Growth of Sobolev norms in linear Schrödinger equations as a dispersive phenomenon

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Abstract

In this paper we consider linear, time dependent Schrödinger equations of the form $\mathrm{i}\partial_t\psi = K_0\psi + V(t)\psi$, where K_0 is a strictly positive selfadjoint operator with discrete spectrum and constant spectral gaps, and $V(t)$ a smooth in time periodic potential. We give sufficient conditions on $V(t)$ ensuring that $K_0 + V(t)$ generates unbounded orbits. The main condition is that the resonant average of $V(t)$, namely the average with respect to the flow of K_0, has a nonempty absolutely continuous spectrum and fulfills a Mourre estimate. These conditions are stable under perturbations. The proof combines pseudodifferential normal form with dispersive estimates in the form of local energy decay.
We apply our abstract construction to the Harmonic oscillator on $\mathbb{R}$ and to the half-wave equation on $\mathbb{T}$; in each case, we provide large classes of potentials which are transporters.

1 Introduction

We consider the abstract linear Schrödinger equation

$$\mathrm{i}\partial_t\psi = K_0\psi + V(t)\psi \tag{1.1}$$

on a scale of Hilbert spaces $\mathcal{H}^r$; here $V(t)$ is a smooth in time 2π-periodic potential and K_0 a selfadjoint, strictly positive operator with compact resolvent, pure point spectrum and constant spectral gaps. We prove some abstract results ensuring, $\forall r > 0$, the existence of solutions $\psi(t)$ whose $\mathcal{H}^r$-norms grow polynomially fast,

$$\|\psi(t)\|_r \geq C_r \langle t\rangle^r, \quad \forall t \gg 1 \ ,$$

whereas their $\mathcal{H}^0$-norms are constant for all times, $\|\psi(t)\|_0 = \|\psi(0)\|_0$ $\forall t$. Here $\langle t\rangle := \sqrt{1+t^2}$. These solutions therefore exhibit weak turbulent behavior in the form of energy cascade towards high frequencies.
We apply our abstract results to two models: the Harmonic oscillator on $\mathbb{R}$ and the half-wave equation on $\mathbb{T}$. In both cases we exhibit large classes of potentials $V(t)$, bounded, smooth and periodic in time, so that the Hamiltonian $K_0 + V(t)$ generates unbounded orbits.

The phenomenon is purely perturbative: for $V = 0$ each norm of each solution is constant for all times. So the central question is the existence of potentials able to transport energy to high-frequencies; we formalize this notion in the following definition:

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Definition 1.1. *We shall say that* $V(t)$ *is a* transporter *if there exists a solution* $\psi(t) \in \mathcal{H}^r$, $r > 0$, *of* (1) *with unbounded growth of norm, i.e.*

$$\limsup_{t\to\infty} \|\psi(t)\|_r = \infty.$$

If this happens for every *nonzero solution we shall say that* $V(t)$ *is a* universal transporter.

Starting with the pioneering work of Bourgain [9], in the last few years there have been several efforts to construct both transporters [18, 23, 62] and universal transporters [6, 50] for different types of Schrödinger equations. All these papers provide explicit examples of potentials, constructed ad hoc for the problem at hand.

The novelty of our result is that we identify sufficient, explicit and robust conditions ensuring $V(t)$ to be a transporter. Precisely, the *resonant average* of $V(t)$, defined as

$$\langle V\rangle := \frac{1}{2\pi}\int_0^{2\pi} e^{\mathrm{i}sK_0}\, V(s)\, e^{-\mathrm{i}sK_0}\, \mathrm{d}s \tag{1.2}$$

must have nontrivial absolutely continuous spectrum in an interval, over which it has to fulfill a Mourre estimate – see (1.1) below (actually we also require that both K_0 and $V(t)$ belong to some abstract graded algebra of pseudodifferential operators, as in [5]).
Our main results prove that any $V(t)$ fulfilling these conditions is a transporter, and so is *any* of its sufficiently small, bounded perturbations, see Theorem 1.8 and 1.9. This shows a sort of "stability of instability", which we believe is a new phenomenon.

Another novelty of the paper is that the mechanism ensuring transport of energy at high frequencies is a dispersive phenomenon in the energy space. Indeed, as we will show, equation (1) is well approximated by the equation $\mathrm{i}\partial_t\psi = \langle V\rangle\psi$ which, under the previous assumptions on $\langle V\rangle$, admits solutions dispersing in the energy space as

$$\|K_0^{-k} e^{-\mathrm{i}t\langle V\rangle} P_c\phi\|_0 \lesssim \langle t\rangle^{-k}\|K_0^k\phi\|_0\ , \quad \forall t\in\mathbb{R}, \tag{1.3}$$

where P_c is a projector on the absolutely continuous spectral space of $\langle V\rangle$. In particular, the Schrödinger flow of $\langle V\rangle$ forces energy to leave any compact set of the frequency space and flow towards infinity, provoking energy cascade. This is the Fourier analogous of the classical mechanism of transport of spatial mass to infinity for Schrödinger equations on euclidean spaces, which goes back to the works of Rauch [55] and Jensen-Kato [42].

The fact that Mourre estimates imply dispersive estimates as above has origin from the work of Sigal-Soffer in quantum scattering theory [59] and it has been extended by many authors (see e.g. [60, 27, 43, 41, 29, 2]), see also the recent results [13, 12, 21].

Previous literature. As we already mentioned, the first result is due to Bourgain [9], who constructed a transporter for the Schrödinger equation on the torus; in this case $V(t)$ is a bounded real analytic function. Delort [18] constructs a transporter for the harmonic oscillator on $\mathbb{R}$, which is a time 2π-periodic pseudodifferential operator of order zero. In [6] we proved that $ax\sin(t)$, $a > 0$, is a universal transporter for the harmonic oscillator on $\mathbb{R}$; in this case the potential is an unbounded operator. In [50] we constructed universal transporters for the abstract equation (1), and applied the result to the harmonic oscillator on $\mathbb{R}$, the half-wave equation on $\mathbb{T}$ and on a Zoll manifold; in all cases the universal transporters are time periodic pseudodifferential operators of order 0. Recently Faou-Raphael [23] constructed a transporter for the harmonic oscillator on $\mathbb{R}$ which is a time dependent function (and not a pseudodifferential operator), and Thomann [62] has constructed a transporter for the harmonic oscillator on the Bargman-Fock space. Liang, Zhao and Zhou [47] and Luo, Liang and Zhao [48] construct transporters for the Harmonic oscillator which are the quantization of polynomial symbols of order at most 2 and are quasi-periodic in time. Recently, we have exploited the results of the present paper to contruct *generic* transporters for the Harmonic oscillator on $\mathbb{R}$ [51]. Finally we recall the long-time growth result [35] for the semiclassical anharmonic oscillator on $\mathbb{R}^d$.

Before closing this introduction, we mention that constructing solutions with unbounded orbits in *nonlinear* Schrödinger-like equations is a big challenge. After the seminal works by Kuksin [45, 46], the breakthrough result by Colliander-Keel-Staffilani-Takaoka-Tao [14] constructed long time unstable orbits for the nonlinear Schrödinger equation on $\mathbb{T}^2$. The methods of [14] have been refined and extended in [32, 33, 36, 31, 30, 28]. However truly unbounded orbits have been constructed only by Gérard-Grellier for the cubic Szegő equation on $\mathbb{T}$ [24, 25], Hani-Pausader-Tzvetkov-Visciglia the cubic NLS on $\mathbb{R} \times \mathbb{T}^2$ [34] and recently by Gérard-Lenzmann for the Calogero-Moser derivative NLS [26].

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1.1 The abstract result

We start with a Hilbert space $\mathcal{H}$, endowed with the scalar product $\langle \cdot, \cdot \rangle$, and a reference operator K_0, which we assume to be selfadjoint, positive, namely such that

$$\langle \psi; K_0 \psi \rangle \geq c_{K_0} \|\psi\|^2 \ , \quad \forall \psi \in D(K_0^{1/2}) \ , \quad c_{K_0} > 0 \ ,$$

and with compact resolvent.
We define as usual a scale of Hilbert spaces by $\mathcal{H}^r := D(K_0^r)$ (the domain of the operator K_0^r) if $r \geq 0$, and $\mathcal{H}^r = (\mathcal{H}^{-r})'$ (the dual space) if $r < 0$. Finally we denote by $\mathcal{H}^{-\infty} = \bigcup_{r \in \mathbb{R}} \mathcal{H}^r$ and $\mathcal{H}^{+\infty} = \bigcap_{r \in \mathbb{R}} \mathcal{H}^r$. We endow $\mathcal{H}^r$ with the natural norm $\|\psi\|_r := \|K_0^r \psi\|_0$, where $\| \cdot \|_0$ is the norm of $\mathcal{H}^0 \equiv \mathcal{H}$. Notice that for any $m \in \mathbb{R}$, $\mathcal{H}^{+\infty}$ is a dense linear subspace of $\mathcal{H}^m$ (this is a consequence of the spectral decomposition of K_0).

Remark 1.2. By the very definition of $\mathcal{H}^r$, the unperturbed flow $e^{-\mathrm{i}tK_0}$ preserves each norm, $\|e^{-\mathrm{i}tK_0}\psi\|_r = \|\psi\|_r \ \forall t \in \mathbb{R}$. Consequently, every orbit of equation (1) with $V(t) = 0$ is bounded.

Following [5], we introduce now a graded algebra $\mathcal{A}$ of operators which mimic some fundamental properties of different classes of pseudodifferential operators. For $m \in \mathbb{R}$ let $\mathcal{A}_m$ be a linear subspace of $\bigcap_{s \in \mathbb{R}} \mathcal{L}(\mathcal{H}^s, \mathcal{H}^{s-m})$ and define $\mathcal{A} := \bigcup_{m \in \mathbb{R}} \mathcal{A}_m$. We notice that the space $\bigcap_{s \in \mathbb{R}} \mathcal{L}(\mathcal{H}^s, \mathcal{H}^{s-m})$ is a Fréchet space equipped with the semi-norms: $\|A\|_{m,s} := \|A\|_{\mathcal{L}(\mathcal{H}^s, \mathcal{H}^{s-m})}$.

We shall need to control the smoothing properties of the operators in the scale $\{\mathcal{H}^r\}_{r \in \mathbb{R}}$. If $A \in \mathcal{A}_m$ then A is more and more smoothing if $m \to -\infty$ and the opposite as $m \to +\infty$. We will say that A is of *order* m if $A \in \mathcal{A}_m$.

Definition 1.3. *We say that $S \in \mathcal{L}(\mathcal{H}^{+\infty}, \mathcal{H}^{-\infty})$ is N-smoothing if $\forall \kappa \in \mathbb{R}$, it can be extended to an operator in $\mathcal{L}(\mathcal{H}^\kappa, \mathcal{H}^{\kappa+N})$. When this is true for every $N \geq 0$, we say that S is a smoothing operator.*

We shall also use the following notations. For $\Omega \subseteq \mathbb{R}^d$ and $\mathcal{F}$ a Fréchet space, we will denote by $C_b^m(\Omega, \mathcal{F})$ the space of C^m maps $f : \Omega \ni x \mapsto f(x) \in \mathcal{F}$ such that, for every seminorm $\| \cdot \|_j$ of $\mathcal{F}$, one has

$$\sup_{x \in \Omega} \|\partial_x^\alpha f(x)\|_j < +\infty \ , \quad \forall \alpha \in \mathbb{N}^d \ : \ |\alpha| \leq m \ . \tag{1.4}$$

If (1.1) is true $\forall m$, we say $f \in C_b^\infty(\Omega, \mathcal{F})$. Similarly we denote by $C^\infty(\mathbb{T}, \mathcal{F})$ the space of smooth maps from the torus $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ to the Fréchet space $\mathcal{F}$. We denote by $C_c^\infty(\mathbb{R}^d, \mathbb{R}_{\geq 0})$ the set of smooth functions with compact support from $\mathbb{R}^d$ to $\mathbb{R}_{\geq 0}$ (hence non-negative). Given two operators $\mathsf{A}, \mathsf{B} \in \mathcal{L}(\mathcal{H})$, we write $\mathsf{A} \leq \mathsf{B}$ with the meaning $\langle \mathsf{A}\varphi, \varphi\rangle \leq \langle \mathsf{B}\varphi, \varphi\rangle \ \ \forall \varphi \in \mathcal{H}$.

The first set of assumptions concerns the properties of $\mathcal{A}_m$:

Assumption I: Pseudodifferential algebra

(i) For each $m \in \mathbb{R}$, $K_0^m \in \mathcal{A}_m$; in particular K_0 is an operator of order one.

(ii) For each $m \in \mathbb{R}$, $\mathcal{A}_m$ is a Fréchet space for a family of filtering semi-norms $\{\wp_j^m\}_{j\geq 0}$ such that the embedding $\mathcal{A}_m \hookrightarrow \bigcap_{s\in\mathbb{R}} \mathcal{L}(\mathcal{H}^s, \mathcal{H}^{s-m})$ is continuous[1].
If $m' \leq m$ then $\mathcal{A}_{m'} \subseteq \mathcal{A}_m$ with a continuous embedding.

(iii) $\mathcal{A}$ is a graded algebra, i.e. $\forall m, n \in \mathbb{R}$: if $A \in \mathcal{A}_m$ and $B \in \mathcal{A}_n$ then $AB \in \mathcal{A}_{m+n}$ and the map $(A, B) \mapsto AB$ is continuous from $\mathcal{A}_m \times \mathcal{A}_n$ into $\mathcal{A}_{m+n}$.

(iv) $\mathcal{A}$ is a graded Lie-algebra[2] : if $A \in \mathcal{A}_m$ and $B \in \mathcal{A}_n$ then the commutator $[A, B] \in \mathcal{A}_{m+n-1}$ and the map $(A, B) \mapsto [A, B]$ is continuous from $\mathcal{A}_m \times \mathcal{A}_n$ into $\mathcal{A}_{m+n-1}$.

(v) $\mathcal{A}$ is closed under perturbation by smoothing operators in the following sense: let A be a linear map: $\mathcal{H}^{+\infty} \to \mathcal{H}^{-\infty}$. If there exists $m \in \mathbb{R}$ such that for every $N > 0$ we have a decomposition $A = A^{(N)} + S^{(N)}$, with $A^{(N)} \in \mathcal{A}_m$ and $S^{(N)}$ is N-smoothing, then $A \in \mathcal{A}_m$.

(vi) If $A \in \mathcal{A}_m$ then also the adjoint operator $A^* \in \mathcal{A}_m$. The duality here is defined by the scalar product $\langle \cdot, \cdot \rangle$ of $\mathcal{H} = \mathcal{H}^0$. The adjoint A^* is defined by $\langle u, Av\rangle = \langle A^* u, v\rangle$ for $u, v \in \mathcal{H}^\infty$ and extended by continuity.

It is well known that classes of pseudodifferential operators satisfy these properties, provided one chooses for K_0 a suitable operator of the right order (see e.g. [39]).

Remark 1.4. One has that $\forall A \in \mathcal{A}_m$, $\forall B \in \mathcal{A}_n$

$$\forall m, s \quad \exists N \ s.t. \ \|A\|_{m,s} \leq C_1 \, \wp_N^m(A) \ , \tag{1.5}$$

$$\forall m, n, j \quad \exists N \ s.t. \ \wp_j^{m+n}(AB) \leq C_2 \, \wp_N^m(A) \, \wp_N^n(B) \ ,$$

$$\forall m, n, j \quad \exists N \ s.t. \ \wp_j^{m+n-1}([A, B]) \leq C_3 \, \wp_N^m(A) \, \wp_N^n(B) \ , \tag{1.6}$$

for some positive constants $C_1(s, m)$, $C_2(m, n, j)$, $C_3(m, n, j)$.

Remark 1.5. Any $A \in \mathcal{A}_m$ with $m < 0$ is a compact operator on $\mathcal{H}$.
Indeed write $A = AK_0^{-m} K_0^m$. Then $AK_0^{-m} \in \mathcal{A}_0$ is a bounded operator on $\mathcal{H}$ (Assumption I (i)–(iii)), whereas $K_0^m \equiv (K_0^{-1})^{-m}$ is compact on $\mathcal{H}$, as K_0^{-1} is a compact operator by assumption.

The second set of assumptions concerns the operator K_0, its spectral structure and an Egorov-like property, also well known for pseudo-differential operators.

Assumption II: Properties of K_0

(i) The operator K_0 has purely discrete spectrum fulfilling

$$\mathrm{spec}(K_0) \subseteq \mathbb{N} + \lambda$$

for some $\lambda \geq 0$.

[1] A family of seminorms $\{\wp_j^m\}_{j\geq 0}$ is called filtering if for any $j_1, j_2 \geq 0$ there exist $k \geq 0$ and $c_1, c_2 > 0$ such that the two inequalities $\wp_{j_1}^m(A) \leq c_1 \wp_k^m(A)$ and $\wp_{j_2}^m(A) \leq c_2 \wp_k^m(A)$ hold for any $A \in \mathcal{A}_m$.

[2] This property will impose the choice of the semi-norms $\{\wp_j^m\}_{j\geq 1}$. We will see in the examples that the natural choice $(\|\cdot\|_{m,s})_{s\geq 0}$ has to be refined.

(ii) For any $m \in \mathbb{R}$ and $A \in \mathcal{A}_m$, the map defined on $\mathbb{R}$ by $\tau \mapsto A(\tau) := \mathrm{e}^{\mathrm{i}\tau K_0} A\, \mathrm{e}^{-\mathrm{i}\tau K_0}$ belongs to $C_b^\infty(\mathbb{R}, \mathcal{A}_m)$ and one has

$$\forall j \quad \exists N \ s.t. \ \sup_{\tau \in \mathbb{R}} \wp_j^m(A(\tau)) \leq C_4\, \wp_N^m(A)$$

for some positive constant $C_4(m, j)$.

Remark 1.6. Assumption II (i) guarantees that $e^{\mathrm{i}2\pi K_0} = e^{\mathrm{i}2\pi\lambda}$. As a consequence, for any operator V, the map $\tau \mapsto e^{\mathrm{i}\tau K_0} V e^{-\mathrm{i}\tau K_0}$ is 2π-periodic.

The last set of assumptions concerns the resonant average $\langle V \rangle$ of the potential $V(t)$ (see (1)) and its spectrum $\sigma(\langle V \rangle)$. Note that if $V(t)$ is selfadjoint $\forall t$, so is $\langle V \rangle$.

Assumption III: Properties of the potential $V(t)$
The operator $V \in C^\infty(\mathbb{T}, \mathcal{A}_0)$, $V(t)$ selfadjoint $\forall t$, and its resonant average $\langle V \rangle$ fulfills:

(i) There exists an interval $I_0 \subset \mathbb{R}$ such that $|\sigma(\langle V \rangle) \cap I_0| > 0$; here $|\cdot|$ denotes the Lebesgue measure.

(ii) *Mourre estimate* over I_0: there exist a selfadjoint operator $A \in \mathcal{A}_1$ and a function $g_{I_0} \in C_c^\infty(\mathbb{R}, \mathbb{R}_{\geq 0})$ with $g_{I_0} \equiv 1$ on I_0 such that

$$g_{I_0}(\langle V \rangle)\, \mathrm{i}[\langle V \rangle, A]\, g_{I_0}(\langle V \rangle) \geq \theta\, g_{I_0}(\langle V \rangle)^2 + K \tag{1.7}$$

for some $\theta > 0$ and K a selfadjoint compact operator.

The operator $g_{I_0}(\langle V \rangle)$ above is defined via functional calculus, see Appendix B.
Following the literature, we shall say that $\langle V \rangle$ is *conjugated to A over I_0*.

Remark 1.7. By Mourre theory [53] $\langle V \rangle$ has, in the interval I_0, a nontrivial absolutely continuous spectrum with finitely many eigenvalues of finite multiplicity and no singular continuous spectrum. In general one cannot exclude the existence of embedded eigenvalues in the absolutely continuous spectrum.[3]

We are ready to state our main results. The first one says that, under the set of assumptions above, $V(t)$ is a transporter in the sense of Definition 1.1:

Theorem 1.8. *Assume that $\mathcal{A}$ is a graded algebra as in* Assumption I, *and that K_0 and $V(t) \in C^\infty(\mathbb{T}, \mathcal{A}_0)$ satisfy* Assumptions II *and* III. *Then $V(t)$ is a transporter for the equation*

$$\mathrm{i}\partial_t \psi = (K_0 + V(t))\psi \ . \tag{1.8}$$

More precisely, for any $r > 0$ there exist a solution $\psi(t)$ of (1.8) *in $\mathcal{H}^r$ and constants $C, T > 0$ such that*

$$\|\psi(t)\|_r \geq C \langle t \rangle^r, \quad \forall t \geq T \ . \tag{1.9}$$

[3] For example consider $H \in \mathcal{L}(L^2(\mathbb{T}))$ given by

$$(Hu)(x) := \cos(x)u(x) + \delta(1 - \delta^{-1}\cos(x))\frac{1}{2\pi}\int_{\mathbb{T}} u(x)\big(1 - \delta^{-1}\cos(x)\big)\,\mathrm{d}x \ , \quad \delta \in \Big(-\frac{1}{2}, \frac{1}{2}\Big) \setminus \{0\} \ .$$

H is selfadjoint, a 1-rank perturbation of the multiplication operator by $\cos(x)$, it has absolutely continuous spectrum in the interval $(-1, 1)$, and δ is an embedded eigenvalue with eigenvector $u(x) \equiv 1$. Moreover H is conjugated to $\sin(x)\frac{\partial_x}{\mathrm{i}} + \frac{\partial_x}{\mathrm{i}}\sin(x)$ over $[-\frac{1}{2}, \frac{1}{2}]$.

