Mathematics Area - PhD course in

Geometry and Mathematical Physics

Quantum Field Theories, Isomonodromic Deformations and Matrix Models

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Abstract

Recent years have seen a proliferation of exact results in quantum field theories, owing mostly to supersymmetric localisation. Coupled with decades of study of dualities, this ensured the development of many novel nontrivial correspondences linking seemingly disparate parts of the mathematical landscape. Among these, the link between supersymmetric gauge theories with 8 supercharges and Painlevé equations, interpreted as the exact RG flow of their codimension 2 defects and passing through a correspondence with two-dimensional conformal field theory, was highly surprising. Similarly surprising was the realisation that three-dimensional matrix models coming from M-theory compute these solutions, and provide a non-perturbative completion of the topological string. Extending these two results is the focus of my work.

After giving a review of the basics, hopefully useful to researchers in the field also for uses besides understanding the thesis, two parts based on published and unpublished results follow. The first is focused on giving Painlevé-type equations for general groups and linear quivers, and the second on matrix models.

Acknowledgments

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I want to thank my collaborator Tomoki Nosaka for the hours upon hours of Skype. I am still in awe of your computational, as well as organisational powers.

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And for the material in this thesis and my humble co-creation, may to the Creator of the Universe be the highest glory.

Contents

Chapter 1

Introduction

1.1 Organisation of the introduction

Caveat lector. This work is, similarly to its material, nonlinear. The chapters of the introduction are not presented in sequential order, and refer one to another often. The main organisational principle is the following small category

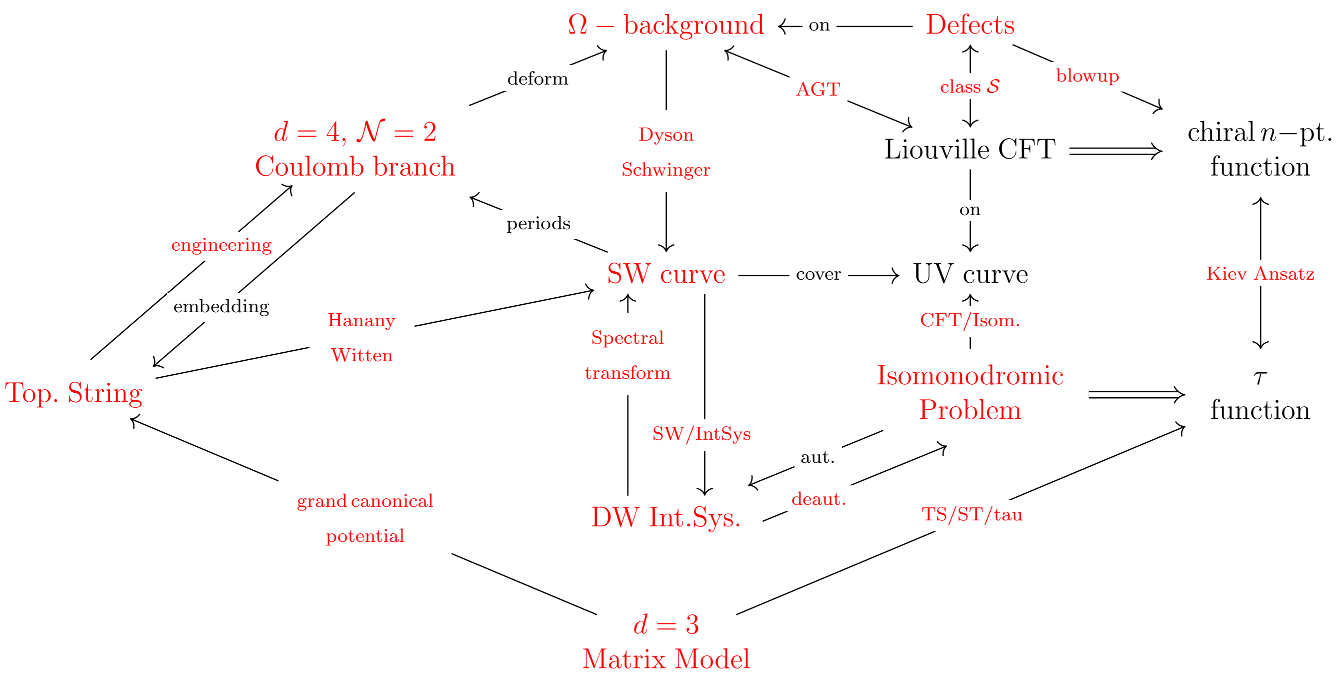


Figure 1.1: Roadmap, clickable.

and the introduction itself serves as a functor, of sorts, to the narrative structure of human cognition. The reader is invited to find a reasonable approximation to a Hamiltonian path through the introduction, or forget about it and just read as presented, or forget about a single path and, aided by the roadmap, use this work as a reference for the future. It is my hope that in this latter function this thesis will serve as a useful review, especially among my younger academic brothers and sisters, and those yet to join us.

After the *solve* of the introduction, the *coagula* of the mostly published material will be presented in two separate sections, fulfilling the alchemical formula.

Finally, a note on style. I have tended to avoid the use of the "scientific we" unless the work features the work of my collaborators or is of an introductory, repetitive character, establishing facts well known to all of "us".

1.2 Supersymmetric gauge theory

The gauge theories encountered in this work are mostly those with 8 supercharges, which in $d = 4$ dimensions means they have extended supersymmetry. This introduces a lot of physically unrealistic features, such as the presence of adjoint matter, and at the same time introduces a tremendous amount of geometry to the theory – the scalars can be seen to locally parametrise a target space with some geometric requirements, the couplings are described by certain "canonical" geometric objects [57]. Usually, if we consider the full array of defects and extended objects, the theory thus written consists of maps of points, lines, surfaces, etc, into a target space – morally a functor of points, with the path integral providing a sort of weighted sum. Although nonsupersymmetric QFT also heavily features geometric structures, I have personally heard many criticisms of this unreality of supersymmetric physics – and this is not even talking about the "dynamically trivial" topological field theories or theories in dimensions not four!

There should be no excuses. The increasing geometrisation of physics might mean that physics is now used to calculate mathematical instead of concrete realities, but this has served to dignify physics – it has taken centre stage, spearheading advancements in mathematics left, right and centre throughout the last century, and it shows no signs of stopping in the current one.

Instead of being viewed through the prism of either "real-world" physics or "imaginary" mathematics – as if imagination is insulting –, in mathematical physics / physical mathematics, more similarly to its sororal mathematics than physics, we find an already well-established science in an unfettered form, growing on its own terms led by intellectual curiosity and creativity, and casting aside the motivation of exploitable material innovation which an obsession on "real world progress" would demand. It is a happy accident that this can all be justified as perhaps leading to physically viable theories after symmetry breaking, but to hope for flying cars as the end result is to miss the point entirely. In the Hilbertianesque programme [205], Gregory Moore also sums up an optimistic attitude necessary for this, and indeed *any*, successful intellectual endeavour

> If a physical insight leads to a significant new result in mathematics, that is considered a success. It is a success just as profound and notable as an experimental confirmation from a laboratory of a theoretical prediction of a peak or trough. For example, the discovery of a new and powerful invariant of four-dimensional manifolds is a vindication just as satisfying as the discovery of a new particle.

This science furthermore incorporates a set of experimental methods deemed of no less importance by our scientific community than the theoretical, found in experimental-mathematical techniques of computer-assisted numerical and symbolic manipulation.

1.2.1 Effective IR dynamics of $\mathcal{N} = 2$ super Yang-Mills

We can interpret the scalar components of chiral multiplets of a Lagrangian $d = 4$ $\mathcal{N} = 1$ theory as a sigma model to a Kähler manifold target space $\mathcal{M}$ with a superpotential as a Morse function, and gauge symmetry being a gauging of a subgroup of the isometry group $\text{Iso}(\mathcal{M})$ [57]. This fixes the chiral-vector interaction, but there is still a tremendous amount of freedom left in choosing a holomorphic superpotential.

Increasing the amount of supersymmetry partially fixes this choice. $\mathcal{N} = 2$ hypermultiplets consist of a pair of $\mathcal{N} = 1$ chiral multiplets while $\mathcal{N} = 2$ vectormultiplets $(\phi, \lambda_\alpha, \bar{\lambda}_{\dot{\alpha}}, A_\mu)$ consist of a $\mathcal{N} = 1$ vectormultiplet and a $\mathcal{N} = 1$ chiral multiplet in the adjoint representation of the gauge group G[1]. Further, matter-matter coupling is forbidden by the extended supersymmetry, so interactions are possible only through the gauge group, i.e. by specifying a matter representation. Quivers, encoding choices of gauge groups and matter representations, are therefore often used to specify the theories completely.

The geometry is no longer Kähler but special Kähler and is fully fixed by a holomorphic Kähler prepotential $\mathcal{F}(z)$. The kinetic term can be written in terms of the $\mathcal{N} = 2$ superspace as

$$\mathcal{L}_{kinetic} = \text{Im} \int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta} \mathcal{F}(\Phi)$$

where $\Phi = \phi + ...$ is a chiral superfield, which makes up the $\mathcal{N} = 2$ vectormultiplet. The Kähler potential is then given by $K = \text{Im}\, \Phi_a^\dagger \partial_{\Phi^a} \mathcal{F}$ and the metric on the moduli space is

$$g_{ab} = \partial_a \partial_b \mathcal{F}$$

The only *renormalisable* choice is quadratic, $\mathcal{F}(x) = (\tau_{UV}/2)x^2$. Here, $\tau_{UV} = 4\pi i/g^2 + \theta/2\pi$ is the complexified gauge coupling. If we explicitly integrate out the auxiliary field present in the superspace formalism, in the case of a pure gauge theory we find the effective potential

$$V(\phi) = \frac{1}{g^2} \operatorname{tr} \left[\phi, \phi^\dagger\right]^2$$

which needs to vanish for the vacuum to be supersymmetric. This means ϕ is valued in the Lie algebra of the maximal torus of G, or the Cartan of $\mathfrak{g} = \text{Lie}(G)$ if G is semisimple. As such, if the scalar vevs are nonzero, interaction terms proportional to $\operatorname{tr}[\phi, A_\mu]^2$ in the Lagrangian will give mass to the vector field and result in W-bosons with masses $\langle \phi \rangle r$, where r is a root of the Lie algebra. We have a breaking $\mathfrak{g} \to U(1)^{\text{rk}G}$. This is called the Coulomb branch. The theory has to be specified in terms of Weyl-invariant combinations of the adjoint scalar vevs, so

$$\mathcal{M} = \text{Spec}[\mathfrak{g}]^G = \text{Spec}\mathbb{C}[u_1, ..., u_{\text{rk}G}]$$

where $u_i = \operatorname{tr} \phi^i$. In the general case with matter, the effective potential is more complicated and involves matter multiplet vevs. One solution of the F and D-term

[1]Unitarity demands positive definiteness of the kinetic term, which can be shown [269, §15.2] to mean that G is (split) reductive, as its Lie algebra $\text{Lie}(G) = \mathfrak{g}$ has to be a direct sum of simple Lie algebras and $U(1)$ subalgebras.

vacuum equations is the same solution except with matter vevs turned off, so it is also called the Coulomb branch. Another phase, called the Higgs branch, is specified by the matter multiplet vevs and adjoint vevs turned off. It receives no quantum corrections and is not the focus of this work.

This splitting of the vacuum moduli space into disjoint Coulomb and Higgs branches is a general feature of $\mathcal{N}=2$ theories, although mixed branches exist for higher rank theories. Physically, these represent different phases of matter. A quark-antiquark pair is created at a distance r and propagating for time T before annihilating can be represented by a Wilson loop taken over a $r\times T$ rectangle. The classical potential $V(r)$ they feel can be extracted from the expectation value of the loop as $T\to\infty$,

$$e^{-TV(r)}\sim\langle \operatorname{tr} Pe^{i\int A}\rangle$$

In the Coulomb phase, $V(r)\propto 1/r$ leads to standard electrodynamics, while in the Higgs phase, $V(r)=\text{const}$. In the dual frame, which we will soon discuss, nothing changes on the Coulomb branch, while monopoles in the Higgs phase experience confinement.

Now consider one-loop renormalisation, leading to

$$\mu\frac{\mathrm{d}g}{\mathrm{d}\mu}=-b\frac{g^2}{16\pi^2}+\mathcal{O}(g^5)\Rightarrow\tau(\mu)=\tau_{UV}-\frac{b}{2\pi i}\log\frac{\mu}{\Lambda_{UV}}=-\frac{b}{2\pi i}\log\frac{\mu}{\Lambda},$$

where $\Lambda=\Lambda_{UV}e^{2\pi\tau_{UV}/b}$ is the emergent dynamical scale. Beyond the perturbative one-loop corrections, due to supersymmetry the only possible corrections are instanton corrections, which modify the gauge coupling according to holomorphic RG flow as

$$\tau(\mu)=-\frac{b}{2\pi i}\log\frac{\mu}{\Lambda}+\sum_{n\geq 0}a_n\left(\frac{\Lambda}{\mu}\right)^{bn}$$

Therefore, we see that RG flow takes us from the microscopic theory corresponding to a quadratic $\mathcal{F}$ to the realms of effective IR dynamics specified by the non-renormalisable general $\mathcal{F}(x)$, as $\tau=\mathcal{F}''$ is the holomorphic function written above. Considering the Coulomb branch, deep IR means we integrate out the massive W-bosons. As such, we are left with the broken gauge group of $\operatorname{rk}G$ copies of $U(1)$. Given scalar vevs $\langle\phi_i\rangle=a_i$, the metric on the IR Coulomb branch is $g_{ij}(a)=\operatorname{Im}\partial_i\partial_j\mathcal{F}=\operatorname{Im}\tau_{ij}(a)$. Clearly, it should be positive to ensure unitarity.

However, there is a tension here as we cannot have a globally defined matrix of gauge coupling τ_{ij} which is both holomorphic and having a positive definite imaginary part[2]. The great insight of Seiberg and Witten [251] was to resolve this in two parts. First of all, we have to drop a global description. Let $a_i^D=\partial_{a_i}\mathcal{F}$ be a dual coordinate on the Coulomb branch. It can be shown that a theory formulated entirely in terms of a^D is a dual magnetic description of the theory, an avatar of Montonen-Olive duality which changes $\tau\mapsto -1/\tau$ and exchanges electric charges and magnetic monopoles. The pair (a,a^D) is called an electric-magnetic frame, and it locally furnishes coordinates on the Coulomb branch. A key insight, however, is that they are multi-valued. The Coulomb branch has to be parameterised in terms of Weyl-invariant combinations of the adjoint scalar vevs a. In the $G=SU(2)$ case, Weyl symmetry is just reflection around the origin, so a good Coulomb branch

[2]Then $\operatorname{Im}\tau$ would be a bounded harmonic function on $\mathbb{R}^2$, so it has to be constant.

parameter is $u = a^2$. But this means a is a square root, and both (a, a^D) develop a nontrivial monodromy around infinity, since $u \mapsto e^{2\pi i} u$ maps $a \mapsto -a$, and we can calculate that $a^D \mapsto -a^D + 2a$. Seiberg and Witten realised that this means the Coulomb branch has singular points.

Second of all, besides the singular points, they recognised that these conditions determine a Riemann period matrix of a Riemann surface. On a Riemann surface Σ_g of genus g, choose a Torelli marking of A- and B-cycles $A_i \cap B_i = \delta_{ij}$, $A_i \cap A_j = B_i \cap B_j = 0$ in $H_*(\Sigma_g, \mathbb{Z})$ and a basis of holomorphic differentials ω_i normalised so that

$$\oint_{A_j} \omega_i = \delta_{ij}, \quad \oint_{B_j} \omega_i = \tau_{ij}^{\text{Period}}$$

Suppose we are given a meromorphic differential $\lambda \in H^1(\Sigma_g)$ depending on a such that $\partial_{a_i} \lambda = \omega_i$ up to an exact 1-form. Then the coordinates can be given as

$$a_i = \oint_{A_i} \lambda, a_j^D = \oint_{B_j} \lambda$$

and the gauge coupling matrix is $\tau_{ij} = \partial_{a_j} a_i^D = \tau_{ij}^{\text{Period}}$. This can be used to reconstruct the prepotential $\mathcal{F}$. Therefore, the theory is solved by giving the algebro-geometric datum (Σ, λ) of a Riemann surface with punctures and a meromorphic one-form. At the punctures, the residue of λ should give the physical masses. These punctures are not to be confused with the punctures on the moduli space $\mathcal{M}$ itself, where certain combinations of charges and magnetic monopoles become massless, so Wilsonian RG flow diverges. Rather, the moduli space $\mathcal{M}$ itself has the geometry of a g-torus fibration over a base $\mathcal{B}$ parameterised by Weyl-invariant polynomials, $\mathcal{M} = \text{Jac}(\Sigma) \to \mathcal{B}$. The tori have local coordinates (a_i, a_i^D). On the singular points of the base, the tori degenerate. The singular points have been classified by Kodaira for elliptic fibrations, and their monodromies are known.

It's not obvious that this is the full solution. Consider a pure theory. For $G = SU(2)$, we need $\text{rk}G = 1$ pair of canonically conjugate coordinates (a, a^D), so a genus $g = 1$ surface which gives us two periods. For rank g, the space of symmetric matrices τ_{ij} has dimension $g(g+1)/2$, those with positive definite imaginary part forming a cone. But the moduli space of genus $g > 2$ Riemann surfaces has dimension $3g-3$. I believe, however, that Novikov's conjecture along with a generalisation of Witten's conjecture [275] solves the issue. Namely, a theta function with period matrix τ_{ij} may be associated to each theory. It is not clear that this theta function may be associated to a Riemann surface. On the other hand, 2-TQFTs have tau functions which solve the KP hierarchy, including the topological string. In this case, the tau function can be identified with the theta function, and a topologically twisted version of the $\mathcal{N} = 2$ theory can be seen as arising from compactifying the topological string. Therefore, by Novikov's conjecture, τ_{ij} is a period matrix of a Riemann surface.

Be as it may, we are mostly interested in the so-called class $\mathcal{S}$ theories, which are built by compactification which features the Seiberg-Witten curve directly. An example is pure $SU(N)$ super Yang-Mills, given in hyperelliptic form as [113]

$$Y^2 = P_N(w)^2 - 4\Lambda^{2N}, \quad P_N = \prod_{i=1}^{N} (w - w_i), \quad Y = \Lambda^N \left(z - \frac{1}{z} \right) \tag{1.1}$$

which is a N-fold covering of the 2-punctured z-plane and a 2-fold covering of the w-plane, and the Seiberg-Witten differential is $\lambda = w\mathrm{d}z/z$. The Weyl-invariant coordinates are the "Chern classes"

$$u_k = (-1)^{k+1} \sum_{i_1<\ldots<i_k} w_{i_1} \cdots w_{i_k}$$

which are the coefficients in the expansion of $P_N(w)$, so we can see how the Seiberg-Witten curve varies along the Coulomb branch. It can be seen that $\partial_{u_k}\lambda = w^{N-k}\mathrm{d}w/Y$, which are the canonical basis of $H^1(\Sigma_g)$ of a hyperelliptic curve. u_1 is a trace and vanishes.

In general, the Seiberg-Witten curve can be written as a ramified covering of another punctured Riemann surface, in the last example given by the z-plane. This base is called the UV curve, because UV data such as hypermultiplet masses are specified by the singularities of the holomorphic differential at those points.

The elliptic fibration structure of the Coulomb moduli space comes alive if the IR theory is further compactified on a circle, $\mathbb{R}^3 \times S^1$, as the $d = 3$ theory itself can be seen as a sigma model on $\mathcal{M}$. The scalars a^i stay, but the gauge field $A^i_\mu \mathrm{d}x^\mu$ decomposes to a 3-dimensional vector $A^i_j \mathrm{d}x^j$ along with a periodic (Peccei-Quinn) scalar coming from the Wilson line $\theta^i_e = \int_{S^1} A^i_\mu \mathrm{d}x^\mu$. But a 3-dimensional vector is dual to a periodic scalar field itself, defined by $\star_3 \mathrm{d}\theta^i_m = \mathrm{d}A^i$. Together, this set of fields describes $\mathcal{M}$ concretely as a torus fibration of a base given by invariants made from the scalar field, which is a complex integrable system. Further, $\mathcal{N} = 4$ $d = 3$ supersymmetry means the target space has a hyperkähler structure. So in this case we recover some facts about the space, although the Seiberg-Witten solution itself is obscure.

1.3 The Omega Deformation

In [221] it was noted that the solution of $\mathcal{N} = 2$ super Yang-Mills on $\mathbb{R}^4$, by which we mean its low-energy prepotential, can be found by deforming the theory with two parameters $\epsilon_{1,2}$, so that the prepotential is recovered in the limit

$$\lim_{\epsilon_{1,2}\to 0} \epsilon_1\epsilon_2 \log Z(\epsilon_{1,2}, a) = \mathcal{F}(a)$$

and $Z(\epsilon_{1,2}, a)$ is the partition function of the deformed theory, which turns out to be the same as the (generating function of the) instanton moduli space integral [207], already known for some years, up to a classical and perturbative part.

The $\mathcal{N} = 2$ supersymmetric algebra in four dimensions has, unlike with more supersymmetry, essentially only one topological twist, the Donaldson-Witten twist. The $\mathcal{N} = 2$ algebra is $U(2)_R = U(1)_R \times SU(2)_R$ R-symmetric. On the other hand, on a spin 4-manifold M we also have an Spin(4) $\cong SU(2)_1 \times SU(2)_2$ action, the Lorentz group. We pick a homomorphism $SU(2)_R \to Spin(4)$. This changes the space of fields, since the extended $\mathcal{N} = 2$ indices and (dotted) spinor indices can be equated by the homomorphism. So, if the vector multiplet was $(\phi, \lambda^i_\alpha, \bar{\lambda}^i_{\dot{\alpha}}, A_\mu)$, with i the extended supersymmetry index, then what used to be the chiral gaugino λ^i_α becomes a vector under the twist, and the antichiral one decomposes into a scalar and an ADS two-form. This general phenomenon means the odd superspace becomes $(\mathbb{C} \oplus$

$(\Lambda^2 TM)^+$) while the even one becomes just the tangent bundle TM. In this way, even spaces which might not have covariantly constant spinors, needed to define supersymmetry, end up with one well-defined scalar supercharge. Often, the energy-momentum tensor of these theories ends up being exact under this supercharge, and therefore, observables closed under it are invariant to changes in the metric. This is the construction of cohomological field theories [276]. The self-dual "fermion" turns out to be a Lagrange multiplier imposing $1/2\varepsilon_{abcde}F_{ce} = -F_{ab}$ while the super Yang-Mills action is

$$S \in \left[\frac{2\pi i\tau}{8\pi^2}\int_M \mathrm{tr}\, F \wedge F\right]$$

in the supercohomology. As such, the evaluation of the path integral reduces to integration on the finite dimensional instanton moduli spaces. There is also an anti-twisted theory which localises on the ASD solutions. This amounts to redefining the coupling τ, however. The presence of matter means the theory localises on certain monopole solutions and defines the Seiberg-Witten invariant of M.

This is all much too restrictive to be of use on the "trivial" space $M = \mathbb{R}^4$. First of all, the chiral ring of observables is somewhat limited by there not being nontrivial cycles, as pointed out by Nekrasov. Secondly, the moduli space of n-instantons, meaning those with second Chern class $c_2(P \to M) = n$, $\mathcal{M}_n$, is non-compact due to point-like instantons. Uhlenbeck-Donaldson compactification adds these points,

$$\bar{\mathcal{M}}_n = \cup_{k=0}^{n} \mathcal{M}_{n-k} \times \mathrm{Sym}^k \mathbb{C}$$

however the space is singular due to reducible connections. The twist appropriate for solving the theory, turns out to be the one introduced by Moore, Nekrasov and Shatashvili [206], although it first appeared in the context of $d = 6$ dimensional super Yang-Mills. The BRST operator is in this case *not* exact, but squares to an isometry of the space. This means we localise on those ADS solutions invariant under a certain torus action, which corresponds to isometries of a deformed $\mathbb{R}^4$, the Omega-deformation. *It is crucial to note that these are the (A)SD connections on the original* $\mathbb{R}^4$*, not* on *the Omega-deformation.* That is, although the Omega-deformation $\mathbb{R}^4_{\epsilon_{1,2}}$ can be given a metric of its own in five dimensions as a twisted $\mathbb{R}^4$ fibration of S^1, the so-called Melvin space, which we could reasonably pullback to the fiber, the localisation involves the flat, run of the mill $\mathbb{R}^4$ metric and its Hodge star. There are no $d = 5$ "instanton particles" involved.

Specifically, we consider $\mathbb{R}^4 = \mathbb{C} \times \mathbb{C}$ and consider the torus action $\mathbb{T} = \mathbb{C}^\times \times \mathbb{C}^\times$. Actually, instantons are (A)SD connections with finite action, which implies the vanishing of the curvature at infinity, or $A \to g^{-1}\partial g$ for some $g \in G$ at infinity. Note that, amazingly, the scalar supercharge is on-shell exactly the BRST differential from BV-quantization of the bare topological action $S \propto c_2(P \to M)$ [253, §2.6]. Therefore, it has to square to an infinitesimal gauge transformation when acting on the vector field A_μ, and we find $\delta^2 A_\mu \propto \nabla_\mu \phi$. Moreover, acting on the scalar field itself we find $\delta^2\phi^\dagger \propto [\phi, \phi^\dagger]$. For concreteness we consider the gauge group $G = U(k)$. In the SW theory, it was supersymmetry which imposed the Coulomb branch breaking of the gauge group to its Cartan $U(k) \to U(1)^k$ as one of the solutions. Here, we find two things:

1. nilpotence of the twisted supercharge demands $[\phi, \phi^\dagger] = 0$

2. ϕ parametrises infinitesimal gauge transformations

Therefore, we have gauge transformations parametrised by the (vacuum expectation) values $\langle\phi\rangle = a$ of the adjoint scalar ϕ. These are rigid gauge transformations and should be interpreted as rotations of the framing at infinity. Although we could think of the gauge group as being *asymptotically* spontaneously broken to the Cartan $U(1)^k$, we are looking at instantons of the *full* gauge group G "in the bulk". However, we will shortly show that we have to localise at fixed points of this torus action. Precisely this will ensure that all our solutions are indeed on the Coulomb branch that is, that the gauge group is not merely asymptotically but everywhere spontaneously broken. This is because the centraliser of the (maximal) torus action in a connected, compact Lie group is precisely the Cartan, which is the torus itself. So our solutions are actually on the torus, as the fixed points of the action have to be in the centraliser. In [215, §4], Nakajima and Yoshioka notice that rank 1 sheaves, in a direct sum of which the instantons split, are easy to study, luckily for the entire mathematical physics community. As we switch the Omega deformation off to get back to Seiberg-Witten theory, we remain with instanton corrections on the Coulomb branch, with broken $U(k) \to U(1)^k$, which also ensures supersymmetry is unbroken. This is in spite of the fact that our single BRST scalar is weaker than full $\mathcal{N} = 2$ supersymmetry, a happy accident of sorts.

The Omega background itself can be described as already mentioned, as a twisted $\mathbb{R}^4$ fibration of S^1, so that the total space the quotient of $\mathbb{C}^2 \times S^1$ under the $\mathbb{Z}$ action

$$n \triangleright (z_1, z_2, t) = (e^{\beta\epsilon_1 n} z_1, e^{\beta\epsilon_2 n} z_2, t + \beta n)$$

where β is the radius of the circle. The four-dimensional theory is the dimensional reduction of this Melvin space to the $\mathbb{R}^4$ fiber, alternatively the $\beta \to 0$ limit when the base shrinks to a point. The Omega background is often denoted by $\mathbb{R}^4_{\epsilon_{1,2}}$. $\epsilon_{1,2}$ are the first Chern class of the $\mathbb{R}^4$ bundle, and we often view them as equivariant parameters. Two main *phases* are the self-dual background where $\epsilon_1 + \epsilon_2 = 0$, which is linked to topological string theory with $g_s = \epsilon_1$, and the Nekrasov-Shatashvili limit $\epsilon_2 = 0$, which is quite singular and has some interesting links to quantum mechanics with $\hbar = \epsilon_1$.

Having deformed the 4 dimensional $\mathcal{N} = 2$ Coulomb branch dynamics to a a twisted TQFT and seen that the super Yang-Mills action reduces to the second Chern class and an exact term, and localises to ASD solutions, and breaks to $U(1)^{\mathrm{rk}G}$ under a torus action, we can convince ourselves that the way forward is to look at the torus $\mathbb{T}$-invariant solutions on the moduli space $\mathcal{M}_{n,G}$ of n-instanton solutions. We also have to take into account quantum fluctuations in the path integral, which don't depend on the instanton number and terminate on the one-loop term due to the supersymmetry. All in all, we have the decomposition of the path integral to

$$Z(\epsilon_{1,2}, a|\Lambda) = Z_{\text{classical}}(\epsilon_{1,2}|\Lambda) Z_{\text{one-loop}}(\epsilon_{1,2}, a) Z_{\text{instanton}}(\epsilon_{1,2}, a|\Lambda)$$

where Λ is the complexified dynamical scale of super Yang-Mills, and

$$Z_{\text{classical}}(\epsilon_{1,2}|\Lambda) = (\Lambda^{2h^\vee})^{\frac{1}{2\epsilon_1\epsilon_2}a^2}, \quad Z_{\text{one-loop}}(\epsilon_{1,2}, a) = \exp\left\{-\sum_{\alpha\in R} \gamma_{\epsilon_1,\epsilon_2}(a\cdot\alpha|\Lambda)\right\}$$

are the tree-level and one-loop terms, explained in [221, §3.10], [253, §2.3], and [215], and the instanton part is the generating function of instanton volumes, futher localised to $\mathbb{T}$-equivariant points using essentially the Duistermaat-Heckman formula,

$$Z_{\text{instanton}}(\epsilon_{1,2}, a|\Lambda) = \sum_{n\geq 0} \Lambda^{2h^\vee n} \int_{\mathcal{M}_{n,G}} 1 = \sum_{n\geq 0} \Lambda^{2h^\vee n} \sum_{p\in(\mathcal{M}_{n,G})^{\mathbb{T}}} \frac{1}{\mathrm{Eu}(T_p\mathcal{M}_{n,G})} \tag{1.2}$$

To actually calculate this, we switch to equivariant localisation [143, 240, 241]. Namely, consider a smooth manifold M with a G-action. If the action is free, we define equivariant cohomology by $H^\bullet_G(M) = H^\bullet(M/G)$ since M/G is smooth. If the action isn't free, we would get stacky points with this construction, so we define $H^\bullet_G(M) = H^\bullet(M \times_G EG)$, where EG is the universal bundle. As an example, for a point we get $H^\bullet_G(\mathrm{pt}) = H^\bullet_G(EG/G) = H^\bullet_G(BG)$. Over $\mathbb{C}$, this is $\mathbb{C}[\mathfrak{g}]^W$, the characteristic classes, given by Weyl-invariant polynomials of the Lie algebra.

In general, we would like to describe it using a complex. On $\alpha \in \Omega^\bullet(M)\otimes\mathbb{C}[\mathfrak{g}]$, the natural G-action is $(g \triangleright \alpha)(x) = \alpha(g^{-1}x)$. Equivariant differential forms satisfy $\alpha(gx) = (g \triangleright \alpha)(x)$, so they are invariant under the G-action. This leads us to consider G-equivariant differential forms, $\Omega^\bullet_G(M) = (\Omega^\bullet(M)\otimes\mathbb{C}[\mathfrak{g}])^G$.

The differential itself is available via the topological twist. Namely, in general we want it to square not to zero but to an isometry generated by a vector field V, and this can be achieved by

$$\mathrm{d}_G = \mathrm{d} + \xi i_V \;\Rightarrow \mathrm{d}_G^2 = \xi\{\mathrm{d}, i_V\} = \xi \pounds_V$$

if $\pounds_V$ is seen to act on $\Omega^\bullet$. Then $H^\bullet_G(M) = \ker \mathrm{d}_G/\mathrm{im}\mathrm{d}_G$.

It turns out integration of G-equivariant top forms is very simple if $G = T$ is a torus. Namely, if $\pi^M : M \to \mathrm{pt}$ is a projection to a point, then integration can be seen as the pushforward to the T-equivariant cohomology $\pi^M_* = H^\bullet_G(M) \to H^\bullet_G(\mathrm{pt})$. Consider the inclusion $i : F \hookrightarrow M$ of T-fixed points into M. The important non-trivial property of the Euler class $\mathrm{Eu}(N_F)$ of the normal bundle to F in M is that it doesn't vanish, and that

$$i^*\Phi_F = \mathrm{Eu}(N_F)$$

where Φ_F is the Thom class, the Poincare dual to F, $i_*\Phi_F = 1$. This lets us write the pushforward π^M_* in terms of of the composition $\pi^F = i \cong \pi^M$ as

$$\pi^M_* = \pi^F_* \frac{i^*}{\mathrm{Eu}(N_F)}$$

This is the Atiyah-Bott integration formula [9]

$$\int_M \alpha = \int_F \frac{i^*\alpha}{\mathrm{Eu}(N_F)}$$

and since F is a set of isolated points, it is equivalent to (1.2) in our case.

This first-principles calculation of $\mathcal{N} = 2$ super Yang-Mills on $\mathbb{R}^4_{\epsilon_1,\epsilon_2}$ can then be done once the moduli space is described.

1.3.1 The ADHM construction

There is a constructive approach to n-instanton solutions on S^4 for classical gauge groups, called the ADHM construction, first tersely presented in [10]. Dimension counting confirms that these are in fact all the solutions. Therefore, it can be used for equivariant localisation, as in [221, 227]. Here we describe $G = U(r)$ instantons.

Instantons in the ADHM construction are reconstructed from a quiver with a superpotential. The two nodes of the quiver are conventionally denoted by

$$N = \mathbb{C}^{\mathrm{rk}G} = \mathbb{C}^r, \quad K = \mathbb{C}^n$$

a quadruple of linear maps

$$(B_1, B_2, I, J), \quad B_{1,2} \in \mathrm{End}(K), I \in \mathrm{Hom}(N, K), J \in \mathrm{Hom}(K, N)$$

and the moment maps, the vanishing of which are usually called the ADHM equations,

$$\begin{aligned} \mu_{\mathbb{R}} &= [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J \\ \mu_{\mathbb{C}} &= [B_1, B_2] + IJ \end{aligned}$$

There is an action of $GL(K) = U(n)$, called the *dual group* $G^\vee$ in this context, so that

$$g \triangleright (B_1, B_2, I, J) = (\mathrm{Ad}_g B_1, \mathrm{Ad}_g B_2, gI, Jg^{-1}), \quad g \in U(n) \tag{1.3}$$

and the action of the torus $\mathbb{T}$ which includes spatial rotations is

$$t \triangleright (B_1, B_2, I, J) = (t_1 B_1, t_2 B_2, It_3^{-1}, t_1 t_2 t_3 J), \quad t_{1,2} = e^{\epsilon_{1,2}} \in \mathbb{C}^\times, t_3 = e^a \in U(1)^r$$

The instanton moduli space is then defined as the hyperkähler quotient

$$\mathcal{M}_{n,r} = \{(B_1, B_2, I, J) | \mu_{\mathbb{R}} = 0, \mu_{\mathbb{C}} = 0\} /// G^\vee$$

and it can be checked by simple component counting that its complex dimension is $2rn$. The actual instanton is constructed from the Dirac operator, with $z_{1,2} \in \mathbb{C}$

$$D^\dagger = \begin{bmatrix} B_1 - z_1 & B_2 - z_2 & I \\ -B_2^\dagger + z_2^* & B_1^\dagger - z_1^* & -J^\dagger \end{bmatrix}$$

Namely, we pick a zero mode $\Psi \in \ker D^\dagger$ normalised to unity, and the ADS connection is then simply $A = \Psi^\dagger \mathrm{d}\Psi$, and all the necessary properties follow from the ADHM equations.

The same construction can be derived from brane dualities in type II string theory [77, 78, 107, 159, 264, 271]. Namely, consider N coincident Dp branes along with k $D(p-4)$ branes ($p \geq 3$). The stack of parallel Dp-branes is described by $p+1$ dimensional $U(N)$ super Yang-Mills theory with 16 supercharges, since fixing of the branes breaks half of the supersymmetry. This is coupled to $9-p$ adjoint scalars which describe the position. The $D(p-4)$ branes can be outside the Dp stack (Coulomb branch) or stuck inside the Dp worldvolume (Higgs branch), or mixed. It turns out, however, that the mass and charge of the trapped $D(p-4)$ brane matches with an instanton in the Dp stack's worldvolume. The Ramond-Ramond form coupling to the $D(p-4)$ branes for an instanton solution $F = \star F$ of the worldvolume theory equals the source of the $D(p-4)$ brane itself

$$\mathrm{tr} \int_{Dp} \mathrm{d}^{p+1} C_{p-3} \wedge F \wedge F = \frac{8\pi^2}{e^2} \mathrm{tr} \int_{D(p-3)} \mathrm{d}^{p-3} C_{p-3}$$

The ADHM variables $(B_{1,2}, I, J)$ can be identified in turn with open strings connecting $D(p-4)$-$D(p-4)$, Dp-$D(p-4)$, and $D(p-4)$-Dp branes (Chan-Paton spaces), and the ADHM equation is the BPS condition for preserving supersymmetry. For $p=0$ one is left with ordinary quantum mechanics, although properly speaking we need higher p to ensure the existence of a moduli space.

This is the whole moduli space. To see why fixed points under the torus action are labelled by Young diagrams, a good piece of intuition comes form comparing this moduli space to the Hilbert scheme of points. This also gives us an insight into how to regularise the moduli space. A Hilbert scheme is, in general, a moduli space of subvarieties, and its definition is one of a representable object of a functor. Namely, for a projective scheme X over a field $k=\bar{k}$, the Hilbert scheme Hilb_X is characterised by the property of the set of k-scheme morphisms $\mathrm{Hom}(U, \mathrm{Hilb}_X)$ being in bijection with closed k-subschemes $Z \subset U \times X$ such that the induced projection $Z \to U$ is flat, and that this is functorial. For $U = \mathrm{Spec}(k)$, Z are really all the closed k-subschemes. Next define, for $p \in \mathbb{Q}[t]$, Hilb^p_X as the same construction but only with those Z with Hilbert polynomial p[3]. Then the Hilbert scheme of n points in X is

$$(X)^{[n]} = \mathrm{Hilb}_X^{p(t)=n}$$

that is, with constant Hilbert polynomial $p(t) = n \in \mathbb{N}$. In general, there is a Hilbert-Chow morphism $(X)^{[n]}_{\mathrm{red.}} \to \mathrm{Sym}^n X$ which is birational for $\dim_{\mathbb{C}} X \leq 2$. For curves, $(X)^{[n]} = \mathrm{Sym}^n X$, while for nonsingular surfaces, the Hilbert-Chow morphism is a resolution of singularities by a theorem of Fogarty. The singularities themselves occur when points are stacked on top of one another, and in dimension one the only datum to describe this is the multiplicity, while in higher dimensions the directions of the limiting process of collision have to be described as fuzzy points. Here we fix our attention to surfaces. Here we follow Theorem 1.14 in [214]. For an ideal $\mathcal{I}$, let $V_{\mathcal{I}} = \mathbb{C}[z_1, z_2]/\mathcal{I}$. Then, as sets,

$$(\mathbb{C}^2)^{[n]} = \{\text{ideals } I \in \mathbb{C}[z_1, z_2] |\, \dim V_{\mathcal{I}} = n\}$$

Define $B_{1,2} \in \mathrm{End}(V_{\mathcal{I}})$ as multiplication by $z_{1,2}$, and $I \mathrm{Hom}(\mathbb{C}, V_{\mathcal{I}})$ as the unit. Fix $V_{\mathcal{I}} \cong \mathbb{C}^n$. We have $[B_1, B_2] = 0$ and $V_{\mathcal{I}} \cong \mathbb{C}[B_1, B_2]I$. This leads to the equivalent description

$$(\mathbb{C}^2)^{[n]} \cong \left\{ (B_{1,2}, I) \;\middle|\; \begin{array}{c} [B_1, B_2] = 0 \\ \mathbb{C}^n = \mathbb{C}[B_1, B_2]I \cdot \mathbb{C} \\ \text{(stability)} \end{array} \right\} / GL(n)$$

with group action analogous to (1.3), but with $GL(n)$. Finally we get to the crux of the matter. The moduli space of torsion free sheaves of rank r and second Chern class n $\mathcal{M}_{r,n}$ on $\mathbb{P}^2$ can be described as a higher rank generalisation of the Hilbert scheme of points,

$$\mathcal{M}_{r,n} \cong \left\{ \left(\begin{array}{l} B_{1,2} \in \mathrm{End}(\mathbb{C}^n), I \in \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^n), \\ J \in \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^r) \end{array} \right) \;\middle|\; \begin{array}{c} [B_1, B_2] + IJ = 0 \\ \mathbb{C}^n = \mathbb{C}[B_1, B_2]I \cdot \mathbb{C} \\ \text{(stability)} \end{array} \right\} / GL(n)$$

[3]Actually, we assign a Hilbert polynomial p_u to each fiber $\pi^{-1}(u)$ of the projection $Z \to U$, but due to flatness this is $u \in U$ independent as long as U is connected.

with group action analogous to (1.3), but with $GL(n)$. By proposition 2.7 in [214], for $r = 1$, $J = 0$, so $(\mathbb{C}^2)^{[n]} \cong \mathcal{M}_{1,n}$. This can be shown by noting that $\mathbb{C}[B_1, B_2]I = \mathbb{C}^n$, so if $Jp(B_1, B_2)I = 0$ for any monomial p, then $J = 0$. Induction on the degree can be used here, with the base step $JI = \operatorname{tr} IJ = -\operatorname{tr}[B_1, B_2] = 0$.

Note that this definition is very similar to the ADHM data. The two main differences are the replacement of the real moment map $\mu_{\mathbb{R}} = 0$ with the stability condition, and a $GL(n)$ quotient instead of a $U(n)$ one. It turns out that imposing the additional conditions and quotienting by a smaller group is almost equivalent to forgetting the "complex structure" and quotienting by the smaller group (note that no adjoints are present in the torsion-free sheaves description). I say almost because there are subtle differences, see [91]. It turns out that, schematically

$$\begin{aligned} \mathcal{M}_{r,n} &= \{(\text{stability}) \text{ and } \mu_{\mathbb{C}} = 0\}/GL(n) \\ &\cong \{\mu_{\mathbb{R}} = \zeta \cdot 1_N \text{ and } \mu_{\mathbb{C}} = 0\}/U(n) \end{aligned}$$

with $\zeta > 0$ is a minimal resolution of the ADHM space

$$\mathcal{M}_{r,n}^{\text{ADHM}} = \{\mu_{\mathbb{R}} = 0 \text{ and } \mu_{\mathbb{C}} = 0\}/U(n)$$

For $r \geq 2$, the smooth locus of $\mathcal{M}_{r,n}^{\text{ADHM}}$ is the set of simultaneously stable and co-stable (ie such that $K = \mathbb{C}[B_1^\dagger, B_2^\dagger]J^\dagger N$) solutions, which don't exist for $r = 1$ since there are no $U(1)$ instantons. Therefore, we have a resolution of the original, singular ADHM space. *How can we ensure not to pick up additional contributions coming from the exceptional divisors?* Both spaces are symplectic manifolds. When localising on the torus-invariant solutions, however, Nekrasov uses the symplectic 2-form of $\mathcal{M}_{r,n}^{\text{ADHM}}$ lifted on the resolved space $\mathcal{M}_{r,n}$, which vanishes on the exceptional divisor by definition, and not the symplectic form of the resolved space itself [221, p. 18].

