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DERIVED EQUIVALENCES FOR THE FLOPS OF TYPE C_2 AND A_4^G VIA MUTATION OF SEMIORTHOGONAL DECOMPOSITION

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ABSTRACT. We give a new proof of the derived equivalence of a pair of varieties connected by the flop of type C_2 in the list of Kanemitsu [13], which is originally due to Segal [22]. We also prove the derived equivalence of a pair of varieties connected by the flop of type A_4^G in the same list. The latter proof follows that of the derived equivalence of Calabi–Yau 3-folds in Grassmannians $\text{Gr}(2, 5)$ and $\text{Gr}(3, 5)$ by Kapustka and Rampazzo [15] closely.

1. INTRODUCTION

Let G be a semisimple Lie group and B a Borel subgroup of G . For distinct maximal parabolic subgroups P and Q of G containing B , three homogeneous spaces G/P , G/Q , and $G/(P \cap Q)$ form the following diagram:

$$\begin{array}{ccc}
 & \mathbf{F} := G/(P \cap Q) & \\
 \swarrow \varpi_- & & \searrow \varpi_+ \\
 \mathbf{P} := G/P & & \mathbf{Q} := G/Q
 \end{array}$$

We write the hyperplane classes of \mathbf{P} and \mathbf{Q} as h and H respectively. By abuse of notation, the pull-back to \mathbf{F} of the hyperplane classes h and H will be denoted by the same symbol. The morphisms ϖ_- and ϖ_+ are projective morphisms whose relative $\mathcal{O}(1)$ are $\mathcal{O}(h)$ and $\mathcal{O}(H)$ respectively. We consider the diagram

$$(1.1) \quad
 \begin{array}{ccccc}
 & & \mathbf{F} & & \\
 & \swarrow \varpi_- & \downarrow \iota & \searrow \varpi_+ & \\
 \mathbf{P} & & \mathbf{V} & & \mathbf{Q} \\
 \downarrow \iota_- & \swarrow \varphi_- & & \searrow \varphi_+ & \downarrow \iota_+ \\
 \mathbf{V}_- & & & & \mathbf{V}_+ \\
 & \swarrow \phi_- & & \searrow \phi_+ & \\
 & & \mathbf{V}_0 & &
 \end{array}$$

where

- \mathbf{V}_- is the total space of $((\varpi_-)_* \mathcal{O}(h + H))^\vee$ over \mathbf{P} ,
- \mathbf{V}_+ is the total space of $((\varpi_+)_* \mathcal{O}(h + H))^\vee$ over \mathbf{Q} ,
- \mathbf{V} is the total space of $\mathcal{O}(-h - H)$ over \mathbf{F} ,
- ι_- , ι_+ , and ι are the zero-sections,
- φ_- and φ_+ are blow-ups of the zero-sections, and
- ϕ_- and ϕ_+ are the affinizations which contract the zero sections.

If \mathbf{V}_- and \mathbf{V}_+ have the trivial canonical bundles, then one expects from [4, Conjecture 4.4] or [16, Conjecture 1.2] that \mathbf{V}_- and \mathbf{V}_+ are derived-equivalent.

When G is the simple Lie group of type G_2 , Ueda [24] used sequence of mutations of semiorthogonal decompositions of $D^b(\mathbf{V})$ obtained by applying Orlov's theorem [20] to the diagram (1.1) to prove the derived equivalence of \mathbf{V}_- and \mathbf{V}_+ . This sequence of mutations in turn follows that of Kuznetsov [18] closely.

In this paper, by using the same method, we give a new proof to the following theorem, which is originally due to Segal [22], where the flop was attributed to Abuaf:

Theorem 1.1. *Varieties connected by the flop of type C_2 are derived-equivalent.*

The term *the flop of type C_2* was introduced in [13], where simple \mathbf{K} -equivalent maps in dimension at most 8 were classified. There are several ways to prove Theorem 1.1. In [22], Segal showed the derived equivalence by using tilting vector bundles. Hara [8] constructed alternative tilting vector bundles and studied the relation between functors defined by him and Segal.

The flop of type A_{2r-2}^G is also in the list of Kanemitsu [13]. It connects \mathbf{V}_- and \mathbf{V}_+ for $\mathbf{P} = \text{Gr}(r-1, 2r-1)$ and $\mathbf{Q} = \text{Gr}(r, 2r-1)$. Similarly, we prove the following theorem:

Theorem 1.2. *Varieties connected by the flop of type A_4^G are derived-equivalent.*

Although the proof of Theorem 1.2 is parallel to that of the derived equivalence of Calabi–Yau complete intersections in $\mathbf{P} = \text{Gr}(2, 5)$ and $\mathbf{Q} = \text{Gr}(3, 5)$ defined by global sections of the equivariant vector bundles dual to \mathbf{V}_- and \mathbf{V}_+ in [15, Theorem 5.7], we write down a full detail for clarity. As explained in [24], the derived equivalence obtained in [15] in turn follows from Theorem 1.2 using matrix factorizations.

We also give a similar proof of derived equivalences for a Mukai flop and a standard flop. For a Mukai flop, Kawamata [16] and Namikawa [19] independently showed the derived equivalence by using the pull-back and the push-forward along the fiber product $\mathbf{V}_- \times_{\mathbf{V}_0} \mathbf{V}_+$. Addington, Donovan, and Meachan [1] introduced a generalization of the functor of Kawamata and Namikawa parametrized by an integer, and discovered that certain compositions of these functors give the \mathbb{P} -twist in the sense of Huybrechts and Thomas [11]. They also considered the case of a standard flop, where the derived equivalence is originally proved by Bondal and Orlov [5]. Our proof is obtained by proceeding the mutation performed in [5] and [1] a little further in a straightforward way. Hara [7] also studied a Mukai flop in terms of non-commutative crepant resolutions.

For a standard flop, Segal [21] showed the derived equivalence by using the grade restriction rule for variation of geometric invariant theory quotients (VGIT) originally introduced by Hori, Herbst, and Page [10]. VGIT method was subsequently developed by Halpern-Leistner [6] and Ballard, Favero, and Katzarkov [2]. It is an interesting problem to develop this method further to prove the derived equivalence for the flop of type C_2 and A_4^G , and a Mukai flop.

Notations and conventions. *We work over an algebraically closed field \mathbf{k} of characteristic 0 throughout this paper. All pull-back and push-forward are derived unless otherwise specified. The complexes underlying $\text{Ext}^\bullet(-, -)$ and $\text{H}^\bullet(-)$ will be denoted by $\mathbf{hom}(-, -)$ and $\mathbf{h}(-)$ respectively.*

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2. FLOP OF TYPE C_2

Let P and Q be the parabolic subgroups of the simple Lie group G of type C_2 associated with the crossed Dynkin diagrams $\times \rightleftarrows$ and $\leftleftarrows \times$. The corresponding homogeneous spaces are the

projective space $\mathbf{P} = \mathbb{P}(V)$, the Lagrangian Grassmannian $\mathbf{Q} = \text{LGr}(V)$, and the isotropic flag variety $\mathbf{F} = \mathbb{P}_{\mathbf{P}}(\mathcal{L}_{\mathbf{P}}^{\perp}/\mathcal{L}_{\mathbf{P}}) = \mathbb{P}_{\mathbf{Q}}(\mathcal{S}_{\mathbf{Q}})$. Here V is a 4-dimensional symplectic vector space, $\mathcal{L}_{\mathbf{P}}^{\perp}$ is the rank 3 vector bundle given as the symplectic orthogonal to the tautological line bundle $\mathcal{L}_{\mathbf{P}} \cong \mathcal{O}_{\mathbf{P}}(-h)$ on \mathbf{P} , and $\mathcal{S}_{\mathbf{Q}}$ is the tautological rank 2 bundle on \mathbf{Q} . Note that \mathbf{Q} is also a quadric hypersurface in \mathbb{P}^4 . Tautological sequences on $\mathbf{Q} = \text{LGr}(V)$ and $\mathbf{F} \cong \mathbb{P}_{\mathbf{Q}}(\mathcal{S}_{\mathbf{Q}})$ give

