

Inhomogeneous Phase of the Chiral Gross-Neveu Model

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There is substantial evidence that the ground state of the 2D chiral Gross-Neveu model, in the presence of a $U(1)$ fermion number chemical potential μ and in the large N limit, is given by a “chiral spiral” phase, namely an inhomogeneous phase with a chiral condensate having a spatially periodic phase. We show that the chiral spiral configuration persists at finite N and $T = 0$ for any $\mu > 0$. Our analysis is based on nonabelian bosonization, that relates the model to a $U(N)_1$ Wess-Zumino-Witten model deformed by current-current interactions. In this description, the appearance of the inhomogeneous phase is surprisingly simple. We also rederive the phase diagram of the large N chiral Gross-Neveu model via a direct diagrammatic computation, finding agreement with previous results in the literature.

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Introduction.—Understanding the phase diagram of strongly interacting matter at finite temperature and density is of fundamental importance in physics, with applications that range from superconductivity to compact astrophysical objects. In particular, inhomogeneous phases where the ground state breaks a subset of the translation symmetries can appear in dense systems—for example, in the context of ordinary superconductivity [1,2] and in one-dimensional metals [3,4]. Inhomogeneous phases have been conjectured to form also in relativistic theories such as quantum chromodynamics (QCD) in a range of temperatures and matter densities. Evidence that cold dense quark matter at large number of colors in QCD might form “standing chiral waves,” i.e., configurations where only a linear combination of chiral symmetry and translations is linearly realized, has been first provided in [5]. Such chiral waves have been subsequently shown to be disfavored in actual QCD (three colors) [6], but the possibility of other inhomogeneous phases in QCD has emerged during the years; see, e.g., Refs. [7–9] and references therein [10].

Because of the limited availability of analytical and numerical tools, it is difficult to rigorously assess the appearance of these phases in QCD. For this reason, it is interesting to understand the phase diagram of simpler theories that qualitatively resemble QCD. A renowned example is the theory of N massless Dirac fermions in

two spacetime dimensions interacting through four-fermion interactions [12]. These theories, like QCD, are uv-free and undergo dynamical mass generation in the ir. One version of the theory, the Gross-Neveu (GN) model, has maximal $O(2N)$ vector symmetry, is gapped, and shows spontaneous breaking of a Z_2 chiral symmetry. Another version of the theory, the chiral Gross-Neveu (cGN) model, enjoys $U(N)$ vector symmetry and $U(1)$ chiral symmetry. The latter theory is gapless in the ir, having in the spectrum a massless compact boson typical of quasi long-range ordered phases in two dimensions [13–15]. The phase diagrams of the GN and the cGN models at large N at finite temperature T and in the presence of a $U(1)$ fermion number chemical potential μ were first studied in [16] and [17], respectively, under the assumption of translational invariance. This assumption and the corresponding results have later been disputed, and revised large N phase diagrams including inhomogeneous phases have been found [18–21]: a crystal phase in the GN model, with a spatially modulated order parameter at low T and sufficiently large μ , and a phase analogous to the chiral waves, dubbed “chiral spiral,” in the cGN model, with a spatially periodic phase with μ -dependent period at low T for any $\mu > 0$ [22]. The results of [18–21] are based on solutions of certain self-consistency equations, and apply in the large N limit. In recent years the appearance of inhomogeneous phases at finite N has started to be analyzed using numerical lattice methods (see, e.g., Refs. [25,26] for the GN model and [27] for the cGN model) [28].

The aim of this Letter is to investigate the fate of the phase diagram of the cGN model at finite N . Using nonabelian bosonization we show that the chiral spiral phase of the cGN model is present at any finite N , at $T = 0$

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for any $\mu > 0$. The chiral wave corresponds to a quasi long-range ordered gapless phase, in contrast to strict long-range order appearing at large N , with a free massless relativistic excitation. In this phase the would-be order parameter is neutral under ir spatial translations that are a linear combination of the original translations and the axial $U(1)$ symmetry; the combination depends on the realization of a certain \mathbf{Z}_N symmetry in the $SU(N)_1$ Wess-Zumino-Witten (WZW) model deformed by a current-current interaction.

