnetic field $\vec{H}$ and at $T = T_c$. The derivative of the magnetization $\vec{M}$ diverge at T_c. The phase transition has no associated latent heat and can be described as a critical phase transition.

The idea of possible application of the RG method in the phase transition context in Statistical Mechanics is connected with the homogeneity of thermodynamic functions near the critical temperature, which was first described in [253], and with the behaviour of correlations functions in the critical region ($T = T_c$), studied by Kadanoff [154]. The important idea is that only long wave length fluctuations in the order parameter, are responsible for the critical singularities. Near to a critical point the correlation length diverges and because the degrees of freedom are correlated over macroscopic distances, the system looks very similar under a transformation that divides the lattice into smaller blocks and replaces the degrees of freedom inside each of them by a block-average. In fact, Kadanoff has brilliantly hypothesized that if the block lattice is considered, then at the critical point the block-lattice Hamiltonian may be reduced to the initial lattice Hamiltonian by scaling transformations. Intuitively, the effects of this kind of averaging are to cut down the correlation length by the scale factor b, where b is the linear dimension of the block and the interactions between the new variables will be different. If the blocks are small then we expect that the interactions between the block variables (spins for a ferromagnet) will be local if the interactions we started with were local. Notice, that the correlation length decreases (by a factor of b) since it is measured in lattice units, which change in our transformation from being the distance between the original lattice points to that between blocks now.

The RG theory of critical phenomena owes its great success to the remarkable works of Wilson. His formulation provided a systematic way of implementing the integration over a finite fraction of degrees of freedom in a system near its critical point, and quantifying the effect on the remaining variables, providing in this way all the mathematical infrastructure explaining scaling and universality. More precisely, the effect of the long wave length fluctuations can be calculated by using self-similarity properties of a critical system under scale transformations, where scaling amounts to the integration over the short wave length components and determining the effect on the long wave components. When this effect is vanishing the scale transformation has a fixed point, which in fact determines the critical singularities that are universal, i.e. insensitive to the details of the molecular interaction. In [257] it was shown how to embody at the microscopic level, Kadanoff's hypothesis of universality into precise differential equations and also how to obtain the explicit block lattice Hamiltonians from initial Ginzburg-Landau Wilson Hamiltonian. He has also found the recursion relations and utilized the scaling transformation and the fixed point method to solve them.

In the real space approach the self-similarity is realized for a microscopic hamiltonian, and scaling now means that the variables of the system are combined to new similar variables. For lattice systems (Ising model) this seems possible. Universality emerges through the properties of the (self similar or) fixed point hamiltonian. It is attractive in the irrelevant (or non-singular) directions and a critical fixed point dictates the singular properties for a whole surface of critical hamiltonians to be the same.

Generally, some problems appear with the above procedure in the applications, but first let us remark that according to what was said above, the RG transformation is a mapping from Hamiltonians or actions defined in a certain phase space, to actions in the same space. More generally, the RG map is defined as a rule (deterministic or stochastic) for generating a configuration ω' of "block spins", given a configuration ω of "original spins". In mathematical language this is given by a probability kernel $T(\omega \to \omega')$, with which one can define a probability distribution $\mu'(\omega')$ of block spins from any given probability distribution $\mu(\omega)$ of original spins [101]

$$\mu'(\omega') = (\mu T)(\omega') \equiv \sum_{\omega} \mu(\omega) T(\omega \to \omega'). \tag{3.5}$$

RG map is therefore defined as a map from measures to measures, in fact in applications we assume it to be defined from Hamiltonians to Hamiltonians[3].

If we represent the initial action as a point in a coupling constant space, it will flow under the RG transformation to another point in the same space, therefore a possibility of a *fixedpoint* of the group action exists, that is to say, the action function which reproduces itself after the RG transformation. Geometrically speaking the fixed point does not flow under the RG transformation. The sequence of theories specified by the couplings generated under successive application of the RG is called the RG flow. The critical behaviour of the model can be gotten from the RG flows [110].

Now, the problems are also, (1) to give the rules relating the new variables to the old ones, and (2) to work out the properties of the relation between the hamiltonians for the new and old variables. Technically, one has problems with determination of the properties of renormalization transformation. It is hoped that approximations, which fail at the critical point for the partition function, may be carried out successfully for a renormalization transformation. This crucially depends on the choice of the renormalization transformation. In fact, many aspects of the renormalization procedure are obscure.

In order to highlight the concepts involved, it is useful to consider a simple example, namely the Ising model[4]. The hamiltonian can be generally written as

$$\mathcal{H}(s) = \sum_{\alpha} K_\alpha s_\alpha, \tag{3.6}$$

where, s_α stands for a spin function of type α, e.g.

$$\begin{aligned} s_h &= \sum_i s_i, \\ s_\epsilon &= \sum_{(ij)} s_i\, s_j, \end{aligned}$$

[3] We will not enter into a discussion of any problems which this assumption can engender; interested reader can consult [101].

[4] Interestingly, the Ising model is not so much a model, as it is a notation now. But more surprisingly in a certain sense it makes its appearance in cosmology too (see [205]).

for the order parameter and for the energy, respectively, where s_i is the spin of site i (and (ij) nearest neighbour (NN) sites). K_α are the coupling constants, namely, K_h is the magnetic field, K_ϵ the NN coupling for the standard Ising model.

The RG transformation is defined as

$$e^{\mathcal{H}'(s')} = \sum_{\{s\}} P(s', s) e^{\mathcal{H}(s')} \tag{3.7}$$

where, $P(s', s)$ is the weight function, satisfying

$$P(s', s) \geq 0, \quad \sum_{\{s'\}} P(s', s) = 1. \tag{3.8}$$

Equation (3.7) is the defining relation for the new hamiltonian and $\mathcal{H}'(s')$ can be written again in the form (3.6), defining new coupling constants K'_α, such that the relation

$$K'_\alpha = K'_\alpha(K_\beta) \tag{3.9}$$

is equivalent with the formal definition (3.7). Many transformations are possible, in fact (3.8) guarantees only that the K'_α are real and the partition functions of $\mathcal{H}$ and $\mathcal{H}'$ are the same (of the same functional form).

The derivative matrix

$$T_{\alpha\beta} = \frac{\partial K'_\alpha}{\partial K_\beta}, \tag{3.10}$$

yields directly the critical exponents.

In the case of an exactly solvable $1d$ Ising model with NN interactions, the averaging process is easily defined to consist of exact integration over every other spin in the chain. This procedure preserves the partition function and the interaction between the thinned-out degrees of freedom remains NN, and moreover is an analytic function of the original NN interactions. The integration procedure can be iterated therefore, and new NN couplings K'_α obtained in terms of the original coupling K_α by the same function as $K' = \mathcal{R}(K)$.

In the case of $2d$ Ising model with NN interactions one can integrate over spins on every, say even site, but this reduction of degrees of freedom induces other types of interactions between the remaining spins, in addition to renormalizing the original NN coupling. The new couplings are analytical functions of the original NN coupling but now, the integration cannot be iterated to yield closed form solutions since at each step longer and longer range interactions get generated.

Now, in the Wilson's formulation of the RG one considers from the outset, a general class of theories defined by the infinite set $\{K_\alpha\}$ of couplings corresponding to interactions of all possible ranges and Hamiltonian's complexity. An appropriate scale transformation $\mathcal{R}_b$, which integrates over a fraction of the original degrees of freedom maps the theory into another one with the renormalized couplings: $K'_\alpha = \mathcal{R}_b(K_\alpha)$. For most models one cannot find out exactly this relationship and must resort to approximate numerical methods. When performing the real-space RG procedure one stipulates that the effective Hamiltonian $\mathcal{H}^{(n+1)}$ must take the same functional form as $\mathcal{H}^{(n)}$, so that the model is exactly the same at every stage, except for a change in the parameters in the effective Hamiltonian (this condition is in almost all cases impossible to meet exactly).

A useful RG transformation has to have a number of properties; we will mention just one, the correlation length decreases under $\mathcal{R}_b : \xi \to \xi' = \xi/b$. It is preserved under the RG transformation if the starting Hamiltonian H[5] has $\xi = 0$ or $\xi = \infty$. Theories with $\xi = 0$ are trivial, e.g. $T = 0$ or $T = \infty$ limit of most statistical mechanics models.

Critical Hamiltonian $\mathcal{H}_c$, the one with the infinite correlation length, when acted on with $\mathcal{R}_b$ produces another critical Hamiltonian, since obviously $\xi' = \xi/b = \infty$. The set of critical points define a hypersurface in the infinite dimensional space $\{K_\alpha\}$. The RG flow on this surface can a) meander randomly b) go to some limit cycle or strange attractor or c) converge to a fixed point $\mathcal{H}^\star$. At the fixed points the renormalized couplings are exactly equal to the original couplings, $K'_\alpha = K_\alpha$, and the theory reproduces itself al all length scales. In the case of critical points only the long distance behaviour is reproduced. In studying the critical phenomena we are interested in RG transformation which has a critical fixed point $\mathcal{H}^\star = \mathcal{R}_b(\mathcal{H}^\star)$ with $\xi = \infty$. Each such fixed point has a basin of attraction, i.e. the set of $\mathcal{H}_c$ that converge to it under $\mathcal{R}_b$, which defines the universality class, since the long distance behaviour of all theories corresponding to these $\mathcal{H}_c$ is governed by the same fixed point. What this means is that the critical (static) exponents are rather insensitive to the details of the system and for a continuous phase transition they depend

[5] We refer to $\mathcal{H}$ as the effective Hamiltonian. For the starting system, the temperature dependence of $\mathcal{H}$ is through $\mathcal{H} = \beta H$, $\beta \sim 1/T$, where H is independent of temperature.

only on: the dimensionality of the system d, its symmetry, the dimensionality of the order parameters, whether the forces are of short or long range. The values of critical exponents can be obtained from the eigenvalue equation for the linearized RG transformation in the vicinity of the fixed point.

We would like now to give an example of the scaling hypothesis in a case of the ferromagnet, represented by Ising model. It says that close to the critical point the singular part of the Gibbs free energy[6] is a generalized homogeneous function of its variables

$$G(\lambda^{a_t}t, \lambda^{a_h}h) = \lambda G(t, h), \tag{3.11}$$

where, a_t, a_h are two parameters (scaling powers), λ is arbitrary and $t = T - T_c$. The scaling laws (algebraic equalities between the critical exponents) can also be derived in the RG approach. We do not present the details here.

For a magnet near its critical point, there are two quantities which measure the deviations from the critical point: a dimension-less magnetic field h and $t = T - T_c/T_c$, and near the critical point $h, t \ll 1$. The statement of scaling is that the powers of the temperature deviation provide a characteristic scale of all physical quantities, e.g. the magnetic field always appears in the theory in the combination h/t^{Δ} (where, Δ is the critical index for the magnetic field). Correspondingly the magnetization appears in the combination m/t^{β}. The content of the scaling hypothesis (which after all should not be called a hypothesis any more) is then that the magnetization appears in the scaling form

$$m(h, t) = t^{\beta} m^{\star}(\frac{h}{t^{\Delta}}). \tag{3.12}$$

If h is of the same order of magnitude as t^{Δ}, then $m^{\star}$ is of order one and m is of order t^{β}. Statements like this can be verified experimentally.

More literature on the subject for interested reader can be found in [20, 79, 34, 131], see also [227].

Before closing this section we offer some comments on the renormalization of fluids. The fluid is believed to be in the same universality class as the Ising model. Conceptually the problem is whether a renormalization on a microscopic level is possible and

[6]The same property holds for all the other thermodynamic potentials.

whether a lattice structure is necessary for a successful renormalization. In [144] a map was constructed by which the fluid properties are associated with a (general) Ising model.

In fact in some respect the fluid is an advantage, as for the fluid the elimination of an infinitesimal fraction of the particles is achieved most easily. On the other hand, the continuous potentials are much more difficult to handle than the discrete interaction constants for a lattice[7].

3.4 Critical phenomena in Cosmology

In this chapter we review the recently discovered critical behaviour in the gravitational collapse situations.

We suspect that these are not the only critical phenomena possible on GR grounds. We will deal with the other possible scenarios in sections 3.6 and 3.5, after we develop the necessary tools to tackle the problem in chapter (3.5). All of them borrow from the previously discussed concepts of the RG.

In our opinion, there are at least a few facts–hints, pointing to the broadly understood criticality in the universe. Firstly the distribution of cosmic structures in the Universe, galaxies and clusters of galaxies, exhibits certain scaling properties [40], namely the two-point correlation function of galaxies, clusters and quasars has a power law[8] behaviour $r^{-\gamma}$, with $\gamma \sim 1.8$, up to present day scales of about $300\,Mpc$ (but with different correlation lengths for different cosmic structures). This fact seems to be a clear hallmark of the phase transition underlying the origin of structure formation.

Scaling behaviour is in general common to systems that obey non-linear dynamical equations with possibly a stochastic driving term, or more generally even to chaotic systems and phenomena (see [214] for the excellent review on chaos in the Einstein equations). In this respect it is interesting to notice that spirals are a dominant type of pattern in spatio-temporal chaos in non-equilibrium systems [122]. A general explanation of the phenomenon is lacking, though many examples[9] show that pattern formation in spatially

[7]This remark will appear of relevance later.

[8]Power laws are essential for scale invariant phenomena, since re-scaling of the variables, e.g. $x \to ax$, does not change the shape of the distribution e.g. $N(ax) = a^{-b}x^{-b}$.

[9]E.g. spirals in thermal convection near the critical point of a fluid, Rayleigh-Bénard convection near

extended non-linear systems near the critical point, specially when chaos is involved, yields spirals. Depending on the value of parameters, the spiral pattern dominates or not and the mechanism by which the pattern conversion occurs seems to be through interaction with defects in the patterns. Their rôle in non-equilibrium patterns is by no means clear; the defect propagation and interaction can be crucial to understanding non-equilibrium patterns and chaos. We have mentioned this fact due to the spiral galaxies which abound in the universe; in our opinion this issue deserves further investigation.

No doubt, the universe is an extremely large dynamical system and moreover it is quite complex, since it contains events and information over a wide range of length- and time-scales.

Recently, a new approach has emerged, namely self-organized criticality (SOC), that might be a mechanism leading to complexity[10]. It is pertinent to systems with many interacting degrees of freedom which operate presumably far from equilibrium. For them small increments in energy input can trigger an arbitrarily large *avalanches* (activity) with power law spatial and temporal distribution functions limited only by the size of the system, whereby they self-organize themselves into a critical state[11] [19].

Trivial remark – the discussed critical phenomena have nothing to do with the phase transitions in gauge theories, connected with particular symmetry breaking mechanisms (Higgs mechanism) in the early universe.

3.4.1 Critical phenomena in GR gravitational collapse

Recently, Choptuik [65] (see also [64]) and Abrahams & Evans [1] discovered numerically new solutions of Einstein equations at the threshold of black hole formation, exhibiting

the thermodynamic critical point. In this case whether the spiral pattern dominates or not, depends on the Prandtl number (related to the viscosity).

[10]Similar ideas built on chaos, scaling, etc. find their way in other branches of science as well, e.g. economics, biology, environmental sciences (see e.g. [213]), though mostly they are only implemented in simple numerical toy-models of sandpile type. This situation is similar to the one that regards fractals – not much is known about the *physics* of fractal dynamics.

[11]As an example, we can give pulsar glitches caused by changes in the speed of neutron star rotation (presumably due to an abrupt change in internal structure). When the changes in frequencies are translated to changes in rotational energy and the cumulative distribution of that energy plotted, it turns out to be a power law.

non-linear dynamical behaviour suggestive of critical phenomena. Before going to the detailed examples we will describe some of their characteristics in general terms.

Critical phenomena become evident as variations in the properties of space-times across a parameter space of space-times. Suppose that a single parameter p_k describes each space-time. There is then a correspondence between parameter spaces $\{\mathcal{G}_k\}$ and the space-times $\{S_k[p_k]\}$. For each $\mathcal{G}_k$ a critical value $p_k^\star$ of the parameter p_k, separates the parameter space into a half-space $\mathcal{G}_k^+$ describing space-times that contain a black hole and the other that do not. The parameters p_k are associated, as can be seen, with the variations in the strength of the gravitational self-interaction. The critical behaviour occurs in space-times that are just on the "edge" of forming a black hole, i.e. when $|p_k - p_k^\star|$ is small. In particular, in $\mathcal{G}_k^+$, the black hole mass was found to fulfill a power-law dependence $|p_k - p_k^\star|^\beta$ with a critical exponent β. For p_k close to $p_k^\star$ each space-time develops a strong field region $\mathcal{R}$, where the gravitational field (and any coupled field) develops an oscillatory character, revealing the existence of (discrete) scaling relations due to which the successive oscillations are echoes of each other on progressively smaller spatial and temporal scales. Also in the case of [65] the universality was demonstrated. This refers to the fact that in case of scalar field collapse the shape of the fields in the critical space-time $S_k[p_k^\star]$ in the strong field region and the value of critical exponent for black hole mass (and the scaling relation) do not depend on which parameter space, $\mathcal{G}_k$ is examined. To put it differently the critical behaviour is generic and independent of the details of the initial data.

3.4.1.1 Scalar field collapse

For massless scalar field (both minimally coupled $\zeta = 0$ and non-minimally coupled) the critical behaviour was found by computer simulation of the collapse of spherically-symmetric wave-packets of scalar field.

The equation of motion is

$$\phi_{;\mu}{}^{;\mu} = \zeta R\phi \tag{3.13}$$

and the line element

$$ds^2 = -\alpha^2(r,t)dt^2 + a^2(r,t)dr^2 + r^2 d\Omega^2, \tag{3.14}$$

where, α is the lapse function and a the radial metric function[12]. Lapse is fixed by the polar time slicing condition and the shift vector $\beta^r = 0$ fixing the spatial coordinate trajectories. This gauge generalizes Schwarzschild coordinates for dynamical space-times, as $a^2 = (1 - 2m(r,t)/r)^{-1}$.

Choptuik introduces auxiliary variables $\Phi = \phi'$, $\Pi = a\dot{\phi}/\alpha$ and solves the following equations ($\zeta = 0$):

$$\dot{\Phi} = (\tfrac{\alpha}{a}\Pi)'$$

$$\dot{\Pi} = \tfrac{1}{r^2}(r^2\tfrac{\alpha}{a}\Phi)'$$

$$\tfrac{\alpha'}{\alpha} - \tfrac{a'}{a} + \tfrac{1-a^2}{r} = 0 \tag{3.15}$$

$$\tfrac{a'}{a} + \tfrac{a^2-1}{2r} - 2\pi r(\Phi^2 + \Pi^2) = 0, \tag{3.16}$$

where a dot stands for $\partial/\partial t$ and a prime for $\partial/\partial r$. The radial coordinate r (covariantly defined) measures proper surface area; time coordinate t has no relevant geometrical interpretation, except as $r \to \infty$ where it measures proper time. In fact, though critical phenomena were discovered via computations in r, t, they are better described in r, T, where T is proper time of an observer fixed at $r = 0$, namely, $T \equiv \int_0^t \alpha(0,\tilde{t})d\tilde{t}$.

The finite difference code is based upon an adaptive-mesh-refinement algorithm which is very well able to resolve very fine spatial and temporal features.

The scalar field ϕ has an initial profile

$$\phi(r,0) = \phi_o r^3 e^{-[(r-r_o)/\Delta]^q} \tag{3.17}$$

and a one-parameter space of solutions is generated from this Cauchy data with a condition on Π that initially the scalar radiation is purely ingoing. The initial data are specified if also the slicing condition (3.15) is solved for α and Hamiltonian constraint (3.16) for a. Once r_o, Δ and q are considered fixed, then ϕ_o serves as a single parameter characterizing the sequence of solutions, and $\mathcal{G}_k = \phi_o$, for a particular k. The parameter ϕ_o is related to the strength of the field's self-interaction. For small ϕ_o, the wave-packet implodes and disperses to infinity, the scalar and gravitational fields decouple (dynamics is described by flat space-time solution of the spherically-symmetric wave equation); for large ϕ_o its

[12] $G = c = 1$ units are used.

implosion leads to a black hole formation. Black hole is detected by monitoring $\frac{2m}{r}$. For black hole space-times' $\frac{2m}{r} \to 1$ (for some specific $r = R_{BH}$), and the mass of black hole $m_{BH} = 2R_{BH}$ can be calculated. A critical value of the parameter $\phi_o^\star$ separates supercritical ($\phi_o > \phi_o^\star$) from subcritical ($\phi_o < \phi_o^\star$) solutions.

In regions of parameter space close to $\phi_o^\star$, Choptuik found that critical features in solutions (close to the critical point) tend to depend linearly on $\ln|\phi_o - \phi_o^\star|$ and therefore exponentially on the initial conditions, and that structures with increasingly finer spatial and temporal scales develop as $\phi_o \to \phi_o^\star$. In terms of two new variables $X = \sqrt{2\pi}r\Phi/a$ and $Y = \sqrt{2\pi}r\Pi/a$ (they are invariant with respect to rescalings of the length and time coordinates ($r \to \kappa r$ and $t \to \kappa t$) and hence to rescaling of the mass of the space-time) one can better describe the echoing and scaling behaviour. This is so because critical dynamics is most naturally expressed in terms of the variables that are form-invariant with respect to these rescalings, which on their own express the absence of any intrinsic mass/length scale in the model. Obviously, the equations to solve and any solutions thereof are invariant under these rescalings.

In solutions close to the critical ones in the strong field region, the scalar field oscillates and the number of oscillations is proportional to $|\ln|\phi_o - \phi_o^\star||$. The conjecture therefore is that every critical solution contains an infinite number of echoes. Let us introduce the logarithmic spatial and temporal coordinates ρ and τ defined as follows

$$\begin{aligned} \rho &= \ln r \\ \tau &= \ln(T^\star - T). \end{aligned} \tag{3.18}$$

(The const $T^\star$ is the finite accumulation time of the echoes in the precisely critical solution and can be determined in solutions close to the critical ones by fitting).

Choptuik finds that approximate scaling relations hold

$$\begin{aligned} X(\rho - \Delta, \tau - \Delta) &= X(\rho, \tau) \\ Y((\rho - \Delta, \tau - \Delta) &= Y(\rho, \tau). \end{aligned} \tag{3.19}$$

Critical dynamics is unique (up to trivial rescalings $r \to kr$, $t \to kt$) and invariant under a *discrete* scaling symmetry. The scale periodicity is similarly conjectured for all other form-invariant quantities, including ϕ, $r^2\phi^{,\mu}\phi_\mu$, $\frac{dm}{dr}$, m/r. Physically it means that for a precisely critical configuration, an infinite series of "echoes" is generated from the recurrence of strong-field evolution on ever decreasing spatio-temporal scales. The scalar field oscillations appear as echoes of one another on scales finer by a factor of $e^{-\Delta} \simeq 1/30$. Critical dynamics "accumulates" at some critical central proper time $T^\star$. In other words, if the radial profiles of X and Y are observed at some time T_1, with a small interval $\delta T_1 = T^\star - T_1$, and again at a second time T_2, with even the smaller interval $\delta T_2 = e^{-\Delta}\delta T_1$ before $T^\star$, then a new detail will have appeared in the later profiles on a finer scale, but upon rescaling radially by a factor of e^{Δ} the new profiles are in fact identical to the earlier ones. Note that the scaling relations observed is always approximate because even in the precisely critical solution the initial oscillations near the outer edge of the strong field region will contain information on the initial data. With each echo this information is washed out and scaling relation holds tighter. Besides, any near-critical solution produces only a finite number of echoes as $T^\star$ is approached before it "decides" whether to form a black hole or not. The value of Δ was found to be universal $\Delta \simeq 3.4$. The profiles of $X(\rho,\tau)$ and $Y(\rho,\tau)$ were also found to be universal, i.e. independent of the family of initial data.

For solutions in the half-spaces $\mathcal{G}_k^+$ (for which $\phi_o \to \phi_o^\star$ from above) a power law has been found for the black hole masses

$$m_{BH} \simeq C|\phi_o - \phi_o^\star|^\beta, \qquad (3.20)$$

with $\beta \simeq 0.37$ (and C family dependent constant). This power-law behaviour was also found to be universal, with possibly weak dependence, if any, on the coupling const ζ. The controversial conjecture one can make based on the above is that a black hole first appears along any sequence at $p = p^\star$ with infinitesimal mass, i.e. the black hole transition point is generically massless. Since the associated oscillations in the scalar field occur on increasingly finer scales, both the "kinetic energy" of ϕ, $\phi^{,\mu}\phi_\mu$ and the scalar curvature of the space-time, get driven to infinite values at the critical event $r = 0, T = T^\star$. The precisely critical space-time is necessarily singular and most likely naked.

3.4.1.2 Axisymmetric gravitational wave collapse

The second example of critical phenomena in GR was demonstrated in [1] in the collapse of axisymmetric gravitational wave-packets. This model is source free, $T^{\mu\nu} = 0$ and less symmetric (one Killing vector) from the Choptuik's example. A dynamical degree of freedom of the gravitational field is necessarily involved here.

Abrahams and Evans compute axisymmetric, asymptotically flat vacuum space-times, using a 3 + 1 formalism. They adopt the maximal time-slicing condition ($K^i_i = 0$; K^i_j is the extrinsic curvature) and the quasi-isotropic spatial gauge, which fix the coordinates. The line element has the form

$$ds^2 = -\alpha^2 dt^2 + \phi^4[e^{2\eta/3}(dr + \beta^r dt)^2 + r^2 e^{2\eta/3}(d\theta + \beta^\theta dt)^2 + e^{-4\eta/3} r^2 \sin^2\theta d\varphi^2], \quad (3.21)$$

where, α is the lapse function, β^r and β^θ are shift vector components, ϕ is the conformal factor, and η is the even-parity "dynamical" metric function. Numerical solutions are computed to the following set of equations

$$\begin{aligned}
&\partial_t\hat{\lambda} = \mathcal{D}_\beta[\hat{\lambda}] - \phi^6(D^r D_r\alpha + 2D^\varphi D_\varphi\alpha) + \alpha\phi^6(R^r_r + 2R^\varphi_\varphi) + \hat{K}^r_\theta/r[r\partial_r\beta^\theta - \partial_\theta(\beta^r/r)],\\
&\partial_t\hat{K}^\varphi_\varphi = \mathcal{D}_\beta[\hat{K}^\varphi_\varphi] - \phi^6 D^\varphi D_\varphi\alpha + \alpha\phi^6 R^\varphi_\varphi,\\
&\partial_t(\hat{K}^r_\theta/r) = \mathcal{D}_\beta[\hat{K}^r_\theta/r - \phi^6 D^r D_\theta\alpha/r + \alpha\phi^6 R^r_\theta/r + (2\hat{\lambda} - 3\hat{K}^\varphi_\varphi)[\partial_\theta(\beta^r/r) - \alpha K^r_\theta/r],\\
&\partial_t\eta = \beta^r\partial_r\eta + \beta^\theta\partial_\theta\eta + \partial_\theta\beta^\theta - \beta^\theta\cot\theta + \alpha\lambda,\\
&\Delta^{(3)}_f\psi = -\frac{1}{4}\psi(\Delta^{(2)}_f\eta + \frac{1}{2}\psi^{-8}e^{-2\eta}\hat{K}^i_j\hat{K}^j_i),\\
&\Delta^{(3)}_f(\alpha\psi) = -\frac{1}{4}\alpha\psi(\Delta^{(2)}_f\eta - \frac{7}{2}A^2 K^i_j K^j_i),\\
&r\partial_r(\beta^r/r) - \partial_\theta\beta^\theta = \alpha(2\lambda - 3K^\varphi_\varphi),\\
&r\partial_r\beta^\theta + \partial_\theta(\beta^r/r) = 2\alpha K^r_\theta/r,
\end{aligned}$$

where, $K^i_j K^j_i = 2\lambda^2 - 6\lambda K^\varphi_\varphi + 6(K^\varphi_\varphi)^2 + 2(K^r_\theta/r)^2$, and the transport operator is defined by $\mathcal{D}_\beta[u] = \frac{1}{r^2}\partial_r[r^2\beta^r_u] + \frac{1}{\sin\theta}\partial_\theta[\sin\theta\beta^\theta u]$; $\lambda = K^r_r + 2K^\varphi_\varphi$, $\hat{K}^i_j = \phi^6 K^i_j$, D_k is the spatial covariant derivative, R^i_j spatial Ricci tensor, $A = \phi^2 e^{\eta/3}$, $B = \phi^2 e^{-2\eta/3}$, $\psi = B^{1/2}$ and $\Delta^{(3)}_f$ and $\Delta^{(2)}_f$ are the three- and two-dimensional flat space Laplacians, respectively.