Besides stumbling upon the correct regularisation of the instanton moduli space, the Hilbert scheme of points approach lets us describe the fixed points under the torus action. We already saw that $B_{1,2}$ are a lift of multiplication by $z_{1,2}$. Correspondingly, spatial rotations act as $B_{1,2} \to e^{\epsilon_{1,2}} B_{1,2}$. By the stability condition, ideals are $\{f(z_1, z_2) \in \mathbb{C}[z_1, z_2] | f(B_1, B_2) = 0\}$. The torus action lifts to the ideals, so that $f(z_1, z_2) \mapsto f(e^{\epsilon_1} z_1, e^{\epsilon_2} z_2) = e^{a\epsilon_1 + b\epsilon_2} f(z_1, z_2)$ are fixed points. These are monomials. However, all monomial ideals can be described by Young diagrams. Drawing a Young diagram λ on the $\mathbb{Z}^2_{>0}$ plane, the we have an isomorphism

$$\begin{aligned} \mathcal{I} &\mapsto \lambda(\mathcal{I}) = \{(i,j) | z_1^i z_2^j \notin \mathcal{I}\} \\ \lambda &\mapsto \mathcal{I}(\lambda) = \{z_1^i z_2^j | (i,j) \notin \lambda\} \end{aligned}$$

and clearly the requirement $\dim \mathbb{C}[z_1, z_2]/\mathcal{I} = n$ translates to the Young diagram having a total of n boxes.

This intuition turns out to be correct. Again, we have a regularised, compact[4] moduli space, and we want to apply equivariant localisation which only picks up torus-fixed points of the ADHM moduli space. Then we can apply the Duistermaat-Heckman formula. The torus-fixed points are such that they stay on the same $U(n)$ orbit, so by writing $g = e^\phi \in U(n)$, $t_3 = e^a \in U(r)$ we get to first order

$$[\phi, B_{1,2}] - \epsilon_{1,2} B_{1,2} = 0, \phi I - Ia = 0, -J\phi + (a - \epsilon_1 - \epsilon_2)J = 0$$

[4]By adding point-like instantons via the Uhlenbeck-Donaldson compactification, as described earlier.

As we already noted, a has to be diagonal, and this can always be arranged by a $U(n)$ adjoint action. If we decompose $I = \oplus I_i$, $J = \oplus J_i$ with respect to the action of the same torus, the last two equations $\phi I_j = I_j a_j$ and $J_j \phi = J_j(a_j - \epsilon_1 - \epsilon_2)$ say I_j, J_j are left and right eigenvectors of ϕ. Since

$$J_i I_j = J_i \frac{1}{a_j} \phi I_j = \frac{a_i - \epsilon_1 - \epsilon_2}{a_j} J_i I_j$$

$JI = 0$ at the fixed points with general epsilon parameters, and $[B_1, B_2] + JI = [B_1, B_2] = 0$ commute. The monomials fixed under the torus action are the eigenvectors

$$\phi(B_1^{s_1} B_2^{s_2} I_i) = (a_i + s_1\epsilon_1 + s_2\epsilon_2) B_1^{s_1} B_2^{s_2} I_i$$

so we have r Young diagrams $\{\lambda_j\}_{j=1}^r$, one for each I_j, parametrising the fixed point As for applying the Duistermaat-Heckman formula, we need the Euler class of the tangent bundle to these fixed points. To add matter to the mix, we need additional bundles [56]. The derivation has been very thoroughly redone in §1.9 and appendix B of [172]. Originally, it was sketched in [189, 221], done rigorously in Theorem 2.11 of [216] similarly as Proposition 5.7 of [214] and expanded to other groups in [253]. The characterisation consists in taking a fixed instanton solution A_μ and deforming it to another ASD connection $A_\mu + \delta A_\mu$. Then trivial gauge transformations are discarded by looking at the Atiyah-Singer complex

$$\Omega^0(\mathbb{C}^2) \times \mathfrak{g} \to \Omega^1(\mathbb{C}^2) \times \mathfrak{g} \to \Omega^{2,+}(\mathbb{C}^2) \times \mathfrak{g}$$

where the first arrow is an infinitesimal gauge transformation $\epsilon \mapsto \nabla_A \epsilon$ and the other is the linearised ASD condition $\delta A \mapsto \nabla_A^+ \delta A$, and rewriting it in terms of the ADHM data, with the first arrow again an infinitesimal gauge transformation and the second linearised ADHM $\delta\mu_{\mathbb{C}}$. The character at the fixed point labelled by a tuple of Young diagrams $\vec{Y}$ is

$$\mathrm{Ch} T_{\vec{Y}} \mathcal{M}_{r,n} = \sum_{i,j=1}^{r} N_{i,j}(t_1, t_2)$$

where $t_{1,2} = e^{\epsilon_{1,2}}$ and

$$N_{i,j}(x,y) = e^{a_j - a_i} \left(\sum_{c \in Y_i} x^{-l_{Y_j}(c)} y^{a_{Y_i}(c)+1} + \sum_{c \in Y_j} x^{l_{Y_i}(c)+1} y^{-a_{Y_j}(c)} \right)$$

Actually, given the character of a bundle in the form $Ch\mathcal{E} = \sum_i n_i e^{x_i}$, we can convert it to $4, 5$ or 6 dimensional formulas by using the index operator defined as

$$I(\mathcal{E}) = \prod_i [x_i]^{n_i}, \quad [x] = \begin{cases} x & \mathbb{R}^4_{\epsilon_{1,2}} \\ 1 - e^{-x} & \mathbb{R}^4_{\epsilon_{1,2}} \times S^1 \\ \theta(e^{-x}|\tau) & \mathbb{R}^4_{\epsilon_{1,2}} \times \mathbb{T}_\tau \end{cases}$$

for theories with 8 supercharges on the indicated spaces, so we can get a lot of mileage from these calculations.

We note that besides this calculation, which realises instantons as the Higgs branch of branes within branes, there is a more recent construction in terms of the Coulomb branch of $3d$ theories with 8 supercharges [66], resulting in a different kind of quiver.

1.3.1.1 qq-character

A related question is what happened to the Seiberg-Witten curve when the Omega-deformation is turned on. It was noted in the context of the BPS/CFT correspondence [222, 223] that Schwinger-Dyson identities imply that

$$y(x) = \langle Y(x) \rangle,$$

the Y-operator is the generating function of the chiral ring operators, satisfies the equation

$$y(x) + \frac{1}{y(q^{-1}x)} = P_N(x|\epsilon_1, \epsilon_2), \quad q = e^{\epsilon_1+\epsilon_2}$$

here concretely for the A_1 quiver theory. The entity on the left hand side is called the qq-character, since it is reminiscent of the $SU(2)$ fundamental character $\chi = y+1/y$. The right hand side is a degree N polynomial in x. In the NS limit, this means that the Seiberg-Witten curve is a difference equation, equivalent to a deformed TQ-relation for an XXX spin chain in $d = 4$ [226].

1.3.2 Nekrasov functions

Here we review the end results, the Nekrasov partition function of interest for this work [56, 88]. Given two partitions $Y_1 = (k_1 \geq k_2 \geq ... \geq k_l > 0)$, $Y_2 = (\tilde{k}_1 \geq \tilde{k}_2 \geq ... \geq \tilde{k}_{\tilde{l}} > 0)$ and a cell $c = (i, j) \in Y_1$ we define the auxiliary functions

$$\begin{aligned}\phi(a, c) &= a + \epsilon_1(i-1) + \epsilon_2(j-1) \\ \xi(a, b, Y_1, Y_2, c) &= a - b + \epsilon_1(\mathrm{leg}(c, Y_1) + 1) - \epsilon_2(\mathrm{arm}(c, Y_2))\end{aligned}$$

the last of which, a deformed hook length, uses $\mathrm{arm}(c, Y) = k_i - j$, $\mathrm{leg}(c, Y^t) = k_j - i$, and finally set

$$E(a, b, Y_1, Y_2) = \prod_{c\in Y_1} \xi(a, b, Y_1, Y_2, c) \left(\epsilon_1 + \epsilon_2 - \xi(a, b, Y_1, Y_2, c)\right)$$

1.3.2.1 The classical gauge groups $SU(n)$, $SO(2n + \chi)$, $Sp(n)$

In the following we consider n partitions $(Y_1, ..., Y_n) = \vec{Y}$ with the total number of boxes k. Then the equivariant volume of the k instanton moduli space for $U(n)$ is given by

$$\begin{aligned}Z^{SU(n)}(\vec{Y}) &= \prod_{i,j=1}^{n} \left(E(\sigma_i, \sigma_j, Y_i, Y_j)\right)^{-1} \\ Z_k^{SU(n)} &= \sum_{|\vec{Y}|=k} Z^{SU(n)}(\vec{Y})\end{aligned} \tag{1.4}$$

and $SU(n)$ is obtained by restricting to the $\sum_k \sigma_k = 0$ slice. For the orthogonal groups, if $\chi \in \{0, 1\}$,

$$Z_k^{SO(2n+\chi)} = \sum_{|\vec{Y}|=k} \prod_{i=1}^{n} \frac{\prod_{c\in Y_i} 4(\phi(\sigma_i, c))^{2\chi}(4\phi(\sigma_i, c)^2 - 1)^2}{\prod_{j=1}^{n} E(\sigma_i - \sigma_j, Y_i, Y_j)^2 E(-\sigma_i - \sigma_j, Y_i^t, Y_j) E(\sigma_i + \sigma_j, Y_i, Y_j^t)} \tag{1.5}$$

The combinatorial expressions for $Sp(n)$ are more involved. Namely, one has to multiply (1.5) by extra factors depending just on the Ω-background parameters[5]. A combinatorial solution is proposed in [170], the issue can also be approached using Jeffrey-Kirwan residues as in [218]. For the self-dual background we can simplify the latter procedure via a $\epsilon_{1,2} = \pm 1 \mp i0$ prescription which renders the pole structure easier to handle. Namely, in that case we need to define two-indexed functions $Z^{Sp(n)}_{2k,l}$ such that

$$Z^{Sp(n)}_{2k,0} = \sum_{|\vec{Y}|=k} \prod_{i=1}^{n} \left(\prod_{c\in Y_i} 4(\phi(\sigma_i,c))^2 (4\phi(\sigma_i,c)^2-1)^2 \right)^{-1}$$
$$\times \left(\prod_{j=1}^{n} E(\sigma_i-\sigma_j,Y_i,Y_j)^2 E(-\sigma_i-\sigma_j,Y_i^t,Y_j) E(\sigma_i+\sigma_j,Y_i,Y_j^t) \right)^{-1}$$

and then the fractional instanton contributions are given as

$$\begin{aligned}
Z^{Sp(n)}_{2k,1} &= \frac{1}{2} Z^{Sp(n)}_{2k,0} \prod_{i=1}^{n} \frac{1}{-\sigma_i^2} \sum_{|\vec{Y}|=k} \prod_{c\in Y_i} \frac{\phi(\sigma_i,c)^4}{(\phi(\sigma_i,c)^2-1)^2} \\
Z^{Sp(n)}_{2k,2} &= \frac{1}{8} Z^{Sp(n)}_{2k,0} \prod_{i=1}^{n} \frac{1}{(\sigma_i^2-1/4)^2} \sum_{|\vec{Y}|=k} \prod_{c\in Y_i} \frac{(\phi(\sigma_i,c)^2-1/4)^2}{(\phi(\sigma_i,c)^2-9/4)^2} \\
Z^{Sp(n)}_{2k,3} &= \frac{1}{144} Z^{Sp(n)}_{2k,0} \prod_{i=1}^{n} \frac{1}{(-\sigma_i^2)(\sigma_i^2-1/4)^2} \sum_{|\vec{Y}|=k} \prod_{c\in Y_i} \frac{\phi(\sigma_i,c)^4(\phi(\sigma_i,c)^2-1/4)^2}{(\phi(\sigma_i,c)^2-1)^2(\phi(\sigma_i,c)^2-9/4)^2} \\
&\quad + \frac{1}{72} Z^{Sp(n)}_{2k,0} \prod_{i=1}^{n} \frac{1}{(-\sigma_i^2)(\sigma_i^2-1)^2} \sum_{|\vec{Y}|=k} \prod_{c\in Y_i} \frac{(\phi(\sigma_i,c)^2-1)^2}{(\phi(\sigma_i,c)^2-4)^2}
\end{aligned}$$

where the summands in the last expressions are due to $V = T^{1/2} + 1 + T^{-1/2}$, $V_2 = T + 1 + T^{-1}$ and $V = T^1 + 1 + T^{-1}$, $V_2 = T^2 + 1 + T^{-2}$ contributions to be put in the character (4.16) of [93], which can be continued further easily. This finally enables one to compute

$$Z^{Sp(n)}_{k} = \sum_{2m+l=k} Z^{Sp(n)}_{2m,l}$$

and it agrees with appendix B of [193]. Further, we can add fundamental matter by adding a factor of

$$\prod_{i=1}^{N_f} \prod_{j=1}^{n} \prod_{c\in Y_j} (\phi(\sigma_j,c)^2 - m_i^2)$$

in the numerators.

[5]In the brane realisation of instanton counting these are usually dubbed *fractional instantons*, stuck at the orientifold plane [93].

1.3.2.2 $SU(2)$ with fundamental matter

Given the partitions $Y_{1,2}, W_{1,2}$ we can define

$$
\begin{aligned}
&Z_{bifund.}(a_1,a_2,Y_1,Y_2,b_1,b_2,W_1,W_2,m) = \\
&\prod_{i,j=1}^{2}\prod_{c\in Y_i}(\xi(a_i-b_j,Y_i,W_j,c)-m)\prod_{c\in W_j}(\epsilon_1+\epsilon_2-\xi(b_j-a_i,W_j,Y_i,c)-m)
\end{aligned}
$$

Further we define,

$$
\begin{aligned}
Z_{adj.}(a_1,a_2,Y_1,Y_2) &= Z_{bifund.}(a_1,a_2,Y_1,Y_2,a_1,a_2,Y_1,Y_2,0)^{-1} \\
Z_{fund.}(a_1,a_2,Y_1,Y_2,m) &= \prod_{i,j=1}^{2}\prod_{c\in Y_i}(\phi(a_i,c)+m)
\end{aligned}
$$

Then for $SU(2)$ with N_f fundamental flavors we have

$$
Z_k(\sigma) = \sum_{|Y_1|+|Y_2|=k}\frac{\prod_{i=1}^{N_f} Z_{fund.}(\sigma,-\sigma,Y_1,Y_2,m_i)}{Z_{adj.}(\sigma,-\sigma,Y_1,Y_2)}
$$

To obtain $U(2)$, replace $(\sigma,-\sigma)$ with (σ_1,σ_2) in the above.

1.3.3 Universal one instanton formula

It was found in [24] that an instanton of topological charge 1 may be constructed by means of an $\mathfrak{sl}_2$ triple corresponding to a long root. This was used to calculate the 1-instanton corrections to the Seiberg-Witten curve [151]. Besides this embedding in the internal degrees of freedom, the instanton has a $\mathbb{C}^2$ of moduli specifying its position, therefore the holomorphic functions on this product space is a $U(1)_{\epsilon_1}\times U(1)_{\epsilon_2}\times W$-module. It is precisely its character that the 5 dimensional uplift of the theory will be calculating, and the 4 dimensional formula may be seen as its "Weyl dimension" analogue, and in

$$
\begin{aligned}
\Lambda^{2h^\vee} Z_1 &= -\frac{1}{\epsilon_1\epsilon_2}\sum_{\boldsymbol{\beta}\text{ long}}\frac{1}{(\epsilon_1+\epsilon_2+\boldsymbol{\beta}\cdot\boldsymbol{a})(\boldsymbol{\beta}\cdot\boldsymbol{a})\prod_{\boldsymbol{\alpha}\cdot\boldsymbol{\beta}^\vee=1}(\boldsymbol{\alpha}\cdot\boldsymbol{a})} \\
&\mapsto \sum_{\boldsymbol{\beta}\text{ long}}\frac{1}{(\boldsymbol{\beta}\cdot\boldsymbol{s})^2\prod_{\boldsymbol{\alpha}\cdot\boldsymbol{\beta}^\vee=1}(\boldsymbol{\alpha}\cdot\boldsymbol{s})}
\end{aligned}
\tag{1.6}
$$

rewritten in ϵ-units [170]. Comparisons with ADHM calculations [93, 193, 227] tend to reveal some sign differences, e.g. $Z_1^{B_n,C_n}|_{(1.6)} = -Z_1^{B_n,C_n}|_{\mathrm{ADHM}}$, which is why the rescalable instanton counting factor of Λ ought to be kept in mind.

1.3.4 Blowup equations

The instanton moduli spaces for exceptional groups lack an ADHM description since their fundamental representation is different from their defining one. Thus to compute higher instanton terms one cannot resort to the usual localisation techniques. An alternative approach, besides the one discussed in this work, is by blowup equations, although these do not give compact expressions such as (1.6). Roughly speaking, blowup relations relate the partition function on the blown-up geometry to the blowdown. The origin $z_1 = z_2 = 0$ of $\mathbb{C}^2$ is replaced by the exceptional divisor, which exchanges the origin as a fixed point to the north and south poles of the divisor, each an ordinary Nekrasov partition function, with their vevs determined by a $\mathbb{Z}$-worth of gluing conditions at the equator [48, §2]. Further, flux can be turned on through the exceptional divisor itself and its d-fold product, giving an observable $\hat{Z}_d$. This observable can be related to the ordinary Nekrasov function, or vanish. A terrific review of how far-reaching the consequences of these relations are, see my older academic brother's thesis [258]. Generalizing the $d = 4$ expression in [216] to general gauge groups as was done for $d = 5$ in [170], except noting that in $d = 4$ the partition function with flux on the exceptional divisor vanishes – so $\hat{Z}_{d=0} = Z$ but $\hat{Z}_{d\geq 1} = 0$ – we obtain

$$\begin{aligned} Z_n(\epsilon_1, \epsilon_2, \boldsymbol{s}) = \frac{1}{n^2\epsilon_1\epsilon_2} \sum_{\substack{\frac{1}{2}\mathbf{m}^2 + i_1 + i_2 = n \\ \mathbf{m}\in Q^\vee,\, i_{1,2}<n}} & \frac{\left(\epsilon_1 i_1 + (\epsilon_1+\epsilon_2) i_2 + \mathbf{m}\cdot\boldsymbol{s} + \frac{1}{2}\mathbf{m}^2(2\epsilon_1+\epsilon_2)\right)}{L(\epsilon_1, \epsilon_1+\epsilon_2, \boldsymbol{s}, \mathbf{m})} \\ & \left(\epsilon_1 i_1 + (\epsilon_1+\epsilon_2)(i_2 - n) + \mathbf{m}\cdot\boldsymbol{s} + \frac{1}{2}\mathbf{m}^2(2\epsilon_1+\epsilon_2)\right) \\ & Z_{i_1}(\epsilon_1, \epsilon_2, \boldsymbol{s}+\epsilon_1\mathbf{m}) Z_{i_2}(-\epsilon_2, \epsilon_1+\epsilon_2, \boldsymbol{s}+(\epsilon_1+\epsilon_2)\mathbf{m}) \end{aligned} \tag{1.7}$$

starting from $Z_0(\epsilon_1, \epsilon_2, \boldsymbol{s}) = 1$, where $L(\epsilon_1, \epsilon_2, \boldsymbol{s}, \mathbf{m}) := \prod_{\boldsymbol{\alpha}\in R} \ell(\epsilon_1, \epsilon_2, \boldsymbol{s}, \mathbf{m}, \boldsymbol{\alpha})$ and

$$\ell(\epsilon_1, \epsilon_2, \boldsymbol{s}, \mathbf{m}, \boldsymbol{\alpha}) = \begin{cases} \prod\limits_{\substack{i,j\geq 0 \\ i+j\leq -\mathbf{m}\cdot\boldsymbol{\alpha}-1}} (-i\epsilon_1 - j\epsilon_2 + \mathbf{m}\cdot\boldsymbol{s}), & \text{if } \mathbf{m}\cdot\boldsymbol{\alpha} < 0, \\ \prod\limits_{\substack{i,j\geq 0 \\ i+j\leq \mathbf{m}\cdot\boldsymbol{\alpha}-2}} ((i+1)\epsilon_1 + (j+1)\epsilon_2 + \mathbf{m}\cdot\boldsymbol{s}), & \text{if } \mathbf{m}\cdot\boldsymbol{\alpha} > 1, \\ 1 & \text{otherwise.} \end{cases} \tag{1.8}$$

Since we are interested in the self-dual background, we see that naively taking it leads to some singular terms in the summands due to the NS limit getting involved, so care must be taken to first preform the summation and then to take the limit. In particular, one can take $\epsilon_1 = 1 + \delta$, $\epsilon_2 = -1$ and then safely send $\delta \to 0$ in the final expression.

Note that blowup equations naturally provide a link between the self-dual and the N.S. Omega-background. This is important from the point of view of exact quantisation [115], surface operators [224] as well as $c = \infty$ irregular conformal blocks [106]. They also provide a link between the self-dual and the $c = -2$ backgrounds, which can be used to derive Painlevé III_3 in tau form [32] as well as the $\mathcal{N} = 2^*$ tau system [25]. The reader may wish to investigate these links further.

1.4 The Seiberg-Witten/Integrable system correspondence

Recall that the Seiberg-Witten solution gives the Coulomb branch of $\mathcal{N} = 2$ $d = 4$ theory the structure of an elliptic fibration, $\mathrm{Jac}(\Sigma_{\mathrm{SW}}) \to \mathcal{B}$. This is explicitly an integrable system. After the first SW curves for $G = SU(N)$ were found [7, 179], in [111] it was noted which integrable system it corresponded – the N-periodic closed Toda chain. In fact, the curve (1.1) can be written as the spectral curve

$$\det(L(z) - w) = 0$$

where $L(z)$ is the Lax operator

$$L(z) = \begin{pmatrix} p_1 & e^{\frac{q_2-q_1}{2}} & 0 & \dots & ze^{\frac{q_1-q_N}{2}} \\ e^{\frac{q_2-q_1}{2}} & p_2 & e^{\frac{q_3-q_2}{2}} & 0 & \\ 0 & e^{\frac{q_3-q_2}{2}} & p_3 & \ddots & \\ \vdots & 0 & \ddots & \ddots & \\ \frac{1}{z}e^{\frac{q_1-q_N}{2}} & & & & p_N \end{pmatrix}$$

This was generalised for any semisimple G to a Toda chain based on G^L [71, 72, 73]. If adjoint matter is present, then the system is elliptic Calogero-Moser [75, 197], and if fundamental matter is present, it is an inhomogenous periodic XXX spin chain [112]. Dimensions can also be increased – see [114] for an overview.

In [75] a link with Hitchin systems was shown. This is elucidated by the class $\mathcal{S}$ construction, where these theories are obtained by compactifying a topologically twisted $d = 6$ $\mathcal{N} = (2, 0)$ theory on $M_4 \times C$. The original theory contains adjoint-valued scalars, and only one survives the twist, and it can be viewed as a holomorphic adjoint-valued one-form $\Phi_z \mathrm{d}z$ on C [99]. The Coulomb branch is parametrised by its higher traces, and a curve $\det_{\text{Adj.}}(\lambda - \Phi_z) = 0$ can be obtained, which corresponds to the SW curve.

1.4.1 Integrable systems and the spectral transform

The Donagi-Witten integrable system is a general feature of Coulomb branches, and most of my work has to do with isomonodromic deformations, which are generalisation of integrable systems. Here we have in mind classical integrable systems, as the study of quantum ones tends to be quite orthogonal to the classical counterparts. A good introduction is [81]. There are several perspectives on integrable systems.

1.4.1.1 Liouville integrability

The most standard one is Liouville integrability via the Hamiltonian perspective. Namely, we consider a $2n$-dimensional phase space M with a non-degenerate closed 2-form $\omega = h^{ij}\mathrm{d}x_i \wedge \mathrm{d}x_j$ in some chart. Introduce a Hamiltonian function $H : M \to \mathbb{C}$ in the usual way via a vector flow ∂_H, $\mathrm{d}H = i_{\partial H}\omega$ along with a Poisson bracket on the sheaf of continuous functions $C(M)$ on M, which is a nondegenerate antisymmetric $\mathbb{C}$-derivation $\{.,.\} : C(M) \times C(M) \to C(M)$ which satisfies the Jacobi identity. In this case it looks like $\{f, g\} = h_{ij}\partial_{x_i} f \partial_{x_j} g$, so that the time evolution with respect to the Hamiltonian H is $\dot{f} = \{H, f\}$. Liouville integrability then means we can find n independent functions F_i in which Poisson-commute with H and are all in involution, $\{F_i, F_j\} = 0$. Locally, we can always introduce Darboux coordinates q_i, p_i such that $\omega = \sum_i \mathrm{d}p_i \wedge \mathrm{d}q_i$ and $\{p_i, q_j\} = \delta_{i,j}$, $\{p_i, p_j\} = \{q_i, q_j\} = 0$, but Liouville integrability is a global statement. As such, the symplectic leaves

F_i = constant folate the phase space. These leaves have the topology of an n-dimensional torus, and we can choose the so-called action-angle variables which correspond to flows around the cycles of the torus: the action variables are just the periods of the fundamental (Liouville) form θ, $\mathrm{d}\theta = \omega$, and the angle variables are their conjugates. The motion is then free.

The construction we described is very much modelled on the cotangent bundle T^*M. Before going on, this would be a suitable place to introduce another example of a symplectic space, coadjoint orbits. Let G be a connected finite dimensional Lie group with Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$. If $\mathfrak{g}^*$ is its dual[6] algebra with pairing $(.,.) : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{C}$, then the adjoint action $\mathrm{Ad} : G \times \mathfrak{g} \to \mathfrak{g}$, $\mathrm{Ad}_g x = gxg^{-1}$, and the coadjoint action is defined via the paring $\rangle \mathrm{Ad}^*_g x, y\rangle = \langle x, \mathrm{Ad}_g y\rangle$. A coadjoint orbit invariant under the Ad^* action $\mathcal{O}_p = \{\mathrm{Ad}^*_g p | g \in G, p \in \mathfrak{g}^*\}$ can be associated to any $p \in \mathfrak{g}^*$. What is interesting about these spaces is that they're all canonically symplectic. If the corresponding infinitesimal version of the coadjoint action is ad_*, so that $\langle \mathrm{ad}_*(Z)x, y\rangle = -\langle x, \mathrm{ad}(Z)y\rangle = -\langle x, [Z, Y]\rangle$, giving a representation of $\mathfrak{g}$ in $\mathfrak{g}^*$, then the natural symplectic form is defined by $\omega_p(\mathrm{ad}_*(x), \mathrm{ad}_*(x)) = \langle p, [x, y]\rangle$. By proposition 1 in [174, §1], this form, called the Kostant-Kirillov symplectic form ω on $\mathcal{O}_p$ is the derivative of $< p, \Theta >$, where $\Theta = g^{-1}\mathrm{d}g$ is the Maurer-Cartan form. With some abuse of notation, we can write $\omega^{KK} = \langle pg^{-1}\mathrm{d}g \wedge g^{-1}\mathrm{d}g\rangle$. On $\mathfrak{g}^*$ itself, the Poisson bracket is $\{X_i, X_j\} = c^k_{ij} X_k$ with c^k_{ij} the structure constants of $\mathfrak{g}$. Far from being abstract, this construction actually gives a purely classical counterpart of a spin-$n/2$ particle. A two-dimensional sphere S^2 with quantised volume $n+1$ and the standard symplectic form can be seen as a coadjoint orbit of $\mathfrak{su}_2$, the geometric quantization of whose n-dimensional unitary irreducible representation gives exactly the spin-$n/2$ particle.

The great value of this construction to integrable systems comes from reformulating the bracket to a factorisation problem $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ into subalgebras along with centrally extending G to the loop group $L^{0+}G = G[[\lambda]]$, $\lambda \in \mathbb{C}^\times$. Consider G as a matrix group for concreteness. The dual loop group is then $L^-G = \lambda^{-1}G[[\lambda^{-1}]]$ with pairing

$$\langle \sum_{n<0} x_n \lambda^{-n}, \sum_{m\geq 0} y_m \lambda^m \rangle = \mathrm{tr}\, \mathrm{Res}_{\lambda=0} \sum_{n<0, m\geq 0} x_n y_m \lambda^{m-n} = \sum_n x_n y_{-1-n}$$

Then it's clear that for this explicit bracket and any $g \in L^{0+}G$, $\langle x, \mathrm{Ad}_g y\rangle = \langle gxg^{-1}, y\rangle = \langle (gxg^{-1})_-, y\rangle$ where $(.)_-$ projects to negative powers of λ, so that

$$\mathrm{Ad}^*_g x = (gxg^{-1})_-$$

Soon we will be looking at the similar case where $L(\lambda) \in \mathcal{O}_A$, $A \in L^-G$, and the infinitesimal flow on the coadjoint orbit is

$$\dot{L}(\lambda) = \mathrm{ad}^*(M)L = [M(\lambda), L(\lambda)] \tag{1.9}$$

which we will call the Lax equation. Besides the actual dynamics, we note that the $L^{0+}G$ and L^-G as we have described them actually fit together into what is usually called the loop group $LG = L^{0+}G \oplus L^-G \cong \mathrm{Hom}(S^1, G)$. As such, this splitting is related to the problem of factorisation of loops, which can further be reformulated

[6]Not in the Langlands' sense.

as a Riemann-Hilbert problem [242]. A classical result is Birkhoff factorisation: any loop $\gamma \in LGL_n$ can be factorised as

$$\gamma = \gamma_- \cdot z^{\boldsymbol{a}} \cdot \gamma_+$$

where $\gamma_\pm \in L^\pm GL_n(\mathbb{C})$ and $z^{\boldsymbol{a}}$ is in the maximal torus i.e. of the form $z^{\boldsymbol{a}} = \mathrm{diag}(z^{a_1}, ..., z^{a_n})$. Loops with $z^{\boldsymbol{a}} = \mathrm{id}$ form a dense open subset of the identity component of LGL_n, and the multiplication $L_1^- \times L^+ \to LGL_n|_{\text{that subset}}$ is a diffeomorphism, where $L_1^- = L^-|_{\gamma(\infty)=1}$.

A beautiful consequence of this factorisation of use in section 1.5.3 is the classical Grothendieck's theorem on holomorphic vector bundles on $\mathbb{P}^1$. Namely, to construct a rank n holomorphic vector bundle, cover $S^2 \cong \mathbb{P}^1$ with two opens $S^2 = U_+ \cup U_-$, $U_\pm = \{z \in \mathbb{C} : |z| \gtrless 1\}$. On $U_\pm$ any bundle is trivial and is locally $U_\pm \times \mathbb{C}^n$. Loops come in when we look at the holomorphic transition functions $\gamma : U_+ \cap U_- \cong S^1 \to GL_n$. Therefore, rewrite $\gamma = \gamma_- \cdot z^{\boldsymbol{a}} \cdot \gamma_+$. If we change coordinates on $U_+ \times \mathbb{C}^n$ by γ_+ and by γ_-^{-1} on $U_- \times \mathbb{C}^n$, the transition function is just $z^{\boldsymbol{a}}$, and so any holomorphic degree n vector bundle on $\mathbb{P}^1$ splits as $\mathcal{O}(a_1) \oplus ... \oplus \mathcal{O}(a_n)$.

1.4.1.2 Lax pairs and the spectral transform

What many people have in mind when they think of classical integrable systems, however, are solitons, the localised, dispersionless travelling wave solutions of non-linear wave equations. This is an infinitely dimensional phase space, and for the shape of a wave to be preserved, intuitively we see the need for infinitely many conserved quantities, consisting of derivatives of all orders which we need to specify its shape[7]. That they be in involution is, however, not an obvious requirement. Further, as the consequence of them being in involution, any one of them defines its own independent flow, so we end up with an infinite hierarchy of flows.

An alternative approach to integrability has the ability to subsume these kinds of systems without going into the subtleties of infinite-dimensional symplectic manifolds is based on the Lax equation (1.9). We can solve it immediately:

$$L(t) = g(t)L(0)g(t)^{-1}, \quad M(t) = g(t)^{-1}\frac{\mathrm{d}g(t)}{\mathrm{d}t}$$

Therefore, integrals of motion are given by traces of powers of L. By itself, this can be written as a compatibility condition of the following equations, if the functions Ψ exist such that

$$L(\lambda, \mu)\Psi(\lambda, \mu) = \mu\Psi(\lambda, \mu), \quad \partial_t\Psi(\lambda, \mu) = M(\lambda, \mu)\Psi(\lambda, \mu)$$

with μ t-independent. This reformulation leads to what I will call Krichever's definition of integrable systems, which is *compatibility conditions of overdetermined systems of linear equations.*

The spectral parameter λ is very important. In the setting of (1.9), it came from the loop group. In general, matching powers of λ, we find a host of nondynamical, constraint equations on $M(\lambda)$, coming from those terms of the commutator

[7] Historically, for the KdV soliton this was very much not obvious, especially as only three were found directly related to some symmetry of the equation, and some speculated there can be no more than seven conserved quantities [67, §3].

unmatched with $\dot{L}(\lambda)$. This reflects a general principle of integrability being given in terms of flat connections with constraints.

The Lax pair can in principle be given by any linear operators. Consider for a moment the finite dimensional case of an $r \times r$ matrix $L(\lambda)$. We define its spectral transform as the algebraic curve

$$\{\det(\mu \mathbb{1}_r - L(\lambda)) = 0\} \subset \mathbb{C}^2$$

It is an r-fold covering of the μ plane, whereas the λ-plane this depends on $L(\lambda)$. It is also possible to invert the spectral transform. Given an r-sheeted genus g hyperelliptic curve Γ with an effective divisor D of degree $g + r - 1$, consider the linear system $\mathcal{L}(-D)$. By Riemann-Roch, its dimension is r if generic, so take a basis $\psi_i \in \mathcal{L}(-D)$ and normalise it at some point $\psi_i(P) = 1$. Consider points $P_i = (\lambda, \mu_i)$ with the same base point λ, and form the matrix $\hat{\psi} = \psi_i(P_j)$. Then it can be shown that $L(\lambda) = \hat{\psi}(\lambda)\mathrm{diag}(\mu_1, ..., \mu_r)\hat{\psi}(\lambda)^{-1}$ is well-defined on the base curve [14, §5.2].

The question of how to describe the KdV hierarchy in Lax form and assign it a spectral curve is more subtle. First of all, the KdV hierarchy can be seen as a reduction of the more general KP hierarchy, so I will describe this more general case. KP is quite special as it, in fact, governs all possible isospectral deformations of a linear operator, by Mulase's theorem.

The KP hierarchy therefore has deep connections to algebraic curves - many of the first known soliton solutions were given in terms of elliptic functions on a seemingly unrelated, emergent curve, and a more general theory was developed by Krichever [181], the systematisation of which can be found in the textbook [21]. Mulase explains this in his excellent review [213]. Namely, every ordinary differential operator defines an algebraic curve from its set of eigenvalues with resolved multiplicities, with the eigenspace as a vector bundle on this curve. Isospectral deformations by definition only deform this vector bundle and keep the base space fixed, and Mulase shows that the moduli of these deformations are isomorphic to the Jacobian of the spectral curve.

Namely, if $P = \partial_x^n + ...$ is a linear differential operator of degree n, and we want to evolve it using a certain set of times $\boldsymbol{t} = (t_1, ..., t_m)$, so that

$$P(\boldsymbol{t})\phi(x, \boldsymbol{t}) = \lambda\phi(x, \boldsymbol{t})$$

then if $L(\boldsymbol{t}) = \partial_x + u_2(x, \boldsymbol{t})\partial_x^{-1} + ...$ is a pseudo-differential operator such that $L(0)^n = P(0)$ and $L(\boldsymbol{t})$ solves the KP hierarchy,

$$\partial_{t_i} L(\boldsymbol{t}) = [Q_i(\boldsymbol{t}), L(\boldsymbol{t})], \quad Q_i(\boldsymbol{t}) = (L^i(\boldsymbol{t}))_+,$$

where $(.)_+$ projects to positive powers of t_i's only, then $P(\boldsymbol{t}) = L(\boldsymbol{t})^n$ solves the isospectral problem. We can also get the explicit KP wave equation

$$3/4u_{yy} - (u_t - 1/4u_{xxx} - 3uu_x)_x = 0$$

from the $i = 2$ and 3 terms of this hierarchy by identifying $u_2 = u$, $t_2 = y$ and $t_3 = t$ and eliminating u_3 and u_4. So, in this case, we have a curve $\det(\lambda \cdot \mathrm{id} - P) = 0$. KdV is, in this treatment, the even-time independent system with $P = \partial_x^2 - u$.

This is the general situation: the Lax operator defines a spectral curve with a vector bundle, and the "original" motion is then interpreted as free motion on the

Jacobian of the spectral curve. The Jacobian is, of course, a torus, so this provides Liouville integrability by construction.

There are more curves we can assign to KP. The differential operators Q_i satisfy the zero-curvature condition coming from the commutativity of the flows, $[\partial_{t_i}, \partial_{t_j}]$, namely

$$\partial_{t_i} Q_j - \partial_{t_j} Q_i = [Q_i, Q_j]$$

and therefore, if our solution is independent of some fixed time t_j, we get the Lax equation $\partial_{t_i} Q_j = [Q_i, Q_j]$, which means the spectral curve $\{\det(\kappa \cdot \mathrm{id} - Q_j) = 0\}$ is independent of all the times.

Instead of the Hamiltonian approach, at this point we prefer the Lax pair approach. With the introduction of the Lax function, namely, our theory of classical integrable systems gains a lot of expressiveness, as we can not only express in one fell swoop systems like

$$L(z) = \begin{cases} \sum_i \frac{u_i}{z-z_i} & \text{in } 0+1, \\ -\partial_x^2 + u(x,t) & \text{in } 1+1, \\ \partial_y - \partial_x^2 + u(x,y,t) & \text{in } 2+1 \text{ dimensions} \end{cases}$$

the latter two examples being the parts of the KdV and the KP hierarchies that correspond directly to the eponymous wave equations, $L(z)|_{\text{here}} = (L^2(\boldsymbol{t}))_+|_{\text{before}} = Q_2$, but also immediately tie it to algebraic curves via the spectral transform, $L(z) \rightsquigarrow \{\det(L(z) - \kappa \cdot \mathrm{id}) = 0\}$.

1.5 String theory realisations

Dirichlet boundary conditions can be introduced to the fundamental superstring's worldsheet theory, forcing its ends to be fixed in space. In the target space, $10-(p+1)$ of these conditions restrict the endpoints to lie on a $(p+1)$-dimensional submanifold of $\mathbb{R}^{1,9}$. This defines an extended object, the Dp-brane, whose fixed existence breaks a certain amount of Poincare invariance, and by extension, supersymmetry. By quantising the oscillator modes of these kinds of strings, its massless spectrum can be seen to be given by a vectormulitplet. In the maximal $D9$-brane case, which breaks the supersymmetry in half to $32/2 = 16$ supercharges, we have the maximally supersymmetric $d = 10$ $\mathcal{N} = 1$ vectormultiplet, which consists of a vector field A_μ and a Majorana-Weyl spinor of positive chirality ψ [165, §2]. Branes with $p < 9$ can then be reached by dimensional reduction. Ignoring the fermions, the main point is that we end up with a vector field A_μ in the bulk, $\mu = 0, ..., p$, while the transverse directions are reduced to $10 - (p + 1)$ scalars which describe the position of the brane. A similar mechanism occurs for charged extended objects in general, outside string theory [256, §5.8,5.9].

The worldvolume theory of Dp-branes is given by an effective action consisting of a Dirac-Born-Infeld plus a Wess-Zumino term. To lowest order in the string length ℓ_s^2, this reduces to the standard[8] super Yang-Mills action describing the vectormultiplet dynamics. The gauge theory coupling is related to the string length and string coupling as $g_{\text{YM}}^{-2} = \ell_s^{3-p} g_s^{-1}$.

[8] This is for a single brane and describes abelian theory – the non-abelian DBI action is not known.

All this is to say that arrangements of weakly-gravitating Dp-branes can be used to realise the appropriate gauge theory, also in any needed phase. This enables us not only to geometrise all the gauge theory datum, but also gives us access to string theory dualities, which act functorially on the gauge theory.

A word of caution is needed about higher dimensional gauge theories. By a naive dimension counting argument, since the gauge field A_μ has mass dimension $[M^{(d-1)/2}]$ in d dimensions, the gauge coupling has $[g^{-2}] = [M^{d-4}]$. This means that for $d > 4$, the theory is nonrenormalisable, and should be seen only as an effective theory. The theory can, however, violate the initial assumption of having a gauge-theory interpretation at all scales – for instance, if in the UV regime the theory is strongly coupled. Therefore, it is possible that there is an interacting UV fixed point. Scale invariance on a quantum level by usual lore means conformal invariance [219]. In six dimensions, there are three such superconformal theories, $\mathcal{N} = (2,0)$, $\mathcal{N} = (1,1)$ and $\mathcal{N} = (1,0)$. The first two have 16 supercharges, and turn out to be related to a different kind of brane, the NS5-brane, whose worldvolume they describe. They are respectively chiral and non-chiral, and occur in type IIA and IIB string theory respectively.

This section is mostly based on the excellent review [107].

1.5.1 IIA: Hanany-Witten

The type of string theory studied determines which Dp-branes are available. Heterotic string theory, for example, has none, and type I string theory can be seen as type II string theory with orientifolds. To engineer $d = 4$ gauge theory, we need a d-dimensional worldvolume, and to have 8 supercharges, we need 2 perpendicular stacks of parallel branes, as each stack halves the amount of supersymmetry. As a general rule, the effective field theory content is governed by the lightest objects in the setup – the lowest-dimensional branes.

The fundamental string is by definition allowed to end on any Dp-brane. However, by using T and S dualities, this can be seen to be dual to branes ending on other branes.

In the original Hanany-Witten setup, $d = 3$ gauge theory was constructed, but the setup can be easily generalised to $d = 4$. The theory is type IIA and the lightest objects in this setup are $D4$-branes extending in the $0,1,2,3,6$ directions. By themselves, they engineer $\mathcal{N} = 2$ $d = 5$ gauge theory. However, they are made to end on a pair of parallel NS5 branes, extending in the $0,1,2,3,4,5$ directions and fixed on x^6_1 and $x^6_2 = x^6_1 + L$. The extension of the $D4$-branes in the 6-th direction is therefore just the interval $[x^6_1, x^6_2]$ of length L.

If $L \to 0$ is small, since the fluctuations of the $D4$-brane along the 6-th dimension have momentum proportional to $1/L$, they decouple – this is the Kaluza-Klein mechanism. The $d = 4$ coupling is $g^{-2}_{4dYM} = L g_{5dYM} = L(\ell_s g_s)^{-1}$. On its own, a $D4$-brane's bosonic massless sector, extending in the $0,1,2,3,6$ directions, is described by a gauge field A_μ, $\mu = 0,1,2,3,6$ and five scalars ϕ^i. Dimensional reduction on the 6-th direction reduces this to a $d = 4$ gauge field A_μ, $\mu = 0,1,2,3$ and six real scalars (ϕ^i, A_6). These six scalars describe the positions of the $D4$ branes in the $4,5,6,7,8,9$ directions. However, in this case the $D4$-brane is forced to end on the NS5-branes, which fixes the $6,7,8,9$ directions, and projects out the associated scalars. Only motion along the $4,5$ directions is unconstrained. This provides the two real scalars needed to form a single complex scalar $\phi = \phi^4 + i\phi^5$ in the $\mathcal{N} = 2$ $d = 4$ vectormultiplet, with bosonic part (ϕ, A_μ).

That we are left with $\mathcal{N} = 2$ $d = 4$ can also be seen by looking at how the Lorentz symmetry breaks,

$$SO(1,9) \to SO(1,3)_{0,1,2,3} \times SO(2)_{4,5} \times SO(2)_{789} \cong SO(1,3) \times U(1) \times SU(2),$$

the latter two groups making up the $\mathcal{N} = 2$ $U(2)_R$ R-symmetry.

