



SCUOLA INTERNAZIONALE SUPERIORE DI STUDI AVANZATI

SISSA Digital Library

Derived Equivalences for the Flops of Type C2 and AG4 via Mutation of Semiorthogonal Decomposition

*Original*

Derived Equivalences for the Flops of Type C2 and AG4 via Mutation of Semiorthogonal Decomposition / Morimura, Hayato. - In: ALGEBRAS AND REPRESENTATION THEORY. - ISSN 1386-923X. - 25:3(2022), pp. 581-594. [10.1007/s10468-021-10036-y]

*Availability:*

This version is available at: 20.500.11767/133550 since: 2023-07-31T08:20:27Z

*Publisher:*

*Published*

DOI:10.1007/s10468-021-10036-y

*Terms of use:*

Testo definito dall'ateneo relativo alle clausole di concessione d'uso

*Publisher copyright*  
Springer

This version is available for education and non-commercial purposes.

note finali coverpage

(Article begins on next page)

# DERIVED EQUIVALENCES FOR THE FLOPS OF TYPE $C_2$ AND $A_4^G$ VIA MUTATION OF SEMIORTHOGONAL DECOMPOSITION

HAYATO MORIMURA

**ABSTRACT.** We give a new proof of the derived equivalence of a pair of varieties connected by the flop of type  $C_2$  in the list of Kanemitsu [13], which is originally due to Segal [22]. We also prove the derived equivalence of a pair of varieties connected by the flop of type  $A_4^G$  in the same list. The latter proof follows that of the derived equivalence of Calabi–Yau 3-folds in Grassmannians  $\mathrm{Gr}(2, 5)$  and  $\mathrm{Gr}(3, 5)$  by Kapustka and Rampazzo [15] closely.

## 1. INTRODUCTION

Let  $G$  be a semisimple Lie group and  $B$  a Borel subgroup of  $G$ . For distinct maximal parabolic subgroups  $P$  and  $Q$  of  $G$  containing  $B$ , three homogeneous spaces  $G/P$ ,  $G/Q$ , and  $G/(P \cap Q)$  form the following diagram:

$$\begin{array}{ccc} & \mathbf{F} := G/(P \cap Q) & \\ \varpi_- \swarrow & & \searrow \varpi_+ \\ \mathbf{P} := G/P & & \mathbf{Q} := G/Q \end{array}$$

We write the hyperplane classes of  $\mathbf{P}$  and  $\mathbf{Q}$  as  $h$  and  $H$  respectively. By abuse of notation, the pull-back to  $\mathbf{F}$  of the hyperplane classes  $h$  and  $H$  will be denoted by the same symbol. The morphisms  $\varpi_-$  and  $\varpi_+$  are projective morphisms whose relative  $\mathcal{O}(1)$  are  $\mathcal{O}(H)$  and  $\mathcal{O}(h)$  respectively. We consider the diagram

$$(1.1) \quad \begin{array}{ccccc} & & \mathbf{F} & & \\ & \swarrow \varpi_- & \downarrow \iota & \searrow \varpi_+ & \\ \mathbf{P} & & \mathbf{V} & & \mathbf{Q} \\ \downarrow \iota_- & \swarrow \varphi_- & & \searrow \varphi_+ & \downarrow \iota_+ \\ \mathbf{V}_- & & & & \mathbf{V}_+ \\ & \swarrow \phi_- & & \searrow \phi_+ & \\ & & \mathbf{V}_0 & & \end{array}$$

where

- $\mathbf{V}_-$  is the total space of  $((\varpi_-)_* \mathcal{O}(h + H))^\vee$  over  $\mathbf{P}$ ,
- $\mathbf{V}_+$  is the total space of  $((\varpi_+)_* \mathcal{O}(h + H))^\vee$  over  $\mathbf{Q}$ ,
- $\mathbf{V}$  is the total space of  $\mathcal{O}(-h - H)$  over  $\mathbf{F}$ ,
- $\iota_-$ ,  $\iota_+$ , and  $\iota$  are the zero-sections,
- $\varphi_-$  and  $\varphi_+$  are blow-ups of the zero-sections, and
- $\phi_-$  and  $\phi_+$  are the affinizations which contract the zero sections.

If  $\mathbf{V}_-$  and  $\mathbf{V}_+$  have the trivial canonical bundles, then one expects from [4, Conjecture 4.4] or [16, Conjecture 1.2] that  $\mathbf{V}_-$  and  $\mathbf{V}_+$  are derived-equivalent.

When  $G$  is the simple Lie group of type  $G_2$ , Ueda [24] used sequence of mutations of semiorthogonal decompositions of  $D^b(\mathbf{V})$  obtained by applying Orlov's theorem [20] to the diagram (1.1) to prove the derived equivalence of  $\mathbf{V}_-$  and  $\mathbf{V}_+$ . This sequence of mutations in turn follows that of Kuznetsov [18] closely.

In this paper, by using the same method, we give a new proof to the following theorem, which is originally due to Segal [22], where the flop was attributed to Abuaf:

**Theorem 1.1.** *Varieties connected by the flop of type  $C_2$  are derived-equivalent.*

The term *the flop of type  $C_2$*  was introduced in [13], where simple K-equivalent maps in dimension at most 8 were classified. There are several ways to prove Theorem 1.1. In [22], Segal showed the derived equivalence by using tilting vector bundles. Hara [8] constructed alternative tilting vector bundles and studied the relation between functors defined by him and Segal.

The flop of type  $A_{2r-2}^G$  is also in the list of Kanemitsu[13]. It connects  $\mathbf{V}_-$  and  $\mathbf{V}_+$  for  $\mathbf{P} = \text{Gr}(r-1, 2r-1)$  and  $\mathbf{Q} = \text{Gr}(r, 2r-1)$ . Similarly, we prove the following theorem:

**Theorem 1.2.** *Varieties connected by the flop of type  $A_4^G$  are derived-equivalent.*

Although the proof of Theorem 1.2 is parallel to that of the derived equivalence of Calabi–Yau complete intersections in  $\mathbf{P} = \text{Gr}(2, 5)$  and  $\mathbf{Q} = \text{Gr}(3, 5)$  defined by global sections of the equivariant vector bundles dual to  $\mathbf{V}_-$  and  $\mathbf{V}_+$  in [15, Theorem 5.7], we write down a full detail for clarity. As explained in [24], the derived equivalence obtained in [15] in turn follows from Theorem 1.2 using matrix factorizations.

We also give a similar proof of derived equivalences for a Mukai flop and a standard flop. For a Mukai flop, Kawamata [16] and Namikawa [19] independently showed the derived equivalence by using the pull-back and the push-forward along the fiber product  $\mathbf{V}_- \times_{\mathbf{V}_0} \mathbf{V}_+$ . Addington, Donovan, and Meachan [1] introduced a generalization of the functor of Kawamata and Namikawa parametrized by an integer, and discovered that certain compositions of these functors give the  $\mathbb{P}$ -twist in the sense of Huybrechts and Thomas [11]. They also considered the case of a standard flop, where the derived equivalence is originally proved by Bondal and Orlov [5]. Our proof is obtained by proceeding the mutation performed in [5] and [1] a little further in a straightforward way. Hara [7] also studied a Mukai flop in terms of non-commutative crepant resolutions.

For a standard flop, Segal [21] showed the derived equivalence by using the grade restriction rule for variation of geometric invariant theory quotients (VGIT) originally introduced by Hori, Herbst, and Page [10]. VGIT method was subsequently developed by Halpern-Leistner [6] and Ballard, Favero, and Katzarkov [2]. It is an interesting problem to develop this method further to prove the derived equivalence for the flop of type  $C_2$  and  $A_4^G$ , and a Mukai flop.