To find Cauchy data for the gravitational field, η and K^r_θ (freely specifiable fields) are taken in the form of a linear ingoing gravitational wave-packet with quadrupolar ($l = 2$)

angular dependence. The general linear $l = 2$ solution is described by a quadrupole moment $I(v)$ of arbitrary profile in advanced time v (or retarded u). The linear solution involves $I(v)$, its first two derivatives, $I^{(1)}(v) \equiv dI/dv$ and $I^{(2)}(v)$, and its integrals, $I^{(-1)}(v) \equiv \int^v dv' I(v')$ and $I^{(-2)}(v)$.

Appropriate expressions for η and K^r_θ have been found

$$\eta = (\frac{I^{(2)}}{r} - 2\frac{I^{(1)}}{r^2}) \sin^2\theta, \tag{3.22}$$

$$\frac{K^r_\theta}{r} = (\frac{I^{(2)}}{r^2} - 3\frac{I^{(1)}}{r^3} + 6\frac{I}{r^4} - 6\frac{I^{(-1)}}{r^5}) \sin 2\theta. \tag{3.23}$$

These Cauchy data will be at least slightly non-linear (depending on the initial amplitude and radius), since a wave-packet of finite amplitude confined within a finite radius will generate a finite mass. In order to find proper data, the exact Hamiltonian and momentum constraints are solved for ϕ, λ and K^φ_φ, subject to the choice above of η and K^r_θ. Nearly linear Cauchy data are still found to generate ingoing solutions. The form of $I(v)$ is chosen, such that a wave-packet has polynomial radial dependence of the form $I^{(-2)}(v) = a\kappa_p L^5[1-(v/L)^2]^6$, for $|v| = |r - r_o| < L$ at $t = 0$. κ_p is a constant, a is an amplitude parameter, L a width parameter and r_o a centering parameter. Each of these might serve as useful parameters of spaces $\mathcal{G}_k$. Initially, L and r_o have been fixed, while a was chosen to parameterize the Cauchy data, i.e. the solutions. In the limit $a \to 0$ the mass of the wave-packet is $M_p^{linear} = a^2 L/(2\pi)$, therefore a strength parameter of the form $\theta(a) = 2\pi M_p/L \simeq a^2$ is a good choice. A wave-packet with $\theta \ll 1$ weakly self-interacts, escaping to infinity virtually unaffected, but for $\theta \geq 1$ a black hole forms with $m_{BH} \to M_p$ as $\theta \to \infty$. The critical value along the sequence was found to be $\theta^\star \simeq 0.80$ ($a^\star \simeq 0.93$).

Here again, supercritical collapse of gravitational wave-packets generates black hole masses that are found to satisfy a power law

$$m_{BH} \simeq C(a - a^\star)^\beta. \tag{3.24}$$

The critical exponent value is also $\beta \simeq 0.37$, presently indistinguishable from that seen in scalar field collapse. Similarly, a scaling relation holds on the gravitational field in the strong field region $\mathcal{R}$. The gravitational field oscillates on progressively finer spatial and temporal scales, as is evident from figure (3.1) [103], which shows radial profiles of η (along

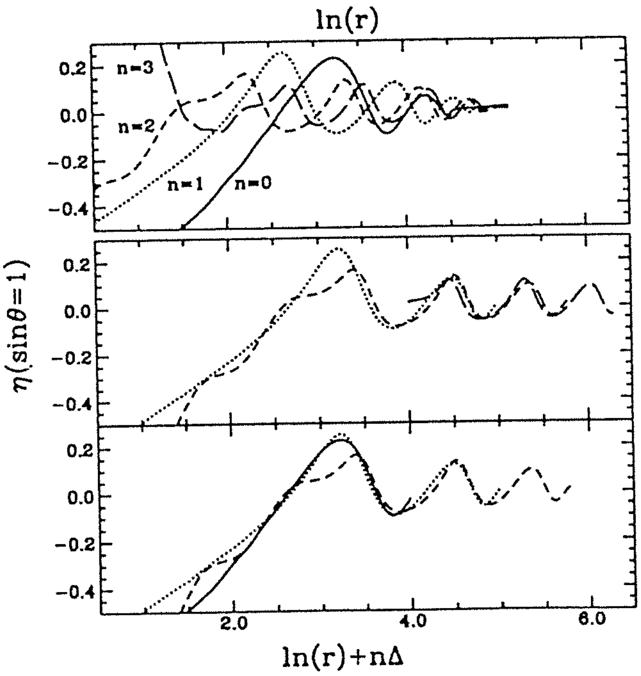

Figure 3.1: Scaling property of a near critical solution of axisymmetric gravitational wave collapse. Radial profiles of the metric function η (along $\theta = \pi/2$) plotted at four times corresponding to alternate maxima of the central value of the lapse function, α_c. The upper panel depicts all four profiles (labelled sequentially $n = 0 \div 3$) plotted versus $\rho = \ln r$. The two lower panels illustrate scaling by overlapping profiles that are shifted by $\rho \rightarrow \rho' = \rho + n\Delta$ with $\Delta \simeq 0.6$. Profiles $n = 0, 1, 2$ are plotted in the bottom panel and $n = 1, 2, 3$ in the middle panel.

the equatorial plane $\theta = \pi/2$). As can be seen, η exhibits an echoing in $\rho = \ln r$ of the form

$$\eta(\rho - \Delta, t_n) \simeq \eta(\rho, t_{n+1}). \tag{3.25}$$

The times t_n are found here, by using the central value of the lapse function $\alpha(t, r = 0)$ as a diagnostic to determine the completion of successive oscillations. The value of the scaling constant Δ was found $\Delta \simeq 0.6$, so a radial scale ratio $e^{\Delta} \simeq 1.8$ differs in this case from the corresponding value $e^{\Delta} \simeq 30$ $(\Delta \simeq 3.4)$ in scalar field collapse. This result having been obtained in numerical simulations with several different resolutions, appears to be robust.

3.4.1.3 Conclusions

The present state of knowledge of critical phenomena in gravitational collapse is rather rudimentary. First of all, we can ask how close the association really is to standard critical phenomena. One would like to have analytic models and more results of numerical simulations of other kinds of space-times, with different symmetries and sources, performed in particular with the adaptive-mesh-refinement scheme for two and three dimensional cases.

As far as the known examples are concerned, one may ask whether they represent the same universality class - probably yes, and on what does it depend - presumably they do depend on dimensionality of the model, number of Killing vectors. Interestingly, it seems that the black hole mass m_{BH} plays a rôle of the order parameter for these critical phenomena[13], like spontaneous magnetization for ferromagnets (below the Curie temperature) or $|\rho - \rho_c|$ for liquid-gas transition in the co-existence region. Indeed, black holes appear only for solutions with $p > p^\star$ and a black hole of infinitesimal mass is conjectured to exist at $p = p^\star$.

Choptuik demonstrated that details inherent in the original data are "washed out" in the strong field region $\mathcal{R}$ in near critical evolution. With each echo as $r \to 0$ and $T \to T^\star$ information may be steadily lost and its rate per echo may depend on the value of Δ. Likewise, an analogue of the correlation length ξ in statistical mechanics systems seems to be now the ratio of the radii of the outer edge of the scaling region, r_{max}, and the inner edge, r_n, of the innermost echo i.e., $\xi \sim r_{max}/r_n \sim e^{n\Delta}$. As $p \to p^\star$ an ever larger region (in terms of the scale r_n) becomes "correlated" with self similar echoes and $\xi \to \infty$, as it should for "critical" systems.

Recently Evans and Coleman [104] reported new research work into critical phenomena in the gravitational collapse. Their model employs a radiation fluid ($\gamma_{adiabatic} = \frac{4}{3}$) in the spherical symmetry. In this case some analytic progress was possible, starting from an *ansatz* of self-similarity (i.e. scale-invariance rather than scale-periodicity). In numerical

[13]Note that there also seem to exist phase transitions (of second order) when we cool down black hole with respect to the corresponding Schwarzschild temperature $T_S = (8\pi m)^{-1}$, by increasing its charge Q and angular momentum J at fixed total mass. The heat capacity c_{JQ} passes then from negative (for a Schwarzschild black hole) to positive values (for Kerr-Newman) through an infinite discontinuity [177].

calculations the power law dependence of black hole mass on $|p - p^{\star}|$ was obtained again with $\beta \approx 0.36$, as well as the evidence for a unique, self-similar critical solution in near-critical computations.

Finally, Strominger and Thorlacius [238] have reported the discovery of universality and mass-scaling in the context of the 2 *dim* semiclassical RST [216] model, which employs null matter as a matter source. This model is used as a simplified model of quantum black hole evaporation and is exactly soluble. It was analytically demonstrated that the system exhibits universal power law mass-scaling at the critical point with $\beta = \frac{1}{2}$. Near critical scaling solution interpreted to describe the formation and evaporation of an arbitrarily small black hole was also found. In this model there is no analogue of the self-similar oscillations.

Some analytic considerations on the subject were offered in [44] and [203], where a spherically symmetric Einstein equations coupled to a massless scalar field were solved assuming self-similar collapse, and a self-similar solution with a critical parameter and critical regime was obtained. In the first paper mentioned, an exact one-parameter (α) family of solutions is shown to exhibit a type of critical behaviour as discussed by Choptuik. For super-critical evolutions a quantity related to the mass of black hole exhibits a power law dependence on α, $m_{BH} \simeq |\alpha - \alpha_{crit}|^{1/2}$. The solution supports the conjecture that black hole formation initially occurs at infinitesimal mass.

The same model is considered in [203], where also Choptuik's scaling relations were derived. Critical behaviour was obtained which exhibits mass evolution $M_h \simeq (\frac{p-1}{8})^{1/2}v$ on the apparent horizon (M_h is the gravitational mass on apparent horizon, v the advanced time and p is the critical parameter and critical regime occurs for $p \to 1$). This equation exhibits mass evolution in some dynamical stage of gravitational collapse, instead of the asymptotic stage (at the future null infinity) $v \to \infty$, as evaluated by Choptuik.

If $X \equiv v/r$ the logarithmic evolution

$$\frac{1}{(p-1)X^2} = \log t - \log r$$

in a neighbourhood of $r = 0$ was derived. So for the self-similar solution the scaling relation $\phi(\log r - \Delta, \log t - \Delta) = \phi(\log r, \log t)$ clearly holds with a continuous parameter

Δ. Choptuik however observed Δ discrete, namely, $n\Delta^*$, $\Delta^* \approx 3.4$, $n = 1, 2, 3, \ldots$. But this is not crucial in the region where $\log r, \log t \gg \Delta^*$. The above in fact holds in the strong-field region when $p \to 1$ as then the apparent horizon is arbitrarily close to $r = 0$, and the spacing of $n\Delta^*$ is unimportant to recover approximately self-similarity outside the apparent horizon near $r = 0$.

In any case the value of the relevant critical exponent obtained in this model is not 0.37. The difference, though slight, is of crucial importance since this number should reflect a *deep property of the gravitational field equations.* Suffice it to notice that the experimental values for the critical exponent β (its explicit definition depends of course on the system considered) range from $0.305 \div 0.37$ for a variety of systems, like binary fluid, He I - He II transition, β–brass, a magnet (Fe) Ni and fluid (Xe), that are believed to belong to the same universality class [34]. It is therefore unlikely that the correct values of critical exponents can be obtained, unless the RG theory is made use of. The situation is roughly the same as it was with the study of phase transitions in condensed matter before the RG theory was available, when the mean field theory calculation was invariably giving critical exponents in form of rational numbers in disagreement with experiment.

3.5 Application of the RG in gravitation

The usual transformations invoking averaging over square blocks are designed mostly having ferromagnetic systems in mind. However, there are many problems suitable for the RG methods but that have not yet been expressed in such a way that they can be solved. They are amongst the hardest problems known in physics, where their difficulty can be traced to a multiplicity of scales [257]. The important issue in the proper application of the RG theory to a particular problem at hand is the choice of variables of the model and the RG map. For each new physical situation one has to "custom-make" the RG map, as Michael Fisher puts it clearly [110]:

> For any given Hamiltonian or class of Hamiltonians there is not just one renormalization group - "the renormalization group" as some people say - but rather there are many that might be introduced, and one must question, for

example, whether the process is best carried out in real space or momentum space and so on. A "good" renormalization group must be "apt" or appropriate for the problem at hand, and it must, in particular, "focus" properly on the critical phenomena of interest.

Some form of the Renormalization Group is active on any system where there are fluctuations present. The nature of the fluctuations does not need to be quantum. For example, they could be thermal fluctuations as in Statistical Mechanics, or they could be due to collisions as in Many-Body perturbed motion in GR, such as the ones due to "frictional" processes in the Universe, be them ones in the epochs of large entropy production [24], or those originating in dynamical friction, or purely chaotic processes due to the many body nature of the gravitational system.

If the universe can be considered as a complex general-relativistic many-body system (GRMBS), the question then arises whether one could apply the methods, much as in statistical mechanics to understand some of its features and undertake the study of its collective behaviour, local morphological problems, etc., by taking advantage of their suitability for the study of this class of problems. A recognition of powerful methods deriving from Renormalization Group, that can be used to study complex systems without loosing their physical picture encourages us to undertake this task. A common feature of many-body systems is that (under certain conditions) they may exhibit condensation-like phenomena (e.g. formation of Cooper pairs in superconductors). One can thus anticipate there would be a possibility of forming extended (elementary excitations) as well as localized states (solitary waves).

In the case of quantum fluctuations we have the Gell-Mann-Low version of the RG, which for discrete iterations (or blockings) is related to the Wilsonian RG. In field theory the use of the Gell-Mann-Low version of the RG is based upon perturbation theory, whilst the Wilson one has a direct geometrical interpretation (as already stressed in the previous section) and is in principle non-perturbative. For thermal fluctuations, we have RG which is similar to the encarnation appearing in the Ising model. In the case of Many-Body perturbed motions we have to develop the ideas.

The central object is the effective Hamiltonian which results from integration of degrees of freedom and fluctuations whose mass is larger than the inverse "Compton" wavelength that one is observing. This converts the RG into a perfect and natural tool for the construction of "bottom-up" scenarios (or simplicity-based) scenarios, where one derives properties of the whole on the basis of knowledge of the laws that control the parts, the components.

The idea is then to compute the effective hamiltonian at different scales and obtain the appropriate recursion relations. How does one actually do this? The strategy requires that one integrates out the fluctuations, and for this the natural scenario is a *path integral*; a path integral where one considers all the possible paths between two (initial and final) configurations, but where two paths differ by fluctuations in the phase space variables of the system. This is a well known and understood tool, whose application to GR-systems is immediate albeit not straightforward. We will consider only systems which can be described by a hamiltonian, although (at this point) we will not say more about the characteristics of the hamiltonian (i.e., whether it is constrained, restricted, etc.). All we care is that one can construct a path integral. If this can be done, then the following program for the computation of the recursion relations (and with them all sorts of critical exponents and phase transition analysis) can be implemented.

In the study of a many-body system, one considers the evolution of the system from an initial state into some final state in the presence of some kind of fluctuations. The unfolding is captured by a *kernel*, which describes how the many body system goes from one particular configuration to another particular configuration in the presence of fluctuations.

The kernel, as is well known, accounts for this transition when the hamiltonian controls the evolution and, in addition, there are present fluctuations, which open–up evolutionary paths that had these fluctuations not been present, would not have been available for the system to run into. From the kernel one can derive an effective hamiltonian that describes the effective form of the interaction between components of the system, while taking into consideration the many–body nature of the system, and whose analysis can give us information about its collective properties.

This is in fact familiar from QFT, namely, it was Feynman's (and Dirac's [75]) genius [109] to realize that the kernel (propagator) of the time evolution operator can be expressed as a sum over all possible paths connecting the points q and q' with weight factor $\exp[\imath S(q', q; T)/\hbar]$, where S is the action, i.e.,

$$\mathbb{K}(q', q; T) = \sum_{all\,paths} A e^{\imath S(q',q;T)/\hbar} \tag{3.26}$$

where, A is a normalization constant.

The same approach, in the sense of an analytical continuation was already known in mathematics due to Wiener in the study of stochastic processes.

Now, the same calculus can be employed to compute the partition function of a classical gravitational system.

The kernel can be computed [108] by using the euclidean formulation of the Feynman path integral, which after a trace (i.e., identification of initial and final states and then summing) yields the partition function of the many-body system. In fact, from the kernel one can compute all the statistical properties of the many-body system: configuration probabilities, free energies, entropies, specific heats, critical exponents, etc. The kernel $\mathbb{K}$, obeys the equation

$$\frac{\partial \mathbb{K}}{\partial \beta} = -\widehat{H}_c \mathbb{K}, \tag{3.27}$$

where, β is proportional to the inverse temperature and $\widehat{H}_c$ is the canonical hamiltonian for the system obtained by applying the principle of correspondence to H_c, which in this setting is just a convenient trick. Likewise, we do not need to deal with the ordering problem since we do not consider quantization of our system. As is well known, the solution to equation (3.27) can be represented by an euclidean path integral, given by

$$\mathbb{K}\left[\beta_f, \beta_i\right] = \int [dq]\,[dp] \exp\left\{-\int_{\beta_i}^{\beta_f} d\tau\, [p\dot{q} + H_c]\right\} \tag{3.28}$$

where, β_i and β_f are the initial and final inverse "temperatures". (This applies to any many body system, independently of whether it is or it is not gravitational.)

There are two qualities of GRMBS which are of interest in connection to the problem we are studying: (i) because of many–body motions, the possibility of chaotic motions and the associated fluctuations exists, and (ii) the hamiltonian of the system weakly vanishes. The first property gives rises to fluctuations in the GRMBS and hence to a full statistical behavior as described by a kernel or a partition function, while (ii) has led to the wide spread belief that the partition function Z of these systems is trivial ($Z \propto \exp -\beta H_c \sim 1$) and therefore unable to provide a useful description of their statistical properties. However, over the last decade, machinery has been developed in Quantum Cosmology which allows one to compute the path integral of equation (3.28) in the case of gravitational systems. The idea is that given a metric $g_{\mu\nu}(x)$ and a matter contents, we perform a $3+1$ splitting, write down the hamiltonian for this GRMBS and represent the kernel by means of a path integral with the constraint incorporated via a Lagrange multiplier. To actually calculate the kernel, we need to either solve the associated "diffusion equation" or resort to approximation techniques, such as steepest descent, perturbation theory, etc. This yields the kernel, an effective hamiltonian and, after taking traces, the partition function.

In the 3+1 split, the canonical hamiltonian of a gravitational system is proportional to the lapse function N: $H_c = NH$, where $H = 0$ is the hamiltonian constraint[14]. Fixing a gauge is equivalent to picking a value for N, with a convenient gauge being the $\dot{N} = 0$ gauge. Performing a simple change of variables in the euclidean action of equation (3.28), and rewriting it as

$$I = \int_{\beta_i N}^{\beta_f N} d\tau \, [p\dot{q} + H], \tag{3.29}$$

we find that when inserted into equations (3.28) and (3.27), it satisfies the following equation[15]

$$\frac{\partial \mathbb{K}}{\partial(\beta N)} = -\widehat{H}\mathbb{K} \tag{3.30}$$

[14] We consider "minisuperspace" *ansatz* which satisfies identically the momentum constraint.

[15] The proof that the equation below is satisfied by $\mathbb{K}$ is independent of the choice of gauge.

from which, as an offshoot, we identify the inverse "temperature" of the GRMBS as[16] being proportional to N. Furthermore, the thermal effective hamiltonian can now be obtained by exponentiating the result of the calculation of the kernel.

It is more interesting to obtain the effective hamiltonian, because one can then more easily see how to "block" the system. Once the system's effective hamiltonian has been derived from the kernel, it is necessary to proceed to "block" the system; this can be performed by "simply" using the formal relationship between the Landau model and the Ising model. In establishing this relationship, we will have to impose and take into account the existence of "restraints" on the dynamical variables; these restraints are typically of the form

$$\text{``}\sum_i s_i^2 = N\text{''}, \tag{3.31}$$

such as the ones that appear in the spherical model.

Having obtained the partition function, it is clear how to obtain the recursion relations.

3.6 Smoothing-out inhomogeneous cosmologies

Contents of this section is based on [61].

Some notions and facts relevant to the contents of this section are given in Appendix C (in particular, a definition of the Gromov distance can be found there), which the reader is encouraged to consult.

General mathematical preliminaries can be found in Appendix A.

3.6.1 Coarse-grained approach in Cosmology

A possible solution to the averaging problem would be to explicitly construct a procedure for carrying out the smoothing process in the full theory. Almost all existing attempts were concerned with the linearized theory (see chapter two), with a possible exception of [55] (see also [149]) and [263].

But our hope is that there is a smart and simpler way to the heart of the problem, borrowing from the known theories and methods of Statistical Mechanics, based on the

[16] A similar result has been found using a different approach by Brown and York [47].

real space Renormalization Group approach to study critical phenomena in lattice models [154, 257, 256] (as was discussed in previous sections of this chapter). Although the usual renormalization transformations invoking averaging over square blocks, are designed mostly having ferromagnetic systems in mind, there are many more problems suitable for the RG methods. These are problems, whose effective degrees of freedom of a physical system are scale dependent and their difficulty can be traced to a multiplicity of scales. The averaging problem in cosmology does belong to such a class.

Often a major step consists in finding a way of looking at things. Therefore it is probably worth stressing again that the problem we face with the averaged description in cosmology is effectively a question of how a system behaves under changes of "scales" and as such, it is most naturally addressed using the RG approach, understood here as a general strategy to handle problems of multiple length scales[17], which allows us to extract the long distance behaviour of the system at hand by making the scale successively coarser.

Some form of the RG is active on any system where there are fluctuations present (they by no means do need to be quantum). This is so, because one can integrate the fluctuations out of the physical quantities of interest, e.g. the partition function, and depending on the scale up to which one is integrating, the same quantities that emerge are different. The functional relation between them provide recursion relation between the physical parameters, the coupling constants, which characterize the physics at each scale, and this is precisely the RG. In cosmology, we have the curvature inhomogeneities and to consistently tackle this problem, we will have to consider a procedure operating on the metric, not only on or apart from the matter present.

To provide even more support in favour of the above idea, that a kind of the RG argument might be applicable to the problem of the inhomogeneous universe, let us notice that one can also be guided by the scaling ideas. Scaling is exhibited (approximately or exactly) by many natural phenomena and mathematical models and is quite useful for making predictions. The universe, namely, the distribution of matter in it, does exhibit certain scaling properties [40], of which the power law behaviour of the two-point correla-

[17] Although the renormalization procedure might seem purely formal, a physical sense can be attached to it, as supported by *universality* and *scaling*.

tion function for galaxies, clusters and quasars is a fair example. Scaling on its own right is deeply understood within the underlying mathematical scaffold which is the RG. These are hints therefore that one can regard the universe as a gravitational dynamical system not far from criticality (understood intuitively, by analogy with e.g. a ferromagnet). Later one can also try to qualify the precise nature of the critical behaviour within the phase transition context, but this will not be of our concern right now.

Real space renormalization techniques are mostly applicable to discretized models, based on a lattice. Therefore we now turn to describing a suitably discretized manifold model we are going to work with. Important difference from the usual lattice models of Statistical Mechanics is that in gravitation the "lattice" (whatever this means) is itself a dynamical variable.

3.6.1.1 Discretized manifold model

The approach taken is that of a (3 + 1) formulation of General Relativity (GR) [12]. Let us suppose we have a differentiable, compact 3−manifold (without a boundary) $\mathcal{M}$. Generally, in this case we will always assume that these manifolds possess certain natural constraints on their diameter and a suitably defined notion of curvature. The point of this requirement is that the manifolds, or more precisely the riemannian structures (classes of isometric Riemannian manifolds with respect to $Diff(\mathcal{M})$), can then be classified according to how they can be covered by small metric balls (to be defined later) and the space of such riemannian structures has some remarkable compactness properties. This is a classical result obtained by M .Gromov [127] (see also [117]). On a set of riemannian structures it is possible to introduce a distance function, the Gromov distance, which roughly speaking enables one to say something about how close particular manifolds are to each other (see figure (3.2)). For the Riemannian manifolds that can be considered close to each other (in the sense of Gromov distance) it is possible to cover them with the balls arranged in similar packing configurations [56].

In order to define such coverings [130], let us parameterize the geodesics by arc-length, and for any point $p \in \mathcal{M}$ let $d_{\mathcal{M}}(x,p)$ denotes the distance function of the generic point x from the chosen one p. Then for any given $\epsilon > 0$ it is always possible to find an ordered

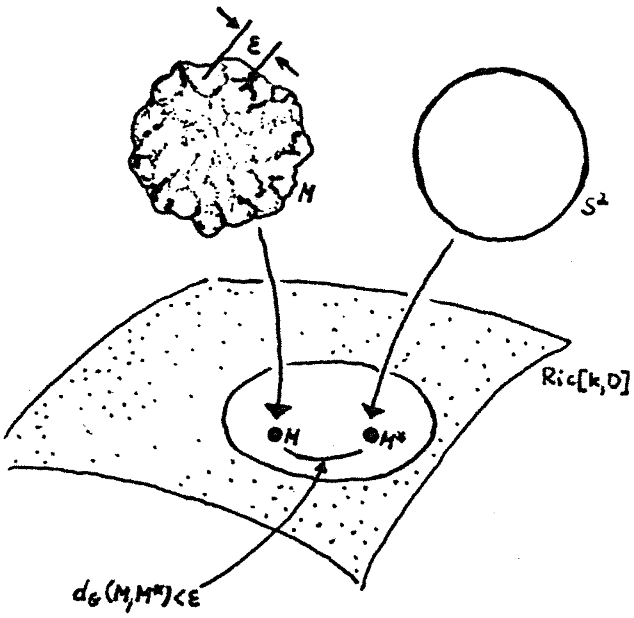

Figure 3.2: For each given ϵ, which is roughly speaking the typical curvature inhomogeneity on a manifold $\mathcal{M}$, one can find a model manifold $\mathcal{M}^*$, such that the Gromov distance between $\mathcal{M}$ and $\mathcal{M}^*$ is smaller than ϵ.

set of points $\{p_1, \ldots, p_N\}$ in $\mathcal{M}$, so that [130]

i) the open metric balls (the geodesic balls) $B_{\mathcal{M}}(p_i, \epsilon) = \{x \in \mathcal{M}| \ d_{\mathcal{M}}(x, p_i) < \epsilon\}$, $i = 1, \ldots, N$, cover $\mathcal{M}$; in other words the collection $\{p_1, \ldots, p_N\}$ is an ϵ-*net* in $\mathcal{M}$.

ii) the open balls $B_{\mathcal{M}}(p_i, \epsilon/2)$, $i = 1, \ldots, N$ are disjoint, i.e., $\{p_1, \ldots, p_N\}$ is a *minimal* ϵ-net in $\mathcal{M}$.