If we have a stack of N $D4$-branes, then the theory thus engineered will be $G = SU(N)$ super Yang-Mills. It is clear from the previous discussion that the $4,5$ positions of the parallel branes are governed by the complex scalar. The Coulomb branch, in which this adjoint scalar has a nonzero vev, is therefore seen in this geometric construction as simply specifying the $4,5$ positions of the parallel branes, specifying how much apart they are.

If instead of $NS5$-branes we had suspended $D4$-branes on $D6$-branes extending in the $0,1,2,3,7,8,9$ directions, we would find that the $6,7,8,9$ locations need scalars to govern them, and this leaves us with exactly a $\mathcal{N} = 2$ $d = 4$ hypermultiplet, with the vectormulitplet projected out. The strings stretching from the N $D4$-branes and the $D6$-branes mean these hypermultiplets are in the fundamental representation of $U(N)$. This is the Higgs branch of the theory.

We can interpolate between these two branches if we add fundamental matter to the original construction. This can be achieved by adding two stacks of semi-infinite $D4$ branes extending in the $0,1,2,3$ directions and along $(-\infty, x_1^6]$ and $[x_2^6, \infty)$ in the 6-th directions. They are, therefore, ending on the same $NS5$-branes. Strings going between a stack of N_L left ones and the stack of N original $D4$-branes will result in hypermultiplets in the $\bar{N}_L$ representation of the $SU(N_L)$ flavour group, and in the N representation of the $SU(N)$ gauge group – since the new branes are semi-infinite, they are taken to be infinitely heavy so their fluctuations are discarded.

The $4,5$ directions of these semi-infinite $D4$-branes are now interpreted as the masses.

Instead of ending at infinity, each[9] new $D4$-brane can be taken to end at a $D6$-brane which we can bring from $x_6 = \pm\infty$ to a finite value. Due to Hanany-Witten moves, these two can be seen to be equivalent. Further, under suitable conditions, the $D6$-branes can go past the $NS5$-branes and realise the Higgs branch we described before.

The same construction can be extended to build quiver gauge theories. Instead of a stack of N $D4$-branes stretched between 2 $NS5$-branes, many stacks of N_i $D4$-branes are arranged to hang in between consecutively placed $NS5$-branes, so that the i-th and $(i+1)$-th stacks share a common $NS5$-brane. The strings stretching from the two stacks result in hypermultiplets in the bifundamental, $\bar{N}_i \otimes N_{i+i}$ representation. Masses now have to do with the relative positions of the stacks in the $4,5$ directions.

In all of this, to obtain a non-gravitating theory the limit $\ell_s, L \to 0$ with g_{YM}^{-2} fixed. Other classical gauge groups can be realised by introducing O-planes.

Notice that the entire Hanany-Witten $NS5 - D4$ system can be seen as actually building the skeleton of the Seiberg-Witten curve of the $\mathcal{N} = 2$ $d = 4$ theory it is

[9] This is the "s-rule" which needs to hold. Namely, a $D6$-brane and a $NS5$-brane cannot be connected by more than one $D4$-brane. Other configurations are not supersymmetric.

describing. For instance, for pure $SU(N)$, the stack of N branes can be interpreted as an N-fold cover of the base.

1.5.2 IIB: Fivebrane webs

The previous $NS5 - D4$ setup becomes $NS5 - D5$ under T-duality [4]. Type IIB fivebranes sit in a (p, q)-multiplet of the $SL(2, \mathbb{Z})$ S-duality group, with $(0, 1)$ corresponding to $NS5$ and $(1, 0)$ to $D5$. These can respectively be taken to extend along the $0, 1, 2, 3, 4, 5$ and the $0, 1, 2, 3, 4, 6$ directions. As before, the effective gauge theory has 8 supercharges – it is the $\mathcal{N} = 1$ $d = 5$ theory, and it describes the shared worldvolume in the $0, 1, 2, 3, 4$ directions. Meanwhile, the details are encoded in the internal dimensions. More specifically, in the $5, 6$ plane.

Due to quantum-mechanical effects, a $D5$ brane cannot end on an $NS5$-brane without it bending. Charge conservation dictates that at vertices where branes meet, we have to impose

$$\sum_i p_i = \sum_i q_i = 0$$

if we fix the orientation in the direction towards the vertex. Further, to preserve 8 supercharges, a (p, q)-fivebrane has to be stretched along a line in the $5, 6$ plane located at

$$\frac{x^6}{x^5} = g_s \frac{p}{q}$$

Gauge theoretic quantities can be easily read from the fivebrane web, see [5]. Of much interest is the polygon dual to the fivebrane web, called the grid diagram – the process of going from the IIA brane web to the type IIB fivebrane web to the grid in one way is illustrated in Figure1.2. In [5, §D], the authors show that if the 5-direction is turned into a circle, so the $d = 5$ theory in question is on $\mathbb{R}^4 \times S^1$, and also if the 10-direction is compactified, then an M-theory realisation may be obtained. It corresponds to an $M5$-brane with worldvolume $\mathbb{R}^{1,3} \times \Sigma$, where Σ is a curve inside the $4, 5, 6, 10$ directions. The curve itself can be obtained from the grid diagram If parametrised by coordinates

$$s = e^{\frac{x^6 + ix^{10}}{R_{10}}}, \quad t = e^{\frac{x^5 + ix^4}{R_4}}$$

it is simply given by

$$F(s, t) = \sum_{(n,m) \text{ in grid}} c_{n,m} s^n t^m$$

The four-dimensional Seiberg-Witten curve is recovered by taking $R_4 \to \infty$.

1.5.3 Topological string: Geometric engineering

Yet another way of obtaining $d = 4$ or 5 gauge theories is by geometric engineering, which means compactifying the type IIA string theory or M-theory respectively on a Calabi-Yau threefold X in the large radius limit and decoupling gravity [139, 168]. This is a different perspective than all the other sections, where we sought the theory as a worldvolume theory of an extended object. Here, we want to take string theory on $\mathbb{R}^{1,3} \times X$ and "integrate out" X.

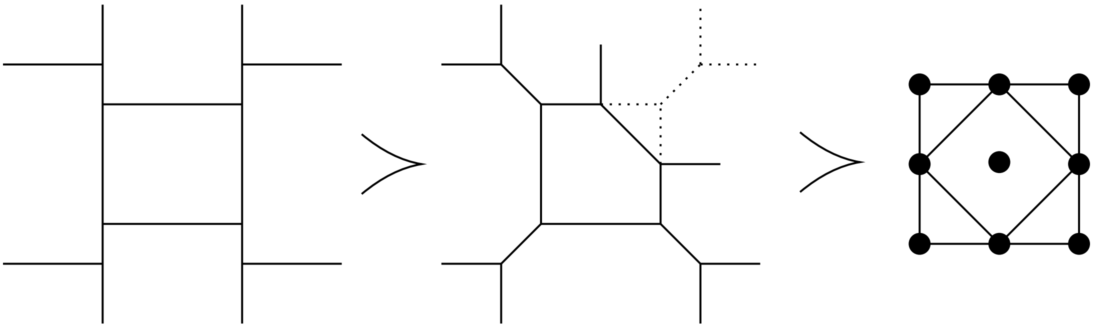


Figure 1.2: The Hanany-Witten $NS5 - D4$ setup for $N_f = 4$ $\mathcal{N} = 2$ $d = 4$ $SU(2)$ super Yang-Mills, along with the fivebrane web featuring a possible flop, and a grid diagram which encodes a Newton polygon.

The main idea is to consider a resolution of an ADE singularity $\mathbb{C}^2/G$, the ALE space. ALE spaces are, of course, non-compact, which turns out to be a simpler case than looking at a compact Calabi-Yau. Its exceptional divisors correspond to simple roots of the Lie algebra $\mathfrak{g}$ corresponding to G, and $D2$-branes can wrap around them, giving us massless states corresponding to $d = 6$ gauge bosons, if we compactify just on the ALE space.

To get to a $d = 4$ gauge theory, the ADE singularity should be fibered over some genus-g curve Σ_g so that the resulting space is a Calabi-Yau threefold. The electric and magnetic Wilson lines of the $d = 6$ gauge theory compactified on Σ_g will give us $4g$ scalars, which turn out to be parts of g adjoint hypermultiplets.

In case of a torus, we get $\mathcal{N} = 4$ $d = 4$ theory, because the fibration turns out to be trivial, and fails to break the supersymmetry from 16 to 8 supercharges. There is a single massless adjoint hypermultiplet, so is expected.

To further break supersymmetry, a nontrivial fibration is needed, and since we don't want adjoint hypermultiplets, the base is taken to be $\mathbb{P}^1$.

This can be generalised. For instance, consider the Riemann surface Σ_g to be holomorphically embedded inside a Calabi-Yau threefold X. The tangent bundle splits, $TX|_{\Sigma_g} = T\Sigma_g \oplus N$, and the normal bundle has to have $c_1(N) = 2g - 2$ to preserve the Calabi-Yau condition. Near Σ_g, X looks like the total space of a rank 2 vector bundle, $\mathrm{tot}(N \to \Sigma_g)$. If $\Sigma_g = \mathbb{P}^1$ then by Grothendieck's theorem from section 1.4.1.1 the vector bundle splits into $\mathcal{O}(-a) \oplus \mathcal{O}(-2 + a) \to \mathbb{P}^1$. In this case, $a = 1$ turns out to correspond to local $\mathbb{P}^1 \times \mathbb{P}^1$, which engineers pure $SU(2)$ $d = 4$ gauge theory.

If what is holomorphically embedded inside X is a closed surface S instead of a curve, the analogous condition for the normal bundle is

$$c_1(N) = -c_1(TS) = c_1(T^*S) = c_1(K_S)$$

where K_S is the anticanonical bundle. Since N is a rank-1 vector bundle, this condition identifies them, $N = K_S$. In particular, one gets all in this way the anticanonical bundles of almost[10] del Pezzo surfaces. These are, in turn, classified by reflexive polytopes [61, §3].

In fact, it turns out that any toric (necessarily local) Calabi-Yau is represented by a toric diagram which is a convex lattice polytope [135]. And here is the main

[10]Generalisation of del Pezzo, see [280] for the definition.

point: this polytope can be identified with the polynomial given by the grid diagram of a (p, q)-fivebrane web [143, 187].

On the other hand, to calculate the topological A model on a local Calabi-Yau threefold, the topological vertex formalism has to be applied directly to its toric diagram [1]. Or, to use the topological B model, topological recursion may be used on the Newton polynomial [52], which can be seen as a mirror curve for a local CY X of the form $\{-w_1w_2 + F(t,s) = 0\} \subset \mathbb{C}^2 \times \mathbb{C}^2_x$.

Finally, to connect with gauge theory, a dictionary between Kähler parameters and gauge-theoretic quantities must be given, and a scaling limit taken, see [44, §3] for a discussion relevant to this work.

1.5.4 M-theory: Class $\mathcal{S}$

There is yet another construction, directly from M-theory, which makes the AGT correspondence explicit. An outstanding review is [87].

A large class of Yang-Mills theories with extended supersymmetry may be constructed from low-energy effective descriptions of M-theoretic compactifications [127, 263, 272]. Especially important are so-called class S theories. Given a simply-laced Lie algebra $\mathfrak{g}$, $\mathcal{S}$ theories, usually labelled $T(\mathfrak{g}, C, D)$, are compactifications of the "strange" maximally superconformal $d = 6$ $\mathcal{N} = (2, 0)$ theory, usually labelled $\mathcal{X}(\mathfrak{g})$, on a punctured Riemann surface C with data D attached to the punctures.

The theory $\mathcal{X}(\mathfrak{g})$ is a superconformal theory thought to describe the worldvolume of $M5$ branes [270, 273], as well as $NS5$ branes in type IIA. It seems the theory itself has to be non-Lagrangian, as the $\mathcal{N} = (2, 0)$ tensor hypermultiplet (ϕ, λ, B) contains the real tensor B with ASD field strength $\star \mathrm{d}B = \mathrm{d}B$. It is unknown if a Lagrangian construction which corresponds to a well defined local QFT exists for this self-duality constraint.

Schematically, a stack of $M5$-branes is wrapping $\mathbb{R}^4 \times C$ or $\mathbb{R}^4_{\epsilon_1,\epsilon_2} \times C$. Compactifying further on C to obtain a $d = 4$ theory, a partial topological twist has to be applied to preserve 8 supercharges. The Coulomb branch, from this point of view, occurs when a stack of N $M5$-branes arranges itself as an N-fold ramified covering of C, $\Sigma \subset T^*C$. The M-theory background itself is $\mathbb{R}^4 \times T^*C \times \mathbb{R}^3$.

The genus of the Riemann surface provides global constraints. For example, it constrains the space of meromorphic functions with perscribed divisors of poles, by Riemann-Roch, and the genera themselves provide us with a Torelli marking consisting of $2g$ cycles. These are, in turn, dual to g cocycles, which correspond to the number of complex structures on a given surface. Indeed, let $\mathcal{M}_{g,n} \cong \{C_{g,n}\}/_{\text{iso}}$ denote the moduli space of the aformentioned surfaces. Then, for a surface $[E] \in \mathcal{M}_{g,n}$, the tangent space $T_{[E]}\mathcal{M}_{g,n}$ may be identified with the cocycles $H^0(E, \Omega^1)$. Physically, these will correspond to deformations on the space of theories - given, e.g., by the renormalization group flow. In the context of isomonodromic deformations, the variations of complex structure will provide us with a set of g times $t_1, \ldots, t_g$.

On the other hand, the punctures serve as boundary conditions - giving monodromy datum around punctures is equivalent to specifying fluxes of fields, i.e. their mass according to Gauß' law, if we recall that a punctured disk is conformally equivalent to a long tube.

The fact that surfaces, 2-real-dimensional objects, know anything about 4-dimenional gauge theory is then a consequence of this compactification, akin to a quantum Fubini's theorem - we may view a theory compactified on $\mathcal{M}^4 \times \mathcal{C}_{g,n}$ as either[11] a 4-dimensional theory with certain datum depending on the Riemann surface, or as a 2-dimensional theory depending on 4-dimensional gauge datum, if we are able to "shrink" either factor freely. Indeed, for the theories of class $\mathcal{S}$, which are the ones that fit the bill, the underlying product space theory is superconformal. We thus get a way to pass from gauge theoretic data to $d = 2$ CFT data, as exepmlified by the AGT correspondence [6] Further, since $C_{g,n}$ is conformally invariant, we may obtain various dualities by stretching parts into long tubes, for example. Playing with such deformations in general will yield dualities between the 4-dimensional gauge theories associated to $\mathcal{C}_{g,n}$, which leads to a "groupoidification" of theory space – $\mathcal{N} = 2$ dualities [96].

The exact details of such correspondences hinge on the construction of Hitchin fibrations. Let us not go into the details here, but just say that the Hitchin systems may be reformulated in terms of flat connections, which are in turn purely global objects and are classified purely by their monodromy data around the aforementioned cycles and punctures, and of course by their representation-theoretic information.

In fact, this is almost enough to fix the whole situation, but, crucially, isn't. A great geometric insight was to explore whether we may change e.g. the locations of punctures while keeping monodromy the same. The answer is yes, and leads to the theory of Painlevé equations, which govern these *isomonodromic* deformations. These equations have the property of not having branch cuts or essential singularities depending on initial conditions, although they may indeed have movable poles and zeroes of any order if allowed by geometry.

1.6 Painlevé equations

Inspired by the discovery of a novel special function arising from a first order algebraic complex ODE in the works of Abel and Jacobi, namely the Weierstraß $\wp$, Paul Painlevé [238] posed the more general problem of

> Déterminer toutes les équations différentielles algébriques du premier ordre, puis du second ordre, puis du troisième ordre, etc., dont l'intégrale générale est uniforme,

where uniform means having a single valued general (not singular) solution; a less strict version is the ability to ensure singlevaluedness, for instance by removing cuts from the domain. Fuchsian equations certainly only have fixed critical points, so only obstacles to this problem are movable critical points in nonlinear ODEs. Therefore the Painlevé Property is conventionally defined as the absence of movable critical points. The old terminology of an integral being a solution implies a global solution, although in practice novel transcendents are given by series solutions around fixed singular points. As these functions are automorphisms of $\mathbb{P}^1$, it is possible to exploit the action of the Möbius group to obtain distinct classes, order by order. For first order, only the Weierstraßtranscendent is new, the main contribution comes from the six so-called Painlevé transcendents defined by second

[11] To be more precise, the BPS states counting is equivalent – whether the full spectrum of a $d = 4$ QFT can be encoded in a $d = 2$ CFT is not known.

order ODEs, the impact of which keeps growing as we enter the second century of their discovery,

$$w'' = R(w', w, z)$$

with R a rational function in w and its derivative, analytic in z in the domain. Besides the six novel transcendents, there are 44[12] other solutions with the Painlevé property, but these can be expressed either in terms of the six, or integrated to first order equations. The Painlevé equations are usually labelled as PVI to PI. For example, PIII is

$$w'' = \frac{w'^2}{w} - \frac{w'}{z} + \frac{\alpha w^2 + \beta}{z} + \gamma w^3 + \frac{\delta}{w}$$

and depends on 4 additional complex parameters $\alpha, \beta, \gamma, \delta$, although we can always set $\gamma = -\delta = 4$ by rescaling w, z. The transcendents, called Painlevé transcendents, have fixed critical points at $\{0, 1, \infty\} \in \mathbb{P}^1$ and moveable first and second order poles. They are called transcendents because their general solutions cannot be rational, algebraic, or depend algebraically on classical transcendental functions [160, §5]. They also cannot be expressed one in terms of another. However, there are discrete Bäcklund flows which express one transcendent in terms of another transcendent of the same type with different parameters or with exchanged critical points. These symmetries are similar to algebraic contingency equations on classical special functions like the hypergeometric one, which is not an accident since as special solutions, ie for particular values of parameters, classical special functions solve

$$\text{PII} \rightsquigarrow \text{ Airy, PIII} \rightsquigarrow \text{ Bessel, PVI} \rightsquigarrow \text{ Hermite-Weber,}$$
$$\text{PV} \rightsquigarrow \text{ Confluent Hypergeometric, PVI} \rightsquigarrow \text{ Hypergeometric,}$$

as on these special points the discrete symmetries simplify [236]. Besides these, there are also algebraic solutions.

Very soon after their stating, Richard Fuchs [92] showed that PVI may be obtained from isomonodromic deformations of a second order linear equation with four regular points, to be described, and Garnier [102] showed how to obtain others by a confluence process, which means "colliding" some of the regular points in a certain pattern. It is in this sense that the Painlevé equations are *integrable* - because they arise as a compatibility condition of an overdetermined linear system [248].

Besides isomonodromy, Fuchs also initiated the study of PVI in terms of what he calls the Legendresche Differentialgleiehung which are solved in terms of elliptic/Abelian integrals, not necessarily over cycles, ie periods, but morally so. In this case, a family of elliptic curves of the form $Y^2 = X(X-1)(X-t)$ over the base $t \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$ are given along with a Picard-Fuchs operator

$$L_t = t(t-1)\partial_t^2 + (2t-1)\partial_t + \frac{1}{4}$$

It annihilates periods but can be shown to be equivalent to PVI when acting on $\int_\infty^{(X,Y)} \mathrm{d}x/y$ if we fix certain boundary conditions. This kind of integrability was what mathematicians of yore had in mind, a solution by quadratures. How this additional elliptic datum ties back to isomonodromic deformations is something

[12] Sometimes literature cites 47

Levin and Olshanetsky elucidated [188]. From the point of view of physics, I cannot help but notice a certain analogy to the Seiberg-Witten curve and its periods (a, a_D) which can be gotten from Picard-Fuchs equations [178], but I am unaware of direct work to tie that geometric approach to the full solution on the Omega-background[13].

What we have described can be summarised as

$$\begin{tikzcd}
\text{Painlevé Property} \arrow[rr, "\text{Manin}"] & & \text{Abelian Integrals} \arrow[dl, "\text{Levin-Olshanetsky}"] \\
& \text{Isomonodromy Deformations} \arrow[ul, "\text{R. Fuchs}"] &
\end{tikzcd}$$

Besides this, we shall also comment on yet another geometric method with phenomenal ability to generalise and unite the study of differential, discrete and multiplicative Painlevé under one framework, based on Okamoto's classification [235] of initial value spaces. We have applied this method to discrete Painlevé equations.

In our work we also encounter higher-rank generalisations of Painlevé equations obtained from the Painlevé-Calogero correspondence. However, the author is unaware of a general theory behind these generalisations, namely the classification of their singularities. We may conjecture that moveable codimension one critical points are absent, but dedicated work is needed for this. Furthermore, we encounter discrete analogues, although for these a general theory of singularity-confinement has been developed [244] with some integrability-related subtleties which led to revision [245].

1.6.1 Isomonodromic deformations

Here we give a brief overview of isomonodromic deformations of Fuchsian systems[14] on a n-punctured sphere, $\mathbb{P}^1 \setminus \{a_1, ..., a_n\}$ with regular singularities. We consider a matrix representation $\rho : \mathfrak{g} \to \mathfrak{sl}(N, \mathbb{C})$[15] and let

$$\partial_z Y(z) = A(z)Y(z), \quad A(z) = \sum_{k=1}^{n} \frac{A_k}{z - a_k}$$

with $A_k \in \rho(\mathfrak{g})$. We assume that the matrices A_k are diagonalisable, which means

$$A_k = G_k \Theta_k G_k^{-1}, \quad \Theta_k \in \mathfrak{h}$$

and additionally assume that the eigenvalues of A_k are distinct (so-called non-resonance). If this is so, in the neighbourhood of each puncture a_k, the solution is

$$Y(z) = \tilde{G}_k(z)(z - a_k)^{L_k}$$

[13]Although the link can be seen to be given by AGT, I am unaware of direct work linking periods to the transcendent.

[14]Isomonodromic deformations are more general, see [265] for a shorter, older review.

[15]Since the Fuchsian system can be viewed as a flatness condition on the singular connection $\partial_z - A(z)$ and there is a compatible notion of a gauge transformation which preserves the equation, $A(z)$ is usually taken to be Lie-algebra-valued.

and $\tilde{G}_\infty(z)(-z)^{-L_\infty}$ around $z=\infty$, where $G_k(z)\in G[[z]]$ and $G_k(a_k)=G_k$. Up to conjugation by G_k, L_k is the logarithm of the monodromy matrix at the point a_k. We also assume $A(z)$ has no singularity at infinity, so that the monodromy at infinity is given trivially by the (inverse) product of the monodromies around the other points, which means $\sum_i A_i = 0$. Without loss of generality, $G_\infty = \mathrm{id}$.

As such, for a loop $\gamma : [0,1] \to \mathbb{P}^1 \setminus \{a_1, ..., a_n\}$ enclosing just one point a_k, we have $Y(\gamma(1)) = Y(\gamma(0))M_k$ on the universal cover, with $M_k = G_k^{-1}e^{2\pi i L_k}G_k$. Since this is a function not of γ but actually of the homotopy class $[\gamma]$, this provides us with a representation

$$\pi_1(\mathbb{P}^1 \setminus \{a_1, ..., a_n\}) \to \mathfrak{sl}_n$$

called the monodromy representation. In general this is then a map from the singular data $\mathrm{SD} = \{(A_k, a_k)\}$ to monodromy data $\mathrm{MD} = \{M_k\}/_\sim$ up to conjugation. This much was known up to David Hilbert's time, so in [137] he asked his 21st problem:

> zeigen, dass es stets eine lineare Differentialgleichung der Fuchsschen Klasse mit gegebenen singulären Stellen und einer gegebenen Monodromiegruppe giebt

In [35] the first counterexample was given, with four 3×3 monodromy matrices that cannot be written as monodromy data of a Fuchsian system. In fact, the codimension of monodromy data not realisable by Fuchsian systems is $(2n-1)(N-1)$ [37, 180]. In interesting cases, $\dim \mathrm{SD} > \dim \mathrm{MD}$ in a locally constant way. Suppose we have an isomonodromic family of Fuchisan systems depending on a parameters a smoothly,

$$\partial_z Y(z) = \sum_{k=1}^n \frac{A_k(a)}{z-a_k}$$

Then we expect the parameters to carve out a submanifold of codimension $\dim \mathrm{SD} - \dim \mathrm{MD}$ in the space of singular data. The local behaviour of this family wrt a is what we mean by isomonodromic deformations. Further, the isomonodromy deformations we have in mind are Schlesinger equations, in which the parameters a are the locations of the marked points themselves. It was in this case that Painlevé VI first turned out in the investigations by Richard Fuchs in 1907 [92], which shows that Hilbert's 21. problem is much more interesting than it seems at first sight. However, we note that if non-resonance is violated, not all isomonodromic deformations are given by Schlesinger equations [36].

Now in [155, 156] there appeared an interpretation of Schlesinger equations via deformation theory as a Maurer-Cartan equation. Namely, if d denotes differentiation with respect to $z, a_1, ..., a_n$, form

$$\Omega = \mathrm{d}YY^{-1}$$

which is single-valued and holomorphic in $\mathbb{P}^1\setminus\{a_1, ..., a_n\}$. In terms of this one-form, we have the characterisation of the Fuchs matrices as the residues $A_k = \mathrm{Res}_{z=a_k}\Omega$. Constant monodromy is in this case equivalent to $\mathrm{d}L_k = 0$. Therefore, near a_k,

$$\Omega = \tilde{G}_k(z)\mathrm{d}\tilde{G}_k(z)^{-1} + \tilde{G}_k(z)L_k\frac{\mathrm{d}(z-a_k)}{z-a_k}\tilde{G}_k(z)^{-1}$$

and then, noting what the residues are, this implies $\Omega = \sum_k A_k \mathrm{d}\log(z-a_k) + \Omega'$. Consider normalising $Y(z=z_0) = \mathrm{id}$. Then $\Omega = 0$ at $z = z_0$, so

$$\Omega = \sum_k A_k \mathrm{d}\log\frac{z-a_k}{z_0-a_k}$$

Looking at the $\mathrm{d}a_k$ component of $\mathrm{d}Y = \Omega Y$ means

$$\partial_{a_k} Y = M_{a_k} Y, \quad M_{a_k} = -\frac{z-z_0}{a_k-z_0}\frac{A_k}{z-a_k}$$

Schlesinger equations are the compatibility conditions of the derivations $[\partial_z, \partial_{a_k}] = 0$, which can be expressed by the Lax-like condition

$$\partial_{a_k} A - \partial_z M_{a_k} = [M_{a_k}, A]$$

Their consistency is the same as asking for $\mathrm{d}^2 = 0$. Therefore,

$$0 = \mathrm{d}^2 Y = \mathrm{d}\Omega \cdot Y + \Omega \wedge \mathrm{d}Y = (\mathrm{d}\Omega - \Omega\wedge\Omega)Y$$

so Ω satisfies the Maurer-Cartan equation. Using the form of Ω, we can write explicitly

$$\mathrm{d}A_i = -\sum_{j\neq i}[A_i, A_j]\mathrm{d}\log\frac{a_i-a_j}{z_0-a_j}$$

which means

$$\partial_{a_i} A_{j\neq i} = \frac{a_j-z_0}{a_i-z_0}\frac{[A_i,A_j]}{a_i-a_j}, \quad \partial_{a_i} A_i = -\sum_{j\neq i}\frac{[A_i,A_j]}{a_i-a_j}$$

and is slightly simplified by taking the reference point $z_0 = \infty$.

Since from the Schlesinger equations the following one-form is closed, we may write it as

$$\mathrm{d}\log\tau = \sum_{i<j}\operatorname{tr} A_i A_j \mathrm{d}\log(a_i - a_j) \tag{1.10}$$

Locally, it defines the Jimbo-Miwa-Ueno tau function τ, and it can be thought of as the generator of the isomonodromic Hamiltonian.

Some generalities may be said about the Schlesinger equations. Namely, seen as an equation on $A_k(a)$, the solutions possess the Painlevé property themselves. Namely, they may be extended to meromorphic functions on $\mathbb{C}^n \setminus \{a_i = a_j, i \neq j\}$ [155, 156, 190]. The divisor of poles is usually called the Malgrange divisor. As can be seen from the definition of the tau function, this divisor coincides with the zeroes of the tau function. By TS/ST, this has connections with exact quantisation conditions.

Consider the first nontrivial group, $A_i \in \mathfrak{sl}_2$. If there are three singular points, Möbius transformations can be used to bring them to $0, 1, \infty$, and the direct and inverse monodromy problems can be solved in terms of the standard hypergeometric equation. However, if we have four points, $\mathbb{P}^1 \setminus \{a_1, a_2, a_3, a_4\}$, the space of singular data has one more dimension than the space of monodromy data. That is, there is a curve in the space of singular data which maps to the same monodromy. By

Möbius transformations, we can fix the points to $0, 1, t, \infty$, with t the cross-ratio of $a_{1,2,3,4}$.

Then, with the z-evolution given by the Fuchsian system, $\partial_z Y = A(z)Y$ and the Schesinger equation in this case for only one moveable parameter t, $\partial_t Y = MY$, the sixth Painlevé equation appears as the compatibility condition $[\partial_z, \partial_t]Y = 0$. In this sense, Painlevé equations are integrable.

However, for $A_i \in \mathfrak{sl}_2$ I have learned of a wonderful trick from A.I. Shchechkin's dissertation [255] I have not been able to trace elsewhere, so we can get to PVI quickly. Namely, recall the "spinor map" $\mathbb{R}^3 \to \mathfrak{sl}_2$ of a vector $\mathbf{x} = (x_1, x_2, x_3)$ to $X = \sum_i x_i \sigma_i$, where $\sigma_{1,2,3}$ are the Pauli matrices. Usually in physics, this map is denoted by switching to spinor indices, $x^\mu = x^{\alpha\dot{\alpha}}$. Consider three such vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}$, mapped to X, Y, Z respectively. Then we can calculate that

$$\operatorname{tr}[X, Y]Z = 4i(\mathbf{x} \times \mathbf{y}) \cdot \mathbf{z} = 4i \det([\mathbf{x}, \mathbf{y}, \mathbf{z}])$$

where the last matrix has the vectors as columns. This is simply giving a concrete expression of the nontrivial element in $H^3(\mathfrak{sl}_2, k) \cong k$ given by the Killing form. Recalling that the inner products are related as $\mathbf{x} \cdot \mathbf{y} = 1/2 \operatorname{tr} XY$, taking the square of the previous identity gives

$$(\operatorname{tr}[X, Y]Z)^2 = -2 \det \begin{pmatrix} \operatorname{tr} X^2 & \operatorname{tr} XY & \operatorname{tr} XZ \\ \operatorname{tr} XY & \operatorname{tr} Y^2 & \operatorname{tr} YZ \\ \operatorname{tr} XZ & \operatorname{tr} YZ & \operatorname{tr} Z^2 \end{pmatrix}$$

Now consider introducing

$$\sigma(t) = t(t-1)\frac{\mathrm{d}\log\tau}{\mathrm{d}t} = (t-1)\operatorname{tr} A_t A_0 + t \operatorname{tr} A_t A_1$$

which is morally the Hamiltonian. Then using the Schlesinger equations we find

$$\dot{\sigma} = \operatorname{tr} A_t A_0 + \operatorname{tr} A_t A_1, \quad t(1-t)\ddot{\sigma} = \operatorname{tr}[A_0, A_t]A_1$$

which together with the monodromies $\theta_i = 1/2 \operatorname{tr} A_i^2$ and $A_\infty = -A_0 - A_1 - A_t$ and the identity we just derived gives the so-called sigma form of PVI,

$$(t(t-1)\ddot{\sigma})^2 = -2 \det \begin{pmatrix} 2\theta_0^2 & t\dot{\sigma} - \sigma & \dot{\sigma} + \theta_0^2 + \theta_t^2 + \theta_1^2 - \theta_\infty^2 \\ t\dot{\sigma} - \zeta & 2\theta_t^2 & (t-1)\dot{\sigma} - \sigma \\ \sigma\dot{\sigma} + \theta_0^2 + \theta_t^2 + \theta_1^2 - \theta_\infty^2 & (t-1)\dot{\sigma} - \sigma & 2\theta_1^2 \end{pmatrix}$$

This is, then a nontrivial relation on the set of singular data, and it turns out that along with it, $\dim \mathcal{M} = \dim \mathrm{SD} = 7$. The transcendent w is then found from the nonlinear transform

$$\sigma = \tfrac{t^2(t-1)^2}{4w(w-1)(w-t)}\left(\dot{w} - \tfrac{w(w-1)}{t(t-1)}\right) - \tfrac{t}{4w}\theta_0^2 + \tfrac{t-1}{4(w-1)}\theta_1^2 - \tfrac{t(t-1)}{4w(w-t)}\theta_t^2 - \tfrac{w+t-1}{4}\theta_\infty^2.$$

Although very much obscure in this presentation, w may be gotten from the 12 component $A(x, t)_{12}$. This confirms what we said about the solution of the Schlesinger equations $A(x, t)$ having the Painlevé property themselves. This also shows that geometrically, the tau function is much better motivated than the actual transcendent, so it is naturally the main player in the Painlevé-Gauge correspondence.

Having obtained the PVI equation, the other Painlevé equations can be obtained by coalescences of the singular points. See [59] for details, and in particular the famous figure 3 therein which shows what this looks like on the 4-punctured sphere. Coalescences should be seen as mappings between the Painlevé equations, and any statement or correspondence about the equations themselves (the objects) should be functorial and include the corresponding mappings.

1.6.2 Initial value spaces and symmetries

There is another geometric approach, initiated by Okamoto [235] and extended to $\mathfrak{q}$-difference equations by Sakai [246]. Sakai noticed that the initial value spaces of both discrete and continuous Painlevé equations may be realised as dP_9, the del Pezzo surface obtained by blowing up $\mathbb{C}^2$ at 9 points. The Cremona isometry group of dP_9 acting on the exceptional divisors is $W(E_9)$, which is an affine Weyl group since $E_9 = E_8^{(1)}$.

This affine Weyl group contains a translation subgroup $\mathbb{Z}^8$ and a finite Coxeter group part generated by reflections. Realising the action of $W(E_8^{(1)})$ or a subgroup as birational transformations on a set of variables leads to a discrete time evolution, which can be seen to be equivalent to the $\mathfrak{q}-$difference Painlevé equations. See [161] for a review.

Besides the symmetry group, the Sakai classification assigns a "surface" root system. For $\mathfrak{q}$-Painlevé, these two pairs are $E_{8-r}^{(1)}/A_r^{(1)}$ for $r = 0, ..., 8$, where $(E_5, E_4, E_3, E_2, E_1) = (D_5, A_4, A_2 \times A_1, A_1 \times A_1, A_1)$ is standard.

In the case of the six continuous Painlevé equations, the discrete flow commutes with the continuous one. In the context of the Painlevé/Gauge correspondence [47, p.30], the un-affinised symmetry group can be interpreted as the flavour symmetry. For instance $D_4 = SO(8)$ is the flavour symmetry of $d = 4$ $\mathcal{N} = 2$ $SU(2)$ gauge theory with four flavours.

In [183, 184, 212] and other works, these symmetries were interpreted as symmetries of the asymptotic values of the quantum curve coming from the TS/Tau correspondence. Since a dictionary between these asymptotics and the parameters of $\mathfrak{q}$-Painlevé equations can be established, the same symmetry group action can be seen to act on these quantum curves. This is, of course, the same statement as in the previous case, because the asymptotic values of the curve correspond to punctures, which correspond to masses. The subtlety is that the symmetries of the quantised curve must be considered.

1.6.2.1 $\mathfrak{q}$-Painlevé/BPS quivers

For the reader's interest, I will point out that a related line of work was successfully launched which associates to a toric Calabi-Yau a certain BPS quiver [89, 281], the cluster algebra associated to whose mutations [109] leads to discrete Painlevé equations [26, 28, 49]. My older academic brother's thesis [204] devotes an entire section to this.

1.6.3 The Painlevé-Calogero correspondence

The first part of my work has roots in Manin's work on the elliptic PVI [191], incidentally also inspired by string theory, in which a mirror to $\mathbb{P}^2$ resulted in the

Picard-Fuchs equation reducing to

$$\frac{\mathrm{d}^2 q}{\mathrm{d}\tau^2} = -\frac{1}{8\pi^2}\wp_\tau(q|\tau)$$

which is deautonomisation of the Calogero-Moser integrable system. The mechanism of deautonomisation was described by Levin and Olshanetsky [188], extended to other Painlevé equations and to higher rank by Takasaki [260, 261] who realised this naturally describes isomonodromic deformations on a one-pointed torus.

De-utonomisation merits its own discussion. First we discuss Levin and Olshanetsky's method which realises isomonodromic deformations in a very "classic" integrable system way, namely by restricting a flatness condition to a symplectic quotient. Then we discuss the very concrete manifestation exploited by Takasaki.

Our ultimate goal is to describe motion on the moduli space of stable curves $\Sigma_{g,n}$, but for now forget the points and consider an unmarked genus-g Riemann surface $\Sigma_{g,n} \mapsto \Sigma_g$ along with a compact complex Lie group G with representation $G \xrightarrow{\rho} Aut(V)$ on a finite dimensional vector space V. Let $\mathfrak{g} \cong T_e G$ be the Lie algebra associated with G. Form the associated bundle $E = P \times_\rho V$ from the principal G-torsor $P \to \Sigma_g$ and consider the space of all connections $\mathrm{Conn}(\Sigma_g) = \{\nabla_\mathcal{A} = \mathrm{d} + \mathcal{A} : \Omega^k(E) \to \Omega^{k+1}(E)\}$, the space of flat connections, $\mathrm{FlatConn}(\Sigma_g) = \{\nabla_\mathcal{A} \in \mathrm{Conn}(\Sigma_g) | \nabla^2_\mathcal{A} = 0\}$ and the space of gauge transformations $\mathcal{G} = \Gamma(\mathrm{Ad}P)$ where $\mathrm{Ad}P = P \times_{\mathrm{Ad}} G$ which naturally acts on connections. The moduli space of flat bundles can be realised in two equivalent ways: as the quotient space $\mathrm{FlatBun}(\Sigma_g) = \mathrm{FlatConn}(\Sigma_g)/\mathcal{G}$ or as the symplectic quotient $\mathrm{Conn}(\Sigma_g)//\mathcal{G}$, which we describe now. Let $\langle .,. \rangle : \mathfrak{g}^2 \to \mathbb{C}$ be the pairing on the Lie algebra induced by the Killing form, nondegenerate by the compactness assumption. Then

$$\omega = \frac{1}{2}\int_{\Sigma_g} \langle \delta\mathcal{A}, \delta\mathcal{A} \rangle$$

is a symplectic form on $\mathrm{Conn}(\Sigma_g)$, with $\delta\mathcal{A} = \Omega^1(\Sigma_g, \mathfrak{g})$. The momentum map simply maps a connection $\delta\mathcal{A}$ to it's field strength $F_\mathcal{A} \in \Omega^2(\Sigma_g, End(P))$, and the equivalence of the two is established. We move on to deformations. First, fix a "base" complex structure on Σ_g, in local coordinates $(z, \bar{z})$, and decompose the connection $d + \mathcal{A} = (\partial + A) \otimes \mathrm{d}z + (\bar{\partial} + \bar{A}') \otimes \mathrm{d}\bar{z}$[16]. Levin and Olshanetsky then introduce chiral deformations and a spectral parameter. First, consider a small change of variables to new coordinates $(w, \bar{w})$ such that

$$w = z - \epsilon(z, \bar{z}), \quad \bar{w} = \bar{z}$$

which means $\partial_w = \partial, \bar{\partial}_w = \bar{\partial} + \mu\partial$, where $\mu = \bar{\partial}\epsilon/(1 - \bar{\partial}\epsilon)(\delta \otimes \mathrm{d}\bar{z}) \in \Omega^{(-1,1)}(\Sigma_g)$ is the Beltrami differential. We assume the deformation ϵ to be small, in particular this means we assume $\mu = \bar{\partial}\epsilon$. The amount of complex moduli is $3g - 3$, so given a basis of complex moduli[17] $\{\mu_l\}_{l=1}^{3g-3}$, we can decompose our transformation as

$$\mu = \sum_{l=1}^{3g-3} \tau_l \mu_l$$

[16] $\partial = \partial_z, \bar{\partial} = \partial_{\bar{z}}$

[17] Properly, global diffeomorphisms of Σ_g induce an equivalence on the Beltrami differentials. This lets us consider the moduli space of deformations. The deformations themselves are then the tangent space to the moduli space, and this is $3g - 3$-dimensional.

The spectral parameter κ is introduced by replacing $\partial \to \kappa\partial$ and to preserve $\partial_{\bar{w}} = \bar{\partial} + \mu\partial$ we also rescale $\mu \to \mu/\kappa$. Then the connection $\nabla_{\mathcal{A}}$ in deformed coordinates becomes

$$(\kappa\partial + A) \otimes (\mathrm{d}w + \bar{\partial}\epsilon \mathrm{d}\bar{w}) + (\bar{\partial} + \bar{A}') \otimes \mathrm{d}\bar{z} = (\kappa\partial_w + A) \otimes \mathrm{d}w + \left(\frac{\mu}{\kappa}(\kappa\partial + A) + \bar{\partial} + \bar{A}'\right) \otimes \mathrm{d}\bar{w}$$

so that, defining $\bar{A} = \bar{A}' + \frac{\mu}{\kappa}A$, the $(0,1)$ component becomes $(\partial_{\bar{w}} + \bar{A}) \otimes \mathrm{d}\bar{w}$. All this interests us because of the Hamiltonian. In these new, deformed coordinates, the symplectic form ω becomes

$$\omega = \frac{1}{2}\int_{\Sigma_g} \langle \delta\mathcal{A}, \delta\mathcal{A} \rangle = \int_{\Sigma_g} \langle \delta A, \delta \bar{A} \rangle - \frac{1}{\kappa}\langle \delta A, A \rangle \delta\mu$$

Recalling the decomposition of μ, we have $3g-3$ Hamiltonians governing complex deformations, $H_l = \frac{1}{2}\int_{\Sigma_g} \langle A, A \rangle \mu_l$, associated with times τ_l. The equations of motion of this system are

$$\partial_{\tau_l}\bar{A} = \frac{1}{\kappa}A\mu_l, \quad \partial_{\tau_l}A = 0$$

which can be seen as the compatibility condition of

$$\begin{aligned} (\kappa\partial_w + A)\psi &= 0 \\ (\partial_{\bar{w}} + \bar{A})\psi &= 0 \\ \kappa\partial_{\tau_l}\psi &= 0, \quad l = 1, ..., 3g-3 \end{aligned}$$

for the Baker-Akhiezer function $\psi \in \Omega^0(\Sigma_g, \mathrm{End}E)$. The last equations signifies that a change of monodromy $\psi \to \psi M$, where M is a representation of the fundamental group $\pi_1(\Sigma_g) \hookrightarrow G$ does not depend on the times τ_l. These equations are, however, trivial on $\mathrm{FlatConn}(\Sigma_g)$. To find nontrivial equations all we do is pick a representative of the $\mathcal{G}$-orbits. We pick a function $f \in C^\infty(\Sigma_g)$ and use it to pick a representative $\bar{L}$ and define its dual L by

$$\begin{aligned} \bar{A} &= f(\bar{\partial} + \mu\partial)f^{-1} + f\bar{L}f^{-1} \\ L &= f^{-1}\kappa\partial f + f^{-1}Af \end{aligned}$$

Finally, defining $M_l = \partial_{\tau_l} f \cdot f^{-1}$ leads to the equations

$$\begin{aligned} (\kappa\partial + L)\psi &= 0 \\ (\bar{\partial} + \mu\partial + \bar{L})\psi &= 0 \\ (\kappa\partial_{\tau_l} + M_l)\psi &= 0, \quad l = 1, ..., 3g-3 \end{aligned}$$

which are the compatibility conditions of

$$\begin{aligned} \kappa\partial_{\tau_l}L - \kappa\partial M_l - [L, M_l] &= 0 \\ \kappa\partial_{\tau_l}\bar{L} - (\bar{\partial} + \mu\partial)M_l - [\bar{L}, M_l] &= L\mu_l \end{aligned}$$

Equivalently, we fixed a particular Baker-Akhiezer function by $\psi \to \psi f$.