We also rederive the phases of the cGN model at large N by a diagrammatic computation of the free energy in the presence of the inhomogeneity.

cGN as $J\bar{J}$ deformation of a WZW model.—The euclidean Lagrangian of the cGN model [12] is

$$\mathcal{L}_{\text{cGN}} = i\psi_{+a}^\dagger \partial_- \psi_+^a - i\psi_{-a}^\dagger \partial_+ \psi_-^a - \frac{\lambda_s}{N} |\psi_{-a}^\dagger \psi_+^a|^2, \quad (1)$$

where ∂_+ and ∂_- are derivatives with respect to complex coordinates $x^+ = (x^1 - ix^2)/2$, $x^- = (x^1 + ix^2)/2$, $a = 1, \dots, N$ (summation over repeated indices is understood) and ψ_\pm^a are the two chiral components of a set of N Dirac fermions. The $O(2N)_+ \times O(2N)_-$ global symmetry of the N free Dirac fermions is broken by the interaction to $(U(N)_V \times U(1)_A)/\mathbf{Z}_2$. The action of the residual group on the fields is

$$\begin{aligned} U(N)_V: \psi_\pm^a &\mapsto \mathcal{V}_b^a \psi_\pm^b, & \mathcal{V} &\in U(N), \\ U(1)_A: \psi_\pm^a &\mapsto e^{\pm i\alpha} \psi_\pm^a, & \alpha &\in \mathbf{R}. \end{aligned} \quad (2)$$

The scalar-scalar quartic interaction appearing in Eq. (1) is not the most generic one with the above symmetries. One can also have the vector-vector interaction [31]:

$$\mathcal{L}_v = \frac{\lambda_v}{N^2} (\psi_{+a}^\dagger \psi_+^a) (\psi_{-b}^\dagger \psi_-^b). \quad (3)$$

With a Fierz transformation the Lagrangian [Eq. (1)], including the term [Eq. (3)], can be formulated as the $U(N)$ generalization [32] of the massless Thirring model [33]:

$$\mathcal{L} = i\psi_{+a}^\dagger \partial_- \psi_+^a - i\psi_{-a}^\dagger \partial_+ \psi_-^a + \frac{\lambda J_+^A J_-^A}{N} + \frac{\lambda' J_+ J_-}{N^2}, \quad (4)$$

where

$$J_\pm^A = \psi_{a\pm}^\dagger (T^A)_b^a \psi_\pm^b, \quad J_\pm = \psi_{a\pm}^\dagger \psi_\pm^a, \quad (5)$$

are the $SU(N)$ and $U(1)$ currents, respectively, with T^A the generators of $SU(N)$ in the fundamental representation, normalized as $\text{Tr}(T^A T^B) = \delta^{AB}/2$. The couplings λ_s , λ_v , λ , and λ' are related by

$$\lambda = 2\lambda_s, \quad \lambda' = \lambda_v + \lambda_s. \quad (6)$$

Using nonabelian bosonization [34–36] (see also Ref. [37]) the theory [Eq. (4)] can be equivalently described as a $J\bar{J}$ deformation of a $U(N)_1$ WZW model [38]. Locally, the degrees of freedom are a $SU(N)$ matrix U and a scalar ϕ of radius $R = \sqrt{N}$ [40] parametrizing the $U(1)$ factor, in terms of which the Lagrangian is

$$\mathcal{L}[U, \phi] = \mathcal{L}_0[U, \phi] + \frac{\lambda J_+^A J_-^A}{N} + \frac{\lambda' J_+ J_-}{N^2}, \quad (7)$$