It is fair to say that as a consequence of the compactness properties of the set of riemannian structures that we consider, for each "length scale ϵ" there exists a finite number of "model" geometries, which describe with an $\epsilon-$approximation any given Riemannian geometry, i.e. given a ball of a certain radius $> \epsilon$ in any Riemannian manifold (with suitable restrictions on their volume, diameter and sectional curvature) there exists a ball metrically similar (up to an ϵ scale) in one of the model geometries, which does not retain the details of the original manifold on scales smaller than ϵ. Roughly speaking ϵ is a measure of the typical curvature inhomogeneity with respect to the model background. Let us stress that this is a highly non-trivial result, in the sense that the metrical properties of the manifolds from an infinite dimensional set, are up to an ϵ scale described by the metrical properties of just a finite number of model Riemannian manifolds.

The ϵ–nets underlying the balls-coverings precisely provide the discretized manifold model. This coarse graining of a manifold according to Gromow is the most natural coarse graining one can think of, pertinent for manifolds with a lower bound to the sectional curvature. This assumption does not limit the generality of our analysis, which is basically motivated by a concrete physical problem, whose nature allows us to deal from the beginning with manifolds, that are already in a certain sense quasi-homogeneous (Cf. comments on the solvability of Ricci-Hamilton flow, later on).

In what follows, when speaking of balls we will always mean geodesic balls here.

3.6.1.2 Averaging procedure

We assume that we have chosen a particular space-like hypersurface Σ of the 4–dimensional manifold $\mathcal{M}$, on which the average of a scalar function $f : \Sigma \to \mathbb{R}$ is given as

$$< f >_{\Sigma(g)} = \frac{\int_\Sigma f d\mu_g}{vol(\Sigma, g)}, \tag{3.32}$$

where $vol(\Sigma, g) = \int_\Sigma d\mu_g$ and μ_g is the Riemannian measure.

By U_ϵ we denote a set of geodesic balls $\{B(x_i, \epsilon)\}, i = 1, \dots, N$, such that $\{x_1, \dots, x_N\}$ is a minimal $\epsilon-$ net in Σ.

Then we can approximate (3.32) as

$$< f >_{U_\epsilon} \simeq \frac{\sum_{\{B_i\}} \int_{B_i} f d\mu_g}{\sum_{\{B_i\}} volB_i}. \tag{3.33}$$

where, we explicitly indicated the dependence of the average on a particular "covering", and $\{B_i\}$ denotes, for simplicity, $\{B(x_i, \epsilon)\}$.

It can easily be seen that for a minimal geodesic balls covering

$$< f >_{\epsilon/2} \; \leq \; < f >_g \; \leq \; < f >_\epsilon, \tag{3.34}$$

holds.

There are certain problems lurking that we have to clear up. Obviously, there are "unwanted" details affecting the average function over the discretized manifold as given by its partition with a collection of geodesic balls, the immediate one being the underlying discretization. The important question to ask is what happens to the average when we change the length scale. Depending on whether we are actually increasing or decreasing it, respectively less or more details of the underlying geometry, will be felt by the average values. The natural philosophy is that over scales big enough, no details are discerned since the homogeneity and isotropy prevails. Indeed, this is the reason why on constant curvature spaces averaging is well defined, since there one can move the balls freely and deform them, but by so doing no new geometric details that measure the inhomogeneities will be felt in the averaged values of quantities we are interested in.

A natural question to ask now is then how the geometry, i.e. curvature inhomogeneities must depend on the scale so that the average over the balls is scale independent, or equivalently, how do we have to deform the geometry in order to achieve the scaling limit, when size of the balls matters no longer.

First we give some details on how we calculate the averages according to (3.33). In order to do this, we employ a preferred system of coordinates on $\{B_i\}$ given by the local diffeomorphism

$$\exp_x : T_x\Sigma \to \Sigma \tag{3.35}$$

i.e. we make use of the exponential mapping

$$\varphi_i \equiv \exp|_{\exp^{-1} B_i \equiv D_i} : D_i \to B_i\,, \tag{3.36}$$

where $D_i = D(x_i, \epsilon)$ is the ball in $T_{x_i}\Sigma$[18].

On D_i we use polar coordinates and pull-back the riemannian measure accordingly

$$\varphi_i{}^*(\mu_g) = \theta(t, x_i)dt \otimes dx_i, \tag{3.37}$$

where, dx_i denotes the canonical measure (euclidean volume form) on the unit sphere $D(x_i, 1) = S_1^2 \subset T_{x_i}\Sigma$ and where dt is the Lebesgue measure on $\mathbb{R}$ $(t \geq 0)$.

[18] An instantaneous observer in $T_x\mathcal{M}$ observes the universe with the help of the exponential mapping, which just means projecting structures from an open neighborhood $U \subset \mathcal{M}$ of x by $\exp_x^{-1}$ and treating them as structures on $T_x\mathcal{M}$. If $u \in T_x\mathcal{M}$, then there exists a unique geodesic curve α, such that $\alpha'(0) = u$; the exponential mapping is defined by $\exp_x(u) = \alpha(1)$ (cf. [219]).

For t small enough one can prove Puiseux' formula

$$\theta(t,x) = t^{n-1}(1 - \frac{1}{3}r(x)t^2 + \mathcal{O}(t^2)) \tag{3.38}$$

where, $n = dim\,\Sigma$ and $r(x)$ is the Ricci curvature $Ric(g)$ (at the point x).

Using this result we have $vol(B_i) = \int_{S_1^2} \theta(t,x)dx_i dt$.

As we can see, the volume of a small geodesic ball in (Σ, g) is given by a power series expansion

$$V_m(r) = V_o(r)(1 + B_2 r^2 + B_4 r^4 + \ldots + B_{2k} r^{2k} + \ldots) \tag{3.39}$$

where, $V_o(r)$ is the volume of the Euclidean ball of the same dimension and radius. $B_2, B_4, \ldots$ are analytic functions on (Σ, g) which can be expressed by means of the curvature tensor and its covariant derivatives (these are all constants if the space is homogeneous)[125].

Further we consider the asymptotic expansion with respect to t [167]

$$\int_{B(x_i,t)} f d\mu_g = \omega_n t^n [f(x_i) + \frac{t^2}{2(n+2)}(\Delta f(x_i) - \frac{R(x_i)}{3} f(x_i)) + \mathcal{O}(t^2)] \tag{3.40}$$

where, ω_n is the volume of the unit ball of $\mathbb{R}^n$, R is the scalar curvature at the center of the ball and Δ the Laplacian operator relative to the manifold.

Substituting $f = 1$ in the above formula we get the asymptotic expansion of the volume of the geodesic ball:

$$vol(B(x_i,t)) = \omega_n t^n (1 - \frac{R(x_i)}{6(n+2)} t^2 + \mathcal{O}(t^2)) \tag{3.41}$$

These formulas are what we need in order to calculate how the average value behaves when we change the scale; specifically let us assume that we change slightly the ball's radius by a small quantity β from its fixed value, say ϵ_o. Therefore we have to calculate

$$< f >_{\epsilon_o+\beta} = \frac{\sum_i [f_i + (\frac{\Delta f_i - R_i f_i/3}{2(n+2)})(\epsilon_o + \beta)^2]}{\sum_i [1 - \frac{R_i}{6(n+2)}(\epsilon_o + \beta)^2]}, \tag{3.42}$$

where, we have introduced somewhat simplified but otherwise obvious notation. Now, we can either take the derivative with respect to β of the above expression and carry out a rather tedious calculation. Otherwise, we notice that upon expanding the expression in β

we get

$$< f >_{\epsilon_o+\beta} \simeq < f >_{\epsilon_o} +2\frac{\sum_i [\frac{\Delta f_i - R_i f_i/3}{2(n+2)}]\epsilon_o}{\sum_i [1-\frac{R_i}{6(n+2)}\epsilon_o^2]}\beta - 2\frac{\sum_j (-\frac{R_j}{6(n+2)}\epsilon_o)}{\sum_j [1-\frac{R_j}{6(n+2)}\epsilon_o^2]} < f >_{\epsilon_o} \beta, \tag{3.43}$$

to leading order. With this the final result is obtained in the following form

$$\frac{1}{\beta}\frac{d}{d\beta} < f >_{\epsilon_o+\beta} = \frac{1}{n+2} < \Delta f >_{\epsilon_o} \epsilon_o + \frac{1}{3(n+2)}[< R >_{\epsilon_o} < f >_{\epsilon_o} - < Rf >_{\epsilon_o}]\epsilon_o \tag{3.44}$$

In the next section we will discuss the consequences of this formula and the connection of our averaging procedure with the Ricci-Hamilton flow.

3.6.2 Fixed point and Ricci-Hamilton flow

The implications of this result, equation (3.44), follow by considering the average $< f >$ as a functional of the metric, and thinkng of the metric g, as a running coupling constant, depending on the cut-off β. It can be easily verified that we can equivalently rewrite the second term on the right hand side of (3.44) as

$$< R >< f > - < Rf >= -D < f > \cdot \frac{\partial g_{ab}}{\partial \beta}, \tag{3.45}$$

where, $D < f > \cdot \partial g_{ab}/\partial\beta$ denotes the formal linearization of the functional $< f > (\epsilon, g)$ in the direction of the symmetric 2−tensor $\partial g_{ab}/\partial\beta$, and where

$$\frac{\partial g_{ab}(\beta)}{\partial \beta} = \frac{2}{3} < R(\beta) >_\beta g_{ab}(\beta) - 2R_{ab}(\beta), \tag{3.46}$$

$R_{ab}(\beta)$ being the components of the Ricci tensor $Ric(g(\beta))$ and $< R(\beta) >_\beta$ the average scalar curvature, given by

$$< R(\beta) >_\beta = \frac{1}{vol(\Sigma)} \int_\Sigma R(\beta) d\mu_\beta. \tag{3.47}$$

Indeed, the linearization of $< f >$ in the direction of the generic 2−tensor $\partial g_{ab}/\partial\beta$ is provided by

$$D < f > \cdot \frac{\partial g_{ab}}{\partial \beta} = \frac{1}{2} < f g^{ab} \frac{\partial}{\partial\beta} g_{ab} > -\frac{1}{2} < f >< g^{ab}\frac{\partial}{\partial\beta} g_{ab} >, \tag{3.48}$$

so (3.45) follows, given the expression (3.46) for $\partial g_{ab}/\partial\beta$.

With these remarks out of the way, we can interpret (3.44) adopting the viewpoint motivated by the RG. First of all, let us remark that there are two length scales at work in (3.44). One is simply the cut-off ϵ, the other is the true geometric (and physical) scale, which affects the definition of the average $< f >$, namely, the typical size of curvature inhomogeneities. The ϵ−length scale can be thought of as equivalent to the "atomic" length scale of definition, related to the lattice spacing, as in statistical mechanics models, whereas the other length scale, much larger than the scale of definition is the scale on which physical phenomena occur, namely it can be thought of as the "correlation length" scale. Our approach is to use (3.44), together with (3.45) and (3.46) in order to compare $< f >_\epsilon$ for different intrinsic length scales. In addition we require that there are no other lengths scales in the problem, and that these two scales join together smoothly.

To be more explicit, let g_1 and g_2 be two different riemannian metrics, corresponding to different riemannian structures, thus to different length scales of the typical inhomogeneities of the geometry. For a given choice of the cut-off ϵ, we let $< f >_\epsilon (g_1)$ and $< f >_\epsilon (g_2)$ denote the ϵ−approximation of the average value of f, associated respectively with g_1 and g_2. If we change the cut-off ϵ, holding fixed the metrics g_1 and g_2, the two expressions for $< f >_\epsilon (g_1)$ and $< f >_\epsilon (g_2)$ flow independently according to their corresponding equation (3.44), which just shows that the average values obviously depend, in a sensible way, on the size of the inhomogeneities of the underlying geometry. This is so, simply because such inhomogeneities have not been ironed out. However, as suggested by rewriting (3.45), we may let the metrics g_1 and g_2 flow with ϵ, according to equation (3.46). If as $\beta \to \infty$, g_1 and g_2 both approach the same constant curvature manifold $(\Sigma, \overline{g})$ (and as we discuss below, this is indeed the case in many circumstances), then $< f >_\epsilon (g_1)$ and $< f >_\epsilon (g_2)$ flow correspondingly, as ϵ is varied ($\beta \to \infty$), to the same value $< \overline{f} >$, which no longer depends on the curvature inhomogeneities. What happened is that by "renormalizing" the metric g the inhomogeneities have been smoothed-out.

Exactly, what this means is that a fixed point of the flow (3.44) has been reached, where

$$\frac{1}{\beta}\frac{d}{d\beta} < f >_{\epsilon_o+\beta} = 0, \tag{3.49}$$

is satisfied[19]. Then (3.46) appears as the beta function for the RG associated with (3.44), if we consider, as we should, the metric g as a cut-off dependent coupling.

[19]This is true only in dimension $n = 3$, as 3−manifolds with constant (sectional) curvature are Einstein manifolds, and vice versa [30]. This means that the Ricci curvature is isotropic and the source term on

The metric flow (3.46) is known as the *Ricci-Hamilton flow* [132], studied in connection with quasi-parabolic flows on manifolds. It has a number of useful properties (apart from being volume preserving, which is simply a consequence of the normalization chosen), namely, any symmetries of $g_{ab}(\beta_o)$ are preserved along the $g_{ab}(\beta)$ flow for all $\beta > \beta_o$, and the limiting metric (if attained) $\overline{g}_{ab} = \lim_{\beta\to\infty} g_{ab}(\beta)$ has constant sectional curvature. Thus equation (3.46), with the initial condition $g_{ab}(0) = g_{ab}$ defines (when solvable) a smooth family of deformations of the initial 3−manifold, deforming it into a 3−space of constant curvature.

The point of the above discussion is therefore that in order to arrive at a fixed point of the RG flow (3.44), the geometry has to be deformed according to the Hamilton flow (3.46). In this setting of the problem the Ricci-Hamilton flow appears naturally and in fact the approach we propose, enables to attach a physical meaning to it within the coarse-graining picture. This element was lacking in [55], where Hamilton's theorem appears rather *ad hoc*. On the other hand, our approach demonstrates that the smoothing issue is deeply connected with the geometry and exhibits how this relationship works.

It must be stressed that the RG flow (3.44) and the associated beta function (3.46) have some characteristics, which are quite peculiar to that particular problem we are addressing. In particular, the metric g, here seen in the rôle of the coupling constant, determines at the same time typical (physical) length scales over which curvature inhomogeneities enter the problem. Ultimately, this is due to the geometrization of gravity, and it is this property that allows to smooth-out the geometry by "renormalizing" the coupling according to (3.46). In this connection, it must also be said that the onset of a regime where the average $< f >$ does not feel any longer curvature inhomogeneities, is not to be associated here with a critical behaviour involving phase transition mechanisms. The occurrence of large length scale correlations is, on a superficial reading, simply due to the (weakly)-parabolic nature of equation (3.46). However, at a deeper level we may regain a point of view more directly connected with critical phenomena. Again the parabolic nature of (3.46) suggests a clue to this. If we linearize it in the direction of h_{ab}, a generic perturbation of a given metric g, then we get a linear (weakly)-parabolic initial value problem for the perturbation h_{ab}. We

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the right hand side of (3.44) vanishes. The average value of the Laplacian vanishes for closed manifolds without boundaries and so is of no concern here.

can interpret this problem in probabilistic terms by considering the linearized Ricci tensor, as a generator of a stochastic process in the space of all perturbations h_{ab} (the space of symmetric 2−tensors). Geometrically speaking, this will generate a Brownian-like motion in the space of riemannian metrics in a (sufficiently small) neighborhood of the metric g. For a given g, the correlation lengths of the resulting fluctuating geometries $(\Sigma, g + \epsilon h)$ would be exponentially damped. Long range correlations are instead expected, if g is an Einstein metric, since the space of inequivalent possible h is in this case collapsed into a finite dimensional one (which otherwise is infinite dimensional).

3.6.2.1 The issue of choice of slice Σ

It must be stressed that Hamilton's initial value problem has global solutions only under certain restrictions, for instance, if the Ricci tensor associated with the metric is a positive bilinear form, then $g(\beta)$ flows with (3.46) to the constant curvature metric on the 3−sphere. Further non-trivial examples have been provided in [150, 133] and [60]. In this connection it might seem that the approach suggested here can be criticized as strongly dependent upon too restrictive conditions on the given metric g. Since g depends on the choice of the slice where we average, the real issue seems to be that our procedure relies too much on the choice of a particular slice.

However, this sort of criticism is rather superficial. It is clear that to ask for an approach capable of smoothing any 3−manifold is something not to be expected on physical grounds. We cannot possibly average over all geometrical fluctuations, from the smallest to the largest ones. What is rather more natural is averaging in the regions where the fluctuations are of comparable size, though not at all necessarily small. In this connection it is remarkable that the same attitude naturally finds its analytic formulation in what is known as Hamilton's program. We already said that the Ricci flow (3.46), while always solvable for sufficiently small β [132], may not yield for a non-singular solution as $\beta \to \infty$. Hamilton noticed that there are patterns in the kind of singularities that may develop as $\beta \to \infty$. Typically the curvature blows up, but in a very regular way (e.g. for $S^1 \times S^2$ with the standard symmetric metric). This has led him to a research program which roughly speaking amounts to saying that any 3−manifold can be decomposed into pieces on which the Ricci-Hamilton flow is global and thereby each of these pieces can be smoothly deformed into an Einstein manifold. Singularities may develop in the regions connecting the smoothable pieces, but such singularities are of a finite number of types and all of a symmetric nature (namely, if they are blown up, they are associated with symmetric manifolds as $S^1 \times S^2$, or $T^2 \times S^1$, etc. [60, 150]).

It may be said that Hamilton's program is an analytic approach to prove Thurston's conjecture, which claims that any closed 3−manifold can be cut into pieces, such that each of them admits a locally homogeneous geometry[20]. It is clear that given the framework of Hamilton's program, the problem of slice dependence in our approach takes on a completely different flavour. Given any (spatial) slice Σ, it can be divided into regions which can be smoothed and regions, where the smoothing may not be possible. The smoothable patches should correspond to regions of the manifold where the geometry does not fluctuate too wildly. The "unsmoothable" patches should instead correspond to the transition regions between any two patches with vastly different patterns of curvature fluctuations.

Obviously, these are heuristic remarks relying on a not proven (yet) conjecture[21]. But in our opinion, it is quite intriguing that motivations coming from a physical problem like the one addressed here, namely, the smoothing of cosmological models, and from geometry, go hand in hand in such a way.

It is also clear that on physical grounds, a proper choice of the slice Σ can be addressed only when we deal with the averaging of sources.

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[20]This decomposition is done by cutting $\mathcal{M}$ along 2-spheres and tori, and gluing 3-balls to the resulting boundary spheres on each piece. The components (the simpler manifolds) are conjectured to admit one and only one of eight possible geometric structures. The situation is quite complicated in reality, firstly, by the fact that there are locally homogeneous structures on 3-manifolds that are not isometric to one of the three constant curvature spaces: S^3, E^3, H^3, and secondly that direct sums of two or more of these (eight) geometries may not admit one of the eight geometries.

Clearly, Thurston's proposition of a geometric classification of the topologies of 3-manifolds is of relevance to theoretical physics. Locally homogeneous models in relativistic cosmology (Bianchi, Kantowski-Sachs) basically make use of the eight Thurston's geometries. (See [244] and for a semi-popular account see also [245]).

[21]Hamilton's program for using the Ricci-Hamilton flow to study Thurston's 3-dimensional geometrization conjecture requires one to understand the Ricci-Hamilton flow of all locally homogeneous geometries on closed 3-manifolds.

The idea to prove this conjecture using Ricci-Hamilton flow is the following: choose an arbitrary metric on $\mathcal{M}$ and deform it via the Ricci-Hamilton flow equation (normalized). One hopes to relate the local singularities of the flow to the manifold decomposition in Thurston's conjecture, and then to show that the Ricci-Hamilton flow of the geometry away from each of the singularities approaches that of a locally homogeneous geometry in each disconnected piece.

3.6.3 Stability of averaging

The relative slopes of $< f >_\epsilon (g_1)$ and $< f >_\epsilon (g_2)$ as $\epsilon \to 0$, and for g_1 in a neighborhood of g_2, are of some relevance to our discussion. In a standard RG analysis such relative slopes are related to critical exponents. Here, as we said the framework is slightly different and such slopes are related to the stability of averaging. Given the averaging procedure for $< f >_\epsilon (g_1)$ as $\epsilon \to 0$, with g_1 "renormalized" according to the Ricci-Hamilton flow (3.46), one can sensibly ask what happens if g_1 is slightly perturbed, namely if we replace it by

$$g_{ab} \to g_{ab} + \delta K_{ab}, \tag{3.50}$$

where K_{ab} is a symmetric bilinear form (a choice of the symbol is quite intentional, since later the above consideration will be applied to the second fundamental form). It is easily seen that if g_1 is scaled according to the Ricci- Hamilton flow (3.46), then K_{ab} gets renormalized according to the linearized Ricci-Hamilton flow, namely (β in the brackets suppressed)

$$\begin{aligned} \frac{\partial}{\partial\beta}K_{ab} &= \frac{2}{3} < R > K_{ab} + \frac{2}{3}g_{ab}[\frac{1}{2} < Rg^{ab}K_{ab} > -\frac{1}{2} < R >< g^{ab}K_{ab} > - \\ &\quad < R^{ab}K_{ab} >] - \Delta_L K_{ab} + 2[div^*(div(K - \frac{1}{2}(Tr\, K)\, g))]_{ab}, \end{aligned} \tag{3.51}$$

with the initial data $K_{ab}(\beta = 0) = K_{ab}$, where, $K \in S^2\Sigma$ is a given symmetric bilinear form, Δ_L is the Lichnerowicz-DeRham Laplacian on bilinear forms,

$$\Delta_L K_{ab} \equiv -\nabla^s\nabla_s K_{ab} + R_{as}K^s_{\ b} + R_{bs}K^s_{\ a} - 2R_{asbt}K^{st} \tag{3.52}$$

and the operators Δ_L, div^*, div and Tr are considered with respect to the flow of metric $(g, \beta) \to g(\beta)$, solution of (3.46). The div (here, minus the usual divergence) is the divergence operator on $S^2\Sigma$, div^* is the L^2-adjoint of div, acting from the space of vector fields on Σ to $S^2\Sigma$ (it can be identified with $\frac{1}{2}[Lie\ derivative]$ of the metric tensor along a vector field).

Note that a $K(\beta)$ solution of the linear (weakly) parabolic initial value problem (3.51) always exists and is unique [132], and represents an infinitesimal deformation of metrics connecting the two neighbouring flows of metrics $g(\beta)$ and $g'(\beta)$ (obtained as solutions of problem (3.46) with initial data $g(\beta = 0) = g$ and $g'(\beta = 0) = g(\beta = 0) + \epsilon K(\beta =$

$0) + \mathcal{O}(\epsilon^2)$, respectively). For what concerns the structure of this solution, one can verify that corresponding to the "trivial" initial datum $K(\beta = 0) = L_X g$ (where $X : \Sigma \to T\Sigma$ is a smooth vector field on Σ), the solution of (3.51) is provided by

$$K_{ab}(\beta) = L_X g_{ab}(\beta). \tag{3.53}$$

This property expresses the $Diff(\Sigma)$ equivariance of the Ricci-Hamilton flow. (Notice that X is β independent).

The above follows by noticing that along the trajectories of the flow $(\beta, g) \to g(\beta)$ solution of (3.46) we have

$$\frac{\partial}{\partial \beta} L_X g_{ab}(\beta) = L_X[\frac{\partial}{\partial \beta} g_{ab}(\beta)] = \frac{2}{3} < R(\beta) >_\beta L_X g_{ab}(\beta) - 2L_X R_{ab}(\beta). \tag{3.54}$$

But the $Diff(\Sigma)$ equivariance of the Ricci tensor, i.e. that fact that $Ric(\varphi^* g) = \varphi^* Ric(g)$ for any smooth diffeomorphism $\varphi : \Sigma \to \Sigma$, implies that

$$L_X R_{ab} = D\, Ric(g) \cdot L_X g_{ab}, \tag{3.55}$$

where, $D\, Ric(g)K$ is the formal linearization of $Ric(g)$, around g, in the direction K:

$$\begin{aligned} D\, Ric(g) \cdot K &\equiv \frac{d}{dt}[Ric(g + tK)]_{t=0} \\ &= \frac{1}{2}\Delta_L K - div^*[div(K - \frac{1}{2}(Tr\, K)g)]. \end{aligned} \tag{3.56}$$

Upon introducing (3.55) in (3.54) we get

$$\frac{\partial}{\partial \beta} L_X g_{ab}(\beta) = \frac{2}{3} < R(\beta) >_\beta L_X g_{ab}(\beta) - 2D\, Ric(g(\beta)) \cdot L_X g_{ab}(\beta). \tag{3.57}$$

It is easily checked that the right hand side of the above expression coincides with the right hand side of (3.51), when this latter is evaluated for $K_{ab}(\beta) = L_X g_{ab}(\beta)$. Hence $L_X g_{ab}(\beta)$ solves the partial differential equation (3.51) and, since for $\beta = 0$, $K_{ab} = L_X g_{ab}$, the uniqueness of any solution of the initial value problem (3.51) implies that $K_{ab}(\beta) = L_X g_{ab}(\beta)$ whenever $K_{ab}(\beta = 0) = L_X g_{ab}$, as stated.

Moreover, if $K(\beta)$ is a solution of (3.51) with initial datum $K(\beta = 0) = K$, then the space average of $Tr\, K(\beta)$ over $(\Sigma, g(\beta))$ is preserved along the flow $(\beta, g) \to g(\beta)$, namely

$$< Tr\, K(\beta) >_\beta = < Tr\, K >_o, \quad 0 \leq \beta < \infty. \tag{3.58}$$

This property of the solutions of (3.51) is an immediate consequence of the volume-preserving character of the Ricci-Hamilton flow.