We also prove a stronger result: namely not only $V(t)$ is a transporter, but also *any* operator sufficiently close to it (in the $\mathcal{A}_0$-topology). Here the precise statement:

Theorem 1.9. *With the same assumptions of Theorem 1.8, there exist $\epsilon_0 > 0$ and $\mathtt{M} \in \mathbb{N}$ such that for any $W \in C^\infty(\mathbb{T}, \mathcal{A}_0)$, $W(t)$ selfadjoint $\forall t$, fulfilling*

$$\sup_{t \in \mathbb{T}} \wp^0_{\mathtt{M}}(W(t)) \leq \epsilon_0, \tag{1.10}$$

then $V(t) + W(t)$ is a transporter for the equation

$$\mathrm{i}\partial_t \psi = \big(K_0 + V(t) + W(t)\big)\psi \ . \tag{1.11}$$

More precisely, for any $r > 0$ there exist a solution $\psi(t)$ in $\mathcal{H}^r$ of (1.9) *and constants $C, T > 0$ such that*

$$\|\psi(t)\|_r \geq C\langle t\rangle^r, \quad \forall t \geq T \ . \tag{1.12}$$

Let us comment the above results.

1. The growth of Sobolev norms of Theorem 1.8 is truly an energy cascade phenomenon; indeed the $\mathcal{H}^0$-norm of any solution of (1.8) is preserved for all times, $\|\psi(t)\|_0 = \|\psi(0)\|_0$, $\forall t \in \mathbb{R}$. This is due to the selfadjointness of $K_0 + V(t)$ (the same is true for solutions of (1.9)).

2. Estimates (1.8), (1.9) provide optimal lower bounds for the speed of growth of the Sobolev norms. Indeed we proved [49] that, under the assumptions above[4], *any* solution of (1.8) or (1.9) fulfills the upper bounds

$$\forall r > 0 \quad \exists\, \widetilde{C}_r > 0\colon \quad \|\psi(t)\|_r \leq \widetilde{C}_r \langle t\rangle^r \, \|\psi(0)\|_r.$$

   Thus, Theorems 1.8, 1.9 construct unbounded solutions with optimal growth.

3. Theorem 1.9 proves robustness of certain type of transporters under small pseudodifferential perturbations. This shows a sort of "stability of instability", which, up to our knowledge, is new in this context.

4. Actually there are infinitely many distinct solutions undergoing growth of Sobolev norms. Their initial data are constructed in a unique way starting from functions belonging to the absolutely continuous spectral subspace of the operator $\langle V\rangle$. We describe such initial data in Corollary 2.16.

5. Energy cascade is a resonant phenomenon; here it happens because $V(t)$ oscillates at frequency $\omega = 1$ which resonates with the spectral gaps of K_0. In [5] we proved that if $V(t) \equiv \mathsf{V}(\omega t)$ is quasiperiodic in time with a frequency vector $\omega \in \mathbb{R}^n$ fulfilling the non-resonant condition

$$\exists \gamma, \tau > 0\colon \quad |\ell + \omega \cdot k| \geq \frac{\gamma}{\langle k\rangle^\tau} \quad \forall \ell, k \in \mathbb{Z} \times \mathbb{Z}^n \setminus \{0\}$$

   (which is violated if $V(t)$ is 2π-periodic) then the Sobolev norms grow at most as $\langle t\rangle^\epsilon$ $\forall \epsilon > 0$. The $\langle t\rangle^\epsilon$-speed of growth is also known for systems with increasing [54, 49, 5] or shrinking [22, 52] spectral gaps and for Schrödinger equation on $\mathbb{T}^d$ with bounded [10, 17, 8] and even unbounded [7] potentials.

[4] in particular the fact that $[K_0, V(t)]$ and $[K_0, V(t) + W(t)]$ are uniformly (in t) bounded operators on the scale $\mathcal{H}^r$

6. In concrete models one can typically prove that if $V(t)$ is sufficiently small in size and oscillates in time with a strongly non resonant frequency ω (typically belonging to some Cantor set of large measure), then all solutions have uniformly in time bounded Sobolev norms. Therefore the stability/instability of the system depends only on the resonance property of the frequency ω. We mention just the recent results [4, 6] which deal with the harmonic oscillator (as we consider it in the applications) and refer to those papers for a complete bibliography.

7. The most delicate assumption to verify is (1.1). In the applications, one can try to construct an escape function for the principal symbol $\langle v\rangle$ of $\langle V\rangle$. This means to find a symbol $a(x,\xi)$ of order 1 and number $\lambda \in \mathbb{R}$ such that the Poisson bracket $\{\langle v\rangle, a\}$ is strictly positive around the energy level λ:

$$\exists c > 0: \quad \{\langle v\rangle, a\} \geq c \qquad \text{in } \{(x,\xi): \ |\langle v\rangle(x,\xi) - \lambda| \leq \delta\} \ .$$

   Then symbolic calculus and sharp Gårding inequality imply that (1.1) holds in the interval $I = (\lambda - \delta, \lambda + \delta)$; see [13] Section 6.2 for details.

Now we briefly describe the main ideas of the proof. The first step is to put system (1) into its resonant pseudodifferential normal form. This is the resonant variant of the normal form developed in [5] for non-resonant systems (and essentially an abstract version of the normal form of Delort [18]); it allows, $\forall N \in \mathbb{N}$, to conjugate equation (1) to the equation

$$\mathrm{i}\partial_t \phi = \big(\langle V\rangle + T_N + R_N(t)\big)\phi \tag{1.13}$$

where T_N is a time independent selfadjoint compact operator and $R_N(t)$ is N-smoothing.

Then we analyze the dynamics of the truncated equation

$$\mathrm{i}\partial_t \phi = \big(\langle V\rangle + T_N\big)\phi \tag{1.14}$$

and prove that it has solutions with decaying negative Sobolev norms and so, by duality, growing positive Sobolev norms. This is the core of the proof; after this step, it is not difficult to construct a solution of the complete equation (1.1) exhibiting energy cascade, exploiting that $R_N(t)$ is regularizing. So let us concentrate on (1.1). The goal is to prove a dispersive estimate of the form (1) with $\langle V\rangle$ replaced by $\langle V\rangle + T_N$. This is delicate because the absolutely continuous spectrum of $\langle V\rangle$ (which exists by Assumption III (i)) could be completely destroyed by adding T_N: a celebrated theorem by Weyl-von Neumann ensures that *any* selfadjoint operator (in a separable Hilbert space) can be perturbed by a compact selfadjoint operator so that its spectrum becomes pure point (see e.g. [44, pag. 525]). This is exactly the situation we want to avoid, as pure point spectrum prevents dispersive estimates. To get around this, we exploit that Mourre estimates are stable under pseudodifferential perturbations. This allows us to prove that $\langle V\rangle + T_N$ fulfills Mourre estimates and thus a dispersive estimate as (1).

We also stress that fulfilling a Mourre estimate seems to be a quite general condition, and in the applications we exhibit large classes of operators which are transporters. For example, for the half wave equation we prove that any operator of the form $\cos(mt)v(x)$ with $v \in C^\infty(\mathbb{T},\mathbb{R})$ and $m \in \mathbb{Z}$ is a transporter provided the m-th Fourier coefficient of $v(x)$ is not zero.

2 Proof of the abstract result

Clearly Theorem 1.9 is stronger than Theorem 1.8 and it includes it in the special case $W(t) \equiv 0$, so we shall only prove Theorem 1.9. The proof is divided in three steps; in the first one we put system (1.9) in its resonant pseudodifferential normal form. In the second one we analyze the dynamics of the effective Hamiltonian and prove the existence of solutions with decaying negative Sobolev norms. The final step is to construct a solution of the complete equation exhibiting growth of Sobolev norms.

2.1 Resonant pseudodifferential normal form

The goal of this section is to put system (1.9) into its resonant pseudodifferential normal form up to an arbitrary N-smoothing operator. In this first step we shall only require Assumptions I and II. It is slightly more convenient to deal with the equation

$$\mathrm{i}\partial_t\psi = \big(K_0 + \mathsf{V}(t)\big)\psi, \qquad \mathsf{V} \in C^\infty(\mathbb{T}, \mathcal{A}_m), \quad m \in \mathbb{R}, \tag{2.1}$$

and then to specify the result for $\mathsf{V}(t) = V(t) + W(t)$ as in (1.9). We define the *averaged operator*

$$\widehat{\mathsf{V}}(t) := \frac{1}{2\pi}\int_0^{2\pi} e^{\mathrm{i}sK_0}\, \mathsf{V}(t+s)\, e^{-\mathrm{i}sK_0}\, \mathrm{d}s \ . \tag{2.2}$$

We shall prove below that $\widehat{\mathsf{V}}(t) \in C^\infty(\mathbb{T}, \mathcal{A}_m)$, see Lemma 2.2.

Proposition 2.1 (Resonant pseudodifferential normal form)**.** *Consider equation* (2.1) *with* $\mathsf{V} \in C^\infty(\mathbb{T}, \mathcal{A}_0)$*,* $\mathsf{V}(t)$ *selfadjoint* $\forall t$*. There exists a sequence* $\{X_j(t)\}_{j\geq 1}$ *of selfadjoint (time-dependent) operators in* $\mathcal{H}$ *with* $X_j \in C^\infty(\mathbb{T}, \mathcal{A}_{1-j})$ *and fulfilling*

$$\forall r \in \mathbb{R},\ \exists c_{r,j}, C_{r,j} > 0: \qquad c_{r,j}\|\varphi\|_r \leq \|e^{\pm\mathrm{i}X_j(t)}\varphi\|_r \leq C_{r,j}\|\varphi\|_r, \qquad \forall t \in \mathbb{R}, \tag{2.3}$$

such that the following holds true. For any $N \geq 1$*, the change of variables*

$$\psi = e^{-\mathrm{i}X_1(t)} \cdots e^{-\mathrm{i}X_N(t)}\varphi \tag{2.4}$$

transforms (2.1) *into the equation*

$$\mathrm{i}\partial_t\varphi = \big(K_0 + Z^{(N)}(t) + \mathsf{V}^{(N)}(t)\big)\varphi \ ; \tag{2.5}$$

here $\mathsf{V}^{(N)} \in C^\infty(\mathbb{T}, \mathcal{A}_{-N})$ *whereas* $Z^{(N)} \in C^\infty(\mathbb{T}, \mathcal{A}_0)$*, it is selfadjoint* $\forall t$*, it fulfills*

$$\mathrm{i}\partial_t Z^{(N)}(t) = [K_0, Z^{(N)}(t)] \tag{2.6}$$

and it has the expansion

$$Z^{(N)}(t) = \widehat{\mathsf{V}}(t) + T^{(N)}(t), \qquad T^{(N)} \in C^\infty(\mathbb{T}, \mathcal{A}_{-1}) \ . \tag{2.7}$$

Here $\widehat{\mathsf{V}}(t)$ *is the averaged operator defined in* (2.1)*.*

In order to prove the proposition we start with some preliminary results. The first regards the properties of the averaged operator $\widehat{\mathsf{V}}(t)$.

Lemma 2.2. *Let* $\mathsf{V} \in C^\infty(\mathbb{T}, \mathcal{A}_m)$*,* $m \in \mathbb{R}$*,* $\mathsf{V}(t)$ *selfadjoint* $\forall t$*. Then the following holds true.*

(i) The averaged operator $\widehat{\mathsf{V}}(t)$ *in* (2.1) *belongs to* $C^\infty(\mathbb{T}, \mathcal{A}_m)$*, it is selfadjoint* $\forall t$*, it commutes with* $\mathrm{i}\partial_t - K_0$*, i.e.* $\mathrm{i}\partial_t\widehat{\mathsf{V}}(t) = [K_0, \widehat{\mathsf{V}}(t)]$ *and*

$$\forall j, \ell \geq 0 \quad \exists\, M \in \mathbb{N},\ C > 0 \ \text{ s.t. } \sup_{t\in\mathbb{T}} \wp_j^m(\partial_t^\ell \widehat{\mathsf{V}}(t)) \leq C \sup_{t\in\mathbb{T}} \wp_M^m(\mathsf{V}(t)) \ . \tag{2.8}$$

(ii) The resonant averaged operator $\langle \mathsf{V} \rangle$*, defined in* (1)*, belongs to* $\mathcal{A}_m$*, it is selfadjoint and*

$$\forall j \geq 0 \quad \exists\, M \in \mathbb{N},\ C > 0 \ \text{ s.t. } \ \wp_j^m(\langle \mathsf{V} \rangle) \leq C \sup_{t \in \mathbb{T}} \ \wp_M^m(\mathsf{V}(t)) \ . \tag{2.9}$$

(iii) One has the chain of identities

$$\widehat{\mathsf{V}}(0) = \langle \mathsf{V} \rangle = e^{\mathrm{i}tK_0}\, \widehat{\mathsf{V}}(t)\, e^{-\mathrm{i}tK_0} = \langle\, \widehat{\mathsf{V}}\, \rangle, \qquad \forall t \in \mathbb{R} \ . \tag{2.10}$$

Proof. (i) The properties $\widehat{\mathsf{V}} \in C^\infty(\mathbb{T}, \mathcal{A}_m)$ and $\widehat{\mathsf{V}}(t)$ selfadjoint $\forall t$ follow from Assumption II and the fact that $\mathsf{V}(t)$ is 2π-periodic in t and selfadjoint $\forall t$. Let us prove it commutes with $\mathrm{i}\partial_t - K_0$. Using

$$\partial_s \left(e^{\mathrm{i}sK_0}\, \mathsf{V}(t+s)\, e^{-\mathrm{i}sK_0}\right) = e^{\mathrm{i}sK_0}\big(\mathrm{i}[K_0, \mathsf{V}(t+s)] + \partial_s \mathsf{V}(t+s)\big)e^{-\mathrm{i}sK_0}$$

and the periodicity of $s \mapsto e^{\mathrm{i}sK_0}\, \mathsf{V}(t+s)\, e^{-\mathrm{i}sK_0}$ (see Remark 1.6), we get

$$\begin{aligned}
\partial_t \widehat{\mathsf{V}}(t) &= \frac{1}{2\pi}\int_0^{2\pi} e^{\mathrm{i}sK_0}\, \partial_t \mathsf{V}(t+s)\, e^{-\mathrm{i}sK_0}\, \mathrm{d}s = \frac{1}{2\pi}\int_0^{2\pi} e^{\mathrm{i}sK_0}\, \partial_s \mathsf{V}(t+s)\, e^{-\mathrm{i}sK_0}\, \mathrm{d}s \\
&= \frac{1}{2\pi\mathrm{i}}\int_0^{2\pi} e^{\mathrm{i}sK_0}\, [K_0, \mathsf{V}(t+s)]\, e^{-\mathrm{i}sK_0}\, \mathrm{d}s = \mathrm{i}^{-1}\, [K_0, \widehat{\mathsf{V}}(t)]
\end{aligned}$$

Estimate (2.2) for $\ell = 0$ follows from Assumption II. For $\ell \geq 1$ we use induction: assume (2.2) is true up to a certain ℓ; using $\partial_t^{\ell+1}\widehat{\mathsf{V}}(t) = -\mathrm{i}\partial_t^\ell [K_0, \widehat{\mathsf{V}}(t)] = -\mathrm{i}[K_0, \partial_t^\ell \widehat{\mathsf{V}}(t)]$, we get $\forall j \in \mathbb{N}$

$$\wp_j^m(\partial_t^{\ell+1}\widehat{\mathsf{V}}(t)) \leq \ \wp_j^m([K_0, \partial_t^\ell \widehat{\mathsf{V}}(t)]) \leq C\wp_{j_1}^m(\partial_t^\ell \widehat{\mathsf{V}}(t)) \leq C\wp_{j_2}^m(\mathsf{V}(t))$$

using also the inductive assumption. This proves (2.2).
(ii) It is clear that $\langle \mathsf{V} \rangle$ is time independent, selfadjoint and in $\mathcal{A}_m$ by Assumption II. Estimate (2.2) follows from Assumption II.
(iii) Clearly $\widehat{\mathsf{V}}(0) = \langle \mathsf{V} \rangle$. Then, as the map $\tau \mapsto e^{\mathrm{i}\tau K_0}\, \mathsf{V}(\tau)\, e^{-\mathrm{i}\tau K_0}$ is 2π-periodic, one has $\forall t \in \mathbb{R}$

$$e^{\mathrm{i}tK_0}\, \widehat{\mathsf{V}}(t)\, e^{-\mathrm{i}tK_0} = \frac{1}{2\pi}\int_0^{2\pi} e^{\mathrm{i}(t+s)K_0}\, \mathsf{V}(t+s)\, e^{-\mathrm{i}(s+t)K_0}\, \mathrm{d}s = \langle \mathsf{V} \rangle \ .$$

Finally, exploiting this last identity, one has $\langle\, \widehat{\mathsf{V}}\, \rangle = \frac{1}{2\pi}\int_0^{2\pi} e^{\mathrm{i}tK_0}\, \widehat{\mathsf{V}}(t)\, e^{-\mathrm{i}tK_0}\mathrm{d}t = \langle \mathsf{V} \rangle$ completing the proof of (2.2). ☐

The second preliminary result regards how to solve the homological equations which appear during the normal form procedure. More precisely we look for a time periodic operator $X(t)$ solving the homological equation

$$\partial_t X(t) + \mathrm{i}[K_0, X(t)] = \mathsf{V}(t) - \widehat{\mathsf{V}}(t), \tag{2.11}$$

where $\widehat{\mathsf{V}}(t)$ is the averaged operator defined in (2.1). This is done in the next lemma.

Lemma 2.3. *Let* $\mathsf{V} \in C^\infty(\mathbb{T}, \mathcal{A}_m)$*,* $m \in \mathbb{R}$*,* $\mathsf{V}(t)$ *selfadjoint* $\forall t$*. The homological equation* (2.1) *has a solution* $X \in C^\infty(\mathbb{T}, \mathcal{A}_m)$ *and* $X(t)$ *is selfadjoint* $\forall t$*.*

Proof. We look for a solution of (2.1) using the method of variation of constants. In particular we take $X(t) = e^{-\mathrm{i}tK_0}\, Y(t)\, e^{\mathrm{i}tK_0}$ for some $Y \in C^\infty(\mathbb{R}, \mathcal{A}_m)$ with $Y(0) = 0$ to be determined. Then X solves (2.1) provided $\partial_t Y(t) = e^{\mathrm{i}tK_0}\, (\mathsf{V}(t) - \widehat{\mathsf{V}}(t))\, e^{-\mathrm{i}tK_0}$, giving

$$Y(t) = \int_0^t e^{\mathrm{i}sK_0}\big(\mathsf{V}(s) - \widehat{\mathsf{V}}(s)\big)\, e^{-\mathrm{i}sK_0}\, \mathrm{d}s.$$

By Lemma 2.2 and Assumption II, $Y \in C^\infty(\mathbb{R}, \mathcal{A}_m)$ and it is selfadjoint $\forall t$. Therefore one gets

$$X(t) = \int_0^t e^{\mathrm{i}(s-t)K_0} \big(\mathsf{V}(s) - \widehat{\mathsf{V}}(s)\big)\, e^{-\mathrm{i}(s-t)K_0}\, \mathrm{d}s.$$

Again $X \in C^\infty(\mathbb{R}, \mathcal{A}_m)$ and it is selfadjoint $\forall t$. We show that $t \mapsto X(t)$ is 2π-periodic. Indeed, using also Remark 1.6, we get

$$\begin{aligned} X(t+2\pi) - X(t) &= e^{-\mathrm{i}tK_0} \int_0^{2\pi} e^{\mathrm{i}sK_0} \big(\mathsf{V}(s) - \widehat{\mathsf{V}}(s)\big)\, e^{-\mathrm{i}sK_0}\, \mathrm{d}s\ e^{\mathrm{i}tK_0} \\ &= 2\pi e^{-\mathrm{i}tK_0} \big(\langle \mathsf{V}\rangle - \langle\, \widehat{\mathsf{V}}\,\rangle\big) e^{\mathrm{i}tK_0} \overset{(2.2)}{=} 0 \end{aligned}$$

proving the claim. □

We are ready to prove Proposition 2.1. During the proof we shall use some results proved in [5] about the flow generated by pseudodifferential operators; we collect them, for the reader's convenience, in Appendix A.

Proof of Proposition 2.1. The proof is inductive on N. Let us start with $N = 1$. We look for a change of variables of the form $\psi = e^{-\mathrm{i}X_1(t)}\varphi$ where $X_1(t) \in C^\infty(\mathbb{T}, \mathcal{A}_0)$ is selfadjoint $\forall t$, to be determined. By Lemma A.1, ψ solves (2.1) iff φ fulfills the Schrödinger equation $\mathrm{i}\partial_t\varphi = H^+(t)\varphi$ with

$$H^+(t) := e^{\mathrm{i}X_1(t)} \big(K_0 + \mathsf{V}(t)\big)\, e^{-\mathrm{i}X_1(t)} - \int_0^1 e^{\mathrm{i}sX_1(t)} \left(\partial_t X_1(t)\right) e^{-\mathrm{i}sX_1(t)}\ \mathrm{d}s\ .$$

Then a commutator expansion, see Lemma A.2, gives

$$H^+(t) = K_0 + \mathrm{i}[X_1(t), K_0] + \mathsf{V}(t) - \partial_t X_1 + \mathsf{V}^{(1)}(t)$$

with $\mathsf{V}^{(1)} \in C^\infty(\mathbb{T}, \mathcal{A}_{-1})$, selfadjoint $\forall t$. By Lemma 2.3, we choose $X_1 \in C^\infty(\mathbb{T}, \mathcal{A}_0)$, selfadjoint $\forall t$, s.t.

$$\mathrm{i}[K_0, X_1(t)] + \partial_t X_1(t) = \mathsf{V}(t) - \widehat{\mathsf{V}}(t)\ ,$$

where $\widehat{\mathsf{V}}(t)$ is the averaged operator in (2.1). With this choice we have

$$H^+(t) = K_0 + Z^{(1)}(t) + \mathsf{V}^{(1)}(t)\ , \quad Z^{(1)}(t) := \widehat{\mathsf{V}}(t)\ .$$

By Lemma 2.2, $Z^{(1)} \in C^\infty(\mathbb{T}, \mathcal{A}_0)$, it is selfadjoint $\forall t$, it commutes with $\mathrm{i}\partial_t - K_0$. The map $e^{-\mathrm{i}X_1(t)}$ fulfills (2.1) thanks to Lemma A.3. This concludes the first step.