**Notations and conventions.** *We work over an algebraically closed field  $\mathbf{k}$  of characteristic 0 throughout this paper. All pull-back and push-forward are derived unless otherwise specified. The complexes underlying  $\text{Ext}^\bullet(-, -)$  and  $H^\bullet(-)$  will be denoted by  $\mathbf{hom}(-, -)$  and  $\mathbf{h}(-)$  respectively.*

**Acknowledgements.** *The author would like to express his gratitude to Kazushi Ueda for guidance and encouragement. The author would like to thank anonymous reviewers for their careful reading of the manuscript and their many suggestions and comments. The author declare that he has no conflict of interest.*

## 2. FLOP OF TYPE $C_2$

Let  $P$  and  $Q$  be the parabolic subgroups of the simple Lie group  $G$  of type  $C_2$  associated with the crossed Dynkin diagrams  $\times \rightleftharpoons \bullet$  and  $\bullet \rightleftharpoons \times$ . The corresponding homogeneous spaces are the

projective space  $\mathbf{P} = \mathbb{P}(V)$ , the Lagrangian Grassmannian  $\mathbf{Q} = \text{LGr}(V)$ , and the isotropic flag variety  $\mathbf{F} = \mathbb{P}_{\mathbf{P}}(\mathcal{L}_{\mathbf{P}}^{\perp}/\mathcal{L}_{\mathbf{P}}) = \mathbb{P}_{\mathbf{Q}}(\mathcal{S}_{\mathbf{Q}})$ . Here  $V$  is a 4-dimensional symplectic vector space,  $\mathcal{L}_{\mathbf{P}}^{\perp}$  is the rank 3 vector bundle given as the symplectic orthogonal to the tautological line bundle  $\mathcal{L}_{\mathbf{P}} \cong \mathcal{O}_{\mathbf{P}}(-h)$  on  $\mathbf{P}$ , and  $\mathcal{S}_{\mathbf{Q}}$  is the tautological rank 2 bundle on  $\mathbf{Q}$ . Note that  $\mathbf{Q}$  is also a quadric hypersurface in  $\mathbb{P}^4$ . Tautological sequences on  $\mathbf{Q} = \text{LGr}(V)$  and  $\mathbf{F} \cong \mathbb{P}_{\mathbf{Q}}(\mathcal{S}_{\mathbf{Q}})$  give

$$(2.1) \quad 0 \rightarrow \mathcal{S}_{\mathbf{Q}} \rightarrow \mathcal{O}_{\mathbf{Q}} \otimes V \rightarrow \mathcal{S}_{\mathbf{Q}}^{\vee} \rightarrow 0$$

and

$$(2.2) \quad 0 \rightarrow \mathcal{O}_{\mathbf{F}}(-h + H) \rightarrow \mathcal{S}_{\mathbf{F}}^{\vee} \rightarrow \mathcal{O}_{\mathbf{F}}(h) \rightarrow 0,$$

where  $\mathcal{S}_{\mathbf{F}} := \varpi_+^* \mathcal{S}_{\mathbf{Q}}$ . We have

$$(\varpi_-)_*(\mathcal{O}_{\mathbf{F}}(H)) \cong ((\mathcal{L}_{\mathbf{P}}^{\perp}/\mathcal{L}_{\mathbf{P}}) \otimes \mathcal{L}_{\mathbf{P}})^{\vee}$$

and

$$(\varpi_+)_*(\mathcal{O}_{\mathbf{F}}(h)) \cong \mathcal{S}_{\mathbf{Q}}^{\vee},$$

whose determinants are given by  $\mathcal{O}_{\mathbf{P}}(2h)$  and  $\mathcal{O}_{\mathbf{Q}}(H)$  respectively. Since  $\omega_{\mathbf{P}} \cong \mathcal{O}_{\mathbf{P}}(-4h)$ ,  $\omega_{\mathbf{Q}} \cong \mathcal{O}_{\mathbf{Q}}(-3H)$ , and  $\omega_{\mathbf{F}} \cong \mathcal{O}_{\mathbf{F}}(-2h - 2H)$ , we have  $\omega_{\mathbf{V}_-} \cong \mathcal{O}_{\mathbf{V}_-}$ ,  $\omega_{\mathbf{V}_+} \cong \mathcal{O}_{\mathbf{V}_+}$ , and  $\omega_{\mathbf{V}} \cong \mathcal{O}_{\mathbf{V}}(-h - H)$ .

Recall from [3] that

$$(2.3) \quad D^b(\mathbf{P}) = \langle \mathcal{O}_{\mathbf{P}}(-2h), \mathcal{O}_{\mathbf{P}}(-h), \mathcal{O}_{\mathbf{P}}, \mathcal{O}_{\mathbf{P}}(h) \rangle,$$

and from [17] (cf. also [14]) that

$$D^b(\mathbf{Q}) = \langle \mathcal{O}_{\mathbf{Q}}(-H), \mathcal{S}_{\mathbf{Q}}^{\vee}(-H), \mathcal{O}_{\mathbf{Q}}, \mathcal{O}_{\mathbf{Q}}(H) \rangle.$$

Since  $\varphi_{\pm}$  are blow-ups along the zero-sections, it follows from [20] that

$$(2.4) \quad D^b(\mathbf{V}) = \langle \iota_* \varpi_-^* D^b(\mathbf{P}), \Phi_-(D^b(\mathbf{V}_-)) \rangle$$

and

$$(2.5) \quad D^b(\mathbf{V}) = \langle \iota_* \varpi_+^* D^b(\mathbf{Q}), \Phi_+(D^b(\mathbf{V}_+)) \rangle,$$

where

$$\Phi_- := ((-) \otimes \mathcal{O}_{\mathbf{V}}(H)) \circ \varphi_-^* : D^b(\mathbf{V}_-) \rightarrow D^b(\mathbf{V})$$

and

$$\Phi_+ := ((-) \otimes \mathcal{O}_{\mathbf{V}}(h)) \circ \varphi_+^* : D^b(\mathbf{V}_+) \rightarrow D^b(\mathbf{V}).$$

By abuse of notation, we use the same symbol for an object of  $D^b(\mathbf{F})$  and its image in  $D^b(\mathbf{V})$  by the push-forward  $\iota_*$ . (2.3) and (2.4) give

$$D^b(\mathbf{V}) = \langle \mathcal{O}_{\mathbf{F}}(-2h), \mathcal{O}_{\mathbf{F}}(-h), \mathcal{O}_{\mathbf{F}}, \mathcal{O}_{\mathbf{F}}(h), \Phi_-(D^b(\mathbf{V}_-)) \rangle.$$

Since  $\omega_{\mathbf{V}} \cong \mathcal{O}_{\mathbf{V}}(-h - H)$ , by mutating the first term to the far right, and then  $\Phi_-(D^b(\mathbf{V}_-))$  one step to the right, we obtain

$$D^b(\mathbf{V}) = \langle \mathcal{O}_{\mathbf{F}}(-h), \mathcal{O}_{\mathbf{F}}, \mathcal{O}_{\mathbf{F}}(h), \mathcal{O}_{\mathbf{F}}(-h + H), \Phi_1(D^b(\mathbf{V}_-)) \rangle,$$

where

$$\Phi_1 := R_{\langle \mathcal{O}_{\mathbf{F}}(-h+H) \rangle} \circ \Phi_-.$$

In the sequel, we will use the following fact.

**Lemma 2.1.** *Given two vector bundles  $\mathcal{E}_{\mathbf{F}}, \mathcal{F}_{\mathbf{F}}$  on  $\mathbf{F}$ , if  $\mathbf{h}(\mathcal{E}_{\mathbf{F}}^{\vee} \otimes \mathcal{F}_{\mathbf{F}}(-h - H)) \simeq 0$ , then we have  $\mathbf{hom}_{\mathcal{O}_{\mathbf{V}}}(\mathcal{E}_{\mathbf{F}}, \mathcal{F}_{\mathbf{F}}) \simeq \mathbf{h}(\mathcal{E}_{\mathbf{F}}^{\vee} \otimes \mathcal{F}_{\mathbf{F}})$ .*

*Proof.* We have

$$\begin{aligned}\mathbf{hom}_{\mathcal{O}_V}(\mathcal{E}_F, \mathcal{F}_F) &\simeq \mathbf{hom}_{\mathcal{O}_V}(\{\mathcal{E}_V(h+H) \rightarrow \mathcal{E}_V\}, \mathcal{F}_F) \\ &\simeq \mathbf{h}(\{\mathcal{E}_F^\vee \otimes \mathcal{F}_F \rightarrow \mathcal{E}_F^\vee \otimes \mathcal{F}_F(-h-H)\}) \\ &\simeq \mathbf{h}(\mathcal{E}_F^\vee \otimes \mathcal{F}_F).\end{aligned}$$