Finally, another relevant property of the initial value problem (3.51) can be stated as follows. If $(\beta, K_{ab}) \to K_{ab}(\beta)$ is the flow solution of (3.51) with initial datum $K_{ab}(\beta = 0) = K_{ab}$, then it can always be written as [176]

$$K_{ab}(\beta) = \hat{K}_{ab}(\beta) + L_{v(\beta)} g_{ab}(\beta) \tag{3.59}$$

where, the bilinear form $\hat{K}_{ab}(\beta)$ and the $\beta-$dependent vector field $v(\beta)$, respectively, are the solutions of the initial value problems:

$$\begin{aligned}&\frac{\partial}{\partial\beta}\hat{K}_{ab} = \frac{2}{3} < R > \hat{K}_{ab} + \frac{2}{3} g_{ab}[\frac{1}{2} < R g^{ab}\hat{K}_{ab} > -\frac{1}{2} < R >< g^{ab}\hat{K}_{ab} > - \\ &< R^{ab}\hat{K}_{ab} >] - \Delta_L \hat{K}_{ab} \\ &\hat{K}_{ab}(\beta = 0) = K_{ab},\end{aligned} \tag{3.60}$$

and

$$\frac{\partial}{\partial\beta} v_a(\beta) = -\nabla^c(\hat{K}_{ca} - \frac{1}{2}\hat{K}^{rs} g_{rs} g_{ca}), \quad v(\beta = 0) = 0. \tag{3.61}$$

To summarize, as $\beta \to \infty$, $K_{ab}(\beta)$ may either approach a Lie derivative term, such as $L_{v(\beta)} g_{ab}(\beta)$, or a non-trivial deformation $\hat{K}_{ab}(\beta)$ [176]. The non-trivial deformation is present only if the corresponding Ricci-Hamilton flow for $g_{ab}(\beta)$ approach an Einstein metric on Σ, which is not isolated. In such a case, (e.g. flat tori) there is a finite dimensional set of such Einstein metrics, and the non-trivial $\hat{K}_{ab}$ simply are the infinitesimal deformations connecting one Einstein metric $\bar{g}_1$ in Σ and infinitesimally non-equivalent one. Also in this case the Lie derivative term may be present. What this simply represents is a reparameterization of the metric $\bar{g}_1$ (under the action of the infinitesimal group of diffeomorphisms generated by v (see equation (3.59)) ("gauge artifact"). This latter Lie derivative term is the only surviving term when $\bar{g}$ is isolated (like in the case of the round 3−sphere. As is known, the round metric $\bar{g}$ on the 3−sphere S^3 is isolated, in the sense that there are not volume-preserving infinitesimal deformations of $\bar{g}$, mapping $\bar{g}$ to another inequivalent constant curvature metric $\bar{g}'$.)

Thus the flow (3.60), (3.61), gives us information on the stability of the averaging prescription for $< f >_\epsilon$ ($\epsilon \to 0$), in the sense that it tells us if the "fixed point" is a critical point or rather a "critical hypersurface". In this latter case, the averaging is rather sensible to the "original metric".

3.6.3.1 RG view point of smoothing-out picture.

It must be said that the question of global behaviour of the RG flow, i.e. its infra-red regime is a difficult problem, to be explored separately in every particular case studied. What can be said here is that we can have complex behaviour in the RG equations simply as "gauge artifacts" and the true complex behaviour with possibly physical consequences. In particular, chaos and limit cycles in RG equations do not seem to lead to unphysical consequences. The question however is whether such behaviour actually occurs in real systems.

As we could see, an important feature of the RG procedure is that it can be iterated. Strictly speaking the inverse of the RG transformation is not defined and RG actually is a semi-group. In other words, in the successive steps of the RG action there is some loss of information, due to the fact that – if we think of the RG transformation as a change of variables, or coordinates – these successive changes will not be one to one or everywhere defined as a rule, otherwise there would be no simplification of the original problem. The semi-group property of the RG is related to a built-in irreversibility of the RG transformation. These characteristics of the RG have led to the conjecture that perhaps an entropy-like function, monotonically increasing along any RG flow, can be defined in any number of dimensions, thus generalizing Zamolodchikov's c–theorem [271], which in $dim = 2$ appears to forbid any of the complex phenomena one could have for non-trivial RG flows. These arguments work for 2–dim Euclidean field theories under certain general assumptions, unitarity being one of them. However gravitational systems are constrained *per se*, i.e. they are presymplectic rather than symplectic and by this virtue the case to investigate is more involved[22]. Notice independently that also in the case of e.g. chaotic RG flow, one would not expect an entropy function to exist, though

[22]That a symplectic structure should be relevant to RG flow was first suggested in [202]. This point is worth exploring in our opinion, especially for gravitation.

here also the degrees of freedom are continually being integrated out, but nevertheless a single well defined fixed point is never approached.

Another interesting parallel supported by the RG approach to our problem is the one we can draw between the universality classes, and what we would accordingly name here, the riemannian universality classes. For a particular universality class, roughly speaking the critical exponents depend only on such features of the system as its dimensionality, symmetry and the dimension of the order parameter, and once the location of the points of phase transition are known, there is a remarkable universality to the behaviour of the system near the critical point, namely, the same large-length-scale physics emerges irrespective of nitty-gritty microscopic details. Here in geometric language, the universality classes would represent the Einstein metrics $\bar{g}$, whether isolated or not. In our case the only *relevant* perturbation is the metric. The rate at which it grows determines the critical exponent ν. There will be various such exponents for each Einstein manifold, and one of them should be possessed by our universe. This is however difficult to estimate even in principle, since in reality we do not know the global topology of the universe. Enough to say that the FLRW metric can be realized for various topologically distinct spaces, in fact infinite number of them for closed and open models [259].

In case, when the Ricci-Hamilton flow for $g_{ab}(\beta)$ approaches an Einstein metric, which is not isolated a phenomenon similar to the $cross-over$ can take place. As a non-trivial deformation is then present, there exist infinitesimal (volume preserving) deformations of $\bar{g}$ mapping it to another inequivalent const curvature metric $\bar{g}'$; by tuning its various multi-critical points and transitions between them can be present. Indeed in such a case one deals with a finite dimensional set of the Einstein metric in the form of a critical surface.

3.6.4 Averaging of the matter

Until now, our discussion has addressed mainly geometrical issues and the function f entering the cut-off dependent averaging $< f >_\epsilon$ was not specified. Now we wish to apply the results of the previous paragraph to the averaging of the matter sources, namely, the matter density μ, the spatial stress tensor s_{ab} and the momentum density j_a, entering in the phenomenological description of the matter energy-momentum tensor with respect to the instantaneous observer comoving with Σ, viz.

$$T_{ab} = \mu u_a u_b + j_a u_b + j_b u_a + s_{ab}, \tag{3.62}$$

where, u is a unit, future directed normal to the slice Σ ($u^a u_a = -1$).

When the cosmological matter fluid is in equilibrium, a unique 4−velocity vector u^a can be defined and thereby the associated local Lorentz rest frame can sensibly be picked up, where the stress-energy tensor takes the perfect fluid form (demanding isotropy only)

$$T_{ab} = \mu u_a u_b + s_{ab}. \tag{3.63}$$

In any other frame n^a an energy flux $\tilde{j}_a = -\tilde{h}_a{}^b T_{bc} n^c$ would appear in the above expression [160], where $\tilde{h}_{ab}$ is the projection tensor, namely, $\tilde{h}_{ab} = g_{ab} + n_a n_b$. The above perfect fluid form form of the energy momentum tensor is assumed to hold in FLRW space-times[23].

In fact, since in the present epoch the universe is mainly matter dominated, i.e. the pressure can be safely neglected, it is in principle justified to start our analysis with

$$T_{ab} = \mu u_a u_b \tag{3.64}$$

form of the stress-energy tensor, which holds in the Lorentz rest frame, i.e. terms involving the momentum density are absent precisely due to the fact of preferred slicing chosen.

In any case, the smoothing of the spatial stress tensor s_{ab} can be carried out by first averaging s_{ab}, at any given point x, over all possible directions, and then by ϵ−averaging the resulting expression over Σ, namely

$$s_{ab} \to < (vol(S^2))^{-1} \int_{S^2} s_x(v,v) d\mu_g >_\epsilon \tag{3.65}$$

where, the integration is over a unit two-sphere S^2.

Since,

$$\int_{S^2} s_x(v,v) d\mu_g = \frac{1}{3} vol(S^2) Tr\, s, \tag{3.66}$$

[23] Nevertheless, even in an exact $k = 0$ or $k = +1$ FLRW universe there can be a non-zero particle flux tilted with respect to the frame of the fundamental observers, even though T_{ab} has the the perfect fluid form. This is possible if the macroscopic quantities (e.g. a particle flux) are derived from kinetic theory with anisotropic distribution function $f(x^i, p^a)$ [97].

we get

$$s_{ab} \rightarrow < p >_{\epsilon}, \tag{3.67}$$

where, p is the pressure associated with s_{ab}.

The smoothing of the matter density μ is naturally accomplished by the mapping

$$\mu \rightarrow \lim_{\epsilon \rightarrow 0,\ g\ flows} < \mu >_{\epsilon} \equiv \bar{\mu}, \tag{3.68}$$

as introduced in section 3.6.1.2, where by $g\ flows$ is understood the "renormalization" of g according to (3.46).

Notice that $\bar{\mu}$ as in (3.68) is the average density over real (and at each step appropriately deformed) geometry, which in the limit ($\beta \rightarrow \infty$) (supposing it exists) is evaluated over the constant curvature manifold. In this sense it is in principle possible to calculate $\bar{\mu}$; as $\epsilon \rightarrow 0$, we rescale also the metric g according to the Ricci-Hamilton flow (3.46) and what happens is that $\bar{\mu}$ is actually provided by

$$\bar{\mu} \equiv \lim_{\beta \rightarrow \infty} < \mu >_{\beta} = \lim_{\beta \rightarrow \infty} \frac{\int \mu \, d\mu_g(\beta)}{vol(\Sigma, g(\beta))}, \tag{3.69}$$

along the solution $g(\beta)$ of (3.46). A differential relation connecting this average $< \mu >_{\beta}$ with the geometry of $(\Sigma, g(\beta))$, can be shown to be the following

$$\frac{\partial < \mu >_{\beta}}{\partial \beta} = < R >_{\beta} < \mu >_{\beta} - < \mu R >_{\beta}, \tag{3.70}$$

with $\bar{\mu} = < \mu >_o$ for $\beta \rightarrow \infty$.

Therefore $\bar{\mu}$, as the average over the real geometry, corresponds to what one would really liked to do in observational cosmology in order to obtain the so-called *empirical* average density $< \mu >_o$. But in reality it is much easier to obtain the empirical average as the average density over the constant curvature manifold – it is this density that enters the Friedmann equation.

Now, for initial inhomogeneous manifold we start with we require Hamiltonian constraint to hold, namely

$$8\pi G \mu = R(g) - K^{ab} K_{ab} + k^2, \tag{3.71}$$

where, $k \equiv K^a{}_a$.

Clearly, upon the averaging considered here, there is no reason why the above constraint should remain valid.

Of course, the second fundamental form K_{ab} is being averaged as well. As it is related to the stability of averaging, there is only one way to smooth it. To actually obtain the deforming flow for K_{ab}, let us notice that the bilinear form $K(\beta) = \lim_{\delta\to 0}[g'(\beta) - g(\beta)]\delta^{-1}$ appears as a "connecting vector" of the flow $g(\beta)$ solution to (3.46), and a nearby flow $g'(\beta)$ with initial condition $g'(0) = g + \delta K$. Therefore the β evolution of $K(\beta)$ is given by the equation (3.51). According to equation (3.59) as $\beta \to \infty$ the solution of this initial value problem $K(\beta)$ approaches a non-trivial deformation $\lim_{\beta\to\infty} \hat{K}(\beta)$ plus a Lie-derivative term $\lim_{\beta\to\infty} L_{v(\beta)} g_{ab}(\beta)$. The former is present only if the Ricci-deformed metric $\bar{g}_{ab} = \lim_{\beta\to\infty} g_{ab}(\beta)$ is not isolated. This latter term may be quite annoying since it gives rise to a lack of isotropy. Since we are mainly interested in FLRW space-times, let us assume that $\bar{g}_{ab}$ is isolated, while in order to take care of the $Diff$-induced shear $\lim_{\beta\to\infty} L_{v(\beta)} g_{ab}(\beta)$ we can average with the linearized Hamilton flow, not the second fundamental form directly, but rather the deformation tensor $H_{ab} \equiv K_{ab} - L_{\mathbf{N}} g_{ab}$, where N is a shift vector field on the manifold Σ, associated with the slicing. By choosing $\mathbf{N} = -\lim_{\beta\to\infty} v(\beta)$ we get rid of the annoying $Diff$-induced shear in $\lim_{\beta\to\infty} K_{ab}(\beta)$. Given this, we see that due to the properties of the Ricci-Hamilton flow we have $\lim_{\beta\to\infty} K_{ab}(\beta) = \frac{1}{3} < k >_o \cdot \bar{g}_{ab}$. The given K is deformed by gradual elimination of its shear $K_{ab} - \frac{1}{3} k g_{ab}$ and the original (position dependent) rate of volume expansion k is being replaced with its corresponding average value.

The validity of constraints was assumed at every smoothing step in [55], but we emphasize again, there is no reason for this to be so, in particular because what we are investigating is the very effect smoothing has on the equations.

Instead, we would like to propose a radically new view. We require the constraint (3.71) to hold on the initial manifold:

$$8\pi G_{bare}\mu = R(g) - K^{ab}K_{ab} + k^2, \qquad (3.72)$$

and we assume it to hold on the final, smoothed-out manifold (and in fact at any step during the deformation) in the following form

$$8\pi\tilde{G}\bar{\mu} = \bar{R}(\bar{g}) - \bar{K}^{ab}\bar{K}_{ab} + \bar{k}^2, \tag{3.73}$$

where, $\tilde{G}$ is the renormalized gravitational constant. This means that we require the functional form of the constraint to be unchanged, i.e. they hold as they are, on condition G is renormalized, measuring this way the response of the quantities entering (3.72) to the deformation inflicted upon averaging. This is motivated by the RG approach, which we decided to employ here from the very beginning.

Notice that equivalently we could have written (3.73) as

$$8\pi G\tilde{\bar{\mu}} = \bar{R} - \bar{K}^{ab}\bar{K}_{ab} + \bar{k}^2, \tag{3.74}$$

and renormalize the smoothed density. However, we prefer to stick to (3.73) since it fits more naturally in the RG picture and cosmological context, where $\bar{\mu} =< \mu >_o$.

Our key assumption (3.73) is not in fact that far-fetched but rather comes about quite naturally, noticing that the Ricci-Hamilton flow is in no way connected to the dynamics that resides in the constraints[24], and renormalization of the gravitational constant, as advocated above is a consequence of "renormalization" (deformation) of the geometry engendered by the Ricci-Hamilton flow. In this sense, the flow generated finally by the averaging (or, coarse graining) can be looked at as a continuous version of generalization of the block-spin transformation, much in the spirit of Wilson's RG approach to lattice models.

Before we explore the consequences of our assumption (3.73), let us remark that running G was also obtained at one-loop level in quantum R^2-gravity models in the UV asymptotically free regime [121]. There it was due to the quantum fluctuations of the geometry, which act over macroscopic distances and the running of the gravitational constant entering the gravitational potential was suggested as a possible solution to the Dark Matter problem. However, in such a case the use of the Wilsonian gravitational potential was not consistent for estimating the infra-red effects. Our approach instead is clearly of different nature and does not require one to assume any fixed background.

[24] For a compact and without a boundary manifold Σ, the dynamics of GR is given entirely by the constraints.

Let us now explore the consequences of (3.73). We will make use of the property of the Ricci-Hamilton flow, namely that the flow $K(\beta)$, solution of (3.51) is such that $\frac{\partial}{\partial\beta} < k(\beta) >_\beta = 0$, i.e. the space average of the trace of second fundamental form remains constant during the deformation. This allows us to write

$$< k >_o^2 = \bar{k}^2, \tag{3.75}$$

since in the limit, the volume expansion is simply a constant.

We also introduce a standard deviation describing the fluctuations of the empirical value of the rate of volume expansion with respect to its conserved value $\bar{k}$,

$$\sigma_o^2 = \frac{< k^2 >_o - < k >_o^2}{< k >_o^2}. \tag{3.76}$$

Equation (3.73) becomes in the limit after extracting a trace free part of K_{ab}

$$8\pi\bar{G}\bar{\mu} = \bar{R} + \frac{2}{3}\bar{k}^2, \tag{3.77}$$

since no residual shear survives; otherwise (3.72) gives

$$\frac{2}{3}k^2 = 8\pi G\mu - R(g) + \tilde{K}^{ab}\tilde{K}_{ab}, \tag{3.78}$$

where, the shear $\tilde{K}_{ab} \equiv K_{ab} - \frac{1}{3}kg_{ab}$ is now present.

Taking into account (3.75) and (3.76) we can rewrite the last equation upon taking the average as

$$\frac{2}{3} < k^2 >_o = \frac{2}{3}\bar{k}^2(\sigma_o^2 + 1) = 8\pi G < \mu >_o - < R >_o + < \tilde{K}^{ab}\tilde{K}_{ab} >_o, \tag{3.79}$$

and substituting it into (3.77) we get

$$8\pi\bar{G}\bar{\mu} = \bar{R} + \frac{8\pi G < \mu >_o - < R >_o + < \tilde{K}^{ab}\tilde{K}_{ab} >_o}{\sigma_o^2 + 1}. \tag{3.80}$$

We can further simplify it slightly by taking $\sigma_o = 0$, as the volume expansion is very nearly constant, to obtain finally

$$8\pi\bar{G} = \frac{\bar{R} - < R >_o + < \tilde{K}^{ab}\tilde{K}_{ab} >_o + 8\pi G < \mu >_o}{\bar{\mu}}. \tag{3.81}$$

It is worth stressing that this equation (3.81) is correct only for the case when the Ricci-Hamilton flow is global. In any case, the Ricci-Hamilton flow is always local and we could have obtained a differential equation instead of (3.81), but we find (3.81) as more suitable for addressing the problem of smoothing in cosmological context and for the line of argument we wish to develop.

The quantities entering (3.81), $\bar{R}- < R >_o$ and $< \tilde{K}^{ab}\tilde{K}_{ab} >_o$ are generally rather small, and on globally smoothable pieces $\bar{R} - < R >_o$ is non-negative, therefore any anisotropies that might be present favour $\tilde{G} \geq G$.

Since $j^a = 0$, (but even in the case in which $j^a \neq 0$) the second constraint is satisfied by the $Diff$-equivariance of Hamilton's flow.

In the case of dust universe of interest here we do not need to consider smoothing-out the spatial stress tensor.

Now, we will comment on the interpretation of the formalism, before turning to a discussion of its physical consequences. The general picture is that we pick up an appropriate initial data set, which when evolved is the space-time to be averaged out. We smooth this data set by means of the deformation via the Ricci-Hamilton flow, into a FLRW initial data set, which has an equivalent interpretation of carrying out a suitable averaging over the set of geodesic balls on the spacelike hypersurface of the initial inhomogeneous space-time, chosen in an optimal way from the point of view of convergence of the Ricci-Hamilton flow. Although, the deformed data do not need to satisfy the constraints, we assumed that the difference can be incorporated into a renormalization of gravitational constant. It then follows that the flow of deformed data generates a one-parameter family of solutions to the field equations (each with its own G), which interpolates between the original space-time and its FLRW smoothed-out counterpart.

We require further that for each t for which the evolution of the data $(g(\beta), K(\beta))$ is defined, the flows $(g_t(\beta), K_t(\beta))$ resulting from evolution equations $\dot{g}_{ab}$, $\dot{K}_{ab}$ are the Ricci-Hamilton flows, with initial conditions $g_t(0) = g_t$, $K_t(0) = K_t$. This last requirement is necessary to tie the smoothed-out model to the inhomogeneous one through the condition $Vol(S^3, \bar{g}_t) = Vol(\Sigma, g_t)$, in the sense that these two models should behave as similarly as possible under their own gravitation. Here, $Vol(\Sigma, g_t)$ denotes the volume of (Σ, g_t) as t varies. Since we assumed that $(g_t(\beta), K_t(\beta))$ are the Ricci-Hamilton flows under the time evolution of the data $(g(\beta), K(\beta))$, the relation about the volumes holds, i.e.

$Vol(\Sigma, g_t(\beta)) = Vol(\Sigma, g_t)$, for each t for which the evolution is defined.

3.6.5 Cosmological implications

In order to be able to correctly model the inhomogeneous universe by its FLRW counterpart, one should consider the term $< \tilde{K}_{ab}\tilde{K}^{ab} >_o$, which takes into account the contribution of cosmological gravitational radiation. It is however difficult to estimate the size of this term, though as we have shown it can affect the dynamics of the universe by modifying the value of the gravitational constant of the smoothed-out cosmological model.

An issue of importance for cosmology is the question on what scale is the FLRW model supposed to describe the universe? Likewise, what averaging scale are we referring to when we give the value of Ω_o? Motivated by this, we can ask what is the behaviour of the average density along the Ricci-Hamilton flow, whether it is monotonic? This question is in fact not as easy to answer as it might seem on its face value, since in our approach changing the radius of geodesic balls implies changing the covering, etc. and a general answer is difficult to obtain. Yet this approach does enable us to draw some qualitative conclusions, based on the results of the last section.

Let us assume that the Friedmann equation valid in flat case is

$$H^2 = \frac{8}{3}\pi\tilde{G}\bar{\mu}, \tag{3.82}$$

with $\bar{\mu}$ referring to baryonic matter. By construction this yields the density parameter equal to one.

Observations seem to suggest that Ω_o is scale dependent (some averaging scale is implicit in any measured value for the density) and $\Omega_o(r)$ is observed to increase for larger volumes, probably reaching $\Omega_o = 1$ for some cosmological smoothing scale. The density parameter is related to the matter density, Hubble constant and gravitational constant by

$$\Omega_o = \frac{8\pi\mu_o}{3H_o^2}G. \tag{3.83}$$

The definition of Ω_o necessarily refers to an idealised, i.e. smoothed background model, for which - taking into account considerations of the previous section - it is given by

$$\bar{\Omega}_o = \frac{8\pi\bar{\mu}_o}{3H_o^2}\tilde{G}. \tag{3.84}$$

It is the product $\tilde{G}\bar{\mu}_o$ that is inferred for different scales (usually with the help of virial theorem) resulting in Ω_o growing with distance. Having $\tilde{G} \geq G$, in effect we can read off that $\bar{\Omega}_o \geq \Omega_o$. We could not expect G to vary significantly over the size of the solar system, but its variation over galactic scales can be of help to understand the variation of the density parameter and the observed flat rotation curves of galaxies, since obviously the renormalized gravitational constant enters into the force law[25].

A well known difficulty of Cold Dark Matter (CDM) models with the usual power spectrum is that they lack enough power at large distances to account for the observed structures. It can be expected that gravitational constant growing with distance can give more power at large scales. This can be verified by studying the effect of the variation of G on the power spectrum, the Jeans' length or the growth of density fluctuations in expanding, almost FLRW universe [28]. Below we will concentrate on the two last options.

In a matter dominated universe with FLRW background, the evolution of density fluctuations is given by the hydrodynamic fluid equations, namely in a static case

$$\frac{\partial^2 \mu(\vec{r},t)}{\partial t^2} = v_s^2 \nabla^2 \mu(\vec{r},t) + < \mu >_o \nabla^2 V(\vec{r}) \, , \tag{3.85}$$

where, $\mu(\vec{r},t)$ is the energy density perturbation, assumed here to be adiabatic, v_s is the sound velocity and $< \mu >_o$ the background energy density. $V(\vec{r})$ is the Newtonian type potential for a scale dependent G.

If the perturbations are taken to be of plane wave form

$$\mu(\vec{r},t) = A \exp[\imath(\vec{k}\cdot\vec{r} - \omega t)] \tag{3.86}$$

the dispersion relation that follows from (3.85) is the following

$$\omega^2 = v_s^2 k^2 - 4\pi \tilde{G} < \mu >_o . \tag{3.87}$$

Consequently, the Jeans' length $\lambda_J = 2\pi/k_J$, where k_J is the root of the right hand side of (3.87) is renormalized.

A comparison with the Jeans' length of CDM models (with $\Lambda = 0$) reveals that the Jeans' length of smoothed-out models is greater than the one of CDM models, since the

[25] Actually, the last equation also implies that the Hubble constant is scale dependent as well.

second term on the right hand side of (3.87) $4\pi\tilde{G} < \mu >_o = 12\pi H_o^2\Omega_o$ is smaller than the corresponding value in CDM models, since now $< \mu >_o$ is only baryonic. This translates into the fact that there is gravitational collapse, and so, more structure at large scales when G varies, than in the $\Omega_o = 1$ CDM models.

Generalizing this analysis to an expanding matter dominated universe, we consider again plane-wave solutions to the hydrodynamic equations describing the evolution of adiabatic energy density perturbations in an expanding fluid. Standard analysis shows that for $\mu(t) = \mu_{in}(\frac{a_{in}}{a(t)})^3\delta(t)$ density perturbations, the rotational modes are negligible, while compressional modes satisfy

$$\ddot{\delta}(t) + 2H\dot{\delta}(t) + \left(\frac{v_s^2 k^2}{a(t)^2} - 4\pi\tilde{G}\bar{\mu}(t)\right)\delta(t) = 0, \tag{3.88}$$

with the expansion rate satisfying the Friedmann equation (3.82). Assuming only that $\tilde{G} = \tilde{G}(r)$ grows with distance, an inspection of the last equation suggests that the collapse would be more effective (gravitational pull stronger), but also the expansion of the universe would be faster, hampering the growth of perturbations.

In the matter dominated universe

$$\bar{\mu}(t) = \mu_{in}\left(\frac{a_{in}}{a(t)}\right)^3 \tag{3.89}$$

and equation (3.88) can be rewritten as

$$\frac{1}{2}\dot{a}^2 + V = 0, \tag{3.90}$$

where, $V = -\frac{4}{3}\pi\mu_{in}a_{in}^3\tilde{G}a^{-1}$, describing a motion of unit mass "particle" in a classical effective potential. For a growing $\tilde{G}$, V tends to be more negative than for constant G, in which case the "particle" takes less time to be further away.

Let us for simplicity assume that for varying $\tilde{G}$ the scale factor evolves like $a(t) \propto t^p$, $p > \frac{2}{3}$. Then $H = pt^{-1}$ and the evolution of perturbations with wavelengths $> \lambda_J$, i.e. satisfying $4\pi\tilde{G}\bar{\mu} \gg v_s^2k^2/a(t)^2$ is described by (3.88) now in the form

$$\ddot{\delta} + 2pt^{-1}\dot{\delta} - \frac{3}{2}p^2t^{-2}\delta = 0. \tag{3.91}$$

The solution is $\delta \propto t^{q_\pm}$, with $q_\pm = \frac{1}{2}[1 - 2p \pm \sqrt{(1-2p)^2 + 6p^2}]$, and a growing mode is given by

$$\delta_+ = A_+ t^{q_+}, \tag{3.92}$$

with $q_+ > \frac{2}{3}$ for $p > \frac{2}{3}$.

To conclude, the effect considered of growing $\tilde{G}$ with scale, is that for a fixed baryonic energy density $\bar{\mu}$, the density perturbations grow faster with time than they would in the classical theory with $G =$ const and no Dark Matter.

As shown in [28], if the variation of G with distance is slow enough at short distances many known features of inflationary scenarios are not affected, which makes this proposal even more interesting, since conclusions of [28] apply here too even though the physical models and effects considered are quite disparate. Clearly more specific models should be examined to carry out a fair confrontation of the consequences in our approach with experimental data.