Autonomisation is then the $\kappa \to 0$ limit when $\tau_l = \tau_l^0 + \kappa t_l$. The compatibility equation reduces to the standard Lax equation

$$\partial_{t_l}L - [L, M_l] = 0$$

and in fact the entire system reduces to the Hitchin integrable system. This can be seen from the behaviour of the connection itself, as the $(1,0)$ component reduces to $(\kappa\partial + A)\otimes \mathrm{d}z \to A\otimes \mathrm{d}z \in \Omega^0(\Sigma_g, \mathfrak{g}\times K)$, which we identify with the Higgs bundle.

Now consider adding n marked points $(x_1, ..., x_n)$ to Σ_g and write $\Sigma_{g,n}$ for the n-pointed curve, and associate to each point x_i a flag variety $\mathrm{Flag}_i = G/P_i$ with P_i a parabolic subgroup[18]. Then we need to restrict the diffeomorphisms of Σ_g to diffeormorphisms which vanish at the marked points, and also restrict the gauge transformations $\mathfrak{G}$ to reduce to the Borel subgroup at the points. Over the cotangent bundle of each flag Flag_i there is a natural affine space, the coadjoint orbit $\mathcal{O}_i = \{p_i = \mathrm{Ad}_g p_i^0 | g\in G, p_i^0 \in \mathfrak{g}^*\}$, which is equipped with the Kostant-Kirillov symplectic form $\omega_i^{\mathrm{KK}} = \langle p_i g^{-1}\delta g \wedge g^{-1}\delta g\rangle = \int_{\Sigma_g} \delta(x_i)\langle p_i g^{-1}\delta g\wedge g^{-1}\delta g\rangle$. We simply add these to the symplectic form

$$\omega \mapsto \omega + \sum_{i=1}^n \omega_i^{\mathrm{KK}}$$

The moduli space of deformations is fibered over the one of Σ_g with fibers isomorphic to $\mathcal{U}\subset\mathbb{C}^n$, and as a result the dimension is increased to $3g-3+n$, giving us times $(\tau_1, ..., \tau_{3g-3}, t_1, ..., t_n)$. The space of connections $\mathrm{Conn}(\Sigma_g)$ is replaced by connections with prescribed singularities

$$\mathrm{Conn}(\Sigma_{g,n}) = \{(A,\bar{A})|\bar{A}|_{\mathcal{U}_i} = 0, A|_{\mathcal{U}_i} = p_i(z_i - x_i)^{-1} + O(1), x_i\in\mathcal{U}_i\}$$

and the flatness condition is replaced by $F_{\mathcal{A}} = \sum_i p_i\delta(x_i)$. As such, the system reduced to the usual G-Hitchin system when $\kappa\to 0$, where the Higgs bundle, coming from A, has prescribed singularities.

For $\mathfrak{sl}_n$, parabolic subalgebras are determined by partitions of n, and the dimensions of the flag varieties are given by the hook formula. In literature on AGT, punctures are often associated directly with Young diagrams.

The Painlevé-Calogero correspondence is the observation that for the one-pointed torus $\mathbb{T}_\tau$, the same Lax pair L, M satisfies both

$$\partial_{t_\tau} L = [L, M_\tau]$$

and $\partial_\tau L + \partial_z M = [L, M_\tau]$, where $\tau = \tau_0 + \kappa t_\tau$. The single point can be fixed to $z = 0$ using $z\mapsto z + c$, $c\in\mathbb{C}$ constant, so the only motion is the deformation of the complex structure τ. Takasaki [260] has noticed that the appropriate Lax pairs have been found by Bordner et al [50, 51]. Following an idea by D'Hoker and Phong [71], non-simply laced Lie algebras receive a slightly modified treatment, with coupling constants dependent on root length. However, we consider in all cases "root-type" Lax pairs with a single coupling constant. For simply-laced algebras with roots R, these are $|R|\times|R|$ matrices

$$L_{\beta\gamma} = \pi\cdot\beta\delta_{\beta\gamma} + ig\sum_{\alpha\in R} x(q\cdot\alpha, z)\delta_{\alpha,\beta-\gamma} + 2x(q\cdot\alpha, 2z)\delta_{2\alpha,\beta-\gamma},$$

$$M_{\beta\gamma} = ig\left(\wp(q\cdot\beta|\tau) + \sum_{\zeta\cdot\beta=1}\wp(q\cdot\gamma|\tau)\right)\delta_{\beta\gamma} + ig\sum_{\alpha\in R} y(q\cdot\alpha, z)\delta_{\alpha,\beta-\gamma} + y(q\cdot\alpha, 2z)\delta_{2\alpha,\beta-\gamma},$$

[18]Recall that a choice of complex structure on G means a decomposition of its Lie algebra $T_eG\cong\mathfrak{g} = \mathfrak{n}^-\oplus\mathfrak{t}\oplus\mathfrak{n}^+$ into negative roots, the Cartan and positive roots. Let $\mathfrak{b} = \mathfrak{t}\oplus\mathfrak{n}^+$ be the Borel subalgebra. The parabolic subalgebras $\mathfrak{p}$ are formed by choosing $\mathfrak{p} = \mathfrak{b}\oplus\mathfrak{n}$, with $\mathfrak{n}\subseteq\mathfrak{n}^-$ possibly empty. We have $\mathfrak{n}^+\subset\mathfrak{b}\subseteq\mathfrak{p}\subseteq\mathfrak{g}$.

where

$$x(u,z)=\frac{\theta_1(z-u|\tau)\theta_1(0|\tau)}{\theta_1(z|\tau)\theta_1(u|\tau)},y(u,z)=\partial_u x(u,z)$$

is the Lamé function, the most important property of which for Takasaki's proof is the heat equation $2\pi i\partial_\tau x+\partial_u\partial_z x=0$. The Hamiltonian coming from L is

$$H=\frac{1}{2}\operatorname{tr}L^2=\frac{1}{2}\pi^2+\frac{g}{2}\sum_{\alpha\in R}\wp(q\cdot\alpha|\tau)$$

up to π,q-independent terms. This is the elliptic Calogero-Moser Hamiltonian. When standard Hamiltonian mechanics are considered, the equations of motion are simply

$$\frac{\mathrm{d}q_j}{\mathrm{d}t}=\pi_j,\quad \frac{\mathrm{d}\pi_j}{\mathrm{d}t}=-g\sum_{\alpha\in R}\wp(q\cdot\alpha|\tau)\alpha$$

However, the equations of motion equivalent to the isomonodromic "Lax" equation $2\pi i\partial_\tau L-\partial_z M=[L,M]$ turn out to be equivalent to formally replacing $\frac{\mathrm{d}}{\mathrm{d}t}\to 2\pi i\frac{\mathrm{d}}{\mathrm{d}\tau}$,

$$2\pi i\frac{\mathrm{d}q_j}{\mathrm{d}\tau}=\pi_j,\quad 2\pi i\frac{\mathrm{d}\pi_j}{\mathrm{d}\tau}=-g\sum_{\alpha\in R}\wp(q\cdot\alpha|\tau)\alpha$$

Manin's Painlevé VI emerges from this discussion from the specialisation to $G=SU(2)$. I have used this correspondence to work with pure gauge theory and not on the torus proper – for this I would like to advertise my older academic brother's terrific work [38, 39, 68].

1.7 The Painlevé-Gauge theory correspondence

In [154] it was noticed that the asymptotics of the PIV tau function at $t=0$ can be written in terms of two integration constants $\sigma,\check{s}$ as

$$\begin{aligned}\tau(t)\sim{}&\text{const.}\,t^{(\sigma^2-\theta_0^2-\theta_t^2)/4}\\&\times\Big(1+\frac{1}{8\sigma^2}(\theta_0^2-\theta_t^2-\sigma^2)(\theta_\infty^2-\theta_1^2-\sigma^2)t\\&-\sum_{\epsilon=\pm1}\frac{\check{s}^\epsilon}{16\sigma^2(1+\epsilon\sigma)^2}(\theta_0^2-(\theta_t-\epsilon\sigma)^2)(\theta_\infty^2-(\theta_1-\epsilon\sigma)^2)t^{1+\epsilon\sigma}+\mathcal{O}(t^2)\Big)\end{aligned}$$

However, notice that the 4-point conformal block in $c=1$ Liouville CFT is

$$B(\Delta_i,\Delta|q)=q^{\Delta-\Delta_1-\Delta_2}\cdot\Big(1+\frac{1}{2\Delta}(\Delta_2-\Delta_1-\Delta)(\Delta_3-\Delta_4-\Delta)q+\mathcal{O}(q^2)\Big)\quad(1.11)$$

Noticing the similarity between the two expressions as "$\Delta\sim\theta^2$", the landmark paper [100] proposed that the complete expansion of the most general PVI tau function at $t=0$ can be written as a multiplicative Zak transform

$$\tau(t)=\text{const.}\sum_{n\in\mathbb{Z}}C(\theta_i,\sigma+n)t^{(\sigma+n)^2-\theta_0^2-\theta_t^2}B(\theta_i^2,(\sigma+\mathbf{n})^2|t)$$

Further coefficients can be calculated by the combinatorics of Nekrasov functions coming from the identification of these conformal blocks with $\mathcal{N}=2$ gauge theory on the Coulomb branch of AGT. Of course, the direct comparison that we sketched was historically just the last step of obtaining a dictionary for the Painlevé-Gauge correspondence. The actual links between isomonodromy and CFT were known before, but it took the prudence of Gamayun, Iogorov and Lisovyy to recognise that they can use the then-recent AGT to calculate the whole tau function.

1.7.1 AGT or CFT/Gauge

Before discussing AGT, I introduce Virasoro conformal blocks. The Virasoro algebra is a central extension of the Witt algebra of DiffS^1 with generators L_m, $m \in \mathbb{Z}$, and central element c with relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m^2(m-1)\delta_{m+n,0}$$

In CFT, these are to be viewed as coming from the conserved energy-momentum tensor $\bar{\partial}T(z) = 0$ of the conformal symmetry [284], so

$$T(z) = \sum_{n\in\mathbb{Z}} \frac{L_n}{z^{n+2}}$$

We naturally have the Borel $\mathfrak{b} = \{L_m | m \geq 0\}$ and define using it the highest weight representation

$$L_m V_\alpha = \delta_{m,0}\Delta_\alpha V_\alpha$$

with the highest weight vector V_α of conformal weight Δ_α. Denote the primary state corresponding to Δ_α by $|\Delta\rangle$. The Verma module is formed by the descendants $V_{\alpha,Y} = L_{-Y}V_\alpha$ where Y is a Young diagram – this can be arranged using the algebra relations. The dimension of the descendant is $\Delta_{\alpha,Y} = \Delta_\alpha + |Y|$. By the state-operator correspondence, operator-product expansions (OPEs) of chiral operators corresponding to the descendants is defined as

$$V_{\alpha_1,Y_1}(z)V_{\alpha_2,Y_2}(z') = \sum_{\alpha_3,Y_3} \frac{C^{\alpha_3,Y_3}_{\alpha_1,Y_1|\alpha_2,Y_2} V_{\alpha_3,Y_3}(z')}{(z-z')^{\Delta_{\alpha_1,Y_1}+\Delta_{\alpha_2,Y_2}-\Delta_{\alpha_3,Y_3}}} \tag{1.12}$$

and the Virasoro algebra then implies that the structure constants C are defined by those of the primary states, $C^{\alpha_3,Y_3}_{\alpha_1,Y_1|\alpha_2,Y_2} = C^{\alpha_3}_{\alpha_1|\alpha_2}\beta^{\Delta_{\alpha_3}}_{\Delta_{\alpha_1},\Delta_{\alpha_2}}(Y_1, Y_2|Y_3)$ where β are purely representation-theoretical objects, and $C^{\alpha_3}_{\alpha_1|\alpha_2}$ depend on the dynamics of the two-dimensional theory. Define the two-point function, or the Shapovalov matrix[19],

$$Q_\Delta(Y_1, Y_2) = \langle\Delta| L_{Y_1}L_{-Y_2} |\Delta\rangle = \langle L_{Y_1}V_\Delta(0)L_{-Y_2}V_\Delta(\infty)\rangle$$

which by the algebra vanishes unless $|Y_1| = |Y_2|$. It lets us express β in terms of the three-point function,

$$\gamma_{\Delta_{\alpha_1},\Delta_{\alpha_2},\Delta_{\alpha_3}}(Y_1, Y_2, Y_3) = \langle L_{-Y_1}V_{\Delta_{\alpha_1}}(0)L_{-Y_2}V_{\Delta_{\alpha_2}}(1)L_{-Y_3}V_{\Delta_{\alpha_3}}(\infty)\rangle$$
$$\sum_{Y} \beta^{\Delta_{\alpha_3}}_{\Delta_{\alpha_1},\Delta_{\alpha_2}}(Y_1, Y_2|Y)Q_{\Delta_{\alpha_3}}(Y, Y_3)$$

[19] To define insertions at $z = \infty$, it is customary to take the scaling limit $\langle V_\alpha(\infty)\cdots\rangle = \lim_{z\to\infty} R^{2\Delta_\alpha}\langle V_\alpha(R)\cdots\rangle$.

which follows from (1.12). We denote the special case when $Y_1 = Y_2 = \varnothing$ by

$$\beta^{\Delta_{\alpha_3}}_{\Delta_{\alpha_1},\Delta_{\alpha_2}}(Y) := \beta^{\Delta_{\alpha_3}}_{\Delta_{\alpha_1},\Delta_{\alpha_2}}(\varnothing,\varnothing|Y), \quad \gamma_{\Delta_{\alpha_1},\Delta_{\alpha_2},\Delta_{\alpha_3}}(Y) := \gamma_{\Delta_{\alpha_1},\Delta_{\alpha_2},\Delta_{\alpha_3}}(\varnothing,\varnothing,Y)$$

For primary fields, general theory gives the three-point function exactly. Namely,

$$\begin{aligned}&\langle V_{\alpha_1,\varnothing}(z_1)V_{\alpha_2,\varnothing}(z_2)V_{\alpha_3,\varnothing}(z_3)\rangle\\ &= \frac{C_{\Delta_{\alpha_1},\Delta_{\alpha_2},\Delta_{\alpha_3}}}{(z_1-z_2)^{\Delta_{\alpha_1}+\Delta_{\alpha_2}-\Delta_{\alpha_3}}(z_1-z_3)^{\Delta_{\alpha_1}+\Delta_{\alpha_3}-\Delta_{\alpha_2}}(z_2-z_3)^{\Delta_{\alpha_2}+\Delta_{\alpha_3}-\Delta_{\alpha_1}}}\end{aligned}$$

Since it can be seen that $[L_{-1}, V_{\alpha,\varnothing}] = \partial_z V_{\alpha,\varnothing}$, this can be used to show that $\gamma_{\Delta_1,\Delta_2,\Delta_3} = \Delta_1 - \Delta_2 + \Delta_3$. Using the OPE twice, the four-point function of primaries can be seen to be

$$\begin{aligned}&\langle V_{\alpha_1,\varnothing}(z_1)V_{\alpha_2,\varnothing}(z_2)V_{\alpha_3,\varnothing}(z_3)V_{\alpha_4,\varnothing}(z_4)\rangle\\ &= \sum_{\alpha_{12},Y_{12},\alpha_{34},Y_{34}} \frac{C^{\alpha_{12},Y_{12}}_{\alpha_1,\varnothing|\alpha_2,\varnothing}(z_2)}{(z_1-z_2)^{\Delta_{\alpha_1}+\Delta_{\alpha_2}-\Delta_{\alpha_{12},Y_{12}}}} \frac{C^{\alpha_{34},Y_{34}}_{\alpha_3,\varnothing|\alpha_4,\varnothing}(z_4)}{(z_3-z_4)^{\Delta_{\alpha_3}+\Delta_{\alpha_4}-\Delta_{\alpha_{34},Y_{34}}}} \langle V_{\alpha_{12},Y_{12}}(z_2)V_{\alpha_{34},Y_{34}}(z_4)\rangle\end{aligned}$$

Setting $(z_1, z_2, z_3, z_4) = (1, \infty, q, 0)$ by a Möbius transformation lets us express this four-point function in terms of the representation-theoretic conformal block. Namely, with some dynamical factors C, we get the s-channel expansion

$$\langle V_{\alpha_1,\varnothing}(1)V_{\alpha_2,\varnothing}(\infty)V_{\alpha_3,\varnothing}(q)V_{\alpha_4,\varnothing}(0)\rangle = \sum_{\alpha} C_{\Delta_{\alpha_1},\Delta_{\alpha_2},\Delta_\alpha} C_{\Delta_\alpha,\Delta_{\alpha_3},\Delta_{\alpha_4}} q^{\Delta_\alpha-\Delta_1-\Delta_q} B(\Delta_{\alpha_i},\Delta_\alpha|q)$$

$$\begin{aligned}B(\Delta_{\alpha_i},\Delta|q) &= \sum_{(Y_1,Y_2)} q^{|Y_1|}\beta^{\Delta}_{\Delta_{\alpha_1},\Delta_{\alpha_2}}(Y)Q_\Delta(Y_1,Y_2)\beta^{\Delta}_{\Delta_{\alpha_3},\Delta_{\alpha_4}}\\ &= \sum_{(Y_1,Y_2)} q^{|Y_1|}\gamma_{\Delta_{\alpha_1},\Delta_{\alpha_2},\Delta}(Y)Q^{-1}_\Delta(Y_1,Y_2)\gamma_{\Delta_{\alpha_3},\Delta_{\alpha_4},\Delta}\end{aligned}$$

Then (1.11) follows by explicit calculation,

$$\begin{aligned}Q_\Delta(\varnothing,\varnothing) &= \langle\Delta|\Delta\rangle = 1\\ Q_\Delta([1],[1]) &= \langle\Delta|\, L_1L_{-1}\,|\Delta\rangle = \langle\Delta|\, L_1L_1 + 2L_0\,|\Delta\rangle = 2\Delta\end{aligned}$$

and the $\gamma_{\Delta_1,\Delta_2,\Delta_3}([1])$ we already have. See [195] for more details. The conformal block itself can be seen to be the square of the norm of a different representation, the Whittaker vector [97].

The exact combinatorial structure of the Virasoro conformal blocks turns out to be given in terms of Nekrasov functions [6, §4.3]. Further, the dynamical 3-point factors in this theory $C_{\Delta_1,\Delta_2,\Delta_3}$ are given by the DOZZ [76, 285] formula and correspond exactly the one-loop contributions [6, §A.2]. The original AGT proposal is given in terms of an n-point function of a Liouville CFT on a Riemann surface $\bar{C}$, which has a chiral and antichiral part, and $\mathcal{N} = 2$ class $\mathcal{S}$ gauge theory on the squashed sphere with UV curve $C = \mathbb{C} \setminus \{z_1, ..., z_n\}$,

$$Z_{S^4_b}(T(\mathfrak{su}(2), C, m)) = \langle V_{\alpha_1}(z_1)\cdots V_{\alpha_n}(z_n)\rangle^{\text{Louville}}_{\bar{C}}$$

which localises to the north and south poles by supersymmetry [125, 239], at which it is given by the theory on the Omega-background. It may, however, be directly related to the Omega-background by the general philosophy of class $\mathcal{S}$ theories and $2d/4d$ dualities. Schematically, the superconformality of the $d = 6$ theory on a product manifold lets us shrink either factor manifold

$$Z^{[\mathbb{R}^4_{\epsilon_1,\epsilon_2}]}(C) = Z(C \times \mathbb{R}^4_{\epsilon_1,\epsilon_2}) = Z^{[C]}(\mathbb{R}^4_{\epsilon_1,\epsilon_2})$$

In this reduced case, there is a direct equality of the instanton part with the conformal block,

$$Z^{\text{inst.}}(a, m_i|q) = B(\Delta_i, \Delta|q)$$

along with a dictionary, $\epsilon_1 : \epsilon_2 = b : 1/b$, $c = 1 + 6Q^2$ where $Q = b + 1/b$ is the background charge of the Louville theory. The dimensions are $\Delta_{\alpha_i} = \alpha_i(Q - \alpha_i)$, and they are linear combinations of the gauge theory masses for $SU(2)$ gauge theory with $N_f = 4$ fundamental flavours,

$$\begin{aligned} m_1 &= \alpha_3 - \alpha_4 + Q/2, \quad & m_2 &= \alpha_1 - \alpha_2 + Q/2, \\ m_3 &= \alpha_1 + \alpha_2 - Q/2, \quad & m_4 &= \alpha_3 + \alpha_4 - Q/2 \end{aligned}$$

while for $\Delta = \alpha(Q - \alpha)$, $\alpha = a + Q/2$, with a the scalar vev. In the more general case of a genus-g UV curve, sewing parameters q associated with thin necks are related to the UV gauge coupling as $q = e^{2\pi i \tau_{UV}}$.

In the higher-rank case, the Virasoro algebra is replaced by W_N-modules.

1.7.2 Isomondromy/CFT

From the definition (1.10), it can be proved [100, §2.2] that the isomonodromic tau function for the n-punctured sphere and Möbius transformations $f(z) = (\alpha z + \beta)/(\gamma z + \delta)$, $\alpha\delta - \beta\gamma \neq 0$ transforms as

$$\tau(f(a)) = \prod_{i=1}^{n} \left[\frac{\partial f}{\partial a}\right]^{-\frac{1}{2}\operatorname{tr} A_i^2} \tau(a)$$

which is also how an n-point function of primary fields with $\Delta_i = \frac{1}{2}\operatorname{tr} A_i^2$ in CFT behaves. For $n = 3$, we explicitly get the tau function as the 3-point function

$$\tau(a_1, a_2, a_3) = \text{const.}(a_1 - a_2)^{\Delta_3 - \Delta_1 - \Delta_2}(a_1 - a_3)^{\Delta_2 - \Delta_1 - \Delta_3}(a_2 - a_3)^{\Delta_1 - \Delta_2 - \Delta_3}$$

This line of work was started in [247], where the Fuchsian system was explicitly solved in terms of the fundamental matrix given by a CFT Ansatz,

$$\Phi_{\alpha\beta} = (z - z_0)\frac{\langle \mathcal{O}_{L_1}(z_1)\cdots\mathcal{O}_{L_n}(z_n)\bar{\psi}_\alpha(z_0)\psi_\beta(z)\rangle}{\langle \mathcal{O}_{L_1}(z_1)\cdots\mathcal{O}_{L_n}(z_n)\rangle}$$

where $\mathcal{O}_{L_i}$ are holonomic fields or twist fields, primaries in a $c = 1$ $d = 2$ CFT with dimensions $\Delta_i = \frac{1}{2}\operatorname{tr} A_i^2$, and $\bar{\psi}, \psi$ are free fermions with OPE

$$\bar{\psi}_\alpha(z_0)\psi_\beta(z) = \frac{\delta_{\alpha\beta}}{z - z_0}$$

This OPE guarantees that $\Phi(z \to z_0) = \mathbb{1}$ is normalised at the reference point z_0. The twist fields $\mathcal{O}_{L_i}$ are constructed to realise the monodromies of the original problem,

$$\mathcal{O}_{L_i}(z_i)\psi_\alpha(z) \sim (z - z_i)^{L_i}\mathcal{O}^{(0)}_{L_i,\alpha}(z_i)$$

with $\mathcal{O}^{(0)}_{L_i,\alpha}$ some local operator. This approach was recently fully developed and even extended to W-algebras in [27, 104, 105]. The tau function is then the n-point function,

$$\tau = \langle \mathcal{O}_{L_1}(z_1) \cdots \mathcal{O}_{L_n}(z_n)\rangle$$

1.7.3 Kiev formula or Painlevé/Gauge

Having expressed the isomonodromic tau function in terms of CFT, AGT enabled Gamayun, Iogorov and Lisovyy [100, 101] to propose that the Painlevé VI tau function, as the isomonodromic tau function corresponding to the 4-punctured sphere with regular singularities, is proportional to the dual or Nekrasov-Okounkov partition function of $N_f = 4$ $SU(2)$ super Yang-Mills on the self-dual Omega-background

$$\tau^{PVI} \propto Z^{\text{N.O.}} = \sum_{n\in\mathbb{Z}} e^{4\pi i\eta n} Z(a+n, m, \epsilon, -\epsilon|t)$$

as the self-dual Omega-background corresponds to $c = 1$ CFT, as well as to extend the correspondence functorially to other Painlevé functions via coalescences. A host of proofs has already been given: directly from representations of Virasoro [29], from Nakajima-Yoshioka blowup relations [32], from the Riemann-Hilbert problem [103].

Extending this to higher-rank isomonodromic problems and general gauge groups G via the Painlevé/Calogero correspondence was a major focus of my work.

1.7.4 Spectral curve

The appearance of the Riemann surface C in all of this merits a brief discussion. The Painlevé/Gauge correspondence identifies the 4-punctured $\mathbb{P}^1$ of the isomonodromic problem with the UV or Gaiotto curve, of which the Seiberg-Witten curve is a double cover of. This is not the SW curve itself. At this point we note a subtlety. On the Hanany-Witten brane realisation of in type IIA string theory , the M-theory curve of $SU(N)$ can be seen either as a "horizontal" 2-fold branched cover of one of the NS5 branes taken as the base with $N + N_f/2$ punctures, or as an $(N + N_f/2)$-fold branched cover of a 2-punctured $D4$. The UV curve corresponds to the latter realisation.

There is one more curve associated to each Fuchsian system, and that is the spectral curve, which is for a $\mathfrak{sl}_2$-valued $A(z)$

$$\Sigma \quad \det(y\mathbb{1} - A(z)) = y^2 - \frac{1}{2}\operatorname{tr} A(z)^2 = 0$$

It is a double-cover of the z-plane, which we already know is the UV or Gaiotto curve, and a canonical meromorphic differential $\lambda = y\mathrm{d}z \in H^1(\Sigma)$. Then it will come to no surprise that in [47] it was noted that this is exactly the Seiberg-Witten curve of $\mathcal{N} = 2$ $SU(2)$ super Yang-Mills with $N_f \leq 4$ flavors, and an explicit dictionary was given mapping all the Painlevé equations to the gauge theories, including the non-gauge Argyres-Douglas theories. The singular behaviour needed to specify λ is the same as the one of $A(z)$, which corresponds to the UV behaviour as masses are attached to punctures. On the other hand, the double cover involves the σ function, which corresponds to the IR Coulomb data.

In this scheme, confluences on the Painlevé side correspond to decouplings of fundamental hypermultiplets, or to Argyres-Douglas scalings.

The spectral curve is not fixed by isomonodromic deformations, but the system can be mapped to an isospectral one which fixes the curve and doesn't involve motion on the moduli space, the Whitham deformations, which correspond to RG flow [113]. This is also physically expected, as the SW curve is a feature of the un-Omega-deformed theory.

1.8 Defects

Defects in a QFT are a set of boundary conditions on the fields and boundary couplings that one imposes on sub-manifolds, fitting, as such, very naturally in the scheme of functional interpretations of QFT [249], like in the Atiyah-Segal formalisation [8, 250]. The AGT correspondence can be extended to defects. On the $d = 4$ gauge theory side, surface operators assign a singular boundary condition to the field normal to the defect [119, 121],

$$A = a\mathrm{d}\phi + ...,$$

where $a \in \mathfrak{t}$ specifies the residual gauge symmetry on the defect by its commutant and ϕ winds around the defect – for instance, the surface can be at $(z_1, z_2 = 0) \subset \mathbb{C}^2$ so $z_2 = re^{i\phi}$ – as well as the monodromy around D by the associated Levi subgroup $L \subset G$. Besides a, the magnetic charge of the defect

$$\exp\left\{\frac{ib}{2\pi}\int_D F\right\}$$

has to be specified. The two parameters are packed in a single complex parameter $\eta \in Q^\vee$ in the coroot lattice. So called *full* surface operators, with $L = T$ the Cartan, can have their monodromy twisted by a central element of $Z(G) \cong P^\vee/Q^\vee$. In fact, they generate the one-form symmetry of super Yang-Mills, which is valued in $Z(G)$ [98, 145].

The surface operator can be seen as an improperly quantised Wilson loop, as $\int_D F = \oint_{\partial D} A$. Therefore, to restore single-valuedness suggests to introduce

$$\tau_\lambda(\eta, \sigma|t) = \sum_{n\in Q^\vee} e^{4\pi i\eta\cdot n} Z(\sigma + \lambda + n, m|t)$$

where $\lambda \in Z(G)$ additionally shifts the scalar vev. This is exactly the Kiev Ansatz. Computations from blowup relations also imply that a surface defect has this form [224].

I obtained Toda-like equations for the RG-flow of these operators, which is the physical interpretation of differential equations they satisfy in t. This system is the radial reduction of a $d = 2$ Toda lattice on the cylinder $\mathbb{C}^\times$, and it arises naturally as the tt^* equations [58] for a Landau-Ginzburg model describing complex deformations of a $Z(G)$ singularity, miniscule flag manifolds [162].

This is explained by looking at a different way to introduce a defect. Rather than purely from the $d = 4$ side, a half-BPS surface defect can be seen as a $d = 2$ theory coupled to the $d = 4$ bulk – a sigma-model. More precisely, this is a $\mathcal{N} = (2, 2)$ $d = 2$ GLSM describing maps $D \to G/L$. For a full surface defect, the target space $G/L = G/T$ is a complete flag variety. Its Hori-Vafa mirror is the GLSM.

From the class $\mathcal{S}$ construction, these defects are $M2$ branes probing the geometry, with pointlike support on the the CFT side in AGT.

1.9 Topological string

Topological string theory on a Calabi-Yau three-fold X encodes Gromov-Witten invariants in the following way. Seen as a sigma model of Riemann surfaces into the target manifold, the partition function can be written as a genus expansion

$$\log Z_X(t, g_s) = \sum_{g\geq 0} g_s^{2g-2} F_g(t) = F(t, g_s)$$

in terms of the genus g free energies $F_g(\mathbf{t})$, where t is a basis of $H^2(X, \mathbb{Z})$. This is a perturbative expansion, as membrane effects $\mathcal{O}(e^{-1/g_s})$ are hidden, and exact calculations can provide the non-perturbative completion. We ignore this issue for now, and reserve the name "perturbative" for something else. The structure of the free energies is such that

$$F_g(t) = \begin{cases} \frac{1}{3!} a_{ijk} t_i t_j t_k + \sum_d N_0^d e^{-d\cdot t}, & g = 0 \\ b_i t_i + \sum_d N_1^d e^{-d\cdot t}, & g = 1 \\ C_g + \sum_d N_g^d e^{-d\cdot t}, & g \geq 2 \end{cases}$$

where a_{ijk} are classical triple intersections in $H^2(X, \mathbb{Z})$ and C_g is the constant map contribution to the free energy. We will call the all-genus sum involving a_{ijk}, b_i and C_g the perturbative part, and distinguish it from the BPS part of the total free energy of the topological string which encodes the rational GW invariants,

$$F^{\mathrm{BPS}}(t, g_s) = \sum_{g\geq 0} \sum_d N_g^d e^{-d\cdot t} g_s^{2g-2} = F(t, g_s) - F^{\mathrm{pert}}(t, g_s)$$

This BPS part be resummed in terms of the integral Gopakumar-Vafa invariants [110]

$$F^{\mathrm{BPS}}(t, g_s) = \sum_{g\geq 0} \sum_d \sum_{w\geq 1} \frac{1}{w} \left(2 \sin \frac{g_s w}{2}\right)^{2g-2} e^{-wd\cdot t}$$

Even though this is a much simpler theory than any full string theory, calculation remains challenging, as we rarely have a firm computational grip of the required invariants. Dualities may, however, simplify the theory enough for direct calculation to be possible - we have elsewhere commented on the A-model partition function on the resolved conifold being given by Chern-Simons theory on S^3, the underlying mechanism being the geometric transition between the resolved and deformed conifold. The generalisation of this argument to toric (and therefore noncompact) Calabi-Yau threefolds is the topological vertex [1]. On the B-model side, we can use topological recursion [52, 84].

These methods have a two-parameter refinement [150], which will on suitable local Calabi-Yau threefolds turn out to correspond to super Yang-Mills theory on the Omega-background [262]. In refined setting, the Gopakumar-Vafa invariants are be generalized to refined BPS invariants $N^d_{j_L,j_R}$ which depend on spins j_L, j_R. They are integral and are related to the usual Gopakumar-Vafa invariants as

$$\sum_{j_L,j_R} \chi_{j_L}(q)(2j_R+1)N^d_{j_L,j_R} = \sum_{g\geq 0} n^d_g \left(q^{1/2}-q^{-1/2}\right)^{2g},$$

where

$$\chi_j(q) = \frac{q^{2j+1}-q^{-2j-1}}{q-q^{-1}}$$

is the $SU(2)$ character for the spin j. Then the BPS part of the refined topological string free energy is given by

$$F^{\mathrm{BPS}}_{\mathrm{ref}}(t,\epsilon_1,\epsilon_2) = \sum_{j_L,j_R}\sum_{w,d\geq 1} \frac{1}{w} N^d_{j_L,j_R} \frac{\chi_{j_L}(q_L^w)\chi_{j_R}(q_R^w)}{(q_1^{w/2}-q_1^{-w/2})(q_2^{w/2}-q_2^{-w/2})} e^{-wd\cdot t},$$

where $q_{1,2} = e^{i\epsilon_{1,2}}$, $q_{L,R} = e^{(\epsilon_1\mp\epsilon_2)/2}$ and the perturbative by

$$F^{\mathrm{pert}}_{\mathrm{ref}}(t,\epsilon_1,\epsilon_2) = \frac{1}{\epsilon_1\epsilon_2}\left(\frac{1}{6}a_{ijk}t_it_jt_k + \left(4\pi^2-(\epsilon_1+\epsilon_2)^2\right)b_i^{\mathrm{NS}}t_i\right) + b_it_i$$

where the b_i^{NS} can be obtained by using mirror symmetry as in [279], so that the full refined topological string free energy is their sum, and has an expansion

$$\begin{aligned} F_{\mathrm{ref}}(t,\epsilon_1,\epsilon_2) &= F^{\mathrm{pert}}_{\mathrm{ref}}(t,\epsilon_1,\epsilon_2) + F^{\mathrm{BPS}}_{\mathrm{ref}}(t,\epsilon_1,\epsilon_2) \\ &= \sum_{n,g\geq 0} (\epsilon_1+\epsilon_2)^{2n}(\epsilon_1\epsilon_2)^{g-1}F_{n,g}(t) \end{aligned}$$

The coefficients $F_{n,g}(t)$ can also be obtained using the holomorphic anomaly equations. Also, there is a subtlety regarding shifting the Kähler parameters t by a certain **B**-field which ensures some pole cancellations, but this can be glossed over as for local $\mathbb{P}^1\times\mathbb{P}^1$, which is mostly the example to keep in mind in the whole introduction, it can be ignored [116, 192]. There are two notable limits. First of all,

$$F_{\mathrm{ref}}(t,\epsilon_1 = g_s,\epsilon_2 = -g_s) = F(t,g_s)$$

gives us back the unrefined topological string. Secondly, the Nekrasov-Shatashvili limit

$$F^{\mathrm{NS}}(t,\hbar) = \lim_{\epsilon_1\to 0}\epsilon_1 F_{\mathrm{ref}}(t,\epsilon_1,\hbar)$$

can be expressed in terms of a new "gauge coupling" $\hbar$ as

$$F^{\mathrm{NS}}(t,\hbar) = \frac{1}{6\hbar}a_{ijk}t_it_jt_k + b_i^{\mathrm{NS}}t_i\hbar + \sum_{j_L,j_R}\sum_{w,d} N^d_{j_L,j_R}\frac{\sin\frac{\hbar w}{2}(2j_L+1)\sin\frac{\hbar w}{2}(2j_R+1)}{2w^2\sin^3\frac{\hbar w}{2}}e^{-wd\cdot t}.$$

It has the expansion

$$F^{\mathrm{NS}}(t,\hbar) = \sum_{n\geq 0} F^{\mathrm{NS}}_n(\mathbf{t})\hbar^{2n-1}.$$

This is the limit featured in ABJM theory. There is an important behaviours which links the refined topological string to $d = 5$ $\mathcal{N} = 1$ gauge theory on the Omega background, provided X engineers it,

$$Z_X(t, \epsilon_1, \epsilon_2) = Z^{\text{Nek.}}(\epsilon_1, \epsilon_2, a, m|q)$$

with an appropriate dictionary between the vevs a and masses m and the Kähler parameters, checked for the self-dual case in [83, 139, 148, 149] and refined [262], although the worldsheet description of the refinement is not yet known. This can be understood in terms of M-theory on $X \times S^1 \times \mathbb{R}_{\epsilon_1,\epsilon_2}$, which reduces to the topological string [74]. The appearance of the 5 dimensional theory is not a surprise if one considers that the toric diagram of X is the same as the (p, q)-fivebrane web in type IIB which realises the same theory.

1.10 Pivot to 3d

1.10.1 Warm-up: GOV correspondence

We note also that the effect of including points is We recall the Gopakumar-Ooguri-Vafa correspondence, which relates $SU(N)$ Chern-Simons theory on S^3 with level k to the closed topological string A-model on the resolved conifold with $g_s = \frac{2\pi}{k+N}$ and complexified Kähler volume of the exceptional divisor of $t = iNg_s = \frac{2\pi i N}{k+N}$. This is a Large N duality, such that $g_s = g_{CS}^2$, with the t' Hooft parameter $\lambda = g_{CS}^2 N$. The result can be verified by writing the exact result

$$Z_{CS}(S^3) = \frac{e^{i\frac{\pi}{2}(N-1)N}}{(k+N)^{N/2}}\sqrt{\frac{k+N}{n}}\prod_{s=1}^{N-1}(2\sin\left(\frac{s\pi}{k+N}\right))^{N-s} = \exp\left\{-\sum_{g\geq 0}\lambda^{2g-2}F_g(t)\right\}$$

and recognizing the conifold free energy [110, 237]. The underlying mechanism of this duality is the geometric transition, an extremal transition connecting two different components of the Calabi-Yau moduli space using a degeneration. This is secretly a closed-open duality, since it can be viewed as a topological sector of open type IIA string theory on N D6 branes wrapping the S^3 inside a deformed conifold being related to the closed A-model on the resolved conifold with flux through the $\mathbb{P}^1$[20]. The branes dissolve into the geometry, and we obtain a gauge-gravity duality.

We note that, besides the deep geometric intuitions behind it, this is a beautiful example of a duality in the sense of relating two differing regimes.

1.10.2 M2 brane worldvolume theory and the ABJM matrix model

The duality above is of considerable interest, yet inexact. To apply the machinery of equivariant localization, we instead look at supersymmetric theories. The

[20] Actually, much can be said about the relation of the open topological string and Chern-Simons theory. The precise link was given by [277], relating open topological strings on the cotangent space of a 3-manifold with N Lagrangian branes wrapping the zero-section to the Chern-Simons theory on the 3-manifold itself. It forms the backbone of the topological vertex formalism, which in principle allows computation of exact topological string partition function for any local Calabi-Yau described by a brane web.

worldvolume theory of M2 branes will naturally be a 3-dimensional theory, with various amounts of supersymmetry depending on the configuration. On the other hand, if a duality linking to gauge theory exists, we expect a 5-dimensional theory with an S^1 fibration, corresponding to the lift to M-theory. This theory of branes was long sought after. Why is M-theory harder to deal with than type II theories? Consider $d = 11$ supergravity as a low-energy model. It has black brane solutions corresponding to $M2$ and $M5$ branes. By counting microstates from entropy calculations, stacks of N such branes have degrees of freedom growing as [177]

$$N\,M2 \sim N^{3/2}, \quad N\,M5 \sim N^3$$

None of these agree with any gauge theory, which scales at large N as N^2, the number of independent components of a unitary matrix, and which agrees with how D-brane microstates scale [176]. $M5$ branes have been the focus of the other parts of this work, as they feature in class $\mathcal{S}$ theories. Here we are interested in the $M2$ brane.

The initial attempt was BLG theory [15, 16, 123], which involved an (almost) 2-Lie algebra structure, a trilinear operator on the scalar fields satisfying a higher analogue of Jacobi's identity, which came about from attempts to generalise Nahm's equation of $D2-D4$ intersections to $M2-M5$ intersections [18]. It was recognised [185] that this theory described the worldvolume theory of 2 $M2$ branes probing a $\mathbb{C}^4\mathbb{Z}$-singularity. As far as I'm aware, the "higher" Lie algebraic datum turned out the be a red herring. The BLG theory also featured maximally supersymmetric, ie $\mathcal{N} = 8$, conformal super Chern-Simons theory with matter, and this led to successful generalisation.

The particular theory is ABJM, a maximally supersymmetric conformal Chern-Simons theory with gauge group $U_k(N_1)\times U_{-k}(N_2)$ and matter in the bifundamental [3, 130, 132]. The target space is $\mathbb{C}^4/\mathbb{Z}_k$, and if $N_1 = N_2 = N$, the theory describes a stack of N M2 branes stuck the A_k singularity; otherwise, we're looking at a system of $\min(N_1, N_2)$ M2 branes and $|N_1 - N_2|$ fractional branes [2]. If $k = 1, 2$, the amount of supersymmetry enhances to $\mathcal{N} = 8$ from the usual $\mathcal{N} = 6$, and for $k = 2$ and $N_1 = N_2 = 2$ we reduce to the BLG model. It is possible to also introduce FI terms as well as masses to the bifundamental chiral multiplets while preserving $N = 2$ supersymmetry by turning on the vevs of the background vectormultiplets of the flavor symmetries [90], but more on that in the more specialised sections.