The iterative step $N \to N+1$ is proved following the same lines, choosing $X_{N+1} \in C^\infty(\mathbb{T}, \mathcal{A}_{-N})$ solving the homological equation

$$\mathrm{i}[K_0, X_{N+1}(t)] + \partial_t X_{N+1}(t) = \mathsf{V}^{(N)}(t) - \widehat{\mathsf{V}^{(N)}}(t)$$

and adding the remark that $e^{\mathrm{i}X_{N+1}} Z^{(N)} e^{-\mathrm{i}X_{N+1}} - Z^{(N)} \in C^\infty(\mathbb{T}, \mathcal{A}_{-N-1})$. So one puts $Z^{(N+1)}(t) := Z^{(N)}(t) + \widehat{\mathsf{V}^{(N)}}(t)$. Note that $\widehat{\mathsf{V}^{(N)}} \in C^\infty(\mathbb{T}, \mathcal{A}_{-N})$ and it commutes with $\mathrm{i}\partial_t - K_0$, so does $Z^{(N)}$. □

It turns out that property (2.1) implies that $e^{\mathrm{i}tK_0}\, Z^{(N)}(t)\, e^{-\mathrm{i}tK_0}$ is time independent. A consequence of this fact is the following corollary.

Corollary 2.4. *Consider equation* (2.1) *with* $\mathsf{V} \in C^\infty(\mathbb{T}, \mathcal{A}_0)$*,* $\mathsf{V}(t)$ *selfadjoint* $\forall t$*. Fix* $N \in \mathbb{N}$ *arbitrary. There exists a change of coordinates* $\mathcal{U}_N(t)$ *unitary in* $\mathcal{H}$ *and fulfilling*

$$\forall r \geq 0 \quad \exists c_r, C_r > 0 \colon \qquad c_r \|\varphi\|_r \leq \|\mathcal{U}_N(t)^{\pm}\varphi\|_r \leq C_r \|\varphi\|_r, \qquad \forall t \in \mathbb{R}, \tag{2.12}$$

such that $\psi(t)$ *is a solution of* (2.1) *if and only if* $\phi(t) := \mathcal{U}_N(t)\psi(t)$ *solves*

$$\mathrm{i}\partial_t \phi = \big(\langle \mathsf{V} \rangle + T_N + R_N(t)\big)\phi \ ;$$

here $\langle \mathsf{V} \rangle$ *is the resonant average of* V *(see* (1)*),* $T_N \in \mathcal{A}_{-1}$ *is time independent and selfadjoint and* $R_N \in C^\infty(\mathbb{T}, \mathcal{A}_{-N})$*.*

Proof. Fix $N \in \mathbb{N}$ and apply Proposition 2.1 to conjugate equation (2.1) to the form (2.1) via the change of variables (2.1). Then we gauge away K_0 by the change of coordinates $\varphi = e^{-\mathrm{i}tK_0}\phi$, getting

$$\mathrm{i}\partial_t \phi = e^{\mathrm{i}tK_0} \left(Z^{(N)}(t) + \mathsf{V}^{(N)}(t)\right) e^{-\mathrm{i}tK_0}\, \phi.$$

Define

$$\mathsf{H}_N := e^{\mathrm{i}tK_0}\, Z^{(N)}(t)\, e^{-\mathrm{i}tK_0}, \qquad R_N(t) := e^{\mathrm{i}tK_0}\, \mathsf{V}^{(N)}(t)\, e^{-\mathrm{i}tK_0}.$$

The operator $R_N \in C^\infty(\mathbb{T}, \mathcal{A}_{-N})$ by Assumption II since $\mathsf{V}^{(N)} \in C^\infty(\mathbb{T}, \mathcal{A}_{-N})$.
Let us now prove that H_N is time independent. We know by Lemma 2.1 that $Z^{(N)}(t)$ commutes with $\mathrm{i}\partial_t - K_0$; therefore

$$\partial_t\big(e^{\mathrm{i}tK_0}\, Z^{(N)}(t)\, e^{-\mathrm{i}tK_0}\big) = e^{\mathrm{i}tK_0} \left(\mathrm{i}[K_0, Z^{(N)}(t)] + \partial_t Z^{(N)}(t)\right) e^{-\mathrm{i}tK_0} = 0$$

and therefore

$$\mathsf{H}_N = e^{\mathrm{i}tK_0}\, Z^{(N)}(t)\, e^{-\mathrm{i}tK_0}|_{t=0} = Z^{(N)}(0) \overset{(2.1)}{=} \widehat{\mathsf{V}}(0) + T^{(N)}(0) \overset{(2.2)}{=} \langle \mathsf{V} \rangle + T^{(N)}(0).$$

So we put $T_N := T^{(N)}(0)$; clearly it belongs to $\mathcal{A}_{-1}$, it is selfadjoint and time independent.

Finally we put $\mathcal{U}_N(t) := e^{\mathrm{i}tK_0}\, e^{\mathrm{i}tX_N(t)} \cdots e^{\mathrm{i}tX_1(t)}$; estimate (2.4) follows from (2.1) and Remark 1.2. ☐

Coming back to the original equation (1.9), we apply Corollary 2.4 with $\mathsf{V} = V + W \in C^\infty(\mathbb{T}, \mathcal{A}_0)$, getting the following result:

Corollary 2.5. *With the same assumptions of Theorem 1.9, the following holds true. Fix* $N \in \mathbb{N}$ *arbitrary. There exists a change of coordinates* $\mathcal{U}_N(t)$*, unitary in* $\mathcal{H}$ *and fulfilling* (2.4) *such that* $\psi(t)$ *is a solution of* (1.9) *if and only if* $\phi(t) := \mathcal{U}_N(t)\psi(t)$ *solves*

$$\mathrm{i}\partial_t \phi = \big(\langle V \rangle + \langle W \rangle + T_N + R_N(t)\big)\phi \tag{2.13}$$

where $T_N \in \mathcal{A}_{-1}$ *is selfadjoint and time independent whereas* $R_N \in C^\infty(\mathbb{T}, \mathcal{A}_{-N})$*.*

2.2 Local energy decay estimates

In the previous section we have conjugated the original equation (1.9) to the resonant equation (2.5). In this section we consider the effective equation obtained removing $R_N(t)$ from (2.5), namely

$$\mathrm{i}\partial_t \varphi = H_N \varphi, \qquad H_N := \langle V \rangle + \langle W \rangle + T_N, \tag{2.14}$$

with $T_N \in \mathcal{A}_{-1}$ of Corollary 2.5. Note that H_N is selfadjoint by Lemma 2.2 and Corollary 2.5. Using Assumption III, we construct a solution of (2.2) with polynomially in time growing Sobolev norms. Actually we will prove the following slightly stronger result, namely the existence of a solution with *decaying negative Sobolev norms*:

Proposition 2.6 (Decay of negative Sobolev norms). *With the same assumptions of Theorem 1.9, consider the operator H_N in* (2.2)*. For any $k \in \mathbb{N}$, there exist a nontrivial solution $\varphi(t) \in \mathcal{H}^k$ of* (2.2) *and $\forall r \in [0, k]$ a constant $C_r > 0$ such that*

$$\|\varphi(t)\|_{-r} \le C_r \langle t \rangle^{-r} \|\varphi(0)\|_r \ , \qquad \forall t \in \mathbb{R} \ . \tag{2.15}$$

Remark 2.7. As H_N is selfadjoint, the conservation of the $\mathcal{H}^0$-norm and duality give

$$\|\varphi(0)\|_0^2 = \|\varphi(t)\|_0^2 \le \|\varphi(t)\|_r \ \|\varphi(t)\|_{-r} \ , \qquad \forall t \in \mathbb{R} \ ,$$

so that (2.6) implies the growth of positive Sobolev norms:

$$\|\varphi(t)\|_r \ge \frac{1}{C_r} \frac{\|\varphi(0)\|_0^2}{\|\varphi(0)\|_r} \langle t \rangle^r \ , \quad \forall t \in \mathbb{R} \ .$$

The rest of the section is devoted to the proof of Proposition 2.6. As we shall see, it follows from a *local energy decay estimate* for the operator H_N, namely a dispersive estimate of the form

$$\|\langle A \rangle^{-k} \, e^{-\mathrm{i} H_N t} \, g_J(H_N) \, \varphi\|_0 \le C_k \langle t \rangle^{-k} \|\langle A \rangle^k g_J(H_N) \varphi\|_0 \ , \qquad \forall t \in \mathbb{R} \tag{2.16}$$

where $A \in \mathcal{A}_1$, $J \subset I_0$ is an interval and $g_J \in C_c^\infty(\mathbb{R}, \mathbb{R}_{\ge 0})$ with $g_J \equiv 1$ on J.

Remark 2.8. Actually estimate (2.2) show the existence of infinitely many solutions of (2.2) with decaying negative Sobolev norms. In particular this happens to any solution with nontrivial initial datum in the (infinite dimensional) set $\operatorname{Ran} E_J(H_N)$, where $E_J(H_N)$ is the spectral projection of H_N corresponding to the interval J.

We will prove 2.2 exploiting Sigal-Soffer minimal velocity estimates [60, 27, 43, 41, 29, 2], which are based on Mourre theory which now we recall.

Mourre theory. Let H be a selfadjoint operator on the Hilbert space $\mathcal{H}$, and denote by $\sigma(\mathsf{H})$ its spectrum. We further denote by $\sigma_d(\mathsf{H})$ its discrete spectrum, $\sigma_{ess}(\mathsf{H})$ its essential spectrum, $\sigma_{pp}(\mathsf{H})$ its pure point spectrum, $\sigma_{ac}(\mathsf{H})$ its absolutely continuous spectrum and $\sigma_{sc}(\mathsf{H})$ its singular spectrum; see e.g. [56] pag. 236 and 231 for their definitions. Furthermore we denote by $E_\Omega(\mathsf{H})$ the spectral projection of H corresponding to the Borel set Ω and by $m_\varphi(\Omega) := \langle E_\Omega(\mathsf{H})\varphi, \varphi \rangle$ the spectral measure associated to $\varphi \in \mathcal{H}$.

Assume a selfadjoint operator A can be found such that $D(\mathsf{A}) \cap \mathcal{H}$ is dense in $\mathcal{H}$. We put

$$\mathrm{ad}^0_{\mathsf{A}}(\mathsf{H}) := \mathsf{H}, \qquad \mathrm{ad}_{\mathsf{A}}(\mathsf{H}) := [\mathsf{H}, \mathsf{A}], \qquad \mathrm{ad}^n_{\mathsf{A}}(\mathsf{H}) := [\mathrm{ad}^{n-1}_{\mathsf{A}}(\mathsf{H}), \mathsf{A}], \quad \forall n \ge 2 \ . \tag{2.17}$$

Consider the following properties:

(M1) For some $\mathtt{N} \ge 1$, the operators $\mathrm{ad}^n_{\mathsf{A}}(\mathsf{H})$ with $n = 1, \dots, \mathtt{N}$, can all be extended to bounded operators on $\mathcal{H}$.

(M2) *Mourre estimate*: there exist an open interval $I \subset \mathbb{R}$ with compact closure and a function $g_I \in C_c^\infty(\mathbb{R}, \mathbb{R}_{\ge 0})$ with $g_I \equiv 1$ on I such that

$$g_I(\mathsf{H}) \, \mathrm{i}[\mathsf{H}, \mathsf{A}] \, g_I(\mathsf{H}) \ge \theta g_I(\mathsf{H})^2 + \mathsf{K} \tag{2.18}$$

for some $\theta > 0$ and K a selfadjoint compact operator on $\mathcal{H}$.

If the estimate (2.2) holds true with $\mathsf{K} = 0$ we shall say that H fulfills a *strict* Mourre estimate. Mourre theorem [53] says the following:

Theorem 2.9 (Mourre)**.** *Assume conditions* (M1) – (M2) *with* $\mathtt{N} = 2$*. In the interval* I*, the operator* H *can have only absolutely continuous spectrum and finitely many eigenvalues of finite multiplicity. If* $\mathsf{K} = 0$*, there are no eigenvalues in the interval* I*, i.e.* $\sigma(\mathsf{H}) \cap I = \sigma_{ac}(\mathsf{H}) \cap I$*.*

Remark 2.10. The version stated here of Mourre theorem is taken from [3, Lemma 5.6] and [15, Theorem 4.7 – 4.9], and it has slightly weaker assumptions compared to [53].

Remark 2.11. Mourre theorem guarantees that $\sigma_{sc}(\mathsf{H}) \cap I = \emptyset$ and, in case $\mathsf{K} = 0$, $\sigma_{pp}(\mathsf{H}) \cap I = \emptyset$. However it does not guarantee that $\sigma(\mathsf{H}) \cap I \neq \emptyset$; in our case we shall verify this property explicitly.

The key point is that if H_N fulfills a *strict* Mourre estimate (namely with $\mathsf{K} = 0$) then one can prove a local energy decay estimate like (2.2) for the Schrödinger flow of H_N. This is a quite general fact which follows exploiting minimal velocity estimates [41] and we prove it for completeness in Appendix C.

So the next goal is to prove that H_N satisfies a strict Mourre estimate over a certain interval $J \subset I_0$. During the proof we will use some standard results from functional calculus; we recall them in Appendix B. We shall also use the following lemma:

Lemma 2.12. *Let* $\mathsf{H} \in \mathcal{L}(\mathcal{H})$ *be selfadjoint. If* $\lambda \in \sigma_{ac}(\mathsf{H})$*, then* $\forall \delta > 0$ *one has*

$$|[\lambda - \delta, \lambda + \delta] \cap \sigma(\mathsf{H})| > 0 \ .$$

Proof. By contradiction, assume that $\exists \delta_0 > 0$ such that $|[\lambda - \delta_0, \lambda + \delta_0] \cap \sigma(\mathsf{H})| = 0$. As $\lambda \in \sigma_{ac}(\mathsf{H})$, there exists $f \in \mathcal{H}$ such that $E_{[\lambda-\delta_0,\lambda+\delta_0]}(\mathsf{H}) f \neq 0$ and the spectral measure $m_f = \langle E(\mathsf{H}) f, f \rangle$ is absolutely continuous. Then

$$0 = m_f([\lambda - \delta_0, \lambda + \delta_0]) = \langle E_{[\lambda-\delta_0,\lambda+\delta_0]}(\mathsf{H}) f, f \rangle = \| E_{[\lambda-\delta_0,\lambda+\delta_0]}(\mathsf{H}) f \|_0^2 > 0$$

giving a contradiction. □

Lemma 2.13. *There exist* $\epsilon_0, \mathtt{M} > 0$ *such that, provided* W *fulfills* (1.9)*, the following holds true:*

(i) There exists an interval $I \subset I_0$ *such that* $|I \cap \sigma(H_N)| > 0$*.*

(ii) H_N *fulfills a strict Mourre estimate over* I*: there exists a function* $g_I \in C_c^\infty(\mathbb{R}, \mathbb{R}_{\geq 0})$ *with* $\operatorname{supp} g_I \subset I_0$*,* $g_I \equiv 1$ *on* I*, and* $\theta' > 0$ *such that*

$$g_I(H_N)\, \mathrm{i}[H_N, A]\, g_I(H_N) \geq \theta' g_I(H_N)^2 \ . \tag{2.19}$$

Here I_0 *is the interval and* A *is the operator of Assumption* III.

Proof. During the proof we shall often use that for $\mathsf{A}, \mathsf{B}, \mathsf{C} \in \mathcal{L}(\mathcal{H})$ and selfadjoints

$$\mathsf{A} \leq \mathsf{B} \quad \Rightarrow \quad \mathsf{CAC} \leq \mathsf{CBC}, \qquad \|\mathsf{A}\|_{\mathcal{L}(\mathcal{H})} \leq a \quad \Rightarrow \quad -a \leq \mathsf{A} \leq a \ . \tag{2.20}$$

To shorten notation we shall put

$$H_0 := \langle V \rangle \ .$$

By Assumption III, H_0 fulfills a Mourre estimate over the interval I_0.
STEP 1: We claim there exists a subinterval $I_1 \subset I_0$ such that:
• I_1 contains only absolutely continuous spectrum of H_0, namely

$$\sigma(H_0) \cap I_1 = \sigma_{ac}(H_0) \cap I_1 \ , \qquad |\sigma(H_0) \cap I_1| > 0 \ ; \tag{2.21}$$

• H_0 fulfills over I_0 a *strict* Mourre estimate: $\exists g_{I_1} \in C_c^\infty(\mathbb{R}, \mathbb{R}_{\geq 0})$, $g_{I_1} \equiv 1$ on I_1, $\operatorname{supp} g_{I_1} \subset I_0$, such that

$$g_{I_1}(H_0)\, \mathrm{i}[H_0, A]\, g_{I_1}(H_0) \geq \frac{\theta}{2} g_{I_1}(H_0)^2 \ . \tag{2.22}$$

To prove this claim, first apply Mourre theorem to H_0 (note that (M1) and (M2) are verified $\forall \mathtt{N} \in \mathbb{N}$ by symbolic calculus and Assumption III), getting that $\sigma(H_0) \cap I_0$ contains only finitely many eigenvalues with finite multiplicity and absolutely continuous spectrum. In particular $|\sigma_{pp}(H_0) \cap I_0| = 0$ and by Assumption III (i) it follows that $|\sigma_{ac}(H_0) \cap I_0| = |\sigma(H_0) \cap I_0| > 0$.
Now we show that, by shrinking enough the interval I_0, a strict Mourre estimate is true. So we take $\lambda_0 \in I_0 \cap (\sigma_{ac}(H_0) \setminus \sigma_{pp}(H_0))$ and a sufficiently small interval $I_1(\overline{\delta}) := (\lambda_0 - \overline{\delta}, \lambda_0 + \overline{\delta}) \subset I_0$, $\overline{\delta} > 0$, which does not contain eigenvalues of H_0; this is possible as the eigenvalues of H_0 in I_0 are finite. Moreover by Lemma 2.12, $|\sigma(H_0) \cap I_1(\delta)| > 0$ for any $\delta > 0$. Now take $\delta \in (0, \overline{\delta})$ and a function $g_\delta \in C_c^\infty(\mathbb{R}, \mathbb{R}_{\geq 0})$ with $\operatorname{supp} g_\delta \subset I_1(\delta)$ and $g_\delta = 1$ on $I_1(\frac{\delta}{2})$. We claim that provided $\delta \in (0, \overline{\delta})$ is sufficiently small

$$\|g_\delta(H_0) K g_\delta(H_0)\|_{\mathcal{L}(\mathcal{H})} \leq \frac{\theta}{2} \ , \tag{2.23}$$

where $\theta > 0$ is the one of Assumption III. Indeed in $I_1(\overline{\delta})$ the spectrum of H_0 is absolutely continuous; this means that $\forall \varphi \in \mathcal{H}$, the vector $\varphi' := E_{I_1(\overline{\delta})}(H_0)\varphi$ belongs to the absolutely continuous subspace of H_0, namely its spectral measure $m_{\varphi'}$ is absolutely continuous w.r.t. the Lebesgue measure. Now, since for any $\varphi \in \mathcal{H}$ one has by functional calculus $g_\delta(H_0) = g_\delta(H_0) E_{I_1(\overline{\delta})}(H_0)$, one has that

$$\|g_\delta(H_0)\varphi\|_0^2 = \|g_\delta(H_0) E_{I_1(\overline{\delta})}(H_0)\varphi\|_0^2 = \int_{\mathbb{R}} g_\delta(\lambda)^2 \, \mathrm{d}m_{\varphi'}(\lambda) \to 0 \quad \text{ as } \delta \to 0$$

by Lebesgue dominated convergence theorem. In particular $g_\delta(H_0) \to 0$ strongly as $\delta \to 0$ and then, being K compact, $g_\delta(H_0)K \to 0$ uniformly as $\delta \to 0$ (see e.g. [1]). Therefore for $\delta \in (0, \overline{\delta})$ sufficiently small (2.2) holds true.
Using the assumption (1.1), (2.2) and (2.2) we deduce that

$$g_\delta(H_0)\, g_{I_0}(H_0)\, \mathrm{i}[H_0, A]\, g_{I_0}(H_0)\, g_\delta(H_0) \geq \theta g_\delta(H_0)\, g_{I_0}(H_0)^2\, g_\delta(H_0) - \frac{\theta}{2} \ ;$$

next apply $g_{\frac{\delta}{2}}(H_0)$ to the right and left of the previous inequality, use again (2.2) and the identity $g_{I_0}(H_0)\, g_\delta(H_0)\, g_{\frac{\delta}{2}}(H_0) = g_{\frac{\delta}{2}}(H_0)$ (which follows from $g_{I_0}\, g_\delta\, g_{\frac{\delta}{2}} = g_{\frac{\delta}{2}}$), to get the strict Mourre estimate (2.2) where $I_1 := I_1(\frac{\delta}{4})$ and $g_{I_1} := g_{\frac{\delta}{2}}$ fulfills $g_{I_1} \equiv 1$ on I_1, $\operatorname{supp} g_{I_1} \subset I_1(\frac{\delta}{2})$. Clearly I_1 fulfills (2.2).
STEP 2: Consider the selfadjoint operator

$$H_{\langle W \rangle} := H_0 + \langle W \rangle \ .$$

We claim there exists a subinterval $I_2 \subseteq I_1$ such that
• I_2 contains only absolutely continuous spectrum of $H_{\langle W \rangle}$, i.e.

$$\sigma(H_{\langle W \rangle}) \cap I_2 = \sigma_{ac}(H_{\langle W \rangle}) \cap I_2 \qquad \text{and} \qquad \left|\sigma(H_{\langle W \rangle}) \cap I_2\right| > 0 \ ; \tag{2.24}$$

• $H_{\langle W \rangle}$ fulfills over I_2 the strict Mourre estimate

$$g_{I_2}\big(H_{\langle W \rangle}\big)\, \mathrm{i}[H_{\langle W \rangle}, A]\, g_{I_2}\big(H_{\langle W \rangle}\big) \geq \frac{\theta}{4} g_{I_2}\big(H_{\langle W \rangle}\big)^2 \tag{2.25}$$

for any $g_{I_2} \in C_c^\infty(\mathbb{R}, \mathbb{R}_{\geq 0})$ with $\operatorname{supp} g_{I_2} \subset I_1$, $g_{I_2} \equiv 1$ on I_2.