3.7 Conclusions and outlook

Whenever the macroscopic characteristics of the system which evolves dynamically are studied, coarse-graining is necessary[26] and relevant, when we are interested in its behaviour over rather short time scales. It basically means some kind of averaging over small but finite volumes of phase space. Coarse-graining does not change the original system, but the coarse grained quantities may behave qualitatively differently from the microscopic ones.

A particular framework to carry this out is in the Renormalization Group formalism. RG calculations are concerned with the asymptotic properties of critical systems, in the sense of infinite size and proximity to the critical point and thus, they predict singular power-law behaviour of the systems with universal exponents and scaling functions.

To put things in perspective, what we demonstrated here is that the same theoretical techniques can be applied to many problems in Condensed Matter, Gravitational and

[26] Another possibility can be long-time averaging, which is usually equivalent to statistical averaging over an *ensemble* of all possible micro-states, that can produce given macroscopic features. This is appropriate for systems in a quasi-static state.

Particle Physics. Among these, we have concentrated on the applications of the Renormalization Group. On the one hand, there is a typical application of the RG to Condensed Matter Physics, in the infra-red limit in the description of critical phenomena and second-order phase transitions, and possibly quantum Hall effect. On the other hand, in Particle Physics, e.g. in QED the RG is applied in the ultra-violet limit to find the Landau pole or in QCD asymptotic freedom. We proposed an application of the RG in classical relativistic cosmology to tackle the smoothing problem, which belongs to the various-length-scales class of problems and due to this, according to K.Wilson [255] can be grasped using the RG methods.

Let us add that another application of the RG can be in Quantum Gravity within the framework of string theory to 2–dimensional theories on the world-sheet, where Zamolodchikov's c–theorem may give insight into the effective string theory that hopefully describes this part of the universe.

Further, RG already found its way in the membrane approach to black holes, where its horizon is described in terms of a dynamical surface. Within path integral approach in the presence of this dynamical boundary, RG arguments yield a description of the dynamics of the horizon by the action of the relativistic bosonic membrane [182]. Also here a classical description of the state of a black hole (non-rotating) is characterized only by the "macroscopic" parameters, such as its mass and charge and there is an implicit coarse-graining over the membrane excitation levels. This membrane approach derived from a RG procedure gives an effective description valid at distances longer than l_{Planck}; the thermodynamic properties derived are in agreement with the standard results.

4 Dynamical systems approach in Cosmology

A dynamical system consists of a phase space which provides with a description of the system's allowed states and a rule which defines the temporal evolution of those states. For differential equations the evolution is continuous, it can also be discrete as for a mapping.

Virtually every model of physical phenomena is a dynamical system and in fact, most of the models are Hamiltonian dynamical systems; GR belongs to the class of *constrained* Hamiltonian systems [135]. Hamiltonian dynamical systems give rise to symplectic mappings. Also, it is worth stressing the fact that the motion of a fluid particle in an incompressible fluid is Hamiltonian, no matter if the fluid motion itself is viscous, or not. Let us add that mappings are in fact more general than differential equations, but also they are easier to study than differential equations.

Typical questions of physical interest are concerned with the long-time stability of orbits and the determination of the regions accessible to the motion. Also problems concerning transport, i.e. the determination of the time for a bunch of trajectories to move from one region of phase space to another, are of interest in physics. Even if the system were not strictly stable, it could be stable in practice if the transport times were longer than the lifetime of the system – this is probably the case with the planetary motions in the solar system, though certainly not so for asteroids.

In practice, the situation is idealized by considering (for positive times) the induced flow on the tangent bundle of the phase space, which is given by the structure of a smooth manifold, and by studying the growth of distances between nearby trajectories[1].

[1] This is normally done with respect to some Riemannian metric in the tangent space as it moves along the trajectory.

A number of definitions is in order, which will be given while proceeding.

4.1 Stability analysis of dust–radiation universe

The material presented below in this chapter is based on [52].

We consider flat and open universe models containing a mixture of cold matter (dust) and radiation interacting only through gravity, and study their stability with respect to linear scalar perturbations. To this end the perturbed universe is considered as a dynamical system, described by coupled differential equations for a gauge-invariant perturbation variable and a relevant background variable.

4.1.1 Introduction

One of the main goals of present - day cosmology is to understand the formation of the structures (galaxies, clusters, superclusters) observed in the universe, while trying to explain why, on large enough scales, this seems to be so well described by the FLRW isotropic models. Given these latter, most theories of structure formation are based on the gravitational instability scenario. At any given epoch, there are perturbations larger than a certain characteristic - time dependent - scale[2]; while perturbations much smaller than this scale oscillate as sound waves, density perturbations on larger scales grow, eventually entering a non - linear regime during the matter dominated epoch, thus forming the observed structures. The mathematical basis for such scenario is the theory of perturbations of FLRW models.

In the inflationary scenario, the perturbations were generated from quantum fluctuations within the Hubble horizon H^{-1}, have evolved classically outside the horizon, and have re-entered it during the radiation or matter dominated epochs (perturbations with larger wavelength re-entering later). During the last decade, observations of the distribution of matter have shown that the scale at which the background homogeneity seems to be reached is larger than what was thought before, being of the order of hundreds of Mpc[3].

[2]This can be the Jeans scale, or the Hubble radius (sometimes loosely referred to as horizon); a more important scale is actually the sound horizon, see [21].

[3]The answer to the question "At which scale in the universe there is a transition to homogeneity?" depends on how the question is posed, i.e. how such a scale is defined. In a very loose sense we can say

Consequently, both the inflationary scenario and the large scale observations motivate the use of a fully relativistic theory of perturbations in FLRW models in order to study the formations of the larger structures.[4] Moreover, while the inflationary scenario seems to favour a flat universe, there is as yet no convincing empirical evidence for a critical density parameter $\Omega_0 = 1$, while a low density $\Omega_0 < 1$ universe is favoured by many observations [68]. For this reason we shall consider perturbations of flat and open models. The density contrast of the larger structures appears to be small enough that a linear perturbation analysis still suffices to describe the evolution of perturbations of the corresponding scale. Even if the density contrast is mildly non - linear, the curvature perturbations are still in the linear regime [22], [107], thus one can imagine the present universe as well described by a linearly perturbed FLRW model at large scales, while non - linearities at smaller scales can be considered smoothed-out in this picture.[5] Then the question arises if in this respect we live in a special epoch - an epoch in which large scale perturbations are still in the linear regime - or if this is a natural output in the theoretical context of perturbed FLRW models. This question is of the same sort as that posed by the "flatness problem": if the geometry of the universe is non–flat, then we live in a special epoch in which the density parameter is still close to unity: $\Omega_0 \sim 1$. It is well known that in a flat dust model there is a perturbation mode that grows unbounded, while in an open dust universe there is a mode that freezes in at an epoch $z \approx \Omega^{-1}$. In both cases the perturbation equation (4.1) is the same for all modes, because for $c_s^2 = w = 0$ (dust) the coefficient β (4.23) does not depend on the wavenumber k. It is usually said that in a dust universe each perturbation evolves as a separate FLRW universe. On the other hand, perturbation scales in a pure radiation model always come within the "sound horizon" [21] and oscillate as sound waves.

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that this scale is measured by the size of the largest structures we see (of the order of 10^2 Mpc), and in a strict sense such a scale does not exist at all if there is not a cut-off in the perturbation spectrum. Various reasonable definitions of such a scale can be given (e.g. specifying a level for the fluctuations, such that we can say that there is a transition to homogeneity at the scale where the fluctuations fall below the specified threshold): cf. [222] and [86].

[4] Newtonian perturbation theory is applicable only for vanishing pressure and at scales much smaller than the horizon.

[5] In general, this raises an issue of defining a proper averaging procedure – problem considered at length in this thesis.

This simple picture gives however rather little information about the generality of this behaviour for the perturbations, and on the stability properties of the perturbed FLRW model. We aim therefore at studying the problem of the stability of FLRW models following an alternative approach: instead of looking for analytic solutions (either exact or approximate, see e.g. [204]) of the perturbation equations, we consider the perturbed universe as a dynamical system, described by coupled differential equations for a gauge-invariant perturbation variable and a relevant background variable. In this approach, the evolution of perturbations is represented by the trajectories in the phase space of the dynamical system, and their final fate is linked to the presence and the nature of the fixed (or stationary) points of the system.

In a certain sense the present work extends that of [85] to include perturbations, although we restrict our analysis to open and flat models only, and we take a vanishing cosmological constant $\Lambda = 0$. Previous works have followed the approach to stability used here [260], [48], but they only considered either a pure radiation or a pure dust model. However, since the analysis here is based on the study of the dynamical system at time-infinity, the pure radiation model does not appear physically significant in this limit, while the stability properties of the dust model are affected by the simplifying assumption of the vanishing of the speed of sound. Therefore here we consider a class of perturbed models containing both dust and radiation as a more realistic description of the real universe. The dust component can be taken to represent Cold Dark Matter (massive weekly interacting particles), while the radiation component represents photons and/or other massless particles such as massless neutrinos, therefore these two fluids interact only through gravity.

Universe models containing a mixture of dust and radiation interacting only through gravity were considered before (e.g. in [137]), as well as perturbations of these models (see e.g. [129] and [204]).

Here, our aim is to study the stability properties of these simple models, considering only the total density perturbation (the single component perturbations are not directly relevant to the evolution of curvature perturbations), assuming adiabatic perturbations.

For the case of flat universe models, as will see, we find that there exists a critical wavenumber k_{EC}, which is an invariant characteristic of the model and is related to the only scale entering the flat models, i.e. the Hubble radius at equi-density of matter and radiation H_E^{-1}. The corresponding critical scale λ_{EC} (λ_E is the perturbation wavelength at equi-density), remarks the transition from stability to instability, but in a way which is more rigorous - from the point of view of the stability analysis - than the Jeans or the Hubble scale. We find that there are actually three regimes for the evolution of fluctuations:

(i) growing large scale perturbations (unstable modes);

(ii) overdamped intermediate scale perturbations;

(iii) damped small scale wave perturbations;

where the transition scale from one regime to another is always of the order of λ_{EC}. Also, we show that λ_{EC} is of the order of the Jeans scale λ_{JE} at equi-density in the same model: however, since $\lambda_{EC} \simeq 2.2\lambda_{JE}$ (today $\lambda_{0C} \simeq 67Mpc$), our analysis shows that there are perturbation modes that decay, despite that their scale λ_E is larger than λ_{JE} at equi-density. Thus the evolution of perturbations in these models depends on their scale, in such a way that smaller scales evolve like in a pure radiation model (case (iii)) and larger scales like in a pure dust model (case (i)), while we found a small intermediate range of scales (case(ii)) for which perturbations are overdamped (critical damping occurs for the transition scale between cases (ii) and (iii)), which is an original feature of the dust–radiation models, and to our knowledge was not known before.

For the case of an open dust–radiation model instead, the evolution of perturbations appears to be dominated by the curvature of the background, and their final state is similar to that they have in a pure dust model, i.e. all the perturbation scales are frozen–in to a constant value. These models appear to be marginally stable [16] with respect to perturbations of any wavelength.

In the following we take $c = 1$, $\kappa = 8\pi G$ (G is the gravitational constant), and we assume a vanishing cosmological constant $\Lambda = 0$.

4.1.2 The dynamical system

We give below a brief outline of the method used to study the stability of FLRW models.

In a recent series of papers, Woszczyna and colleagues considered the dynamics of Newtonian [262] and relativistic [260] linearly perturbed universe models, studying the stability of these dynamical systems. The analysis of the relativistic case was however affected by a certain assumption on the allowed range for a scale parameter k (q in [260] and in [48]) and by a misinterpretation of the perturbation variables, as was shown in [48].

We shall focus now on relativistic perturbations of flat and open FLRW models containing two fluids coupled only through gravity: dust and radiation. Open and flat FLRW models expand for an infinite amount of time, and therefore one can apply standard stability criteria (see e.g. [15] and [16], and [11]). Specifically, one can establish if a cosmic dynamical system, i.e. a perturbed cosmological model, is: *stable* (in the sense of Lyapunov) around the unperturbed state, which roughly speaking means that a small change in the initial conditions do not produce a big change in the following evolution, i.e. in this context an initial small inhomogeneity do not grow; *asymptotically stable*, i.e. the unperturbed state is stable *and* the perturbation decay so that the cosmic system returns to the original homogeneous state; *marginally stable*, when is stable but not asymptotically stable; *unstable*, when the perturbations grow with time.

In general, in the gauge- invariant approach to cosmological perturbations the scalar[6] density perturbation variable (we shall consider specific perturbation measures later) satisfy a second order (in some time variable) differential equation. If one restricts the attention to the harmonic component X of the perturbation, and assumes that this is adiabatic (see section 4.1.3.2), its evolution is given by a homogeneous ordinary differential equation

$$\ddot{X} + \alpha(t)\dot{X} + \beta(t)X = 0 \,, \tag{4.1}$$

where t here is proper time, and the dot indicates a derivative with respect to t. When it is not possible to find a simple solution to (4.1), a qualitative analysis of its properties is useful in order to determine the late - time behaviour of the perturbations[7]. The

[6] It is standard to call scalar perturbations those related with density perturbations describing the clumping of matter (e.g. see [234]). In this paper we shall consider only these perturbations, as the only relevant to the problem of stability of the universe.

[7] One can obviously find numerical solutions, but the study of the phase space allows us to obtain general conclusions for a generic set of initial conditions.

coefficients α and β are functions given by the background dynamics, but in general their time dependence cannot be explicitly determined. Therefore, it is useful to think of α and β as known functions of one or more parameters, and add to (4.1) the evolution equation for the parameters in order to have an autonomous system. A sensible choice followed in [260] is given by the density parameter Ω, which in a FLRW model satisfies the equation

$$\dot{\Omega} = \Omega(\Omega - 1)(\tfrac{1}{3} + w)\Theta \, , \tag{4.2}$$

where $\Theta = 3H = 3\dot{a}/a$ is the expansion of the cosmic fluid, a is the FLRW scale factor, H is the Hubble parameter, and $w = p/\mu$ is the ratio of the pressure to the energy density. It is useful to change the independent variable from the proper time t to a function of the scale factor a; with the choice of [8] $\tau = \ln a^3$, ($\frac{d\tau}{dt} = \Theta$) equations (4.1) and (4.2) give

$$X'' + \psi X' + \xi X = 0 \, , \tag{4.3}$$

$$\Omega' = \Omega(\Omega - 1)(\tfrac{1}{3} + w) \, , \tag{4.4}$$

and in general

$$w' = -(1 + w)\left[c_s^2(w) - w\right] \tag{4.5}$$

gives the evolution of w, and in the previous equations the prime refers to the derivative with respect to $\tau = \ln a^3$.

In a single fluid FLRW model the dynamics is fixed by an equation of state $p = w\mu$, with $w = const$ (e.g. $w = 0$ dust, $w = 1/3$ radiation), and in this case $c_s^2 = \dot{p}/\dot{\mu}$ is the speed of sound, and $c_s^2 = w$. In dealing with two or more fluids however, $w' \neq 0$, and $c_s^2 \neq w$ is no longer a proper speed of sound unless the fluids are coupled, and ψ and ξ are given by

$$\psi(w, \Omega) = \frac{\dot{\Theta}}{\Theta^2} + \frac{1}{\Theta}\alpha(w, \Omega) = -\frac{1}{3} - \frac{1}{6}(1 + 3w)\Omega + \frac{1}{\Theta}\alpha(w, \Omega) \, , \tag{4.6}$$

$$\xi(w, \Omega) = \frac{1}{\Theta^2}\beta(w, \Omega) \, , \tag{4.7}$$

[8] There are various typos in [260]: there, the power appearing here in the definition of τ is missed in equation (3), and consequently a factor 1/3 is missed in equation (9).

while the second step in (4.6) is given by the Raychaudhuri equation

$$\dot{\Theta} + \tfrac{1}{3}\Theta^2 + \tfrac{1}{2}\kappa\mu(1+3w) = 0 \,, \tag{4.8}$$

governing the evolution of Θ (see e.g. [81]). In the simplest case of one single fluid [260], [48] with $w = const$ we have a third order autonomous system, given by (4.3) and (4.4) (or a corresponding pair of first order equations); also, equation (4.4) forms an autonomous first order subsystem in this case. In the most general case we have a fourth order autonomous system, with (4.5) as autonomous subsystem (as we shall see in section 4.1.3.1, $c_s^2 = c_s^2(w)$ is fixed once the fluid components are specified). The order of the system can however be reduced. First, it turns out that in the practical case (see section 4.1.3.1) either w or Ω can be eliminated; second, here we are not really interested in the evolution-law of X, but rather in its qualitative behavior. Because of this, we can achieve a further dimensional reduction of the phase space passing to the Riccati equation corresponding to (4.3). Introducing $Y = X'$ and

$$\mathcal{U} = Y/X = X'/X \,, \tag{4.9}$$

$$\mathcal{R} = \sqrt{X^2+Y^2} \,, \tag{4.10}$$

we pass in the new phase space $\{\mathcal{R}, \mathcal{U}, w, \Omega\}$, where

$$\frac{\mathcal{R}'}{\mathcal{R}} = \frac{\mathcal{U}(1-\xi-\psi\mathcal{U})}{\mathcal{U}^2+1} \,, \tag{4.11}$$

$$\mathcal{U}' = -\mathcal{U}^2 - \psi\mathcal{U} - \xi \,, \tag{4.12}$$

while the evolution of Ω and w is still given by (4.4) and (4.5). The variable $\mathcal{R}$ represent an "amplitude" of the perturbation and it is not directly relevant to the present analysis. Since (4.12) and (4.4), (4.5) form an autonomous subsystem, one can restrict the analysis to the phase space $\{\mathcal{U}, w, \Omega\}$; moreover, as we said above, in practical cases we can restrict our attention either to $\{\mathcal{U}, w\}$ or to $\{\mathcal{U}, \Omega\}$. The relevant variable here is $\mathcal{U}$: when it is positive we have either a growing density enhancement or an increasing energy deficit, while $\mathcal{U} < 0$ indicates that the inhomogeneity is decreasing (note that from its definition (4.9) $\mathcal{U}$ is a tangent in the original phase space $\{X, Y\}$, thus $-\infty < \mathcal{U} < \infty$). One is therefore interested in the nature of the fixed (or stationary) points (if any) on the $\Omega = 0$ or the $w = 0$ axis (the final state of the cosmological dynamical system). These correspond to the real roots of the right hand side of (4.12). A stable node on the $\mathcal{U} > 0$ semi-axis will indicate that the given perturbation mode will indefinitely grow, thus indicating instability of the underlying cosmic system; a stable node on the $\mathcal{U} < 0$ semi-axis will indicate that the given perturbation mode will finally decay, i.e. the system is asymptotically stable with respect to that perturbation; a stable node on $\mathcal{U} = 0$ means that the perturbation will asymptotically approach a constant value: the cosmic system is then marginally stable (see [16]). Finally, if there are no stationary points on the $\Omega = 0$ (or the $w = 0$) axis, the perturbation maintains a sound wave character at any time. This situation corresponds to complex roots of the right hand side of (4.12) with negative real part, and we shall see in section 4.1.4.4 that these roots are also the eigenvalues of the system given by (4.12), (4.5): therefore the cosmic system is stable against these perturbations.

A discussion of the system (4.12), (4.4) can be found in [260] (and reference therein); a discussion of the flaws of the application of the analysis given in [260] to the perturbation equations of [21] and [93], [94], [50] (see also [261]) is given in [48].

4.1.3 Dynamics of the dust - radiation models

We shall now apply the general method outlined in the previous section to the case of uncoupled dust and radiation. In the following, we shall normalize the scale factor at equi-density of dust and radiation, introducing $S = a/a_E$. An analysis of the phase–space of FLRW models containing dust and radiation has been recently given in [85] (see also [137] and [204]); here we simply review well known results, that are needed for the perturbation analysis, with emphasis on a useful parameterization.

4.1.3.1 The background

Since the two fluids are uncoupled, we have separate energy conservation, with $\mu_d = \frac{1}{2}\mu_E S^{-3}$ and $\mu_r = \frac{1}{2}\mu_E S^{-4}$, where μ_E is the *total* energy density at equi-density. Then the total energy is also conserved, with density $\mu = \frac{1}{2}\mu_E(S^{-3} + S^{-4})$, while the total pressure is that of the radiation component, $p = p_r = \frac{1}{3}\mu_r = \frac{1}{6}\mu_E S^{-4}$; therefore

$$w = \frac{1}{3(S+1)} \,. \tag{4.13}$$

For two non-interacting fluids the quantity $c_s^2 = \frac{\dot{p}}{\dot{\mu}}$ is only formally the speed of sound: for dust and radiation we have

$$c_s^2 = \frac{4}{3(4+3S)} \,. \tag{4.14}$$

At equi-density $S = 1$, $w = 1/6$, and $c_s^2 = 4/21$. Clearly, (4.13) can be inverted to give

$$S = \frac{1-3w}{3w} \,, \tag{4.15}$$

and then the expansion of the universe model we consider is parameterized by w, with w $(1/3 \geq w \geq 0)$ varying from a pure radiation dominated $(t \to 0)$ to a pure matter dominated $(t \to \infty)$ phase.

Note that (4.13), with (4.14) is in practice an integral for (4.5); also, given (4.13) we can integrate (4.2):

$$\Omega = \frac{\Omega_E(S+1)}{\Omega_E(S+1) + 2(1-\Omega_E)S^2} \,; \tag{4.16}$$

since today $S_0 = a_0/a_E = 1 + z_E \gg 1$, the density parameter at equi-density is $\Omega_E \simeq \left(1 + \frac{(1-\Omega_0)}{2S_0\Omega_0}\right)^{-1}$. Clearly, equation (4.16) can also be inverted to parameterize the expanding model with Ω. However in the following we shall find convenient the parameterization in w, which will allow us to give a unified treatment of flat and open models: from (4.15) and (4.14) we have

$$c_s^2 = \frac{4w}{3(1+w)} \,, \tag{4.17}$$

and this with (4.5) give

$$w' = w\left(w - \tfrac{1}{3}\right) \,, \tag{4.18}$$

where hereafter the prime stands for the derivative with respect to $\tau = \ln S^3$.

From (4.18) and (4.4) we get

$$\Omega = \frac{3\Omega_E w}{3\Omega_E w + 2(1-\Omega_E)(1-3w)^2} \,, \tag{4.19}$$

which can be used in expressions for ξ and ψ to obtain these coefficients as functions of w only, thus reducing the effective phase space needed for the stability analysis to $\{\mathcal{U}, w\}$, i.e. that of the plane autonomous system (4.12), (4.18). We note also that the system (4.18), (4.4) that describes the evolution of the background universe model is a plane autonomous subsystem in the full space $\{\mathcal{U}, \Omega, w\}$: this will always be a property of a dynamical system describing a perturbed universe, since the basic assumption of the perturbation analysis is that of neglecting the backreaction of the perturbations on the dynamics of the background model.

Finally, the total energy density is given by

$$\mu = \frac{27}{2}\mu_E \frac{w^3}{(1-3w)^4} \,. \tag{4.20}$$

This obviously also follows from the conservation equation $\mu' = -\mu(1+w)$, integrated using (4.18); and $\mu \to \infty$ as $w \to 1/3$, $\mu \to 0$ as $w \to 0$.

4.1.3.2 The perturbed model

We shall now consider the dynamics of the perturbed dust - radiation models.

First, we have to specify a perturbation measure X, i.e. the variable appearing in (4.1) and (4.3). Here, we shall focus on density perturbations, and we shall take $\Delta = a^2({}^{(3)}\nabla^2\mu)/\mu$ as our fundamental variable, as was originally defined in [94] (see also [261]) following the covariant approach to perturbation introduced in [93].

This covariant quantity is an exact measure of density inhomogeneity, it is scalar and locally defined. With respect to a FLRW background Δ is a gauge - invariant variable that, once expanded at first order in the perturbations is proportional to the Bardeen variable ε_m (see [50]); thus its components with respect to an orthonormal set of scalar harmonic functions are proportional to the density perturbation in the comoving gauge (see [21]; for a comprehensive treatment of perturbations in this gauge, see [169], see also [51]).

In the following, we shall restrict our attention to the harmonic components of Δ, where the scalar harmonics Q are defined by

$${}^{(3)}\nabla^2 Q = -\frac{k^2}{a^2} Q \,, \tag{4.21}$$

${}^{(3)}\nabla^2$ is the Laplace operator in the 3-surface of constant curvature and $k \geq 0$ for $K = 0$, but $k \geq 1$ for $K = -1$ [136], [173]. Thus in the flat case the wavenumber k is simply related to the physical scale of the perturbation i.e. its wavelength $\lambda = 2\pi a/k$, but this is not the case for the open models. However k can always be taken as invariantly characterizing the scale of the perturbation, and we shall do so.

Since the harmonic component $\Delta^{(k)}$ of Δ and ε_m are just proportional [50], in the following the perturbation measure X appearing in (4.1) and (4.3) can be identified either with $\Delta^{(k)}$ or with ε_m.

In the case of a mixture of dust and radiation, in general the evolution equation for X is coupled to the evolution equation for an entropy perturbation variable (see [161]; [81]; [178] and also [204]). However the coupling is important only at small scales: here we shall only focus on perturbations at scales of the order of or larger than the Hubble radius H_E^{-1} at equi-density, thus we shall restrict to purely adiabatic modes, i.e. solutions of (4.1), neglecting the coupling with the entropy perturbation.

In general the coefficients α and β in (4.1) are given by (see [94]; [50]; [204])

$$\alpha = (2 + 3c_s^2 - 6w)H \; , \tag{4.22}$$

$$\beta = -\left[\left(\tfrac{1}{2} + 4w - \tfrac{3}{2}w^2 - 3c_s^2\right)\kappa\mu + 12(c_s^2 - w)\frac{K}{a^2}\right] + c_s^2\frac{k^2}{a^2} \; ; \tag{4.23}$$

using (4.17) for the dust - radiation background we get

$$\alpha = 2\frac{(1-3w^2)}{(1+w)}H \; , \tag{4.24}$$

$$\beta = -\left[\frac{3}{2}\left(1 + \frac{w^2(5-3w)}{(1+w)}\right)\Omega H^2 + 4\frac{w(1-3w)}{(1+w)}\frac{K}{a^2}\right] + \frac{4w}{3(1+w)}\frac{k^2}{a^2} \; . \tag{4.25}$$

From this, we obtain

$$\psi(w,\Omega) = \frac{1}{6}(1+3w)(1-\Omega) + \frac{1-6w-15w^2}{6(1+w)} \; , \tag{4.26}$$

$$\xi(w,\Omega) = -\frac{1}{6}\left(1 + \frac{w^2(5-3w)}{1+w}\right)\Omega + \frac{4w(1-3w)}{9(1+w)}(1-\Omega) + \Xi_K(w,k) \; , \tag{4.27}$$

where, $\Xi_K(w,k) = \frac{4w}{27(1+w)}\frac{k^2}{a^2H^2}$ is a function that takes a different form, as function of w, depending on the curvature K: in an open universe $a^2H^2 = (1-\Omega)^{-1}$, but in the flat case we cannot use the Friedmann equation to substitute for a^2H^2. Rather, we use it to substitute for $3H^2 = \kappa\mu$, and μ is given by (4.20), while here $a = Sa_E$ and S is given by (4.15). Thus

$$\begin{array}{lll} K=0 & \Xi = \frac{(1-3w)^2}{1+w}\Xi_0 k^2\,, & \Xi_0 = \frac{8}{81a_E^2H_E^2}\,, \\ K=-1 & \Xi = \frac{4w}{27(1+w)}(1-\Omega)k^2\,, & \end{array} \tag{4.28}$$

where, $3H_E^2 = \kappa\mu_E$, and for open models we can substitute for $\Omega = \Omega(w)$ from (4.19) in all the previous expressions. It is clear from (4.28) that in the space $\{\mathcal{U}, w, \Omega\}$ the function Ξ_K is not continuous (for $k \neq 0$) on the plane $\Omega = 1$, except on the line $w = 1/3$, $\Omega = 1$; this discontinuity of Ξ_K will play an important rôle in the behaviour of perturbations.