$$(2.1) \quad 0 \rightarrow \mathcal{S}_{\mathbf{Q}} \rightarrow \mathcal{O}_{\mathbf{Q}} \otimes V \rightarrow \mathcal{S}_{\mathbf{Q}}^{\vee} \rightarrow 0$$

and

$$(2.2) \quad 0 \rightarrow \mathcal{O}_{\mathbf{F}}(-h + H) \rightarrow \mathcal{S}_{\mathbf{F}}^{\vee} \rightarrow \mathcal{O}_{\mathbf{F}}(h) \rightarrow 0,$$

where $\mathcal{S}_{\mathbf{F}} := \varpi_+^* \mathcal{S}_{\mathbf{Q}}$. We have

$$(\varpi_-)_*(\mathcal{O}_{\mathbf{F}}(H)) \cong ((\mathcal{L}_{\mathbf{P}}^{\perp}/\mathcal{L}_{\mathbf{P}}) \otimes \mathcal{L}_{\mathbf{P}})^{\vee}$$

and

$$(\varpi_+)_*(\mathcal{O}_{\mathbf{F}}(h)) \cong \mathcal{S}_{\mathbf{Q}}^{\vee},$$

whose determinants are given by $\mathcal{O}_{\mathbf{P}}(2h)$ and $\mathcal{O}_{\mathbf{Q}}(H)$ respectively. Since $\omega_{\mathbf{P}} \cong \mathcal{O}_{\mathbf{P}}(-4h)$, $\omega_{\mathbf{Q}} \cong \mathcal{O}_{\mathbf{Q}}(-3H)$, and $\omega_{\mathbf{F}} \cong \mathcal{O}_{\mathbf{F}}(-2h - 2H)$, we have $\omega_{\mathbf{V}_-} \cong \mathcal{O}_{\mathbf{V}_-}$, $\omega_{\mathbf{V}_+} \cong \mathcal{O}_{\mathbf{V}_+}$, and $\omega_{\mathbf{V}} \cong \mathcal{O}_{\mathbf{V}}(-h - H)$.

Recall from [3] that

$$(2.3) \quad D^b(\mathbf{P}) = \langle \mathcal{O}_{\mathbf{P}}(-2h), \mathcal{O}_{\mathbf{P}}(-h), \mathcal{O}_{\mathbf{P}}, \mathcal{O}_{\mathbf{P}}(h) \rangle,$$

and from [17] (cf. also [14]) that

$$D^b(\mathbf{Q}) = \langle \mathcal{O}_{\mathbf{Q}}(-H), \mathcal{S}_{\mathbf{Q}}^{\vee}(-H), \mathcal{O}_{\mathbf{Q}}, \mathcal{O}_{\mathbf{Q}}(H) \rangle.$$

Since φ_{\pm} are blow-ups along the zero-sections, it follows from [20] that

$$(2.4) \quad D^b(\mathbf{V}) = \langle \iota_* \varpi_-^* D^b(\mathbf{P}), \Phi_-(D^b(\mathbf{V}_-)) \rangle$$

and

$$(2.5) \quad D^b(\mathbf{V}) = \langle \iota_* \varpi_+^* D^b(\mathbf{Q}), \Phi_+(D^b(\mathbf{V}_+)) \rangle,$$

where

$$\Phi_- := ((-) \otimes \mathcal{O}_{\mathbf{V}}(H)) \circ \varphi_-^* : D^b(\mathbf{V}_-) \rightarrow D^b(\mathbf{V})$$

and

$$\Phi_+ := ((-) \otimes \mathcal{O}_{\mathbf{V}}(h)) \circ \varphi_+^* : D^b(\mathbf{V}_+) \rightarrow D^b(\mathbf{V}).$$

By abuse of notation, we use the same symbol for an object of $D^b(\mathbf{F})$ and its image in $D^b(\mathbf{V})$ by the push-forward ι_* . (2.3) and (2.4) give

$$D^b(\mathbf{V}) = \langle \mathcal{O}_{\mathbf{F}}(-2h), \mathcal{O}_{\mathbf{F}}(-h), \mathcal{O}_{\mathbf{F}}, \mathcal{O}_{\mathbf{F}}(h), \Phi_-(D^b(\mathbf{V}_-)) \rangle.$$

Since $\omega_{\mathbf{V}} \cong \mathcal{O}_{\mathbf{V}}(-h - H)$, by mutating the first term to the far right, and then $\Phi_-(D^b(\mathbf{V}_-))$ one step to the right, we obtain

$$D^b(\mathbf{V}) = \langle \mathcal{O}_{\mathbf{F}}(-h), \mathcal{O}_{\mathbf{F}}, \mathcal{O}_{\mathbf{F}}(h), \mathcal{O}_{\mathbf{F}}(-h + H), \Phi_1(D^b(\mathbf{V}_-)) \rangle,$$

where

$$\Phi_1 := R_{(\mathcal{O}_{\mathbf{F}}(-h+H))} \circ \Phi_-.$$

In the sequel, we will use the following fact.

Lemma 2.1. *Given two vector bundles $\mathcal{E}_{\mathbf{F}}, \mathcal{F}_{\mathbf{F}}$ on \mathbf{F} , if $\mathbf{h}(\mathcal{E}_{\mathbf{F}}^{\vee} \otimes \mathcal{F}_{\mathbf{F}}(-h - H)) \simeq 0$, then we have $\mathbf{hom}_{\mathcal{O}_{\mathbf{V}}}(\mathcal{E}_{\mathbf{F}}, \mathcal{F}_{\mathbf{F}}) \simeq \mathbf{h}(\mathcal{E}_{\mathbf{F}}^{\vee} \otimes \mathcal{F}_{\mathbf{F}})$.*

Proof. We have

$$\begin{aligned} \mathbf{hom}_{\mathcal{O}_V}(\mathcal{E}_F, \mathcal{F}_F) &\simeq \mathbf{hom}_{\mathcal{O}_V}(\{\mathcal{E}_V(h+H) \rightarrow \mathcal{E}_V\}, \mathcal{F}_F) \\ &\simeq \mathbf{h}(\{\mathcal{E}_F^\vee \otimes \mathcal{F}_F \rightarrow \mathcal{E}_F^\vee \otimes \mathcal{F}_F(-h-H)\}) \\ &\simeq \mathbf{h}(\mathcal{E}_F^\vee \otimes \mathcal{F}_F). \end{aligned}$$

□

Note that the canonical extension of $\mathcal{O}_F(h)$ by $\mathcal{O}_F(-h+H)$ associated with

$$\begin{aligned} \mathbf{hom}_{\mathcal{O}_V}(\mathcal{O}_F(h), \mathcal{O}_F(-h+H)) &\simeq \mathbf{h}(\mathcal{O}_F(-2h+H)) \\ &\simeq \mathbf{h}((\varpi_+)_* \mathcal{O}_F(-2h) \otimes \mathcal{O}_Q(H)) \\ &\simeq \mathbf{h}(\mathcal{O}_Q[-1]) \\ &\simeq \mathbf{k}[-1] \end{aligned}$$

is given by the short exact sequence (2.2). By mutating $\mathcal{O}_F(-h+H)$ one step to the left, $\mathcal{O}_F(-h)$ to the far right, and then $\Phi_1(D^b(\mathbf{V}_-))$ one step to the right, we obtain

$$D^b(\mathbf{V}) = \langle \mathcal{O}_F, \mathcal{S}_F^\vee, \mathcal{O}_F(h), \mathcal{O}_F(H), \Phi_2(D^b(\mathbf{V}_-)) \rangle,$$

where

$$\Phi_2 := R_{(\mathcal{O}_F(H))} \circ \Phi_1.$$

One can easily see that $\mathcal{O}_F(h)$ and $\mathcal{O}_F(H)$ are orthogonal, so that

$$(2.6) \quad D^b(\mathbf{V}) = \langle \mathcal{O}_F, \mathcal{S}_F^\vee, \mathcal{O}_F(H), \mathcal{O}_F(h), \Phi_2(D^b(\mathbf{V}_-)) \rangle.$$