Imamura and Kimura [146] showed how more general circular quiver superChern-Simons theories with bifundamental matter describe stacks of branes.

$$\cdots\cdots\; U(N_{i-1})_{k_{i-1}} \rightleftarrows U(N_i)_{k_i} \rightleftarrows U(N_{i+1})_{k_{i+1}} \;\cdots\cdots$$

For these kinds of theories, supersymmetric localisation [22, 163] can be used to calculate the partition functions exactly. For instance, for the theory on S^3, the rules are such that for each node we get an integration with a

$$U(N_i)_{k_i} \rightsquigarrow \frac{1}{N_i!}\int \frac{\mathrm{d}^{N_i}\sigma_j}{(2\pi)^{N_i}} \frac{e^{\frac{ik_i}{4\pi}\sum_j \sigma_j^2}\prod_{p<q}^{N_i}\left(2\sinh\frac{\sigma_p-\sigma_q}{2}\right)^2}{\cdots}$$

factor in the numerator and for each bifundamental

$$U(N_i)_{k_i} \rightleftarrows U(N_{i+1})_{k_{i+1}} \quad \rightsquigarrow \frac{\cdots}{\prod_{p=1}^{N_i}\prod_{q=1}^{N_{i+1}} 2\cosh\frac{\sigma_p^{(1)}-\sigma_q^{(2)}}{2}}$$

The brane configuration whose worldvolume these theories describe can be presented in type IIB string theory in terms of stacks of N_i $D3$ branes and $(1, n_i)5$-branes as [3, 23, 175] The Chern-Simons level of the node corresponding to the i-th stack is

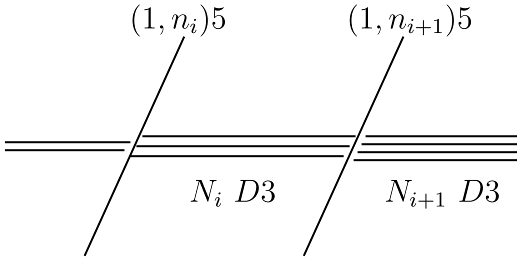


Figure 1.3: The type IIB setup for a $\mathcal{N} = 3$ super Chern-Simons necklace quiver.

the difference of the $D5$ stacks of the $(p, q)5$-branes they are suspended on. This realisation makes it apparent that we need

$$\sum_i k_i = 0.$$

In any case, the rules outlined above realise the partition function of stacks of $M2$ branes to a matrix model. The ABJM matrix model for $U(N)_k \times U(N + M)_{-k}$ is, ignoring some overall signs,

$$Z_{k,M}(N) = \frac{1}{N!(N+M)!} \int \frac{\mathrm{d}^N\sigma}{(2\pi)^N} \frac{\mathrm{d}^{N+M}\tilde{\sigma}}{(2\pi)^{N+M}} e^{\frac{ik}{4\pi}\left(\sum_j \sigma_j^2 - \sum_j \tilde{\sigma}_j^2\right)} \frac{\prod_{p<q}^{N}\left(2\sinh\frac{\sigma_p-\sigma_q}{2}\right)^2 \prod_{p<q}^{N+M}\left(2\sinh\frac{\tilde{\sigma}_p-\tilde{\sigma}_q}{2}\right)^2}{\prod_p^N \prod_q^{N+M}\left(2\cosh\frac{\sigma_p-\tilde{\sigma}_q}{2}\right)^2}$$

Consider $M = 0$ for now. We will use this case to illustrate the Fermi gas formalism. Using the Cauchy determinant formula,

$$\frac{\prod_{p<q} 2\sinh\frac{x_p-x_q}{2} \prod_{p<q} 2\sinh\frac{y_p-y_q}{2}}{\prod_{p,q} 2\cosh\frac{x_p-y_q}{2}} = \det_{p,q} \frac{1}{2\cosh\frac{x_p-x_q}{2}}$$

we may write

$$Z_{k,0}(N) = \frac{1}{N!^2} \int \frac{\mathrm{d}^N\sigma}{(2\pi)^N} \frac{\mathrm{d}^N\tilde{\sigma}}{(2\pi)^N} \det_{p,q}\left[\frac{e^{\frac{ik}{4\pi}\sigma_p^2 - \frac{ik}{4\pi}\tilde{\sigma}_q^2}}{2\cosh\frac{\sigma_p-\tilde{\sigma}_q}{2}}\right] \det_{p,q}\left[\frac{1}{2\cosh\frac{\sigma_q-\tilde{\sigma}_p}{2}}\right]$$

We can use the Andréief or Gram or Heine identity next to get "rid" of one integration,

$$\int \prod_{i=1}^{N} \mathrm{d}\mu(x_i) \det_{i,k}[f_i(x_k)] \det_{j,k}[g_j(x_k)] = N! \det_{i,j}\left[\int \mathrm{d}\mu(x) f_i(x) g_j(x)\right]$$

leading to

$$Z_{k,0}(N) = \frac{1}{N!} \int \frac{\mathrm{d}^N\sigma}{(2\pi)^N} \det_{p,q}\left[\frac{\mathrm{d}\tilde{\sigma}}{2\pi} \frac{e^{\frac{ik}{4\pi}\sigma_p^2 - \frac{ik}{4\pi}\tilde{\sigma}_q^2}}{2\cosh^2\frac{\sigma_p-\tilde{\sigma}_q}{2}}\right]$$

The Fermi gas formalism [192] interprets the determinand in terms of a quantum mechanical operator,

$$Z_{k,0}(N) = \frac{1}{N!}\int \frac{\mathrm{d}^N x}{(2\pi)^N} \det_{i,j} \langle x_i|\,\hat\rho\,|x_j\rangle\,, \quad \tilde\rho = \frac{1}{2\cosh\frac{\hat p}{2}} e^{-\frac{i}{4\pi k}\hat x^2} \frac{1}{2\cosh\frac{\hat p}{2}} e^{\frac{i}{4\pi k}\hat x^2}$$

where $[\hat x, \hat p] = 2\pi i k$ are canonical position and momentum operators, and the identity essentially follows from

$$\langle x_i|\,\frac{1}{2k\cosh\frac{\hat p}{2}}\,|x_j\rangle = \int \frac{\mathrm{d}p}{2\pi k} \frac{e^{\frac{i}{2\pi k}(x_i - x_j)p}}{2\cosh\frac{p}{2}}$$

The name Fermi gas follows from the observation that if $\hat\rho = e^{-\hat H}$ then this is the partition function of N free fermions with

$$\hat H = \log 2\cosh\frac{\hat p}{2} + \log 2\cosh\frac{\hat x}{2}$$

as the one-particle Hamiltonian at $\beta = 1$, which can be seen to reduce to the harmonic oscillator in a scaling limit of small $\hat p, \hat x$. Crucially, the inverse of $\hat\rho$ turns out to be the Newton polygon corresponding to the mirror of the topological string on local $\mathbb{P}^1\times\mathbb{P}^1$ with the two Kähler parameters, the sizes of the $\mathbb{P}^1$'s, equal. Under T-duality, this is exactly the Hanany-Witten geometry which realises the Coulomb branch of $\mathcal{N} = 2$ $d = 4$ $SU(2)$ super Yang-Mills as the IR theory. To see this, note that under a similarity transformation $\hat\rho \mapsto e^{-\frac{i}{4\pi k}\hat p^2}\hat\rho e^{\frac{i}{4\pi k}\hat x^2} = (2\cosh\hat p/2)^{-1}(2\cosh\hat x/2)^{-1}$,

$$\hat\rho^{-1} = e^{\pi i k/2} e^{\hat p/2 + \hat x/2} + e^{\hat p/2 - \hat x/2} + e^{-\pi i k/2} e^{-\hat p/2 + \hat x/2} + e^{-\hat p/2 - \hat x/2}$$

The road to the TS/ST correspondence as well as the identification with the topological string comes from the grand partition function,

$$\Xi_{k,0}(\mu) = \sum_{N\geq 0} e^{N\mu} Z_{k,0}(N) = \sum_{d_n \geq 0}\prod_{n=1}^{\infty} e^{\mu d_n n}\frac{1}{d_n!}\left(\frac{(-1)^{n-1}}{n}\,\mathrm{tr}\,\hat\rho^n\right)^{d_n}$$
$$\prod_{n=0}^{\infty}\exp\left(\frac{(-1)^{n-1}}{n} e^{\mu n}\,\mathrm{tr}\,\hat\rho^n\right) = \exp\mathrm{tr}\log(1 + e^{\mu}\hat\rho) = \det(1 + e^{\mu}\hat\rho)$$

where the second equality comes from decomposing the permutations coming from the integral of an $N \times N$ determinant into cycles. The end result relates the zero-divisor of $\Xi_{k,0}(\mu)$ to the quantisation of the one-particle Hamiltonian $\hat H$.

Of more interest is the case of unequal ranks, $M > 0$, which describes a theory with fractional branes. There are two different approaches, which we mention in brief.

The first of these is called the closed-string formalism [12, 140, 200]. Here, the goal is to "absorb" the fractional branes into the background geometry, modifying the $M = 0$ operator $\hat\rho$ to $\hat\rho_M$. It can be shown that, if we define the potential

$$V(x) = \left(e^{\frac{x}{2}} + e^{\frac{x}{2} + \pi i M}\right)^{-1} \prod_{s=-\frac{M-1}{2}}^{\frac{M-1}{2}} \tanh\frac{x + 2\pi i s}{2k}$$

then the matrix model, normalised by the $N = 0$ partition function[21], becomes

$$\frac{Z_k(N, N+M)}{Z_k(0, M)} = \frac{1}{N!}\int \frac{\mathrm{d}^N x}{(4\pi k)^N}\prod_{p<q}^{N}\tanh^2\frac{x_p - x_q}{2}\prod_p V(x_p) = \frac{1}{N!}\int \frac{\mathrm{d}x}{2\pi}\det_{i,j}\langle x_i|\,\hat\rho_M\,|x_j\rangle$$

with $\langle x|\,\hat\rho_M\,|y\rangle = V^{1/2}(x)(2k\cosh\frac{x-y}{2k})V^{1/2}(y)$. The explicit inverse can be obtained by using Fadeev's quantum dilogarithm [167], and reduces to

$$\hat\rho_M^{-1}(-1)^M = e^{\pi ik/2 - 2\pi iM}e^{\hat p/2+\hat x/2} + e^{\hat p/2-\hat x/2} + e^{-\pi ik/2}e^{-\hat p/2+\hat x/2} + e^{-\hat p/2-\hat x/2}$$

which can be seen to be the same curve with a different Kähler parameter. The spectral determinant is now

$$\Xi_{k,M}(\mu) = \sum_{N\geq 0} e^{\mu N} Z_k(N, N+M) = Z_k(0, M)\det(1 + e^{\mu}\hat\rho_M)$$

The other way, called the open-string formalism [200] is to keep $\hat\rho$ the same and calculate the corrections,

$$\Xi_{k,M}(\mu) = \Xi_{k,0}(\mu)\det(H(M))$$

where $H(M)$ is a finite-dimensional matrix, whose elements turn out to be computable recursively in terms of traces $\operatorname{tr}\hat\rho^n$ using the TWPY formalism [243, 266].

I would like to thank my co-worker Tomoki Nosaka for explaining these things to me. A good review is [131].

1.10.3 The grand potential and the nonperturbative topological string

ABJM can be seen to provide the exact, nonperturbative completion of the topological string. In M-theory, the fundamental string and the D-brane descend from the same object, the M2 brane, and the two effects of worldsheet and D2 instantons get unified in terms of M2 branes wrapping Calabi-Yau three-cycles [20]. In the 't Hooft limit, with k fixed and N large, ABJM is dual to M-theory on $AdS \times (S^7/\mathbb{Z}_k)$. As explained in [130], $M2$ branes on $S^7/\mathbb{Z}_k$ can wrap the three-cycle $S^3/\mathbb{Z}_k$ coming from the S^3 fibration of S^7, and this corresponds to worldsheet instantons on $\mathbb{P}^1$ in type IIA, but they can also wrap 3-cycles in $H_3(S^7/\mathbb{Z}_k, \mathbb{Z}) = \mathbb{Z}_k$ which corresponds to "$D2$ instantons" wrapping the Lagrangian submanifold $\mathbb{RP}^3$. This connection to the topological string can be made concrete by defining the grand potential,

$$\sum_{n\in\mathbb{Z}} e^{J(\mu+2\pi in,k)} = \Xi_{k,0}(\mu)$$

which we here consider just for $M = 0$. The same potential may be obtained by $J^{\text{naive}}(\mu, k) = \log\Xi_{k,0}(\mu)$ and dropping an oscillatory part which restores the $2\pi i$ periodicity in μ. The grand potential splits as

$$J(\mu, k) = J^{\text{pert}}(\mu^{\text{eff}}, k) + J^{\text{np}}(\mu^{\text{eff}}, k)$$

[21] This is equal to the $U(M)$ Chern-Simons partition function, $Z_k(0, M) = k^{-M/2}\prod_{s=1}^{M-1}(2\sin\frac{\pi s}{2})^{M-s}$ [274]

where the relation between μ and μ^{eff} will be explained shortly. The perturbative part is cubic in μ [80, 126, 134, 192, 200]

$$J^{\text{pert}}(\mu,k) = \frac{C_k}{3}\mu^3 + B_k\mu^2 + A_k,$$
$$C_k = \frac{2}{\pi^2 k^2}, \quad B_k = \frac{1}{3k} - \frac{k}{12},$$
$$A_k = \frac{2\zeta(3)}{\pi^2 k}\left(1 - \frac{k^3}{16}\right) + \frac{k^2}{\pi^2}\int_0^\infty \mathrm{d}x \frac{x}{e^{kx}-1}\log\left(1-e^{-2x}\right)$$

For integral k, the last integration can be performed exactly. The effect of $M \neq 0$ here is to modify only B_k quadratically in M. Because $Z_k(N)$ can be recovered from $J(\mu,k)$, the purely perturbative part gives

$$Z_k^{\text{pert}}(N) = \int_{i\mathbb{R}} \frac{\mathrm{d}\mu}{2\pi i} e^{J^{\text{pert}}(k,\mu)-\mu N} = C_k^{-1/3} e^{A_k} \mathrm{Ai}\left(C_k^{-1/3}(N - B_k)\right)$$

which confirms the $N^{3/2}$ scaling of the free energy along with corrections, as $\mathrm{Ai}(x) \sim e^{-\frac{2}{3}x^{3/2}} 2^{-1}\pi^{-1/2}x^{-1/4}$ for $x \to +\infty$, or indeed from a saddle-point analysis which gives $\mu \sim \sqrt{N/C_k}$. In fact, the actual coefficient tells us slightly more, including the volume of the compact manifold Y in the gravity dual $AdS^4 \times Y$ [136].

The non-perturbative part is given by

$$J^{\text{np}}(\mu,k) = F(t^{\text{eff}}, g_s) + \frac{1}{2\pi i}\frac{\partial}{\partial g_s}\left(g_s F^{\text{NS}}(\frac{t^{\text{eff}}}{g_s}, \frac{1}{g_s})\right)$$

for local $\mathbb{P}^1 \times \mathbb{P}^1$, where the two Kähler parameters are $t_{\pm}^{\text{eff}} = 4\mu^{\text{eff}}/k \mp \pi i$. Note that this explains the origin of the effective chemical potential μ^{eff}. Namely, as seen from the Fermi gas formalism, ABJM theory quantises the mirror curve to local $\mathbb{P}^1 \times \mathbb{P}^1$. The periods of the mirror curve $H(x,p) = -1 + e^x + e^p + z_1 e^{-x} + z_2 e^{-p} = 0$ are related to the Kähler parameters via the quantum mirror map which relates

$$Q_{\pm} = e^{-t_{\pm}^{\text{eff}}/g_s} = z_{\pm} e^{\Pi_A}$$

and $z_{\pm}$ can be identified with the bare Kähler parameters as $z_{\pm} = e^{-t_{\pm}/g_s}$. This total free energy satisfies the HMO pole-cancellation mechanism [132], so that order by order in e^{-T} there are no poles in g_s. Both the NS part of the grand potential and the effective Kähler parameters t^{eff} encode nonperturbative terms of the form e^{-t/g_s}, coming from the quantum periods from the B-model side.

1.11 The TS/ST/tau correspondence

The results of this section build up on and generalise the correspondence of the ABJM theory with topological string on local $\mathbb{P}^1 \times \mathbb{P}^1$, which engineers $d = 4$ $\mathcal{N} = 2$ super Yang-Mills. My attempt was to explain the correspondence by building it up using an easy toy model, mainly based on the work of Marcos Mariño and colleagues.

1.11.1 TS/ST

Consider a B-model mirror curve for a local Calabi-Yau threefold X of the form $\{-w_1 w_2 + W_X(e^p, e^x) = 0\} \subset \mathbb{C}^2 \times \mathbb{C}^2_x$. Here and elsewhere, we restrict our focus to W_X Newton polynomials

$$W_X(e^p, e^x) = \sum_{i,j} \mu_{i,j} e^{ip+jx}$$

We consider only the zero-section

$$W_X(e^p, e^x) = 0$$

This is an algebraic curve in e^p, e^x. As of date of writing, the theory we describe is exactly known for several cases, some of which are infinite families, but not in full generality. By exactly known, I mean exactly calculable, a property which currently hinges on several properties of Fadeev's quantum dilogarithm – eiven enough numerical ingenuity, however, no obstacle to general calculations is immediately obvious. *In specialibus generalia quaerimus*, therefore we restrict to the genus-1 case, the first nontrivial one[22]. The concrete example I have in mind that of X as local $\mathbb{P}^2$, ie $X = K_{\mathbb{P}^2} = \mathrm{tot}(\mathcal{O}(-3) \to \mathbb{P}^2)$[23], and

$$W_X(e^p, e^x) = e^p + e^x + e^{-p-x} + z^{-\frac{1}{3}} = 0$$

This is an elliptic curve with a single modulus corresponding to the interior point. To motivate the discussion, let us calculate the classical periods

$$\Pi_{A,B} = \oint_{A,B} p\, \mathrm{d}x, \quad p_\pm = \log\left\{-z^{-\frac{1}{3}} - e^x \pm \sqrt{(z^{-\frac{1}{3}} + e^x)^2 - 4e^x}\right\}$$

For now, we consider the regime $z \to 0$, which means

$$e^p + e^x + e^{-p-x} \to \infty$$

under appropriate scaling. Correspondingly, let us identify the A-cycle with a loop around the vanishing e^x. Consequently, the A-period becomes, with a rescaling of convenience[24],

$$\Pi_A = \frac{3}{2} \operatorname*{Res}_{e^x=0} [p_+ - p_-] = \log z - 6z + 45z^2 - 560z^3 + \mathcal{O}(z^4)$$

The B-period is harder to evaluate – the reader can find it in [278]. It yields

$$\Pi_B = \log^2 z + 2\log z \left(-6z + 45z^2 - 560z^3 + ...\right) - 18z + \frac{423}{2} z^2 - 2672z^3 + \mathcal{O}(z^4)$$

Armed with the periods, we can calculate the genus-0 prepotential $F_0(t)$ defined implicitly via

$$\frac{\mathrm{d}F_0}{\mathrm{d}t} = \frac{1}{6}\Pi_B, \quad t = \Pi_A$$

[22]Later we comment on genus 0 as a limiting case of a more general family.
[23]Calabi-Yau, since $c_1(\mathcal{O}(-3)) + c_1(\mathbb{P}^2) = -3 + 3 = 0$.
[24]Periods are projective.

to find

$$\begin{aligned} F_0(t) &= \frac{t^3}{18} + 3e^{-t} - \frac{45}{8}e^{-2t} + \frac{244}{9}e^{-3t} - \frac{12333}{4^3}e^{-4t} \\ &= \frac{1}{6}a_{111}t^3 + \sum_{d\geq 1} N_{d,0}e^{-dt} \end{aligned}$$

where the numbers $N_{d,0} \in \mathbb{Q}$ are the genus 0 Gromov-Witten invariants of local $\mathbb{P}^2$, a list of which can be found in the appendix of [62]. Properly, these numbers are of interest in the A-model of the topological string, and what we calculate is maps from a genus g Riemann surface Σ_g to the Calabi-Yau 3-fold X. In this topological string theory limit, the maps are holomorphic, and the t is a holomorphic Kähler modulus, in this instance the only one. What we've done is calculate the genus 0 version of the more general free energy in the large radius limit,

$$F_g(t) = \sum_{\text{hol.} f:\Sigma_g \to X} e^{-\int_{\Sigma_g} f^*(\omega)} = \sum_{d\geq 1} N_{d,g}e^{-dt}$$

with $\omega \in H^2(X;\mathbb{C})$ the Kähler class. What is, of course, interesting about this calculation is that it involved periods of an elliptic curve associated to X as a section of its *mirror* Calabi-Yau. We've seen this here in a very concrete manner, as z is the complex modulus of the mirror CY, related nonlinearly to the Kähler modulus t. It's important at this point to stress that the large radius limit was involved, which was $t >> 0$,$z << 1$. The individual prepotentials can be organised as an all-genus sum as in section 1.9

$$F(g_s, t) = \sum_{g\geq 0} F_g(t) g_s^{2g-2}$$

A key insight comes from the combinatorial argument

$$F_g(t) \sim (2g)!, \quad g >> 1$$

which indicates that the above sum is an asymptotic series. General arguments on resurgence give rise to the expectations of this being rather a trans-series:

$$F(g_s, t) = \sum_{g\geq 0} F_g(t) g_s^{2g-2} + \mathcal{O}(e^{-1/g_s})$$

Several remarks are in order. Firstly, only the first term has, as yet, a combinatorial interpretation in terms of counting holomorphic curves (GW invariants). The second term is non-perturbative and counts BPS effects. Secondly, the general form of the series strongly suggests we view the higher genus corrections as being, in a sense, *quantum corrections*. We are interested in curve quantisation in the sense described in [120]. That is, to the elliptic curve we associate the operators

$$\begin{aligned} x, p \mapsto \hat{x}, \hat{p}, \quad [\hat{x}, \hat{p}] = i\hbar \cdot \mathrm{id},\ \hbar > 0 \\ W_X(x,p) = 0 \mapsto W_X(x, -i\hbar\partial_x)\psi(x,\hbar) = 0 \end{aligned}$$

The idea is to repeat the previous calculation to get all the higher-genus corrections, after an appropriate identification of g_s and the Planck constant. At this point we can solve it perturbatively using a WKB-type Ansatz

$$\psi(x,\hbar) \sim \frac{1}{\sqrt{p(x,\hbar)}} e^{\frac{i}{\hbar}\int^x \mathrm{d}\tilde{x} p(\tilde{x},\hbar)},$$
$$p(x,\hbar) = p(x) + \sum_{n\geq 1} \hbar^{2n} p_n(x)$$

The leading term is just the classical period, from $W_X(x, p(x)) = 0$, so it makes sense to define quantum periods using $p(x,\hbar)$,

$$\Pi_{A,B}(z,\hbar) = \oint_{A,B} \mathrm{d}x p(x,\hbar) = \Pi_{A,B}(z) + \mathcal{O}(\hbar)$$

and consequently

$$\frac{\mathrm{d}F_\hbar}{\mathrm{d}t} = \frac{1}{6}\Pi_B(z,\hbar), \quad t = \Pi_A(z,\hbar)$$

This almost works. In fact, what we get from this process is the prepotential in the Nekrasov-Shatashvili limit [202],

$$F_\hbar(t) = \sum_{n\geq 0} \hbar^{2n} F^{N.S.}(z)$$

which we discuss elsewhere. In a sense, this is orthogonal to the series we wanted. Since $g_s = \epsilon_1$, where $\epsilon_1 = -\epsilon_2$ and $\hbar = \epsilon_1$ where $\epsilon_2 = 0$, we could conclude that in this case,

$$g_s \sim \frac{1}{\hbar}$$

This means that the solution is again perturbative. Note that WKB is in general an asymptotic solution. These issues can, however, be fixed if we recognise that we've not really been doing quantisation, since we had no Hilbert space. In fact, to be less sloppy, we consider both our operator and our state vector.

For the Hilbert space we take the canonical choice $L^2(\mathbb{R})$. The curve, now promoted to an operator we're looking at is of the form

$$W_X(e^{\hat{p}}, e^{\hat{x}}) = \sum_{i,j} \mu_{i,j} e^{i\hat{p}+j\hat{x}}$$

once we adopt Weyl ordering. Here $e^{\hat{x}}, e^{\hat{p}}$ are unbounded, self-adjoint operators on $L^2(\mathbb{R})$, but and it's not clear that their bilinear combination is self-adjoint on the Hilbert space. In general it isn't, however, for our example and many others, it is self-adjoint [116, 186]. This, on its own, is a big surprise. In a sense, a conjecture is lurking here: for W_X which define mirror Calabi-Yaus, the associated operator is self-adjoint on $L^2(\mathbb{R})$. What is interesting is that the associated equations are a difference equations, in our case

$$\psi(x+i\hbar) + \psi(x) + e^{i\hbar}e^x\psi(x-i\hbar) = z^{-1/3}\psi(x)$$

Turning back to $\psi(x,\hbar)$ itself, trying to ensure it's square-integrable leads to the Heisenberg quantisation condition

$$\Pi_B(z,\hbar) = 2\pi\hbar\left(n+\frac{1}{2}\right), n\in\mathbb{N}_0$$

However, we know that, classically, $\Pi_B = \log^2 z + \ldots$. Consider writing $-z^{-1/3} = e^E$. The quantisation condition translates to

$$E_n \sim n^{1/2}$$

and the equation itself to

$$\hat{O}_{K_{\mathbb{P}^2}} \psi_n = e^{E_n} \psi_n, \quad \hat{O}_{K_{\mathbb{P}^2}} = e^{\hat{p}} + e^{\hat{x}} + e^{-\hat{p}-\hat{x}}$$

With this estimate, we come to the crux of the TS/ST correspondence of [117], which equates a strongly-coupled quantum spectral problem to the exact non-perturbative completion of the topological string. Namely, let $\hat{\rho} = \hat{O}^{-1}$ be the *inverse* mirror curve operator. Then, due to the estimate on the eigenvalues of $\hat{O}^{-1}$, for every $l \geq 1$,

$$\mathrm{Tr}_{\mathcal{H}} \, \hat{\rho}^l = \sum_{n \geq 0} e^{-lE_n} < \infty$$

In other words, $\hat{\rho}$ is trace-class. Crucial to the conjecture are the so-called fermionic spectral traces

$$Z(N, \hbar) = \mathrm{Tr}_{\wedge^N \mathcal{H}} \underbrace{\hat{\rho} \wedge \ldots \wedge \hat{\rho}}_{\text{N times}} = \sum_{\{m_l | \sum_l l m_l = N\}} \prod_l \frac{(-1)^{(l-1)m_l} (\mathrm{Tr}_{\mathcal{H}} \, \hat{\rho}^l)^{m_l}}{m_l! l^{m_l}}$$

which naturally arise in the context of ABJM. Then we form the spectral determinant

$$\Xi(\kappa) := \det(\mathbb{1} + \kappa\hat{\rho}) = \prod_n \left(1 + \kappa e^{-E_n}\right) = 1 + \sum_{N \geq 1} Z(N, \hbar) \kappa^N$$

The main claim of [116, 186] is that the spectral determinant is entire in $\kappa \in \mathbb{C}$. As with the rest of the claims of this chapter, this must be established on a case by case basis as a more general theory is lacking.

1.11.2 Conjectures on limiting behaviours

The spectral determinant admits three different limiting behaviours, its behaviour there governed by distinct expansions. The first of these is the large-radius limit. Let $\kappa = e^\mu$. Then, consider the scaling limit

$$\mu \to \infty, \quad \hbar \to \infty, \quad \frac{\mu}{\hbar} := t \text{ fixed}$$

Then in this double scaling limit,

$$\log \Xi(\kappa) \sim \sum_{g \geq 0} F_g(t) \hbar^{2g-2}$$

plus oscillatory contributions, where $F_g(t)$ is the genus-g GW invariant. This corresponds to the large-radius limit of the CY geometry, which corresponded to $z >> 1$ in our toy example. The second conjecture regards the so-called orbifold or dual magnetic limit,

$$N \to \infty, \quad \hbar \to \infty, \quad \frac{N}{\hbar} := \lambda \text{ fixed}$$

under which the coefficients $Z(N,\hbar)$ of $\Xi(\kappa)$ are conjectured to behave as

$$\log Z(N,\hbar) \sim \sum_{g\geq 0} F_g^D(\lambda)\hbar^{2-2g}$$

where

$$F_g^D(\lambda) = c_g\lambda^{2-2g} + \sum_{n\geq 0} F_{n,g}^D\lambda^n$$

are the dual GW invariants, defined around the "conifold point" $\lambda = 0$ where the Kähler parameter of the A model vanishes. The first term is a pole, c_g being a combinatorial factor. This is a resummation of the standard GW invariant generating function. How are we to interpret this from the point of view of our toy geometry, from the B model side? We present a geometric argument first. Using the birational equivalence of an elliptic curve with its Jacobian, $C \cong \mathrm{Jac}(C)$, we can write the curve $x^2y + y^2x + z^{-1/3}xy + 1 = 0$ in its Weierstrass normal form as

$$y^2 = x^3 + ax + b = x^3 - \frac{1+24z}{48z^{4/3}}x + \frac{1+36z+216z^2}{864z^2}$$

and calculate the discriminant

$$\Delta = -16(4a^3 + 27b^2) = -27 - z^{-1}$$

Therefore there is a distinguished point in the moduli space, $z = -1/27$, at which the curve becomes singular. The physical argument is as follows: in terms of z, the Yukawa coupling has a pole

$$\left(\frac{\mathrm{d}t}{\mathrm{d}z}\right)^2 \frac{\mathrm{d}^3F}{\mathrm{d}t^3} = \left(-\frac{1}{z^3} + \frac{18}{z^2} + \frac{378}{z} + ...\right)\left(-\frac{1}{z} + 3z + 63z^2 + ...\right) = \frac{-1}{3z^3(1+27z)} + \mathcal{O}(z)$$

at $z = -1/27$. At this point, the fermions acquire infinite mass, and the theory is purely magnetic. Finally, the third distinguished point on the moduli space of our toy CY is $z \to \infty$, when the curve becomes $y^2 = x^3 + 1/4$ and the CY becomes $\mathbb{C}^3/\mathbb{Z}^3$. This is the "orbifold point", and in fact this is just the expansion in small $\kappa = -z^{-1/3}$ from the original definition of the spectral determinant. Besides the conjectural asymptotics, there is an exact conjecture

$$\Xi_X(e^\mu) = \sum_{n\in\mathbb{Z}} e^{J_X(\mu+2\pi i n,\hbar)}$$

for any such local CY X, where J_X is called the grand potential and is an involved resummation of refined GW invariants along with BPS corrections. For $\hbar = 2\pi$, symmetry enhancement turns the RHS into a theta function, and proofs become possible [259].

1.11.3 Enter Dilog

The main strength of this formalism is the ability to exactly calculate the spectral determinant. Our ability to do so so hinges on being able to invert the mirror curve and obtain $\hat{\rho}$, as well as performing the subsequent integral. To follow the calculation as in [166, 167], we introduce Fadeev's quantum dilogarithm [85, 86], which is morally a deformation of the exponential of the ordinary dilogarithm $\sum_n x^n/n^2$,

$$\phi_b(x) := \exp \int_{\mathbb{R}+i0} \frac{e^{-2ixy}}{4\sinh by \sinh \frac{y}{b}} \frac{dy}{y}, \quad |\operatorname{Im}\{x\}| < \operatorname{Re}\left\{\frac{b+b^{-1}}{2}\right\}$$

which circles around the pole at $y = 0$ to the upper half-plane. The relevant property here is

$$\frac{\phi_b(x - \frac{ib}{2})}{\phi_b(x + \frac{ib}{2})} = 1 + e^{2\pi bx}$$

Of course, we're dealing with an operator with canonically conjugate variables $[\hat{x}, \hat{y}] = i\hbar$. With some hindsight, we should consider for now a linearly transformed set $[\hat{q}, \hat{p}] = 2\pi ib^2$. Then, using Taylor's theorem,

$$e^{\alpha\hat{p}} f(\hat{q}) e^{-\alpha\hat{p}} = f(\hat{q} - 2\pi ib^2\alpha)$$

we can write the quantum dilog's recurrence relation as

$$1 + e^{\hat{q}} = \frac{1}{\phi_b(\frac{\hat{q}}{2\pi b} + \frac{ib}{2})} \phi_b(\frac{\hat{q}}{2\pi b} - \frac{ib}{2}) = e^{\frac{\hat{p}}{2}} \frac{1}{\phi_b(\frac{\hat{q}}{2\pi b})} e^{-\frac{\hat{p}}{2}} e^{-\frac{\hat{p}}{2}} \phi_b(\frac{\hat{q}}{2\pi b}) e^{\frac{\hat{p}}{2}}$$

More strenuous exercise yields the identity we need to invert the mirror curve, namely

$$e^{-\frac{\hat{q}}{6}} \frac{1}{\phi_b(\frac{\hat{q}}{2\pi b} + \frac{2ib}{3})} e^{\frac{\hat{p}}{3}} (1 + e^{-\hat{p}}) e^{-\frac{\hat{q}}{6}} \phi_b(\frac{\hat{q}}{2\pi b} - \frac{2ib}{3}) = e^{\hat{x}} + e^{\hat{p}} + e^{-\hat{x}-\hat{p}} = \hat{\mathcal{O}}$$

Finally, we can calculate the matrix element of the inverse mirror curve,

$$\begin{aligned}
\langle q_1 | \, \hat{\rho} \, | q_2 \rangle &= \langle q_1 | \, F_1(\hat{q}) \frac{e^{\frac{2\hat{p}}{3}}}{1 + e^{\hat{p}}} F_2(\hat{q}) \, | q_2 \rangle \\
&= F_1(q_1) F_2(q_2) \int \frac{\mathrm{d}z}{4\pi^2 b^2} \frac{e^{\frac{2z}{3}}}{1 + e^z} e^{\frac{iz}{2\pi b^2}(q_1 - q_2)} \\
&= F_1(q_1) F_2(q_2) \frac{1}{4\pi b^2} \frac{1}{\cosh\left[\frac{q_1 - q_2}{2b^2} - \frac{i\pi^2}{6}\right]}
\end{aligned}$$

Finally, by the Cauchy determinant formula, we can calculate

$$Z(N, \hbar) = \frac{1}{N!} \int \mathrm{d}^N x \det_{i,j} \langle q_i | \, \hat{\rho} \, | q_j \rangle = \frac{1}{N!} \int \frac{\mathrm{d}^N x}{(2\pi)^N} e^{\hbar \sum_i \operatorname{Re}\{V(x_i)\}} \frac{\prod_{i<j} 2\sinh[2] \frac{x_1 - x_2}{2}}{\prod_{i,j} 2\cosh\left[\frac{x_1 - x_2}{2} - \frac{i\pi^2}{6}\right]}$$

and check the conjectures hold. The result is a deformed $O(2)$ matrix model, and its spectral curve again is a level set of $e^x + e^y + e^{-x-y}$ [283] .

1.11.4 Further comments

Other examples are, conjecturally, all the anticanonical bundles of almost del Pezzo surfaces. There are, in fact, only 16 possibilities, since all such nonlocal CY 3-folds are built from reflective polygons. Indeed, toric geometry may be used here [65, Prop. 8.2.7.]: if we want a global section of $\omega = X_1^{-1}\mathrm{d}X_1 \wedge ... \wedge X_n^{-1}\mathrm{d}X_n$, with $\{X_i\}_i$

the torus coordinates corresponding to a particular choice of basis of the lattice, this is the same as $\mathcal{O}(-\sum_i D_i)$ being trivial, with D_i the toric divisors. This is equivalent to $\sum_i D_i \sim 0$, which is equivalent to the existence of a u such that $\sum_i \langle u, v_i \rangle D_i = 0$, but this finally equivalent to $\langle u, v_i \rangle = 0 \ \ \forall i$, meaning all the v_i are coplanar. Of course, coplanarity ensures non-compactness. The fact that we need to look at anticanonical bundles further restricts this to be a reflexive polytope by the Batyrev-Borisov theorem. These are all genus 1 curves, since they necessarily have a single interior point. However, the conifold may be reached by a limiting procedure [128]. The standard procedure to build mirror curves then follows. For simplicity's sake, we focused on dP_0 here, however, more can be said for $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$. Indeed, in a series of works [44, 45, 46, 231] my coauthors have described the links of the spectral determinant with the tau function of Painlevé III_3, Seiberg-Witten theory of pure $SU(2)$, and further generalisations to $\mathfrak{q}$-difference equations and 5d gauge theory on $\mathbb{R}^4 \times S^1$. In this special case, we can see that quantisation of the curve actually reduces to solving the Pöschl-Teller potential,

$$E\psi = \left(e^{\hat{p}} + e^{-\hat{p}} + e^{\hat{x}} + e^{-\hat{x}}\right)\psi = \psi(x+i\hbar)+\psi(x-i\hbar)+2\cosh x\psi(x) \approx -\hbar^2\psi''+2\cosh x\psi$$

as well as the simple harmonic oscillator when x is small. This can be generalised further to anharmonic oscillators. Namely, we start from the topological string (e^p, e^x), take the geometrical engineering limit to get to the Seiberg-Witten curve (e^p, x) and finally use Argyres-Douglas scaling to get quantum mechanics with anharmonic potentials (p, x). Provided the quantum mirror curve can be inverted, this translates the difficulties of anharmonic potentials to an easier calculation in the topological string, followed by nontrivial limits.

1.11.5 TS/Tau

The main interest in this work is, however, the conjecture

$$\Xi \sim \tau$$

for appropriate polytopes and isomonodromic systems. For X being the local $\mathbb{P}^1 \times \mathbb{P}^1$, it was shown in [46] that the spectral determinant $\Xi(z)$ satisfies the $\mathfrak{q}$-Painlevé III_3 equation in tau form. In this case, Ξ is explicitly given by the ABJM theory, along with a dictionary which links $\mathfrak{q}$-shifts to rank-deformations. By shrinking the M-theory circle, the spectral determinant reduces to the topological string and the $\mathfrak{q}$-difference equation reduces to standard Painlevé III_3 [44] and expresses Seiberg-Witten theory as a Fermi gas, explicitly in terms of a matrix model. Since the large-N expansion calculates the theory in the magnetic frame, one application of this presentation is to probe the multi-monopole point as done in [69].

More generally, the TS/ST correspondence suggests a link between the spectral determinant of the quantum Seiberg-Witten curve and the five dimensional NO partition function [46]. Via geometric engineering [5, 169], M-theory compactified on a local Calabi-Yau threefold defines a five dimensional $\mathcal{N} = 1$ gauge theory, whose Seiberg-Witten curve is identified with the mirror curve, while the topological string partition function gets related to the Nekrasov partition function in a self-dual Omega background [221].

The TS/ST correspondence was extended to higher genus curves [63, 64], which turn out to be related to $G = SU(N)$ $\mathcal{N} = 1$ theory on $\mathbb{R}^4_{\epsilon_1,\epsilon_2} \times S^1$, and it was shown that in this case the TS/tau correspondence holds as well [45]. For X corresponding to $SU(N)$, the $Y^{N,0}$ geometry, a matrix model realisation was found by my co-author Tomoki Nosaka [231] from mass-deformed ABJM.

Chapter 2

Generalized Painlevé equations

2.1 Extending the Painlevé/Gauge correspondence

In the introduction we have seen how the Painlevé/Gauge correspondence means the Kiev Ansatz for $d = 4$ $\mathcal{N} = 2$ $G = SU(2)$ super Yang-Mills theory on a self-dual Omega-background gives the general solution of the corresponding Painlevé equation in tau form.

Interpreted as a surface operator, we can associate two different tau functions τ_0,τ_1 to the pure $SU(2)$ theory, since $|Z(G)| = 2$. The shifts involved mean that $\tau_1(\sigma|t) = \tau_0(\sigma + \frac{1}{2}|t)$, which can be seen as a Bäcklund transformation. It turns out they satisfy PIII_3 in Toda form

$$\partial^2_{\log t} \log \tau_0 = -t^{\frac{1}{2}} \frac{\tau_1^2}{\tau_0^2},$$
$$\partial^2_{\log t} \log \tau_1 = -t^{\frac{1}{2}} \frac{\tau_0^2}{\tau_1^2}.$$

which can be interpreted as surface operator RG flow. The first part of this section is based on joint work with my advisors, Alessandro Tanzini and Giulio Bonelli [42, 43]. We worked to extend this Toda representation of the pure $SU(2)$ Kiev Ansatz to a more general group G. This was possible by starting with the Painlevé/Calogero correspondence, which has a natural extension for any gauge group G, and is known that appropriate Lax pairs of the elliptic Calogero-Moser systems furnish the Seiberg-Witten curves by the SW/Integrable system correspondence. When deautonoimised, they should therefore lift the theory to the Omega-background. The resulting isomonodromic problem on the torus involves the whole root system. This gets simplified if the Inosemtzev limit is taken, which takes the Calogero-Moser problem to the Toda system involving only the extended positive roots of the Langlands dual Lie algebra. At the same time, the torus geometry degenerates to that corresponding to the pure gauge theory by AGT.

My key insight was to put the appropriate shifted Kiev Ansätze to those nodes which correspond to elements of $Z(G)$, which are the cominuscule weights. The A_n system is therefore an anomaly, as every node has this property. What the other nodes then represent is still a mystery – they seem to be composite surface operators. In any case, they can be expressed in terms of the special nodes I selected. Eliminating these middle-nodes leaves us with usually a single equation on the tau functions of the cominiscule nodes. The form of the Kiev Ansatz is then enough for us to recursively solve for all of the equivariant volumes of instanton moduli spaces, even when an ADHM description is unavailable, which is the case for the exceptional algebras.

I have also generalised this to the 5-punctured sphere, which corresponds to a certain degeneration of the Garnier system. The exact relation between these Toda equations and the Gariner system is however unknown, and represents a promising direction for future work.

In the second part I present some additional work. It is shown how to generalise the results of the first part to $\mathfrak{q}$-difference equations which correspond to $d = 5$ $\mathcal{N} = 1$ gauge. The relation with the blowup formulas is discussed. Finally, I show how to extend the $G = U(2)$ results (not $SU(2)$!) to an arbitrary amount of fundamental hypermultiplets, including greater than four.

2.2 Surface operator flow

In this part we study the partition function of $\mathcal{N} = 2$ super Yang Mills theories with general simple gauge group G in presence of a surface defect. As mentioned in 1.8, the latter is described by a two-dimensional $\mathcal{N} = (2, 2)$ gauged linear σ-model living on the defect and coupled to the bulk four-dimensional theory. This implies that the defect partition function obeys tt^* equations [58], which for the theories under consideration correspond to a de-autonomized Toda system. The defect partition function is vector-valued according to the set of admissible boundary conditions, labeled by the roots of the affine Langlands dual Lie algebra $(\hat{\mathfrak{g}})^\vee$[121]. The deautonomization corresponds to studying the gauge theory in the self-dual Ω-background $(\epsilon, -\epsilon)$. The limit $\epsilon \to 0$ reproduces the classical Seiberg-Witten theory [252] which is known to be described by the autonomous Toda chain of type $(\hat{G})^\vee$ [111, 197] by the SW/Integrable systems correspondence.

The system of equations we study is the radial reduction of tt^*-equations which describes complex deformations of a $Z(G)$-singularity, $Z(G)$ being the center of the gauge group. These are the equations of *non-autonomous* twisted affine Toda chain of type $(\hat{G})^\vee$, where $(\hat{G})^\vee$ is the Langlands dual of the untwisted affine Kac-Moody algebra $\hat{G}$. In order to clarify the appeareance of the Langlands dual group, we start from the analysis of the surface operators in the $\mathcal{N} = 2^*$ theory in terms of the de-autonomized Calogero system, whose limit to super Yang-Mills naturally produces the relevant root system. Each node of the resulting affine Dynkin diagram defines a surface operator, the associated τ-function being its vacuum expectation value. A special rôle is played by the surface operators associated to the affine nodes. These are simple surface operators whose monodromy is twisted by elements of the center of the gauge group $Z(G)$. As such, they are bounded by fractional 't Hooft lines and generate the corresponding one-form symmetry of the gauge theory. This is manifest as a $Z(G)$-symmetry of the τ-system and will be used to simplify its solution. Our analysis will be based on the observation that the surface operators associated to affine nodes are described in a perturbative regime of the bulk gauge theory and as such the partition function of the theory in their presence admits the Ansatz (2.6).

The time flow of the non-autonomous system corresponds in the gauge theory to the renormalisation group flow, the time playing the rôle of the gauge coupling constant. The system of equations produces recurrence relations for the coefficients of expansion in the gauge coupling (2.6) thus providing a new effective algorithm to calculate instanton contributions for all classical groups G. Actually, general recursion formulae based on bilinear relations can be provided for the A, B and D groups, while for gauge group of type C, E, F and G a case by case analysis is needed.

On the mathematical side, the τ-functions we provide the general solution at the canonical rays for the Jimbo-Miwa-Ueno isomonodromic deformation problem [155, 156] on the sphere with two-irregular punctures for all classical groups, which to the best of our knowledge was not known in the previous literature.

The recursion relations we obtain are indeed different from the blow-up equations of [216] and further elaborated in [171]. The latter necessarily involve the knowledge of the partition function in different Ω-backgrounds, which makes the recursion relations and the results from blow-up equations more involved and difficult to handle. Nonetheless, we expect a relation between the two approaches to follow from surface defects blow-up relations. The isomonodromic τ-function for the sphere with four regular punctures was obtained in a similar way from $SU(2)$ gauge theory with $N_f = 4$ in [153]. An analogous analysis for general classical groups is still missing in the literature.