where

$$\begin{aligned} \mathcal{L}_0[U, \phi] &= \frac{1}{8\pi} \partial_+ \phi \partial_- \phi + \frac{1}{8\pi} \text{Tr}(\partial_+ U^\dagger \partial_- U + \partial_- U^\dagger \partial_+ U) \\ &+ \mathcal{L}_{\text{WZ}}^{SU(N)_1}, \end{aligned} \quad (8)$$

is the Lagrangian of the undeformed WZW model, with $\mathcal{L}_{\text{WZ}}^{SU(N)_1}$ being the level $k = 1$ $SU(N)$ Wess-Zumino term, which is the boundary term of the three-dimensional topological Lagrangian:

$$\frac{1}{12\pi} \epsilon^{ijk} \text{Tr}(U^\dagger \partial_i U U^\dagger \partial_j U U^\dagger \partial_k U), \quad (9)$$

and

$$\begin{aligned} J_+^A &= \frac{i}{2\pi} \text{Tr}(U^\dagger (\partial_+ U) T^A), & J_-^A &= \frac{i}{2\pi} \text{Tr}[(\partial_- U) U^\dagger T^A], \\ J_\pm &= -\frac{\sqrt{N}}{4\pi} \partial_\pm \phi, \end{aligned} \quad (10)$$

are the bosonized $SU(N)$ and $U(1)$ currents.

The $J_+^A J_-^A$ deformation breaks the $(SU(N)_+ \times SU(N)_-)/\mathbf{Z}_N^V$ symmetry of the $SU(N)_1$ WZW model,

$$SU(N)_+ \times SU(N)_-: U \mapsto g_- U g_+^\dagger, \quad g_\pm \in SU(N), \quad (11)$$

to $(SU(N)_V/\mathbf{Z}_N^V) \times \mathbf{Z}_N^A$. Here, \mathbf{Z}_N^V denotes a transformation in the center of the diagonal group $SU(N)_V$, which leaves U invariant, while \mathbf{Z}_N^A denotes a transformation in the center of only one of the two \pm factors (it does not matter which one) and acts as a phase on U . Explicitly,

$$\begin{aligned} SU(N)_V/\mathbf{Z}_N^V: U &\mapsto V U V^\dagger, & V &\in SU(N), \\ \mathbf{Z}_N^A: U &\mapsto e^{2\pi i k/N} U, & k &\in \mathbf{Z}_N. \end{aligned} \quad (12)$$

In the compact boson sector instead, the operator $J_+ J_-$ is not an interaction, as

$$\frac{\lambda' J_+ J_-}{N^2} = \frac{\lambda'}{16\pi^2 N} \partial_+ \phi \partial_- \phi. \quad (13)$$

The compact scalar ϕ and the $SU(N)$ matrix U are completely decoupled in the Lagrangian [Eq. (7)]. However, since $U(N) \simeq [SU(N) \times U(1)]/\mathbf{Z}_N$, to obtain the deformed $U(N)_1$ model from Eq. (7) we further need to gauge the diagonal \mathbf{Z}_N symmetry between \mathbf{Z}_N^A and a \mathbf{Z}_N^ϕ subgroup of the shift symmetry of the compact boson. A similar subtlety arises in the identification of the deformed $U(N)_1$ model with the original fermionic theory [Eq. (4)]. Since the fermionic theory depends on the spin structures while the bosonic one does not, to obtain a full equivalence one needs to gauge a \mathbf{Z}_2 symmetry (see, e.g., Refs. [41–43]). The gauging of these discrete symmetries will not affect the calculation of the free energy on \mathbf{R}^2 —where we do not have twisted sectors—as a function of μ . However both the gauging of \mathbf{Z}_N and of \mathbf{Z}_2 can play a role in the study of the thermal partition function, because of the nontrivial topology of the background. It would be interesting to study these issues in more detail.