By mutating $\Phi_2(D^b(\mathbf{V}_-))$ one step to the left, and then $\mathcal{O}_F(h)$ to the far left, we obtain

$$D^b(\mathbf{V}) = \langle \mathcal{O}_F(-H), \mathcal{O}_F, \mathcal{S}_F^\vee, \mathcal{O}_F(H), \Phi_3(D^b(\mathbf{V}_-)) \rangle,$$

where

$$\Phi_3 := L_{(\mathcal{O}_F(h))} \circ \Phi_2.$$

We have

$$\mathbf{hom}_{\mathcal{O}_V}(\mathcal{O}_F, \mathcal{S}_F^\vee) \simeq \mathbf{h}(\mathcal{S}_F^\vee) \simeq V^\vee,$$

and the dual of (2.1) shows that the kernel of the evaluation map $\mathcal{O}_F \otimes V^\vee \rightarrow \mathcal{S}_F^\vee$ is $\mathcal{S}_F \cong \mathcal{S}_F^\vee(-H)$. By mutating \mathcal{S}_F^\vee one step to the left, we obtain

$$(2.7) \quad D^b(\mathbf{V}) = \langle \mathcal{O}_F(-H), \mathcal{S}_F^\vee(-H), \mathcal{O}_F, \mathcal{O}_F(H), \Phi_3(D^b(\mathbf{V}_-)) \rangle.$$

By comparing (2.7) with (2.5), we obtain a derived equivalence

$$\Phi := \Phi_+^! \circ \Phi_3: D^b(\mathbf{V}_-) \xrightarrow{\sim} D^b(\mathbf{V}_+),$$

where

$$\Phi_+^!(-) := (\varphi_+)_* \circ ((-) \otimes \mathcal{O}_V(-h)): D^b(\mathbf{V}) \rightarrow D^b(\mathbf{V}_+)$$

is the left adjoint functor of Φ_+ .

3. FLOP OF TYPE A_4^G

Let P and Q be the parabolic subgroups of the simple Lie group G of type A_4 associated with the crossed Dynkin diagrams $\bullet \rightarrow \bullet \rightarrow \bullet$ and $\bullet \rightarrow \bullet \rightarrow \bullet$. The corresponding homogeneous spaces are the Grassmannians $\mathbf{P} = \text{Gr}(2, V)$, $\mathbf{Q} = \text{Gr}(3, V)$, and the partial flag variety $\mathbf{F} = \mathbb{P}_{\mathbf{P}}(\wedge^2 \mathcal{Q}_{\mathbf{P}}^{\vee}) = \mathbb{P}_{\mathbf{Q}}(\wedge^2 \mathcal{S}_{\mathbf{Q}})$. Here V is a 5-dimensional vector space, $\mathcal{Q}_{\mathbf{P}}^{\vee}$ is the dual of the universal quotient bundle on \mathbf{P} , and $\mathcal{S}_{\mathbf{Q}}$ is the tautological rank 3 bundle on \mathbf{Q} . We have

$$(\varpi_-)_*(\mathcal{O}_{\mathbf{F}}(H)) \cong \wedge^2 \mathcal{Q}_{\mathbf{P}}$$

and

$$(\varpi_+)_*(\mathcal{O}_{\mathbf{F}}(h)) \cong \wedge^2 \mathcal{S}_{\mathbf{Q}}^{\vee},$$

whose determinants are given by $\mathcal{O}_{\mathbf{P}}(2h)$ and $\mathcal{O}_{\mathbf{Q}}(2H)$ respectively. Since $\omega_{\mathbf{P}} \cong \mathcal{O}_{\mathbf{P}}(-5h)$, $\omega_{\mathbf{Q}} \cong \mathcal{O}_{\mathbf{Q}}(-5H)$, and $\omega_{\mathbf{F}} \cong \mathcal{O}_{\mathbf{F}}(-3h - 3H)$, we have $\omega_{\mathbf{V}_-} \cong \mathcal{O}_{\mathbf{V}_-}$, $\omega_{\mathbf{V}_+} \cong \mathcal{O}_{\mathbf{V}_+}$ and $\omega_{\mathbf{V}} \cong \mathcal{O}_{\mathbf{V}}(-2h - 2H)$.

First, we adapt several lemmas in [15] to our situation. To distinguish vector bundles which are obtained as a pull-back to \mathbf{F} from \mathbf{P} or \mathbf{Q} , we put tilde on the pull-back from \mathbf{Q} . By abuse of notation, we use the same symbol for an object of $D^b(\mathbf{F})$ and its image in $D^b(\mathbf{V})$ by the push-forward ι_* .

Lemma 3.1. $\text{hom}_{\mathcal{O}_{\mathbf{V}}}(\widetilde{\mathcal{Q}}_{\mathbf{F}}, \mathcal{O}_{\mathbf{F}}(h + aH)) \simeq 0$ for integers $-4 \leq a \leq -2$.

Proof. We have

$$\text{hom}_{\mathcal{O}_{\mathbf{V}}}(\widetilde{\mathcal{Q}}_{\mathbf{F}}, \mathcal{O}_{\mathbf{F}}(h + aH)) \simeq \mathbf{h}(\widetilde{\mathcal{Q}}_{\mathbf{F}}^{\vee}(h + aH)) \simeq 0,$$

where the first and the second isomorphisms follow from Lemma 2.1, Borel-Bott-Weil theorem and [15, Lemma 5.1] respectively. \square

Similarly, one can deduce Lemma 3.2 and Lemma 3.3 below from [15, Lemma 5.2, Lemma 5.3] by checking that $\mathcal{O}_{\mathbf{F}}((a - 1)H)$, $\mathcal{E}_{\mathbf{F}}^{\vee} \otimes \mathcal{E}'_{\mathbf{F}}((a - 1)h - 2H)$, and $\widetilde{\mathcal{F}}_{\mathbf{F}}^{\vee} \otimes \widetilde{\mathcal{F}}'_{\mathbf{F}}(-2h + (a - 1)H)$ are acyclic as an object of $D^b(\mathbf{F})$.

Lemma 3.2. $\text{hom}_{\mathcal{O}_{\mathbf{V}}}(\mathcal{O}_{\mathbf{F}}, \mathcal{O}_{\mathbf{F}}(h + aH)) \simeq 0$ for integers $-3 \leq a \leq -1$.

Lemma 3.3. Let $\mathcal{E}_{\mathbf{F}}, \mathcal{E}'_{\mathbf{F}}$ be the pull-back to \mathbf{F} of vector bundles $\mathcal{E}, \mathcal{E}'$ on \mathbf{P} , and let $\widetilde{\mathcal{F}}_{\mathbf{F}}, \widetilde{\mathcal{F}}'_{\mathbf{F}}$ be the pull-back to \mathbf{F} of vector bundles $\mathcal{F}, \mathcal{F}'$ on \mathbf{Q} . Then we have $\text{hom}_{\mathcal{O}_{\mathbf{V}}}(\mathcal{E}_{\mathbf{F}}, \mathcal{E}'_{\mathbf{F}}(ah - H)) \simeq 0$ and $\text{hom}_{\mathcal{O}_{\mathbf{V}}}(\widetilde{\mathcal{F}}_{\mathbf{F}}, \widetilde{\mathcal{F}}'_{\mathbf{F}}(-h + aH)) \simeq 0$ for all integers a .