To prove the claim, we exploit that $\langle W \rangle \in \mathcal{A}_0$ is a small bounded perturbation of H_0, fulfilling, by (1.4), (2.2)

$$\exists M_0 \in \mathbb{N}, C_0 > 0\colon \qquad \|\langle W \rangle\|_{\mathcal{L}(\mathcal{H})} \leq C_0 [W]_{M_0}, \tag{2.26}$$

where we denoted

$$[W]_M := \sup_{t \in \mathbb{T}} \wp_M^0(W(t)) \ .$$

First let us prove that $\sigma(H_{\langle W \rangle}) \cap I_1 \neq \emptyset$. Take again the same $\lambda_0 \in \sigma(H_0) \cap I_1$ as in the previous step. We claim that

$$\operatorname{dist}\big(\lambda_0, \sigma(H_{\langle W \rangle})\big) \leq C_0\, [W]_{M_0}. \tag{2.27}$$

If $\lambda_0 \in \sigma(H_{\langle W \rangle})$ this is trivial. So assume that λ_0 belongs to the resolvent set of $H_{\langle W \rangle}$. As $\lambda_0 \in \sigma(H_0)$, by Weyl criterion $\exists (f_n)_{n \geq 1} \in \mathcal{H}$ with $\|f_n\|_0 = 1$ such that $\|(H_0 - \lambda_0) f_n\|_0 \to 0$ as $n \to \infty$. Then $\forall n \geq 1$

$$\begin{aligned} 1 = \|f_n\|_0 &= \|(H_{\langle W \rangle} - \lambda_0)^{-1} (H_{\langle W \rangle} - \lambda_0) f_n\|_0 \leq \frac{1}{\operatorname{dist}\big(\lambda_0, \sigma(H_{\langle W \rangle})\big)} \|(H_{\langle W \rangle} - \lambda_0) f_n\|_0 \\ &\overset{(2.2)}{\leq} \frac{1}{\operatorname{dist}\big(\lambda_0, \sigma(H_{\langle W \rangle})\big)} \Big(\|(H_0 - \lambda_0) f_n\|_0 + C_0 [W]_{M_0} \Big) \end{aligned}$$

which proves (2.2) passing to the limit $n \to \infty$. Then, provided $[W]_{M_0}$ is sufficiently small, (2.2) implies that $\operatorname{dist}\big(\lambda_0, \sigma(H_{\langle W \rangle})\big) < \delta/8$. From this we learn that (recall $I_1 = (\lambda_0 - \frac{\delta}{4}, \lambda_0 + \frac{\delta}{4})$)

$$\sigma(H_{\langle W \rangle}) \cap I_1 \neq \emptyset \ . \tag{2.28}$$

Next we prove the Mourre estimate (2.2); we shall work perturbatively from (2.2). First

$$g_{I_1}(H_0)\, \mathrm{i}[H_{\langle W \rangle}, A]\, g_{I_1}(H_0) = g_{I_1}(H_0)\, \mathrm{i}[H_0, A]\, g_{I_1}(H_0) + g_{I_1}(H_0)\, \mathrm{i}[\langle W \rangle, A]\, g_{I_1}(H_0);$$

we bound the first term in the right hand side above from below using (2.2). Concerning the second term, we use

$$\exists\, M_1 \in \mathbb{N},\, C_1 > 0\colon \quad \|\mathrm{i}[\langle W \rangle, A]\|_{\mathcal{L}(\mathcal{H})} \leq C_1 [W]_{M_1} \tag{2.29}$$

(by (1.4), (1.4), (2.2)) and the inequalities (2.2) to bound it from above getting

$$g_{I_1}(H_0)\, \mathrm{i}[\langle W \rangle, A]\, g_{I_1}(H_0) \geq -C_1\, [W]_{M_1}\, g_{I_1}(H_0)^2 \ .$$

Therefore we find

$$g_{I_1}(H_0)\, \mathrm{i}[H_{\langle W \rangle}, A]\, g_{I_1}(H_0) \geq \left(\frac{\theta}{2} - C_1 [W]_{M_1} \right) g_{I_1}(H_0)^2 \ . \tag{2.30}$$

By (2.2) we can take an open interval $I_2 \subset I_1$ with

$$\sigma(H_{\langle W \rangle}) \cap I_2 \neq \emptyset \ ; \tag{2.31}$$

take also $g_{I_2} \in C_c^\infty(\mathbb{R}, \mathbb{R}_{\geq 0})$ with $\operatorname{supp} g_{I_2} \subseteq I_1$ and $g_{I_2} \equiv 1$ on I_2; remark that $g_{I_1} g_{I_2} = g_{I_2}$. Now we wish to replace $g_{I_1}(H_0)$ by $g_{I_2}(H_{\langle W \rangle})$ in (2.2), thus getting the claimed estimate (2.2). So write

$$\begin{aligned} g_{I_2}(H_{\langle W \rangle})\, \mathrm{i}[H_{\langle W \rangle}, A]\, g_{I_2}(H_{\langle W \rangle}) &= g_{I_2}(H_{\langle W \rangle})\, g_{I_1}(H_{\langle W \rangle})\, \mathrm{i}[H_{\langle W \rangle}, A]\, g_{I_1}(H_{\langle W \rangle})\, g_{I_2}(H_{\langle W \rangle}) \\ &= g_{I_2}(H_{\langle W \rangle})\, g_{I_1}(H_0)\, \mathrm{i}[H_{\langle W \rangle}, A]\, g_{I_1}(H_0)\, g_{I_2}(H_{\langle W \rangle}) && (2.32) \\ &+ g_{I_2}(H_{\langle W \rangle}) \Big(\big(g_{I_1}(H_{\langle W \rangle}) - g_{I_1}(H_0)\big)\, \mathrm{i}[H_{\langle W \rangle}, A]\, g_{I_1}(H_0) && (2.33) \\ &+ g_{I_1}(H_{\langle W \rangle})\, \mathrm{i}[H_{\langle W \rangle}, A]\, \big(g_{I_1}(H_{\langle W \rangle}) - g_{I_1}(H_0)\big)\Big)\, g_{I_2}(H_{\langle W \rangle}) && (2.34) \end{aligned}$$

Again we estimate (2.2) from below and the other lines from above. First

$$(2.2) \overset{(2.2)}{\geq} \left(\frac{\theta}{2} - C_1[W]_{M_1}\right) g_{I_2}(H_{\langle W\rangle})\, g_{I_1}(H_0)^2\, g_{I_2}(H_{\langle W\rangle}) \ . \tag{2.35}$$

We still have to bound from below $g_{I_2}(H_{\langle W\rangle})\, g_{I_1}(H_0)^2\, g_{I_2}(H_{\langle W\rangle})$. To proceed we use that $g_{I_1}(H_{\langle W\rangle}) - g_{I_1}(H_0)$ is small in size, being bounded, via Lemma B.6 and (2.2), by

$$\|g_{I_1}(H_{\langle W\rangle}) - g_{I_1}(H_0)\|_{\mathcal{L}(\mathcal{H})} \leq C\,[W]_{M_0} \ . \tag{2.36}$$

We deduce, using $g_{I_1} g_{I_2} = g_{I_2}$, estimates (2.2) and (2.2), the bound

$$g_{I_2}(H_{\langle W\rangle})\, g_{I_1}(H_0)^2\, g_{I_2}(H_{\langle W\rangle}) \geq \big(1 - C[W]_{M_0}\big)\, g_{I_2}(H_{\langle W\rangle})^2 \ .$$

Thus we estimate line (2.2) from below using (2.2) and the previous estimate, concluding

$$(2.2) \geq \left(\frac{\theta}{2} - C_1[W]_{M_1}\right) \big(1 - C[W]_{M_0}\big)\, g_{I_2}(H_{\langle W\rangle})^2. \tag{2.37}$$

Next consider lines (2.2), (2.2). We use the bound (see (2.2))

$$\|[H_{\langle W\rangle}, A]\|_{\mathcal{L}(\mathcal{H}^0)} \leq C\big(1 + [W]_{M_1}\big) \ ,$$

and (2.2) to get

$$(2.2) + (2.2) \geq -C\,[W]_{M_0}\,(1 + [W]_{M_1})\, g_{I_2}(H_{\langle W\rangle})^2. \tag{2.38}$$

Putting together (2.2) and (2.2) we finally find

$$g_{I_2}(H_{\langle W\rangle})\, \mathrm{i}[H_{\langle W\rangle}, A]\, g_{I_2}(H_{\langle W\rangle}) \geq \left(\frac{\theta}{2} - C([W]_{M_1} + [W]_{M_0} + [W]_{M_0}\,[W]_{M_1})\right) g_{I_2}(H_{\langle W\rangle})^2.$$

Thus, provided (1.9) holds true for $\mathtt{M}$ sufficiently large and ϵ_0 sufficiently small, the strict Mourre estimate (2.2) follows. Mourre theorem implies that

$$\sigma(H_{\langle W\rangle}) \cap I_2 = \sigma_{ac}(H_{\langle W\rangle}) \cap I_2 \ .$$

Using (2.2) and Lemma 2.12 we deduce that $\big|\sigma(H_{\langle W\rangle}) \cap I_2\big| > 0$, proving (2.2) .
STEP 3: Finally consider the operator

$$H := H_N = H_0 + \langle W\rangle + T_N = H_{\langle W\rangle} + T_N \ .$$

We claim that, with the same interval I_2 of the previous step:
- one has

$$|\sigma(H) \cap I_2| > 0 \ . \tag{2.39}$$

- H fulfills a Mourre estimate over I_2, i.e.

$$g_{I_2}\big(H\big)\, \mathrm{i}[H, A]\, g_{I_2}\big(H\big) \geq \frac{\theta}{4} g_{I_2}\big(H\big)^2 + K \tag{2.40}$$

with K a compact operator.

We shall constantly use that any pseudodifferential operator of strictly negative order is a compact operator on $\mathcal{H}$ (see Remark 1.5); in particular $T_N \in \mathcal{A}_{-1}$ is compact.

By Weyl theorem $\sigma_{ess}(H) = \sigma_{ess}(H_{\langle W\rangle})$ and therefore

$$\sigma(H) \cap I_2 \supseteq \sigma_{ess}(H) \cap I_2 = \sigma_{ess}(H_{\langle W\rangle}) \cap I_2 = \sigma(H_{\langle W\rangle}) \cap I_2 \ ,$$

since $\sigma_d(H_{\langle W\rangle}) \cap I_2 = \emptyset$ having $H_{\langle W\rangle}$ no eigenvalues in I_2. Then (2.2) follows by (2.2).

To prove (2.2) we work perturbatively from (2.2). Again first we compute

$$g_{I_2}\big(H_{\langle W\rangle}\big)\, \mathrm{i}[H, A]\, g_{I_2}\big(H_{\langle W\rangle}\big) = g_{I_2}\big(H_{\langle W\rangle}\big)\, \mathrm{i}[H_{\langle W\rangle}, A]\, g_{I_2}\big(H_{\langle W\rangle}\big) + g_{I_2}\big(H_{\langle W\rangle}\big)\, \mathrm{i}[T_N, A]\, g_{I_2}\big(H_{\langle W\rangle}\big) \ ;$$

we estimate the first term in the r.h.s. above by (2.2), whereas the second term is a compact operator since $[T_N, A] \in \mathcal{A}_{-1}$. We obtain

$$g_{I_2}\big(H_{\langle W\rangle}\big)\, \mathrm{i}[H, A]\, g_{I_2}\big(H_{\langle W\rangle}\big) \geq \frac{\theta}{4} g_{I_2}\big(H_{\langle W\rangle}\big)^2 + K_1 \tag{2.41}$$

with K_1 a compact operator. Now we must replace $g_{I_2}\big(H_{\langle W\rangle}\big)$ with $g_{I_2}(H)$. We write

$$g_{I_2}(H)\, \mathrm{i}[H, A]\, g_{I_2}(H) = g_{I_2}(H_{\langle W\rangle})\, \mathrm{i}[H, A]\, g_{I_2}(H_{\langle W\rangle}) \tag{2.42}$$

$$+ \big(g_{I_2}(H) - g_{I_2}(H_{\langle W\rangle})\big)\, \mathrm{i}[H, A]\, g_{I_2}(H_{\langle W\rangle}) + g_{I_2}(H)\, \mathrm{i}[H, A] \big(g_{I_2}(H) - g_{I_2}(H_{\langle W\rangle})\big) \tag{2.43}$$

This time we use that $g_{I_2}(H) - g_{I_2}(H_{\langle W\rangle})$ is a compact operator, see Lemma B.6. Thus

$$(2.2) \overset{(2.2)}{\geq} \frac{\theta}{4} g_{I_2}\big(H_{\langle W\rangle}\big)^2 + K_1 = \frac{\theta}{4} g_{I_2}(H)^2 + K_2$$

where K_1, K_2 are compact operators. Similarly, using that $\mathrm{i}[H, A] \in \mathcal{A}_0$ is a bounded operator, we deduce that (2.2) is a compact operator. Estimate (2.2) follows.

Final Step: By (2.2), (2.2), the operator H fulfills Assumption III over the interval I_2. Proceeding as in Step 1, we produce a subinterval $I \subset I_2$ such that

$$|I \cap \sigma(H)| > 0 \ , \qquad I \cap \sigma(H) = I \cap \sigma_{ac}(H)$$

over which H fulfills the strict Mourre estimate (2.13), concluding the proof of Lemma 2.13. □

The previous result has proved the existence of an interval I over which H_N fulfills a strict Mourre estimate. This implies that H_N fulfills dispersive estimates in the form of local energy decay. In the literature there are various variants of this result, thus in Appendix C we state and prove the one we apply here.

Corollary 2.14. *Fix $k \in \mathbb{N}$. For any interval $J \subset I$, any function $g_J \in C_c^\infty(\mathbb{R}, \mathbb{R}_{\geq 0})$ with* $\operatorname{supp} g_J \subset I$, *$g_J \equiv 1$ on J, there exists a constant $C_k > 0$ such that*

$$\|\langle A\rangle^{-k}\, e^{-\mathrm{i}H_N t}\, g_J(H_N)\, \varphi\|_0 \leq C_k \langle t\rangle^{-k} \|\langle A\rangle^k g_J(H_N)\varphi\|_0 \ , \quad \forall t \in \mathbb{R} \ , \quad \forall \varphi \in \mathcal{H}^k \ . \tag{2.44}$$

Moreover J can be chosen so that $|J \cap \sigma(H_N)| > 0$ and $\sigma(H_N) \cap J = \sigma_{ac}(H_N) \cap J$.

Proof. Apply Theorem C.1, noting that condition (M1) at page 12 is trivially satisfied $\forall n \in \mathbb{N}$ as $\mathrm{ad}_A^n(H_N) \in \mathcal{A}_0 \subset \mathcal{L}(\mathcal{H})$, whereas the whole point of Lemma 2.13 was to verify (M2). This gives estimate (2.14). The right hand side is finite for $\varphi \in \mathcal{H}^k$ by Lemma 2.15 below, which ensures that $g_J(H_N)\varphi \in \mathcal{H}^k$. Finally note that, since $|I \cap \sigma(H_N)| > 0$, it is certainly possible to choose $J \subset I$ so that $|J \cap \sigma(H_N)| > 0$; as H_N fulfills a strict Mourre estimate over I, its spectrum in this interval is absolutely continuous, so the same is true in J.

□

Lemma 2.15. *For any $k \in \mathbb{N}$, $g_J(H_N)$ extends to a bounded operator $\mathcal{H}^k \to \mathcal{H}^k$.*

Proof. As $K_0^k\, g_J(H_N)\, K_0^{-k} = g_J(H_N) - [g_J(H_N), K_0^k]K_0^{-k}$, it is clearly sufficient to show that $[g_J(H_N), K_0^k]K_0^{-k}$ is bounded on $\mathcal{H}$. The adjoint formula (B) gives

$$[g_J(H_N), K_0^k]K_0^{-k} = \sum_{j=1}^{k} c_{k,j}\, \mathrm{ad}_{K_0}^j(g_J(H_N))\, K_0^{-j} \ ;$$

then it is enough to show that $\mathrm{ad}_{K_0}^j(g_J(H_N)) \in \mathcal{L}(\mathcal{H})$. As $\mathrm{ad}_{K_0}^j(H_N)$ is a bounded operator $\forall j$ (symbolic calculus), the result is an immediate application of Lemma B.5. □

We finally prove Proposition 2.6.

Proof of Proposition 2.6. First we show that for any $k \in \mathbb{N}$, there exists $C_{2k} > 0$ such that

$$\|e^{-\mathrm{i}tH_N} g_J(H_N)\varphi\|_{-2k} \leq C_{2k}\, \langle t\rangle^{-2k}\, \|g_J(H_N)\varphi\|_{2k}, \qquad \forall t \in \mathbb{R}, \quad \forall \varphi \in \mathcal{H}^{2k}\ . \tag{2.45}$$

This follows from Corollary 2.14 with $k \rightsquigarrow 2k$. Indeed, as $A \in \mathcal{A}_1$, the operator $\langle A\rangle^{2k} = (1 + A^2)^k \in \mathcal{A}_{2k}$ and therefore, by symbolic calculus, $K_0^{-2k}\langle A\rangle^{2k}$ and $\langle A\rangle^{2k}K_0^{-2k}$ belong to $\mathcal{A}_0 \subset \mathcal{L}(\mathcal{H})$. Then

$$\begin{aligned}\|e^{-\mathrm{i}tH_N} g_J(H_N)\varphi\|_{-2k} &\leq \|K_0^{-2k}\, \langle A\rangle^{2k}\|_{\mathcal{L}(\mathcal{H})}\, \|\langle A\rangle^{-2k} e^{-\mathrm{i}tH_N} g_J(H_N)\varphi\|_0 \\ &\leq C_{2k}\langle t\rangle^{-2k}\|\langle A\rangle^{2k} g_J(H_N)\varphi\|_0 \\ &\leq C_{2k}\langle t\rangle^{-2k}\|\langle A\rangle^{2k} K_0^{-2k}\|_{\mathcal{L}(\mathcal{H})}\|g_J(H_N)\varphi\|_{2k}\end{aligned}$$

proving (2.2). Then linear interpolation with the equality $\|e^{-\mathrm{i}tH_N}\varphi_0\|_0 = \|\varphi_0\|_0$ $\forall t$ gives $\forall r \in [0, 2k]$

$$\|e^{-\mathrm{i}tH_N} g_J(H_N)\varphi\|_{-r} \leq C_r\langle t\rangle^{-r}\|g_J(H_N)\varphi\|_r\ , \quad \forall t \in \mathbb{R}\ , \quad \forall \varphi \in \mathcal{H}^r\ .$$

Finally we show that this estimate is not trivial, namely $\exists \varphi \in \mathcal{H}^k$ with $g_J(H_N)\varphi \neq 0$. But since $|J \cap \sigma(H_N)| > 0$ and $\sigma(H_N) \cap J = \sigma_{ac}(H_N) \cap J$, one has that $g_J(H_N)\mathcal{H} \neq \{0\}$, and by density so is $g_J(H_N)\mathcal{H}^k$. □

2.3 Proof of Theorem 1.9

We are finally in position of proving Theorem 1.9. Recall that in Corollary 2.5 we have conjugated equation (1.9) to (2.5) with a change of variables bounded $\mathcal{H}^r \to \mathcal{H}^r$ uniformly in time, whereas in Proposition 2.6 we have constructed a solution of the effective equation $\mathrm{i}\partial_t\psi = H_N\psi$ with decaying negative Sobolev norms, therefore with growing positive Sobolev norms. The last step is to construct a solution of the full equation (2.5) with growing Sobolev norms. To achieve this, we exploit that the perturbation $R_N(t)$ is N-smoothing (Definition 1.3).
So to proceed we fix the parameters. First fix $r > 0$, then choose $N, k \in \mathbb{N}$ such that

$$N \geq 2r + 2, \quad k \geq N - r. \tag{2.46}$$

Apply Corollary 2.5 with such N, producing the operators T_N, $R_N(t)$ and conjugating (1.9) to (2.5). By Proposition 2.6, $\exists\, \varphi_0 \in \mathcal{H}^k$ such that $\varphi(t) := e^{-\mathrm{i}tH_N}\varphi_0$ fulfills $\forall \mathtt{r} \in [0, k]$:

$$\|\varphi(t)\|_{-\mathtt{r}} \leq C_{\mathtt{r},N}\langle t\rangle^{-\mathtt{r}}\, \|\varphi_0\|_{\mathtt{r}}, \qquad \forall t \in \mathbb{R}\ . \tag{2.47}$$

We look for an exact solution $\phi(t)$ of (2.5) of the form $\phi(t) = \varphi(t) + u(t)$, i.e. $u(t)$ has to satisfy

$$\mathrm{i}\partial_t u = \big(H_N + R_N(t)\big)u + R_N(t)\varphi(t).$$

Denoting by $U_N(t,s)$ the linear propagator of $H_N + R_N(t)$, we choose

$$u(t) := \mathrm{i} \int_t^{+\infty} U_N(t,s)\, R_N(s)\, \varphi(s)\, \mathrm{d}s.$$

We estimate the $\mathcal{H}^r$ norm of $u(t)$. As

$$\sup_t \|[H_N + R_N(t),\, K_0]\|_{\mathcal{L}(\mathcal{H}^m)} < C_m < \infty\ , \qquad \forall m \in \mathbb{R},$$

Theorem 1.5 of [49] guarantees that the propagator $U_N(t,s)$ extends to a bounded operator $\mathcal{H}^r \to \mathcal{H}^r$ fulfilling[5]

$$\forall r > 0 \quad \exists\, C_r > 0\colon \qquad \|U_N(t,s)\|_{\mathcal{L}(\mathcal{H}^r)} \leq C_r\, \langle t-s\rangle^r, \quad \forall t,s \in \mathbb{R}\ .$$

This estimate, the smoothing property $R_N(t)\colon \mathcal{H}^{r-N} \to \mathcal{H}^r$ and (2.3) with $\mathtt{r} := N - r \in [0,k]$ give

$$\begin{aligned} \|u(t)\|_r &\leq C_r \int_t^{+\infty} \langle t-s\rangle^r \|R_N(s)\,\varphi(s)\|_r\, \mathrm{d}s \leq C_r \int_t^{+\infty} \langle t-s\rangle^r\, \|\varphi(s)\|_{-(N-r)}\, \mathrm{d}s \\ &\leq C_{r,N}\, \|\varphi_0\|_{N-r} \int_t^{+\infty} \langle t-s\rangle^r\, \frac{1}{\langle s\rangle^{N-r}}\, \mathrm{d}s \leq C_{r,N}\, \|\varphi_0\|_k \langle t\rangle^{-1}\ . \end{aligned}$$

In particular the $\mathcal{H}^r$ norm of $u(t)$ decreases to 0 as $t \to \infty$. Then $\phi(t) = \varphi(t) + u(t)$ fulfills

$$\|\phi(t)\|_r \geq \|\varphi(t)\|_r - \|u(t)\|_r \geq c_r \frac{\|\varphi_0\|_0^2}{\|\varphi_0\|_r} \langle t\rangle^r - C_{r,N} \|\varphi_0\|_k \langle t\rangle^{-1} \geq C\langle t\rangle^r\ , \quad \forall |t| \geq T\,,$$

where we used (2.3) with $\mathtt{r} = r$ and Remark 2.7.
Finally we get a solution of the original equation (1.9) putting $\psi(t) = \mathcal{U}_N(t)^{-1}\phi(t)$, recall Proposition 2.4. The operator $\mathcal{U}_N(t)$ fulfills (2.4), thus $\psi(t)$ has polynomially growing Sobolev norms as (1.9), concluding the proof of Theorem 1.9.