□

Note that the canonical extension of  $\mathcal{O}_F(h)$  by  $\mathcal{O}_F(-h+H)$  associated with

$$\begin{aligned}\mathbf{hom}_{\mathcal{O}_V}(\mathcal{O}_F(h), \mathcal{O}_F(-h+H)) &\simeq \mathbf{h}(\mathcal{O}_F(-2h+H)) \\ &\simeq \mathbf{h}((\varpi_+)_* \mathcal{O}_F(-2h) \otimes \mathcal{O}_Q(H)) \\ &\simeq \mathbf{h}(\mathcal{O}_Q[-1]) \\ &\simeq \mathbf{k}[-1]\end{aligned}$$

is given by the short exact sequence (2.2). By mutating  $\mathcal{O}_F(-h+H)$  one step to the left,  $\mathcal{O}_F(-h)$  to the far right, and then  $\Phi_1(D^b(\mathbf{V}_-))$  one step to the right, we obtain

$$D^b(\mathbf{V}) = \langle \mathcal{O}_F, \mathcal{S}_F^\vee, \mathcal{O}_F(h), \mathcal{O}_F(H), \Phi_2(D^b(\mathbf{V}_-)) \rangle,$$

where

$$\Phi_2 := R_{\langle \mathcal{O}_F(H) \rangle} \circ \Phi_1.$$

One can easily see that  $\mathcal{O}_F(h)$  and  $\mathcal{O}_F(H)$  are orthogonal, so that

$$(2.6) \quad D^b(\mathbf{V}) = \langle \mathcal{O}_F, \mathcal{S}_F^\vee, \mathcal{O}_F(H), \mathcal{O}_F(h), \Phi_2(D^b(\mathbf{V}_-)) \rangle.$$

By mutating  $\Phi_2(D^b(\mathbf{V}_-))$  one step to the left, and then  $\mathcal{O}_F(h)$  to the far left, we obtain

$$D^b(\mathbf{V}) = \langle \mathcal{O}_F(-H), \mathcal{O}_F, \mathcal{S}_F^\vee, \mathcal{O}_F(H), \Phi_3(D^b(\mathbf{V}_-)) \rangle,$$

where

$$\Phi_3 := L_{\langle \mathcal{O}_F(h) \rangle} \circ \Phi_2.$$

We have

$$\mathbf{hom}_{\mathcal{O}_V}(\mathcal{O}_F, \mathcal{S}_F^\vee) \simeq \mathbf{h}(\mathcal{S}_F^\vee) \simeq V^\vee,$$

and the dual of (2.1) shows that the kernel of the evaluation map  $\mathcal{O}_F \otimes V^\vee \rightarrow \mathcal{S}_F^\vee$  is  $\mathcal{S}_F \cong \mathcal{S}_F^\vee(-H)$ . By mutating  $\mathcal{S}_F^\vee$  one step to the left, we obtain

$$(2.7) \quad D^b(\mathbf{V}) = \langle \mathcal{O}_F(-H), \mathcal{S}_F^\vee(-H), \mathcal{O}_F, \mathcal{O}_F(H), \Phi_3(D^b(\mathbf{V}_-)) \rangle.$$

By comparing (2.7) with (2.5), we obtain a derived equivalence

$$\Phi := \Phi_+^! \circ \Phi_3 : D^b(\mathbf{V}_-) \xrightarrow{\sim} D^b(\mathbf{V}_+),$$

where

$$\Phi_+^!(-) := (\varphi_+)_* \circ ((-) \otimes \mathcal{O}_V(-h)) : D^b(\mathbf{V}) \rightarrow D^b(\mathbf{V}_+)$$

is the left adjoint functor of  $\Phi_+$ .

### 3. FLOP OF TYPE $A_4^G$

Let  $P$  and  $Q$  be the parabolic subgroups of the simple Lie group  $G$  of type  $A_4$  associated with the crossed Dynkin diagrams  $\bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet$  and  $\bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet$ . The corresponding homogeneous spaces are the Grassmannians  $\mathbf{P} = \text{Gr}(2, V)$ ,  $\mathbf{Q} = \text{Gr}(3, V)$ , and the partial flag variety  $\mathbf{F} = \mathbb{P}_{\mathbf{P}}(\wedge^2 \mathcal{Q}_{\mathbf{P}}^{\vee}) = \mathbb{P}_{\mathbf{Q}}(\wedge^2 \mathcal{S}_{\mathbf{Q}})$ . Here  $V$  is a 5-dimensional vector space,  $\mathcal{Q}_{\mathbf{P}}^{\vee}$  is the dual of the universal quotient bundle on  $\mathbf{P}$ , and  $\mathcal{S}_{\mathbf{Q}}$  is the tautological rank 3 bundle on  $\mathbf{Q}$ . We have

$$(\varpi_-)_*(O_{\mathbf{F}}(H)) \cong \wedge^2 \mathcal{Q}_{\mathbf{P}}$$

and

$$(\varpi_+)_*(O_{\mathbf{F}}(h)) \cong \wedge^2 \mathcal{S}_{\mathbf{Q}}^{\vee},$$

whose determinants are given by  $O_{\mathbf{P}}(2h)$  and  $O_{\mathbf{Q}}(2H)$  respectively. Since  $\omega_{\mathbf{P}} \cong O_{\mathbf{P}}(-5h)$ ,  $\omega_{\mathbf{Q}} \cong O_{\mathbf{Q}}(-5H)$ , and  $\omega_{\mathbf{F}} \cong O_{\mathbf{F}}(-3h - 3H)$ , we have  $\omega_{V_-} \cong O_{V_-}$ ,  $\omega_{V_+} \cong O_{V_+}$  and  $\omega_V \cong O_V(-2h - 2H)$ .

First, we adapt several lemmas in [15] to our situation. To distinguish vector bundles which are obtained as a pull-back to  $\mathbf{F}$  from  $\mathbf{P}$  or  $\mathbf{Q}$ , we put tilde on the pull-back from  $\mathbf{Q}$ . By abuse of notation, we use the same symbol for an object of  $D^b(\mathbf{F})$  and its image in  $D^b(\mathbf{V})$  by the push-forward  $\iota_*$ .

**Lemma 3.1.**  $\text{hom}_{O_V}(\widetilde{\mathcal{Q}}_{\mathbf{F}}, O_{\mathbf{F}}(h + aH)) \simeq 0$  for integers  $-4 \leq a \leq -2$ .

*Proof.* We have

$$\text{hom}_{O_V}(\widetilde{\mathcal{Q}}_{\mathbf{F}}, O_{\mathbf{F}}(h + aH)) \simeq \mathbf{h}(\widetilde{\mathcal{Q}}_{\mathbf{F}}^{\vee}(h + aH)) \simeq 0,$$

where the first and the second isomorphisms follow from Lemma 2.1, Borel-Bott-Weil theorem and [15, Lemma 5.1] respectively.  $\square$

Similarly, one can deduce Lemma 3.2 and Lemma 3.3 below from [15, Lemma 5.2, Lemma 5.3] by checking that  $O_{\mathbf{F}}((a - 1)H)$ ,  $\mathcal{E}_{\mathbf{F}}^{\vee} \otimes \mathcal{E}'_{\mathbf{F}}((a - 1)h - 2H)$ , and  $\widetilde{\mathcal{F}}_{\mathbf{F}}^{\vee} \otimes \widetilde{\mathcal{F}}'_{\mathbf{F}}(-2h + (a - 1)H)$  are acyclic as an object of  $D^b(\mathbf{F})$ .

**Lemma 3.2.**  $\text{hom}_{O_V}(O_{\mathbf{F}}, O_{\mathbf{F}}(h + aH)) \simeq 0$  for integers  $-3 \leq a \leq -1$ .

**Lemma 3.3.** Let  $\mathcal{E}_{\mathbf{F}}, \mathcal{E}'_{\mathbf{F}}$  be the pull-back to  $\mathbf{F}$  of vector bundles  $\mathcal{E}, \mathcal{E}'$  on  $\mathbf{P}$ , and let  $\widetilde{\mathcal{F}}_{\mathbf{F}}, \widetilde{\mathcal{F}}'_{\mathbf{F}}$  be the pull-back to  $\mathbf{F}$  of vector bundles  $\mathcal{F}, \mathcal{F}'$  on  $\mathbf{Q}$ . Then we have  $\text{hom}_{O_V}(\mathcal{E}_{\mathbf{F}}, \mathcal{E}'_{\mathbf{F}}(ah - H)) \simeq 0$  and  $\text{hom}_{O_V}(\widetilde{\mathcal{F}}_{\mathbf{F}}, \widetilde{\mathcal{F}}'_{\mathbf{F}}(-h + aH)) \simeq 0$  for all integers  $a$ .

The parallel result to the following lemma was tacitly used in [15].