The parallel result to the following lemma was tacitly used in [15].

Lemma 3.4. As an object of $D^b(\mathbf{V})$, $\mathcal{O}_{\mathbf{F}}, \widetilde{\mathcal{Q}}_{\mathbf{F}}, \mathcal{S}_{\mathbf{F}}$, and $\mathcal{S}_{\mathbf{F}}^{\vee}$ are left orthogonal to $\widetilde{\mathcal{S}}_{\mathbf{F}}^{\vee}(h - 2H)$, $\widetilde{\mathcal{S}}_{\mathbf{F}}^{\vee}(h - 2H)$, $\mathcal{O}_{\mathbf{F}}(2h - 2H)$, and $\mathcal{Q}_{\mathbf{F}}$ respectively.

Lemma 3.5 below and the tautological sequence show that $R_{\mathcal{O}_{\mathbf{F}}}\widetilde{\mathcal{Q}}_{\mathbf{F}}^{\vee} \simeq \widetilde{\mathcal{S}}_{\mathbf{F}}^{\vee}$ and $R_{\mathcal{O}_{\mathbf{F}}}\mathcal{S}_{\mathbf{F}} \simeq \mathcal{Q}_{\mathbf{F}}$ in $D^b(\mathbf{V})$.

Lemma 3.5. $\text{hom}_{\mathcal{O}_{\mathbf{V}}}(\widetilde{\mathcal{Q}}_{\mathbf{F}}^{\vee}, \mathcal{O}_{\mathbf{F}}) \simeq V$ and $\text{hom}_{\mathcal{O}_{\mathbf{V}}}(\mathcal{S}_{\mathbf{F}}, \mathcal{O}_{\mathbf{F}}) \simeq V$.

Again, both Lemma 3.4 and Lemma 3.5 follow from Lemma 2.1 and Borel-Bott-Weil theorem. Lemma 3.6 below and the exact sequences

$$0 \rightarrow \mathcal{O}_{\mathbf{F}}(h - H) \rightarrow \mathcal{Q}_{\mathbf{F}} \rightarrow \widetilde{\mathcal{Q}}_{\mathbf{F}} \rightarrow 0$$

and

$$0 \rightarrow \mathcal{S}_{\mathbf{F}} \rightarrow \widetilde{\mathcal{S}}_{\mathbf{F}} \rightarrow \mathcal{O}_{\mathbf{F}}(h - H) \rightarrow 0$$

obtained in [15] show that $R_{\mathcal{O}_{\mathbf{F}}(h-H)}\widetilde{\mathcal{Q}}_{\mathbf{F}} \simeq \mathcal{Q}_{\mathbf{F}}[1]$ and $L_{\mathcal{O}_{\mathbf{F}}(-h+H)}\widetilde{\mathcal{S}}_{\mathbf{F}}^{\vee} \simeq \mathcal{S}_{\mathbf{F}}^{\vee}$ in $D^b(\mathbf{V})$.

Lemma 3.6. $\mathbf{hom}_{\mathcal{O}_V}(\tilde{\mathcal{Q}}_F, \mathcal{O}_F(h-H)) \simeq \mathbf{k}[-1]$ and $\mathbf{hom}_{\mathcal{O}_V}(\mathcal{O}_F(-h+H), \tilde{\mathcal{F}}_F^\vee) \simeq \mathbf{k}$.

Proof. We have

$$\mathbf{hom}_{\mathcal{O}_V}(\tilde{\mathcal{Q}}_F, \mathcal{O}_F(h-H)) \simeq \mathbf{h}(\tilde{\mathcal{Q}}_F^\vee(h-H)) \simeq \mathbf{k}[-1],$$

where the isomorphisms follow from Lemma 2.1 and Borel-Bott-Weil theorem. Similarly, we have

$$\mathbf{hom}_{\mathcal{O}_V}(\mathcal{O}_F(-h+H), \tilde{\mathcal{F}}_F^\vee) \simeq \mathbf{h}(\tilde{\mathcal{F}}_F^\vee(h-H)) \simeq \mathbf{k}.$$

□

Recall from [17] (cf. also [14])

$$D^b(\mathbf{P}) = \langle \mathcal{S}_{\mathbf{P}}(-2h), \mathcal{O}_{\mathbf{P}}(-2h), \mathcal{S}_{\mathbf{P}}(-h), \mathcal{O}_{\mathbf{P}}(-h), \dots, \mathcal{S}_{\mathbf{P}}(2h), \mathcal{O}_{\mathbf{P}}(2h) \rangle,$$

and

$$(3.1) \quad D^b(\mathbf{Q}) = \langle \mathcal{O}_{\mathbf{Q}}, \mathcal{Q}_{\mathbf{Q}}, \mathcal{O}_{\mathbf{Q}}(H), \mathcal{Q}_{\mathbf{Q}}(H), \dots, \mathcal{O}_{\mathbf{Q}}(4H), \mathcal{Q}_{\mathbf{Q}}(4H) \rangle.$$

Since φ_{\pm} are blow-ups along the zero-sections, it follows from [20] that

$$(3.2) \quad D^b(\mathbf{V}) = \langle \iota_* \varpi_-^* D^b(\mathbf{P}), \iota_* \varpi_-^* D^b(\mathbf{P})(h+H), \Phi_-(D^b(\mathbf{V}_-)) \rangle$$

and

$$(3.3) \quad D^b(\mathbf{V}) = \langle \iota_* \varpi_+^* D^b(\mathbf{Q}), \iota_* \varpi_+^* D^b(\mathbf{Q})(h+H), \Phi_+(D^b(\mathbf{V}_+)) \rangle,$$

where

$$\Phi_- := ((-) \otimes \mathcal{O}_V(2H)) \circ \varphi_-^* : D^b(\mathbf{V}_-) \rightarrow D^b(\mathbf{V})$$

and

$$\Phi_+ := ((-) \otimes \mathcal{O}_V(2h)) \circ \varphi_+^* : D^b(\mathbf{V}_+) \rightarrow D^b(\mathbf{V}).$$

We write $\mathcal{O}_{i,j} := \mathcal{O}_F(ih + jH)$. (3.1) and (3.3) give a semiorthogonal decomposition of the form

$$D^b(\mathbf{V}) = \langle \mathcal{O}_{0,0}, \tilde{\mathcal{Q}}_{0,0}, \mathcal{O}_{0,1}, \tilde{\mathcal{Q}}_{0,1}, \mathcal{O}_{0,2}, \tilde{\mathcal{Q}}_{0,2}, \mathcal{O}_{0,3}, \tilde{\mathcal{Q}}_{0,3}, \mathcal{O}_{0,4}, \tilde{\mathcal{Q}}_{0,4}, \\ \mathcal{O}_{1,1}, \tilde{\mathcal{Q}}_{1,1}, \mathcal{O}_{1,2}, \tilde{\mathcal{Q}}_{1,2}, \mathcal{O}_{1,3}, \tilde{\mathcal{Q}}_{1,3}, \mathcal{O}_{1,4}, \tilde{\mathcal{Q}}_{1,4}, \mathcal{O}_{1,5}, \tilde{\mathcal{Q}}_{1,5}, \Phi_+(D^b(\mathbf{V}_+)) \rangle.$$