We can also prove the existence of infinitely many solutions undergoing growth of Sobolev norms.

Corollary 2.16. *There are infinitely many distinct solutions of equation* (1.9) *with growing Sobolev norms.*

Proof. We fix $r > 0$ and choose N, k as in (2.3). From the previous proof, it follows that any initial data of the form

$$\psi(0) := (\mathrm{Id} + \mathcal{K}_0)\varphi\ , \qquad \mathcal{K}_t\varphi := \mathrm{i} \int_t^{+\infty} U_N(t,s) R_N(s) e^{-\mathrm{i}sH_N} \varphi\, \mathrm{d}s, \qquad t \geq 0,$$

with $\varphi \in \operatorname{Ran} g_J(H_N) \cap \mathcal{H}^k$, gives rise to a solution with growing Sobolev norms (see also Remark 2.8). Here J is the interval of Corollary 2.14. In particular, as $|J \cap \sigma(H_N)| > 0$ and $\sigma(H_N) \cap J = \sigma_{ac}(H_N) \cap J$, the set $\operatorname{Ran} g_J(H_N)$ has infinite dimension. Let us prove that $\mathrm{Id} + \mathcal{K}_0$

[5] apply the theorem with $\tau = 0$ and note that in that paper we defined $\|\psi\|_r \equiv \|K_0^{r/2}\psi\|_0$, therefore the estimate in that paper reads explicitly $\|K_0^{r/2} U_N(t,s)\psi\|_0 \leq C_r\, \langle t-s\rangle^{r/2} \|K_0^{r/2}\psi\|_0$

is injective. Assume there are $\varphi_1 \neq \varphi_2 \in \operatorname{Ran} g_J(H_N) \cap \mathcal{H}^k$ with $(\mathrm{Id} + \mathcal{K}_0)\varphi_1 = (\mathrm{Id} + \mathcal{K}_0)\varphi_2$. Put $u_j(t) := \mathcal{K}_t \varphi_j$, $j = 1, 2$; arguing as in the previous proof one has $\|u_j(t)\|_r \to 0$ as $t \to \infty$. Then $\mathcal{U}_N(t)^{-1}(e^{-\mathrm{i}tH_N}\varphi_j + u_j(t))$, $j = 1, 2$, both solve (1.9) and have the same initial datum, so they are the same solution $\psi(t)$ of equation (1.9). Then

$$\|\varphi_1 - \varphi_2\|_0 \leq C_r \|\mathcal{U}_N^{-1}(t) e^{-\mathrm{i}tH_N}(\varphi_1 - \varphi_2)\|_r \leq C_r \big(\|u_1(t)\|_r + \|u_2(t)\|_r\big) \to 0$$

as $t \to \infty$. Hence $\varphi_1 = \varphi_2$. □

3 Applications

In the following section we apply Theorem 1.9 to the harmonic oscillator on $\mathbb{R}$ and the half-wave equation on $\mathbb{T}$. In both cases we construct transporters which are stable under small, time periodic, pseudodifferential perturbations.

3.1 Harmonic oscillator on $\mathbb{R}$

Consider the quantum harmonic oscillator

$$\mathrm{i}\partial_t \psi = \frac{1}{2}(-\partial_x^2 + x^2)\psi + V(t, x, D)\psi, \quad x \in \mathbb{R}. \tag{3.1}$$

Here $K_0 := \frac{1}{2}\left(-\partial_x^2 + x^2\right)$ is the quantum Harmonic oscillator, the scale of Hilbert spaces is defined as usual by $\mathcal{H}^r = \operatorname{Dom}(K_0^r)$, and the base space $(\mathcal{H}^0, \langle \cdot, \cdot \rangle)$ is $L^2(\mathbb{R}, \mathbb{C})$ with its standard scalar product. The perturbation V is chosen as the Weyl quantization of a symbol belonging to the following class:

Definition 3.1. *A function f is a* symbol of order $\rho \in \mathbb{R}$ *if $f \in C^\infty(\mathbb{R}_x \times \mathbb{R}_\xi, \mathbb{C})$ and $\forall \alpha, \beta \in \mathbb{N}_0$, there exists $C_{\alpha,\beta} > 0$ such that*

$$|\partial_x^\alpha \, \partial_\xi^\beta f(x, \xi)| \leq C_{\alpha,\beta} \; (1 + |x|^2 + |\xi|^2)^{\rho - \frac{\beta+\alpha}{2}} \; .$$

We will write $f \in S^\rho_{\mathrm{har}}$.

We endow S^ρ_{har} with the family of seminorms

$$\wp_j^\rho(f) := \sum_{|\alpha|+|\beta| \leq j} \; \sup_{(x,\xi)\in\mathbb{R}^2} \frac{\left|\partial_x^\alpha \, \partial_\xi^\beta f(x,\xi)\right|}{(1 + |x|^2 + |\xi|^2)^{\rho - \frac{\beta+\alpha}{2}}} \; , \qquad j \in \mathbb{N} \cup \{0\} \; .$$

Such seminorms turn S^ρ_{har} into a Fréchet space. If a symbol f depends on additional parameters (e.g. it is time dependent), we ask that all the seminorms are uniform w.r.t. such parameters. To a symbol $f \in S^\rho_{\mathrm{har}}$ we associate the operator $f(x, D)$ by standard Weyl quantization

$$\Big(f(x, D)\psi\Big)(x) := \frac{1}{2\pi} \iint_{y,\xi\in\mathbb{R}} \mathrm{e}^{\mathrm{i}(x-y)\xi} \, f\left(\frac{x+y}{2}, \xi\right) \, \psi(y) \, \mathrm{d}y\mathrm{d}\xi \; .$$

Definition 3.2. *We say that $F \in \mathcal{A}_\rho$ if it is a pseudodifferential operator with symbol of class S^ρ_{har}, i.e., if there exists $f \in S^\rho_{\mathrm{har}}$ and S smoothing (in the sense of Definition 1.3) such that $F = f(x, D_x) + S$.*

Remark 3.3. With our numerology, the symbol of the harmonic oscillator K_0 is of order 1, $\frac{1}{2}(x^2 + \xi^2) \in S^1_{\mathrm{har}}$, and not of order 2 as typically in the literature.

As an application of the abstract theorems, we describe a class of operators which are transporters. This class, which we call *smooth Töplitz operators*, is easily described in terms of their matrix elements, which we now introduce. We denote by $\{\mathbf{e}_n\}_{n\in\mathbb{N}}$ the Hermite basis, formed by the (orthonormal) eigenvectors of the Harmonic oscillator K_0:

$$K_0\mathbf{e}_n = \left(n - \frac{1}{2}\right)\mathbf{e}_n, \quad \|\mathbf{e}_n\|_0 = 1, \quad n \in \mathbb{N}\ . \tag{3.2}$$

To each operator $\mathsf{H} \in \mathcal{L}(\mathcal{H})$ we associate its *matrix* $(\mathsf{H}_{mn})_{m,n\in\mathbb{N}}$ with respect to the Hermite basis, whose elements are given by

$$\mathsf{H}_{mn} := \langle \mathsf{H}\,\mathbf{e}_n, \mathbf{e}_m\rangle\ , \qquad \forall m,n \in \mathbb{N}\ . \tag{3.3}$$

Remark 3.4. If H is selfadjoint, so is its matrix $(\mathsf{H}_{mn})_{m,n\in\mathbb{N}}$, in particular $\mathsf{H}_{mn} = \overline{\mathsf{H}_{nm}}$.

Definition 3.5 (Smooth Töplitz operators)**.** *A linear operator* $\mathsf{H} \in \mathcal{L}(\mathcal{H})$ *is said a* Töplitz operator *if the entries of its matrix are constant along each diagonal, i.e.*

$$\mathsf{H}_{m_1n_1} = \mathsf{H}_{m_2n_2}, \quad \forall m_1,n_1,m_2,n_2 \in \mathbb{N}\colon \quad m_1 - n_1 = m_2 - n_2\ . \tag{3.4}$$

A Töplitz operator is said smooth *if its matrix elements decay fast off diagonal, i.e.* $\forall N > 0$, $\exists C_N > 0$ *such that*

$$|\mathsf{H}_{mn}| \leq \frac{C_N}{\langle m-n\rangle^N}\ , \qquad \forall m,n \in \mathbb{N}\ . \tag{3.5}$$

Example 3.6. The shift operators S and its adjoint S^* are defined on the Hermite functions $\{\mathbf{e}_n\}_{n\geq 1}$ by

$$S\mathbf{e}_n = \mathbf{e}_{n+1}\ , \quad \forall n \in \mathbb{N}\ , \qquad S^*\mathbf{e}_n = \begin{cases} 0 & \text{if } n = 1 \\ \mathbf{e}_{n-1} & \text{if } n \geq 2 \end{cases}\ . \tag{3.6}$$

The action of S (and of S^*) is extended on all $\mathcal{H}$ by linearity, giving $S\psi = \sum_{n\geq 1}\psi_n\mathbf{e}_{n+1}$, where we defined $\psi_n := \langle \psi, \mathbf{e}_n\rangle$ for $n \geq 1$. Their matrices are given by

$$(S_{mn})_{m,n\in\mathbb{N}} = \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & 1 & 0 & \\ & & \ddots & \ddots \end{pmatrix}, \qquad (S^*_{mn})_{m,n\in\mathbb{N}} = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & \ddots \end{pmatrix},$$

from which it is clear that both S and S^* are smooth Töplitz operators.

We prove in the following that any smooth Töplitz operator is actually a pseudodifferential operator in $\mathcal{A}_0$, see Lemma 3.10.

As an application of the abstract theorems, we show that any smooth Töplitz operator becomes a transporter for the Harmonic oscillator once it is multiplied by an appropriate scalar time periodic function.

Theorem 3.7. *Let* $\mathsf{V}(x,D)$ *be a selfadjoint and smooth Töplitz operator (see Definition 3.5). Take* $m,n \in \mathbb{N}$, $m > n$, *such that the matrix element*

$$\mathsf{V}_{m-n} := \langle \mathsf{V}(x,D)\,\mathbf{e}_n, \mathbf{e}_m\rangle \neq 0\ .$$

Then

$$V(t,x,D) := \cos((m-n)t)\,\mathsf{V}(x,D) \tag{3.7}$$

is a transporter for (3.1)*. More precisely,* $\forall r \geq 0$ *there exist a solution* $\psi(t) \in \mathcal{H}^r$ *of* (3.1) *and constants* $C, T > 0$ *such that*

$$\|\psi(t)\|_r \geq C\langle t\rangle^r, \quad \forall t > T.$$

The theorem follows applying Theorem 1.8. So we check that Assumptions I-III are fulfilled. Regarding Assumption I, it is the usual Weyl calculus for symbols in $S^\rho_{\rm har}$, see e.g. [58]. Concerning Assumption II, one has $\sigma(K_0) = \{n - \frac{1}{2}\}_{n\in\mathbb{N}}$. Furthermore Egorov theorem for the Harmonic oscillator [38] states that the map $t \mapsto e^{itK_0}\mathsf{A}e^{-itK_0} \in C^\infty(\mathbb{T}, \mathcal{A}_\rho)$ for any $\mathsf{A} \in \mathcal{A}_\rho$ (use also the periodicity of the flow of K_0). This can be seen e.g. by remarking that the symbol of $e^{itK_0}\mathsf{A}e^{-itK_0}$ is $a \circ \phi^t_{\rm har}$, where $a \in S^\rho_{\rm har}$ is the symbol of A and $\phi^t_{\rm har}$ is the time t flow of the harmonic oscillator; explicitly

$$\left(a \circ \phi^t_{\rm har}\right)(x, \xi) = a(x\cos t + \xi \sin t, -x \sin t + \xi \cos t) \ .$$

Verification of Assumption III. First we show that smooth Töplitz operators belong to $\mathcal{A}_0$. We exploit Chodosh's characterization [11], which we now recall. Define the discrete difference operator $\triangle$ on a function $M : \mathbb{N}\times\mathbb{N} \to \mathbb{C}$ by

$$(\triangle M)(m, n) := M(m + 1, n + 1) - M(m, n) \ ,$$

and its powers $\triangle^\gamma$, $\gamma \in \mathbb{N}$, by $\triangle$ applied γ-times.

Definition 3.8 (Symbol matrix)**.** *A function $M : \mathbb{N}\times\mathbb{N} \to \mathbb{C}$ will be said to be a* symbol matrix of order ρ *if for any $\gamma \in \mathbb{N}_0$, $N \in \mathbb{N}$, there exists $C_{\gamma,N} > 0$ such that*

$$|(\triangle^\gamma M)(m, n)| \leq C_{\gamma,N}\frac{(1 + m + n)^{\rho-|\gamma|}}{\langle m - n\rangle^N} \ , \quad \forall m, n \in \mathbb{N} \ . \tag{3.8}$$

The connection between pseudodifferential operators and symbol matrices is given by Chodosh's characterization:

Theorem 3.9 ([11])**.** *An operator H belongs to $\mathcal{A}_\rho$ if and only if its matrix $M^{(\mathsf{H})}(m, n) := \mathsf{H}_{mn}$ (as defined in* (3.1)*) is a symbol matrix of order ρ.*

As a direct consequence we have the following result:

Lemma 3.10. *Any smooth Töplitz operator is a pseudodifferential operator in $\mathcal{A}_0$.*

Proof. We use Theorem 3.9. Let H be smooth Töplitz and put $M^{(\mathsf{H})}(m, n) := \mathsf{H}_{mn}$. Then (3.8) holds with $\rho = \gamma = 0$ by (3.5). By (3.5) one has $\triangle M^{(\mathsf{H})} = 0$; so (3.8) holds also $\forall \gamma \geq 1$. □

In particular $V(t, x, D) = \cos((m - n)t)\mathsf{V}(x, D)$ belongs to $C^\infty(\mathbb{T}, \mathcal{A}_0)$, which is the first required property of Assumption III.

Remark 3.11. The shift operators S, S^*, defined in (3.6), belong to $\mathcal{A}_0$ being smooth Töplitz. Also their (integer) powers S^k, S^{*k}, given for $k \in \mathbb{N}$ by

$$S^k\mathbf{e}_n = \mathbf{e}_{n+k} \ , \quad \forall n \in \mathbb{N} \ , \qquad S^{*k}\mathbf{e}_n = \begin{cases} 0 & \text{if } n \leq k \\ \mathbf{e}_{n-k} & \text{if } n \geq k + 1 \end{cases}$$

are smooth Töplitz, so in $\mathcal{A}_0$.

Next we compute the resonant average of $V(t, x, D)$.

Lemma 3.12. *Let $V(t, x, D)$ as in* (3.7)*. Its resonant average $\langle V\rangle$ (see* (1)*) is*

$$\langle V\rangle = \frac{1}{2}\left(\mathsf{V}_k\, S^k + \overline{\mathsf{V}_k}\, S^{*k}\right) \ , \qquad k := m - n \in \mathbb{N} \ , \tag{3.9}$$

where $S \in \mathcal{A}_0$ is defined in (3.6) *and $\mathsf{V}_k := \mathsf{V}_{m-n} := \langle \mathsf{V}\,\mathbf{e}_n, \mathbf{e}_m\rangle \in \mathbb{C}$.*

Proof. For $\ell \in \mathbb{N}$, denote by $\Pi_\ell \varphi := \langle \varphi, \mathbf{e}_\ell \rangle \, \mathbf{e}_\ell$ the projector on the Hermite function $\mathbf{e}_\ell$. Clearly

$$e^{\mathrm{is}K_0}\, \Pi_\ell = \Pi_\ell \, e^{\mathrm{is}K_0} = e^{\mathrm{is}(\ell - \frac{1}{2})}\, \Pi_\ell, \quad \forall \ell \in \mathbb{N} \ .$$

From now on we simply write $\mathsf{V} \equiv \mathsf{V}(x, D)$. Using this identity and writing $\mathrm{Id} = \sum_{\ell \geq 1} \Pi_\ell$ we get

$$e^{\mathrm{is}K_0}\, \mathsf{V}\, e^{-\mathrm{is}K_0} = \sum_{j,\ell \geq 1} e^{\mathrm{is}(j-\ell)} \Pi_j \, \mathsf{V} \, \Pi_\ell = \sum_{j,\ell \geq 1} e^{\mathrm{is}(j-\ell)} \, \langle \cdot, \mathbf{e}_\ell \rangle \, \langle \mathsf{V}\mathbf{e}_\ell, \mathbf{e}_j \rangle \, \mathbf{e}_j \ .$$

Now we compute, with $k := m - n \in \mathbb{N}$,

$$\begin{aligned} \langle V \rangle &= \frac{1}{2\pi} \int_0^{2\pi} \cos(ks)\, e^{\mathrm{is}K_0}\, \mathsf{V}\, e^{-\mathrm{is}K_0} \mathrm{d}s = \sum_{j,\ell \geq 1} \langle \mathsf{V}\mathbf{e}_\ell, \mathbf{e}_j \rangle \, \langle \cdot, \mathbf{e}_\ell \rangle \, \mathbf{e}_j \, \frac{1}{2\pi} \int_0^{2\pi} \cos(ks)\, e^{\mathrm{is}(j-\ell)}\, \mathrm{d}s \\ &= \frac{1}{2} \sum_{\ell \geq 1} \langle \mathsf{V}\mathbf{e}_\ell, \mathbf{e}_{\ell+k} \rangle \langle \cdot, \mathbf{e}_\ell \rangle \, \mathbf{e}_{\ell+k} + \frac{1}{2} \sum_{\ell \geq k+1} \langle \mathsf{V}\mathbf{e}_\ell, \mathbf{e}_{\ell-k} \rangle \langle \cdot, \mathbf{e}_\ell \rangle \, \mathbf{e}_{\ell-k} = \frac{1}{2} \mathsf{V}_k \, S^k + \frac{1}{2} \overline{\mathsf{V}_k} \, S^{*k} \end{aligned}$$

where in the last line we used $\mathsf{V}_{-k} = \langle \mathsf{V}\mathbf{e}_\ell, \mathbf{e}_{\ell-k} \rangle = \overline{\langle \mathsf{V}\mathbf{e}_{\ell-k}, \mathbf{e}_\ell \rangle} = \overline{\mathsf{V}_k}$ being V selfadjoint and smooth Töplitz (see Remark 3.4). ☐

Now define the selfadjoint operator

$$A := \frac{\mathsf{V}_k}{\mathrm{i}} \, (K_0 + \frac{1}{2}) \, S^k - \frac{\overline{\mathsf{V}_k}}{\mathrm{i}} \, S^{*k} \, (K_0 + \frac{1}{2}) - \frac{\overline{\mathsf{V}_k}}{\mathrm{i}} \, (K_0 + \frac{1}{2}) \, S^{*k} + \frac{\mathsf{V}_k}{\mathrm{i}} \, S^k \, (K_0 + \frac{1}{2}) \ , \tag{3.10}$$

which belongs to $\mathcal{A}_1$ by symbolic calculus as $K_0 \in \mathcal{A}_1$ and $S, S^* \in \mathcal{A}_0$ (see Remark 3.11). The next lemma verifies Assumption III.

Lemma 3.13. *Assume that* $\mathsf{V}_k \neq 0$. *The following holds true:*

(i) The spectrum of the operator $H_0 := \langle V \rangle$ *fulfills* $\sigma(H_0) \supseteq [-|\mathsf{V}_k|, \, |\mathsf{V}_k|]$.

(ii) For any interval $I_0 \subset [-|\mathsf{V}_k|, \, |\mathsf{V}_k|]$, *any* $g_{I_0} \in C_c^\infty(\mathbb{R}, \mathbb{R}_{\geq 0})$ *with* $g_{I_0} \equiv 1$ *over* I_0 *and* $\operatorname{supp} g_{I_0} \subset [-|\mathsf{V}_k|, \, |\mathsf{V}_k|]$, *there exist* $\theta > 0$ *and* K *compact operator such that*

$$g_{I_0}(H_0) \, \mathrm{i}[H_0, A] \, g_{I_0}(H_0) \geq \theta \, g_{I_0}(H_0)^2 + \mathsf{K} \ .$$

Here A *is defined in* (3.1).