**Lemma 3.4.** As an object of  $D^b(\mathbf{V})$ ,  $O_{\mathbf{F}}, \widetilde{\mathcal{Q}}_{\mathbf{F}}, \mathcal{S}_{\mathbf{F}}$ , and  $\mathcal{S}_{\mathbf{F}}^{\vee}$  are left orthogonal to  $\widetilde{\mathcal{S}}_{\mathbf{F}}^{\vee}(h - 2H)$ ,  $\widetilde{\mathcal{S}}_{\mathbf{F}}^{\vee}(h - 2H)$ ,  $O_{\mathbf{F}}(2h - 2H)$ , and  $\mathcal{Q}_{\mathbf{F}}$  respectively.

Lemma 3.5 below and the tautological sequence show that  $R_{O_{\mathbf{F}}} \widetilde{\mathcal{Q}}_{\mathbf{F}}^{\vee} \simeq \widetilde{\mathcal{S}}_{\mathbf{F}}^{\vee}$  and  $R_{O_{\mathbf{F}}} \mathcal{S}_{\mathbf{F}} \simeq \mathcal{Q}_{\mathbf{F}}$  in  $D^b(\mathbf{V})$ .

**Lemma 3.5.**  $\text{hom}_{O_V}(\widetilde{\mathcal{Q}}_{\mathbf{F}}^{\vee}, O_{\mathbf{F}}) \simeq V$  and  $\text{hom}_{O_V}(\mathcal{S}_{\mathbf{F}}, O_{\mathbf{F}}) \simeq V$ .

Again, both Lemma 3.4 and Lemma 3.5 follow from Lemma 2.1 and Borel-Bott-Weil theorem. Lemma 3.6 below and the exact sequences

$$0 \rightarrow O_{\mathbf{F}}(h - H) \rightarrow \mathcal{Q}_{\mathbf{F}} \rightarrow \widetilde{\mathcal{Q}}_{\mathbf{F}} \rightarrow 0$$

and

$$0 \rightarrow \mathcal{S}_{\mathbf{F}} \rightarrow \widetilde{\mathcal{S}}_{\mathbf{F}} \rightarrow O_{\mathbf{F}}(h - H) \rightarrow 0$$

obtained in [15] show that  $R_{O_{\mathbf{F}}(h-H)} \widetilde{\mathcal{Q}}_{\mathbf{F}} \simeq \mathcal{Q}_{\mathbf{F}}[1]$  and  $L_{O_{\mathbf{F}}(-h+H)} \widetilde{\mathcal{S}}_{\mathbf{F}}^{\vee} \simeq \mathcal{S}_{\mathbf{F}}^{\vee}$  in  $D^b(\mathbf{V})$ .

**Lemma 3.6.**  $\mathrm{hom}_{\mathcal{O}_V}(\widetilde{\mathcal{Q}}_F, \mathcal{O}_F(h-H)) \simeq \mathbf{k}[-1]$  and  $\mathrm{hom}_{\mathcal{O}_V}(\mathcal{O}_F(-h+H), \widetilde{\mathcal{F}}_F^\vee) \simeq \mathbf{k}$ .

*Proof.* We have

$$\mathrm{hom}_{\mathcal{O}_V}(\widetilde{\mathcal{Q}}_F, \mathcal{O}_F(h-H)) \simeq \mathbf{h}(\widetilde{\mathcal{Q}}_F^\vee(h-H)) \simeq \mathbf{k}[-1],$$

where the isomorphisms follow from Lemma 2.1 and Borel-Bott-Weil theorem. Similarly, we have

$$\mathrm{hom}_{\mathcal{O}_V}(\mathcal{O}_F(-h+H), \widetilde{\mathcal{F}}_F^\vee) \simeq \mathbf{h}(\widetilde{\mathcal{F}}_F^\vee(h-H)) \simeq \mathbf{k}.$$

□

Recall from [17] (cf. also [14])

$$D^b(\mathbf{P}) = \langle \mathcal{S}_P(-2h), \mathcal{O}_P(-2h), \mathcal{S}_P(-h), \mathcal{O}_P(-h), \dots, \mathcal{S}_P(2h), \mathcal{O}_P(2h) \rangle,$$

and

$$(3.1) \quad D^b(\mathbf{Q}) = \langle \mathcal{O}_Q, \mathcal{Q}_Q, \mathcal{O}_Q(H), \mathcal{Q}_Q(H), \dots, \mathcal{O}_Q(4H), \mathcal{Q}_Q(4H) \rangle.$$

Since  $\varphi_\pm$  are blow-ups along the zero-sections, it follows from [20] that

$$(3.2) \quad D^b(\mathbf{V}) = \langle \iota_* \varpi_-^* D^b(\mathbf{P}), \iota_* \varpi_-^* D^b(\mathbf{P})(h+H), \Phi_-(D^b(\mathbf{V}_-)) \rangle$$

and

$$(3.3) \quad D^b(\mathbf{V}) = \langle \iota_* \varpi_+^* D^b(\mathbf{Q}), \iota_* \varpi_+^* D^b(\mathbf{Q})(h+H), \Phi_+(D^b(\mathbf{V}_+)) \rangle,$$

where

$$\Phi_- := ((-) \otimes \mathcal{O}_V(2H)) \circ \varphi_-^* : D^b(\mathbf{V}_-) \rightarrow D^b(\mathbf{V})$$

and

$$\Phi_+ := ((-) \otimes \mathcal{O}_V(2h)) \circ \varphi_+^* : D^b(\mathbf{V}_+) \rightarrow D^b(\mathbf{V}).$$

We write  $\mathcal{O}_{i,j} := \mathcal{O}_F(ih + jH)$ . (3.1) and (3.3) give a semiorthogonal decomposition of the form

$$D^b(\mathbf{V}) = \langle \mathcal{O}_{0,0}, \widetilde{\mathcal{Q}}_{0,0}, \mathcal{O}_{0,1}, \widetilde{\mathcal{Q}}_{0,1}, \mathcal{O}_{0,2}, \widetilde{\mathcal{Q}}_{0,2}, \mathcal{O}_{0,3}, \widetilde{\mathcal{Q}}_{0,3}, \mathcal{O}_{0,4}, \widetilde{\mathcal{Q}}_{0,4}, \\ \mathcal{O}_{1,1}, \widetilde{\mathcal{Q}}_{1,1}, \mathcal{O}_{1,2}, \widetilde{\mathcal{Q}}_{1,2}, \mathcal{O}_{1,3}, \widetilde{\mathcal{Q}}_{1,3}, \mathcal{O}_{1,4}, \widetilde{\mathcal{Q}}_{1,4}, \mathcal{O}_{1,5}, \widetilde{\mathcal{Q}}_{1,5}, \Phi_+(D^b(\mathbf{V}_+)) \rangle.$$

Since  $\omega_V \cong \mathcal{O}_V(-2h-2H)$ , by mutating the first five terms to the far right, and then  $\Phi_+(D^b(\mathbf{V}_+))$  five steps to the right, we obtain

$$D^b(\mathbf{V}) = \langle \widetilde{\mathcal{Q}}_{0,2}, \mathcal{O}_{0,3}, \widetilde{\mathcal{Q}}_{0,3}, \mathcal{O}_{0,4}, \widetilde{\mathcal{Q}}_{0,4}, \mathcal{O}_{1,1}, \widetilde{\mathcal{Q}}_{1,1}, \mathcal{O}_{1,2}, \widetilde{\mathcal{Q}}_{1,2}, \mathcal{O}_{1,3}, \\ \widetilde{\mathcal{Q}}_{1,3}, \mathcal{O}_{1,4}, \widetilde{\mathcal{Q}}_{1,4}, \mathcal{O}_{1,5}, \widetilde{\mathcal{Q}}_{1,5}, \mathcal{O}_{2,2}, \widetilde{\mathcal{Q}}_{2,2}, \mathcal{O}_{2,3}, \widetilde{\mathcal{Q}}_{2,3}, \mathcal{O}_{2,4}, \Phi_1(D^b(\mathbf{V}_+)) \rangle,$$

where

$$\Phi_1 := R_{\langle \mathcal{O}_{2,2}, \widetilde{\mathcal{Q}}_{2,2}, \mathcal{O}_{2,3}, \widetilde{\mathcal{Q}}_{2,3}, \mathcal{O}_{2,4} \rangle} \circ \Phi_+.$$