Since $\omega_V \cong \mathcal{O}_V(-2h-2H)$, by mutating the first five terms to the far right, and then $\Phi_+(D^b(\mathbf{V}_+))$ five steps to the right, we obtain

$$D^b(\mathbf{V}) = \langle \tilde{\mathcal{Q}}_{0,2}, \mathcal{O}_{0,3}, \tilde{\mathcal{Q}}_{0,3}, \mathcal{O}_{0,4}, \tilde{\mathcal{Q}}_{0,4}, \mathcal{O}_{1,1}, \tilde{\mathcal{Q}}_{1,1}, \mathcal{O}_{1,2}, \tilde{\mathcal{Q}}_{1,2}, \mathcal{O}_{1,3}, \\ \tilde{\mathcal{Q}}_{1,3}, \mathcal{O}_{1,4}, \tilde{\mathcal{Q}}_{1,4}, \mathcal{O}_{1,5}, \tilde{\mathcal{Q}}_{1,5}, \mathcal{O}_{2,2}, \tilde{\mathcal{Q}}_{2,2}, \mathcal{O}_{2,3}, \tilde{\mathcal{Q}}_{2,3}, \mathcal{O}_{2,4}, \Phi_1(D^b(\mathbf{V}_+)) \rangle,$$

where

$$\Phi_1 := R_{\langle \mathcal{O}_{2,2}, \tilde{\mathcal{Q}}_{2,2}, \mathcal{O}_{2,3}, \tilde{\mathcal{Q}}_{2,3}, \mathcal{O}_{2,4} \rangle} \circ \Phi_+.$$

One can easily see that $\mathcal{O}_{1,1}$ is orthogonal to $\mathcal{O}_{0,3}$, $\tilde{\mathcal{Q}}_{0,3}$, $\mathcal{O}_{0,4}$, and $\tilde{\mathcal{Q}}_{0,4}$ by Lemma 3.1 and Lemma 3.2, so that

$$D^b(\mathbf{V}) = \langle \tilde{\mathcal{Q}}_{0,2}, \mathcal{O}_{1,1}, \mathcal{O}_{0,3}, \tilde{\mathcal{Q}}_{0,3}, \mathcal{O}_{0,4}, \tilde{\mathcal{Q}}_{0,4}, \tilde{\mathcal{Q}}_{1,1}, \mathcal{O}_{1,2}, \tilde{\mathcal{Q}}_{1,2}, \mathcal{O}_{1,3}, \\ \tilde{\mathcal{Q}}_{1,3}, \mathcal{O}_{2,2}, \mathcal{O}_{1,4}, \tilde{\mathcal{Q}}_{1,4}, \mathcal{O}_{1,5}, \tilde{\mathcal{Q}}_{1,5}, \tilde{\mathcal{Q}}_{2,2}, \mathcal{O}_{2,3}, \tilde{\mathcal{Q}}_{2,3}, \mathcal{O}_{2,4}, \Phi_1(D^b(\mathbf{V}_+)) \rangle.$$

By mutating $\tilde{\mathcal{Q}}_{0,2}$, $\tilde{\mathcal{Q}}_{1,3}$, $\tilde{\mathcal{Q}}_{1,1}$, and $\tilde{\mathcal{Q}}_{2,2}$ one step to the right, we obtain by $\tilde{\mathcal{Q}}_{1,1} \cong \tilde{\mathcal{Q}}_{1,2}^\vee$, Lemma 3.5, and Lemma 3.6

$$D^b(\mathbf{V}) = \langle \mathcal{O}_{1,1}, \mathcal{Q}_{0,2}, \mathcal{O}_{0,3}, \tilde{\mathcal{Q}}_{0,3}, \mathcal{O}_{0,4}, \tilde{\mathcal{Q}}_{0,4}, \mathcal{O}_{1,2}, \tilde{\mathcal{F}}_{1,2}^\vee, \tilde{\mathcal{Q}}_{1,2}, \mathcal{O}_{1,3}, \\ \mathcal{O}_{2,2}, \mathcal{Q}_{1,3}, \mathcal{O}_{1,4}, \tilde{\mathcal{Q}}_{1,4}, \mathcal{O}_{1,5}, \tilde{\mathcal{Q}}_{1,5}, \mathcal{O}_{2,3}, \tilde{\mathcal{F}}_{2,3}^\vee, \tilde{\mathcal{Q}}_{2,3}, \mathcal{O}_{2,4}, \Phi_1(D^b(\mathbf{V}_+)) \rangle.$$

By mutating $O_{1,2}$ and $O_{2,3}$ four steps to the left, we obtain by Lemma 3.1, Lemma 3.2, and Lemma 3.6

$$D^b(\mathbf{V}) = \langle O_{1,1}, \mathcal{Q}_{0,2}, O_{1,2}, O_{0,3}, \mathcal{Q}_{0,3}, O_{0,4}, \widetilde{\mathcal{Q}}_{0,4}, \widetilde{\mathcal{F}}_{1,2}^V, \widetilde{\mathcal{Q}}_{1,2}, O_{1,3}, \\ O_{2,2}, \mathcal{Q}_{1,3}, O_{2,3}, O_{1,4}, \mathcal{Q}_{1,4}, O_{1,5}, \widetilde{\mathcal{Q}}_{1,5}, \widetilde{\mathcal{F}}_{2,3}^V, \widetilde{\mathcal{Q}}_{2,3}, O_{2,4}, \Phi_1(D^b(\mathbf{V}_+)) \rangle.$$

One can easily see that $\widetilde{\mathcal{F}}_{1,2}^V$ is orthogonal to $O_{0,4}$ and $\widetilde{\mathcal{Q}}_{0,4}$ by Lemma 3.4, so that

$$D^b(\mathbf{V}) = \langle O_{1,1}, \mathcal{Q}_{0,2}, O_{1,2}, O_{0,3}, \mathcal{Q}_{0,3}, \widetilde{\mathcal{F}}_{1,2}^V, O_{0,4}, \widetilde{\mathcal{Q}}_{0,4}, \widetilde{\mathcal{Q}}_{1,2}, O_{1,3}, \\ O_{2,2}, \mathcal{Q}_{1,3}, O_{2,3}, O_{1,4}, \mathcal{Q}_{1,4}, \widetilde{\mathcal{F}}_{2,3}^V, O_{1,5}, \widetilde{\mathcal{Q}}_{1,5}, \widetilde{\mathcal{Q}}_{2,3}, O_{2,4}, \Phi_1(D^b(\mathbf{V}_+)) \rangle.$$

By mutating $O_{0,3}$ and $O_{1,4}$ two steps to the right, $O_{1,3}$ and $O_{2,4}$ three steps to the left, and then $O_{0,4}$ and $O_{1,5}$ two steps to the right, we obtain by Lemma 3.5 and Lemma 3.6

$$D^b(\mathbf{V}) = \langle O_{1,1}, \mathcal{Q}_{0,2}, O_{1,2}, \mathcal{S}_{0,3}, \mathcal{S}_{1,2}^V, O_{0,3}, O_{1,3}, \mathcal{S}_{0,4}, \mathcal{S}_{1,3}^V, O_{0,4}, \\ O_{2,2}, \mathcal{Q}_{1,3}, O_{2,3}, \mathcal{S}_{1,4}, \mathcal{S}_{2,3}^V, O_{1,4}, O_{2,4}, \mathcal{S}_{1,5}, \mathcal{S}_{2,4}^V, O_{1,5}, \Phi_1(D^b(\mathbf{V}_+)) \rangle.$$