Proof. (*i*) Let $\mathtt{f}(\rho) := \mathrm{Re}(\mathsf{V}_k \, e^{-\mathrm{i}\rho k})$. We shall prove that $\mathtt{f}(\rho) \in \sigma(H_0) \ \forall \rho \in \mathbb{R}$, from which the claim follows. As H_0 is selfadjoint, it is enough to construct a Weyl sequence for $\mathtt{f}(\rho)$, i.e. a sequence $(\psi^{(n)})_{n \geq 1}$ with $\|\psi^{(n)}\|_0 = 1 \ \forall n$ and $\|(H_0 - \mathtt{f}(\rho))\psi^{(n)}\|_0 \to 0$ as $n \to \infty$. We put

$$\psi^{(n)} := \frac{1}{\sqrt{n}} \sum_{\ell=1}^{n} e^{\mathrm{i}\rho\ell} \mathbf{e}_\ell \ .$$

Then $\|\psi^{(n)}\|_0 = 1 \ \forall n$ and a direct computation shows that for $n > k$

$$H_0 \psi^{(n)} = \frac{1}{\sqrt{n}} \frac{\overline{\mathsf{V}}_k}{2} e^{\mathrm{i}\rho k} \sum_{m=1}^{k} e^{\mathrm{i}\rho m} \, \mathbf{e}_m + \frac{1}{\sqrt{n}} \mathtt{f}(\rho) \sum_{m=k+1}^{n-k} e^{\mathrm{i}\rho m} \, \mathbf{e}_m + \frac{1}{\sqrt{n}} \frac{\mathsf{V}_k}{2} e^{-\mathrm{i}\rho k} \sum_{m=n-k+1}^{n+k} e^{\mathrm{i}\rho m} \, \mathbf{e}_m \ .$$

Thus one finds a constant $C_k > 0$ such that

$$\|(H_0 - \mathtt{f}(\rho))\psi^{(n)}\|_0 \leq \frac{C_k}{\sqrt{n}} \to 0 \quad \text{as } n \to \infty \ ,$$

proving that $\psi^{(n)}$ is a Weyl sequence; by Weyl criterium $\mathtt{f}(\rho) \in \sigma(H_0)$.
(ii) First note that, by (3.1) and (3.6), one has $\forall k \in \mathbb{N}$

$$[S^k, K_0] = -kS^k, \qquad [S^{*k}, K_0] = kS^{*k}, \qquad [S^{*k}, S^k] = \Pi_{\leq k} \qquad (3.11)$$
$$S^k S^{*k} = \mathrm{Id} - \Pi_{\leq k} \qquad S^{*k} S^k = \mathrm{Id}$$

where $\Pi_{\leq k} := \sum_{\ell=1}^k \Pi_\ell$ is the projector on the Hermite modes with index $\leq k$. Using (3.1) a direct computation gives

$$\begin{aligned} \mathrm{i}[H_0, A] &= k\big(2|\mathsf{V}_k|^2 - \mathsf{V}_k^2 S^{2k} - \overline{\mathsf{V}}_k^2 S^{*2k} - |\mathsf{V}_k|^2 \Pi_{\leq k}\big) + 2|\mathsf{V}_k|^2 (K_0 + \frac{1}{2})\Pi_{\leq k} \\ &= 4k\big(|\mathsf{V}_k|^2 - H_0^2\big) + 2|\mathsf{V}_k|^2 (K_0 + \frac{1}{2} - k)\Pi_{\leq k} \ . \end{aligned} \qquad (3.12)$$

Clearly $\mathsf{K} := 2|\mathsf{V}_k|^2(K_0 + \frac{1}{2} - k)\Pi_{\leq k}$ is compact, being finite rank.
Next put $\tilde{f}(\lambda) = 4k(|\mathsf{V}_k|^2 - \lambda^2)$ getting $\forall \varphi \in \mathcal{H}$

$$\langle g_{I_0}(H_0)\, \mathrm{i}[H_0, A]\, g_{I_0}(H_0)\varphi, \varphi\rangle = \langle g_{I_0}(H_0)\, \tilde{f}(H_0)\, g_{I_0}(H_0)\varphi, \varphi\rangle + \langle g_{I_0}(H_0)\, \mathsf{K}\, g_{I_0}(H_0)\varphi, \varphi\rangle \ . \qquad (3.13)$$

Note that $\tilde{f}$ is strictly positive in the interior of $[-|\mathsf{V}_k|, |\mathsf{V}_k|]$; we put

$$\theta := \inf\{\tilde{f}(\lambda) \colon \lambda \in \operatorname{supp} g_{I_0}\} > 0 \ .$$

With this information we apply the spectral theorem and get

$$\begin{aligned} \langle g_{I_0}(H_0)\, \tilde{f}(H_0)\, g_{I_0}(H_0)\varphi, \varphi\rangle &= \int_{\lambda \in \sigma(H_0)} g_{I_0}(\lambda)^2\, \tilde{f}(\lambda)\, \mathrm{d}m_\varphi(\lambda) \\ &\geq \theta \int_{\lambda \in \sigma(H_0)} g_{I_0}(\lambda)^2\, \mathrm{d}m_\varphi(\lambda) = \theta \|g_{I_0}(H_0)\varphi\|_0^2 \ . \end{aligned}$$

This estimate and (3.1) proves that H_0 fulfills a Mourre estimate over I_0. □

To conclude this section, we recall that in [50] it is proved that the pseudodifferential operator

$$V(t, x, D) := e^{-\mathrm{i}tK_0}\,(S + S^*)\, e^{\mathrm{i}tK_0} \qquad (3.14)$$

is a universal transporter (see Definition 1.1). Using the abstract Theorem 1.9 we prove its stability under perturbations of class $C^\infty(\mathbb{T}, \mathcal{A}_0)$:

Theorem 3.14. *Consider equation* (3.1) *with* $V(t, x, D)$ *defined in* (3.1). *There exist* $\epsilon_0, \mathtt{M} > 0$ *such that* $\forall W \in C^\infty(\mathbb{T}, \mathcal{A}_0)$ *with* $\sup_t \wp_{\mathtt{M}}^0(W(t, x, D)) \leq \epsilon_0$*, the operator* $V(t, x, D) + \epsilon W(t, x, D)$ *is a transporter. More precisely* $\forall r > 0$ *there exist a solution* $\psi(t) \in \mathcal{H}^r$ *of* $\mathrm{i}\partial_t \psi = \big(\frac{-\partial_x^2 + x^2}{2} + V(t, x, D) + W(t, x, D)\big)\psi$ *and constants* $C, T > 0$ *such that*

$$\|\psi(t)\|_r \geq C\langle t\rangle^r, \quad \forall t \geq T \ .$$

Proof. We verify Assumption III. Clearly $V(t, x, D) \in C^\infty(\mathbb{T}, \mathcal{A}_0)$ and $\langle V\rangle = S + S^*$, so it has the form (3.12) with $k = 1$ and $\mathsf{V}_1 = 2$. Lemma 3.13 implies that $\langle V\rangle$ fulfills a Mourre estimate. □

Remark 3.15. Actually formula (3.1) in case $\langle V\rangle = S + S^*$ gives

$$\mathrm{i}[\langle V\rangle, A] = 4(4 - \langle V\rangle^2) \ , \tag{3.15}$$

so $\langle V\rangle$ fulfills a strict Mourre estimate. Moreover (3.15) implies that $\mathrm{ad}^p_{\mathrm{i}\langle V\rangle}(A) = 0 \ \forall p \geq 2$, and using also (A.2), (B) and the arguments in Appendix A of [50] we get that

$$\|e^{-\mathrm{i}t\langle V\rangle}\varphi\|_k^2 \geq c_k \langle e^{\mathrm{i}t\langle V\rangle} A^{2k} e^{-\mathrm{i}t\langle V\rangle}\varphi, \varphi\rangle = \widetilde{c}_k t^{2k} \langle (\mathrm{i}[\langle V\rangle, A])^{2k} \varphi, \varphi\rangle + \mathcal{O}(t^{2k-1}) \ .$$

Since $4 - \langle V\rangle^2$ is injective[6], the flow $e^{-\mathrm{i}t\langle V\rangle}\varphi$ has unbounded trajectories for any nontrivial initial data. Hence the same holds for the solutions of $\mathrm{i}\partial_t u = (H_0 + V(t,x,D))u$ and $V(t,x,D)$ in (3.1) is a universal transporter.

3.2 Half-wave equation on $\mathbb{T}$

The half-wave equation on $\mathbb{T}$ is given by

$$\mathrm{i}\partial_t \psi = |D|\psi + V(t,x,D)\psi \ , \qquad x \in \mathbb{T} \ . \tag{3.16}$$

Here $|D|$ is the Fourier multiplier defined by

$$|D|\psi := \sum_{j\in\mathbb{Z}} |j|\, \psi_j \, e^{\mathrm{i}jx} \ , \qquad \psi_j := \frac{1}{2\pi}\int_{\mathbb{T}} \psi(x) e^{-\mathrm{i}jx} \mathrm{d}x \ ,$$

whereas $V(t,x,D)$ is a pseudodifferential operator of order 0. In this case $K_0 := |D| + 1$, the scale of Hilbert spaces defined as $\mathcal{H}^r = \mathrm{Dom}\,(K_0^r)$ coincides with standard Sobolev spaces on the torus $H^r(\mathbb{T})$, and the base space $(\mathcal{H}^0, \langle\cdot,\cdot\rangle)$ is $L^2(\mathbb{T},\mathbb{C})$ with its standard scalar product. In this setting we shall use pseudodifferential operators with periodic symbols, belonging to the following class:

Definition 3.16. *A function $a(x,\xi)$ is a* periodic symbol of order $\rho \in \mathbb{R}$ *if $a \in C^\infty(\mathbb{T}_x \times \mathbb{R}_\xi, \mathbb{C})$ and for any $\alpha, \beta \in \mathbb{N}_0$, there exists a constant $C_{\alpha\beta} > 0$ such that*

$$\left|\partial_x^\alpha \, \partial_\xi^\beta \, a(x,\xi)\right| \leq C_{\alpha\beta} \, \langle\xi\rangle^{\rho-\beta}, \quad \forall x \in \mathbb{T}, \, \forall \xi \in \mathbb{R} \, .$$

We will write $a \in S^\rho_{\mathrm{per}}$. We also put $S^{-\infty}_{\mathrm{per}} := \bigcap_{\rho\in\mathbb{R}} S^\rho_{\mathrm{per}}$ the class of smoothing symbols.

We endow S^ρ_{per} with the family of seminorms

$$\wp_j^\rho(a) := \sum_{|\alpha|+|\beta|\leq j} \ \sup_{(x,\xi)\in\mathbb{T}\times\mathbb{R}} \left|\partial_x^\alpha \, \partial_\xi^\beta a(x,\xi)\right| \, \langle\xi\rangle^{-\rho+\beta} \ , \qquad j \in \mathbb{N}_0 \ .$$

Such seminorms turn S^ρ_{per} into a Fréchet space. If a symbol a depends on additional parameters (e.g. it is time dependent), we ask that all the seminorms are uniform w.r.t. such parameters. To a symbol $a \in S^\rho_{\mathrm{per}}$ we associate its quantization $a(x,D)$ acting on a 2π-periodic function $u(x) = \sum_{j\in\mathbb{Z}} u_j e^{\mathrm{i}jx}$ as

$$a(x,D)u := \mathrm{Op}(a)[u] := \sum_{j\in\mathbb{Z}} a(x,j)\, u_j \, e^{\mathrm{i}jx} \, .$$

[6] it is unitarily equivalent, via the map $\mathbf{e}_n \mapsto \sin(ny) \ \forall n \in \mathbb{N}$, to the multiplication operator by $4\sin(y)^2$ on the space $L^2_{\mathrm{odd}}(\mathbb{T})$ of L^2 odd functions on $\mathbb{T}$

Remark 3.17. Given a symbol $a(\xi)$ independent of x, then $\mathrm{Op}(a)$ is the Fourier multiplier operator $a(D)u = \sum_{j\in\mathbb{Z}} a(j)\, u_j\, e^{\mathrm{i}jx}$. If instead the symbol $a(x)$ is independent of ξ, then $\mathrm{Op}(a)$ is the multiplication operator $\mathrm{Op}(a)u = a(x)u$.

Definition 3.18. *We say that* $A \in \mathcal{A}_\rho$ *if* $A = \mathrm{Op}(a)$ *with* $a \in S^\rho_{\mathrm{per}}$.

Example 3.19. The operator $|D| \in \mathcal{A}_1$ with symbol given by $\mathtt{d}(\xi) := |\xi|\chi(\xi)$ where χ is an even, positive smooth cut-off function satisfying $\chi(\xi) = 0$ for $|\xi| \le \frac{1}{5}$, $\chi(\xi) = 1$ for $|\xi| \ge \frac{2}{5}$ and $\partial_\xi \chi(\xi) > 0$ $\forall \xi \in (\frac{1}{5}, \frac{2}{5})$.
Also the Fourier projectors $\Pi_\pm$ and Π_0 defined by

$$\Pi_+ u := \sum_{j\ge 1} u_j\, e^{\mathrm{i}jx}, \qquad \Pi_- u := \sum_{j\le -1} u_j\, e^{\mathrm{i}jx}, \qquad \Pi_0 u := u_0 \tag{3.17}$$

are pseudodifferential operators. In particular $\Pi_\pm = \mathrm{Op}(\pi_\pm) \in \mathcal{A}_0$ and $\Pi_0 = \mathrm{Op}(\pi_0) \in \mathcal{A}_{-\infty}$, where $\pi_\pm, \pi_0$ are a smooth partition of unity, $\pi_+(\xi) + \pi_-(\xi) + \pi_0(\xi) = 1$ $\forall \xi$, fulfilling

$$\pi_+(\xi) = \begin{cases} 1 & \text{if } \xi \ge \frac{4}{5} \\ 0 & \text{if } \xi \le \frac{3}{5} \end{cases}, \quad \pi_-(\xi) = \begin{cases} 1 & \text{if } \xi \le -\frac{4}{5} \\ 0 & \text{if } \xi \ge -\frac{3}{5} \end{cases}, \quad \pi_0(\xi) = \begin{cases} 1 & \text{if } |\xi| \le \frac{3}{5} \\ 0 & \text{if } |\xi| \ge \frac{1}{5} \end{cases}.$$

In this setting we prove that any multiplication operator, multiplied by an appropriate time periodic function, becomes a transporter. Here the result.

Theorem 3.20. *Let* $v \in C^\infty(\mathbb{T}, \mathbb{R})$. *Choose* $j \in \mathbb{Z} \setminus \{0\}$ *such that the Fourier coefficient* $v_j \neq 0$. *Then the selfadjoint operator*

$$V(t, x) := \cos(jt)\, v(x)$$

is a transporter. More precisely, $\forall r > 0$ *there exist a solution* $\psi(t) \in \mathcal{H}^r$ *of* $\mathrm{i}\partial_t \psi = \big(|D| + V(t,x)\big)\psi$ *and constants* $C, T > 0$ *such that*

$$\|\psi(t)\|_r \ge C\langle t\rangle^r, \quad \forall t > T.$$

The theorem follows from Theorem 1.8. So first we put ourselves in the setting of the abstract theorem and rewrite (3.2) as

$$\mathrm{i}\partial_t \psi = K_0 \psi + \widetilde{V}(t,x)\psi, \qquad \widetilde{V}(t,x) := \cos(jt)v(x) - 1 \in \mathcal{A}_0 . \tag{3.18}$$

Again we check Assumptions I-III. Regarding Assumption I, it is the usual pseudodifferential calculus for periodic symbols, see e.g. [57].

Verification of Assumption II. One has $\sigma(K_0) = \{n\}_{n\in\mathbb{N}}$. To prove Assumption II (ii) we use the identity $e^{-\mathrm{i}tK_0}\mathsf{A}\, e^{\mathrm{i}tK_0} = e^{-\mathrm{i}t|D|}\mathsf{A}\, e^{\mathrm{i}t|D|}$ and Egorov theorem for $|D|$, see e.g. [61, Theorem 4.3.6]. Actually we need also the following version of Egorov theorem.

Lemma 3.21. *Let* $a \in S^\rho_{\mathrm{per}}$, $\rho \in \mathbb{R}$. *Then*

$$e^{\mathrm{i}t|D|}\, \mathrm{Op}(a)\; e^{-\mathrm{i}t|D|} = \mathrm{Op}(a(x+t,\xi))\, \Pi_+ + \mathrm{Op}(a(x-t,\xi))\, \Pi_- + R(t) \tag{3.19}$$

where $\Pi_\pm$ *are defined in* (3.19) *and* $R(t) \in C^\infty(\mathbb{T}, \mathcal{A}_{\rho-1})$.
If $\mathrm{Op}(a)$ *is selfadjoint, so is* $e^{\mathrm{i}t|D|}\mathrm{Op}(a)\; e^{-\mathrm{i}t|D|}$, $\forall t$.

Proof. The classical Egorov theorem for the half-Laplacian $|D|$ says that

$$e^{\mathrm{i}t|D|}\,\mathrm{Op}\,(a)\;e^{-\mathrm{i}t|D|} = \mathrm{Op}\left(a\circ\phi^t_{\mathtt{d}}(x,\xi)\right) + \widetilde{R}(t)$$

where $\phi^t_{\mathtt{d}}(x,\xi)$ is the time t flow of the classical Hamiltonian $\mathtt{d}(\xi) = |\xi|\chi(\xi)$ (the symbol of $|D|$) and $\widetilde{R}(t) \in C^\infty(\mathbb{R}, \mathcal{A}_{\rho-1})$, see e.g. [61, Theorem 4.3.6].
We compute more explicitly $a\circ\phi^t_{\mathtt{d}}(x,\xi)$. The Hamiltonian equations of $\mathtt{d}(\xi)$ and its flow $\phi^t_{\mathtt{d}}$ are given by

$$\begin{cases}\dot{x} = \partial_\xi \mathtt{d}(\xi) = \mathtt{d}'(\xi)\\ \dot{\xi} = -\partial_x \mathtt{d}(\xi) = 0\end{cases}, \qquad \phi^t_{\mathtt{d}}(x,\xi) = (x + t\mathtt{d}'(\xi),\, \xi)\ .$$

As $\mathtt{d}'(\xi) = 1$ for $\xi \geq \frac{2}{5}$ and $\mathtt{d}'(\xi) = -1$ for $\xi \leq -\frac{2}{5}$, we write

$$(a\circ\phi^t_{\mathtt{d}})(x,\xi) = a(x+t,\xi)\,\pi_+(\xi) + a(x-t,\xi)\,\pi_-(\xi) + a(x+t\mathtt{d}'(\xi),\xi)\,\pi_0(\xi)\ .$$

As $\pi_0 \in S^{-\infty}_{\mathrm{per}}$, the operator $\mathrm{Op}\,(a(x+t\mathtt{d}'(\xi),\xi)\,\pi_0(\xi)) \in C^\infty(\mathbb{R},\mathcal{A}_{-\infty})$. Moreover by symbolic calculus

$$\mathrm{Op}\,(a(x\pm t,\xi)\,\pi_\pm(\xi)) = \mathrm{Op}\,(a(x\pm t,\xi))\,\Pi_\pm + R_\pm(t), \quad R_\pm(t) \in C^\infty(\mathbb{R},\mathcal{A}_{\rho-1})\ .$$

Formula (3.21) follows with $R(t) := \widetilde{R}(t) + R_+(t) + R_-(t) + \mathrm{Op}\,(a(x+t\mathtt{d}'(\xi),\xi)\,\pi_0(\xi))$. We claim that $R(t)$ is periodic in time. This follows by difference since both $e^{\mathrm{i}t|D|}\,\mathrm{Op}\,(a)\;e^{-\mathrm{i}t|D|}$ and $\mathrm{Op}\,(a(x\pm t,\xi))\,\Pi_\pm$ are periodic in t (recall that the symbol $a(x,\xi)$ is periodic in x).
Finally as $e^{\pm\mathrm{i}t|D|}$ are unitary, the claim on the selfadjointness of $e^{\mathrm{i}t|D|}\,\mathrm{Op}\,(a)\;e^{-\mathrm{i}t|D|}$ follows. □

Verification of Assumption III. First we compute $\langle\widetilde{V}\rangle$.

Lemma 3.22. *The resonant average $\langle\widetilde{V}\rangle \in \mathcal{A}_0$ of $\widetilde{V}$ (defined in* (3.2)*) is given by*

$$\langle\widetilde{V}\rangle = \mathtt{v}(x) - 1 + R, \qquad \mathtt{v}(x) := \mathrm{Re}(v_j e^{\mathrm{i}jx}) \tag{3.20}$$

and $R \in \mathcal{A}_{-1}$ is selfadjoint.

Proof. First remark that, as $e^{\mathrm{i}tK_0} = e^{\mathrm{i}t|D|}e^{\mathrm{i}t}$,

$$\langle\widetilde{V}\rangle = \frac{1}{2\pi}\int_0^{2\pi} e^{\mathrm{i}s|D|}\,\widetilde{V}(s)\,e^{-\mathrm{i}s|D|}\mathrm{d}s = \frac{1}{2\pi}\int_0^{2\pi}\cos(js)\,e^{\mathrm{i}s|D|}\,v(x)\,e^{-\mathrm{i}s|D|}\,\mathrm{d}s - 1\ . \tag{3.21}$$

We compute $e^{\mathrm{i}s|D|}\,v(x)\,e^{-\mathrm{i}s|D|}$ with the aid of Lemma 3.21, getting

$$e^{\mathrm{i}s|D|}\,v(x)\,e^{-\mathrm{i}s|D|} = v(x+s)\,\Pi_+ + v(x-s)\,\Pi_- + \widetilde{R}(s), \tag{3.22}$$

where $\widetilde{R}(s) \in C^\infty(\mathbb{T},\mathcal{A}_{-1})$. Then, recalling that $v_j = \overline{v_{-j}}$ being $v(x)$ real valued,

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\cos(js)\,e^{\mathrm{i}s|D|}\,v(x)\,e^{-\mathrm{i}s|D|}\,\mathrm{d}s &\overset{(3.2)}{=} \sum_{\sigma=\pm}\frac{1}{2\pi}\int_0^{2\pi}\cos(js)\,v(x\sigma s)\,\mathrm{d}s\,\Pi_\sigma + \frac{1}{2\pi}\int_0^{2\pi}\cos(js)\widetilde{R}(s)\mathrm{d}s\\ &= \mathrm{Re}\left(v_j e^{\mathrm{i}jx}\right)\,(\Pi_+ + \Pi_-) + \frac{1}{2\pi}\int_0^{2\pi}\cos(js)\widetilde{R}(s)\mathrm{d}s\\ &= \mathrm{Re}\left(v_j e^{\mathrm{i}jx}\right) + R\end{aligned}$$

where $R := \frac{1}{2\pi}\int_0^{2\pi}\cos(js)\widetilde{R}(s)\mathrm{d}s - \mathrm{Re}\left(v_j e^{\mathrm{i}jx}\right)\Pi_0 \in \mathcal{A}_{-1}$. Together with (3.2), this proves (3.22).
Finally R is selfadjoint by difference, since both $\langle\widetilde{V}\rangle$ and $\mathtt{v}(x) - 1$ are selfadjoint operators. □

Define the selfadjoint operator

$$A := \mathtt{w}(x)\,\frac{\partial_x}{\mathrm{i}} + \frac{\partial_x}{\mathrm{i}}\,\mathtt{w}(x)\ , \qquad \mathtt{w}(x) := \mathrm{Im}(v_j e^{\mathrm{i}jx}) \tag{3.23}$$

belonging to $\mathcal{A}_1$. The next lemma verifies Assumption III.