One can easily see that  $\mathcal{O}_{1,1}$  is orthogonal to  $\mathcal{O}_{0,3}$ ,  $\widetilde{\mathcal{Q}}_{0,3}$ ,  $\mathcal{O}_{0,4}$ , and  $\widetilde{\mathcal{Q}}_{0,4}$  by Lemma 3.1 and Lemma 3.2, so that

$$D^b(\mathbf{V}) = \langle \widetilde{\mathcal{Q}}_{0,2}, \mathcal{O}_{1,1}, \mathcal{O}_{0,3}, \widetilde{\mathcal{Q}}_{0,3}, \mathcal{O}_{0,4}, \widetilde{\mathcal{Q}}_{0,4}, \widetilde{\mathcal{Q}}_{1,1}, \mathcal{O}_{1,2}, \widetilde{\mathcal{Q}}_{1,2}, \mathcal{O}_{1,3}, \\ \widetilde{\mathcal{Q}}_{1,3}, \mathcal{O}_{2,2}, \mathcal{O}_{1,4}, \widetilde{\mathcal{Q}}_{1,4}, \mathcal{O}_{1,5}, \widetilde{\mathcal{Q}}_{1,5}, \widetilde{\mathcal{Q}}_{2,2}, \mathcal{O}_{2,3}, \widetilde{\mathcal{Q}}_{2,3}, \mathcal{O}_{2,4}, \Phi_1(D^b(\mathbf{V}_+)) \rangle.$$

By mutating  $\widetilde{\mathcal{Q}}_{0,2}$ ,  $\widetilde{\mathcal{Q}}_{1,3}$ ,  $\widetilde{\mathcal{Q}}_{1,1}$ , and  $\widetilde{\mathcal{Q}}_{2,2}$  one step to the right, we obtain by  $\widetilde{\mathcal{Q}}_{1,1} \cong \widetilde{\mathcal{Q}}_{1,2}^\vee$ , Lemma 3.5, and Lemma 3.6

$$D^b(\mathbf{V}) = \langle \mathcal{O}_{1,1}, \mathcal{Q}_{0,2}, \mathcal{O}_{0,3}, \widetilde{\mathcal{Q}}_{0,3}, \mathcal{O}_{0,4}, \widetilde{\mathcal{Q}}_{0,4}, \mathcal{O}_{1,2}, \widetilde{\mathcal{F}}_{1,2}^\vee, \widetilde{\mathcal{Q}}_{1,2}, \mathcal{O}_{1,3}, \\ \mathcal{O}_{2,2}, \mathcal{Q}_{1,3}, \mathcal{O}_{1,4}, \widetilde{\mathcal{Q}}_{1,4}, \mathcal{O}_{1,5}, \widetilde{\mathcal{Q}}_{1,5}, \mathcal{O}_{2,3}, \widetilde{\mathcal{F}}_{2,3}^\vee, \widetilde{\mathcal{Q}}_{2,3}, \mathcal{O}_{2,4}, \Phi_1(D^b(\mathbf{V}_+)) \rangle.$$

By mutating  $O_{1,2}$  and  $O_{2,3}$  four steps to the left, we obtain by Lemma 3.1, Lemma 3.2, and Lemma 3.6

$$D^b(\mathbf{V}) = \langle O_{1,1}, \mathcal{Q}_{0,2}, O_{1,2}, O_{0,3}, \mathcal{Q}_{0,3}, O_{0,4}, \widetilde{\mathcal{Q}}_{0,4}, \widetilde{\mathcal{F}}_{1,2}^\vee, \widetilde{\mathcal{Q}}_{1,2}, O_{1,3}, \\ O_{2,2}, \mathcal{Q}_{1,3}, O_{2,3}, O_{1,4}, \mathcal{Q}_{1,4}, O_{1,5}, \widetilde{\mathcal{Q}}_{1,5}, \widetilde{\mathcal{F}}_{2,3}^\vee, \widetilde{\mathcal{Q}}_{2,3}, O_{2,4}, \Phi_1(D^b(\mathbf{V}_+)) \rangle.$$

One can easily see that  $\widetilde{\mathcal{F}}_{1,2}^\vee$  is orthogonal to  $O_{0,4}$  and  $\widetilde{\mathcal{Q}}_{0,4}$  by Lemma 3.4, so that

$$D^b(\mathbf{V}) = \langle O_{1,1}, \mathcal{Q}_{0,2}, O_{1,2}, O_{0,3}, \mathcal{Q}_{0,3}, \widetilde{\mathcal{F}}_{1,2}^\vee, O_{0,4}, \widetilde{\mathcal{Q}}_{0,4}, \widetilde{\mathcal{Q}}_{1,2}, O_{1,3}, \\ O_{2,2}, \mathcal{Q}_{1,3}, O_{2,3}, O_{1,4}, \mathcal{Q}_{1,4}, \widetilde{\mathcal{F}}_{2,3}^\vee, O_{1,5}, \widetilde{\mathcal{Q}}_{1,5}, \widetilde{\mathcal{Q}}_{2,3}, O_{2,4}, \Phi_1(D^b(\mathbf{V}_+)) \rangle.$$

By mutating  $O_{0,3}$  and  $O_{1,4}$  two steps to the right,  $O_{1,3}$  and  $O_{2,4}$  three steps to the left, and then  $O_{0,4}$  and  $O_{1,5}$  two steps to the right, we obtain by Lemma 3.5 and Lemma 3.6

$$D^b(\mathbf{V}) = \langle O_{1,1}, \mathcal{Q}_{0,2}, O_{1,2}, \mathcal{S}_{0,3}, \mathcal{S}_{1,2}^\vee, O_{0,3}, O_{1,3}, \mathcal{S}_{0,4}, \mathcal{S}_{1,3}^\vee, O_{0,4}, \\ O_{2,2}, \mathcal{Q}_{1,3}, O_{2,3}, \mathcal{S}_{1,4}, \mathcal{S}_{2,3}^\vee, O_{1,4}, O_{2,4}, \mathcal{S}_{1,5}, \mathcal{S}_{2,4}^\vee, O_{1,5}, \Phi_1(D^b(\mathbf{V}_+)) \rangle.$$

By mutating  $O_{1,1}$  to the far right, and then  $\Phi_1(D^b(\mathbf{V}_+))$  one step to the right, we obtain

$$D^b(\mathbf{V}) = \langle \mathcal{Q}_{0,2}, O_{1,2}, \mathcal{S}_{0,3}, \mathcal{S}_{1,2}^\vee, O_{0,3}, O_{1,3}, \mathcal{S}_{0,4}, \mathcal{S}_{1,3}^\vee, O_{0,4}, O_{2,2}, \\ \mathcal{Q}_{1,3}, O_{2,3}, \mathcal{S}_{1,4}, \mathcal{S}_{2,3}^\vee, O_{1,4}, O_{2,4}, \mathcal{S}_{1,5}, \mathcal{S}_{2,4}^\vee, O_{1,5}, O_{3,3}, \Phi_2(D^b(\mathbf{V}_+)) \rangle,$$

where

$$\Phi_2 := R_{\langle O_{3,3} \rangle} \circ \Phi_1.$$

By Lemma 3.2, Lemma 3.3, and Lemma 3.4, we obtain

$$D^b(\mathbf{V}) = \langle \mathcal{Q}_{0,2}, O_{1,2}, \mathcal{S}_{1,2}^\vee, O_{2,2}, \mathcal{S}_{0,3}, O_{0,3}, O_{1,3}, \mathcal{S}_{1,3}^\vee, \mathcal{Q}_{1,3}, O_{2,3}, \\ \mathcal{S}_{2,3}^\vee, O_{3,3}, \mathcal{S}_{0,4}, O_{0,4}, \mathcal{S}_{1,4}, O_{1,4}, O_{2,4}, \mathcal{S}_{2,4}^\vee, \mathcal{S}_{1,5}, O_{1,5}, \Phi_2(D^b(\mathbf{V}_+)) \rangle.$$