4.1.4 Results

We shall now summarize the results we obtained from the analysis of the dynamical system given by (4.12), (4.18) and (4.4), with coefficients ψ and ξ given by (4.26) and (4.27), first restricting to the flat models and then considering the open ones.

4.1.4.1 The flat models

The flat perturbed models are described by the subsystem (4.12), (4.18) substituting $\Omega = 1$ in ψ (4.26) and ξ (4.27) and using Ξ for $K = 0$ given in (4.28). From this latter we see that for flat models there is a characteristic scale that, as we shall see, is related to the late time behaviour of the perturbations: this is the Hubble radius at equi-density H_E^{-1}. It is therefore convenient to define

$$k_E = \frac{k}{a_E H_E} = 2\pi\frac{H_E^{-1}}{\lambda_E}\,, \tag{4.29}$$

which represents a wavenumber normalized at equi-density: $k_E = 2\pi \Leftrightarrow \lambda_E = H_E^{-1}$ i.e. $k_E = 2\pi$ corresponds to a perturbation wavelength that enters the horizon at equivalence epoch.

Stationary points eventually exist on the $w = 1/3$ and $w = 0$ axis, with

$$\mathcal{U}_\pm = \tfrac{1}{2}\left(-\psi_w \pm \sqrt{\psi_w^2 - 4\xi_w}\right)\,, \tag{4.30}$$

where ψ_w, ξ_w (with $w = 0, 1/3$) correspond to the stationary values of ψ and ξ.

The stationary points for the system (4.12), (4.18) are given in Table 4.1.4.1.

POINT I: UNSTABLE NODE $w = 1/3, \ \ \mathcal{U}_- = -\frac{1}{3}$	POINT II: SADDLE $w = 1/3, \ \ \mathcal{U}_+ = \frac{2}{3}$
$\lambda_{\mathcal{U}} = 1$ $\lambda_w = \frac{1}{3}$	$\lambda_{\mathcal{U}} = -1$ $\lambda_w = \frac{1}{3}$
POINT III: SADDLE $w = 0, \ \ \mathcal{U}_- = -\frac{1}{12} - \frac{1}{2}\lambda_{\mathcal{U}}$	POINT IV: STABLE NODE $w = 0, \ \ \mathcal{U}_+ = -\frac{1}{12} - \frac{1}{2}\lambda_{\mathcal{U}}$
$\lambda_{\mathcal{U}} = \sqrt{\frac{2}{3}\left(\frac{25}{24} - \frac{k_E^2}{k_{EC}^2}\right)}$ $\lambda_w = -\frac{1}{3}$	$\lambda_{\mathcal{U}} = -\sqrt{\frac{2}{3}\left(\frac{25}{24} - \frac{k_E^2}{k_{EC}^2}\right)}$ $\lambda_w = -\frac{1}{3}$

Table 4.1: Flat models: the four stationary points with the corresponding eigenvalues along the axis $\mathcal{U}$ and w and their nature; Point III and IV exist only for $k_E \leq \frac{5}{2\sqrt{6}} k_{EC}$.

As we have explained at the end of section 4.1.2, the condition for having growing perturbations is given by the appearance of a stable node on the positive side of the $\mathcal{U}$ axis. From (4.30) we see that we need $\xi_0 \leq 0$ in order to have $\mathcal{U}_+ \geq 0$, i.e.

$$k_E \leq k_{EC} \ , \qquad k_{EC} = \frac{3\sqrt{3}}{4} \, . \tag{4.31}$$

Moreover, stationary points only exist for $\psi_w^2 - 4\xi_w \geq 0$, which is always satisfied only for $w = 1/3$. For $w = 0$ we have $\psi_0^2 - 4\xi_0 = \frac{2}{3}(\frac{25}{24} - k_E^2/k_{EC}^2)$, i.e. stationary points exist on this axis for perturbations of wavelength at equivalence $\lambda_E \geq \frac{2\sqrt{6}}{5}\lambda_{EC} < \lambda_{EC}$. We see from Table 4.1.4.1 that this is the same condition for the reality of the eigenvalues, a result that follows from the fact that w' does not depend on $\mathcal{U}$. From the values of these eigenvalues we have that: Point I is an unstable node, Point II is a saddle, Point III is a saddle, Point IV is a stable node.

When for this latter we have $\mathcal{U}_+ > 0$, i.e. for k_E smaller than the *critical* wavenumber k_{EC}, generic perturbations with physical wavelength $\lambda_E > \lambda_{EC}$ *grow unbounded*, with the critical perturbation scale λ_{EC} corresponding to k_{EC} in (4.31), given by $\lambda_{EC} = \frac{8\pi}{3\sqrt{3}} H_E^{-1}$, i.e. $\lambda_{EC} > H_E^{-1}$: this situation is illustrated in Fig. 4.1. Note in this figure the special *saddle trajectories*: the one ending in $\mathcal{U}_-$ represents the purely decaying mode, while the other (ending in $\mathcal{U}_+$ as the generic trajectory) represents the purely growing mode[9]

[9]The terminology *growing* and *decaying* is purely conventional: it is adopted here because it is standard in the literature to refer in this way to the two modes, as they are effectively growing and decaying e.g. in a flat pure dust model. We stress again that the "growing" mode is actually growing only when $\mathcal{U}_+ > 0$.

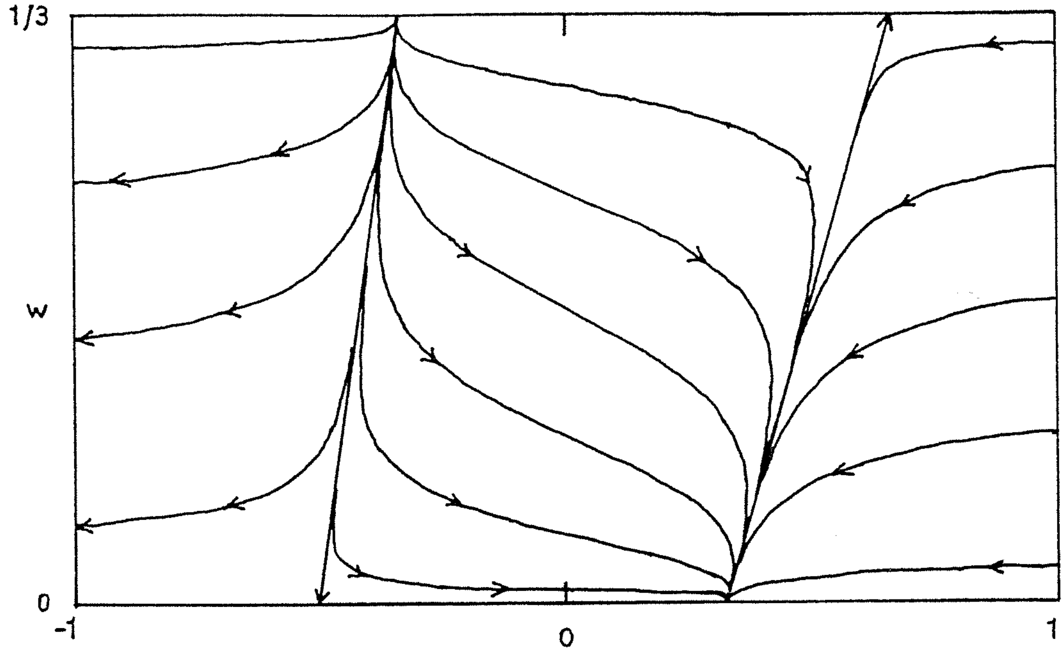

Figure 4.1: Flat models: phase space for the evolution of large scales $\lambda_E > \lambda_{EC}$ that grow unbounded ($\mathcal{U}_+ > 0$). Here and in the other figures dots represent the stationary points, while $0 \leq w \leq 1/3$ and $-1 \leq \mathcal{U} \leq 1$.

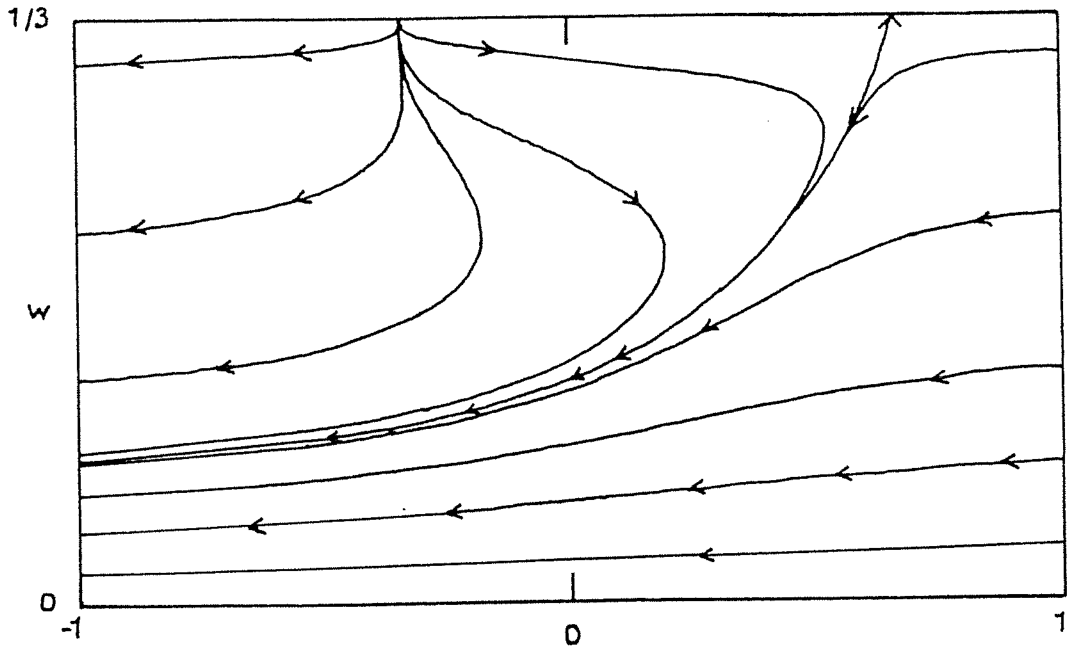

Figure 4.2: Flat models: phase space for the evolution of the damped wave modes on small scales $\lambda_E < \frac{2\sqrt{6}}{5} < \lambda_{EC}$ ($\mathcal{U}_\pm \in \Im$).

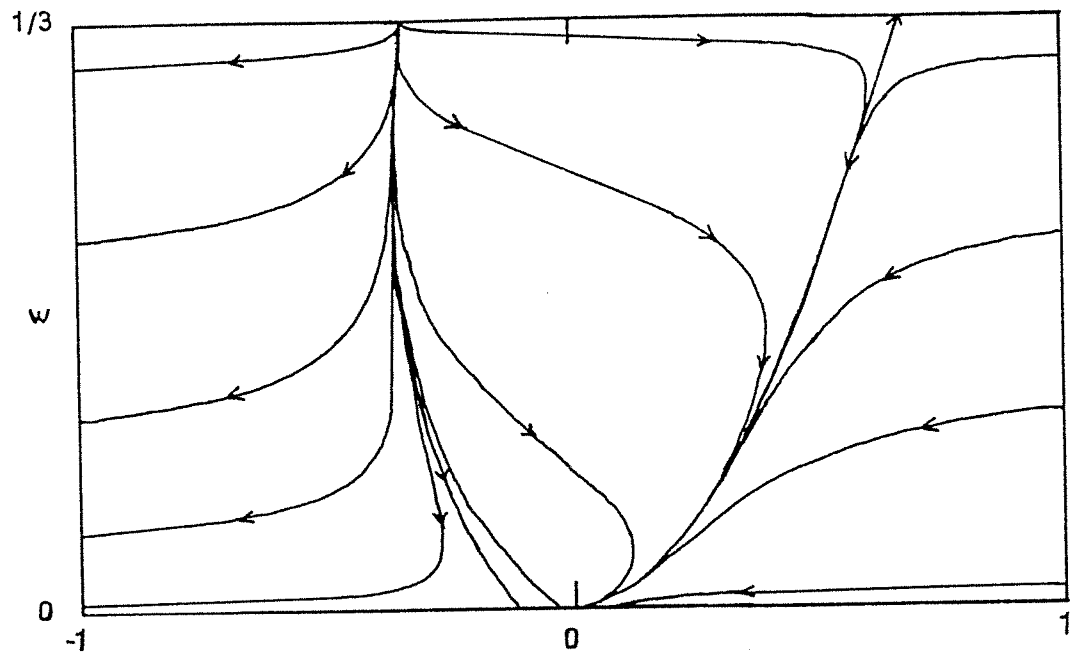

Figure 4.3: Flat models: phase space for the overdamped modes ($\mathcal{U}_+ < 0$) on intermediate scales $\frac{2\sqrt{6}}{5}\lambda_{EC} < \lambda_E < \lambda_{EC}$.

(the attractor); the generic trajectory represents a linear combination of the two modes solutions of (4.3). Also, we remind that the variable $\mathcal{U}$ is a tangent in the original phase space $\{X, X'\}$, so that the trajectories in the figures that start from Point I and go to the left of the saddle trajectory exit the figure on the left boundary, and re-enter it from the right ending in Point IV.

Clearly, when Point IV does not exist, i.e. for perturbations with wavelength at equivalence $\lambda_E < \frac{2\sqrt{6}}{5}\lambda_{EC}$, we are considering perturbations that always *oscillate* as "sound waves" (see Fig. 4.2); we have checked through a direct stability analysis of the system (4.37) below (see section 4.1.4.4) that the amplitude of these modes decay. Again, trajectories that exit from the left re-enter from the right: for $\lambda_E < \frac{2\sqrt{6}}{5}\lambda_{EC}$ however this happens many times, and indeed it is this fact that characterizes the oscillatory behaviour of these modes in the $\{\mathcal{U}, w\}$ plane.

Also, for scales $k_{EC} < k_E < \frac{5}{2\sqrt{6}}k_{EC}$ we have $\mathcal{U}_+ < 0$ for Point IV, i.e. the stable node is located on the negative side of the $\mathcal{U}$ axis: thus perturbation scales in this range are overdamped: they decay without oscillating, as it appears from Fig. 4.3. Again, the saddle trajectories represent the two modes for (4.3), and the trajectories going out from

INSTABILITY	STABILITY	
$\lambda > \lambda_{EC}$ grow unbounded	$\frac{2\sqrt{6}}{5}\lambda_{EC} < \lambda_E < \lambda_{EC}$ overdamped	$\lambda_E < \frac{2\sqrt{6}}{5}\lambda_{EC}$ damped waves

Table 4.2: Flat models: summary of the three different evolution regimes for different perturbation wavelengths.

the left of Fig. 4.3, re-enter from the right, ending in Point IV.

Finally, we point out that for $k_E = \frac{5}{2\sqrt{6}}k_{EC}$ the two stationary points on the $\mathcal{U}$ axis (the saddle and the stable node in the second line of Table 4.1.4.1) coincide, i.e. this is a *fold bifurcation point* (see e.g. [16]). The corresponding scale $\lambda_E = \frac{2\sqrt{6}}{5}\lambda_{EC} = \frac{16\sqrt{2}}{15}\pi H_E^{-1} \simeq 4.7 H_E^{-1}$ is quite larger than the Hubble radius at equi-density; thus we can consider scales $\lambda_E < \frac{2\sqrt{6}}{5}\lambda_{EC}$, as in Fig. 4.2, such that still $\lambda_E > H_E^{-1}$: this partially justifies our assumption of purely adiabatic perturbations (cf. [197]).

Thus we have three different evolution regimes for the adiabatic perturbation modes of a mixture of uncoupled dust and radiation in a flat universe, depending on their wavelength:

(i) large scale perturbations that grow unbounded, giving instability;

(ii) intermediate scale perturbations that are overdamped, i.e. decaying without oscillating;

(iii) small scale damped perturbations which oscillate like sound waves while their amplitude decays;

as summarized in Table 4.2.

4.1.4.2 The open models

The open models are in principle described by the full 3-dimensional system given by (4.12), (4.18) and (4.4), with trajectories in the $\{\mathcal{U}, w, \Omega\}$ space. However, given $\Omega = \Omega(w, \Omega_E)$ (4.19), we can substitute for Ω in the expressions for ψ (4.26), ξ (4.27) and Ξ for $K = -1$ (4.28), and restrict our analysis to the 2-dimensional system (4.12), (4.18). In doing this, we are selecting one particular open model in the class parameterized by Ω_E:

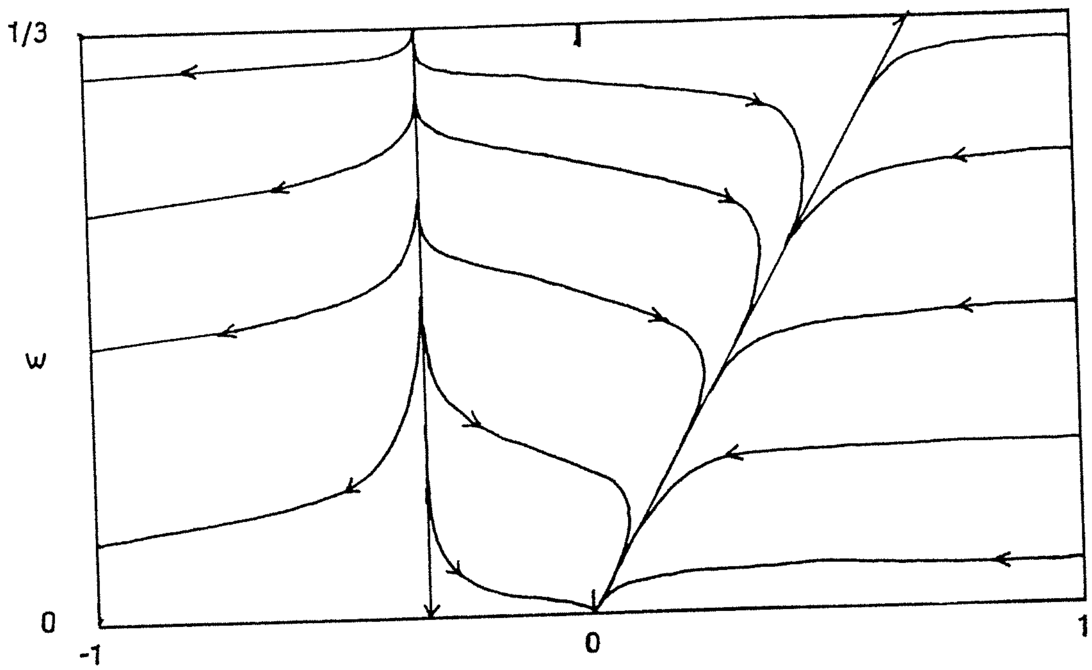

Figure 4.4: Open models: the phase space topology is independent of the wavelength, and all perturbations tend to a constant value ($\mathcal{U}_+ = 0$).

there is no loss of generality in doing this, as the dynamical properties of the models in this class (as specified by the character of the stationary points in the corresponding phase space) do not depend on Ω_E, i.e. for a given wavenumber k the phase space evolution of all the models in the class is qualitatively the same. From the point of view of the geometry of the phase space $\{\mathcal{U}, w, \Omega\}$ we are looking at the trajectories in the 2-dimensional surface specified by Ω_E in this space: we shall then consider the projection of this surface with its trajectories in the $\{\mathcal{U}, w\}$ plane, as depicted in Fig. 4.4.

It is clear from this figure and Table 4.3 that, contrary to what we have seen for the flat models, the existence, position and the nature of Points I–IV do not depend on the wavenumber k, so that all open models share the same dynamical history. This depends on the vanishing of the function Ξ in the limit $w \to 0$, which also implies $\Omega \to 0$ (because we are moving on the surface specified by Ω_E). Note however that if we do not substitute for Ω from (4.19) in the 3-dimensional system, and we take the limit $\Omega \to 0$, this does not give the 2-dimensional system for flat models: this fact is due to the discontinuity (remarked at the end of section 4.1.4.1) of the function Ξ on the surface $\Omega = 1$, where Ξ for $K = 0$ do not vanish in general, except for $w = 1/3$ or $k = 0$. Somehow this fact can be seen as an example of "fragility" in cosmology [69].

From table 4.2 and Fig. 4.4 we see that Point IV is a stable node located at $\mathcal{U} = 0$. The generic trajectory ends up in this point, either directly from Point I, or first going out from the left boundary of the picture and then re-entering from the right. Again, the saddle trajectory starting form Point I and ending in Point III represents the evolution of the purely decaying mode, and the saddle trajectory starting from Point III and ending in Point IV represents the purely growing mode (attractor). The fact that Point IV is located in $\mathcal{U} = 0$ means that (by definition of $\mathcal{U}$) all the perturbation modes evolve up to a constant value, and then freeze–in. Then we can say that open models are stable but not asymptotically stable, as the generic perturbation modes do not decay, so they are marginally stable.

A direct analysis of the system (4.37) below (section 4.1.4.4) shows indeed that the appearance of a fixed point on $\mathcal{U} = 0$ for the system (4.12), (4.18) corresponds to the vanishing of one of the eigenvalue for the corresponding fixed point in the phase space $\{X, X', w\}$, while the other two eigenvalues are negative. Thus there is a whole line of fixed marginally stable points.

While this happens only for $k_E = k_{EC}$ in flat models, it is a generic characteristic for any k in open universes. A comparison of Fig. 4.4 and Fig. 1a in [48] shows that open dust-radiation models and open pure dust models share the same stability properties. It appears that the curvature of the background dominates also the evolution of the perturbations: for dust radiation models there is a conspiracy between curvature and pressure, such that curvature has an opposite effect on small and large scale: with respect to the behaviour in flat models, it avoids the oscillation of small scales perturbations ($\lambda_E < \frac{2\sqrt{6}}{5}\lambda_{EC}$) and damps the growth of those on large scales ($\lambda_E > \frac{2\sqrt{6}}{5}\lambda_{EC}$).

4.1.4.3 Comparison with Jeans instability

We can apply the Jeans instability criterion directly to equation (4.1): this implies that gravitational collapse of a given perturbation mode will occur if $\beta < 0$, i.e. if $k < k_J$, where for the dust–radiation models β is given by (4.25), and the corresponding k_J is

$$\frac{k_J^2}{a^2} = \frac{3(1+w)}{4w} \cdot \left[\frac{3}{2}\left(1 + \frac{w^2(5-3w)}{1+w}\right)\Omega H^2 + \frac{4w(1-3w)}{1+w}\frac{K}{a^2}\right]. \qquad (4.32)$$

POINT I: UNSTABLE NODE	POINT II: SADDLE
$w=\frac{1}{3},\ \mathcal{U}_-=-\frac{1}{3}$	$w=\frac{1}{3},\ \mathcal{U}_+=\frac{2}{3}$
$\lambda_{\mathcal{U}}=1$ $\lambda_w=\frac{1}{3}$	$\lambda_{\mathcal{U}}=-1$ $\lambda_w=\frac{1}{3}$
POINT III: SADDLE	POINT IV: STABLE NODE
$w=0,\ \mathcal{U}_-=-\frac{1}{3}$ $\lambda_{\mathcal{U}}=\frac{1}{3}$ $\lambda_w=-\frac{1}{3}$	$w=0,\ \mathcal{U}_+=0$ $\lambda_{\mathcal{U}}=-\frac{1}{3}$ $\lambda_w=-\frac{1}{3}$

Table 4.3: Open models: the four stationary points with the corresponding eigenvalues along the axis $\mathcal{U}$ and w and their nature.

At equi-density ($w = 1/6$) this gives

$$\frac{k_{JE}^2}{a_E^2} = \frac{279}{32}\Omega_E H_E^2 + \frac{3}{2}(\Omega_E - 1)H_E^2 \; ; \tag{4.33}$$

in the real universe $\Omega_E \approx 1$ and the contribution to k_{JE} from the curvature term (the last in the equation above) is completely negligible. Thus in a flat universe ($K = 0$) we have

$$\lambda_{JE} \equiv \frac{2\pi a_E}{k_{JE}} = \frac{8\pi}{3}\sqrt{\frac{2}{31}}H_E^{-1} \tag{4.34}$$

for the Jeans scale at equi-density. Then a comparison with the critical scale λ_{EC} defined in section 4.1.4.1 gives

$$\lambda_{JE} = \sqrt{\frac{6}{31}}\lambda_{EC} \; , \tag{4.35}$$

i.e. for flat models the stability criteria used in section 4.1.4.1 give a critical scale λ_{EC} for instability which is larger than the corresponding Jeans scale at equi-density by a factor of 2, i.e. $\lambda_{EC} \approx 2.3\,\lambda_{JE}$. The fact that the values of these two scales are relatively close appears physically meaningful, and in a way obvious, since both scales are somehow defined through the same differential equation. However, a comparison of (4.35) with the analysis of section 4.1.4.1 shows that there are perturbations with $\lambda_E > \lambda_{JE}$ that decay: those with wavelength $\lambda_{JE} < \lambda_E < \frac{2\sqrt{6}}{5}\lambda_{EC}$ are damped oscillation, while those with $\lambda_{EC} > \lambda_E > \frac{2\sqrt{6}}{5}\lambda_{EC}$ are overdamped. It is immediate to show that today the critical scale corresponding to k_{EC} is

$$\lambda_{0C} = \frac{4\pi}{3}\sqrt{\frac{2}{3}(1+z_E)^{-1}H_0^{-1}} \; , \tag{4.36}$$

i.e. about $67.3\,h^{-1}Mpc$ in a flat universe with $1 + z_E \approx 4.310^{-5}h^{-2}$ (see e.g. [162]). Even if this value may be an artifact of our simplifying assumptions, e.g. the fact that we have neglected isocurvature modes at all times and we have treated radiation as a perfect fluid even at small scales, we believe that the discrepancy between our critical scale and the Jeans scale in the same universe model is a general feature that deserves further investigation in order to consider possible effects for models of structure formation in the universe.

Finally, it is interesting to consider the limit $w \to 0$ of k_J. For flat models, one gets from (4.32) that in this limit $k_J = \frac{3\sqrt{3}}{4}a_E H_E$, i.e. we recover k_{EC} (4.31) in this limit. However the same limit for open models gives a value for k_J that: *a)* is real only for $\Omega_E > 16/25$, a result that, although satisfied in the real universe, appears spurious for the theory; *b)* gives the false impression that there could be growing and oscillating modes also for open models, contrary to what we have shown in the previous section.