By mutating $O_{1,1}$ to the far right, and then $\Phi_1(D^b(\mathbf{V}_+))$ one step to the right, we obtain

$$D^b(\mathbf{V}) = \langle \mathcal{Q}_{0,2}, O_{1,2}, \mathcal{S}_{0,3}, \mathcal{S}_{1,2}^V, O_{0,3}, O_{1,3}, \mathcal{S}_{0,4}, \mathcal{S}_{1,3}^V, O_{0,4}, O_{2,2}, \\ \mathcal{Q}_{1,3}, O_{2,3}, \mathcal{S}_{1,4}, \mathcal{S}_{2,3}^V, O_{1,4}, O_{2,4}, \mathcal{S}_{1,5}, \mathcal{S}_{2,4}^V, O_{1,5}, O_{3,3}, \Phi_2(D^b(\mathbf{V}_+)) \rangle,$$

where

$$\Phi_2 := R_{(O_{3,3})} \circ \Phi_1.$$

By Lemma 3.2, Lemma 3.3, and Lemma 3.4, we obtain

$$D^b(\mathbf{V}) = \langle \mathcal{Q}_{0,2}, O_{1,2}, \mathcal{S}_{1,2}^V, O_{2,2}, \mathcal{S}_{0,3}, O_{0,3}, O_{1,3}, \mathcal{S}_{1,3}^V, \mathcal{Q}_{1,3}, O_{2,3}, \\ \mathcal{S}_{2,3}^V, O_{3,3}, \mathcal{S}_{0,4}, O_{0,4}, \mathcal{S}_{1,4}, O_{1,4}, O_{2,4}, \mathcal{S}_{2,4}^V, \mathcal{S}_{1,5}, O_{1,5}, \Phi_2(D^b(\mathbf{V}_+)) \rangle.$$

By mutating $\Phi_2(D^b(\mathbf{V}_+))$ ten steps to the left, and then last ten terms to the far left, we obtain

$$D^b(\mathbf{V}) = \langle \mathcal{S}_{0,1}^V, O_{1,1}, \mathcal{S}_{-2,2}, O_{-2,2}, \mathcal{S}_{-1,2}, O_{-1,2}, O_{0,2}, \mathcal{S}_{0,2}^V, \mathcal{S}_{-1,3}, O_{-1,3}, \\ \mathcal{Q}_{0,2}, O_{1,2}, \mathcal{S}_{1,2}^V, O_{2,2}, \mathcal{S}_{0,3}, O_{0,3}, O_{1,3}, \mathcal{S}_{1,3}^V, \mathcal{Q}_{1,3}, O_{2,3}, \Phi_3(D^b(\mathbf{V}_+)) \rangle,$$

where

$$\Phi_3 := L_{(\mathcal{S}_{2,3}^V, O_{3,3}, \mathcal{S}_{0,4}, O_{0,4}, \mathcal{S}_{1,4}, O_{1,4}, O_{2,4}, \mathcal{S}_{2,4}^V, \mathcal{S}_{1,5}, O_{1,5})} \circ \Phi_2.$$

By Lemma 3.3, we obtain

$$D^b(\mathbf{V}) = \langle \mathcal{S}_{0,1}^V, O_{1,1}, \mathcal{S}_{-2,2}, O_{-2,2}, \mathcal{S}_{-1,2}, O_{-1,2}, O_{0,2}, \mathcal{S}_{0,2}^V, \mathcal{Q}_{0,2}, O_{1,2}, \\ \mathcal{S}_{1,2}^V, O_{2,2}, \mathcal{S}_{-1,3}, O_{-1,3}, \mathcal{S}_{0,3}, O_{0,3}, O_{1,3}, \mathcal{S}_{1,3}^V, \mathcal{Q}_{1,3}, O_{2,3}, \Phi_3(D^b(\mathbf{V}_+)) \rangle.$$

By mutating $\mathcal{Q}_{0,2}$ and $\mathcal{Q}_{1,3}$ two steps to the left, the first two terms to the far right, and then $\Phi_3(D^b(\mathbf{V}_+))$ two steps to the right, we obtain by $\mathcal{S}_{0,0}^V \simeq \mathcal{S}_{1,0}$, Lemma 3.4, and Lemma 3.6

$$(3.4) \quad D^b(\mathbf{V}) = \langle \mathcal{S}_{-2,2}, O_{-2,2}, \mathcal{S}_{-1,2}, O_{-1,2}, \mathcal{S}_{0,2}, O_{0,2}, \mathcal{S}_{1,2}, O_{1,2}, \mathcal{S}_{2,2}, O_{2,2}, \\ \mathcal{S}_{-1,3}, O_{-1,3}, \mathcal{S}_{0,3}, O_{0,3}, \mathcal{S}_{1,3}, O_{1,3}, \mathcal{S}_{2,3}, O_{2,3}, \mathcal{S}_{3,3}, O_{3,3}, \Phi_4(D^b(\mathbf{V}_+)) \rangle,$$

where

$$\Phi_4 := R_{(\mathcal{S}_{2,3}^V, O_{3,3})} \circ \Phi_3.$$

By comparing (3.4) with (3.2), we obtain a derived equivalence

$$\Phi := \Phi_-^! \circ \Phi_4: D^b(\mathbf{V}_+) \xrightarrow{\sim} D^b(\mathbf{V}_-),$$

where

$$\Phi_-^!(-) := (\varphi_-)_* \circ ((-) \otimes \mathcal{O}_{\mathbf{V}}(-2H)) : D^b(\mathbf{V}) \rightarrow D^b(\mathbf{V}_-)$$

is the left adjoint functor of Φ_- .

4. MUKAI FLOP

For $n \geq 2$, let P and Q be the maximal parabolic subgroups of the simple Lie group of type A_n associated with the crossed Dynkin diagrams $\times \bullet \cdots \bullet$ and $\bullet \cdots \bullet \times$. The corresponding homogeneous spaces are the projective spaces $\mathbf{P} = \mathbb{P}V$, $\mathbf{Q} = \mathbb{P}V^\vee$, and the partial flag variety $\mathbf{F} = \mathbf{F}(1, n; V)$, where V is an $(n+1)$ -dimensional vector space. Since $\omega_{\mathbf{P}} \cong \mathcal{O}(-(n+1)h)$, $\omega_{\mathbf{Q}} \cong \mathcal{O}(-(n+1)H)$, and $\omega_{\mathbf{F}} \cong \mathcal{O}(-nh - nH)$, we have $\omega_{\mathbf{V}_-} \cong \mathcal{O}_{\mathbf{V}_-}$, $\omega_{\mathbf{V}_+} \cong \mathcal{O}_{\mathbf{V}_+}$, and $\omega_{\mathbf{V}} \cong \mathcal{O}(-(n-1)h - (n-1)H)$.

Lemma 4.1. $\mathcal{O}_{\mathbf{F}}(-ih + jH)$ and $\mathcal{O}_{\mathbf{F}}(-(i+1)h + (j-1)H)$ are acyclic for $1 \leq j \leq n-1$ and $1 \leq i \leq n-j$.

Proof. Since $j-n \leq -i \leq -1$ and $j-n-1 \leq -i-1 \leq -2$, the derived push-forward of $\mathcal{O}_{\mathbf{F}}(-ih + jH)$ and $\mathcal{O}_{\mathbf{F}}(-(i+1)h + (j-1)H)$ vanish by [9, Exercise III.8.4] unless $i = n-1$ and $j = 1$, in which case the acyclicity of $\mathcal{O}_{\mathbf{F}}(-nh)$ is obvious. \square

Lemma 4.2. $\mathrm{hom}_{\mathcal{O}_{\mathbf{V}}}(\mathcal{O}_{\mathbf{F}}(ih - jH), \mathcal{O}_{\mathbf{F}}) \simeq 0$ for $1 \leq j \leq n-1$ and $1 \leq i \leq n-j$.