By mutating  $\Phi_2(D^b(\mathbf{V}_+))$  ten steps to the left, and then last ten terms to the far left, we obtain

$$D^b(\mathbf{V}) = \langle \mathcal{S}_{0,1}^\vee, O_{1,1}, \mathcal{S}_{-2,2}, O_{-2,2}, \mathcal{S}_{-1,2}, O_{-1,2}, O_{0,2}, \mathcal{S}_{0,2}^\vee, \mathcal{S}_{-1,3}, O_{-1,3}, \\ \mathcal{Q}_{0,2}, O_{1,2}, \mathcal{S}_{1,2}^\vee, O_{2,2}, \mathcal{S}_{0,3}, O_{0,3}, O_{1,3}, \mathcal{S}_{1,3}^\vee, \mathcal{Q}_{1,3}, O_{2,3}, \Phi_3(D^b(\mathbf{V}_+)) \rangle,$$

where

$$\Phi_3 := L_{\langle \mathcal{S}_{2,3}^\vee, O_{3,3}, \mathcal{S}_{0,4}, O_{0,4}, \mathcal{S}_{1,4}, O_{1,4}, O_{2,4}, \mathcal{S}_{2,4}^\vee, \mathcal{S}_{1,5}, O_{1,5} \rangle} \circ \Phi_2.$$

By Lemma 3.3, we obtain

$$D^b(\mathbf{V}) = \langle \mathcal{S}_{0,1}^\vee, O_{1,1}, \mathcal{S}_{-2,2}, O_{-2,2}, \mathcal{S}_{-1,2}, O_{-1,2}, O_{0,2}, \mathcal{S}_{0,2}^\vee, \mathcal{Q}_{0,2}, O_{1,2}, \\ \mathcal{S}_{1,2}^\vee, O_{2,2}, \mathcal{S}_{-1,3}, O_{-1,3}, \mathcal{S}_{0,3}, O_{0,3}, O_{1,3}, \mathcal{S}_{1,3}^\vee, \mathcal{Q}_{1,3}, O_{2,3}, \Phi_3(D^b(\mathbf{V}_+)) \rangle.$$

By mutating  $\mathcal{Q}_{0,2}$  and  $\mathcal{Q}_{1,3}$  two steps to the left, the first two terms to the far right, and then  $\Phi_3(D^b(\mathbf{V}_+))$  two steps to the right, we obtain by  $\mathcal{S}_{0,0}^\vee \simeq \mathcal{S}_{1,0}$ , Lemma 3.4, and Lemma 3.6

$$(3.4) \quad D^b(\mathbf{V}) = \langle \mathcal{S}_{-2,2}, O_{-2,2}, \mathcal{S}_{-1,2}, O_{-1,2}, \mathcal{S}_{0,2}, O_{0,2}, \mathcal{S}_{1,2}, O_{1,2}, \mathcal{S}_{2,2}, O_{2,2}, \\ \mathcal{S}_{-1,3}, O_{-1,3}, \mathcal{S}_{0,3}, O_{0,3}, \mathcal{S}_{1,3}, O_{1,3}, \mathcal{S}_{2,3}, O_{2,3}, \mathcal{S}_{3,3}, O_{3,3}, \Phi_4(D^b(\mathbf{V}_+)) \rangle,$$

where

$$\Phi_4 := R_{\langle \mathcal{S}_{2,3}^\vee, O_{3,3} \rangle} \circ \Phi_3.$$

By comparing (3.4) with (3.2), we obtain a derived equivalence

$$\Phi := \Phi_-^\dagger \circ \Phi_4: D^b(\mathbf{V}_+) \xrightarrow{\sim} D^b(\mathbf{V}_-),$$



where

$$\Phi_-^!(-) := (\varphi_-)_* \circ ((-) \otimes \mathcal{O}_V(-2H)) : D^b(\mathbf{V}) \rightarrow D^b(\mathbf{V}_-)$$

is the left adjoint functor of  $\Phi_-$ .

#### 4. MUKAI FLOP

For  $n \geq 2$ , let  $P$  and  $Q$  be the maximal parabolic subgroups of the simple Lie group of type  $A_n$  associated with the crossed Dynkin diagrams  $\times \cdots \bullet$  and  $\bullet \cdots \times$ . The corresponding homogeneous spaces are the projective spaces  $\mathbf{P} = \mathbb{P}V$ ,  $\mathbf{Q} = \mathbb{P}V^\vee$ , and the partial flag variety  $\mathbf{F} = \mathbf{F}(1, n; V)$ , where  $V$  is an  $(n+1)$ -dimensional vector space. Since  $\omega_{\mathbf{P}} \cong \mathcal{O}(-(n+1)h)$ ,  $\omega_{\mathbf{Q}} \cong \mathcal{O}(-(n+1)H)$ , and  $\omega_{\mathbf{F}} \cong \mathcal{O}(-nh - nH)$ , we have  $\omega_{\mathbf{V}_-} \cong \mathcal{O}_{\mathbf{V}_-}$ ,  $\omega_{\mathbf{V}_+} \cong \mathcal{O}_{\mathbf{V}_+}$ , and  $\omega_{\mathbf{V}} \cong \mathcal{O}(-(n-1)h - (n-1)H)$ .

**Lemma 4.1.**  $\mathcal{O}_{\mathbf{F}}(-ih + jH)$  and  $\mathcal{O}_{\mathbf{F}}(-(i+1)h + (j-1)H)$  are acyclic for  $1 \leq j \leq n-1$  and  $1 \leq i \leq n-j$ .

*Proof.* Since  $j-n \leq -i \leq -1$  and  $j-n-1 \leq -i-1 \leq -2$ , the derived push-forward of  $\mathcal{O}_{\mathbf{F}}(-ih + jH)$  and  $\mathcal{O}_{\mathbf{F}}(-(i+1)h + (j-1)H)$  vanish by [9, Exercise III.8.4] unless  $i = n-1$  and  $j = 1$ , in which case the acyclicity of  $\mathcal{O}_{\mathbf{F}}(-nh)$  is obvious.  $\square$

**Lemma 4.2.**  $\mathrm{hom}_{\mathcal{O}_V}(\mathcal{O}_{\mathbf{F}}(ih - jH), \mathcal{O}_{\mathbf{F}}) \simeq 0$  for  $1 \leq j \leq n-1$  and  $1 \leq i \leq n-j$ .

*Proof.* We have

$$\mathrm{hom}_{\mathcal{O}_V}(\mathcal{O}_{\mathbf{F}}(ih - jH), \mathcal{O}_{\mathbf{F}}) \simeq \mathbf{h}(\{\mathcal{O}_{\mathbf{F}}(-ih + jH) \rightarrow \mathcal{O}_{\mathbf{F}}(-(i+1)h + (j-1)H)\}),$$

which vanishes by Lemma 4.1.  $\square$

Recall from [3] that

$$(4.1) \quad D^b(\mathbf{P}) = \langle \mathcal{O}_{\mathbf{P}}, \mathcal{O}_{\mathbf{P}}(h), \dots, \mathcal{O}_{\mathbf{P}}(nh) \rangle$$

and

$$(4.2) \quad D^b(\mathbf{Q}) = \langle \mathcal{O}_{\mathbf{Q}}, \mathcal{O}_{\mathbf{Q}}(H), \dots, \mathcal{O}_{\mathbf{Q}}(nH) \rangle.$$

Since  $\varphi_{\pm}$  are blow-ups along the zero-sections, it follows from [20] that

$$(4.3) \quad D^b(\mathbf{V}) = \langle \iota_* \varpi_-^* D^b(\mathbf{P}), \dots, \iota_* \varpi_-^* D^b(\mathbf{P}) \otimes \mathcal{O}_V((n-2)H), \Phi_-(D^b(\mathbf{V}_-)) \rangle$$

and

$$(4.4) \quad D^b(\mathbf{V}) = \langle \iota_* \varpi_+^* D^b(\mathbf{Q}), \dots, \iota_* \varpi_+^* D^b(\mathbf{Q}) \otimes \mathcal{O}_V((n-2)h), \Phi_+(D^b(\mathbf{V}_+)) \rangle,$$

where

$$\Phi_- := ((-) \otimes \mathcal{O}_V((n-1)H)) \circ \varphi_-^* : D^b(\mathbf{V}_-) \rightarrow D^b(\mathbf{V})$$

and

$$\Phi_+ := ((-) \otimes \mathcal{O}_V((n-1)h)) \circ \varphi_+^* : D^b(\mathbf{V}_+) \rightarrow D^b(\mathbf{V}).$$