Lemma 3.23. *Assume that* $v_j \neq 0$. *The following holds true:*

(i) The operator $H_0 := \langle \widetilde{V} \rangle$ *has spectrum* $\sigma(H_0) \supseteq [-|v_j| - 1,\, |v_j| - 1] =: I$.

(ii) For any interval $I_0 \subset I$, *any* $g_{I_0} \in C_c^\infty(\mathbb{R}, \mathbb{R}_{\geq 0})$ *with* $g_{I_0} \equiv 1$ *over* I_0 *and* $\mathrm{supp}\, g_{I_0} \subset I$, *there exist* $\theta > 0$ *and a compact operator* K *such that*

$$g_{I_0}(H_0)\,\mathrm{i}[H_0, A]\,g_{I_0}(H_0) \geq \theta\, g_{I_0}(H_0)^2 + \mathsf{K}\ .$$

Here A *is defined in* (3.2).

Proof. During the proof we shall use that any operator in $\mathcal{A}_{-1}$ is compact. Moreover we shall simply denote any compact operator by K, which can change from line to line.
(i) By Lemma 3.22, H_0 is a compact perturbation of the multiplication operator by $\mathtt{v}(x) - 1$, whose spectrum coincides with I. Then by Weyl's theorem

$$\sigma(H_0) \supseteq \sigma_{ess}(H_0) = \sigma_{ess}(\mathtt{v}(x) - 1) = I\ .$$

(ii) First notice that, as $\mathtt{v}(x) = \mathrm{Re}(v_j e^{\mathrm{i}jx})$ and $\mathtt{w}(x) = \mathrm{Im}(v_j e^{\mathrm{i}jx})$, one has the identities

$$\mathtt{v}(x)^2 + \mathtt{w}(x)^2 = |v_j|^2, \qquad \mathtt{v}'(x) = -j\,\mathtt{w}(x)\ . \tag{3.24}$$

Next we compute

$$\begin{aligned} \mathrm{i}[H_0, A] &= \mathrm{i}[\mathtt{v}(x) - 1 + R, A] = -2\mathtt{w}(x)\,\mathtt{v}'(x) + \mathrm{i}[R, A] \\ &\overset{(3.2)}{=} 2j\big(|v_j|^2 - \mathtt{v}(x)^2\big) + \mathsf{K} = 2j\big(|v_j|^2 - (H_0 + 1 - R)^2\big) + \mathsf{K} \\ &= 2j\big(|v_j|^2 - (H_0 + 1)^2\big) + \mathsf{K}\ . \end{aligned}$$

Putting $f(\lambda) := 2j\big(|v_j|^2 - (\lambda + 1)^2\big)$, we get $\forall \varphi \in \mathcal{H}$

$$\langle g_{I_0}(H_0)\,\mathrm{i}[H_0, A]\,g_{I_0}(H_0)\varphi, \varphi\rangle = \langle g_{I_0}(H_0)\,f(H_0)\,g_{I_0}(H_0)\varphi, \varphi\rangle + \langle \mathsf{K}\varphi, \varphi\rangle. \tag{3.25}$$

Now we notice that $f(\lambda)$ is positive in the interior of I; so we put

$$\theta := \inf\{f(\lambda):\quad \lambda \in \mathrm{supp}\, g_{I_0}\} > 0\ .$$

With this information we apply the spectral theorem, getting, as in the previous section,

$$\langle g_{I_0}(H_0)\,f(H_0)\,g_{I_0}(H_0)\varphi, \varphi\rangle \geq \theta \int_{\lambda \in I} g_{I_0}(\lambda)^2\ \mathrm{d}m_\varphi(\lambda) = \theta \|g_{I_0}(H_0)\varphi\|_0^2\ .$$

This together with (3.2) establishes the Mourre estimate over I_0. ◻

A Flows of pseudodifferential operators

In this appendix we collect some known results about the flow generated by pseudodifferential operators belonging to the algebra $\mathcal{A}$. The setting is the same as [5] and we refer to that paper for the proofs. The first result describes how a Schrödinger equation is changed under a change of variables induced by the flow of a pseudodifferential operator, see Lemma 3.1 of [5]:

Lemma A.1. *Let $H(t)$ be a time dependent selfadjoint operator, and $X(t)$ be a selfadjoint family of operators. Assume that $\psi(t) = e^{-\mathrm{i}X(t)}\varphi(t)$ then*

$$\mathrm{i}\partial_t \psi = H(t)\psi \qquad \Longleftrightarrow \qquad \mathrm{i}\partial_t\varphi = H^+(t)\varphi$$

where

$$H^+(t) := e^{\mathrm{i}X(t)}\, H(t)\, e^{-\mathrm{i}X(t)} - \int_0^1 e^{\mathrm{i}sX(t)}\, (\partial_t X(t))\, e^{-\mathrm{i}sX(t)}\ \mathrm{d}s\ .$$

The next property we shall need is the Lie expansion of $e^{\mathrm{i}X}\, A\, e^{-\mathrm{i}X}$ in operators of decreasing order, see Lemma 3.2 of [5]:

Lemma A.2. *Let $X \in \mathcal{A}_\rho$ with $\rho < 1$ be a symmetric operator. Let $A \in \mathcal{A}_m$ with $m \in \mathbb{R}$. Then $e^{\mathrm{i}\tau X}\, A\, e^{-\mathrm{i}\tau X}$ is selfadjoint and for any $M \geq 1$ we have*[7]

$$e^{\mathrm{i}\tau X}\, A\, e^{-\mathrm{i}\tau X} = \sum_{\ell=0}^{M} \frac{\tau^\ell}{\mathrm{i}^\ell\, \ell!}\mathrm{ad}_X^\ell(A) + R_M(\tau, X, A)\ , \qquad \forall \tau \in \mathbb{R}\ , \tag{A.1}$$

where $R_M(\tau, X, A) \in \mathcal{A}_{m-(M+1)(1-\rho)}$.
In particular $\mathrm{ad}_X^\ell(A) \in \mathcal{A}_{m-\ell(1-\rho)}$ and $e^{\mathrm{i}\tau X}\, A\, e^{-\mathrm{i}\tau X} \in \mathcal{A}_m$, $\forall \tau \in \mathbb{R}$.

The last result concerns boundedness properties of the operator $e^{-\mathrm{i}\tau X}$, see Lemma 3.3 of [5]:

Lemma A.3. *Assume that $X(t)$ is a family of selfadjoint operators in $\mathcal{A}_1$ s.t.*

$$\sup_{t\in\mathbb{R}} \wp_j^1(X(t)) < \infty\ , \quad \forall j \geq 1\ .$$

Then $e^{-\mathrm{i}\tau X(t)}$ extends to an operator in $\mathcal{L}(\mathcal{H}^r)$ $\forall r \in \mathbb{R}$, and moreover there exist $c_r, C_r > 0$ s.t.

$$c_r\|\psi\|_r \leq \|e^{-\mathrm{i}\tau X(t)}\psi\|_r \leq C_r\|\psi\|_r\ , \qquad \forall t \in \mathbb{R}\ , \quad \forall \tau \in [0,1]\ .$$

B Functional calculus

In this section we collect some known results about functional calculus of selfadjoint operators which are used thorough the paper. We begin recalling Helffer-Sjöstrand formula [37], following the presentation of [16].

Definition B.1. *A function $f \in C^\infty(\mathbb{R}, \mathbb{C})$ will be said to belong to the class S^ρ, $\rho \in \mathbb{R}$, if $\forall m \in \mathbb{N}_0$, $\exists C_m > 0$ such that*

$$\left|\frac{\mathrm{d}^m}{\mathrm{d}x^m} f(x)\right| \leq C_m \langle x\rangle^{\rho - m}, \quad \forall x \in \mathbb{R}\ .$$

[7] in [5] we have defined $\mathrm{ad}_X(A) = \mathrm{i}[X, A]$ rather than (2.2); so we formulate the next result with the current notation

As usual we set the seminorms

$$\wp^{\rho}_{m}(f) := \sum_{0\le j\le m} \sup_{x\in\mathbb{R}} \left|\frac{\mathrm{d}^m f(x)}{\mathrm{d}x^m}\right| \langle x\rangle^{-\rho+m} , \qquad m \in \mathbb{N}_0 .$$

Given $f \in S^{\rho}$, we define its *almost analytic extension* as follows: for any $N \in \mathbb{N}$, put

$$\widetilde{f}_N : \mathbb{R}^2 \to \mathbb{C}, \qquad \widetilde{f}_N(x,y) := \left(\sum_{\ell=0}^{N} f^{(\ell)}(x)\frac{(\mathrm{i}y)^{\ell}}{\ell!}\right) \tau\left(\frac{y}{\langle x\rangle}\right)$$

where $\tau \in C^{\infty}(\mathbb{R}, \mathbb{R}_{\ge 0})$ is a cut-off function fulfilling $\tau(s) = 1$ for $|s| \le 1$ and $\tau(s) = 0$ for $|s| \ge 2$. It is well known [16] that the choice of N and of the cut-off function τ are by no means critical, and even other choices of $\widetilde{f}_N$ are possible (see e.g. [19]). The following properties are true [16]: let $f \in S^{\rho}$ with $\rho < 0$, then

$$\widetilde{f}_N|_{\mathbb{R}} = f, \qquad \operatorname{supp} \widetilde{f}_N \subset \{x + \mathrm{i}y : \quad x \in \operatorname{supp} f, \quad |y| \le 2\langle x\rangle\} ,$$

$$\left|\frac{\partial \widetilde{f}_N(x,y)}{\partial \overline{z}}\right| \le C_N \langle x\rangle^{\rho-N-1} |y|^N, \qquad \frac{\partial \widetilde{f}_N}{\partial \overline{z}} := \left(\frac{\partial \widetilde{f}_N}{\partial x} + \mathrm{i}\frac{\partial \widetilde{f}_N}{\partial y}\right)$$

$$\int_{\mathbb{R}^2} \left|\frac{\partial \widetilde{f}_N(z)}{\partial \overline{z}}\right| |\mathrm{Im}\,(z)|^{-p-1}\, \mathrm{d}\overline{z}\wedge \mathrm{d}z \le C_N\, \wp^{\rho}_{N+2}(f), \qquad \forall p = 0,\dots,N, \tag{B.1}$$

where $z = x + \mathrm{i}y$ and $\mathrm{d}\overline{z} \wedge \mathrm{d}z$ is the Lebesgue measure on $\mathbb{C}$.
Given H a selfadjoint operator and $f \in S^{\rho}$, $\rho < 0$, the *Helffer-Sjöstrand formula* defines $f(\mathsf{H})$ as

$$f(\mathsf{H}) := \frac{\mathrm{i}}{2\pi}\int_{\mathbb{R}^2} \frac{\partial \widetilde{f}_N(z)}{\partial \overline{z}} (z-\mathsf{H})^{-1}\mathrm{d}\overline{z}\wedge \mathrm{d}z = -\frac{1}{\pi}\int_{\mathbb{R}^2} \frac{\partial \widetilde{f}_N(z)}{\partial \overline{z}} (z-\mathsf{H})^{-1}\mathrm{d}x\, \mathrm{d}y . \tag{B.2}$$

Theorem B.2 ([16]). *Let $f \in S^{\rho}$, $g \in S^{\mu}$ with $\rho, \mu < 0$ and H a selfadjoint operator. Then*

(i) The operator $f(\mathsf{H})$ is independent of N and of the cut-off function τ.

(ii) The integral in (B) *is norm convergent and $\|f(\mathsf{H})\|_{\mathcal{L}(\mathcal{H})} \le \|f\|_{L^{\infty}}$.*

(iii) $f(\mathsf{H})\, g(\mathsf{H}) = (fg)(\mathsf{H})$.

(iv) $\overline{f}(\mathsf{H}) = f(\mathsf{H})^*$.

(v) If $f \in C^{\infty}_c$ has support disjoint from $\sigma(\mathsf{H})$, then $f(\mathsf{H}) = 0$.

(vi) If $z \notin \mathbb{R}$ and $f_z(x) := (z - x)^{-1}$ for all $x \in \mathbb{R}$, then $f_z \in S^{-1}$ and $f_z(\mathsf{H}) = (z - \mathsf{H})^{-1}$.

Remark B.3. Given $f \in S^{\rho}$, $\rho < 0$ and H selfadjoint, the operator $f(\mathsf{H})$ defined via Helffer-Sjöstrand formula coincides with the classical definition given by the spectral theorem, namely

$$f(\mathsf{H}) = \int_{\mathbb{R}} f(\lambda)\, \mathrm{d}E(\lambda)$$

where $\mathrm{d}E(\lambda)$ is the spectral resolution of H. For a proof, see e.g. [20], Theorem 8.1.

Next we recall expansion formulas for commutators. We start from the basic identities

$$\mathrm{ad}^n_{\mathsf{A}}(\mathsf{PQ}) = \sum_{k=0}^{n} \binom{n}{k} \mathrm{ad}^{n-k}_{\mathsf{A}}(\mathsf{P})\, \mathrm{ad}^{k}_{\mathsf{A}}(\mathsf{Q}), \qquad [\mathsf{P}, \mathsf{A}^n] = \sum_{j=1}^{n} c_{n,j}\, \mathrm{ad}^{j}_{\mathsf{A}}(\mathsf{P})\, \mathsf{A}^{n-j} . \tag{B.3}$$

For the next lemma see e.g. [19, Lemma C.3.1] or [40, Appendix B].

Lemma B.4 (Commutator expansion formula)**.** *Let $k \in \mathbb{N}$ and A, B selfadjoint operators with*

$$\|\mathrm{ad}_{\mathsf{A}}^{j}(\mathsf{B})\|_{\mathcal{L}(\mathcal{H})} < \infty, \qquad \forall\, 1 \leq j \leq k\ .$$

Let $f \in S^{\rho}$ with $\rho < 0$, then one has the right and left commutator expansions

$$[\mathsf{B}, f(\mathsf{A})] = \sum_{j=1}^{k-1} \frac{1}{j!} f^{(j)}(\mathsf{A})\, \mathrm{ad}_{\mathsf{A}}^{j}(\mathsf{B}) + R_k(f, \mathsf{A}, \mathsf{B}) \tag{B.4}$$

$$= \sum_{j=1}^{k-1} \frac{(-1)^{j-1}}{j!}\, \mathrm{ad}_{\mathsf{A}}^{j}(\mathsf{B})\, f^{(j)}(\mathsf{A}) + \widetilde{R}_k(f, \mathsf{A}, \mathsf{B}) \tag{B.5}$$

where the operators $R_k, \widetilde{R}_k$ fulfill

$$\|R_k(f, \mathsf{A}, \mathsf{B})\|_{\mathcal{L}(\mathcal{H})}, \quad \|\widetilde{R}_k(f, \mathsf{A}, \mathsf{B})\|_{\mathcal{L}(\mathcal{H})} \leq C_N\, \wp^{\rho}_{k+2}(f)\, \|\mathrm{ad}_{\mathsf{A}}^{k}(\mathsf{B})\|_{\mathcal{L}(\mathcal{H})}\ . \tag{B.6}$$

Lemma B.5. *Let $k \in \mathbb{N}$ and A, H selfadjoint operators such that*

$$\|\mathrm{ad}_{\mathsf{A}}^{j}(\mathsf{H})\|_{\mathcal{L}(\mathcal{H})} < \infty, \qquad \forall\, 1 \leq j \leq k\ . \tag{B.7}$$

Let $g \in S^{\rho}$ with $\rho < 0$. Then

$$\|\mathrm{ad}_{\mathsf{A}}^{j}(g(\mathsf{H}))\|_{\mathcal{L}(\mathcal{H})} < \infty \quad \forall 1 \leq j \leq k\ .$$

Proof. Take $N \geq k$ and use Helffer-Sjöstrand formula to write

$$\mathrm{ad}_{\mathsf{A}}^{j}(g(\mathsf{H})) = \frac{\mathrm{i}}{2\pi} \int_{\mathbb{R}^2} \frac{\partial \widetilde{g}_N(z)}{\partial \overline{z}}\, \mathrm{ad}_{\mathsf{A}}^{j}\big((z - \mathsf{H})^{-1}\big)\, \mathrm{d}\overline{z} \wedge \mathrm{d}z. \tag{B.8}$$

As $\mathrm{ad}_{\mathsf{A}}\big((z-\mathsf{H})^{-1}\big) = (z-\mathsf{H})^{-1}\, \mathrm{ad}_{\mathsf{A}}(\mathsf{H})\, (z-\mathsf{H})^{-1}$, by induction one gets for $j = 1, \ldots, k$

$$\mathrm{ad}_{\mathsf{A}}^{j}\big((z-\mathsf{H})^{-1}\big) = \sum_{\ell=1}^{j} \sum_{\substack{k_1+\cdots+k_\ell = j \\ k_1, \ldots, k_\ell \geq 1}} c^{\ell, j}_{k_1 \cdots k_\ell}\, (z-\mathsf{H})^{-1}\, \mathrm{ad}_{\mathsf{A}}^{k_1}(\mathsf{H})\, (z-\mathsf{H})^{-1} \mathrm{ad}_{\mathsf{A}}^{k_2}(\mathsf{H}) \cdots (z-\mathsf{H})^{-1}\, \mathrm{ad}_{\mathsf{A}}^{k_\ell}(\mathsf{H})\, (z-\mathsf{H})^{-1}$$

Using (B.5) and the estimate $\|(z-\mathsf{H})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq |\mathrm{Im}\,(z)|^{-1}$, $\forall z \in \mathbb{C} \setminus \mathbb{R}$, one has for $j = 1, \ldots, k$

$$\|\mathrm{ad}_{\mathsf{A}}^{j}\big((z-\mathsf{H})^{-1}\big)\|_{\mathcal{L}(\mathcal{H})} \leq \sum_{\ell=1}^{j} C_\ell\, |\mathrm{Im}\,(z)|^{-\ell-1}, \quad \forall z \in \mathbb{C} \setminus \mathbb{R}.$$

Inserting this estimate into (B) and using (B) we bound for any $j = 1, \ldots, k$

$$\|\mathrm{ad}_{\mathsf{A}}^{j}(g_J(\mathsf{H}))\|_{\mathcal{L}(\mathcal{H})} \lesssim \sum_{\ell=1}^{j} \int_{\mathbb{R}^2} \left| \frac{\partial \widetilde{g}_N(z)}{\partial \overline{z}} \right|\, |\mathrm{Im}\,(z)|^{-\ell-1} \mathrm{d}\overline{z} \wedge \mathrm{d}z \lesssim \wp^{\rho}_{N+2}(g) < \infty\ .$$

□

Lemma B.6. *Let $g \in C_c^{\infty}(\mathbb{R}, \mathbb{R})$. Let $\mathsf{H}, \mathsf{B} \in \mathcal{L}(\mathcal{H})$ be selfadjoint. Then $\exists C > 0$ such that*

$$\|g(\mathsf{H}+\mathsf{B}) - g(\mathsf{H})\|_{\mathcal{L}(\mathcal{H})} \leq C \|\mathsf{B}\|_{\mathcal{L}(\mathcal{H})}\ .$$

If B is compact on $\mathcal{H}$, so is $g(\mathsf{H}+\mathsf{B}) - g(\mathsf{H})$.

Proof. Take $N \geq 1$. Using Helffer-Sjöstrand formula and the resolvent identity we obtain

$$g(\mathsf{H}+\mathsf{B}) - g(\mathsf{H}) = \frac{\mathrm{i}}{2\pi}\int_{\mathbb{R}^2} \frac{\partial \widetilde{g}_N(z)}{\partial \overline{z}} \big(z - (\mathsf{H}+\mathsf{B})\big)^{-1} \mathsf{B}\,(z-\mathsf{H})^{-1}\, \mathrm{d}\overline{z}\wedge \mathrm{d}z\ .$$

Then use $\|\big(z-(\mathsf{H}+\mathsf{B})\big)^{-1}\|_{\mathcal{L}(\mathcal{H})}$, $\|(z-\mathsf{H})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq |\mathrm{Im}(z)|^{-1}$ for $z \in \mathbb{C}\setminus\mathbb{R}$ and (B).
If B is compact then $\big(z-(\mathsf{H}+\mathsf{B})\big)^{-1}\mathsf{B}\,(z-\mathsf{H})^{-1}$ is a compact operator for any $z \in \mathbb{C}\setminus\mathbb{R}$. ☐

C Local energy decay estimates

In this section we prove a local energy decay estimate starting from Mourre estimate. The result is essentially known but we could not find in the literature a statement exactly as the one we use in the paper, so we include here a proof, which follows closely the one of Lemma 4.1 of [29]. In this part we do not require pseudodifferential properties of the operators. We shall assume conditions (M1) and (M2) at page 12.

Theorem C.1 (Local energy decay estimate)**.** *Fix $k \in \mathbb{N}$ and assume* (M1)–(M2) *with* $\mathtt{N} \geq 4k+2$ *and* $\mathsf{K}=0$. *Then for any interval $J \subset I$, any function $g_J \in C_c^\infty(\mathbb{R},\mathbb{R}_{\geq 0})$ with* $\operatorname{supp} g_J \subset I$, $g_J = 1$ *on J, there exists $C>0$ such that*

$$\|\langle \mathsf{A}\rangle^{-k} e^{-\mathrm{i}\mathsf{H}t}\, g_J(\mathsf{H})\,\psi\|_0 \leq C\langle t\rangle^{-k}\|\langle\mathsf{A}\rangle^k\, g_J(\mathsf{H})\psi\|_0, \quad \forall t\in\mathbb{R}, \tag{C.1}$$

for any ψ such that the r.h.s. is finite.