In particular we study the cases of twisted affine Lie algebrae and linear quiver theories. We find that the τ function for the twisted affine Lie algebra BC_1 interestingly satisfies the radial reduction of Bullough-Dodd equations, and it is related to the v.e.v. of surface defect in $\mathcal{N} = 2$ $SU(2)$ gauge theory with one massless hypermultiplet in the fundamental representation. We also study the BC_2 for which we do not have at present a gauge theory interpretation.

We conjecture bilinear relations satisfied by the τ-functions of $SU(2)$ linear quiver theories which can be obtained from M-theory compactification on a Riemann sphere with two irregular punctures and $n-2$ regular ones [96]. From the mathematical viewpoint these τ functions describe isomonodromic deformations of $SL(2,\mathbb{C})$ flat connections on the very same Riemann sphere, and can be obtained from a suitable confluence limit of the Garnier system on the Riemann sphere with $n+2$ regular singularities. While the bilinear equations we write govern just the isomonodromic flow in the moduli of the two irregular punctures, we observe that a general solution can be found by imposing consistency of the system in suitable asymptotic limits. It would be interesting to complete the τ-system with the equations governing the dependence on the moduli of the regular punctures and study the relation of the results we find with bilinear systems for the τ-functions of the Garnier system [267].

2.2.1 Isomonodromic deformations

In this section we derive the relevant equations on the τ-functions for the Toda system related to any simple classical group. These are derived starting from the elliptic case in which Langlands duality is manifest.

On the one-pointed torus, $\mathcal{C}_{1,1} \cong \mathbb{T}_\tau \cong \mathbb{C}/\mathbb{Z} \oplus \tau\mathbb{Z}$, where $\tau \in \mathcal{M} \cong (\mathbb{H}_+)^{\mathrm{PSL}(2,\mathbb{Z})} \cup \{\sqrt{-1}\infty\}$ denotes a complex structure and corresponds to the isomonodromic time τ[1]. The isomonodromic system is given by a Fuchsian system together with an

[1] Confusion with the same label used for τ functions will not arise; in any case, we will soon be

isomonodromic flow

$$\partial_z \Phi(z) = L(z)\Phi(z), \quad (2\pi i)\partial_\tau \Phi(z) = -M(z)\Phi(z)$$

where $z \in \mathcal{C}_{1,1}$ [260]. The related autonomous integrable system is the elliptic Calogero-Moser system [182] which in gauge theory corresponds to $\mathcal{N} = 2^*$. The reason for starting with an extra adjoint hypermultiplet as opposed to the pure theory is that the limit to pure theory gives the context as to why the Langland dual extended root system plays a rôle, since these are the only roots whose contributions survives in the decoupling limit to the de-autonomized Toda system.

The deautonomized elliptic Calogero-Moser system can be formulated for any complex simple Lie algebra $\mathfrak{g}$ of finite rank k, whose root system we realize in a finite dimensional $\mathbb{C}$-vector space V equipped with an explicit basis $\{e_i\}_{i=1,\dots,\dim V}$, so the root system is $R \subseteq V$. We identify $V^\vee \cong V$ using the canonical product. Let $\boldsymbol{\varphi} : \mathcal{M} \to V$ be a vector valued function satisfying the *deautonomized* elliptic Calogero-Moser system

$$(2\pi\sqrt{-1})\partial_\tau^2 \boldsymbol{\varphi} = -\frac{M^2}{2}\sum_{\boldsymbol{\alpha}\in R} \wp'(\boldsymbol{\alpha}\cdot\boldsymbol{\varphi}|\tau)\boldsymbol{\alpha}$$

where $\wp'(z|\tau)$ denotes the z-derivative of the Weierstrass elliptic function, and M is the mass of the adjoint hypermultiplet. There is a well-defined autonomization procedure which maps the isomonodromic to the integrable system [260]. These deautonomized systems are quite non-trivial. Indeed, in [191] the so-called elliptic sixth Painlevé transcendent was defined as the solution to the equation $\partial_\tau^2 z = -\wp'(z|\tau)/(8\pi)^2$, and this is the simplest such system, corresponding to the Lie algebra $\mathfrak{g} = A_1$. Let us briefly recall how the autonomization procedure works. Essentially, here we need to pass from the full problem formulated on the moduli space $\mathcal{M}$ of the one-punctured torus with complex structure τ, $\mathbb{T}_\tau$, to its tangent space at some fixed complex structure τ_0, $T\mathcal{M}|_{\mathbb{T}_{\tau_0}} \cong H^0(\mathbb{T}_{\tau_0}, \Omega^1) \cong \mathbb{C}$. As described in [188], we take $\tau = \tau_0 + \epsilon t$, $\partial_\tau \mapsto \epsilon\partial_{\tau_0}$, and take the $\epsilon \to 0$ limit, perhaps ridding ourselves of some convenient $2\pi i$ factors as well. In the context of gauge theory, this limit corresponds to turning off the Omega-background.

Let $\boldsymbol{\rho}^\vee$ and $h^\vee$ denote the dual Weyl vector and dual Coxeter number, respectively. The decoupling of the hypermultiplet which brings to pure $\mathcal{N} = 2$ Super Yang-Mills or non-conformal AGT [196] is the Inosemtsev limit, achieved by setting

$$\tau = \frac{1}{2\pi\sqrt{-1}}\log\left(\frac{\Lambda}{M}\right)^{2h^\vee}$$

$$\boldsymbol{\varphi} \mapsto \boldsymbol{\varphi} + \frac{1}{2\pi\sqrt{-1}}\frac{1}{h^\vee}\log\left(\frac{\Lambda}{M}\right)^{2h^\vee}\cdot\boldsymbol{\rho}^\vee$$

and then sending $M \to \infty$. $\Lambda \in \mathbb{C}$ plays the role of the time.

To perform the limit, we quote the q-series of the relevant elliptic function [257], which can be proved using the Lipschitz summation formula [282, § 2.2],

$$(2\pi\sqrt{-1})^{-3}\wp'(z|\tau) = \sum_{n\geq 0} q^n w_z \frac{1+q^n w_z}{(1-q^n w_z)^3} - \sum_{n\geq 1} q^n w_{-z}\frac{1+q^n w_{-z}}{(1-q^n w_{-z})^3}$$

ditching this time variable for another

where $q = e^{2\pi\sqrt{-1}\tau}$ is the so-called *nome* and $w_z = e^{2\pi\sqrt{-1}z}$. To perform the limit, first note that we may restrict ourselves to positive roots, as $\wp'(-\boldsymbol{\alpha}, \mathbf{Q}|\tau)(-\boldsymbol{\alpha}) = \wp'(\boldsymbol{\alpha}, \mathbf{Q}|\tau)\boldsymbol{\alpha}$. Second, examine the powers of $(\Lambda/M)^2$, and use the properties of positive simple coroots and the longest coroot given by the level function, as follows. Following [70], define the *level* function $\ell : R \to \mathbb{R}$ by $\boldsymbol{\alpha} \mapsto \ell(\boldsymbol{\alpha}) := \langle \boldsymbol{\rho}^\vee, \boldsymbol{\alpha}\rangle$, where $\boldsymbol{\rho}^\vee := \frac{1}{2}\sum_{j=1}^r \boldsymbol{\alpha}_j^\vee \in \mathfrak{g}^\vee$ is the dual Weyl vector. Then, $\ell(\boldsymbol{\alpha}^\vee) = 1$ if and only if $\boldsymbol{\alpha} \in \Delta_+$, and $\ell(\boldsymbol{\alpha}^\vee) = h^\vee - 1$ if and only if $\boldsymbol{\alpha} = \boldsymbol{\theta}^\vee$, where $h^\vee$ is the dual Coxeter number. Examining the terms remaining after the limit, we see we have contributions either from positive simple coroots, or from $\boldsymbol{\theta}^\vee$.

The elliptic system reduces to a trigonometric one, and only the roots corresponding to the dual extended root system survive, namely the ones whose affine Cartan matrix got transposed. The significance of the dual affine system to SW theory is well-known [73, 197]. The resulting system is

$$\partial^2_{\log t}\boldsymbol{\varphi} = -t^{1/h^\vee} \sum_{\boldsymbol{\alpha}\in\hat{\Delta}_+} \boldsymbol{\alpha}^\vee e^{\boldsymbol{\alpha}^\vee\cdot\boldsymbol{\varphi}} \tag{2.1}$$

where $t := \Lambda^{2h^\vee}$, $\hat{\Delta}_+ = \{\boldsymbol{\theta}\} \cup \Delta_+$ are the extended positive roots, and we redefined $(2\pi i)\boldsymbol{\varphi} \mapsto \boldsymbol{\varphi}$ for simplicity. Once the asymptotic form of the solution is specified, the solution can be found by series expansion with a non zero (possibly infinite) convergence radius. The natural choice is to start with the homogenous solution and let $\boldsymbol{\varphi} = \mathbf{a} + \log t \cdot \mathbf{b} + \boldsymbol{\xi}$ for constant $\mathbf{a}$ and $\mathbf{b}$. The prefactor $t^{1/h^\vee}$ can be eliminated by setting $\mathbf{b} = \boldsymbol{\sigma} - \frac{1}{h^\vee}\boldsymbol{\rho}^\vee$. After this, a solution in terms of a power series in t and $\{t^{\sigma_i}\}_{i=1}^k$ can be found recursively from

$$\partial^2_{\log t}\boldsymbol{\xi} = \boldsymbol{\theta}^\vee e^{-\boldsymbol{\theta}^\vee\cdot\mathbf{a}} t^{1-\boldsymbol{\theta}^\vee\cdot\boldsymbol{\sigma}} e^{-\boldsymbol{\theta}\cdot\boldsymbol{\xi}} - \sum_{\boldsymbol{\alpha}\in\Delta_+} \boldsymbol{\alpha}^\vee e^{\boldsymbol{\alpha}^\vee\cdot\mathbf{a}} t^{\boldsymbol{\alpha}^\vee\cdot\boldsymbol{\sigma}} e^{\boldsymbol{\alpha}^\vee\cdot\boldsymbol{\xi}}$$

from which we see that to ensure convergence, $\boldsymbol{\sigma} \in \mathcal{W}^\vee_{\text{fund.}}$, the fundamental Weyl alcove of the dual root system, as

$$\left.\begin{aligned} 1 - \boldsymbol{\theta}^\vee\cdot\boldsymbol{\sigma} &> 0 \\ \boldsymbol{\alpha}_i^\vee\cdot\boldsymbol{\sigma} &> 0,\ i = 1, ..., r \end{aligned}\right\} \Rightarrow \boldsymbol{\sigma} \in \mathcal{W}^\vee_{\text{fund.}}$$

Therefore, solutions are in bijection with points of $\mathcal{W}^\vee_{\text{fund.}}$.

The choice of the affine root is not unique if outer automorphisms of the affine Dynkin diagram exist. For the simplest case A_1, there is one root which we realize as $\boldsymbol{\alpha} = (1, -1)$ and the automorphism is the reflection around the origin. Then we have $\boldsymbol{\rho} = 1/2\cdot\alpha = (1/2, -1/2)$ and $h^\vee = 2$, so

$$\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \boldsymbol{\sigma} - \frac{1}{h^\vee}\boldsymbol{\rho}^\vee = \begin{pmatrix} \sigma_1 - 1/4 \\ \sigma_2 + 1/4 \end{pmatrix} \mapsto \begin{pmatrix} b_2 \\ b_1 \end{pmatrix} = \begin{pmatrix} \sigma_2 + 1/4 \\ \sigma_1 - 1/4 \end{pmatrix}$$

The effect of the reflection is $\sigma_1 \mapsto \sigma_2 + 1/2$, $\sigma_2 \mapsto \sigma_1 - 1/2$. We should really be specializing to the $\mathfrak{sl}_2$ slice $\sigma_1 + \sigma_2 = 0$, which we often neglect to make expressions simpler; setting $\sigma = \sigma_1 = -\sigma_2$, however, we see that this is really the Bäcklund transformation $\tau(\sigma|t) \mapsto \tau(1/2 - \sigma|t)$ of Painlevé III_3, analyzed in detail in [30]. In A_n, cyclic transformations may be seen to shift $\boldsymbol{\sigma}$ by fundamental weights. We use this redundant to solve the system, since as we will see it reduces the order of the equations drastically.

2.2.2 The Hirota relations

For any $\boldsymbol{\alpha} \in \hat{\Delta}_+$ we define the formal power series $\tau_{\boldsymbol{\alpha}} \in \mathbb{C}\left[\left[t, t^{\sigma_1}, ..., t^{\sigma_k}\right]\right]$ associated to $\boldsymbol{\varphi}$ as a solution to the following equation

$$\partial^2_{\log t} \log \tau_{\boldsymbol{\alpha}}(\boldsymbol{\varphi}, t) = t^{\frac{1}{h_{\mathfrak{g}}}} e^{\boldsymbol{\alpha}^\vee \cdot \boldsymbol{\varphi}} \tag{2.2}$$

up to constant and logarithmic terms.

We claim that the τ functions generate the Hamiltonian, in the sense that they satisfy

$$\sum_{\boldsymbol{\alpha} \in \hat{\Delta}_+} \partial_{\log t} \log \tau_{\boldsymbol{\alpha}} = h^\vee t\, \mathcal{H}$$

up to a constant, where

$$t\, \mathcal{H}(\boldsymbol{\varphi}, \boldsymbol{\pi}, t) = \frac{1}{2}\boldsymbol{\pi}^2 + t^{\frac{1}{h^\vee}} \sum_{\boldsymbol{\alpha} \in \hat{\Delta}_+} e^{\boldsymbol{\alpha}^\vee \cdot \boldsymbol{\varphi}}$$

is the Hamiltonian of the deautonomised system.Indeed, we check that

$$\begin{aligned}
(t\partial_t)\,(t\mathcal{H}) &= (t\partial_t)\left(\frac{\boldsymbol{\pi}^2}{2} + t^{\frac{1}{h^\vee}} \sum_{\boldsymbol{\alpha} \in \hat{\Delta}_+} e^{\boldsymbol{\alpha}^\vee \cdot \boldsymbol{\varphi}}\right) \\
&= \boldsymbol{\pi} \cdot \partial_{\log t} \boldsymbol{\pi} + \left(t^{\frac{1}{h^\vee}} \sum_{\boldsymbol{\alpha} \in \hat{\Delta}_+} \boldsymbol{\alpha}^\vee e^{\boldsymbol{\alpha}^\vee \cdot \boldsymbol{\varphi}}\right) \cdot \partial_{\log t} \boldsymbol{\varphi} + \frac{1}{h^\vee} t^{\frac{1}{h^\vee}} \sum_{\boldsymbol{\alpha} \in \hat{\Delta}_+} e^{\boldsymbol{\alpha}^\vee \cdot \boldsymbol{\varphi}}
\end{aligned}$$

and since the first two terms vanish on-shell, so the claim follows. Equipped with these τ functions, we note that (2.1) can be rewritten as

$$\partial^2_{\log t}\left(\boldsymbol{\varphi} + \sum_j \boldsymbol{\alpha}_j^\vee \log \tau_{\boldsymbol{\alpha}_j}\right) = 0$$

We can integrate this and then reconstruct the solution $\boldsymbol{\varphi}$ in terms of τ functions from the components in the expansion $\boldsymbol{\varphi} = \sum_i \varphi_i e_i$, namely

$$\varphi_i = c_{1,i} + c_{2,i} \log t - \log \prod_{\boldsymbol{\alpha} \in \hat{\Delta}_+} \left[\tau_{\boldsymbol{\alpha}}(\boldsymbol{\varphi})\right]^{\boldsymbol{\alpha} \cdot e_i}$$

where the integration constants $c_{1,2}$ follow from the ambiguity in the definition of the τ functions. Feeding back into (2.2), the isomonodromic system may be reformulated purely in terms of the τ functions as

$$\partial^2_{\log t} \log \tau_{\boldsymbol{\alpha}} = -t^{\frac{1}{h^\vee}} \prod_{\boldsymbol{\beta} \in \hat{\Delta}_+} \left[\tau_{\boldsymbol{\alpha}}\right]^{-\boldsymbol{\beta}^\vee \cdot \boldsymbol{\alpha}^\vee} \tag{2.3}$$

were the minus sign in the R.H.S. of (2.3) is obtained by a rescaling the time variable as $t \to e^{\sqrt{-1}\pi h^\vee} t$. We will find it useful to rewrite this expression in terms of a logarithmic Hirota derivative defined as

$$D^2(f) = f^2 \partial^2_{\log t} \log f = f \partial^2_{\log t} f - (\partial_{\log t} f)^2$$

and satisfying

$$D^2(f^n) = 2nf^{2(n-1)}D^2(f^n)$$
$$D^2(f\cdot g) = f^2D^2(g) + g^2D^2(f)\,.$$

We can then rewrite the system as

$$\tau_{\boldsymbol{\alpha}}{}^{\boldsymbol{\alpha}^\vee\cdot\boldsymbol{\alpha}^\vee-2}D^2(\tau_{\boldsymbol{\alpha}}) = -t^{\frac{1}{h^\vee}}\prod_{\boldsymbol{\beta}\neq\boldsymbol{\alpha}}[\tau_{\boldsymbol{\beta}}]^{-\boldsymbol{\beta}^\vee\cdot\boldsymbol{\alpha}^\vee} \tag{2.4}$$

where the factor of $\tau_{\boldsymbol{\beta}}{}^{\boldsymbol{\beta}^\vee\cdot\boldsymbol{\beta}^\vee}$ has been extracted from the product and carried to the other side and the definition of the Hirota derivative was used. Moreover, using the first Hirota derivative identity, we get

$$\tau_{\boldsymbol{\alpha}}{}^{\boldsymbol{\alpha}^\vee\cdot\boldsymbol{\alpha}^\vee-2}D^2(\tau_{\boldsymbol{\alpha}}) = \frac{2}{\boldsymbol{\alpha}^\vee\cdot\boldsymbol{\alpha}^\vee}D^2([\tau_{\boldsymbol{\alpha}}]^{\frac{\boldsymbol{\alpha}^\vee\cdot\boldsymbol{\alpha}^\vee}{2}}) = -t^{\frac{1}{h^\vee}}\prod_{\boldsymbol{\beta}\neq\boldsymbol{\alpha}}[\tau_{\boldsymbol{\beta}}]^{\frac{\boldsymbol{\beta}^\vee\cdot\boldsymbol{\beta}^\vee}{2}(-\boldsymbol{\alpha}^\vee\cdot\frac{2\boldsymbol{\beta}^\vee}{\boldsymbol{\beta}^\vee\cdot\boldsymbol{\beta}^\vee})}$$

and redefining $[\tau_{\boldsymbol{\alpha}}]^{\frac{\boldsymbol{\alpha}^\vee\cdot\boldsymbol{\alpha}^\vee}{2}}\mapsto\tau_{\boldsymbol{\alpha}}$ for every root $\boldsymbol{\alpha}$ we finally get the equation

$$D^2(\tau_{\boldsymbol{\alpha}}) = -\frac{\boldsymbol{\alpha}^\vee\cdot\boldsymbol{\alpha}^\vee}{2}t^{1/h^\vee}\prod_{\boldsymbol{\beta}\in\hat{\Delta},\boldsymbol{\beta}\neq\boldsymbol{\alpha}}[\tau_{\boldsymbol{\beta}}]^{-\boldsymbol{\alpha}^\vee\cdot\boldsymbol{\beta}}\,, \tag{2.5}$$

The above redefinition leaves unchanged the τ-functions corresponding to miniscule coweights. By the ambiguity in the constants of integration, both (2.3) and (2.4) may be modified by a constant or a power of t. Further, we will be writing $D^4 := D^2\circ D^2$, $D^{2n} := D^2\circ D^{2n-2}$. Finally, the τ function associated to the constant solution $\boldsymbol{\varphi}_0 = \mathbf{a}$ is immediate from (2.2),

$$\tau_{\boldsymbol{\alpha}}(\boldsymbol{\varphi}_0, t) = \exp\Big\{(h^\vee)^2t^{\frac{1}{h^\vee}}e^{\boldsymbol{\alpha}\cdot\boldsymbol{a}}\Big\}$$

Eq. (2.5) is the de-autonomization of the τ-form of the standard Toda integrable system. From [111, 197] it is known that this governs the classical Seiberg-Witten (SW) theory [251]. The de-autonomization is induced by coupling the theory to a self-dual Ω-background $(\epsilon_1,\epsilon_2) = (\epsilon,-\epsilon)$ [47]. In the autonomous limit $\epsilon\to 0$, the relevant τ-functions boil down to Riemann θ-functions on the classical SW curve [39]. These were used to provide recursion relations on the coefficients of the expansion of the SW prepotential in [82].

The actual form of equations (2.5) depends on the Dynkin diagram. In particular, these reduce to bilinear equations for the classical groups A, B and D, which we solve via general recursion relations. Instead, for C, E, F and G groups the equations of the τ-system are of higher order and must be studied by a case by case analysis. The τ-system displays a finite symmetry generated by the center of the group G, namely

$\mathfrak{g}$	A_n	B_n	C_n	D_{2n}	D_{2n+1}	E_n	F_4	G_2
$Z(G)$	$\mathbb{Z}_{n+1}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2\times\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_{9-n}$	1	1

The center is isomorphic to the coset of the affine coweight lattice by the affine coroot lattice, and coincides with the automorphism group of the affine Dynkin diagram. As in [2], the coweights, and by extension the lattice cosets, corresponding to these nodes are the minuscule coweights. We recall that a representation of $\mathfrak{g}$ is minuscule if all its weights form a single Weyl-orbit. This remark will be crucial to solve the τ-system.

The τ-functions corresponding to the affine nodes, namely the ones which can be removed from the Dynkin diagram while leaving behind that of an irreducible simple Lie algebra, play a special rôle. In the gauge theory interpretation of the Introduction, these are related to simple surface operators associated to elements of the center $Z(G)$, and are bounded by fractional 't Hooft lines. As such, they are the generators of the one-form symmetry of the corresponding gauge theory, [98]. Since their magnetic charge is defined modulo the magnetic root lattice, a natural Ansatz for their expectation value is

$$\tau_{\boldsymbol{\alpha}_{\mathrm{aff}}}(\boldsymbol{\sigma},\boldsymbol{\eta}|\kappa_{\mathfrak{g}}t)=\sum_{\mathbf{n}\in Q^{\vee}_{\mathrm{aff}}} e^{2\pi i\boldsymbol{\eta}\cdot\mathbf{n}}t^{\frac{1}{2}(\boldsymbol{\sigma}+\mathbf{n})^2}B(\boldsymbol{\sigma}+\mathbf{n}|t) \tag{2.6}$$

where $B(\boldsymbol{\sigma}|t)=B_0(\boldsymbol{\sigma})\sum_{i\geq 0}t^iZ_i(\boldsymbol{\sigma})$ with $Z_0(\boldsymbol{\sigma})\equiv 1$ and $Q^{\vee}_{\mathrm{aff}}=\boldsymbol{\lambda}^{\vee}_{\mathrm{aff}}+Q^{\vee}$, $Q^{\vee}$ being the coroot lattice equipped with the canonical inner product normalized such that the norm of the short coroots is 2, and $(\boldsymbol{\lambda}^{\vee}_{\mathrm{aff}},\boldsymbol{\alpha})=\delta_{\boldsymbol{\alpha}\mathrm{aff},\boldsymbol{\alpha}}$ for any non-extended simple root $\boldsymbol{\alpha}$. The constant $\kappa_{\mathfrak{g}}=(-n_{\mathfrak{g}})^{r_{\mathfrak{g},s}}$, where $n_{\mathfrak{g}}$ is the ratio of the squares of long vs. short roots and $r_{\mathfrak{g},s}$ is the number of short simple roots. For simply laced, all roots are long and $\kappa_{\mathfrak{g}}=1$.

In the A_n case, (2.6) is known as the Kiev Ansatz. In particular, in the A_1 case, it was used to give the general solution of Painlevé III_3 equation in [152] and further analysed in [203]. It was crucial for these results to identify the expansion coefficients of (2.6) with the full Nekrasov partition function in the self-dual Ω-background. We will now show that this still holds for general classical groups. More precisely, this follows upon the identification $\boldsymbol{\sigma}=\mathbf{a}/\epsilon$, where $\mathbf{a}$ is the Cartan parameter of the gauge theory. Let us remark that the variables $\boldsymbol{\eta},\boldsymbol{\sigma}\in Q^{\vee}$ are the integration constants of the second order differential equations (2.5) and correspond to the initial position and velocity of the de-autonomized Toda particle.

Let us set now the boundary conditions which we impose to the solutions of equations (2.5). We consider the asymptotic behaviour of the solutions at $t\to 0$ and $\boldsymbol{\sigma}\to\infty$ as

$$\log(B_0)\sim -\frac{1}{4}\sum_{\mathbf{r}\in R}(\mathbf{r}\cdot\boldsymbol{\sigma})^2\log{(\mathbf{r}\cdot\boldsymbol{\sigma})^2} \tag{2.7}$$

up to quadratic and log-terms.

Notice that the τ-system knows itself the one-loop exactness of the $\mathcal{N}=2$ gauge theory! Indeed, if one chooses a more general ansatz for the Wilsonian effective action as $\log(B_0)\sim\sum_{\mathbf{r}\in R}c_{n,m}(\mathbf{r}\cdot\boldsymbol{\sigma})^{2n}\log{((\mathbf{r}\cdot\boldsymbol{\sigma})^2)^m}$, then the consistency of the equation itself implies that $(n,m)=(1,1)$ and $(n,m)=(2,0)$ are the only allowed terms.

We will show that the solution of (2.5) which satisfies the above asymptotic condition is

$$B_0(\boldsymbol{\sigma})=Z_{1-loop}(\boldsymbol{\sigma})\equiv\prod_{\mathbf{r}\in R}\frac{1}{G(1+\mathbf{r}\cdot\boldsymbol{\sigma})} \tag{2.8}$$

[2] Bourbaki [Lie gps Ch. VIII §7]

where $G(z)$ is the Barnes' G-function and R is the adjoint representation of the group G. The expansion of the above function matches the one-loop gauge theory result upon the appropriate identification of the log-branch. This reads, in the gauge theory variables, as $\ln\left[\sqrt{-1}\mathbf{r}\cdot\mathbf{a}/\Lambda\right]\in\mathbb{R}$ and in the A_n case matches the canonical Stokes rays obtained in [118]. Eq.(2.8) corresponds to the 1-loop term in the self-dual Ω-background. To see this more clearly, recall the perturbative Coleman-Weinberg 1-loop term for a massless hypermultiplet in 4 dimensions with IR regulator μ,

$$\mathcal{F}_{1\ \mathrm{loop}}(\sigma)=\frac{3}{4}\operatorname{tr}\sigma^2-\frac{1}{4}\operatorname{tr}\sigma^2\log\left(\frac{\sigma}{\mu}\right)^2=\int_\mu^\infty\frac{\mathrm{d}s}{s^3}\operatorname{tr}e^{-s\sigma}+\mathcal{O}(\frac{1}{\mu^2})$$

where the trace is taken in the relevant representation. In the self-dual Ω-background with parameter ϵ, this gets deformed to

$$\int_\mu^\infty\frac{\mathrm{d}s}{s}\frac{-\epsilon^2\cdot\operatorname{tr}e^{-s\sigma}}{(1-e^{s\epsilon})(1-e^{-s\epsilon})}$$

which we can write in terms of the Barnes' G function by using its Lévy-Khintchine type representation valid for $|z|<1$

$$\frac{1}{G(1+z)}=\exp\left\{-\frac{\log2\pi-1}{2}z+\frac{1+\gamma}{2}z^2-\int_0^\infty\frac{\mathrm{d}s}{s}\frac{e^{-zs}-1+zs-\frac{1}{2}s^2z^2}{(1-e^s)(1-e^{-s})}\right\}$$

where the substractions in the integrand serve as an infrared regulator. For a general gauge group in the Coulomb phase, tracing over the Cartan yields

$$e^{\mathcal{F}_{\text{1-loop}}}=\exp\left\{\int_\mu^\infty\frac{\mathrm{d}s}{s}\frac{-\epsilon^2\cdot\operatorname{tr}_R e^{-s\boldsymbol{\sigma}}}{(1-e^{s\epsilon})(1-e^{-s\epsilon})}\right\}\rightsquigarrow\prod_{\boldsymbol{\alpha}\in R}\frac{1}{G(1+\boldsymbol{\sigma}\cdot\boldsymbol{\alpha})}=:B_0(\boldsymbol{\sigma})$$

The most important property of this expression is that, given some $\boldsymbol{\beta}\in R^\vee$,

$$B_0(\boldsymbol{\sigma}+\boldsymbol{\beta})=B_0(\boldsymbol{\sigma})\prod_{\substack{\boldsymbol{\alpha}\in R,n\geq1\\ \boldsymbol{\alpha}\cdot\boldsymbol{\beta}=n}}\frac{(-1)^{\lfloor n/2\rfloor}\Gamma(-\boldsymbol{\alpha}\cdot\boldsymbol{\sigma})^n}{\Gamma(\boldsymbol{\alpha}\cdot\boldsymbol{\sigma})^n(\boldsymbol{\alpha}\cdot\boldsymbol{\sigma})^n\prod_{k=1}^{n-1}(\boldsymbol{\alpha}\cdot\boldsymbol{\sigma}+k)^{2n-2k}}\tag{2.9}$$

where we can pick only positive n's since the product runs over the whole root system.

2.2.3 The operators Y^n

Besides the Hirota derivatives, we will make frequent use of the operators recursively defined as

$$\begin{aligned}&Y^1(f)=f\\&Y^2(f)=D^2(f)\\&Y^n(f)=(Y^{n-2}(f))^{-1}D^2(Y^{n-1}(f)),\quad n\geq2\end{aligned}\tag{2.10}$$

Besides being shorthands, their utility consists in the property that they act on a formal power series as

$$Y^n(\sum_i y_it^{x_i})=\sum_{i_1,\ldots,i_n}\prod_{j=1}^n y_{i_j}t^{x_{i_j}}\prod_{k<l=1}^n(x_{i_k}-x_{i_l})^2\tag{2.11}$$

It is easy to calculate by hand that is true for $n = 2$. For $n > 2$, suppose we only have a total of n distinct exponents x_{i_n} in the sum $\sum_i y_i t^{x_i}$. Then if we assume (2.11) holds for $n-1$, Y^{n-1} will by assumption have only ${}_nC_{n-1} = n$ terms, differing by one pair of indices. Without loss of generality, consider two terms with the last index labelled differently. That is, let $x_{\text{same}} := \sum_{k=1}^{n-2} x_{i_k}$ and $x_{i_{n-1}} \neq x_{i_n}$. By assumption of the induction we have that in applying Y^{n-1} to $\sum_i y_i t^{x_i}$ we end up with two different terms

$$c_1\, t^{x_{\text{same}}+x_{n-1}}, c_2\, t^{x_{\text{same}}+x_n} \in Y^{n-1}(\sum_i y_i t^{x_i})$$

where the coefficients $c_{1,2}$ are

$$c_1 = y_{i_{n-1}} \prod_{k=1}^{n-2} (x_{i_k} - x_{i_{n-1}})^2 \cdot c_{\text{same}}$$
$$c_2 = y_{i_n} \prod_{k=1}^{n-2} (x_{i_k} - x_{i_n})^2 \cdot c_{\text{same}}$$

where

$$c_{\text{same}} = \prod_{j=1}^{n-2} y_{i_j} \prod_{k<l=1}^{n-2} (x_{i_k} - x_{i_l})^2$$

comes from the exponents purely inside x_{same}. Considering the application of D^2 to just those two terms we obtain

$$D^2 \left(c_1 c_{\text{same}} t^{x_{\text{same}}+x_{i_{n-1}}} + c_2 c_{\text{same}} t^{x_{\text{same}}+x_{i_n}} \right) = c_1 c_2 c_{\text{same}}^2 (x_{i_{n-1}} - x_{i_n})^2 t^{2x_{\text{same}}+x_{i_{n-1}}+x_{i_n}}$$

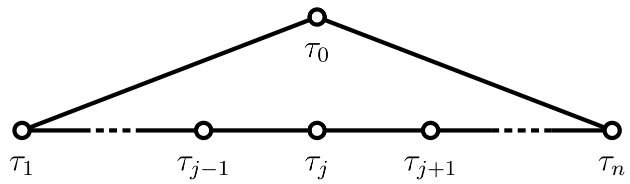
which we can write as

$$\begin{aligned} &\prod_{j=1}^{n-2} y_{i_j} t^{x_{i_j}} \prod_{k<l=1}^{n-2} (x_{i_k} - x_{i_l})^2 \cdot \prod_{j=1}^{n} y_{i_j} t^{x_{i_j}} \prod_{k<l=1}^{n} (x_{i_k} - x_{i_l})^2 \\ &\quad = \prod_{j=1}^{n-2} y_{i_j} t^{x_{i_j}} \prod_{k<l=1}^{n-2} (x_{i_k} - x_{i_l})^2 \cdot Y^n(\sum_i y_i t^{x_i}) \end{aligned}$$

and in the last line we've used (2.11) in this particular case of only n distinct exponents. The term in front explicitly lacks the pair of indices we chose. Now if we extend by linearly to all such pairs, we see that we have shown that $D^2(Y^{n-1}(f)) = Y^{n-2}(f)Y^n(f)$. But now we are done since we can reduce the general case of more than n distinct exponents to this one by multilinearity.

2.2.4 The Lie algebra Zoo

2.2.4.1 A_n



The A_n case is the simplest but already illustrates most of the ideas of our analysis. The simplification in this case comes from the fact that every node of the extended Dynkin diagram corresponds to a miniscule (co)weight and that the resulting equations are strictly bilinear, none of which are true in general for different algebras.

We realise the roots using an orthonormal basis $\{e_i\}$ of $\mathbb{R}^{n+1}$ as $\{\pm(e_i - e_j)\}$ for $i \neq j$. The algebra is simply laced so the coroot lattice is the root lattice and is $Q^\vee = Q = \{\sum_{i=1}^{n+1} c_i e_i | \sum_{i=1}^{n+1} c_i = 0\}$, while the fundamental weights

$$\boldsymbol{\lambda}_i = (1^i, 0^{n+1-i}) - \frac{i}{n+1}(1^{n+1}),$$

are all minuscule. Here $(1^p, 0^{n+1-p})$ stands for a vector whose first p entries are 1 and the remaining entries vanish, while in (1^{n+1}) all entries are 1. Moreover we label the τ-functions as $\tau_{\boldsymbol{\alpha}_j} \equiv \tau_j$ and identify $\tau_j = \tau_{n+1+j}$ periodically. Then the τ-system can be written succinctly as

$$D^2(\tau_j) = -t^{\frac{1}{n+1}} \tau_{j-1}\tau_{j+1}. \tag{2.12}$$

Due to the $\mathbb{Z}_{n+1}$ outer automorphism group of the Dynkin diagram, each of the nodes of A_n can be taken as the affine one so that the corresponding τ-functions can be expressed through the Kiev Ansatz (2.6). Therefore, all the τ-functions are determined by a single one, say τ_0, as $\tau_j = \tau_0|_{Q \mapsto Q_j}$. Owing to the $\mathbb{Z}_{n+1}$ symmetry, it is enough to solve (2.12) corresponding to $j = 0$. Henceforth we adopt the shorthand $f(y \pm x) \equiv f(y + x)f(y - x)$. The Ansatz (2.6) for τ_0 reads

$$\tau_0(\boldsymbol{\sigma}, \boldsymbol{\eta}|t) = \sum_{\mathbf{n} \in Q,\, i \geq 0} e^{2\pi\sqrt{-1}\mathbf{n}\cdot\boldsymbol{\eta}} t^{\frac{1}{2}(\boldsymbol{\sigma}+\mathbf{n})^2 + i} B_0(\boldsymbol{\sigma} + \mathbf{n}) Z_i(\boldsymbol{\sigma} + \mathbf{n}) \tag{2.13}$$

It is necessary to discuss the effect of shifting the lattice Q by $\boldsymbol{\lambda}_i$, since we explicitly write the shifts $\boldsymbol{\lambda}_i$ and simplify them, instead of considering three different lattices. In the Ansatz, the initial conditions $\boldsymbol{\eta}, \boldsymbol{\sigma} \in \mathbb{R}^{n+1}$, along with the whole root system and all $\mathbf{n} \in Q$ as such, are orthogonal to the (1^{n+1}) direction. Therefore, $\boldsymbol{\sigma} \cdot \boldsymbol{\lambda}_1 = \boldsymbol{\sigma} \cdot e_1$, $\boldsymbol{\sigma} \cdot \boldsymbol{\lambda}_n = -\boldsymbol{\sigma} \cdot e_1$. Therefore, $B_0(\boldsymbol{\sigma})$ and $Z_i(\boldsymbol{\sigma})$ will be fixed by functional and recursive relations which involve only $\pm e_1$, as only such inner products enter. Therefore, we will write, e.g. $B_0(\boldsymbol{\sigma} + \mathbf{n} + \boldsymbol{\lambda}_1) = B_0(\boldsymbol{\sigma} + \mathbf{n} + e_1)$. With these and some other simplifications, Inserting the Kiev Ansatz (2.13) into (2.12) gives us

$$\sum_{\substack{\mathbf{n}_1,\mathbf{n}_2 \in Q \\ i_1, i_2 \geq 0}} e^{2\pi\sqrt{-1}(\mathbf{n}_1+\mathbf{n}_2)\cdot\boldsymbol{\eta}} t^{\frac{1}{2}\mathbf{n}_1^2 + \frac{1}{2}\mathbf{n}_2^2 + i_1 + i_2 + \boldsymbol{\sigma}\cdot(\mathbf{n}_1+\mathbf{n}_2)} \left(\frac{1}{2}\mathbf{n}_1^2 - \frac{1}{2}\mathbf{n}_2^2 + i_1 - i_2 + \boldsymbol{\sigma}\cdot(\mathbf{n}_1 - \mathbf{n}_2)\right)^2$$
$$\times B_0(\boldsymbol{\sigma} + \mathbf{n}_1) B_0(\boldsymbol{\sigma} + \mathbf{n}_2) Z_{i_1}(\boldsymbol{\sigma} + \mathbf{n}_1) Z_{i_2}(\boldsymbol{\sigma} + \mathbf{n}_2)$$
$$= -\sum_{\substack{\mathbf{m}_1,\mathbf{m}_2 \in Q \\ j_1, j_2 \geq 0}} e^{2\pi\sqrt{-1}(\mathbf{m}_1+\mathbf{m}_2)\cdot\boldsymbol{\eta}} t^{1 + \frac{1}{2}\mathbf{m}_1^2 + \frac{1}{2}\mathbf{m}_2^2 + e_1\cdot(\mathbf{m}_1 - \mathbf{m}_2) + j_1 + j_2 + \boldsymbol{\sigma}\cdot(\mathbf{m}_1+\mathbf{m}_2)} \times$$
$$B_0(\boldsymbol{\sigma} + \mathbf{m}_1 + e_1) B_0(\boldsymbol{\sigma} + \mathbf{m}_2 - e_1) Z_{j_1}(\boldsymbol{\sigma} + \mathbf{m}_1 + e_1) Z_{j_2}(\boldsymbol{\sigma} + \mathbf{m}_2 - e_1)$$

This is solved as a power series in $t, t^{\sigma_1}, ..., t^{\sigma_n}$. To fix $B_0(\boldsymbol{\sigma})$, we look at the lowest order. The lowest order is linear in t and produces the quadratic constraint

$$\frac{1}{2}\mathbf{n}_1^2 + \frac{1}{2}\mathbf{n}_2^2 + i_1 + i_2 = 1 + \frac{1}{2}\mathbf{m}_1^2 + \frac{1}{2}\mathbf{m}_2^2 + e_1 \cdot (\mathbf{m}_1 - \mathbf{m}_2) + j_1 + j_2 = 1 \tag{2.14}$$

as well as $n+1$ linear constraints on the root lattice variables $(\mathbf{n}_1, \mathbf{n}_2)$ and $(\mathbf{m}_1, \mathbf{m}_2)$. Let us fix $p, q \in \{0, ...n+1\}$, $p \neq q$ and look for terms with $t^{\sigma_p - \sigma_q}$. The linear constrains are $\mathbf{n}_1 + \mathbf{n}_2 = \mathbf{m}_1 + \mathbf{m}_2 = e_p - e_q$. Up to Weyl reflections, the only solution to the above mentioned constraints is given by $\mathbf{n}_1 = e_p - e_q$, $\mathbf{n}_2 = 0$ and $\mathbf{m}_1 = e_p - e_1$, $\mathbf{m}_2 = -e_q + e_1$, with is and js in (2.14) vanishing, leading to the functional equation

$$\left(1 + (e_p - e_q)\cdot\boldsymbol{\sigma}\right)^2 B_0(\boldsymbol{\sigma} + e_p - e_q) B_0(\boldsymbol{\sigma}) = -B_0(\boldsymbol{\sigma} + e_p) B_0(\boldsymbol{\sigma} - e_q)\,. \tag{2.15}$$

Now we suppose that $B_0(\boldsymbol{\sigma}) = f(\boldsymbol{\sigma}) \prod_{\mathbf{r}\in R} \frac{1}{G(1+\mathbf{r}\cdot\boldsymbol{\sigma})}$. First of all we show that of ratios of Γ-functions which arise from manipulating the Barnes' G-functions cancels. Namely, consider, for $\boldsymbol{\beta} \in Q^\vee + \boldsymbol{\lambda}^\vee$ in a general Lie algebra

$$\hat{\Gamma}(\boldsymbol{\beta}) := \prod_{\substack{\boldsymbol{\alpha}\in R, n\geq 1 \\ \boldsymbol{\alpha}\cdot\boldsymbol{\beta} = n}} \left(\frac{\Gamma[-\boldsymbol{\alpha}\cdot\boldsymbol{\sigma}]}{\Gamma[\boldsymbol{\alpha}\cdot\boldsymbol{\sigma}]}\right)^n$$

which is the product of Γ-functions in (2.9). Noting that

$$\begin{aligned}
\hat{\Gamma}(\boldsymbol{\beta}) &= \prod_{\substack{\boldsymbol{\alpha}\in R \\ \boldsymbol{\alpha}\cdot\boldsymbol{\beta}\geq 0}} \left(\frac{\Gamma[-\boldsymbol{\alpha}\cdot\boldsymbol{\sigma}]}{\Gamma[\boldsymbol{\alpha}\cdot\boldsymbol{\sigma}]}\right)^{\boldsymbol{\alpha}\cdot\boldsymbol{\beta}} \\
&= \prod_{\substack{\boldsymbol{\alpha}\in R \\ \boldsymbol{\alpha}\cdot\boldsymbol{\beta}\geq 0}} \left(\frac{1}{\Gamma[\boldsymbol{\alpha}\cdot\boldsymbol{\sigma}]}\right)^{\boldsymbol{\alpha}\cdot\boldsymbol{\beta}} \left(\frac{1}{\Gamma[-\boldsymbol{\alpha}\cdot\boldsymbol{\sigma}]}\right)^{-\boldsymbol{\alpha}\cdot\boldsymbol{\beta}} = \prod_{\boldsymbol{\alpha}\in R} \left(\frac{1}{\Gamma[\boldsymbol{\alpha}\cdot\boldsymbol{\sigma}]}\right)^{\boldsymbol{\alpha}\cdot\boldsymbol{\beta}}
\end{aligned}$$

we get

$$\hat{\Gamma}(\boldsymbol{\beta}_1)\hat{\Gamma}(\boldsymbol{\beta}_2) = \hat{\Gamma}(\boldsymbol{\beta}_1 + \boldsymbol{\beta}_2)\,.$$

In particular if $\sum_k \boldsymbol{\beta}_k = \sum_k \boldsymbol{\gamma}_k$, which corresponds to the linear constrains,

$$\prod_k \hat{\Gamma}(\boldsymbol{\beta}_k) = \hat{\Gamma}(\sum_k \boldsymbol{\beta}_k) = \hat{\Gamma}(\sum_k \boldsymbol{\gamma}_k) = \prod_k \hat{\Gamma}(\boldsymbol{\gamma}_k)$$

Therefore, these products of Γ-functions cancels from all formulas, as we obtain all of them by matching equal powers of $t^{\sigma_1}, ..., t^{\sigma_n}$. This discussion is valid for all Lie algebras. For the A_n case, the LHS of (2.15), after discarding products of Γ-functions, becomes

$$\frac{(1 + \sigma_p - \sigma_q)^2 f(\boldsymbol{\sigma} + e_p - e_q) f(\boldsymbol{\sigma})}{-(\sigma_p - \sigma_q)^2 (1 + \sigma_p - \sigma_q)^2 \prod\limits_{\boldsymbol{\beta}\cdot(e_p - e_q) = 1} \boldsymbol{\beta}\cdot\boldsymbol{\sigma}} = -\frac{f(\boldsymbol{\sigma} + e_p - e_q) f(\boldsymbol{\sigma})}{(\sigma_p - \sigma_q)^2 \prod\limits_{p\neq k\neq q} (\sigma_p^2 - \sigma_k^2)(\sigma_k^2 - \sigma_q^2)}\,.$$

This has to equal the RHS

$$-\frac{f(\boldsymbol{\sigma} + e_p)}{\prod\limits_{k\neq p} (\sigma_p^2 - \sigma_k^2)} \frac{f(\boldsymbol{\sigma} - e_q)}{\prod\limits_{k\neq q} (\sigma_k^2 - \sigma_q^2)}\,.$$

Simple arithmetics converts this to $f(\boldsymbol{\sigma} + e_p - e_q) f(\boldsymbol{\sigma}) = f(\boldsymbol{\sigma} + e_p) f(\boldsymbol{\sigma} - e_q)$ which implies that f is periodic on the lattice. The asymptotic condition (2.7) reads as $f \sim 1$ when $\boldsymbol{\sigma} \to \infty$, so that $f = 1$.