4.1.4.4 Metric and curvature perturbations

In the previous sections we have given the results of the analysis of the dynamical system $\{\mathcal{U}, w\}$ (4.12), (4.18) for flat and open models, and inferred conclusions on the evolution of density perturbations, represented by the harmonic component X (see section 4.1.3.2). As we have already pointed out, there is a particular relation between the location and character of fixed points in the phase space $\{\mathcal{U}, w\}$, and the character of the corresponding point in the original phase space $\{X, X', w\}$[10]. Indeed, it is immediate that the original system ($Y \equiv X'$)

$$\left\{\begin{array}{lcl} X' & = & Y \\ Y' & = & -\psi Y - \xi X \\ w' & = & w\left(w - \frac{1}{3}\right) \end{array}\right. \qquad (4.37)$$

admits only two fixed points for $\xi \neq 0$: Point A $\equiv \{X = 0, Y = 0, w = 1/3\}$ and Point B $\equiv \{X = 0, Y = 0, w = 0\}$. The first represents an unperturbed pure radiation model, and the second a pure dust model, while the line connecting them is the mixed background model we are more interested in here. Then it is easy to see that the eigenvalues at these points along the principal directions in the $\{X, Y\}$ planes equate the roots $\mathcal{U}_\pm$ of (4.12),

[10]For open models we are using the function $\Omega = \Omega(w, \Omega)$ (4.19), so that the further dimension Ω in the phase space is suppressed.

i.e. $\lambda_\pm = \mathcal{U}_\pm$. Point A is the same for flat and open models ($\lambda_- = -1/3$, $\lambda_+ = 2/3$, $\lambda_w = 1/3$), while (4.27), (4.28) and the analysis in section 4.1.4.2 shows that for open models $\xi = 0$ for $w = 0$, and thus Point B degenerates into a line (the $Y = 0$ axis): $\lambda_+ = \mathcal{U}_+ = 0$ in this case, so that open models are marginally stable. For flat models, the same happens for $k_E = k_{EC}$, as already pointed out.

Having clarified the relation between the roots $\mathcal{U}_\pm$ of the system (4.12), (4.18) and the eigenvalues at the fixed points of (4.37), we can now turn to the asymptotic evolution of X. It is clear from the definition of $\mathcal{U} = Y/X$ that around the roots $\mathcal{U}_\pm$ the evolution of X is given by

$$X' = \mathcal{U}_\pm X\ , \quad \Rightarrow \quad X_\pm \sim S^{3\mathcal{U}_\pm}\ . \tag{4.38}$$

When $\mathcal{U}_+ = 0$ we have $X_+ = const.$ (cf. Table 4.3); for $w \to 0$ we recover the well known constant mode for matter dominated open models: here the same mode is found for the critical scale $k_E = k_{EC}$ in flat models.

In Tables 4.4 and 4.5 we give the asymptotic behaviour for X, for a metric perturbation Φ_N and a dimensionless curvature perturbation scalar E/Θ^2. In particular, in Table 4.4 we consider flat models in the limit $w \to 0$, giving the asymptotic behaviour of various variables as functions of the scale factor S and in order of increasing wavelength λ_E. In Table 4.5 we give the asymptotic solutions of open models around Points I–IV: as for $w \to 1/3$ the universe is radiation dominated and also $\Omega \to 1$, then $S \sim t^{\frac{1}{2}}$ in this limit, and the asymptotic solutions around Points I and II are in common with flat models. In the limit $w \to 0$ the universe models are matter dominated, and then in flat models $S \sim t^{\frac{2}{3}}$, while in open models $S \sim t$.

In the following we shall outline the relation between the gauge–invariant metric potential Φ_N, the curvature variable E, and the density perturbation Δ (see section 4.1.3.2): more details can be found in [50] and references therein. Let E_{ab} be the electric part of the Weyl tensor[11] C_{acbd}: $E_{ab} \equiv C_{acbd}u^c u^d$, where u^a is the 4-velocity of matter; then to first order

$$a^{2\,(3)}\nabla^{b\,(3)}\nabla^a E_{ab} = \frac{\kappa\mu}{3}\Delta\ , \tag{4.39}$$

where ${}^{(3)}\nabla_a$ is a covariant derivative orthogonal to u^a. The analogue of E_{ab} in Newtonian

[11] Latin indices are 4-dimensional (0, 1, 2, 3), and greek indices 3-dimensional (1, 2, 3).

QUANTITY	$\lambda_E < \frac{5}{2\sqrt{6}}\lambda_{EC}$	$\lambda_E = \frac{5}{2\sqrt{6}}\lambda_{EC}$	$\lambda_{EC} > \lambda_E > \frac{5}{2\sqrt{6}}\lambda_{EC}$	$\lambda_E = \lambda_{EC}$	$\lambda_E \gg \lambda_{EC}$
$X_\pm$	$S^{-\frac{1}{4}\pm\imath Q}$	$S^{-\frac{1}{4}}$	$S^{-\frac{1}{4}\pm P}$	$S^{-\frac{1}{2}}$, *const.*	$S^{-\frac{3}{2}}$, S
$\Phi_{N\pm}$	$S^{-\frac{5}{4}\pm\imath Q}$	$S^{-\frac{5}{4}}$	$S^{-\frac{5}{4}\pm P}$	$S^{-\frac{3}{2}}$, S^{-1}	$S^{-\frac{5}{2}}$, *const.*
$E_\pm/\Theta^2$	$S^{-\frac{1}{4}\pm\imath Q}$	$S^{-\frac{1}{4}}$	$S^{-\frac{1}{4}\pm P}$	$S^{-\frac{1}{2}}$, *const.*	$S^{-\frac{3}{2}}$, S

Table 4.4: Flat models: asymptotic behaviour for $w \longrightarrow 0$ (i.e. around Points III–IV) of X, Φ_N and E/Θ^2, in order of increasing wavelengths, as function of the scale factor S ($S \sim t^{\frac{2}{3}}$ for $w \longrightarrow 0$). The decaying ($-$, Point III) and growing ($+$, Point IV) modes are distinguished either by the $\pm$ or presented in order: they coincide for the critical damping scale $\lambda_E = \frac{5}{2\sqrt{6}}\lambda_{EC}$. $Q = \left[\frac{3}{2}\left(k_E^2/k_{EC}^2 - \frac{25}{24}\right)\right]^{\frac{1}{2}}$, and $P = \left[\frac{3}{2}\left(\frac{25}{24} - k_E^2/k_{EC}^2\right)\right]^{\frac{1}{2}}$.

QUANTITY	POINT I	POINT II	POINT III	POINT IV
X	S^{-1}	S^2	S^{-1}	*const.*
Φ_N	S^{-3}	*const.*	S^{-2}	S^{-1}
$\frac{E}{\Theta^2}$	S^{-1}	S^2	S^{-2}	S^{-1}

Table 4.5: Open models: asymptotic behaviour around Points I–IV for X, Φ_N and E/Θ^2, as function of the scale factor S. For $w \longrightarrow 1/3$ $S \sim t^{\frac{1}{2}}$, and Point I and II are in common with flat models, while for $w \longrightarrow 0$ $S \sim t$. Points I and III correspond to the decaying mode, and II and IV to the growing mode.

theory is the tidal field $E_{\alpha\beta} = \nabla_{\alpha\beta}\phi$, where ϕ is the Newtonian potential and $\nabla_{\alpha\beta}\phi \equiv \nabla^2\phi - \frac{1}{3}\delta_{\alpha\beta}\phi$. Then it is possible to show that to linear order a similar formula holds in relativistic perturbation theory, i.e. $E_{\alpha\beta} = \nabla_{\alpha\beta}\Phi_N$ for the scalar part of E_{ab} (all the $\{0,0\}$ and $\{0,\alpha\}$ components are second order). Moreover the field Φ_N, which play here the rôle of a gauge–invariant analogue of the Newtonian potential, is just $\Phi_N = \frac{1}{2}(\Phi_A - \Phi_H)$, where Φ_A and Φ_H are the gauge invariant metric perturbations defined by Bardeen [21] ($\Phi_A = -\Phi_H$ for perfect fluids). Then using $E_{\alpha\beta} = \nabla_{\alpha\beta}\Phi_N$ in (4.39), the harmonic decomposition (4.21) and $\Delta^{(k)} = -k^2\varepsilon_m$, we get equation (4.3) of Bardeen [21]:

$$2(3K - k^2)\Phi_N = \kappa a^2 \mu \varepsilon_m \; . \tag{4.40}$$

Then this relation can be used to determine the asymptotic evolution of the gauge–invariant metric potential Φ_N, and also that of amplitude $E = \frac{1}{2}\sqrt{E_a{}^b E_b{}^a}$ of the tidal field E_{ab}: indeed from a comparison of (4.39) and (4.40) the harmonic components of E and Φ_N are related by

$$E = \frac{1}{2} a^{-2} k^2 \Phi_N \; . \tag{4.41}$$

Then it is usual to consider a dimensionless scalar to measure the relative dynamical significance of a given field using Θ to take into account the expansion (see e.g. [123]): thus in the case of E we consider E/Θ^2, for which we have

$$\frac{E}{\Theta^2} \sim \frac{a^{-2}\Phi_N}{\Theta^2} \sim \frac{\kappa\mu}{\Theta^2}\varepsilon_m \sim \Omega\varepsilon_m \; . \tag{4.42}$$

Hence, for example for a flat model $\Phi_N \sim$ *const.* for very large scales, but the relative amplitude of the tidal field $E/\Theta^2 \sim \Delta \sim a$ grows unbounded (cf. [80]).

4.1.5 Conclusions

We have considered the stability properties of FLRW models with uncoupled Cold Matter (dust) and radiation, and the perturbed models were considered as dynamical systems described by an evolution equation for a gauge–invariant density perturbation variable coupled with the equations governing the evolution of relevant background variables: the pressure–energy density ratio $w = p/\mu$ and the density parameter Ω. For the subsystem describing flat models, given by $\Omega = 1$, we deal with a planar autonomous system, and for the open models we have shown that we can also restrict the analysis to a planar system for each particular value of the density parameter at equi-density Ω_E.

The analysis of flat and open models gives different results: flat models admit unstable perturbation modes, while open models are marginally stable with respect to perturbations, irrespective of their scale, ie. the perturbations freeze–in at a constant value. Qualitatively, both results can be expected on the basis of the results of a more traditional study of the behaviour of the perturbations in simple models. The final fate of the perturbations in open models appears to be dominated by the background curvature, which governs the background expansion at late times: therefore the stability properties of the dust–radiation models are the same as those of a pure dust model, irrespective of the size of the perturbation and of the radiation content of the given model.

Instead, we find more interesting features for flat models: in this case each model has a characteristic critical invariant wavenumber k_{EC} which depends on the proportion of matter and radiation (e.g. at present), and the corresponding scale λ_{EC} determines the transition from stable to unstable modes. The scale λ_{EC} is of the order of the Hubble radius H_E^{-1} at the equi-density of matter and radiation, H_E^{-1} being the only scale entering the background model. The present value of this transition scale is $\lambda_{0C} = (1 + z_E)\lambda_{EC}$ and depends on the actual present proportion of matter and radiation, for which it is of the order of $67Mpc$. Therefore the stability properties of dust–radiation flat models are a mixture of the properties of pure dust and pure radiation models, being well known that perturbations in a flat dust models grow unbounded irrespective of their size, while all perturbation scales in a radiation model always enter an oscillatory (sound waves) regime after they enter the sound horizon (e.g. see [21]).

However, we actually find a structure in the stability properties of flat dust–radiation models which is more interesting than expected, because we find three different regimes for the evolution of the perturbations. Perturbations on scales $\lambda_E > \lambda_{EC}$ grow unbounded (unstable modes), while perturbations in the range $\frac{2\sqrt{6}}{5}\lambda_{EC} < \lambda_E < \lambda_{EC}$ are overdamped; finally, perturbations on scales $\lambda_E < \frac{2\sqrt{6}}{5}\lambda_{EC}$ are damped, i.e. they oscillate as "sound waves" while decaying.

The stability properties of perturbed FLRW models that we have found, should be taken *cum grano salis* with respect to the problem of structure formation in the universe.

For example, it is clear that the marginal stability we have found for open models should not be taken as implying that structures cannot form in an open universe: in fact it can be expected that this stability could be broken by non–linearity. In other words, here we have found that open models (see also [48]) are marginally stable against *linear* perturbations; in going beyond this level of approximation we can expect that instability will be switched on by non–linearity.

Also, we have assumed adiabatic perturbations at all times, neglecting isocurvature modes that are in principle important at small scales and late times, and we have neglected photon diffusion, treating radiation as a perfect fluid irrespective of the scale of the perturbation (in particular, the oscillatory behaviour of small scales in flat models is probably an artifact of this assumption). However, we believe that these assumptions should not question the validity of our main results for flat models: *a)* in a given flat universe model (including the assumptions about the matter content and how to treat it) there are perturbations that decay despite the fact that their wavelength at equi-density λ_E is larger than the Jeans scale λ_{JE}; *b)* the today size of the critical scale we have found is of the order of $67\,Mpc$, a fact that perhaps deserves further investigation regarding its implication for structure formation in the universe. Also, while in standard flat CDM models the fluctuations stop to oscillate when the universe enters the matter dominated era, we have found that small scale fluctuations continue to be damped even at late times. It will be therefore interesting to make a proper comparison between the power spectrum of standard CDM and the one that could be derived from the analysis given here (normalizing the spectrum at large scales), eventually considering also the isocurvature modes that, as said above, are expected to modify the small scale fluctuations behaviour. This sort of analysis could lead to interesting results, giving less power at small scales in comparison with the standard CDM results. However we cannot go beyond this speculative level at this stage, and we leave this for a future work.

Finally, we remark that a comparison of the stability properties of dust–radiation models with the observed small amplitude of the large scale density fluctuations seems to suggest that if the spatial curvature of the universe vanishes, then we live in a special epoch in which these perturbations are still in a linear regime of growth, while if we live in a universe with negative spatial curvature the smallness of the large scale perturbations is a characteristic of the model at all times and the description of the universe as an open FLRW model is appropriate at any epoch.

Thus open models are special (at a given time) on average, because the density parameter Ω depends on time (this is the flatness problem), but not from the local point of view, because large scale structures are frozen-in, while the reverse is true for flat models, because $\Omega = 1$ at all times, but large scale perturbations grow unbounded in these models.

5 Outlook and directions for future research

Following the conclusions of chapter three we would like to advocate that the application of RG in cosmology is a line of research worth pursuing. It seems at the end to be connected with critical phenomena, chaos, self-organized criticality (SOC), fractals, flicker noise, and in particular, the notion of entropy for gravitational systems and complexity. The interconnections are not yet understood nor appreciated, and many issues are not understood, e.g. the flicker noise ("$1/f$") (e.g. [196]) is still one of the great mysteries of physics and only now have some models displaying SOC been shown to exhibit this kind of noise.

Particularly interesting are, in our opinion, critical phenomena. Critical phenomena are after all always manifested macroscopically since phase transitions are collective phenomena (arising from interactions between quasi-particles of the system), with the presence of non-linearities of great importance. It would therefore be very interesting and valuable to study the connections between critical phenomena, Macroscopic Gravity and the Renormalization Group. Indeed, the correlation tensors in MG theory are functions of a space-time point, but once the many points correlation functions (whose coincidence limit gives the local correlation tensors) are introduced in a non-equilibrium theory of macroscopic gravitational processes, one can look for solutions exhibiting critical behaviour.

Another point is to understand how things become organized into complex structures in the universe. We can conjecture that some "laws" of organization (self-organization)[1]

[1] L. Smolin, private communication.

or information-processing are necessary to describe not only the quantity, but also the organization of the information within a system. As proposed in [62], one could describe complexity of a state by the running-time or entropy production of the shortest possible program needed to compute the state. These ideas, relevant to a definition of randomness and chaos, in general, have not yet played any important rôle in the understanding of cosmology.

Expanding further considerations of this chapter, let us notice a useful connection between gravitation and turbulence at high Reynolds number, statistical field theory and critical phenomena. Firstly, there is an analogue between classical gravitation, turbulence and field theories with an infra-red attractive RG fixed point. This is the situation for field theories corresponding to critical lattice spin systems, and in particular the Ising model. The following list can be drawn up:

Turbulence	**Critical Phenomena**	**Gravitation**
space separation	wavenumber	separation (geodesic)
viscosity	temperature	3−metric
energetic length-scale	UV cut-off (or, inverse lattice-spacing)	coarse-graining scale
dissipation wavenumber	correlation length	correlation length
velocity correlation function	spin correlation function	two-point correlation fn, etc.
intermittency exponent	correlation exponent	correlation exponent

The analogies between turbulence and critical phenomena have been pointed out e.g. in [73] (see also [105] recently) and we will not discuss them here. The analogy with gravitation has not been, to our knowledge, spelled out.

Above, the rôles of space and wavenumber for turbulence and critical phenomena are interchanged, because turbulence as opposed to critical phenomena, exhibits short distance scaling believed to be generic and essentially independent of the large scale statistics or driving mechanisms.

Secondly, let us point out for completeness, an analogy that can be made between turbulence and field theories with an ultra-violet attractive fixed point [105], and Quantum Gravity:

Turbulence	**Field Theory**	**Quantum Gravity**
space separation	space separation	separation
viscosity (or Kolmogorov scale)	lattice-spacing	"discretization"-spacing[2]
energetic length-scale	correlation length	correlation length $\sim \frac{1}{G}$
Kolmogorov wavenumber	UV cut-off = inverse lattice spacing	UV cut-off
velocity correlation function	field-theoretic Green function	correlation function on the lattice

Let us remark that field theories such as UV asymptotically free QCD exhibits scaling at short distances, just as turbulence does. Likewise, QG models studied so far (mainly numerically studied dynamically triangulated QG models), exhibit similar finite-size scaling, and in dim= 4 a second order phase transition most probably takes place from the "smooth" to the "crumpled" phase [134, 4, 5, 2, 3, 58] (the last one is an analytical approach)[3].

Let us also remark that in the first analogy the inverse rôle of large and small scales arises from the different character of "cascade" in the two cases. In the cascade picture, there is a transfer of excitation on the average from the large turbulent eddies to the small ones by a stepwise process, which is chaotic in nature and entails a loss of memory of the large-scale statistics. Wilson has emphasized [254] that there is also a "cascade of fluctuations" in critical phenomena. Droplet fluctuations nucleated at the lattice-scale in the critical state can grow to the size of the correlation length. But now the details of the lattice structure are lost instead and the scale-invariant distributions of the large "droplets" are universal. In this connection, let us notice that interestingly enough there is a "hierarchy of structures" in the universe that is governed by relativistic gravitation field theory, which in certain sense has in it a cascade picture too.

Now the challenge would be to really understand these analogies (and differences). While concrete problems may be highly non-trivial, e.g. how to define a suitable analog of the Migdal decimation or Kadanoff block-spin transformation for curved dynamical lattices (see [153] for an attempt in 2 dimensions). In any case, the overall conceptual picture sketched here could be, we hope, of help for further research. It is after all one

[2] E.g. size of simplicial complexes in triangulations or edge links as in Regge calculus [211].

[3] For a good introduction to this subject see [72] (cf. also [43]).

more example of the great unity and pluralism in theoretical physics [100], and this is what makes it even more interesting.

Another issue worth pondering is the status of Macroscopic Gravity theory and we would like to offer now some comments on this point. Clearly, any attempts at constructing relativistic cosmological models are likely to involve "averaging" of some kind, both on observational and theoretical grounds. A systematic way to approach this problem would be to start with the Einstein field equations and derive the corresponding averaged macroscopic equations as a Lagrangian theory of the equilibrium gravitational macroscopic phenomena. MG theory is a possible solution to this problem destined to give a consistent description of gravitating continuous media, in particular, the universe filled with matter.

MG offers also a solution to another problem, namely that of correlation functions. In modern mathematical physics and field theory an important problem is that of a description of collective field phenomena, be they classical or quantum, in geometric theories such as Yang-Mills or General Relativity. These problems require one to take into account the field correlations in the non-linear theories on a curved space-time in a non-perturbative way. Such geometric correlation functions are introduced in MG as well as the structure equations describing the macroscopic geometry and macroscopic (averaged) field equations. A similar problem can be pointed out in quantum gravity, where one of the main tasks to be solved would be to find a geometric structure underlying quantum gravitational phenomena. In order to achieve this one should find a geometric setting for the introduction and description of quantum correlation functions (propagators, vertices, etc.) defined as quantum averages.

The next step dictated by the logic of the development of the theory is to formulate it as a many-point theory in Hamiltonian form [270] and consider its generalization to the case of a statistical *ensemble* averaging and look at the possibilities of formulating a geometric theory of gravitational critical phenomena. First of all, the many-point generalization of space-time averaging should be developed, which is expected to give a two-point function for the average product of two functions, etc., unlike in the presently available scheme where all average values are prescribed to one point of the averaging region when averaging of products of functions is carried out.

Further, the space-time averaging can be considered in a $3+1$ foliation of space-time, where a possible non–commutativity of a projection of a space-time average on a $3-$space and $3-$space averaging can be expected, in order to identify the field variables in $3+1$ splitting.

Finally, one of the aims would be also to express all geometric fields as statistical and probability distributions over (pseudo-)Riemannian space-time and define proper statistical *ensemble* averaging for them, which would involve dealing with the question of equivalence of the space averaging and the *ensemble* averaging, of relevance for any kind of ergodicity assumptions that can be expected to be involved in some way.

An alternative way to perform the averaging was recently put forward by Zeeman [272], who adopted the $\epsilon-$smoothing[4] in applying the related Fokker-Planck equations as a way to "stabilize" the resultant dynamical system. It would therefore be interesting to compare the outcomes of this approach and Macroscopic Gravity [243].

Clearly the strength of any theory lies in its physical applications and testable predictions. Macroscopic Gravity is quite a young theory and research is under way to study the properties of "macroscopic–FLRW" models, with one non–vanishing correlation tensor and obtain particular predictions in these models concerning the speed of sound, the Sachs-Wolfe effect, etc., in order that a proper confrontation of the theory with reality can be made [207].

To gain a thorough understanding of "macroscopic cosmology" will require much work. In particular, the rôle of the correlation tensors in the evolution of the universe and large scale structure formation is not known presently.

[4]Not to be confused with the $\epsilon-$smoothing as introduced in section 3.6.

Appendix A
Some useful notions from Riemannian Geometry

We recall here some basic concepts in Riemannian Geometry [30, 117] (see also [23] for a concise overview).

Let $\mathcal{M}$ be a smooth C^∞, Hausdorff, connected, oriented, compact n-dimensional manifold without boundary and let g be a Riemannian metric on $\mathcal{M}$, i.e., a smoothly varying family of inner products G_x on the tangent spaces $T_x\mathcal{M}$, $x \in \mathcal{M}$.

A metric g on $\mathcal{M}$ is called an *Einstein* metric if the Ricci curvature $Ric(g) = \lambda g$ for some constant λ. By normalization, one can always assume to be in one of the three cases: $Ric(g) = g$ (when $\lambda > 0$), $Ric(g) = 0$ ($\lambda = 0$) or $Ric(g) = -g$ (when $\lambda < 0$). We use the term "Einstein manifolds" for Riemannian manifolds of constant Ricci curvature.

Let $c : [0, a] \to \mathcal{M}$ be a curve, and $0 = a_o < a_1 < \ldots < a_n = a$ be a partition of $[0, a]$ such that $c_{|[a_i, a_{i+1}]}$ is of class C^1. The *length of* c is defined by

$$L(c) = \sum_{i=0}^{n-1} \int_{a_i}^{a_{i+1}} |c'(t)| dt, \tag{A.1}$$

where, $|c'(t)| = \sqrt{g(c'(t), c'(t))}$.

The length of a curve does not depend on the choice of a regular parameterization.

The *Riemannian distance* between two points x and y in $\mathcal{M}$ is defined to be the infimum of the length (w.r.t. g) of the curves from x to y. The *diameter* D of $(\mathcal{M}, g)$ is the diameter of $\mathcal{M}$ for the Riemannian distance.

The *geodesics* are the curves which satisfy the Euler-Lagrange equation of the problem of minimization of the energy of a curve. In particular, given any point x in $\mathcal{M}$ and any unit vector $u \in T_x\mathcal{M}$, there is (locally) one and only one geodesic $c_{x,u}$ parameterized by arc length t, such that $c_{x,u}(0) = x$ and $\dot{c}_{x,u}(0) = u$ (such a geodesic is defined for all values of t when $\mathcal{M}$ is closed).

We define the *exponential map* $\exp_x : T_x\mathcal{M} \to \mathcal{M}$, by $\exp_x(tu) = c_{x,u}(t)$, for any $t \geq 0$ and any unit tangent vector u. The exponential map is a local diffeomorphism from a neighbourhood of 0 in $T_x\mathcal{M}$ to a neighbourhood of x in M, its derivative at 0 is the identity map.

An *isometry* f between two Riemannian manifolds $(\mathcal{M}, g)$ and $(\mathcal{N}, h)$ is a smooth map $f : \mathcal{M} \to \mathcal{N}$ whose derivative induces isometries between the tangent spaces, with respect to the inner products g and h, respectively. In particular, the two Riemannian manifolds are isometric if there exists some diffeomorphism $f : \mathcal{M} \to \mathcal{N}$, which transfers h into g, i.e., $f^*h = g$.

A Riemannian *structure* is a class of isometric Riemannian manifolds. In other words, if $Riem(\mathcal{M})$ denotes the set of Riemannian metrics on $\mathcal{M}$, the set of Riemannian structures on $\mathcal{M}$ is the quotient $Riem(\mathcal{M})/Diff(\mathcal{M})$ of $\mathcal{M}$ by the group of diffeomorphisms $Diff(\mathcal{M})$ of $\mathcal{M}$.

The various notions of *curvature* measure how the exponential maps differ from being isometries (at least locally). Let P be a 2-plane in $T_x\mathcal{M}$. Given a small enough r, consider the image under the exponential map $\exp_x$ of a circle of radius r and centre 0 in the plane P. This is a closed curve in $\mathcal{M}$ with length $L(r)$. When $r \to 0$ we have Puiseux' formula:

$$L(r) = 2\pi r(1 - \frac{1}{6}\sigma(x, P)r^2 + \mathcal{O}(r^3)). \tag{A.2}$$

The number $\sigma(x, P)$ is called the *sectional curvature* of the 2-plane P at x (see [111] for a lucid exposition).

An oriented Riemannian manifold is also equipped with a natural *Riemannian measure* v_g, whose expression in a local coordinate system $\{x_i\}$ is $\det(g_{ij})^{\frac{1}{2}}dx$, where dx is the

Lebesgue measure and where $g_{ij} = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$. The volume of $(\mathcal{M}, g)$ is denoted by $V(g) = \int_{\mathcal{M}} dv_g$.

We can write the pull-back $\exp_x^* v_g$ of the Riemannian measure v_g by the exponential map in polar coordinates in $T_x\mathcal{M}$ by $\exp_x^* v_g = \Theta_x(t, u) dt du$, where $t \geq 0$, dt is the Lebesgue measure on $\mathbb{R}_+$, u is a unit vector and du is the canonical measure on the unit sphere.

When $t \to 0$, we have

$$\Theta_x(t, u) = t^{n-1}(1 - \frac{1}{6}\rho_x(u)t^2 + \mathcal{O}(t^3)). \tag{A.3}$$

The number $\rho_x(u)$ is a quadratic form on $T_x\mathcal{M}$ which defines a symmetric bilinear form called the *Ricci curvature*, $Ric(g)$ of $\mathcal{M}$ at the point x.