We write  $\mathcal{O}_{i,j} := \mathcal{O}_{\mathbf{F}}(ih + jH)$ . (4.1) and (4.3) give a semiorthogonal decomposition of the form

$$D^b(\mathbf{V}) = \langle \mathcal{A}_0, \Phi_-(D^b(\mathbf{V}_-)) \rangle$$

where  $\mathcal{A}_0$  is given by

$$(4.5) \quad \begin{array}{ccccccc} \mathcal{O}_{0,0} & \mathcal{O}_{1,0} & \cdots & \mathcal{O}_{n-2,0} & \mathcal{O}_{n-1,0} & \mathcal{O}_{n,0} & \\ & \mathcal{O}_{1,1} & \cdots & \mathcal{O}_{n-2,1} & \mathcal{O}_{n-1,1} & \mathcal{O}_{n,1} & \mathcal{O}_{n+1,1} \\ & & \ddots & \vdots & \vdots & \vdots & \vdots \\ & & & \mathcal{O}_{n-2,n-2} & \mathcal{O}_{n-1,n-2} & \mathcal{O}_{n,n-2} & \mathcal{O}_{n+1,n-2} \cdots \mathcal{O}_{2n-2,n-2}. \end{array}$$

Note from Lemma 4.2 that there are no morphisms from right to left in (4.5). Since  $\omega_V \cong O_{-(n-1),-(n-1)}$ , by mutating first

$$\begin{array}{cccc} O_{0,0} & O_{1,0} & \cdots & O_{n-2,0} \\ & O_{1,1} & \cdots & O_{n-2,1} \\ & & \ddots & \vdots \\ & & & O_{n-2,n-2} \end{array}$$

to the far right, and then  $\Phi_-(D^b(\mathbf{V}_-))$  to the far right, we obtain

$$D^b(\mathbf{V}) = \langle \mathcal{A}_1, \Phi_1(D^b(\mathbf{V}_-)) \rangle,$$

where

$$\Phi_1(D^b(\mathbf{V}_-)) := R_{\langle O_{n-1,n-1}, \dots, O_{2n-3,2n-3} \rangle} \circ \Phi_-$$

and  $\mathcal{A}_1$  is given by

$$\begin{array}{ccccccc} O_{n-1,0} & O_{n,0} & & & & & \\ O_{n-1,1} & O_{n,1} & O_{n+1,1} & & & & \\ \vdots & \vdots & \vdots & \ddots & & & \\ O_{n-1,n-2} & O_{n,n-2} & O_{n+1,n-2} & \cdots & O_{2n-3,n-2} & O_{2n-2,n-2} & \\ O_{n-1,n-1} & O_{n,n-1} & O_{n+1,n-1} & \cdots & O_{2n-3,n-1} & & \\ & O_{n,n} & O_{n+1,n} & \cdots & O_{2n-3,n} & & \\ & & O_{n+1,n+1} & \cdots & O_{2n-3,n+1} & & \\ & & & \ddots & \vdots & & \\ & & & & O_{2n-3,2n-3} & & \end{array}$$

By mutating  $\Phi_1(D^b(\mathbf{V}_-))$  one step to the left, and then  $O_{2n-2,n-2}$  to the far left, we obtain

$$(4.6) \quad D^b(\mathbf{V}) = \langle \mathcal{A}_2, \Phi_2(D^b(\mathbf{V}_-)) \rangle,$$

where

$$\Phi_2(D^b(\mathbf{V}_-)) := L_{O_{2n-2,n-2}} \circ \Phi_1$$

and  $\mathcal{A}_2$  is given by

$$\begin{array}{ccccccc} O_{n-1,-1} & & & & & & \\ O_{n-1,0} & O_{n,0} & & & & & \\ O_{n-1,1} & O_{n,1} & O_{n+1,1} & & & & \\ \vdots & \vdots & \vdots & \ddots & & & \\ O_{n-1,n-2} & O_{n,n-2} & O_{n+1,n-2} & \cdots & O_{2n-3,n-2} & & \\ O_{n-1,n-1} & O_{n,n-1} & O_{n+1,n-1} & \cdots & O_{2n-3,n-1} & & \\ & O_{n,n} & O_{n+1,n} & \cdots & O_{2n-3,n} & & \\ & & O_{n+1,n+1} & \cdots & O_{2n-3,n+1} & & \\ & & & \ddots & \vdots & & \\ & & & & O_{2n-3,2n-3} & & \end{array}$$

By comparing (4.6) with (4.2) and (4.4), we obtain a derived equivalence

$$\Phi := (\varphi_+)_* \circ ((-) \otimes O_{-(2n-2),0}) \circ \Phi_2 : D^b(\mathbf{V}_-) \xrightarrow{\sim} D^b(\mathbf{V}_+).$$

## 5. STANDARD FLOP

For  $n \geq 1$ , let  $P$  and  $Q$  be the maximal parabolic subgroups of the semisimple Lie group  $G = \mathrm{SL}(V) \times \mathrm{SL}(V^\vee)$  associated with the crossed Dynkin diagram  $\times \cdots \oplus \cdots$  and  $\cdots \oplus \cdots \times$ . The corresponding homogeneous spaces are the projective spaces  $\mathbf{P} = \mathbb{P}V$ ,  $\mathbf{Q} = \mathbb{P}V^\vee$ , and their product  $\mathbf{F} = \mathbb{P}V \times \mathbb{P}V^\vee$ . Since  $\omega_{\mathbf{P}} \cong \mathcal{O}(-(n+1)h)$ ,  $\omega_{\mathbf{Q}} \cong \mathcal{O}(-(n+1)H)$ , and  $\omega_{\mathbf{F}} \cong \mathcal{O}(-(n+1)h - (n+1)H)$ , we have  $\omega_{\mathbf{V}_-} \cong \mathcal{O}_{\mathbf{V}_-}$ ,  $\omega_{\mathbf{V}_+} \cong \mathcal{O}_{\mathbf{V}_+}$ , and  $\omega_{\mathbf{V}} \cong \mathcal{O}(-nh - nH)$ .

**Lemma 5.1.**  $\mathrm{hom}_{\mathcal{O}_{\mathbf{V}}}(\mathcal{O}_{\mathbf{F}}(ih - jH), \mathcal{O}_{\mathbf{F}}) \simeq 0$  for  $1 \leq j \leq n-1$  and  $1 \leq i \leq n-j$ .

*Proof.* We have

$$\mathrm{hom}_{\mathcal{O}_{\mathbf{V}}}(\mathcal{O}_{\mathbf{F}}(ih - jH), \mathcal{O}_{\mathbf{F}}) \simeq \mathbf{h}(\{\mathcal{O}_{\mathbf{F}}(-ih + jH) \rightarrow \mathcal{O}_{\mathbf{F}}(-(i+1)h + (j-1)H)\}),$$

which vanishes for  $1 \leq i \leq n-j \leq n-1$ . □

It follows from [20] that

$$(5.1) \quad D^b(\mathbf{V}) = \langle \iota_* \varpi_-^* D^b(\mathbf{P}), \dots, \iota_* \varpi_-^* D^b(\mathbf{P}) \otimes \mathcal{O}((n-1)(h+H)), \Phi_-(D^b(\mathbf{V}_-)) \rangle$$

and

$$(5.2) \quad D^b(\mathbf{V}) = \langle \iota_* \varpi_+^* D^b(\mathbf{Q}), \dots, \iota_* \varpi_+^* D^b(\mathbf{Q}) \otimes \mathcal{O}((n-1)(h+H)), \Phi_+(D^b(\mathbf{V}_+)) \rangle,$$

where

$$\Phi_- := (-) \otimes \mathcal{O}_{\mathbf{V}}(n(h+H)) \circ \varphi_-^*: D^b(\mathbf{V}_-) \rightarrow D^b(\mathbf{V})$$

and

$$\Phi_+ := (-) \otimes \mathcal{O}_{\mathbf{V}}(n(h+H)) \circ \varphi_+^*: D^b(\mathbf{V}_+) \rightarrow D^b(\mathbf{V}).$$

We write  $\mathcal{O}_{i,j} := \mathcal{O}_{\mathbf{F}}(ih + jH)$ . (4.1) and (5.1) give a semiorthogonal decomposition of the form

$$D^b(\mathbf{V}) = \langle \mathcal{A}_0, \Phi_-(D^b(\mathbf{V}_-)) \rangle$$

where  $\mathcal{A}_0$  is given by

$$(5.3) \quad \begin{array}{ccccccc} \mathcal{O}_{0,0} & \mathcal{O}_{1,0} & \cdots & \mathcal{O}_{n-2,0} & \mathcal{O}_{n-1,0} & \mathcal{O}_{n,0} & \\ & \mathcal{O}_{1,1} & \cdots & \mathcal{O}_{n-2,1} & \mathcal{O}_{n-1,1} & \mathcal{O}_{n,1} & \mathcal{O}_{n+1,1} \\ & & \ddots & \vdots & \vdots & \vdots & \vdots \\ & & & \mathcal{O}_{n-2,n-2} & \mathcal{O}_{n-1,n-2} & \mathcal{O}_{n,n-2} & \mathcal{O}_{n+1,n-2} \cdots \mathcal{O}_{2n-2,n-2} \\ & & & & \mathcal{O}_{n-1,n-1} & \mathcal{O}_{n,n-1} & \mathcal{O}_{n+1,n-1} \cdots \mathcal{O}_{2n-2,n-1} \mathcal{O}_{2n-1,n-1}. \end{array}$$