The higher order terms in $t, t^{\sigma_1}, ..., t^{\sigma_n}$ provide the recursion relations

$$k^2 Z_k(\boldsymbol{\sigma}) = -\sum_{\substack{\mathbf{n}^2+j_1+j_2=k \\ \mathbf{n}\in e_1+Q,\, j_{1,2}<k}} \frac{B_0(\boldsymbol{\sigma}\pm\mathbf{n})}{B_0(\boldsymbol{\sigma})^2} Z_{j_2}(\boldsymbol{\sigma}-\mathbf{n}) Z_{j_1}(\boldsymbol{\sigma}+\mathbf{n})$$
$$+\sum_{\substack{\mathbf{n}^2+i_1+i_2=k \\ \mathbf{n}\in Q,\, i_{1,2}<k}} (i_1-i_2+2\mathbf{n}\cdot\boldsymbol{\sigma})^2 \frac{B_0(\boldsymbol{\sigma}\pm\mathbf{n})}{B_0(\boldsymbol{\sigma})^2} Z_{i_1}(\boldsymbol{\sigma}+\mathbf{n}) Z_{i_2}(\boldsymbol{\sigma}-\mathbf{n}),$$

where $B_0(\boldsymbol{\sigma})$ is given by (2.8). In particular, $k=1$ gives the simple expression

$$Z_1(\boldsymbol{\sigma}) = -\sum_{i=1}^{n+1} \frac{B_0(\boldsymbol{\sigma}\pm e_i)}{B_0(\boldsymbol{\sigma})^2} = (-1)^{n+1}\sum_{i=1}^{n+1}\frac{1}{\prod_{j\neq i}(\sigma_i-\sigma_j)^2}.$$

Upon abbreviating $\sigma_{ij}=\sigma_i-\sigma_j$, the $k=2$ term gives

$$Z_2(\boldsymbol{\sigma}) = -\frac{1}{4}\sum_{i=1}^{n+1}\frac{B_0(\boldsymbol{\sigma}\pm e_i)}{B_0(\boldsymbol{\sigma})^2}[Z_1(\boldsymbol{\sigma}+e_i)+Z_1(\boldsymbol{\sigma}-e_i)]+\sum_{i<j}^{n+1}(\sigma_i-\sigma_j)^2\frac{B_0(\boldsymbol{\sigma}\pm(e_i-e_j))}{B_0(\boldsymbol{\sigma})^2}$$

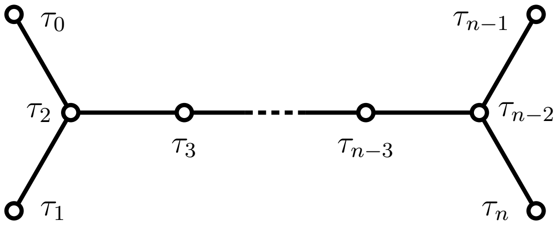
which we can write as

$$\begin{aligned}
Z_2(\boldsymbol{\sigma}) &= \frac{1}{4}\sum_i \frac{1}{\prod_{j\neq i}\sigma_{ij}^2}\left(\sum_k \frac{1}{\prod_{l\neq k}(\sigma_{kl}+\delta_{kl}-\delta_{li})^2}+\frac{1}{\prod_{l\neq k}(\sigma_{kl}-\delta_{kl}+\delta_{li})^2}\right)\\
&\quad -\sum_{i<j}\frac{1}{(\sigma_{ij}+1)^2(\sigma_{ij}-1)^2\sigma_{ij}^2\prod_{i\neq k\neq j}\sigma_{ik}^2\sigma_{jk}^2}\\
&= \frac{1}{4}\sum_i\frac{1}{\prod_{k\neq i}\sigma_{ki}^2\cdot\prod_{k\neq i}(\sigma_{ki}-1)^2}+\frac{1}{4}\sum_i\frac{1}{\prod_{k\neq i}\sigma_{ki}^2\cdot\prod_{k\neq i}(\sigma_{ki}+1)^2}\\
&\quad +\sum_{i<j}\frac{\sigma_{ij}^2}{(\sigma_{ij}^2-1)^2}\frac{1}{\prod_{k\neq i}\sigma_{ki}^2\cdot\prod_{k\neq j}\sigma_{kj}^2}
\end{aligned}$$

where in the second step we cancelled the off-diagonal terms in the double product, to simplify the comparison with Nekrasov formulae for $k=2$ for $\epsilon_1=-\epsilon_2=1$. Indeed, the three sums above correspond exactly to $Z^{SU(n+1)}(\vec{Y})$ of (1.4) with $\vec{Y}$ having two boxes $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ or two boxes $\square\square$ in the i-th position and the last double sum is over $\vec{Y}$ such that one box is in the i-th and another in the j-th position. These are all the possible tuples $\vec{Y}$ such that $|\vec{Y}|=2$.
To summarise, the above coincide with one and two instanton contributions to the $SU(n+1)$ Nekrasov partition function as computed from supersymmetric localization [221, 225]. Let us remark that the use of the τ-system (2.12) provides a completely independent tool to compute all instanton corrections just starting from the asymptotic behaviour (2.7). This procedure extends to all classical groups.

2.2.4.2 B_n, D_n

Due to our strategy of solving the problem by attaching Kiev Ansätze to nodes corresponding to minuscule coweights, we treat the algebras B_n and D_n simultaneously. The coroot lattices are likewise the same, so the only difference between B_n to D_n is the asymptotic condition the extra roots of B_n impose.



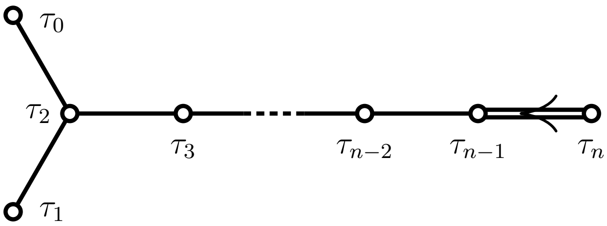
D_n is a simply laced algebra, whose coroot lattice is the checkerboard lattice $Q = Q^\vee = \{\sum_{i=1}^n c_i e_i | \sum_{i=1}^n c_i \in 2\mathbb{Z}\}$. In this section we consider only $n \geq 4$ and leave the special cases of $n = 2, 3$ to a separate section. There are four minuscule weights, $\boldsymbol{\lambda}_0 = (0^n)$, $\boldsymbol{\lambda}_1 = (1, 0^{n-1})$, $\boldsymbol{\lambda}_{n-1} = ((\frac{1}{2})^{n-1}, -\frac{1}{2})$, $\boldsymbol{\lambda}_n = ((\frac{1}{2})^{n-1}, +\frac{1}{2})$ and these correspond to the "legs" of the affine diagram. Whatever the rank we consider, we always have the consistency conditions

$$D^2(\tau_0) = D^2(\tau_1), \quad D^2(\tau_{n-1}) = D^2(\tau_n) \tag{2.16}$$

which immediately follow from the equations $D^2(\tau_0) = -t^{1/2n}\tau_2$, $D^2(\tau_1) = -t^{1/2n}\tau_2$ and the analogue ones at the other end of the diagram. The second consistency condition is morally just the first one with σ shifted by $((\frac{1}{2})^n)$. In the special case $n = 4$ we have a further equality, due to the enhanced symmetry of D_4,

$$D^2(\tau_0) = D^2(\tau_1) = D^2(\tau_3) = D^2(\tau_4).$$

Practically, however, the first condition is sufficient to solve the problem.



B_n is not simply laced. In addition to the roots of the corresponding D_n, $\{e_i \pm e_j\}_{i \neq j}$, it has shorter roots $\{e_i\}$. This is the first case, however, in which we have to worry about looking at the Langlands dual algebra, and send each root to the coroot via $R \ni \boldsymbol{\alpha} \mapsto 2\boldsymbol{\alpha}/(\boldsymbol{\alpha} \cdot \boldsymbol{\alpha}) \in R^\vee$. Therefore, the extended Dynkin diagram above has reversed arrows compared to the usual, since the roots $\{e_i\}$ get rescaled to $\{2e_i\}$. The coroot lattice is still the checkerboard lattice $Q^\vee = \{\sum_{i=1}^n c_i e_i | \sum_{i=1}^n c_i \in 2\mathbb{Z}\}$ of D_n, and the two minuscule weights are $\boldsymbol{\lambda}_0^\vee = (0^n)$ and $\boldsymbol{\lambda}_1^\vee = (1, 0^{n-1})$, corresponding to the "antennae" of the new diagram, provided $n > 3$. The additional $\mathbb{Z}_2$ symmetry of D_n is broken. The τ-system coincides with that of D_{n+1}, with the modification that (i) there is no τ_{n+1} node and (ii) that

$$D^2(\tau_{n-1}) = -2t^{\frac{1}{2n-1}}\tau_{n-2}\tau_n, \quad D^2(\tau_n) = -t^{\frac{1}{2n-1}}\tau_{n-1}^2.$$

The case $n = 3$ is discussed separately along with the algebra C_2 in section 2.2.4.7. We limit the present discussion to $n > 3$ so the analysis proceeds as for D_n, except we can only consider the first equation in (2.16). This unifies the approach to both D_n and B_n. Explicitly, inserting (2.6) and $\tau_1(\boldsymbol{\sigma}|t) = \tau_0(\boldsymbol{\sigma} + \boldsymbol{\lambda}_1|t)$ into the first equation of (2.16) we get

$$\begin{aligned}
&\sum_{\substack{\mathbf{n}_1,\mathbf{n}_2\in Q^\vee\\ i_1,i_2\geq 0}} e^{2\pi\sqrt{-1}(\mathbf{n}_1+\mathbf{n}_2)\cdot\boldsymbol{\eta}} t^{\frac{1}{2}\mathbf{n}_1^2+\frac{1}{2}\mathbf{n}_2^2+i_1+i_2+\boldsymbol{\sigma}\cdot(\mathbf{n}_1+\mathbf{n}_2)}\\
&\qquad\left(\frac{1}{2}\mathbf{n}_1^2-\frac{1}{2}\mathbf{n}_2^2+i_1-i_2+\boldsymbol{\sigma}\cdot(\mathbf{n}_1-\mathbf{n}_2)\right)^2\\
&\qquad B_0(\boldsymbol{\sigma}+\mathbf{n}_1)B_0(\boldsymbol{\sigma}+\mathbf{n}_2)Z_{i_1}(\boldsymbol{\sigma}+\mathbf{n}_1)Z_{i_2}(\boldsymbol{\sigma}+\mathbf{n}_2)\\
=&\sum_{\substack{\mathbf{m}_1,\mathbf{m}_2\in Q^\vee\\ j_1,j_2\geq 0}} e^{2\pi\sqrt{-1}(\mathbf{m}_1+\mathbf{m}_2)\cdot\boldsymbol{\eta}} t^{1+\frac{1}{2}\mathbf{m}_1^2+\frac{1}{2}\mathbf{m}_2^2+\boldsymbol{\lambda}_1\cdot(\mathbf{m}_1+\mathbf{m}_2)+j_1+j_2+\boldsymbol{\sigma}\cdot(\mathbf{m}_1+\mathbf{m}_2+2\boldsymbol{\lambda}_1)}\\
&\qquad\left(\frac{1}{2}\mathbf{m}_1^2-\frac{1}{2}\mathbf{m}_2^2+j_1-j_2+(\boldsymbol{\sigma}+\boldsymbol{\lambda}_1)\cdot(\mathbf{m}_1-\mathbf{m}_2)\right)^2\\
&\qquad B_0(\boldsymbol{\sigma}+\mathbf{m}_1+\boldsymbol{\lambda}_1)B_0(\boldsymbol{\sigma}+\mathbf{m}_2+\boldsymbol{\lambda}_1)Z_{j_1}(\boldsymbol{\sigma}+\mathbf{m}_1+\boldsymbol{\lambda}_1)Z_{j_2}(\boldsymbol{\sigma}+\mathbf{m}_2+\boldsymbol{\lambda}_1)
\end{aligned}$$

In the following, $p,q=1,...,n$, $p\neq q$, and following the discussion in the previous section, we look for the lowest terms in powers of t and $\{t^{\sigma_i}\}$. Explicitly, the term to consider is $t^{1+\boldsymbol{\sigma}\cdot(e_p+e_q)}$, which we got by putting $\mathbf{n}_1=e_p+e_q$ and $\mathbf{n}_2=0$ on the LHS, up to symmetry. To get this term we need to impose $\mathbf{m}_1=e_p-e_1$, $\mathbf{m}_2=e_q-e_1$ on the RHS, with all i's and j's vanishing. The functional equation we get, analogous to (2.15), is

$$(1+(e_p+e_q)\cdot\boldsymbol{\sigma})^2B_0(\boldsymbol{\sigma})B_0(\boldsymbol{\sigma}+e_p+e_q)=\left((e_p-e_q)\cdot\boldsymbol{\sigma}\right)^2B_0(\boldsymbol{\sigma}+e_p)B_0(\boldsymbol{\sigma}+e_q)\,.$$

The two cases are distinguished by the different asymptotic conditions (2.7) the root systems impose. Indeed, we have

$$\begin{aligned}
B_0^{[D_n]}(\boldsymbol{\sigma})&=\prod_{i<j}^n\frac{1}{G(1\pm\sigma_i\pm\sigma_j)}\\
B_0^{[B_n]}(\boldsymbol{\sigma})&=\left(\prod_{k=1}^n\frac{1}{G(1\pm\sigma_k)}\right)B_0^{[D_n]}(\boldsymbol{\sigma})
\end{aligned}$$

One can show that the large σ asymptotics of these different solutions are consistent with the full τ system, not only the reduced consistency condition we are considering. Next, since the equation and the Ansatz are the same, the recursion relations are as well, and turn out to be

$$\begin{aligned}
k^2Z_k(\boldsymbol{\sigma})=&\sum_{\substack{(\mathbf{n}-\boldsymbol{\lambda}_1)^2+j_1+j_2=k\\ \mathbf{n}\in\boldsymbol{\lambda}_1+Q^\vee,j_{1,2}<k}}Z_{j_1}(\boldsymbol{\sigma}+\mathbf{n})Z_{j_2}(\boldsymbol{\sigma}-\mathbf{n})\left(j_1-j_2+2\mathbf{n}\cdot\boldsymbol{\sigma}\right)^2\frac{B_0(\boldsymbol{\sigma}\pm\mathbf{n})}{B_0(\boldsymbol{\sigma})^2}\\
&-\sum_{\substack{\mathbf{n}^2+i_1+i_2=k\\ \mathbf{n}\in Q^\vee,\,i_{1,2}<k}}Z_{j_1}(\boldsymbol{\sigma}+\mathbf{n})\times Z_{j_2}(\boldsymbol{\sigma}-\mathbf{n})\left(i_1-i_2+2\mathbf{n}\cdot\boldsymbol{\sigma}\right)^2\frac{B_0(\boldsymbol{\sigma}\pm\mathbf{n})}{B_0(\boldsymbol{\sigma})^2}
\end{aligned}\tag{2.17}$$

This result is in line with the contour integral formulae for the relevant Nekrasov partition functions. Indeed the poles in the D_n and B_n cases are the same, but with different residues, as noticed in [193]. From the above recursion relation we can compute the 1-instanton terms

$$Z_1(\boldsymbol{\sigma}) = \sum_{k=1}^{n} 4\sigma_k^2 \frac{B_0(\boldsymbol{\sigma} \pm e_k)}{B_0(\boldsymbol{\sigma})^2} = \begin{cases} \sum_{k=1}^n \frac{-4}{\prod\limits_{j\neq k}(\sigma_k^2-\sigma_j^2)^2}, & B_n \\ \sum_{k=1}^n \frac{4\sigma_k^2}{\prod\limits_{j\neq k}(\sigma_k^2-\sigma_j^2)^2}, & D_n \end{cases}$$

and the 2-instantons

$$\begin{aligned} Z_2(\boldsymbol{\sigma}) = & \sum_{\boldsymbol{\alpha}\in Q^\vee, \boldsymbol{\alpha}^2=2} \frac{-1}{(\boldsymbol{\alpha}\cdot\boldsymbol{\sigma})^2((\boldsymbol{\alpha}\cdot\boldsymbol{\sigma})^2-1)^2 \prod\limits_{\boldsymbol{\beta}\cdot\boldsymbol{\alpha}=1}(\boldsymbol{\beta}\cdot\boldsymbol{\sigma})^2} \\ & + \sum_{k=1}^{n} \frac{Z_1(\boldsymbol{\sigma}+e_k)(\sigma_k+\frac{1}{2})^2 + Z_1(\boldsymbol{\sigma}-e_k)(\sigma_k-\frac{1}{2})^2}{\prod\limits_{\boldsymbol{\beta}\cdot e_k=\pm 1}(\boldsymbol{\beta}\cdot\boldsymbol{\sigma})} \end{aligned}$$

and so on. These are easily compared to the instanton counting from the introduction 1.3.2, and the appendix of [193] where the results were first presented.

2.2.4.3 $D_2 = A_1 \times A_1$

An interesting thing about (2.17) is that it generalizes to lower n. Explicitly, under the isomorphism

$$\begin{aligned} \sigma_1^{[D_2]} &= (\sigma_1+\sigma_2)^{[A_1\times A_1]} \\ \sigma_2^{[D_2]} &= (\sigma_1-\sigma_2)^{[A_1\times A_1]} \end{aligned} \tag{2.18}$$

we find

$$\begin{aligned} Z_1(\boldsymbol{\sigma})^{[D_2]} &= Z_1(\sigma_1)^{[A_1]} + Z_1(\sigma_2)^{[A_1]} \\ Z_2(\boldsymbol{\sigma})^{[D_2]} &= Z_2(\sigma_1)^{[A_1]} + 2Z_1(\sigma_1)^{[A_1]}Z_1(\sigma_2)^{[A_1]} + Z_2(\sigma_2)^{[A_1]} \end{aligned}$$

That this continues can be confirmed by the recursion relations or instanton counting. Together with (2.18) which splits $Q_{D_2} \cong Q_{A_1} \times Q_{A_1}$ this suggests

$$\tau_0^{[D_2]} = (\tau_0^{[A_1]})^2, \quad \tau_1^{[D_2]} = (\tau_1^{[A_1]})^2$$

Since $D^2(\tau_0^{[A_1]}) = -t^{1/2}(\tau_1^{[A_1]})^2$ and $D^2(\tau_1^{[A_1]}) = -t^{1/2}(\tau_0^{[A_1]})^2$,

$$D^2((\tau_0^{[A_1]})^2) = 2(\tau_0^{[A_1]})^2 D^2((\tau_0^{[A_1]})^2) = 2t^{-1/2}D^2((\tau_1^{[A_1]})^2)t^{1/2}\tau_1^{[A_1]} = D^2((\tau_1^{[A_1]})^2)$$

So (2.16) is valid in this case as well. From the isomonodromic viewpoint, a linear quiver such as $A_1 \times A_1$ corresponds to the degeneration of the sphere with 5 points where we have two complex deformations, the Garnier system. Up to now, we have not considered masses. However, due to this identification, we expect an identification between D_n with one fundamental flavor and the $SU(2) \times SU(2)$ quiver with one bifundamental. Indeed, we find

$$\sum_{i\geq 0} Z_i^{[D_2]}(b_1-b_2, b_1+b_2, m)t^i = e^{4t}\sum_{i\geq 0} Z_i^{[A_1\times A_1]}(b_1, b_2, m)(-1)^i t^i$$

This agrees with [138].

2.2.4.4 $D_3 = A_3$

Paralleling the previous discussion, there is a linear isomorphism of D_3 and A_3. Their extended root systems are the same, and from (2.3) for D_3 we can obtain (2.16), (2.17) since

$$D^2(\tau_0) = -t^{1/4}\tau_2\tau_3 = D^2(\tau_1)$$

so the equations are likewise the same.

2.2.4.5 C_n

Here there is a potential issue of normalizing the roots, so we must make note of our conventions. In writing (2.6) we have stressed that the bilinear form is fixed by demanding $|\boldsymbol{\alpha}|^2 = 2$ for all long roots $\boldsymbol{\alpha}$. If we decide to choose roots of C_n as $\{D_n \text{ roots}\} \cup \{\pm 2e_i\}$, clearly $|2e_i|^2 = 4$. So we should normalize them as $\{\pm\frac{1}{\sqrt{2}}e_i \pm \frac{1}{\sqrt{2}}e_j\} \cup \{\pm\sqrt{2}e_i\}$. The dual lattice is then $Q^\vee = \sqrt{2}\mathbb{Z}^n$. In literature, the factors of $\sqrt{2}$ are sometimes avoided, which can be accommodated in this approach by rescaling time and working with

$$\tau_i = \sum_{\mathbf{m}\in\mathbb{Z}^n+\lambda_i^\vee} e^{\mathbf{m}\cdot\boldsymbol{\eta}} t^{\frac{1}{2}\sum_{i=1}^n(\sigma_i+m_i)^2} B(\boldsymbol{\sigma}+\mathbf{m}|\sqrt{t}) \tag{2.19}$$

The minuscule weights are $\boldsymbol{\lambda}_0 = \mathbf{0}$ and $\boldsymbol{\lambda}_n = ((\frac{1}{\sqrt{2}})^n)$. Bilinear relations are only available for $n = 1, 2$, where accidental isomorphisms map the algebras to those already considered. We explore the lower ranks explicitly up to and including C_4.

As for the analysis of the higher order algebras, these produce more complicated recurrence relations to be solved by a case by case analysis, unlike in the A, B, D types which allow for a unified treatment. We performed explicit checks for C_5 and C_6 up to one-instanton, again in agreement with [193].

2.2.4.6 C_1

This is the simplest case, in fact isomorphic to A_1. The coroot lattice is $Q^\vee = \sqrt{2}\mathbb{Z}$, $\lambda_1 = 1/\sqrt{2}$, and the equations are formally the same as A_1,

$$D^2(\tau_0) = -t^{\frac{1}{2}}\tau_1^2, \quad D^2(\tau_1) = -t^{\frac{1}{2}}\tau_0^2$$

2.2.4.7 C_2

For the subsequent rank, the lattice is $Q^\vee = \sqrt{2}\mathbb{Z}^2$, $\boldsymbol{\lambda} = [(\frac{1}{\sqrt{2}})^2]$. The full system

$$D^2(\tau_0) = -t^{\frac{1}{3}}\tau_1, \quad D^2(\tau_1) = -2t^{\frac{1}{3}}\tau_0^2\tau_2^2, \quad D^2(\tau_2) = -t^{\frac{1}{3}}\tau_1$$

leads to the single equation

$$D^2(\tau_0) = D^2(\tau_1)$$

As with the rank 1 case, there is an accidental isomorphism at this level, namely, $C_2 \cong B_2$, i.e. $\mathfrak{sp}_2 \cong \mathfrak{so}_5$, leading to the same equation. The isomorphism is realised by

$$2\sigma_1^{[C_2]} = (\sigma_1 + \sigma_2)^{[B_2]} \quad 2\sigma_2^{[C_2]} = (\sigma_1 - \sigma_2)^{[B_2]} .$$

As such, we find a recurrence relation for the equivariant volumes of the instanton moduli space which resembles the other recurrence relations we have already found, but it does not generalize to higher rank and pertains only to C_2.

One easily finds that

$$B_0(\boldsymbol{\sigma}) = \frac{1}{G(1 \pm \sqrt{2}\sigma_1)G(1 \pm \sqrt{2}\sigma_1)G(1 \pm \frac{1}{\sqrt{2}}(\sigma_1 \pm \sigma_2))}$$

as well as the simple recurrence relation which we can write as

$$\begin{aligned}2(\frac{k}{2})^2 Z_k(\boldsymbol{\sigma}) = & \sum_{\frac{1}{2}\mathbf{m}+\boldsymbol{\lambda}\cdot\mathbf{m}+j_1+j_2=\frac{k-1}{2}} (j_1 - j_2 + 2(\boldsymbol{\lambda}+\mathbf{m})\cdot\boldsymbol{\sigma})^2 \\ & Z_{2j_1}(\boldsymbol{\sigma}+\boldsymbol{\lambda}+\mathbf{m})Z_{2j_2}(\boldsymbol{\sigma}-\boldsymbol{\lambda}-\mathbf{m})\frac{B_0(\boldsymbol{\sigma}\pm(\boldsymbol{\lambda}+\mathbf{m}))}{B_0(\boldsymbol{\sigma})^2} \\ & - \sum_{\substack{\mathbf{n}^2+i_1+i_2=\frac{k}{2} \\ i_{1,2}<k/2}} (i_1 - i_2 + 2\mathbf{n}\cdot\boldsymbol{\sigma})^2 \, Z_{2i_1}(\boldsymbol{\sigma}+\mathbf{n})Z_{2i_2}(\boldsymbol{\sigma}-\mathbf{n})\frac{B_0(\boldsymbol{\sigma}\pm\mathbf{n})}{B_0(\boldsymbol{\sigma})^2}\end{aligned}$$

2.2.4.8 C_3

In higher ranks one gets higher order relations among the τ-functons. In particular, while for C_n with n even the central node is seen to be invariant, n being odd presents an interesting challenge. In the following, $Q^\vee = \sqrt{2}\mathbb{Z}^3$, and $\boldsymbol{\lambda}_3 = ((\frac{1}{\sqrt{2}})^3)$, and the τ-system is

$$\begin{aligned}D^2(\tau_0) &= -t^{\frac{1}{4}}\tau_1 \\ D^2(\tau_1) &= -2t^{\frac{1}{4}}\tau_0^2\tau_2 \\ D^2(\tau_2) &= -2t^{\frac{1}{4}}\tau_1\tau_3^2 \\ D^2(\tau_3) &= -t^{\frac{1}{4}}\tau_2\end{aligned} \tag{2.20}$$

By multiplying (2.20) by τ_0^2, we obtain

$$\tau_0^2 D^2(\tau_3) = -t^{\frac{1}{4}}\tau_0^2\tau_2 = \frac{1}{2}D^2(\tau_1) = \frac{1}{2}t^{-\frac{1}{2}}D^4(\tau_0)$$

Dividing by τ_0 and using the Y operators defined in 2.2.3 we rewrite this as the cubic system

$$Y^3(\tau_0) = 2t^{1/2}\tau_0 D^2(\tau_3)$$

Inserting (2.19) we obtain

$$\sum_{\substack{\mathbf{n}_{1,2,3}\in\sqrt{2}\mathbb{Z}^3\\ i_{1,2,3}\in\mathbb{N}_0}}\prod_{k=1}^{3} e^{2\pi i\boldsymbol{\eta}\cdot\mathbf{n}_k}t^{\frac{1}{2}(\boldsymbol{\sigma}+\mathbf{n}_k)^2+i_k}B_0(\boldsymbol{\sigma}+\mathbf{n}_k)Z_{i_k}(\boldsymbol{\sigma}+\mathbf{n}_k)$$
$$\frac{1}{3!}\prod_{k_1<k_2}\left(\frac{1}{2}\mathbf{n}_{k_1}^2+i_{k_1}-\frac{1}{2}\mathbf{n}_{k_2}^2-i_{k_2}+(\mathbf{n}_{k_1}-\mathbf{n}_{k_2})\cdot\boldsymbol{\sigma}\right)^2$$
$$=2t^{1/2}\sum_{\substack{\mathbf{m}_1\in\sqrt{2}\mathbb{Z}^3\\ \mathbf{m}_{2,3}\in\sqrt{2}\mathbb{Z}^3+\boldsymbol{\lambda}_3\\ j_{1,2,3}\in\mathbb{N}_0}}\prod_{k=1}^{3} e^{2\pi i\boldsymbol{\eta}\cdot\mathbf{n}_k}t^{\frac{1}{2}(\boldsymbol{\sigma}+\mathbf{n}_k)^2+i_k}B_0(\boldsymbol{\sigma}+\mathbf{n}_k)Z_{i_k}(\boldsymbol{\sigma}+\mathbf{n}_k)$$
$$\left(\frac{1}{2}\mathbf{m}_2^2+j_2-\frac{1}{2}\mathbf{m}_3^2-j_3+(\mathbf{m}_2-\mathbf{m}_3)\cdot(\boldsymbol{\sigma}+\boldsymbol{\lambda}_3)\right)^2$$

Then, seeing that $2\times\frac{1}{2}\boldsymbol{\lambda}_3^2=\frac{3}{2}$ and rewriting $\mathbf{m}_1=\mathbf{m}_1^{(0)}$ $\mathbf{m}_{2,3}=\mathbf{m}_{2,3}^{(0)}+\boldsymbol{\lambda}_3$ where $\mathbf{m}_{1,2,3}^{(0)}\in\sqrt{2}\mathbb{Z}^3$, we reduce to the constraints

$$\sum_{k=1}^{3}\frac{1}{2}\mathbf{n}_i^2+i_k=2+\boldsymbol{\lambda}_3\cdot(\mathbf{m}_2^{(0)}+\mathbf{m}_3^{(0)})+\sum_{k=1}^{3}\frac{1}{2}(\mathbf{m}_k^{(0)})^2+j_k \tag{2.21}$$
$$\sum_{k=1}^{3}\mathbf{n}_i=2\boldsymbol{\lambda}_3+\sum_{k=1}^{3}\mathbf{m}_i^{(0)} \tag{2.22}$$

Let p_1,p_2,p_3 be a permutation of $\{1,2,3\}$. We consider factors of $t^{\sqrt{2}\boldsymbol{\sigma}\cdot(e_{p_1}+e_{p_2})+2}$, in other words (2.21)=2 and (2.22)= $\sqrt{2}(e_{p_1}+e_{p_2})$. For the LHS we find the solutions $\mathbf{n}_1=\sqrt{2}(e_{p_1}+e_{p_2})$, $\mathbf{n}_2=\mathbf{n}_3=\mathbf{0}$ and permutations thereof, for which the LHS vanishes due to degeneracy, and $\mathbf{n}_1=\sqrt{2}e_{p_1}$, $\mathbf{n}_2=\sqrt{2}e_{p_1}$, $\mathbf{n}_3=\mathbf{0}$. For the RHS, there are two solutions $\mathbf{m}_1^{(0)}=\mathbf{m}_2^{(0)}=\mathbf{0}$, $\mathbf{m}_3^{(0)}=-\sqrt{2}e_{p_3}$, and $\mathbf{m}_1^{(0)}=\mathbf{m}_3^{(0)}=\mathbf{0}$, $\mathbf{m}_2^{(0)}=-\sqrt{2}e_{p_3}$. We are led then to the equation

$$\begin{aligned}(1+\sqrt{2}\sigma_{p_1})^2(1+\sqrt{2}\sigma_{p_2})^2(\sigma_{p_1}-\sigma_{p_2})^2B_0(\boldsymbol{\sigma}+\sqrt{2}e_{p_1})B_0(\boldsymbol{\sigma}+\sqrt{2}e_{p_2})&\\ =4\sigma_{p_3}^2B_0(\boldsymbol{\sigma}+\boldsymbol{\lambda}_3)B_0(\boldsymbol{\sigma}+\boldsymbol{\lambda}_3-\sqrt{2}e_{p_3})&\end{aligned}$$

Using (2.9) we find on the LHS

$$(\sigma_{p_1}-\sigma_{p_2})^2\frac{2}{\sigma_{p_1}^2(\sigma_{p_1}^2-\sigma_{p_2}^2)(\sigma_{p_1}^2-\sigma_{p_3}^2)}\frac{2}{\sigma_{p_2}^2(\sigma_{p_1}^2-\sigma_{p_2}^2)(\sigma_{p_2}^2-\sigma_{p_3}^2)}$$

and on the RHS

$$4\sigma_{p_3}^2\frac{1}{\sigma_{p_1}\sigma_{p_2}\sigma_{p_3}(\sigma_{p_1}+\sigma_{p_2})(\sigma_{p_1}+\sigma_{p_3})(\sigma_{p_2}+\sigma_{p_3})}\frac{1}{\sigma_{p_1}\sigma_{p_2}(-\sigma_{p_3})(\sigma_{p_1}+\sigma_{p_2})(\sigma_{p_1}-\sigma_{p_3})(\sigma_{p_2}-\sigma_{p_3})}$$

as there are no roots $\beta^\vee$ such that $\beta^\vee\cdot\boldsymbol{\lambda}_3=2$. Due to this an equality, we get

$$B_0(\boldsymbol{\sigma})=\prod_{i=1}^{3}\frac{1}{G(1\pm\sqrt{2}\sigma_i)}\prod_{i<j=1}^{3}\frac{1}{G(1\pm\frac{1}{\sqrt{2}}(\sigma_i\pm\sigma_i))}$$

Keeping (2.21)=2, but letting (2.22)=$\sqrt{2}e_{p_1}$, we find one nonvanishing solution for the LHS, $\mathbf{n}_1 = \sqrt{2}e_{p_1}$, $\mathbf{n}_2 = \mathbf{n}_3 = 0$, and $i_2 = 1$ or $i_3 = 1$, with the rest zero. This leads to the term

$$-\frac{4}{(\sigma_{p_1} \pm \sigma_{p_2})(\sigma_{p_1} \pm \sigma_{p_3})} Z_1(\boldsymbol{\sigma})$$

On the RHS we can describe the four solutions as the two couples $\mathbf{m}_1 = 0$, $\mathbf{m}_2 = 1/\sqrt{2}e_{p_1} \pm 1/\sqrt{2}(e_{p_2} + e_{p_3})$ and $\mathbf{m}_3 = 1/\sqrt{2}e_{p_1} \mp 1/\sqrt{2}(e_{p_2} + e_{p_3})$ and $\mathbf{m}_1 = 0$, $\mathbf{m}_2 = 1/\sqrt{2}e_{p_1} + 1/\sqrt{2}(\pm e_{p_2} \mp e_{p_3})$ and $\mathbf{m}_3 = 1/\sqrt{2}e_{p_1} + 1/\sqrt{2}(\mp e_{p_2} + \pm e_{p_3})$. This gives the RHS

$$-\frac{16}{\sigma_1^2\sigma_2^2\sigma_3^2(\sigma_{p_1} \pm \sigma_{p_2})(\sigma_{p_1} \pm \sigma_{p_3})}$$

so that $Z_1(\boldsymbol{\sigma}) = \frac{4}{\sigma_1^2\sigma_2^2\sigma_3^2}$, which is indeed the 1 $Sp(6)$ instanton equivariant volume, with the v.e.v.'s rescaled by $\sqrt{2}$ factors.

Continuing to two instantons, we have to collect $t^{\sqrt{2}\boldsymbol{\sigma}\cdot e_{p_1}+3}$ terms, as we find that $t^{\sqrt{2}\boldsymbol{\sigma}\cdot(e_{p_1}+e_{p_2})+3}$ ones don't involve Z_2 and lead to an identity involving shifts of Z_1 and rational functions. The structure of solutions is more involved. By picking $(p_1, p_2, p_3) = (1, 2, 3)$ for readability, we find the relation $\left(\sqrt{2}\sigma_1 - 1\right)^2\sigma_2^2\sigma_3^2 Z_2(\boldsymbol{\sigma}) =$

$$\begin{aligned}
&= \sum_{(w_1,w_2)\in\{(-1,-1),(-1,1),(1,-1)\}} \frac{2\left(\sigma_1 + \sigma_2 w_1 + \sigma_3 w_2\right)^2}{\sigma_1^2\left(\sqrt{2}\sigma_1 + 1\right)^2\left(\sigma_1 + \sigma_2 w_1\right)^2\left(\sigma_1 + \sigma_3 w_2\right)^2\left(\sigma_2 w_1 + \sigma_3 w_2\right)^2} + 2Z_1\left(\boldsymbol{\sigma}\right) \\
&+ \sum_{(w_1,w_2)\in\{(-1,-1),(-1,1),(1,-1),(1,1)\}} \frac{\left(\sqrt{2}\sigma_2 w_1 + \sqrt{2}\sigma_3 w_2 + 1\right)^2 Z_1\left(\sigma_1 + \frac{1}{\sqrt{2}}, \sigma_2 + \frac{w_1}{\sqrt{2}}, \sigma_3 + \frac{w_2}{\sqrt{2}}\right)}{2\left(\sigma_2 w_1 + \sigma_3 w_2\right)^2} \\
&+ \sum_{w=\pm1} \frac{2\sigma_2^2}{\sigma_3^2\left(\sigma_2^2 - \sigma_3^2\right)^2\left(1 - \sqrt{2}\sigma_3 w\right)^2\left(\sigma_3 w + \sigma_1\right)^2} + \frac{2\sigma_3^2}{\sigma_2^2\left(\sigma_2^2 - \sigma_3^2\right)^2\left(1 - \sqrt{2}\sigma_2 w\right)^2\left(\sigma_2 w + \sigma_1\right)^2} \\
&+ \sum_{w=\pm1} -\frac{2\sigma_3^2\left(\sqrt{2}\sigma_1 + \sqrt{2}\sigma_2 w + 1\right)^2}{\left(\sqrt{2}\sigma_1 + 1\right)^2\sigma_2^2\left(\sigma_2^2 - \sigma_3^2\right)^2\left(\sigma_1 - \sigma_2 w\right)^2\left(\sqrt{2}\sigma_2 w + 1\right)^2} - \frac{2\sigma_2^2\left(\sqrt{2}\sigma_1 + \sqrt{2}\sigma_3 w + 1\right)^2}{\left(\sqrt{2}\sigma_1 + 1\right)^2\sigma_2^2\left(\sigma_2^2 - \sigma_3^2\right)^2\left(\sigma_1 - \sigma_3 w\right)^2\left(\sqrt{2}\sigma_3 w + 1\right)^2} \\
&- \frac{32\sigma_2^2}{\left(\sqrt{2}\sigma_1 + 1\right)^2\left(1 - 2\sigma_3^2\right)^2\left(\sigma_2^2 - \sigma_3^2\right)^2} - \frac{32\sigma_3^2}{\left(\sqrt{2}\sigma_1 + 1\right)^2\left(1 - 2\sigma_2^2\right)^2\left(\sigma_2^2 - \sigma_3^2\right)^2} - \frac{1}{4}\left(\sqrt{2}\sigma_1 + 2\right)^2\sigma_2^2\sigma_3^2 Z_1\left(\boldsymbol{\sigma}\right) Z_1\left(\sigma_1 + \sqrt{2}, \sigma_2, \sigma_3\right)
\end{aligned}$$

which gives the correct 2-instanton equivariant volume compared to instanton counting, although in a vastly different presentation.