If $\{u, e_2, \ldots, e_n\}$ is an orthonormal basis in $T_x\mathcal{M}$ and if P_i is the 2-plane spanned by u and e_i, we have the formula

$$Ric(g)(u, u) = \sum_{i=2}^{n} \sigma(x, P_i), \tag{A.4}$$

so the Ricci quadratic form is essentially a sum of sectional curvatures.

A Riemannian metric g on a compact 3-manifold $\mathcal{M}$ is defined to be *locally homogeneous*, if and only if for every pair (x, y) of points of $\mathcal{M}$, there exist neighbourhoods U_x of x and V_y of y, such that there is an isometry $\psi : U_x \to V_y$ with $\psi(x) = y$.

Generally, these local isometries do not extend to isometries of the whole space $(\mathcal{M}, g)$. If the local isometries do extend, then the geometry is defined to be homogeneous, i.e., $(\mathcal{M}, g)$ is *homogeneous* if for every pair of points x, y in $\mathcal{M}$, there exists an isometry $\phi : \mathcal{M} \to \mathcal{M}$ with $\phi(x) = y$. In this case the group of isometries of $\mathcal{M}$ acts transitively. For every locally homogeneous geometry the universal cover is homogeneous. We say then that the locally homogeneous geometry is modeled by the homogeneous geometry.

Appendix B
The Ricci-Hamilton Flow

The dimensionality $n = 3$, unless explicitly stated otherwise.

Let $Riem(\mathcal{M})$ denote the space (infinite dimensional) of smooth Riemannian metrics on $\mathcal{M}$ (this set has a natural structure of Fréchet manifold); and $S^2\mathcal{M}$ the set of smooth bilinear forms on $\mathcal{M}$.

The diffeomorphisms act by pull-back, i.e., $Diff(\mathcal{M}) : S^2\mathcal{M} \to S^2\mathcal{M}$.

The Riemannian structure underlying $(\mathcal{M}, g)$ is described by the orbit O_g of metric g, in $Riem(\mathcal{M})$ under the action of $Diff(\mathcal{M})$, it is defined as

$$O_g \equiv \{g' \in Riem(\mathcal{M}) \,|\, g' = \varphi^* g \text{ for some } \phi \in Diff(\mathcal{M})\}. \tag{B.5}$$

The tangent space to $Riem(\mathcal{M})$ at a given g, i.e., $T_g Riem(\mathcal{M})$ is interpreted as the set of infinitesimal deformations of the given g, and is isomorphic to $S^2\mathcal{M}$. In particular, it contains as the subspace a tangent space to the orbit O_g, which is an image in $S^2\mathcal{M}$ of the linear differential operator $L : \Xi \to S^2\mathcal{M}$, $X \to L_X g$, where L_X is Lie differentiation along the vector field X, and Ξ the space of vector fields on $\mathcal{M}$ (Ξ has an interpretation of the Lie algebra of $Diff(\mathcal{M})$).

One can show (using the decomposition theorems [25], see also [54] for a clear account), that we have the L^2−orthogonal splitting

$$T_g Riem(\mathcal{M}) \simeq Im\, L \oplus Ker\, L^*, \tag{B.6}$$

where $\oplus$ stands for the orthogonal sum with respect to the global scalar product on $\mathcal{M}$, defined by

$$(h, h')_g \equiv \int_{\mathcal{M}} h_{ik} h'_{lm} g^{il} g^{km} dv_g, \tag{B.7}$$

for each h, $h' \in S^2\mathcal{M}$.

Formula (B.6) can be rewritten as

$$T_g Riem(\mathcal{M}) \simeq Im\, div^* \oplus Ker\, div, \tag{B.8}$$

where, $(div\, S)_i \equiv -g^{jk}\nabla_k S_{ij}$, $(div^* x)_{ij} \equiv \frac{1}{2}(\nabla_i x_j + \nabla_j x_i)$, because the formal L^2-adjoint of L with respect to $(\,,\,)_g$, $L^* : S^2\mathcal{M} \to \Xi$ is (minus) twice the divergence operator on $S^2\mathcal{M}$.

The geometrical interpretation of (B.8) tells us that any infinitesimal deformation $h \in S^2\mathcal{M}$, can be decomposed into a longitudinal deformation $h_{lon} \in Im\, div^*$, mapping g into, say, g' within the same orbit, i.e., $g \to g' \in O_g$, and a transversal one $h_{tran} \in Ker\, div$, $g \to g'' \notin O_g$, which takes g to the other orbit and provides thus an infinitesimally deformed new Riemannian structure on $\mathcal{M}$.

On $Riem(\mathcal{M})$ there is a naturally defined $Ric(g)$-generated field of non-trivial infinitesimal deformations provided by associating with the given metric g, the tensor field $[Ric(g) - kgR(g)]$, where k is any real number. This follows by noticing that for any k, $[Ric(g) - kgR(g)]$ is never tangent, at g, to the orbit O_g, unless it vanishes [5]. In other words, the deformation $Ric(g) - kgR(g)$ mapping $g \to g'$, such that infinitesimaly close Riemannian metric $g' = g - \xi[Ric(g) - kgR(g)] + \mathcal{O}(\xi^2)$, defines a new Riemannian structure on $\mathcal{M}$, since such $g' \notin O_g$ (for a proof of this fact see [42]).

Now one can investigate the question of existence and behaviour of the integral curves (if any) of this vector field[6].

The answer was given by R. Hamilton [132], who showed that the flow associated with the $Riem(\mathcal{M})$ vector field $g \to -2Ric(g)$ is the local flow of metrics in $Riem(\mathcal{M})$ and

[5] Ricci tensor $Ric(g) : Riem(\mathcal{M}) \to S^2\mathcal{M}$; $R(g)$ stands for the scalar curvature.

[6] It is not evident *a priori* that this $Ric(g)$-generated deformations patch together to define a local (or possibly global) flow of metrics in $Riem(\mathcal{M})$ due to the Fréchet structure of $Riem(\mathcal{M})$.

moreover it is global, on condition the Ricci tensor associated with the metric is a positive bilinear form.

By deforming or smoothing flow of metrics, we mean a curve $g_{ab}(\beta)$, such that $g_{ab}(0)$ is the original given metric, and $g_{ab}(\beta)$ becomes smooth for $\beta \to \infty$. To see how it comes about, consider the general infinitesimal deformation of a metric

$$g_{ab} \to g_{ab} + \Delta\beta h_{ab}, \tag{B.9}$$

where, h_{ab} is any symmetric rank two tensor. Since, in appropriate coordinates, the leading term in the Ricci curvature $Ric(g)$ is $\nabla^c\nabla_c g_{ab}$, a natural choice for h_{ab} is

$$h_{ab} = -2Ric(g). \tag{B.10}$$

Writing this in the form of a differential equation, and adding a term responsible for preserving the volume of $(\mathcal{M}, g_{ab})$ along the flow, results in the Ricci-Hamilton flow equation (B.11).

Theorem 1 *Let $(\mathcal{M}, g)$ be a closed (compact and without boundary) Riemannian 3-manifold, such that its Ricci tensor, $Ric(g)$, is a positive definite bilinear form (i.e., $[Ric(g)]_{ab}v^a v^b > 0\ \forall v \neq 0$ vector field), then the given metric g can be uniformly deformed into a constant curvature metric $\overline{g}$.*

In this case, the universal simply connected cover of $\mathcal{M}$ is the 3-sphere S^3 and the pull back of $\overline{g}$ to S^3 via the covering map $S^3 \to \mathcal{M}$ is the standard metric[7]*.*

The one-parameter flow of metrics on $\mathcal{M}$ $(g, \beta) \to g(\beta)$ (with $\beta \geq 0$ the deformation parameter) realizing the above deformation is the unique solution to the weakly parabolic initial value problem

$$\frac{\partial}{\partial\beta} g_{ab}(\beta) = \frac{2}{3} < R(\beta) >_\beta g_{ab}(\beta) - 2R_{ab}(\beta), \tag{B.11}$$

with the initial data $g_{ab}(\beta = 0) = g_{ab}$ $(a, b = 1, 2, 3)$, where $R_{ab}(\beta)$ are the components of the Ricci tensor $Ric(g(\beta))$, and $< R(\beta) >_\beta$ denotes the average scalar curvature

$$< R(\beta) >_\beta = \frac{1}{Vol(\mathcal{M}, g(\beta))} \int_{\mathcal{M}} R(\beta) dv_\beta \tag{B.12}$$

[7]The theorem in fact, forces $\mathcal{M}$ to be topologically S^3/Γ, i.e., S^3 possibly quotiented by a discrete group.

The Ricci-Hamilton flow equation is a heat-like equation (weakly parabolic) and results in a smoothing deformation of the initial data $g(x, \beta = 0) = g(x)$.

Before examining some of the properties of the Ricci-Hamilton flow we would like to recall the strategy underlying the proof of Hamilton's theorem.

In fact, one can equivalently deal with a simpler initial value problem than (B.11) for what concerns most of the analysis involved in proving theorem 1. Equation (B.11) is not strictly parabolic since the Ricci tensor (thought of as a second order differential operator) is not elliptic. This is a consequence of its $Diff(\mathcal{M})$ equivariance, i.e. $Ric(\varphi^* g) = \varphi^* Ric(g)$, for any smooth diffeomorphism $\varphi : \mathcal{M} \to \mathcal{M}$. There is also associated with (B.11) a natural integrability condition in the form of contracted Bianchi identities, $div[Ric(g(\beta)) - \frac{1}{2} g(\beta) R(g(\beta))] = 0$, which has to hold true for any β for which the flow $g(\beta)$ exists. Up to this integrability condition (B.11) is parabolic and its local solvability can be handled by means of Nash-Moser implicit function theorem. According to the results of De Turck [74] we can associate with (B.11) a manifestly parabolic initial value problem at the expense of a clever use of the mentioned above $Diff(\mathcal{M})$ equivariance, namely

$$\frac{\partial}{\partial \beta} g_{ab}(\beta) = -2R_{ab}(\beta),$$

with the initial condition $g_{ab}(\beta = 0) = g_{ab}$. This simplifies the proof of local solvability for (B.11). Let us remark that the local part of Hamilton's theorem does not rely either on the positivity of the Ricci tensor or the three-dimensionality of $\mathcal{M}$. But apparently both these requirements are needed in order to globalize the local flow $(g, \beta) \to g(\beta)$. To see this, notice that the positivity of the Ricci tensor if assumed initially is preserved along the flow. This fact allows to get *a priori* estimates showing that the solution of (B.11) exists for all $0 \leq \beta < +\infty$. Moreover, the sign restriction on $Ric(g)$ yields a proof that as $\beta \to \infty$, the three eigenvalues of the Ricci tensor, at the generic point $x \in \mathcal{M}$, converge to a common value. It is then proved that this common value is a constant (positive).

As $\beta \to \infty$, $g(\beta)$ approaches the constant curvature metric $\bar{g}$ on S^3 uniformly, in fact the convergence is even exponential [132].

Now we will discuss some of the properties of the Ricci-Hamilton flow.

Firstly, the flow $(g, \beta) \to g(\beta)$ preserves the total volume of $(\mathcal{M}, g)$: $Vol(\mathcal{M}, g(\beta)) =$

$Vol(\mathcal{M}, g)$ for $0 \leq \beta < \infty$.

To prove this, we use the fact that along the trajectories of the flow $g(\beta)$ solution to (B.11) we have

$$\frac{\partial}{\partial \beta} dv_\beta = \frac{1}{2}\left(g^{ab}\frac{\partial}{\partial \beta} g_{ab}\right) dv_\beta = [< R(\beta) >_\beta - R(\beta)] dv_\beta,$$

so that

$$\frac{\partial}{\partial \beta} Vol(\mathcal{M}, g(\beta)) \equiv \frac{\partial}{\partial \beta}\int_{\mathcal{M}} dv_\beta = \int_{\mathcal{M}} [< R(\beta) >_\beta - R(\beta)] dv_\beta = 0.$$

The second group of properties follow upon examining the formal linearization of (B.11) around a given solution $g(\beta)$.

The linearized Ricci-Hamilton flow evolves a given infinitesimal deformation yielding a β-parameterised family of vectors $h(\beta) \in$ T Riem($\mathcal{M}$), connecting two neighbouring flows of metrics.

Upon the formal linearization of the initial value problem (B.11) around a given solution $g(\beta)$, we obtain (the β's in the brackets suppressed)

$$\begin{aligned}\frac{\partial}{\partial \beta} h_{ab} &= \frac{2}{3} < R > h_{ab} + \frac{2}{3} g_{ab}[\frac{1}{2} < R g^{ab} h_{ab} > - \frac{1}{2} < R >< g^{ab} h_{ab} > - \\ & < R^{ab} h_{ab} >] - \Delta_L h_{ab} + 2[div^*(div(h - \frac{1}{2}(Trh)\, g))]_{ab}, \qquad \text{(B.13)}\end{aligned}$$

with the initial data $h_{ab}(\beta = 0) = h_{ab}$, where, $h \in S^2\mathcal{M}$ is a given symmetric bilinear form, Δ_L is the Lichnerowicz-DeRham Laplacian on bilinear forms, and the operators Δ_L, div^*, div and Tr are considered with respect to the flow of metric $(g, \beta) \to g(\beta)$, solution of (B.11). The div (here, minus the usual divergence) is the divergence operator on $S^2\mathcal{M}$, div^* is the L^2 adjoint of div, acting from the space of vector fields on $\mathcal{M}$ to $S^2\mathcal{M}$ (it can be identified with $\frac{1}{2}$[Lie derivative] of the metric tensor along a vector field).

Note that a $h(\beta)$ solution of the initial value problem (B.13) always exists and is unique, and evolves a given infinitesimal deformation yielding a β-parameterised family of vectors $h(\beta)$ in T Riem($\mathcal{M}$) connecting two neighbouring flows of metrics $g(\beta)$ and $g'(\beta)$ (obtained as solutions of problem (B.11) with initial data $g(\beta = 0) = g$ and $g'(\beta = 0) = g(\beta = 0) + \epsilon h(\beta = 0) + \mathcal{O}(\epsilon^2)$, respectively).

The solution to (B.13) has a basic property expressing the $Diff(\mathcal{M})$ equivariance of the Ricci-Hamilton flow, namely, a trivial deformation $h_{ab} = L_X g_{ab}$ (where $X : \mathcal{M} \rightarrow T\mathcal{M}$ is a smooth vector field on $\mathcal{M}$) is always mapped by (B.13) into a trivial deformation, in other words, the solution to the linearized Ricci-Hamilton initial value problem is determined up to the infinitesimal diffeomorphism.

This result implies that if X is a Killing vector field for the given $(\mathcal{M}, g)$, then it remains so along the trajectories of the flow $(\beta, g) \rightarrow g(\beta)$. In other words, all the symmetries which the original metric g may be endowed with are preserved by the Ricci-Hamilton flow.

The natural problem to address, would be next that of generalizing Hamilton's theorem as much as possible.

At this point we refer the reader to the original literature [132] (see also [56] for a comprehensive review); for the understanding, why there is a positivity requirement on the Ricci tensor, and to what extend it can be weakened - from the point of view of solvability the initial-value problem (B.11), as well as of the topological obstructions to positive Ricci curvature - see [56] (see also [57]).

Let us stress that the positivity condition on the Ricci tensor apparently is not a necessary one for the Ricci-Hamilton flow to be global [149, 60]. As explicitly proved in [60], the 3-torus T^3 provides a non-trivial example of Ricci-Hamilton flow (with the Ricci tensor being non-positive), such that Hamilton's initial-value problem admits a global solution[8].

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[8]The T^3 cosmology, with a 3-space in form of a 3-torus is the simplest inhomogeneous empty universe [193].

Appendix C
Gromov space of bounded geometries

Consider two Riemannian manifolds, and let $i_1(\mathcal{M}_1), i_2(\mathcal{M}_2)$ stand for two isometric embeddings of $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively, in some metric space (A, d).

A *Hausdorff distance* in (A, d) between $i_1(\mathcal{M}_1)$, $i_2(\mathcal{M}_2)$ can be introduced as follows:

$$\begin{aligned} d_H^A[i_1(\mathcal{M}_1), i_2(\mathcal{M}_2)] &= \inf\{\epsilon > 0 |\ U_\epsilon(i_1(\mathcal{M}_1)) \supset i_2(\mathcal{M}_2), \\ &\quad U_\epsilon(i_2(\mathcal{M}_2)) \supset i_1(\mathcal{M}_1)\}, \end{aligned} \tag{C.14}$$

where, the ϵ-neighbourhood $U_\epsilon(i_i(\mathcal{M}_i))$ of $i_i(\mathcal{M}_i)$, $i = 1, 2$ is defined as

$$U_\epsilon(i_i(\mathcal{M}_i)) = \{z \in A |\ d(z, i_i(\mathcal{M}_i)) \leq \epsilon\}. \tag{C.15}$$

The Hausdorff distance thus defined is the lower bound of the ϵ, such that $i_1(\mathcal{M}_1)$ is contained in the ϵ-neighbourhood of $i_2(\mathcal{M}_2)$, and vice versa.

The *Gromov distance* $d_G(\mathcal{M}_1, \mathcal{M}_2)$ provides a natural generalization of the Hausdorff distance, and it is defined as the lower bound of the Hausdorff distances, as A varies in the set of metric spaces, and i_1, i_2 vary in the set of all isometric embeddings of $\mathcal{M}_1$ and $\mathcal{M}_2$ in (A, d).

The Gromov distance provides us with a sense of geometric nearness among Riemannian structures, which is related to a classification of Riemannian manifolds, according to how they can be covered by small geodesic balls. Coverings with the balls packed in similar configurations are possible for Riemannian manifolds that can be considered close to each other in the sense of Gromov distance.

In particular, the Gromov distance between two compact manifolds is always finite, and $d_G(\mathcal{M}_1, \mathcal{M}_2) = 0$, when the two manifolds are isometric.

Let us introduce the following class of Riemannian structures:

for $k \in \mathbb{R}$ and $D \in \mathbb{R}_+$, let $Ric[n, k, D]$ denote the space of isometry classes of closed, connected n-dimensional Riemannian manifolds $(\mathcal{M}, g)$ (without any pre-assumption on their topology) with Ricci curvature $Ric(g) \geq (n-1)kg$ and diameter $\leq D$.

Recall that if we define

$$k(x) \equiv \inf\{\inf Ric(u,u)|\, u \in T_x\mathcal{M},\, |u_x| = 1\}, \tag{C.16}$$

the lower bound of the Ricci tensor of $\mathcal{M}$ is defined as the lower bound of $k(x)$ as x varies in $\mathcal{M}$.

The best such $k = k(g)$ is just the lowest eigenvalue of the Ricci curvature $Ric(g)$. It is a fundamental numerical invariant of a compact Riemannian manifold.

For any manifold $\mathcal{M} \in Ric[n, k, D]$, it is possible to introduce the covering by geodesic balls, providing a coarse classification of Riemannian structures occurring in $Ric[n, k, D]$.

For any given $\epsilon > 0$, it is always possible to find an ordered set of points $\{p_1, \ldots, p_N\}$ in $\mathcal{M}$ from the above class, so that:

i) the balls $B_{\mathcal{M}}(p_i, \epsilon) = \{x \in \mathcal{M}|\, d(x, p_i) \leq \epsilon\}$, $i = 1, \ldots, N$ (where, $d(\,,\,)$ denotes the distance function of $\mathcal{M}$) cover $\mathcal{M}$, i.e., the collection $\{p_1, \ldots, p_N\}$ is an ϵ-*net* in $\mathcal{M}$.

ii) the open balls $B_{\mathcal{M}}(p_i, \epsilon/2)$, $i = 1, \ldots, N$ are disjoint, i.e., $\{p_1, \ldots, p_N\}$ is a *minimal* ϵ-net in $\mathcal{M}$.

A *filling function* $N_\epsilon^{(o)}(\mathcal{M})$ of the covering is defined as the function, which associates with $\mathcal{M}$ the maximum number of geodesic balls realizing a minimal ϵ-net in $\mathcal{M}$.

Any minimal net is characterized by its *intersection pattern*, defined as the set of indices pairs

$$I_\epsilon(\mathcal{M}) \equiv \{(i,j)|\, i,j = 1, \ldots, N|\, B(p_i, \epsilon) \cap B(p_j, \epsilon) \neq \emptyset\} \tag{C.17}$$

Any two manifolds $\mathcal{M}_1, \mathcal{M}_2 \in Ric[n,k,D]$ with minimal ϵ-nets $\{p_1, \ldots, p_N\}$, and $\{q_1, \ldots, q_N\}$, respectively, are said to be equivalent, if and only if $N = L$ and if they have the same intersection pattern, i.e., if the equivalence relations

$$N_\epsilon^{(o)}(\mathcal{M}_1) = N_\epsilon^{(o)}(\mathcal{M}_2) \tag{C.18}$$

$$I_\epsilon(\mathcal{M}_1) = I_\epsilon(\mathcal{M}_2), \tag{C.19}$$

are true (up to combinatorial isomorphism).

In fact, the above relations partition $Ric[n,k,D]$ into disjoint equivalence classes, whose finite number can be estimated in terms of the parameters n, k, D.

Two Riemannian manifolds in $Ric[n,k,D]$ get closer and closer to each other in d_G, if we can cover them with finer and finer minimal ϵ-nets of geodesic balls with the same intersection patterns.

In order to have $d_G(\mathcal{M}_1, \mathcal{M}_2) < \epsilon$, for any two compact Riemannian manifolds, it is sufficient to show that there exist an $\epsilon/2$ lattice in $\mathcal{M}_1$ and an $\epsilon/2$ lattice in $\mathcal{M}_2$, and two isometric embeddings $i_j : \mathcal{M}_j \to Z$ in some metric space (Z, d), such that the distance between the corresponding points of the embedded lattices is $< \epsilon/2$.

When discussing the convergence of a sequence $\{\mathcal{M}_i\}$ of Riemannian manifolds with respect to Gromov distance d_G, there is no need to refer to isometric embeddings in metric spaces. The sequence $\{\mathcal{M}_i\}$ admits a convergent subsequence, if and only if $\forall\epsilon > 0$, $\exists$ a number $N_\epsilon^{(o)}$ providing for each i, an upper bound to the maximum number of disjoint geodesic balls of radius ϵ, filling up each $\mathcal{M}_i$, i.e., $N_\epsilon(\mathcal{M}_i) \leq N_\epsilon^{(o)}$, for each i.

Stated differently, the convergence with respect to d_G is related to a uniform control of the "geometric size" of the manifolds $\mathcal{M}_i$, as indicated by the number of balls of a given radius that is needed to fill up each $\mathcal{M}_i$ in the considered sequence. For example, in the case of a sequence of compact surfaces of bounded curvature converging in d_G to S^2, each of them can be filled up by $N_\epsilon(\mathcal{M}_i)$ maximum number of disjoint geodesic balls of radius ϵ, such that $N_\epsilon(\mathcal{M}_i) \leq N_\epsilon^{(o)}(S^2)\ \forall i$.

A metric space E is said to be *precompact*, if $\forall\epsilon > 0$, $\exists$ a finite (open) covering B_j of E, such that the sets B_j have diameter $< \epsilon$. Equivalently, $\forall\epsilon > 0$, there exist a finite set

$F \subset E$, such that $d(x, F) < \epsilon, \forall x \in E$.

A stronger notion is that of compactness, yielded by the closure of the considered space.

Theorem 2 *The set $Ric[n, k, D]$ of isometry classes of compact manifolds, with the Ricci tensor satisfying $Ric(g) \geq (n-1)kg$ and diameter $\leq D$, ($k \in \mathbb{R}, D \in \mathbb{R}_+$), is precompact when endowed with the Gromov distance d_G.*

This theorem, due to M.Gromov, states that in the set of closed Riemannian manifolds (with Ricci curvature bounded below, diameter bounded above) there is a subset, let us call it $\tilde{Ric}$, containing for each ϵ, a finite number of Riemannian manifolds $\tilde{\mathcal{M}}_j$, such that for any $\mathcal{M} \in Ric[n, k, D]$, we have $d_G(\mathcal{M}, \tilde{\mathcal{M}}_i) < \epsilon$ for some $\tilde{\mathcal{M}}_i \in \tilde{Ric}$.

What this means is that for each "length scale ϵ", there exists a finite number of "model" geometries, which describes with an ϵ-approximation any given Riemannian geometry. Given a ball of a certain radius $> \epsilon$ in any Riemannian manifold $\mathcal{M}$ in $Ric[n, k, D]$, there exists a ball metrically similar (up to an ϵ scale) in one of the "model" geometries, which does not retain the details of the original manifold on scales smaller than ϵ. Roughly speaking, ϵ is a measure of the typical curvature inhomogeneity with respect to the model–background.

Let us stress that this is a highly non-trivial result, in the sense that the metrical properties of the manifolds in the infinite dimensional set $Ric[n, k, D]$[9] are up to an ϵ scale described by the metrical properties of just a finite number of "model" Riemannian manifolds.

However, since $Ric[n, k, D]$ is only precompact, and not compact we can have for instance a situation where a sequence of manifolds in $Ric[n, k, D]$ converges under d_G, to a manifold of lower dimension[10], or to a space with singularities[11].

Therefore below we will limit ourselves to the subset of $Ric[n, k, D]$, generated by those Riemannian manifolds with sectional curvatures bounded in absolute value.

[9] This set is of infinite dimension because a point $\mathcal{M}$ in its interior, remains on the set under small perturbations of the metric, so locally it is in principle as complicated as the set of all Riemannian metrics.

[10] The dimension of a manifold is not continuous for the topology defined by d_G.

[11] Phenomena of this kind are, for example the pinching of a geodesic in a torus.

Theorem 3 *The set $(\tilde{Ric}[n,k,D], d_G)$ of Riemannian structures having diameter $\leq D$, volume $\geq V$ and sectional curvatures bounded below in absolute value, is compact.*

In certain sense, one can think of the "model" manifolds $\tilde{\mathcal{M}}_i \in \tilde{Ric}$ as the "smoothed out" counterparts of the manifolds in $\tilde{Ric}[n,k,D]$.

A connection can be made between the above theorems (due to M.Gromov) and the Ricci-Hamilton flow [56].

Hamilton's flow associated with a 3−geometry satisfying the rather weak conditions of theorem 2, evolves in a set $Ric[n,k,D]$ which is precompact when endowed with d_G. As is known, precompactness of a set is not a condition strong enough for yielding a globalization of a local (smooth) flow which evolves in it. But if $Ric(g) > 0$ is required for a closed 3−manifold $(\mathcal{M}, g)$ then the associated Hamilton's initial value problem (B.11) defines a flow $(g,\beta) \to g(\beta)$, $0 \leq \beta < \infty$, in the compact set $\tilde{Ric}[n,k,D]$. The positivity requirement for $Ric(g)$, needed for the global version of Hamilton's theorem is a convenient way of controlling the growth of the diameter of the manifolds $(\mathcal{M}, g(\beta))$, which discriminates between the permanence of the Ricci-Hamilton flow in the compact set of smooth geometries versus the possibility of leaving this set and evolving toward a singular geometry.

More details can be found in [127], [117] and [58, 59].

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