Proof. Take $\chi(\xi) := \frac{1}{2}(1-\tanh\xi)$. Put $\eta(\xi) := \frac{1}{\sqrt{2}\cosh\xi}$ and note that

$$\chi' = -\eta^2, \qquad \left|\eta^{(m)}(\xi)\right| \leq C_m\,\eta(\xi), \quad \forall \xi\in\mathbb{R}, \ \ \forall m\in\mathbb{N}. \tag{C.2}$$

Next we set for $a \in \mathbb{R}$, $s \geq 1$ and $\vartheta := \frac{\theta}{2}$ (with θ of (M2))

$$\mathsf{A}_{t,s} := \frac{1}{s}\big(\mathsf{A} - a - \vartheta t\big)$$

and define via functional calculus the operators $\chi(\mathsf{A}_{t,s})$ and $\eta(\mathsf{A}_{t,s})$; both are bounded and selfadjoint on $\mathcal{H}$. To shorten the notation, from now on we write $\chi_{t,s} \equiv \chi(\mathsf{A}_{t,s})$, $\eta_{t,s} \equiv \eta(\mathsf{A}_{t,s})$, $g_J \equiv g_J(\mathsf{H})$ and $\psi_t := e^{-\mathrm{i}t\mathsf{H}}\psi$. Note that $e^{-\mathrm{i}\mathsf{H}t}g_J(\mathsf{H})\psi = g_J(\mathsf{H})e^{-\mathrm{i}\mathsf{H}t}\psi \equiv g_J\psi_t$.
The starting point of the proof is an energy estimate for the quantity $\|(\chi_{t,s})^{\frac12}\, g_J\psi_t\|_0$. We have

$$\frac{d}{dt}\|(\chi_{t,s})^{\frac12}\, g_J\psi_t\|_0^2 = \frac{d}{dt}\langle \chi_{t,s}g_J\psi_t,\, g_J\psi_t\rangle = \frac{\vartheta}{s}\|\eta_{t,s}\, g_J\, \psi_t\|_0^2 + \langle \mathrm{i}[\mathsf{H},\chi_{t,s}]\, g_J\psi_t,\, g_J\psi_t\rangle. \tag{C.3}$$

To evaluate the right hand side we shall use the commutator formulas in Lemma B.4, the identity

$$\mathrm{ad}^j_{\mathsf{A}_{t,s}}(\mathsf{H}) = \frac{1}{s^j}\mathrm{ad}^j_{\mathsf{A}}(\mathsf{H})\ , \quad \forall s\geq 1, \ \ \forall 1\leq j\leq \mathtt{N} \tag{C.4}$$

and the fact that all the operators $\mathrm{ad}^j_{\mathsf{A}}(\mathsf{H})$ are bounded $\forall 1 \leq j \leq \mathtt{N}$ by (M1). The goal now is to estimate the second term in the right hand side of (C). For an arbitrary $f \in \mathcal{H}$ we write

$$\langle \mathrm{i}[\mathsf{H},\chi_{t,s}]f,f\rangle \overset{(\mathrm{B.4}),(\mathrm{C})}{=} -\frac{1}{s}\langle \eta^2_{t,s}\mathrm{i}[\mathsf{H},\mathsf{A}]f,f\rangle + \sum_{j=2}^{\mathtt{N}-1}\frac{1}{j!}\frac{1}{s^j}\langle \chi^{(j)}_{t,s}\,\mathrm{i}\,\mathrm{ad}^j_{\mathsf{A}}(\mathsf{H})f,f\rangle + \frac{1}{s^{\mathtt{N}}}\langle R_{\mathtt{N}}f,f\rangle \tag{C.5}$$

where $\chi^{(j)}_{t,s} := \chi^{(j)}(\mathsf{A}_{t,s})$ and the remainder $R_{\mathtt{N}}$ fulfills the estimate (see (B.4))

$$\|R_{\mathtt{N}}\|_{\mathcal{L}(\mathcal{H})} \le C_{\mathtt{N}}\, \|\mathrm{ad}^{\mathtt{N}}_{\mathsf{A}}(\mathsf{H})\| \le C_{\mathtt{N}}. \tag{C.6}$$

Note that the constant $C_{\mathtt{N}} > 0$ in the previous estimate is uniform in $a \in \mathbb{R}$. In the following we shall simply denote by $R_{\mathtt{N}}$ any bounded operator fulfilling an estimate like (C).
Consider now the first term in the expansion (C) above. This time we use the left expansion (B.4) and write

$$\begin{aligned}\frac{1}{s}\langle \eta^2_{t,s}\, \mathrm{i}[\mathsf{H},\mathsf{A}]f, f\rangle = \frac{1}{s}\langle\, \mathrm{i}[\mathsf{H},\mathsf{A}]\, \eta_{t,s} f,\, \eta_{t,s} f\rangle \\ + \sum_{j=2}^{\mathtt{N}-1}\frac{(-1)^{j-1}}{(j-1)!}\frac{1}{s^j}\langle\, \mathrm{i}\,\mathrm{ad}^j_{\mathsf{A}}(\mathsf{H})\, \eta^{(j-1)}_{t,s} f, \eta_{t,s} f\rangle + \frac{1}{s^{\mathtt{N}}}\langle R_{\mathtt{N}} f, f\rangle\end{aligned} \tag{C.7}$$

where $R_{\mathtt{N}}$ is estimated as in (C). Consider now the second term in (C). From $\chi' = -\eta^2$, we have by functional calculus $\chi^{(j)}(\mathsf{A}_{t,s}) = \sum_{\ell=1}^{j} c_{\ell j}\, \eta^{(j-\ell)}(\mathsf{A}_{t,s})\, \eta^{(\ell)}(\mathsf{A}_{t,s})$. Thus we get that

$$\begin{aligned}\frac{1}{s^j}\langle \chi^{(j)}_{t,s}\, \mathrm{i}\,\mathrm{ad}^j_{\mathsf{A}}(\mathsf{H}) f, f\rangle \overset{\text{(B.4)}}{=} \frac{1}{s^j}\sum_{\ell=1}^{j} c_{\ell j}\langle\, \mathrm{i}\,\mathrm{ad}^j_{\mathsf{A}}(\mathsf{H})\, \eta^{(\ell)}_{t,s} f, \eta^{(j-\ell)}_{t,s} f\rangle \\ + \sum_{\ell=1}^{j}\sum_{n=1}^{\mathtt{N}-j-1}\frac{c_{\ell j n}}{s^{j+n}}\langle\, \mathrm{i}\,\mathrm{ad}^{j+n}_{\mathsf{A}}(\mathsf{H})\, \eta^{(\ell+n)}_{t,s} f,\, \eta^{(j-\ell)}_{t,s} f\rangle + \frac{1}{s^{\mathtt{N}}}\langle R_{\mathtt{N}} f, f\rangle\end{aligned} \tag{C.8}$$

By (C), (C), (C) we have found that $\langle \mathrm{i}[\mathsf{H}, \chi_{t,s}]f, f\rangle$ is a sum of terms of the form

$$\langle \mathrm{i}[\mathsf{H}, \chi_{t,s}]f, f\rangle = -\frac{1}{s}\langle\, \mathrm{i}[\mathsf{H},\mathsf{A}]\, \eta_{t,s} f,\, \eta_{t,s} f\rangle + \sum_{j=2}^{\mathtt{N}-1}\frac{1}{s^j}\sum_{n,\ell,m}\langle\, R_n\, \eta^{(\ell)}_{t,s} f, \eta^{(m)}_{t,s} f\rangle + \frac{1}{s^{\mathtt{N}}}\langle R_{\mathtt{N}} f, f\rangle$$

where $R_n, R_{\mathtt{N}}$ are bounded operators. Furthermore, from the second of (C) and the spectral theorem, we bound

$$\left|\langle\, R_n\, \eta^{(\ell)}_{t,s} f, \eta^{(m)}_{t,s} f\rangle\right| \le C\, \|\eta_{t,s} f\|^2_0 \ . \tag{C.9}$$

We thus obtain, for any $f \in \mathcal{H}$ and $s \ge 1$, the estimate

$$\langle \mathrm{i}[\mathsf{H}, \chi_{t,s}]f, f\rangle \le -\frac{1}{s}\langle\, \mathrm{i}[\mathsf{H},\mathsf{A}]\, \eta_{t,s} f,\, \eta_{t,s} f\rangle + \frac{C_{\mathtt{N}}}{s^2}\|\eta_{t,s} f\|^2_0 + \frac{C_{\mathtt{N}}}{s^{\mathtt{N}}}\|f\|^2_0 \ .$$

Now we evaluate such inequality at $f = g_J \psi_t$, getting

$$\langle \mathrm{i}[\mathsf{H}, \chi_{t,s}] g_J\psi_t,\, g_J\psi_t\rangle \le -\frac{1}{s}\langle\, \mathrm{i}[\mathsf{H},\mathsf{A}]\, \eta_{t,s} g_J\psi_t,\, \eta_{t,s} g_J\psi_t\rangle + \frac{C_{\mathtt{N}}}{s^2}\|\eta_{t,s} g_J\psi_t\|^2_0 + \frac{C_{\mathtt{N}}}{s^{\mathtt{N}}}\|g_J\psi_t\|^2_0 \ . \tag{C.10}$$

The next step is to prove that the first term in the right hand side above has a sign, up to higher order terms in s^{-j}. This is the point where the Mourre estimate (M2) comes into play. To see this, we analyze

$$\langle\, \mathrm{i}[\mathsf{H},\mathsf{A}]\, \eta_{t,s}\, g_J\psi_t,\, \eta_{t,s}\, g_J\psi_t\rangle \equiv \langle\, \mathrm{i}[\mathsf{H},\mathsf{A}]\, \eta_{t,s}\, g_I\, g_J\psi_t,\, \eta_{t,s} g_I\, g_J\psi_t\rangle \tag{C.11}$$

where we used that $g_J g_I = g_J$. Next we commute and expand in commutators $\eta_{t,s} g_I$:

$$\eta_{t,s} g_I = g_I \eta_{t,s} + [\eta_{t,s}, g_I] \overset{\text{(B.4)}}{=} g_I\, \eta_{t,s} + \sum_{j=1}^{\mathtt{N}-2}\frac{c_j}{s^j}\,\mathrm{ad}^j_{\mathsf{A}}(g_I(\mathsf{H}))\, \eta^{(j)}_{t,s} + \frac{1}{s^{\mathtt{N}-1}}\widetilde{R}_{\mathtt{N}-1}\ ; \tag{C.12}$$

note that Lemma B.5 assures that the operators $\mathrm{ad}_{\mathsf{A}}^{j}(g_I(\mathsf{H}))$ are bounded $\forall j = 1, \dots, \mathtt{N}$, so is the operator $\widetilde{R}_{\mathtt{N}-1}$ which fulfills

$$\|\widetilde{R}_{\mathtt{N}-1}\|_{\mathcal{L}(\mathcal{H})} \leq C_{\mathtt{N}}\, \mathrm{ad}_{\mathsf{A}}^{\mathtt{N}-1}(g_I(\mathsf{H})) < \infty \ . \tag{C.13}$$

Again in the following we shall denote by $\widetilde{R}_{\mathtt{N}-1}$ any operator fulfilling an estimate like (C). Inserting the expansion (C) into (C) one gets, with $w := g_J \psi_t$,

$$(\mathrm{C}) = \langle \mathrm{i}[\mathsf{H}, \mathsf{A}]\, g_I\, \eta_{t,s} w,\ g_I\, \eta_{t,s} w \rangle + \sum_{j=1}^{\mathtt{N}-2} \frac{c_j}{s^j} \sum_{n,\ell,m} \langle R_n\, \eta_{t,s}^{(\ell)} w,\ \eta_{t,s}^{(m)} w \rangle + \frac{1}{s^{\mathtt{N}-1}} \langle \widetilde{R}_{\mathtt{N}-1} w, w \rangle$$

where each term of the form $\langle R_n\, \eta_{t,s}^{(\ell)} w,\ \eta_{t,s}^{(m)} w \rangle$ fulfills an estimate like (C).
It is finally time to use the strict Mourre estimate: by assumption (M2) we have for $s \geq 1$

$$\langle \mathrm{i}[\mathsf{H}, \mathsf{A}]\, g_I\, \eta_{t,s} w,\ g_I\, \eta_{t,s} w \rangle \geq \theta \| g_I\, \eta_{t,s} w\|_0^2 \ .$$

Using again the expansion (C) and estimates (C), (C) we get therefore

$$\langle \mathrm{i}[\mathsf{H}, \mathsf{A}]\, g_I\, \eta_{t,s} w,\ g_I\, \eta_{t,s} w \rangle \geq \theta \|\, \eta_{t,s} g_I w\|_0^2 - \frac{C_{\mathtt{N}}}{s} \|\eta_{t,s} w\|_0^2 - \frac{C_{\mathtt{N}}}{s^{\mathtt{N}-1}} \|w\|_0^2. \tag{C.14}$$

This proves that the first term in the right hand side of (C) has a sign; we proceed from (C) and using inequality (C) (recall $w = g_J \psi_t$) we get

$$\langle \mathrm{i}[\mathsf{H}, \chi_{t,s}] g_J \psi_t,\, g_J \psi_t \rangle \leq -\frac{\theta}{s} \|\eta_{t,s}\, g_J \psi_t\|_0^2 + \frac{C_{\mathtt{N}}}{s^2} \|\eta_{t,s}\, g_J \psi_t\|_0^2 + \frac{C_{\mathtt{N}}}{s^{\mathtt{N}}} \|\, g_J \psi_t\|_0^2 \ . \tag{C.15}$$

We come back to the estimate (C) of $\|(\chi_{t,s})^{\frac{1}{2}}\, g_J \psi_t\|_0$. We finally obtain, with $\vartheta = \frac{\theta}{2}$ and $s \geq 1$ sufficiently large,

$$\begin{aligned} \frac{d}{dt} \|(\chi_{t,s})^{\frac{1}{2}}\, g_J \psi_t\|_0^2 &\overset{(\mathrm{C})}{\leq} \frac{1}{s} (\vartheta - \theta)\, \|\eta_{t,s}\, g_J \psi_t\|_0^2 + \frac{C_{\mathtt{N}}}{s^2} \|\eta_{t,s}\, g_J \psi_t\|_0^2 + \frac{C_{\mathtt{N}}}{s^{\mathtt{N}}} \|\, g_J \psi_t\|_0^2 \\ &\leq \frac{1}{s} \left(-\frac{\theta}{2} + \frac{C_{\mathtt{N}}}{s} \right) \|\eta_{t,s}\, g_J \psi_t\|_0^2 + \frac{C_{\mathtt{N}}}{s^{\mathtt{N}}} \|\, g_J \psi_t\|_0^2 \end{aligned}$$

So, for $s \geq 1$ sufficiently large, the first term in the right hand side above is negative and, using also that $e^{-\mathrm{i}t\mathsf{H}}$ is unitary and commutes with $g_J \equiv g_J(\mathsf{H})$, we get

$$\frac{d}{dt} \|(\chi_{t,s})^{\frac{1}{2}}\, g_J \psi_t\|_0^2 \leq \frac{C_{\mathtt{N}}}{s^{\mathtt{N}}} \|\, g_J \psi_0\|_0^2 \ .$$

Integrating this inequality between 0 and t we find $\forall t > 0$

$$\|\chi^{\frac{1}{2}} \Big(\frac{\mathsf{A} - a - \vartheta t}{s} \Big)\, g_J \psi_t\|_0^2 \leq \|\chi^{\frac{1}{2}} \Big(\frac{\mathsf{A} - a}{s} \Big)\, g_J \psi\|_0^2 + \frac{C_{\mathtt{N}}\, t}{s^{\mathtt{N}}} \|\, g_J \psi\|_0^2,$$

uniformly for $a \in \mathbb{R}$ and $s \geq 1$ sufficiently large. We evaluate this inequality at $a = -\frac{\vartheta}{2} t$ and $s = \sqrt{t}$, obtaining for $t \geq 1$ sufficiently large, the *minimal velocity estimate*

$$\|\chi^{\frac{1}{2}} \Big(\frac{\mathsf{A} - \frac{\vartheta}{2} t}{\sqrt{t}} \Big)\, g_J \psi_t\|_0 \leq \|\chi^{\frac{1}{2}} \Big(\frac{\mathsf{A} + \frac{\vartheta}{2} t}{\sqrt{t}} \Big)\, g_J \psi\|_0 + C_{\mathtt{N}}\, t^{-\frac{\mathtt{N}}{4} + \frac{1}{2}} \|\, g_J \psi\|_0 \ . \tag{C.16}$$

To conclude, take $k \in \mathbb{N}$ and consider $\|\langle \mathsf{A}\rangle^{-k}\, g_J\, \psi_t\|_0$. Clearly

$$\|\langle \mathsf{A}\rangle^{-k}\, g_J\, \psi_t\|_0 \leq \|\chi^{\frac12}\Big(\frac{\mathsf{A}-\frac{\vartheta}{2}t}{\sqrt{t}}\Big)\,\langle \mathsf{A}\rangle^{-k}\, g_J\, \psi_t\|_0 \tag{C.17}$$

$$+\,\|\Big(1-\chi^{\frac12}\Big(\frac{\mathsf{A}-\frac{\vartheta}{2}t}{\sqrt{t}}\Big)\Big)\,\langle \mathsf{A}\rangle^{-k}\, g_J\, \psi_t\|_0 \tag{C.18}$$

We estimate first (C). By Theorem B.2 (ii) we have

$$\|\Big(1-\chi^{\frac12}\Big(\frac{\mathsf{A}-\frac{\vartheta}{2}t}{\sqrt{t}}\Big)\Big)\,\langle \mathsf{A}\rangle^{-k}\|_{\mathcal{L}(\mathcal{H})} \leq \sup_{\lambda\in\mathbb{R}}\left|\Big(1-\chi^{\frac12}\Big(\frac{\lambda-\frac{\vartheta}{2}t}{\sqrt{t}}\Big)\Big)\,\langle \lambda\rangle^{-k}\right| \leq C_k\langle t\rangle^{-k}.$$

To prove the last inequality, use that for $\lambda \geq \frac{\vartheta}{4}t$ then $\langle\lambda\rangle^{-k} \leq \langle t\rangle^{-k}$, whereas when $\lambda < \frac{\vartheta}{4}t$ then, being $\lambda \mapsto 1-\chi^{\frac12}(\lambda)$ monotone increasing and exponentially decaying at $-\infty$,

$$1-\chi^{\frac12}\Big(\frac{\lambda-\frac{\vartheta}{2}t}{\sqrt{t}}\Big) \leq 1-\chi^{\frac12}\Big(-\frac{\vartheta}{4}\sqrt{t}\Big) \leq C\left(e^{-\frac{\vartheta}{4}\sqrt{t}}\right)^{\frac12} \leq C_k\langle t\rangle^{-k}.$$

Next we estimate (C) using the minimal velocity estimate. As $\langle \mathsf{A}\rangle^{-k}$ is a bounded operator,

$$\begin{aligned}
\|\chi^{\frac12}\Big(\frac{\mathsf{A}-\frac{\vartheta}{2}t}{\sqrt{t}}\Big)\,\langle \mathsf{A}\rangle^{-k}\, g_J\, \psi_t\|_0 &\leq \|\chi^{\frac12}\Big(\frac{\mathsf{A}-\frac{\vartheta}{2}t}{\sqrt{t}}\Big)\, g_J\, \psi_t\|_0 \\
&\overset{(C)}{\leq} \|\chi^{\frac12}\Big(\frac{\mathsf{A}+\frac{\vartheta}{2}t}{\sqrt{t}}\Big)\, g_J\psi\|_0 + C_{\mathtt{N}}\, t^{-\frac{\mathtt{N}}{4}+\frac12}\|\, g_J\psi\|_0 \\
&\leq \|\chi^{\frac12}\Big(\frac{\mathsf{A}+\frac{\vartheta}{2}t}{\sqrt{t}}\Big)\,\langle \mathsf{A}\rangle^{-k}\|_{\mathcal{L}(\mathcal{H})}\,\|\langle \mathsf{A}\rangle^{k}\, g_J\psi\|_0 + C_{\mathtt{N}}\, t^{-\frac{\mathtt{N}}{4}+\frac12}\|\, g_J\psi\|_0
\end{aligned}$$

Again we have

$$\|\chi^{\frac12}\Big(\frac{\mathsf{A}+\frac{\vartheta}{2}t}{\sqrt{t}}\Big)\,\langle \mathsf{A}\rangle^{-k}\|_{\mathcal{L}(\mathcal{H})} \leq \sup_{\lambda\in\mathbb{R}}\left|\chi^{\frac12}\Big(\frac{\lambda+\frac{\vartheta}{2}t}{\sqrt{t}}\Big)\,\langle \lambda\rangle^{-k}\right| \leq C_k\langle t\rangle^{-k},$$

since for $\lambda \leq -\frac{\vartheta}{4}t$ one has $\langle\lambda\rangle^{-k} \leq C\langle t\rangle^{-k}$, whereas in case $\lambda > -\frac{\vartheta}{4}t$, as $\lambda \mapsto \chi^{\frac12}(\lambda)$ is monotone decreasing exponentially fast at $+\infty$, one has

$$\chi^{\frac12}\Big(\frac{\lambda+\frac{\vartheta}{2}t}{\sqrt{t}}\Big) \leq \chi^{\frac12}\Big(\frac{\vartheta}{4}\sqrt{t}\Big) \leq C\left(e^{-\frac{\vartheta}{4}\sqrt{t}}\right)^{\frac12} \leq C_k\langle t\rangle^{-k}.$$

Altogether, from (C), (C) we have proved that for $t \geq 1$ sufficiently large,

$$\begin{aligned}
\|\langle \mathsf{A}\rangle^{-k}\, g_J\, \psi_t\|_0 &\leq C_k\langle t\rangle^{-k}\|g_J\psi_t\|_0 + C_k\langle t\rangle^{-k}\|\langle \mathsf{A}\rangle^{k} g_J\psi\|_0 + C_{\mathtt{N}}\, t^{-\frac{\mathtt{N}}{4}+\frac12}\|\, g_J\psi\|_0 \\
&\leq C_k\langle t\rangle^{-k}\,\|\langle \mathsf{A}\rangle^{k} g_J\psi\|_0
\end{aligned}$$

provided $\mathtt{N} = 4k+2$. This proves the estimate (C.1) for $t \geq 1$ sufficiently large, and it is also clearly true for t in any bounded interval.

□

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