Note from Lemma 5.1 that there are no morphisms from right to left in (5.3). Since  $\omega_{\mathbf{V}} \cong \mathcal{O}_{\mathbf{V}}(-nh - nH)$ , by mutating first

$$\begin{array}{cccc} \mathcal{O}_{0,0} & \mathcal{O}_{1,0} & \cdots & \mathcal{O}_{n-2,0} \\ & \mathcal{O}_{1,1} & \cdots & \mathcal{O}_{n-2,1} \\ & & \ddots & \vdots \\ & & & \mathcal{O}_{n-2,n-2} \end{array}$$

to the far right, and then  $\Phi_-(D^b(\mathbf{V}_-))$  to the far right, we obtain

$$D^b(\mathbf{V}) = \langle \mathcal{A}_1, \Phi_1(D^b(\mathbf{V}_-)) \rangle,$$

where

$$\Phi_1(D^b(\mathbf{V}_-)) := R_{\langle \mathcal{O}_{n,n}, \dots, \mathcal{O}_{2n-2,2n-2} \rangle} \circ \Phi_-$$

and  $\mathcal{A}_1$  is given by

$$\begin{array}{ccccccc}
O_{n-1,0} & O_{n,0} & & & & & \\
O_{n-1,1} & O_{n,1} & O_{n+1,1} & & & & \\
\vdots & \vdots & \vdots & \ddots & & & \\
O_{n-1,n-1} & O_{n,n-1} & O_{n+1,n-1} & \cdots & O_{2n-2,n-1} & O_{2n-1,n-1} & \\
& O_{n,n} & O_{n+1,n} & \cdots & O_{2n-2,n} & & \\
& & O_{n+1,n+1} & \cdots & O_{2n-2,n+1} & & \\
& & & \ddots & \vdots & & \\
& & & & O_{2n-2,2n-2} & & 
\end{array}$$

By mutating  $\Phi_1(D^b(\mathbf{V}_-))$  one step to the left, and then  $O_{2n-1,n-1}$  to the far left, we obtain

$$(5.4) \quad D^b(\mathbf{V}) = \langle \mathcal{A}_2, \Phi_2(D^b(\mathbf{V}_-)) \rangle,$$

where

$$\Phi_2(D^b(\mathbf{V}_-)) := L_{O_{2n-1,n-1}} \circ \Phi_1$$

and  $\mathcal{A}_2$  is given by

$$\begin{array}{ccccccc}
O_{n-1,-1} & & & & & & \\
O_{n-1,0} & O_{n,0} & & & & & \\
O_{n-1,1} & O_{n,1} & O_{n+1,1} & & & & \\
\vdots & \vdots & \vdots & \ddots & & & \\
O_{n-1,n-1} & O_{n,n-1} & O_{n+1,n-1} & \cdots & O_{2n-2,n-1} & & \\
& O_{n,n} & O_{n+1,n} & \cdots & O_{2n-2,n} & & \\
& & O_{n+1,n+1} & \cdots & O_{2n-2,n+1} & & \\
& & & \ddots & \vdots & & \\
& & & & O_{2n-2,2n-2} & & 
\end{array}$$

By comparing (5.4) with (4.2) and (5.2), we obtain a derived equivalence

$$\Phi := (\varphi_+)_* \circ ((-) \otimes O_{-(2n-1),0}) \circ \Phi_2 : D^b(\mathbf{V}_-) \xrightarrow{\sim} D^b(\mathbf{V}_+).$$

*Remark 5.1.* The way of presenting our proof in Section 4 and 5 is called chess game by some authors [12, 23].

## REFERENCES

- [1] Addington, N., Donovan, W., Meachan, C.: Mukai flops and  $\mathbb{P}$ -twists. *J. Reine. Angew. Math.* 748, 227-240 (2019)
- [2] Ballard, M., Favero, D., Katzarkov, L.: Variation of geometric invariant theory quotients and derived categories, *J. Reine. Angew. Math.* 746, 235-304 (2019)
- [3] Beilinson, A.A.: Coherent sheaves on  $\mathbb{P}^n$  and problems in linear algebra. *Funct. Anal. Appl.* + 12, 214-216 (1978)
- [4] Bondal, A., Orlov, D.: Derived categories of coherent sheaves. *Proceedings of the International Congress of Mathematicians. Vol. II (Beijing, 2002)*, Higher Ed. Press, Beijing, pp. 47-56 (2002)
- [5] Bondal, A., Orlov, D.: *Semiorthogonal decomposition for algebraic varieties*, arXiv:alg-geom/9506012 (1995)
- [6] Halpern-Leistner, D.: The derived category of a GIT quotient. *J. Am. Math. Soc.* 28(3), 871-912 (2015)
- [7] Hara, W.: Non-commutative crepant resolution of minimal nilpotent orbit closure of type A and Mukai flops. *Adv. Math.* 318, 355-410 (2017)
- [8] Hara, W.: *On derived equivalence for Abuef flop: mutation of non-commutative crepant resolutions and spherical twists*, arXiv:1706.04417 (2018)
- [9] Hartshorne, R.: *Algebraic geometry*. Springer-Verlag (1977)

- [10] Herbst, M., Hori, K., Page, D.: *Phases of  $N=2$  theories in  $1+1$  dimensions with boundary*, arXiv:0803.2045 (2008)
- [11] Huybrechts, D., Thomas, R.: P-objects and autoequivalences of derived categories. *Math. Res. Lett.* 13(1), 87-98 (2006)
- [12] Jiang, Q., Leung, C.: *Derived category of projectivization and flops*, arXiv:1811.12525 (2018)
- [13] Kanemitsu, A.: *Mukai pairs and simple  $K$ -equivalence*, arXiv:1812.05392 (2018)
- [14] Kapranov, M.: On the derived categories of coherent sheaves on some homogenous spaces. *Invent. Math.* 92(3), 479-508 (1988)
- [15] Kapustka, M., Rampazzo, M.: Torelli problem for Calabi-Yau threefolds with GLSM description, *Commun. Num. Theor. Phys.* 13(4), 725-761 (2019).
- [16] Kawamata, Y.:  $D$ -equivalence and  $K$ -equivalence. *J. Differ. Geom.* 61(1), 147-171 (2002)
- [17] Kuznetsov, A.: Exceptional collections for Grassmannians of isotropic lines. *Proc. Lond. Math. Soc.* 97(1), 155-182 (2008)
- [18] Kuznetsov, A.: Derived equivalence of Ito–Miura–Okawa–Ueda Calabi–Yau 3-folds, *J. Math. Soc. Jpn.* 70(3), 1007-1013 (2018)
- [19] Namikawa, Y.: Mukai flops and derived categories, *J. Reine. Angew. Math.* 560, 65-76 (2003)
- [20] Orlov, D.: Projective bundles, monoidal transformations, and derived categories of coherent sheaves. *Izv. Ross. Akad. Nauk Ser. Mat.* 56(4), 852-862 (1992)
- [21] Segal, E.: Equivalence between GIT quotients of Landau–Ginzburg B-models. *Commun. Math. Phys.* 304(2), 411-432 (2011)
- [22] Segal, E.: A new 5-fold flop and derived equivalence. *Bull. Lond. Math. Soc.* 48(3), 533-538 (2016)
- [23] Thomas, R.: Notes on HPD. *Algebraic Geometry: Salt Lake City 2015*. AMS (2018)
- [24] Ueda, K.: G2-Grassmannians and derived equivalences. *Manuscripta Math.* 159(3-4), 549-559 (2019)

GRADUATE SCHOOL OF MATHEMATICAL SCIENCES, THE UNIVERSITY OF TOKYO, 3-8-1 KOMABA, MEGURO-KU, TOKYO, 153-8914, JAPAN.

*Email address:* morimura@ms.u-tokyo.ac.jp, hmorimur@sissa.it

*Current address:* SISSA, via Bonomea 265, 34136 Trieste, Italy.