Proof. We have

$$\mathrm{hom}_{\mathcal{O}_{\mathbf{V}}}(\mathcal{O}_{\mathbf{F}}(ih - jH), \mathcal{O}_{\mathbf{F}}) \simeq \mathbf{h}(\{\mathcal{O}_{\mathbf{F}}(-ih + jH) \rightarrow \mathcal{O}_{\mathbf{F}}(-(i+1)h + (j-1)H)\}),$$

which vanishes by Lemma 4.1. \square

Recall from [3] that

$$(4.1) \quad D^b(\mathbf{P}) = \langle \mathcal{O}_{\mathbf{P}}, \mathcal{O}_{\mathbf{P}}(h), \dots, \mathcal{O}_{\mathbf{P}}(nh) \rangle$$

and

$$(4.2) \quad D^b(\mathbf{Q}) = \langle \mathcal{O}_{\mathbf{Q}}, \mathcal{O}_{\mathbf{Q}}(H), \dots, \mathcal{O}_{\mathbf{Q}}(nH) \rangle.$$

Since φ_{\pm} are blow-ups along the zero-sections, it follows from [20] that

$$(4.3) \quad D^b(\mathbf{V}) = \langle \iota_* \varpi_-^* D^b(\mathbf{P}), \dots, \iota_* \varpi_-^* D^b(\mathbf{P}) \otimes \mathcal{O}_{\mathbf{V}}((n-2)H), \Phi_-(D^b(\mathbf{V}_-)) \rangle$$

and

$$(4.4) \quad D^b(\mathbf{V}) = \langle \iota_* \varpi_+^* D^b(\mathbf{Q}), \dots, \iota_* \varpi_+^* D^b(\mathbf{Q}) \otimes \mathcal{O}_{\mathbf{V}}((n-2)h), \Phi_+(D^b(\mathbf{V}_+)) \rangle,$$

where

$$\Phi_- := ((-) \otimes \mathcal{O}_{\mathbf{V}}((n-1)H)) \circ \varphi_-^* : D^b(\mathbf{V}_-) \rightarrow D^b(\mathbf{V})$$

and

$$\Phi_+ := ((-) \otimes \mathcal{O}_{\mathbf{V}}((n-1)h)) \circ \varphi_+^* : D^b(\mathbf{V}_+) \rightarrow D^b(\mathbf{V}).$$

We write $\mathcal{O}_{i,j} := \mathcal{O}_{\mathbf{F}}(ih + jH)$. (4.1) and (4.3) give a semiorthogonal decomposition of the form

$$D^b(\mathbf{V}) = \langle \mathcal{A}_0, \Phi_-(D^b(\mathbf{V}_-)) \rangle$$

where \mathcal{A}_0 is given by

$$(4.5) \quad \begin{array}{cccccccc} \mathcal{O}_{0,0} & \mathcal{O}_{1,0} & \cdots & \mathcal{O}_{n-2,0} & \mathcal{O}_{n-1,0} & \mathcal{O}_{n,0} & & \\ & \mathcal{O}_{1,1} & \cdots & \mathcal{O}_{n-2,1} & \mathcal{O}_{n-1,1} & \mathcal{O}_{n,1} & \mathcal{O}_{n+1,1} & \\ & & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots \\ & & & \mathcal{O}_{n-2,n-2} & \mathcal{O}_{n-1,n-2} & \mathcal{O}_{n,n-2} & \mathcal{O}_{n+1,n-2} & \cdots & \mathcal{O}_{2n-2,n-2}. \end{array}$$

Note from Lemma 4.2 that there are no morphisms from right to left in (4.5). Since $\omega_{\mathbf{V}} \cong \mathcal{O}_{-(n-1),-(n-1)}$, by mutating first

$$\begin{array}{cccc} \mathcal{O}_{0,0} & \mathcal{O}_{1,0} & \cdots & \mathcal{O}_{n-2,0} \\ & \mathcal{O}_{1,1} & \cdots & \mathcal{O}_{n-2,1} \\ & & \ddots & \vdots \\ & & & \mathcal{O}_{n-2,n-2} \end{array}$$

to the far right, and then $\Phi_-(D^b(\mathbf{V}_-))$ to the far right, we obtain

$$D^b(\mathbf{V}) = \langle \mathcal{A}_1, \Phi_1(D^b(\mathbf{V}_-)) \rangle,$$

where

$$\Phi_1(D^b(\mathbf{V}_-)) := R_{(\mathcal{O}_{n-1,n-1}, \dots, \mathcal{O}_{2n-3,2n-3})} \circ \Phi_-$$

and \mathcal{A}_1 is given by

$$\begin{array}{cccccc} \mathcal{O}_{n-1,0} & \mathcal{O}_{n,0} & & & & \\ \mathcal{O}_{n-1,1} & \mathcal{O}_{n,1} & \mathcal{O}_{n+1,1} & & & \\ \vdots & \vdots & \vdots & \ddots & & \\ \mathcal{O}_{n-1,n-2} & \mathcal{O}_{n,n-2} & \mathcal{O}_{n+1,n-2} & \cdots & \mathcal{O}_{2n-3,n-2} & \mathcal{O}_{2n-2,n-2} \\ \mathcal{O}_{n-1,n-1} & \mathcal{O}_{n,n-1} & \mathcal{O}_{n+1,n-1} & \cdots & \mathcal{O}_{2n-3,n-1} & \\ & \mathcal{O}_{n,n} & \mathcal{O}_{n+1,n} & \cdots & \mathcal{O}_{2n-3,n} & \\ & & \mathcal{O}_{n+1,n+1} & \cdots & \mathcal{O}_{2n-3,n+1} & \\ & & & \ddots & \vdots & \\ & & & & \mathcal{O}_{2n-3,2n-3} & \end{array}$$

By mutating $\Phi_1(D^b(\mathbf{V}_-))$ one step to the left, and then $\mathcal{O}_{2n-2,n-2}$ to the far left, we obtain

$$(4.6) \quad D^b(\mathbf{V}) = \langle \mathcal{A}_2, \Phi_2(D^b(\mathbf{V}_-)) \rangle,$$

where

$$\Phi_2(D^b(\mathbf{V}_-)) := L_{\mathcal{O}_{2n-2,n-2}} \circ \Phi_1$$

and \mathcal{A}_2 is given by

$$\begin{array}{cccccc} \mathcal{O}_{n-1,-1} & & & & & \\ \mathcal{O}_{n-1,0} & \mathcal{O}_{n,0} & & & & \\ \mathcal{O}_{n-1,1} & \mathcal{O}_{n,1} & \mathcal{O}_{n+1,1} & & & \\ \vdots & \vdots & \vdots & \ddots & & \\ \mathcal{O}_{n-1,n-2} & \mathcal{O}_{n,n-2} & \mathcal{O}_{n+1,n-2} & \cdots & \mathcal{O}_{2n-3,n-2} & \\ \mathcal{O}_{n-1,n-1} & \mathcal{O}_{n,n-1} & \mathcal{O}_{n+1,n-1} & \cdots & \mathcal{O}_{2n-3,n-1} & \\ & \mathcal{O}_{n,n} & \mathcal{O}_{n+1,n} & \cdots & \mathcal{O}_{2n-3,n} & \\ & & \mathcal{O}_{n+1,n+1} & \cdots & \mathcal{O}_{2n-3,n+1} & \\ & & & \ddots & \vdots & \\ & & & & \mathcal{O}_{2n-3,2n-3} & \end{array}$$

By comparing (4.6) with (4.2) and (4.4), we obtain a derived equivalence

$$\Phi := (\varphi_+)_* \circ ((-) \otimes \mathcal{O}_{-(2n-2),0}) \circ \Phi_2 : D^b(\mathbf{V}_-) \xrightarrow{\sim} D^b(\mathbf{V}_+).$$

5. STANDARD FLOP

For $n \geq 1$, let P and Q be the maximal parabolic subgroups of the semisimple Lie group $G = \mathrm{SL}(V) \times \mathrm{SL}(V^\vee)$ associated with the crossed Dynkin diagram $\times \cdots \oplus \cdots$ and $\cdots \oplus \cdots \times$. The corresponding homogeneous spaces are the projective spaces $\mathbf{P} = \mathbb{P}V$, $\mathbf{Q} = \mathbb{P}V^\vee$, and their product $\mathbf{F} = \mathbb{P}V \times \mathbb{P}V^\vee$. Since $\omega_{\mathbf{P}} \cong \mathcal{O}(-(n+1)h)$, $\omega_{\mathbf{Q}} \cong \mathcal{O}(-(n+1)H)$, and $\omega_{\mathbf{F}} \cong \mathcal{O}(-(n+1)h - (n+1)H)$, we have $\omega_{\mathbf{V}_-} \cong \mathcal{O}_{\mathbf{V}_-}$, $\omega_{\mathbf{V}_+} \cong \mathcal{O}_{\mathbf{V}_+}$, and $\omega_{\mathbf{V}} \cong \mathcal{O}(-nh - nH)$.