The value of λ' is an RG invariant because it only changes the radius of the compact boson. The coupling λ , instead, runs and becomes strong in the ir. Equivalently, the couplings λ_s, λ_v must both run, with $\beta_{\lambda_v} = -\beta_{\lambda_s}$ [44]. The coupling λ_v at finite N has then to be included, as it is anyway generated radiatively.

The impossibility of a trivially gapped theory could be anticipated by the presence of a 't Hooft anomaly between the $U(1)_V$ and $U(1)_A$ currents. In the bosonized version of the theory, this 't Hooft anomaly is reproduced by the compact scalar, where $U(1)_A$ and $U(1)_V$ act as a shift on ϕ and on its dual, respectively.

Chiral spiral at finite N .—Let us introduce a chemical potential for the $U(1)_V$ charge by adding to the Lagrangian [Eq. (4)] the term

$$\mathcal{L}_\mu = \mu(\psi_{+a}^\dagger \psi_+^a + \psi_{-a}^\dagger \psi_-^a). \quad (14)$$

Upon bosonization, this maps simply to

$$\mathcal{L}_\mu = -\mu \frac{\sqrt{N}}{2\pi} \partial_1 \phi. \quad (15)$$

The Eq. (15) term, which does not depend on the $SU(N)$ degrees of freedom, provides an expectation value for $\partial_1 \phi$. By adding Eq. (15) to the Lagrangian [Eq. (7)] we get that the effective action for ϕ is minimized on configurations with

$$\langle \partial_1 \phi \rangle = 2\mu\sqrt{N} \left(1 + \frac{\lambda'}{2\pi N}\right)^{-1} \equiv 2\mu'\sqrt{N}. \quad (16)$$

The difference of the zero-temperature free energy density per flavor between the configuration with and without the expectation value for $\partial_1 \phi$ is

$$\delta F = -\frac{\mu^2}{2\pi} \left(1 + \frac{\lambda'}{2\pi N}\right)^{-1}, \quad (17)$$

showing that the former is favored on \mathbf{R}^2 .

Using the bosonization identity $\psi_{-a}^\dagger \psi_+^b = U_a^b e^{i\phi/\sqrt{N}}$ [34–36] (omitting a scheme-dependent renormalization mass scale), the two-point function of the fermion bilinear in terms of the $SU(N)$ and free scalar degrees of freedom is

$$\begin{aligned} \langle \psi_{-a}^\dagger \psi_+^a(x) \psi_{+b}^\dagger \psi_-^b(0) \rangle &= \langle \text{Tr} U(x) \text{Tr} U^\dagger(0) \rangle \\ &\times e^{2i\mu'x^1} \langle e^{\frac{\delta\phi(x)}{\sqrt{N}}} e^{-\frac{\delta\phi(0)}{\sqrt{N}}} \rangle, \end{aligned} \quad (18)$$

where $\delta\phi$ denotes excitations of ϕ around Eq. (16) and we have used the decoupling of the two sectors to factorize the correlator.

Let us now assume that the operator $\text{Tr} U$ has a non-vanishing expectation value (at zero temperature) in the $SU(N)_1$ theory deformed by the current-current interaction. Then, in the limit $|x| \rightarrow \infty$, Eq. (18) approaches

$$|\langle \text{Tr} U \rangle|^2 e^{2i\mu'x^1} |x|^{-\frac{2}{N(1+\lambda'/2\pi N)}}, \quad (19)$$

that is, it decays with powerlike behavior times an oscillating factor. The latter is a consequence of the chiral spiral configuration, whereas the former is the hallmark of quasi long-range order. Only a diagonal subgroup of the $U(1)_A$ symmetry and spatial translations preserves the would-be order parameter $\text{Tr} U e^{2i\mu'x^1}$. Recalling that the $\psi_{-a}^\dagger \psi_+^a$ bilinear carries $U(1)_A$ charge $+2$, this subgroup is a $U(1)_A$ transformation of parameter α accompanied by a translation with parameter $\delta x^1 = -\alpha/\mu'$. The condensation of $\text{Tr} U$ breaks completely the global center symmetry \mathbf{Z}_N^A of the $SU(N)_1$ theory. In the next section we will see that indeed in the large N limit the fermion bilinear gets a chiral spiral condensate (in the strict sense), giving evidence in favor of this assumption. If the \mathbf{Z}_N^A symmetry is unbroken in the $SU(N)_1$ theory ($\langle \text{Tr} U \rangle = 0$), an operator of the cGN theory with a spatially modulated expectation value must contain only vertex operators of the form $e^{ikN\phi/\sqrt{N}}$, with $k \in \mathbf{Z}$. This ensures that the operator is invariant under \mathbf{Z}_N^ϕ , without the need of a compensating factor charged under \mathbf{Z}_N^A from the $SU(N)_1$ sector (recall that the diagonal subgroup of these two \mathbf{Z}_N is gauged, so any physical operator must be neutral under it). For instance, the quasi long-range order could be detected in the two-point function of the N th power of the fermion bilinear $(\psi_{-a}^\dagger \psi_+^a)^N$. In this case the $U(1)_A$ transformation of parameter α needs to be accompanied by a translation with parameter $\delta x^1 = -N\alpha/\mu'$. There are also intermediate possibilities in which only a nontrivial subgroup $\mathbf{Z}_{N'}^A \subset \mathbf{Z}_N^A$ is preserved, when N is a multiple of N' .

The vacuum is then in a so-called ‘‘chiral spiral’’ configuration, where only a linear combination of the

$U(1)_A$ symmetry and of spatial translations preserves the would-be order parameter. Such combination depends on the realization of the \mathbf{Z}_N^A symmetry in the vacuum of the deformed $SU(N)_1$ theory. Excitations on top of the chiral spiral are gapless and have a relativistic dispersion relation.

These results apply in \mathbf{R}^2 , namely at $T = 0$ and at infinite spatial length. It is reasonable to expect that some chiral spiral configuration persists for $T > 0$ for sufficiently small temperatures in a thermal circle, when fermions are taken antiperiodic along the time direction. Discrete symmetries are, however, always restored in two dimensions at $T > 0$. As a result, $\langle \text{Tr} U \rangle$ vanishes and the chiral spiral cannot be detected by looking at the fermion bilinear but possibly only at \mathbf{Z}_N^A -invariant order parameters. Similarly, on a spatial circle of length L chiral spirals can possibly occur only for quantized values of the step. It would be interesting to establish more firmly the fate of the chiral spiral configuration for both $T > 0$ and finite spatial length L at finite N .

Chiral spiral at large N .—The cGN model in the large N limit is conveniently studied by introducing a complex Hubbard-Stratonovich (HS) field Δ rather than reformulating the theory as a $J\bar{J}$ deformation of a WZW model. At large N the \mathbf{Z}_N^A symmetry can be spontaneously broken at finite T , because the usual no-go theorems do not apply in this limit [45]. In this section we derive the critical temperature T_c of the cGN model, reproducing the result of [18–21] with a different method. To this end, we compute the free energy density per flavor (which for simplicity is called free energy in what follows) both for a homogeneous condensate and for an inhomogeneous one, assumed to have a chiral spiral form analogous to the one found above at finite N and $T = 0$. We show that at low temperatures $T < T_c$ the latter minimizes the free energy also at large N ; for $T > T_c$ the symmetry-preserving configuration, in which fermions are massless, is recovered.

We take the 't Hooft limit $N \rightarrow \infty$ with λ_s, λ_v fixed. The cGN model at large N is recovered for $\lambda_v = 0$. The free energy of the cGN model at large N is given by

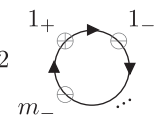
$$F = \frac{\overline{|\Delta|^2}}{\lambda_s} - \text{Tr} \log (\not{\partial} + \Delta P_+ + \Delta^* P_-), \quad (20)$$

where $\overline{|\Delta|^2}$ denotes the spacetime average of the square modulus of the condensate, and $P_\pm = (1 \pm \gamma_*)/2$ denote chiral projectors, γ_* being the 2D chirality matrix.