Lemma 5.1. $\mathrm{hom}_{\mathcal{O}_{\mathbf{V}}}(\mathcal{O}_{\mathbf{F}}(ih - jH), \mathcal{O}_{\mathbf{F}}) \simeq 0$ for $1 \leq j \leq n-1$ and $1 \leq i \leq n-j$.

Proof. We have

$$\mathbf{hom}_{\mathcal{O}_{\mathbf{V}}}(\mathcal{O}_{\mathbf{F}}(ih - jH), \mathcal{O}_{\mathbf{F}}) \simeq \mathbf{h}(\{\mathcal{O}_{\mathbf{F}}(-ih + jH) \rightarrow \mathcal{O}_{\mathbf{F}}(-(i+1)h + (j-1)H)\}),$$

which vanishes for $1 \leq i \leq n-j \leq n-1$. □

It follows from [20] that

$$(5.1) \quad D^b(\mathbf{V}) = \langle \iota_* \varpi_-^* D^b(\mathbf{P}), \dots, \iota_* \varpi_-^* D^b(\mathbf{P}) \otimes \mathcal{O}((n-1)(h+H)), \Phi_-(D^b(\mathbf{V}_-)) \rangle$$

and

$$(5.2) \quad D^b(\mathbf{V}) = \langle \iota_* \varpi_+^* D^b(\mathbf{Q}), \dots, \iota_* \varpi_+^* D^b(\mathbf{Q}) \otimes \mathcal{O}((n-1)(h+H)), \Phi_+(D^b(\mathbf{V}_+)) \rangle,$$

where

$$\Phi_- := (-) \otimes \mathcal{O}_{\mathbf{V}}(n(h+H)) \circ \varphi_-^* : D^b(\mathbf{V}_-) \rightarrow D^b(\mathbf{V})$$

and

$$\Phi_+ := (-) \otimes \mathcal{O}_{\mathbf{V}}(n(h+H)) \circ \varphi_+^* : D^b(\mathbf{V}_+) \rightarrow D^b(\mathbf{V}).$$

We write $\mathcal{O}_{i,j} := \mathcal{O}_{\mathbf{F}}(ih + jH)$. (4.1) and (5.1) give a semiorthogonal decomposition of the form

$$D^b(\mathbf{V}) = \langle \mathcal{A}_0, \Phi_-(D^b(\mathbf{V}_-)) \rangle$$

where \mathcal{A}_0 is given by

$$(5.3) \quad \begin{array}{cccccccc} \mathcal{O}_{0,0} & \mathcal{O}_{1,0} & \cdots & \mathcal{O}_{n-2,0} & \mathcal{O}_{n-1,0} & \mathcal{O}_{n,0} & & \\ & \mathcal{O}_{1,1} & \cdots & \mathcal{O}_{n-2,1} & \mathcal{O}_{n-1,1} & \mathcal{O}_{n,1} & \mathcal{O}_{n+1,1} & \\ & & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots \\ & & & \mathcal{O}_{n-2,n-2} & \mathcal{O}_{n-1,n-2} & \mathcal{O}_{n,n-2} & \mathcal{O}_{n+1,n-2} & \cdots & \mathcal{O}_{2n-2,n-2} \\ & & & & \mathcal{O}_{n-1,n-1} & \mathcal{O}_{n,n-1} & \mathcal{O}_{n+1,n-1} & \cdots & \mathcal{O}_{2n-2,n-1} & \mathcal{O}_{2n-1,n-1} \end{array}$$

Note from Lemma 5.1 that there are no morphisms from right to left in (5.3). Since $\omega_{\mathbf{V}} \cong \mathcal{O}_{\mathbf{V}}(-nh - nH)$, by mutating first

$$\begin{array}{cccc} \mathcal{O}_{0,0} & \mathcal{O}_{1,0} & \cdots & \mathcal{O}_{n-2,0} \\ & \mathcal{O}_{1,1} & \cdots & \mathcal{O}_{n-2,1} \\ & & \ddots & \vdots \\ & & & \mathcal{O}_{n-2,n-2} \end{array}$$

to the far right, and then $\Phi_-(D^b(\mathbf{V}_-))$ to the far right, we obtain

$$D^b(\mathbf{V}) = \langle \mathcal{A}_1, \Phi_1(D^b(\mathbf{V}_-)) \rangle,$$

where

$$\Phi_1(D^b(\mathbf{V}_-)) := R_{\langle \mathcal{O}_{n,n}, \dots, \mathcal{O}_{2n-2,2n-2} \rangle} \circ \Phi_-$$

and \mathcal{A}_1 is given by

$$\begin{array}{ccccccc}
O_{n-1,0} & O_{n,0} & & & & & \\
O_{n-1,1} & O_{n,1} & O_{n+1,1} & & & & \\
\vdots & \vdots & \vdots & \ddots & & & \\
O_{n-1,n-1} & O_{n,n-1} & O_{n+1,n-1} & \cdots & O_{2n-2,n-1} & O_{2n-1,n-1} & \\
& O_{n,n} & O_{n+1,n} & \cdots & O_{2n-2,n} & & \\
& & O_{n+1,n+1} & \cdots & O_{2n-2,n+1} & & \\
& & & \ddots & \vdots & & \\
& & & & O_{2n-2,2n-2} & &
\end{array}$$

By mutating $\Phi_1(D^b(\mathbf{V}_-))$ one step to the left, and then $O_{2n-1,n-1}$ to the far left, we obtain

$$(5.4) \quad D^b(\mathbf{V}) = \langle \mathcal{A}_2, \Phi_2(D^b(\mathbf{V}_-)) \rangle,$$

where

$$\Phi_2(D^b(\mathbf{V}_-)) := L_{O_{2n-1,n-1}} \circ \Phi_1$$

and \mathcal{A}_2 is given by

$$\begin{array}{ccccccc}
O_{n-1,-1} & & & & & & \\
O_{n-1,0} & O_{n,0} & & & & & \\
O_{n-1,1} & O_{n,1} & O_{n+1,1} & & & & \\
\vdots & \vdots & \vdots & \ddots & & & \\
O_{n-1,n-1} & O_{n,n-1} & O_{n+1,n-1} & \cdots & O_{2n-2,n-1} & & \\
& O_{n,n} & O_{n+1,n} & \cdots & O_{2n-2,n} & & \\
& & O_{n+1,n+1} & \cdots & O_{2n-2,n+1} & & \\
& & & \ddots & \vdots & & \\
& & & & O_{2n-2,2n-2} & &
\end{array}$$

By comparing (5.4) with (4.2) and (5.2), we obtain a derived equivalence

$$\Phi := (\varphi_+)_* \circ ((-) \otimes O_{-(2n-1),0}) \circ \Phi_2 : D^b(\mathbf{V}_-) \xrightarrow{\sim} D^b(\mathbf{V}_+).$$

Remark 5.1. The way of presenting our proof in Section 4 and 5 is called chess game by some authors [12, 23].

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