Under a $U(1)_A$ transformation, the HS field $\Delta = \rho e^{i\theta}$ transforms as $\theta \mapsto \theta + 2\alpha$, $\alpha \in \mathbf{R}$. At large N we can then identify $\theta = \phi/\sqrt{N}$, where ϕ is the compact scalar in the bosonization of the model. In terms of the HS field Δ , the chiral spiral configuration [Eq. (16)] reads $\Delta = M e^{2iqx}$, where $q = \mu$ at large N (at fixed λ_s and $\lambda_v = 0$), and we

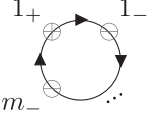
denote $x = x^1$. However we keep q generic, and compute the free energy in these configurations.

We can perform a perturbative expansion in the coupling λ_s , or equivalently in M . Neglecting irrelevant constant terms, we have

$$\begin{aligned} F(M, q) &= \frac{M^2}{\lambda_s} + \sum_{n=1}^{\infty} \frac{\text{Tr}[-\not{\partial}^{-1} M (e^{2iqx} P_+ + e^{-2iqx} P_-)]^n}{n} \\ &= \frac{M^2}{\lambda_s} - \sum_{m=1}^{\infty} \frac{1}{2m} \times 2 \times \text{Diagram} \end{aligned} \quad (21)$$


We can interpret the traces diagrammatically as follows. Oriented lines denote massless fermion propagators. Each insertion of P_\pm , denoted by \oplus, \ominus respectively, brings an insertion of mass M and spatial momentum $\mp 2q$. By momentum conservation in the loop there has to be equal number of P_\pm insertions, which must be alternated because $P_\pm \gamma^\mu P_\pm = 0$, therefore only terms with even $n = 2m$ contribute. Invariance under $\oplus \leftrightarrow \ominus$ exchange gives the extra factor of 2.

Letting $p_{1\mp} = p_1 \mp 2q$, at $T = \mu = 0$, one has

$$\begin{aligned} &\sum_{m=1}^{\infty} \frac{1}{m} \times \text{Diagram} \\ &= \int \frac{d^2 p}{(2\pi)^2} \log \left(1 + \frac{M^2 (p_1^2 + p_2^2 - q^2 + 2iqp_2)}{(p_{1-}^2 + p_2^2)(p_{1+}^2 + p_2^2)} \right). \end{aligned} \quad (22)$$


At $T > 0$ we need to replace $\int (dp_2/2\pi) \rightarrow T \sum_{p_2}$, with $p_2 = \pi(2n+1)T$, and at $\mu \neq 0$, $p_2 \rightarrow p_2 + i\mu$. After p_1 integration we have

$$F(M, q) = \frac{M^2}{\lambda_s} - T \sum_{p_2} \left[\sqrt{M^2 + p_2^2(q)} - \sqrt{p_2^2(q)} \right], \quad (23)$$

where $p_2(q) = \pi(2n+1)T + i(\mu - q)$. This expression contains divergences and needs to be regularized. We can use the $T = \mu = 0$ gap equation

$$\frac{1}{\lambda_s} = \int_{-\Lambda}^{\Lambda} \frac{dp_2}{2\pi} \int \frac{dp_1}{2\pi} \frac{1}{p^2 + M_0^2} \approx \frac{1}{2\pi} \log \left(\frac{2\Lambda}{M_0} \right), \quad (24)$$

to trade the dependence on the bare coupling λ_s for the uv cutoff Λ . Here, M_0 is the vacuum expectation value of $|\Delta|$ at $T = \mu = 0$. Similarly, we regularize the sum by putting a cutoff $p_{2,\max} = 2\pi n_{\max} T$ over the Matsubara frequencies and relate it to the uv cutoff by letting $\Lambda = p_{2,\max}$. The resulting expression is now finite in the limit $n_{\max} \rightarrow \infty$, and we obtain

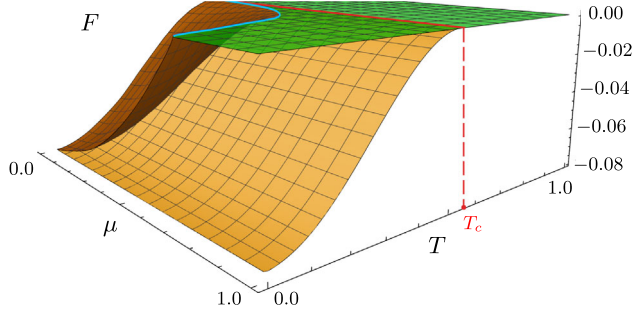


FIG. 1. The large N free energy F as a function of μ and T , in units of M_0 . At low T , the chiral spiral configuration (yellow) is favored with respect to the homogeneous configuration (brown) and the chirally symmetric phase (green). At $T = T_c$, there is a second order phase transition (red line) dividing the chiral spiral phase from the chirally symmetric phase. Assuming homogeneity one finds a different phase transition line (blue line).

$$F(M, q) = \frac{M^2}{2\pi} \left[\log \left(\frac{4\pi T}{M_0} \right) + \text{Re} \psi \left(\frac{1}{2} + i \frac{\mu - q}{2\pi T} \right) \right] - T \sum_{p_2} \left[\sqrt{p_2^2(q) + M^2} - \sqrt{p_2^2(q)} - \frac{M^2}{2\sqrt{p_2^2(q)}} \right], \quad (25)$$

where ψ is the digamma function.

We compare the minimum of the free energy over the whole parameter space, $F_{\text{cs}} = \min_{M, q} F(M, q)$, with the one computed assuming translational invariance, $F_{\text{hom}} = \min_M F(M, 0)$. For any given value of M , minimization with respect to q of the former is achieved at $q = \mu$, in agreement with Eq. (16). The μ dependence drops completely in F_{cs} , and $F_{\text{cs}} = F_{\text{hom}}(\mu = 0)$. The residual minimizations over M are performed numerically and the result is plotted as a function of (μ, T) in Fig. 1. The chiral spiral configuration $\Delta = M(T)e^{2i\mu x}$ is always favored with respect to the homogeneous configuration $\Delta = M(\mu, T)$; moreover, one has $M(T) = 0$, i.e., the symmetric massless phase is recovered, for $T \geq T_c = M_0 e^\gamma / \pi$, where a second order phase transition occurs [46]. Inhomogeneous vacuum configurations of this kind were already found in [18–21] at large N by using a Hartree-Fock approach and by solving the gap equation for inhomogeneous condensates. Here, we provided a streamlined computation of the free energy based on a resummation of Feynman diagrams, and our findings are in agreement with these previous works.

More recently, inhomogeneous phases of the cGN model at finite N have been found using lattice simulations (also for small but nonvanishing T). Though it is not clear to us to what extent these lattice results persist when the continuum and infinite volume limits are taken, it is worth noticing that the different slopes of the two linear fits in Fig. 8 of [27] ($\langle \partial_1 \phi \rangle / \sqrt{N}$ as a function of μ in our notation) is in qualitative agreement with our analytic finding [Eq. (16)],

which predicts that the linear slope decreases as N decreases.

Summarizing, using nonabelian bosonization, we have found that inhomogeneous phases persist for any finite N and any μ in the cGN model at $T = 0$. We have also presented a simple derivation of the phase diagram of the large N cGN model as a function of μ and T and reproduced the findings of [18–21].

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