2.2.4.9 C_4

In this case n is even, so under shifts, the middle node gets mapped to itself, up to some power of t as required by asymptotics. The relevant lattice is $Q^\vee = \sqrt{2}\mathbb{Z}^4$, the shift $\boldsymbol{\lambda}_4 = ((\frac{1}{\sqrt{2}})^4)$, and the full system is

$$D^2(\tau_0) = -t^{\frac{1}{5}}\tau_1, \quad D^2(\tau_1) = -2t^{\frac{1}{5}}\tau_0^2\tau_2, \quad D^2(\tau_2) = -2t^{\frac{1}{5}}\tau_1\tau_3$$
$$D^2(\tau_3) = -2t^{\frac{1}{5}}\tau_2\tau_4^2, \quad D^2(\tau_4) = -t^{\frac{1}{5}}\tau_3$$

We can eliminate the middle node tau function τ_2 from the following

$$D^4(\tau_0) = -2t^{-\frac{1}{5}}\tau_0^2\tau_2, \quad D^4(\tau_4) = -2t^{-\frac{1}{5}}\tau_2\tau_4^2$$

to write

$$\tau_4 Y^3(\tau_0) = \tau_0 Y^3(\tau_4)\,.$$

We can repeat the calculation in the previous section, this time in short. Inserting (2.19) we obtain

$$
\begin{aligned}
&\sum_{\substack{\mathbf{n}_1\in\sqrt{2}\mathbb{Z}^4+\boldsymbol{\lambda}_4\\ \mathbf{n}_{2,3,4}\in\sqrt{2}\mathbb{Z}^3\\ i_{1,2,3,4}\in\mathbb{N}_0}} \prod_{k=1}^{4} e^{2\pi i\boldsymbol{\eta}\cdot\mathbf{n}_k} t^{\frac{1}{2}(\boldsymbol{\sigma}+\mathbf{n}_k)^2+i_k} B_0(\boldsymbol{\sigma}+\mathbf{n}_k) Z_{i_k}(\boldsymbol{\sigma}+\mathbf{n}_k)\\
&\prod_{k_1<k_2=2}^{4}\left(\frac{1}{2}\mathbf{n}_{k_1}^2+i_{k_1}-\frac{1}{2}\mathbf{n}_{k_2}^2-i_{k_2}+(\mathbf{n}_{k_1}-\mathbf{n}_{k_2})\cdot\boldsymbol{\sigma}\right)^2\\
=&\sum_{\substack{\mathbf{m}_1\in\sqrt{2}\mathbb{Z}^4\\ \mathbf{m}_{2,3,4}\in\sqrt{2}\mathbb{Z}^4+\boldsymbol{\lambda}_4\\ j_{1,2,3,4}\in\mathbb{N}_0}} \prod_{k=1}^{4} e^{2\pi i\boldsymbol{\eta}\cdot\mathbf{n}_k} t^{\frac{1}{2}(\boldsymbol{\sigma}+\mathbf{n}_k)^2+i_k} B_0(\boldsymbol{\sigma}+\mathbf{n}_k) Z_{i_k}(\boldsymbol{\sigma}+\mathbf{n}_k)\\
&\prod_{k_1<k_2=2}^{4}\left(\frac{1}{2}\mathbf{m}_{k_1}^2+j_{k_1}-\frac{1}{2}\mathbf{m}_{k_2}^2-j_{k_2}+(\mathbf{m}_{k_1}-\mathbf{m}_{k_2})\cdot\boldsymbol{\sigma}\right)^2
\end{aligned}
$$

Then, as $2\times\frac{1}{2}\boldsymbol{\lambda}_4^2=2$, we decompose the vectors in terms of the coroot lattice as $\mathbf{n}_1=\mathbf{n}_1^{(0)}+\boldsymbol{\lambda}_4$, $\mathbf{n}_{2,3,4}=\mathbf{n}_{2,3,4}^{(0)}$, $\mathbf{m}_1=\mathbf{m}_1^{(0)}$ $\mathbf{m}_{2,3,4}=\mathbf{m}_{2,3}^{(0)}+\boldsymbol{\lambda}_4$ which implies the constraints

$$\boldsymbol{\lambda}_4\cdot\mathbf{n}_1^{(0)}+\sum_{k=1}^{4}\frac{1}{2}(\mathbf{n}_i^{(0)})^2+i_k=2+\boldsymbol{\lambda}_4\cdot(\mathbf{m}_2^{(0)}+\mathbf{m}_3^{(0)}+\mathbf{m}_4^{(0)})+\sum_{k=1}^{4}\frac{1}{2}(\mathbf{m}_k^{(0)})^2+j_k \tag{2.23}$$

$$\sum_{k=1}^{4}\mathbf{n}_i^{(0)}=2\boldsymbol{\lambda}_4+\sum_{k=1}^{4}\mathbf{m}_i^{(0)} \tag{2.24}$$

Let p_1,p_2,p_3,p_4 be a permutation of $\{1,2,3,4\}$. To obtain the functional equations for the one-loop term we consider factors of $t^{\sqrt{2}\boldsymbol{\sigma}\cdot(e_{p_1}+e_{p_2})+2}$, (2.23)=2 and (2.24)= $\sqrt{2}(e_{p_1}+e_{p_2})$. For the LHS the only nonvanishing solutions are $\mathbf{n}_1^{(0)}=0$ and $\mathbf{n}_{2,3,4}$ permutations of $\{\sqrt{2}e_{p_1},\sqrt{2}e_{p_2},0\}$, while on the RHS the only nonvanishing ones are $\mathbf{m}_1^{(0)}=0$ and $\mathbf{m}_{2,3,4}^{(0)}$ permutations of $\{-\sqrt{2}e_{p_3},-\sqrt{2}e_{p_4},0\}$. Some factors cancel, leading to

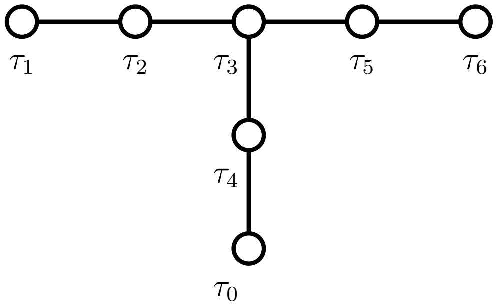
$$
\begin{aligned}
&2(1+\sqrt{2}\sigma_{p_1})^2(1+\sqrt{2}\sigma_{p_2})^2(\sigma_{p_1}-\sigma_{p_2})^2 B_0(\boldsymbol{\sigma}+\sqrt{2}e_{p_1})B_0(\boldsymbol{\sigma}+\sqrt{2}e_{p_2})\\
&\quad=4\sigma_{p_3}^2(\sigma_{p_3}-\sigma_{p_4})^2\sigma_{p_4}^2 B_0(\boldsymbol{\sigma}+\boldsymbol{\lambda}_3-\sqrt{2}e_{p_3})B_0(\boldsymbol{\sigma}+\boldsymbol{\lambda}_3-\sqrt{2}e_{p_4})
\end{aligned}
$$

We checked that (2.8) satisfies this relation also in this case. To find the one-instanton term, we need to collect factors of $t^{\sqrt{2}\boldsymbol{\sigma}\cdot e_{p_1}+2}$. The solutions on the LHS are either with all i's vanishing, that is with $\mathbf{n}_1^{(0)}=-\mathbf{n}_k^{(0)}=-\sqrt{2}e_p$ for any $k,p\in\{2,3,4\}$ and the remaining two vectors equal to $\sqrt{2}e_{p_1}$ and zero respectively, or with one out of $i_{2,3,4}$ being 1 with a single vector – of index different from both 1 and from the index of the is – being equal to $\sqrt{2}e_p$. On the RHS, j's vanish, $\mathbf{m}_1^{(0)}=0$ and the rest are a permutation of $\{\sqrt{2}e_{\tilde{p}_2},\sqrt{2}e_{\tilde{p}_3}+\sqrt{2}e_{\tilde{p}_4},0\}$ with $\tilde{p}_{2,3,4}$ a permutation of $\{p_2,p_3,p_4\}$. After some cancellation of rational functions, we find the correct one-instanton term $Z_1(\boldsymbol{\sigma})=-8/\sigma_1^2\sigma_2^2\sigma_3^2\sigma_4^2$.

One continues similarly up to higher order. We have checked agreement with instanton counting until four instantons.

2.2.4.10 E_6

Even though the equations presented up to this point were novel, the instanton volumes were able to be obtained by means of instanton counting as in the introduction 1.3.2. We now turn to non-classical Lie algebras and describe novel ways of obtaining instanton volumes where instanton counting is unavailable. We note that yet another way to obtain them is via blowup relations, likewise conjectural at time of writing, and these serve as a cross-check. They are also described in the introduction 1.3.2. Computationally, however, the equations we find are quicker, because they only involve instanton volumes at the same Ω-background. We begin with the simplest simply laced exceptional Lie algebra, E_6.



Due to the similarities of the root systems of the E-type algebras, we will give a brief overview of E_8 at this point and describe the others as its reductions. The root system is the union of D_8 roots $\{e_i \pm e_j\}_{i\neq j}$ and $(x_1, ..., x_8) \in \mathbb{R}^8$ of length 2 such that all $x_i \in \mathbb{Z} + \frac{1}{2}$ and $\sum_i x_i$ is even. The coroot lattice can be obtained from two cosets of the D_8 one as $Q^{[E_8]} = Q^{[D_8]} \cup \left(Q^{[E_8]} + ((\frac{1}{2})^{n-1}, -\frac{1}{2})\right)$. E_6 is then obtained by projecting all of the roots to have the last three coordinates equal, $(x_1, ..., x_5, x_6, x_6, x_6)$. Clearly, this forces the D_8-type roots to an embedding of D_5, with the last three coordinates zero. Unlike E_8, which is unimodular and has no minuscule coweights, E_6 has three: $\boldsymbol{\lambda}_0 = 0$, $\boldsymbol{\lambda}_1 = (1, 0^4, (-\frac{1}{3})^3)$, and $\boldsymbol{\lambda}_6 = (0^5, (-\frac{2}{3})^3)$. The Dynkin diagram exhibits an outer $\mathbb{Z}_3$ symmetry. For this exceptional algebra we obtain the τ-system

$$\tau_4 = -t^{-\frac{1}{12}} D^2(\tau_0), \quad \tau_2 = -t^{-\frac{1}{12}} D^2(\tau_1), \quad \tau_5 = -t^{-\frac{1}{12}} D^2(\tau_6) \tag{2.25}$$

$$D^2(\tau_3) = -t^{\frac{1}{12}} \tau_2 \tau_4 \tau_5$$

$$-t^{\frac{1}{12}} \tau_3 = \tau_0^{-1} D^2(\tau_4) = \tau_1^{-1} D^2(\tau_2) = \tau_6^{-1} D^2(\tau_5) \tag{2.26}$$

Focusing on the legs with τ_0 and τ_6, inserting (2.25) in the last equation (2.26) and using the operators defined in (2.10) gives us

$$Y^3(\tau_0) = Y^3(\tau_6) \tag{2.27}$$

The Kiev Ansatz we insert likewise has $\boldsymbol{\sigma}, \boldsymbol{\eta} \in \mathbb{C}^8$, but with the last three components restricted to be same. The equation to be solved becomes

$$\sum_{\substack{\mathbf{n}_{1,2,3}\in Q\\ i_{1,2,3}\in\mathbb{N}}} \prod_{k=1}^{3} e^{2\pi i\boldsymbol{\eta}\cdot\mathbf{n}_k} t^{\frac{1}{2}(\boldsymbol{\sigma}+\mathbf{n}_k)^2+i_k} B_0(\boldsymbol{\sigma}+\mathbf{n}_k) Z_{i_k}(\boldsymbol{\sigma}+\mathbf{n}_k)$$
$$\prod_{k_1<k_2}\left(\frac{1}{2}\mathbf{n}_{k_1}^2+i_{k_1}-\frac{1}{2}\mathbf{n}_{k_2}^2-i_{k_2}+(\mathbf{n}_{k_1}-\mathbf{n}_{k_2})\cdot\boldsymbol{\sigma}\right)^2$$
$$=\sum_{\substack{\mathbf{m}_{1,2,3}\in Q\\ j_{1,2,3}\in\mathbb{N}}} t^2\cdot\prod_{k=1}^{3} e^{2\pi i\boldsymbol{\eta}\cdot\mathbf{m}_k} t^{\frac{1}{2}(\boldsymbol{\sigma}+\mathbf{m}_k)^2+\boldsymbol{\lambda}_6\cdot(\boldsymbol{\sigma}+\mathbf{m}_k)+i_k} B_0(\boldsymbol{\sigma}+\mathbf{m}_k+\boldsymbol{\lambda}_6) Z_{j_k}(\boldsymbol{\sigma}+\mathbf{m}_k+\boldsymbol{\lambda}_6)$$
$$\prod_{k_1<k_2}\left(\frac{1}{2}\mathbf{m}_{k_1}^2+j_{k_1}-\frac{1}{2}\mathbf{m}_{k_2}^2-j_{k_2}+(\mathbf{m}_{k_1}-\mathbf{m}_{k_2})\cdot(\boldsymbol{\sigma}+\boldsymbol{\lambda}_6)\right)^2$$

To get the lowest order equations which specify B_0, let $p_1, ...p_5$ be a permutation of $\{1, ..., 5\}$ and let $\boldsymbol{\delta} := ((\frac{1}{2})^8)$. Then looking at the coefficients of $t^{2+\boldsymbol{\sigma}\cdot(2e_{p_1}+e_{p_2}+e_{p_3})}$ gives the equation

$$\begin{aligned}(1+\sigma_{p_1}+\sigma_{p_2})^2(1+\sigma_{p_1}+\sigma_{p_3})^2(\sigma_{p_2}-\sigma_{p_3})^2 B_0(\boldsymbol{\sigma})B_0(\boldsymbol{\sigma}+e_{p_1}+e_{p_2})B_0(\boldsymbol{\sigma}+e_{p_1}+e_{p_3}) = \\ ((\boldsymbol{\delta}-e_{p_2}-e_{p_3})\cdot\boldsymbol{\sigma})^2((\boldsymbol{\delta}-e_{p_2}-e_{p_3}-e_{p_4}-e_{p_5})\cdot\boldsymbol{\sigma})^2(\sigma_{p_4}+\sigma_{p_5})^2\times \\ B_0(\boldsymbol{\sigma}+\boldsymbol{\delta}+\boldsymbol{\lambda})B_0(\boldsymbol{\sigma}+\boldsymbol{\delta}+\boldsymbol{\lambda}-e_{p_4}-e_{p_5})B_0(\boldsymbol{\sigma}+e_{p_1}-\boldsymbol{\lambda}/2)\end{aligned}$$

The solution satisfying the asymptotic behaviour (2.7) is

$$B_0^{[E_6]} = \prod_{i<j=1}^{5}\frac{1}{G(1\pm\sigma_i\pm\sigma_j)}\prod_{\substack{\varepsilon_i=\pm1\\ \prod_{i=1}^8\varepsilon_i=1,\\ \varepsilon_6=\varepsilon_7=\varepsilon_8}}\frac{1}{G(1+\frac{1}{2}\sum_{i=1}^{8}\varepsilon_i\sigma_i)}$$

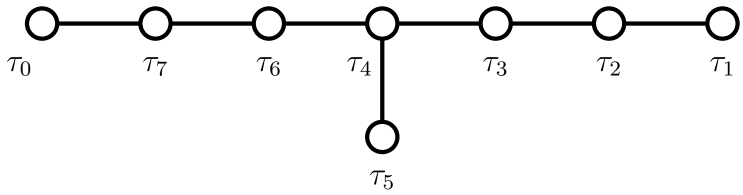
We also solved the recurrence relation arising from (2.27) up to three instantons. For one instanton, our results agree with the ones of [170], and for two instantons they agree with the blowup formula. Three instantons proved to be too computationally intensive to check using the blowup formula, however it obeys the expected large-$\boldsymbol{\sigma}$ limit described in appendix B. The one instanton contribution follows most easily by looking at the coefficients of $t^{2+e_{p_1}+e_{p_2}}$, where we obtain

$$\begin{aligned}&3!(\sigma_{p_1}+\sigma_{p_3})^2(1+\sigma_{p_1}+\sigma_{p_2})^2 B_0(\boldsymbol{\sigma})^2 B_0(\boldsymbol{\sigma}+e_{p_1}+e_{p_2})Z_1(\boldsymbol{\sigma})\\ &=\sum_{\substack{\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3+3\boldsymbol{\lambda}\\ =e_{p_1}+e_{p_2}\\ \frac{1}{2}\mathbf{n}_k^2+\boldsymbol{\lambda}\cdot\mathbf{n}_k=0}}\prod_{i<j}^{3}(\boldsymbol{\sigma}\cdot(\mathbf{n}_i-\mathbf{n}_j))^2\prod_{i=1}^{3}B_0(\boldsymbol{\sigma}+\mathbf{n}_i)\\ &-\sum_{\substack{\mathbf{n}_1+\mathbf{n}_2\\ =e_{p_1}+e_{p_2}\\ \mathbf{n}_1^2=\mathbf{n}_2^2=2}} B_0(\boldsymbol{\sigma})(\boldsymbol{\sigma}\cdot(\mathbf{n}_1-\mathbf{n}_2))^2(\boldsymbol{\sigma}\cdot\mathbf{n}_1)^2(\boldsymbol{\sigma}\cdot\mathbf{n}_2)^2 B_0(\boldsymbol{\sigma}+\mathbf{n}_1)B_0(\boldsymbol{\sigma}+\mathbf{n}_2)\end{aligned}$$

The higher instanton expressions are too cumbersome and unenlighting to display here[3].

[3] These can be provided privatly to any interested reader.

2.2.4.11 E_7



The roots of E_7 are obtained by projecting the E_8 ones to have the last two coordinates equal, $(x_1, ..., x_5, x_6, x_7, x_7)$. E_7 has two minuscule coweights $\boldsymbol{\lambda}_0 = 0$ and $\boldsymbol{\lambda}_1 = (1, 0^5, (-\frac{1}{2})^2)$. The Dynkin diagram exhibits an outer $\mathbb{Z}_2$ symmetry. For this exceptional algebra we obtain the τ-system

$$\begin{gathered}
\tau_7 = -t^{-\frac{1}{18}} D^2(\tau_0), \quad \tau_2 = -t^{-\frac{1}{18}} D^2(\tau_1) \\
\tau_6 = -t^{-\frac{1}{6}} \tau_0^{-1} D^4(\tau_0), \quad \tau_3 = -t^{-\frac{1}{6}} \tau_1^{-1} D^4(\tau_1) \\
\tau_2^{-1} D^2(\tau_3) = -t^{-\frac{1}{18}} \tau_4 = \tau_7^{-1} D^2(\tau_6)
\end{gathered} \tag{2.28}$$

When we rewrite (2.28) in terms of the single equation, the powers of $t^{\frac{1}{18}}$ drop out to give

$$\frac{1}{D^2(\tau_0)} D^2\left(\frac{D^4(\tau_0)}{\tau_0}\right) = \frac{1}{D^2(\tau_1)} D^2\left(\frac{D^4(\tau_1)}{\tau_1}\right)$$

Here we recognize an operator defined in (2.10), which enables us to write

$$Y^4(f) = \frac{1}{D^2(f)} D^2\left(\frac{D^4(f)}{f}\right) \quad \Rightarrow \quad Y^4(\tau_0) = Y^4(\tau_1),$$

The Kiev Ansatz we insert likewise has $\boldsymbol{\sigma}, \boldsymbol{\eta} \in \mathbb{C}^8$, but with the last two components restricted to be same. The equation to be solved becomes

$$\begin{aligned}
&\sum_{\substack{\mathbf{n}_{1,2,3,4} \in Q \\ i_{1,2,3,4} \in \mathbb{N}}} \prod_{k=1}^{4} e^{2\pi i \boldsymbol{\eta} \cdot \mathbf{n}_k} t^{\frac{1}{2}(\boldsymbol{\sigma}+\mathbf{n}_k)^2 + i_k} B_0(\boldsymbol{\sigma} + \mathbf{n}_k) Z_{i_k}(\boldsymbol{\sigma} + \mathbf{n}_k) \\
&\prod_{k_1 < k_2} \left(\frac{1}{2}\mathbf{n}_{k_1}^2 + i_{k_1} - \frac{1}{2}\mathbf{n}_{k_2}^2 - i_{k_2} + (\mathbf{n}_{k_1} - \mathbf{n}_{k_2}) \cdot \boldsymbol{\sigma} \right)^2 \\
= &\sum_{\substack{\mathbf{m}_{1,2,3,4} \in Q \\ j_{1,2,3,4} \in \mathbb{N}}} t^3 \cdot \prod_{k=1}^{4} e^{2\pi i \boldsymbol{\eta} \cdot \mathbf{m}_k} t^{\frac{1}{2}(\boldsymbol{\sigma}+\mathbf{m}_k)^2 + \boldsymbol{\lambda}_1 \cdot (\boldsymbol{\sigma}+\mathbf{m}_k) + i_k} B_0(\boldsymbol{\sigma} + \mathbf{m}_k + \boldsymbol{\lambda}_1) Z_{j_k}(\boldsymbol{\sigma} + \mathbf{m}_k + \boldsymbol{\lambda}_1) \\
&\prod_{k_1 < k_2} \left(\frac{1}{2}\mathbf{m}_{k_1}^2 + j_{k_1} - \frac{1}{2}\mathbf{m}_{k_2}^2 - j_{k_2} + (\mathbf{m}_{k_1} - \mathbf{m}_{k_2}) \cdot (\boldsymbol{\sigma} + \boldsymbol{\lambda}_1) \right)^2
\end{aligned}$$

With regards to the linear and quadratic constraints obtained from comparing exponents of t, $\{t^{\sigma_i}\}_i$, this is similar to C_4. In, $\boldsymbol{\lambda}_1^2 = \frac{3}{2}$, so in the analogue of (2.23) we end up with $\frac{4}{2}\boldsymbol{\lambda}_1^2 = 3$ in pure powers of t. Likewise, we have even powers of τ_1, and $2\boldsymbol{\lambda}_1 \in Q$. Both of these lead to well defined analogues of (2.23) and (2.24). The lowest possible order in t is t^3. If we pick $p_1, ..., p_6$ to be a permutation of $\{1, ..., 6\}$, looking at powers of $t^{3+\boldsymbol{\sigma}\cdot(2e_{p_1}+2e_{p_2}+2e_{p_3})}$ we get

$$\begin{aligned}&B_0(\boldsymbol{\sigma})\prod_{i<j}^{3}(1+\sigma_{p_i}+\sigma_{p_j})^2(\sigma_{p_i}-\sigma_{p_j})^2 B_0(\boldsymbol{\sigma}+e_{p_i}+e_{p_j})\\ =\ & B_0(\boldsymbol{\sigma}+\boldsymbol{\delta}-e_1-e_{p_4}-e_{p_5}-e_{p_6})\prod_{i<j}^{3}(-\delta_{i,1}+\sigma_{p_i}\pm\sigma_{p_j})^2\prod_{i=1}^{3}B_0(\boldsymbol{\sigma}+\boldsymbol{\delta}-e_1-e_{p_{i+3}})\end{aligned}$$

with $\boldsymbol{\delta}:=((\frac{1}{2})^8)$ as above. Clearly, the only lattice points satisfying the quadratic constraint while summing up to $2(e_{p_1}+e_{p_2}+e_{p_3})$ are $e_{p_1}+e_{p_2}, e_{p_1}+e_{p_3}, e_{p_2}+e_{p_3}$ and zero, while the ones on the shifted lattice, which can be inferred from the above equation, are similarly unique up to permutation. The solution satisfying the asymptotic behaviour (2.7) is

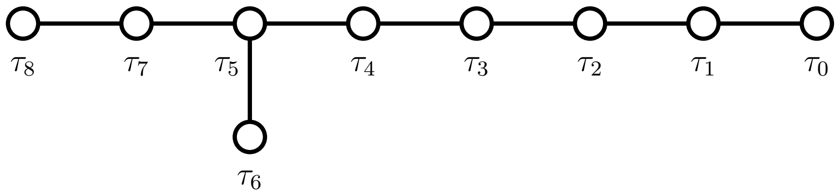
$$B_0^{[E_7]}=\prod_{i<j=1}^{6}\frac{1}{G(1\pm\sigma_i\pm\sigma_j)}\prod_{\substack{\varepsilon_i=\pm1\\ \prod_{i=1}^{8}\varepsilon_i=1,\\ \varepsilon_7=\varepsilon_8}}\frac{1}{G(1+\frac{1}{2}\sum_{i=1}^{8}\varepsilon_i\sigma_i)}$$

The one instanton contribution follows most easily by looking at the coefficients of $t^{3+2\sigma_{p_1}+\sigma_{p_2}+\sigma_{p_3}}$, where we obtain

$$\begin{aligned}4!(\sigma_{p_2}-\sigma_{p_3})^2(1+\sigma_{p_1}+\sigma_{p_2})^2(\sigma_{p_1}+\sigma_{p_2})^2(1+\sigma_{p_1}+\sigma_{p_3})^2(\sigma_{p_1}+\sigma_{p_3})^2&\\ B_0(\boldsymbol{\sigma})^2B_0(\boldsymbol{\sigma}+e_{p_1}+e_{p_2})B_0(\boldsymbol{\sigma}+e_{p_1}+e_{p_3})Z_1(\boldsymbol{\sigma})&\\ =\sum_{\substack{\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3+\mathbf{n}_4+4\boldsymbol{\lambda}\\ =2e_{p_1}+e_{p_2}+e_{p_3}\\ \frac{1}{2}\mathbf{n}_k^2+\boldsymbol{\lambda}\cdot\mathbf{n}_k=0}}\prod_{i<j}^{4}(\boldsymbol{\sigma}\cdot(\mathbf{n}_i-\mathbf{n}_j))^2\prod_{i=1}^{4}B_0(\boldsymbol{\sigma}+\mathbf{n}_i)&\\ -\sum_{\substack{\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3\\ =2e_{p_1}+e_{p_2}+e_{p_3}\\ \mathbf{n}_1^2=\mathbf{n}_2^2=\mathbf{n}_3^2=2}}B_0(\boldsymbol{\sigma})\prod_{i<j}^{3}(\boldsymbol{\sigma}\cdot(\mathbf{n}_i-\mathbf{n}_j))^2\prod_{i=1}^{3}(\boldsymbol{\sigma}\cdot\mathbf{n}_i)^2B_0(\boldsymbol{\sigma}+\mathbf{n}_i)&\end{aligned}$$

This can be compared with the general one instanton term, most easily when we specialize all variables except one; for example, leaving intact σ_7 yields a ratio of a degree 50 and a degree 66 polynomial in $\mathbb{C}[\sigma_7]$. Comparing other powers, i.e. $t^{3+e_{p_1}+e_{p_2}}$ and t^3 yields different expressions for $Z_1(\boldsymbol{\sigma})$. To obtain the two instanton term, we can look at $t^{4+2e_{p_1}+e_{p_2}+e_{p_3}}$. Similarly to the previous case, we obtain a ratio of a degree 166 to one of 198 in $\mathbb{C}[\sigma_7]$. The large-σ limit conforms to the expected limit from appendix B.

2.2.4.12 E_8

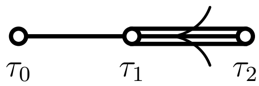


For the exceptional algebra E_8 we obtain the system

$$Y^6(\tau_0) = Y^3(\tau_8) \tag{2.29}$$
$$\tau_6 D^2(\tau_8) = Y^7(\tau_0) \tag{2.30}$$
$$D^2(\tau_6) = Y^6(\tau_0) \tag{2.31}$$

Here, τ_8 needs to be determined from (2.29), and then fed into (2.31), once τ_6 has been eliminated using (2.30). As the algebra with the largest root system, it was not practical to explicit calculations for the above E_8 system.

2.2.4.13 G_2



G_2 is a non-simply laced exceptional algebra. As can be seen from the (dual) extended Dynkin diagram, eliminating the node corresponding to τ_2 leaves us with a copy of A_2, which is a subalgebra which we previously embedded into a hyperplane orthogonal to $(1,1,1)$ in $\mathbb{R}^3$. We will use the same embedding for G_2, with $\sigma_1+\sigma_2+\sigma_3=0$. Besides the 6 roots of A_2, G_2 has 6 other roots of the form $e_{p_1}+e_{p_2}-2e_{p_3}$ for $p_{1,2,3}$ permutations of $\{1,2,3\}$. In the normalization where G_2's longest roots have length 2, the coroot lattice is the span $Q^\vee = \mathbb{Z}\frac{1}{\sqrt{3}}(-2,1,1)\oplus\mathbb{Z}\sqrt{3}(1,-1,0)$ - we are not aware of a simpler definition. The τ-system is

$$D^2(\tau_0) = -t^{\frac{1}{6}}\tau_1 \tag{2.32}$$
$$D^2(\tau_1) = -t^{\frac{1}{6}}\tau_0\tau_2 \tag{2.33}$$
$$D^2(\tau_2) = -3t^{\frac{1}{6}}\tau_1 \tag{2.34}$$

By using (2.32) to eliminate τ_1 from (2.33) and then using (2.33) to eliminate τ_2 from (2.34), the τ-system reduces to the single equation

$$D^2(\tau_0^{-1}D^4(\tau_0)) = 3t(D^2(\tau_0))^3$$

which can be simplified to

$$Y^4(\tau_0) = 3t(D^2(\tau_0))^2$$

since the Kiev Ansatz (2.6) implies $D^2(\tau_0)\neq 0$. We insert

$$\tau_0(\boldsymbol{\sigma},\boldsymbol{\eta}|t) = \sum_{\mathbf{n}\in Q^\vee} e^{2\pi i\boldsymbol{\eta}\cdot\mathbf{n}}\left(\frac{-t}{3}\right)^{\frac{1}{2}(\boldsymbol{\sigma}+\mathbf{n})^2} B_0\left(\boldsymbol{\sigma}+\mathbf{n}\middle|\frac{-t}{3}\right)$$

and after a rescaling $t \mapsto -3t$ we obtain the equation

$$\sum_{\substack{\mathbf{n}_{1,2,3,4}\in Q^\vee \\ i_{1,2,3,4}\in\mathbb{N}_0}} \prod_{k=1}^{4} e^{2\pi i \boldsymbol{\eta}\cdot\mathbf{n}_k} t^{\frac{1}{2}(\boldsymbol{\sigma}+\mathbf{n}_k)^2+i_k} B_0(\boldsymbol{\sigma}+\mathbf{n}_k) Z_{i_k}(\boldsymbol{\sigma}+\mathbf{n}_k)$$
$$\Bigg(\frac{1}{4!}\prod_{k_1<k_2}(\frac{1}{2}\mathbf{n}_{k_1}^2+i_{k_1}-\frac{1}{2}\mathbf{n}_{k_2}^2-i_{k_2}+(\mathbf{n}_{k_1}-\mathbf{n}_{k_2})\cdot\boldsymbol{\sigma})^2$$
$$-\frac{9}{4}t(\frac{1}{2}\mathbf{n}_1^2+i_1-\frac{1}{2}\mathbf{n}_2^2-i_2+(\mathbf{n}_1-\mathbf{n}_2)\cdot\boldsymbol{\sigma})^2$$
$$(\frac{1}{2}\mathbf{n}_3^2+i_3-\frac{1}{2}\mathbf{n}_4^2-i_4+(\mathbf{n}_3-\mathbf{n}_4)\cdot\boldsymbol{\sigma})^2\Bigg) = 0\,.$$

The coefficients of $t^{3+\boldsymbol{\sigma}\cdot(\frac{4}{\sqrt{3}},-\frac{2}{\sqrt{3}},-\frac{2}{\sqrt{3}})}$ are the lowest order powers which give the functional equation for $B_0(\boldsymbol{\sigma})$. Instead of a quartic relation we find that it simplifies to the following quadratic one

$$\left(\frac{2\sigma_1-\sigma_2-\sigma_3}{\sqrt{3}}+1\right)^2 B_0(\boldsymbol{\sigma})B_0\left(\boldsymbol{\sigma}+\frac{1}{\sqrt{3}}(2,-1,-1)\right) = \left(\frac{\sigma_2-\sigma_3}{\sqrt{3}}\right)^2\left(\frac{\sigma_1+\sigma_2-2\sigma_3}{\sqrt{3}}\right)^2$$
$$\times\left(\frac{\sigma_1-2\sigma_2+\sigma_3}{\sqrt{3}}\right)^2\left(\frac{\sigma_1+\sigma_2-2\sigma_3}{\sqrt{3}}+1\right)^2\left(\frac{\sigma_1-2\sigma_2+\sigma_3}{\sqrt{3}}+1\right)^2$$
$$\times B_0\left(\boldsymbol{\sigma}+\frac{1}{\sqrt{3}}(1,-2,1)\right)B_0\left(\boldsymbol{\sigma}+\frac{1}{\sqrt{3}}(1,1,-2)\right)$$

By imposing (2.7), these are solved by $B_0^{[G_2]}(\boldsymbol{\sigma}) =$

$$\prod_{i<j}^{3}\frac{1}{G(1\pm\frac{1}{\sqrt{3}}(\sigma_i-\sigma_j))}\prod_{\substack{ijk\\ cyclic}}^{3}\frac{1}{G(1\pm\frac{1}{\sqrt{3}}(2\sigma_i-\sigma_j-\sigma_k))}$$

However, such a simplification doesn't apply to the higher orders or different powers of t, $\{t^{\sigma_i}\}$. The 1-instanton contribution is obtained by considering the coefficient of the next order $t^{3+\boldsymbol{\sigma}\cdot(\sqrt{3},0,-\sqrt{3})}$ term: curiously, all $B_0(\boldsymbol{\sigma})$ factors drop out and we obtain just

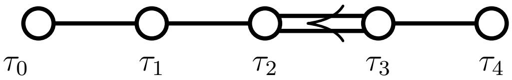
$$Z_1(\boldsymbol{\sigma})^{[G_2]} = -\frac{486}{(\sigma_1+\sigma_2-2\sigma_3)^2(\sigma_1-2\sigma_2+\sigma_3)^2(-2\sigma_1+\sigma_2+\sigma_3)^2} \overset{\sigma_3=-\sigma_1-\sigma_2}{=} -\frac{3}{2\sigma_1^2\sigma_2^2(\sigma_1+\sigma_2)^2}$$

This expression is in agreement with (4.39) in [170] when $Q=0$ and their $a_1 = (\sigma_1-\sigma_2)/\sqrt{6}$, $a_2=(\sigma_1+\sigma_2)/\sqrt{3}$ as well as with the blowup-formula from 1.3.4. The next order in t, $t^{4+\boldsymbol{\sigma}\cdot(\sqrt{3},0,-\sqrt{3})}$, gives the 2-instanton term $Z_2(\boldsymbol{\sigma})^{[G_2]}|_{\sigma_3=-\sigma_1-\sigma_2} =$

$$\frac{3\left(9\sigma_1^4\left(6\sigma_2^2+1\right)+18\sigma_1^3\left(6\sigma_2^3+\sigma_2\right)+3\sigma_1^2\left(18\sigma_2^4+9\sigma_2^2-2\right)+6\sigma_1\sigma_2\left(3\sigma_2^2-1\right)+\left(1-3\sigma_2^2\right)^2\right)}{\sigma_1^2\left(1-3\sigma_1^2\right)^2\sigma_2^2\left(1-3\sigma_2^2\right)^2(\sigma_1+\sigma_2)^2\left(1-3(\sigma_1+\sigma_2)^2\right)^2}$$

which agrees with the expression obtained from the blowup formula of the introduction 1.3.4. $Z_3(\boldsymbol{\sigma})$ can be obtained by looking at $t^{5+\boldsymbol{\sigma}\cdot(4,5,-1)/\sqrt{3}}$, although it is much to cumbersome to display. At this point, comparison with the blowup formula again becomes impossible, and we have to be content with checking that the large-$\boldsymbol{\sigma}$ limit of appendix B is correct.

2.2.4.14 F_4



F_4 is another non-simply laced unimodular exceptional algebra. As such, we have to express the system of equations in terms of the single tau function τ_0 associated to the extended node. From the system

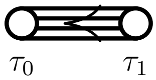
$$D^2(\tau_0) = -t^{1/9}\tau_1, \quad D^2(\tau_1) = -t^{1/9}\tau_0\tau_2, \quad D^2(\tau_2) = -t^{1/9}\tau_1\tau_3$$
$$D^2(\tau_3) = -t^{1/9}\tau_2^2\tau_4, \quad D^2(\tau_4) = -t^{1/9}\tau_3$$

we obtain the single equation

$$D^2(\frac{Y^5(\tau_0)}{Y^3(\tau_0)}) = -8t^2 Y^4(\tau_0)$$

Like for E_8, we leave it as it is, being beyond our computational power.

2.2.5 Twisted affine Lie algebras: radial Bullough-Dodd and BC_1



We consider the *twisted* affine Lie algebra, called either $A_{2n}^{(2)}$ or BC_n, with roots of three different lengths inherited from a folding of affine D_{2n+2}; the roots are $\pm e_k$, $\pm e_j \pm e_k$ as well as $\pm 2e_k$ of lengths $1, 2, 4$ respectively[4]. The simplest case $n = 1$ is slightly exceptional in this regard. Indeed, it comes from a quotient of affine D_4 by its order 4 automorphism and possessing no middle roots. It gives us

$$D^2(\tau_0) = -\frac{1}{2}t^{1/3}\tau_1$$
$$D^2(\tau_1) = -2t^{1/3}\tau_0^4$$

We redefine $t \mapsto 32^{-4}t^2$ from which we obtain the single equation

$$Y^3(\tau_0) = -6t^2\tau_0^3 \tag{2.35}$$

suitable for inserting an Ansatz analogous to the one used for the A_1 case,

$$\tau_0(\sigma, \eta|t) = \sum_{n\in\mathbb{Z}, i\in\mathbb{N}_0} e^{2\pi i\eta n} t^{(\sigma+n)^2+i} B_0(\sigma+n) Z_i(\sigma+n), \tag{2.36}$$

yielding from (2.35) the equation

$$\sum_{\substack{n_{1,2,3}\in\mathbb{Z}\\ i_{1,2,3}\in\mathbb{N}}} \prod_{k=1}^{3} e^{2\pi i\eta n_k} t^{(\sigma+n_k)^2+i_k} B_0(\sigma+n_k) Z_{i_k}(\sigma+n_k)$$
$$\left(\prod_{k_1<k_2} (n_{k_1}^2 + i_{k_1} - n_{k_2}^2 - i_{k_2} + 2(n_{k_1} - n_{k_2})\sigma)^2 + 6t^2\right) = 0\,.$$

[4] Note that the BC nomenclature refers to both sets of roots $\pm e_k$ and $\pm 2e_k$, which are peculiar to algebras of B_n and C_n respectively, being present along with the usual roots of D_n algebras.

The lowest order terms are the ones proportional to t^2. For terms proportional to the $6t^2$, the solution has no shifts or instanton numbers, while for rest of the the only nonvanishing possibilities are $n_1 = 1, n_2 = -1, n_3 = 0$ and permutations. After a cancellation of a $B_0(\sigma)$ factor, this leads to

$$4\sigma^2 \left(1 - 4\sigma^2\right)^2 B_0(\sigma \pm 1) = B_0(\sigma)^2 \tag{2.37}$$

At this point we have to discuss what kind of asymptotics would be suitable. Notice that we have the roots ± 2. If we identify these with the ones of the usual SU(2) adjoint representation, then the roots ± 1 correspond to the fundamental representation. Both representations are obtained from the folding of a pure D_4 Super Yang-Mills theory. As the parent theory has no mass parameters, it is natural to consider asymptotic conditions corresponding to an SU(2) with one massless fundamental flavor, i.e.

$$\log(B_0) \sim \frac{1}{4}(\sigma)^2 \log{(\sigma)}^2 - \frac{1}{4}(2\sigma)^2 \log{(2\sigma)}^2$$

We find that the solution to equation (2.37) with the appropriate asymptotics is

$$B_0(\sigma) = \frac{G(1 \pm \sigma)}{G(1 \pm 2\sigma)} \tag{2.38}$$

Further, by looking at $t^{2k+2\sigma}$ terms in (2.2.5), we find that there is a unique term proportional to

$$(2k-1)^2(1-2k+2\sigma)^2(1+2\sigma)^2 B_0(\sigma+1)B_0(\sigma)Z_{2k-1}(\sigma)$$

which comes from a factor from the Kiev Ansatz with $t^{1+2\sigma}$, one with no shifts, and another with only an instanton contribution t^{2k-1}. All the other terms are necessarily combinations of terms proportional to $Z_{2k'-1}$ with $k' < k$. To get $t^{2k+2\sigma}$ have to solve

$$\begin{aligned} &n_1 + n_2 + n_3 = 1 \\ &n_1^2 + n_2^2 + n_3^2 + i_1 + i_2 + i_3 + (2) = 2k \end{aligned}$$

The first equation, however, implies $n_1^2 + n_2^2 + n_3^2$ is odd, and so one of the i's always has to be odd and accordingly $Z_{2k-1} = 0$. Indeed, for $k = 1$ there is only one such term possible and we find that it has to vanish due to (2.2.5), so by induction we can conclude that

$$Z_{odd}(\sigma) = 0.$$

For the rest we find

$$\begin{aligned}
Z_2(\sigma) &= -\frac{3}{2^2\left(1-4\sigma^2\right)^2} \\
Z_4(\sigma) &= \frac{9\left(4\sigma^2+1\right)}{2^7\sigma^2\left(1-4\sigma^2\right)^2\left(9-4\sigma^2\right)^2} \\
Z_6(\sigma) &= -\frac{576\sigma^6 - 2160\sigma^4 + 5324\sigma^2 + 75}{2^9\sigma^2\left(1-4\sigma^2\right)^4\left(9-4\sigma^2\right)\left(25-4\sigma^2\right)} \\
Z_8(\sigma) &= \frac{3\left(4608\sigma^{10} - 78336\sigma^8 + 482560\sigma^6 - 615824\sigma^4 + 243742\sigma^2 + 62475\right)}{2^{16}\sigma^2\left(1-\sigma^2\right)^2\left(1-4\sigma^2\right)^4\left(9-4\sigma^2\right)^2\left(25-4\sigma^2\right)^2\left(49-4\sigma^2\right)^2}
\end{aligned}$$

We can recognize here exactly the SU(2) Nekrasov functions with one massless flavor in the fundamental representation. Note that, starting from PIII_2 in the form

$$\ddot{q} = \frac{\dot{q}^2}{q(t)} - \frac{\dot{q}}{t} - \frac{(1-2\theta)\dot{q}^2}{t} + q^3 - \frac{2}{t^2}$$

and setting $q = t^{-2/3}\exp\{w\}, \theta = 1/2$ gives us

$$\partial^2_{\log t} w = t^{2/3}(e^{2w} - 2e^{-w})$$

Setting $w = a + 2(\sigma - 1/6)\log t + X(t)$, since $\rho/h^\vee = 1/6$, gives us

$$\partial^2_{\log t} X = e^{2a}t^{4\sigma}e^{2X} - 2e^{a}t^{1-2\sigma}e^{-X}$$

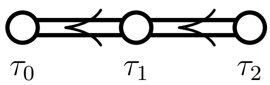
This cannot be obtained by directly applying (2.1) to this affine root system. Instead we must start from D_4 and require a solution of the form $\phi_1 = \phi_4 = 0, \phi_3 = -\phi_2$ Let us comment on other interesting directions to investigate further and look at the equation for the surviving degree of freedom. This is exactly the folding which gives the diagram BC_1 depicted at the beginning of this subsection. Solving the equation we find

$$X(\sigma, a|t) = 2\log\tau_0(\sigma, \tilde{a}|it) - \log\tau_1(\sigma, a|t)$$

where $\partial^2_{\log t}\log\tau_1(\sigma, a|t) = e^{2a}t^{4\sigma}e^{2X}$, normalized such that $\tau_1(\sigma, a|0) = 1$, τ_0 is (2.36) with perturbative term (2.38), first four instanton terms we found and the initial condition

$$\tilde{a} = a - \log\left(\frac{\Gamma(1-2\sigma)^2\Gamma(\sigma)}{\Gamma(2\sigma)^2\Gamma(1-\sigma)}\right) - i\pi\sigma + \frac{i\pi}{2}$$

Note that this is different from the tau function defined in (2.25), (2.29) of [101].



The slightly more general case of $n = 2$ gives us the system

$$\begin{aligned} D^2(\tau_0) &= -\frac{1}{2}t^{1/5}\tau_1 \\ D^2(\tau_1) &= -t^{1/5}\tau_0^2\tau_2 \\ D^2(\tau_2) &= -2t^{1/5}\tau_1^2 \end{aligned}$$

from which we obtain the single equation

$$\tau_0^2 Y^4(\tau_0) - (Y^3(\tau_0))^2 = -\frac{1}{2}t\tau_0^4 D^2(\tau_0)$$

The lattice is $Q^\vee = \mathbb{Z}^2$, rescaled by a factor of $\sqrt{2}$, as the underlying finite root system is C_2. Examining the lowest order terms we find that from the Ansatz, where the rescaling is taken care of by fractional t,

$$\tau_0(\boldsymbol{\sigma}, \boldsymbol{\eta}|t) = \sum_{\mathbf{n}\in\mathbb{Z}^2,\, i\in\mathbb{N}_0/2} e^{2\pi\sqrt{-1}\mathbf{n}\cdot\boldsymbol{\eta}} t^{\frac{1}{2}(\boldsymbol{\sigma}+\mathbf{n})^2+i} B_0(\boldsymbol{\sigma}+\mathbf{n}) Z_{2i}(\boldsymbol{\sigma}+\mathbf{n})$$

we find that $Z_1(\boldsymbol{\sigma})$ should vanish again. Looking at the lowest order gives us the equation

$$
\begin{aligned}
16\,(2\sigma_1+1)^{\,2}\,(\sigma_1-\sigma_2)^{\,2}\sigma_2^2\,(\sigma_1+\sigma_2)^{\,2}\,(2\sigma_2-1)^{\,2}\,(2\sigma_2+1)^{\,2}&\\
\times B_0(\boldsymbol{\sigma})B_0(\boldsymbol{\sigma}+e_1)B_0(\boldsymbol{\sigma}+e_2)B_0(\boldsymbol{\sigma}-e_2)&\\
=-\,(2\sigma_1+1)^{\,2}B_0(\boldsymbol{\sigma})^3B_0(\boldsymbol{\sigma}+e_1)&
\end{aligned}
$$

with the solution

$$
B_0(\boldsymbol{\sigma})=\frac{G(1\pm\sigma_1)G(1\pm\sigma_2)}{G(1\pm 2\sigma_1)G(1\pm 2\sigma_2)G(1\pm\sigma_1\pm\sigma_2)}
$$

and further we find

$$
Z_2(\boldsymbol{\sigma})=\frac{24\sigma_1^4-32\sigma_2^2\sigma_1^2-4\sigma_1^2+24\sigma_2^4-4\sigma_2^2+1}{2\,(2\sigma_1-1)^{\,2}\,(2\sigma_1+1)^{\,2}\,(\sigma_1-\sigma_2)^{\,2}\,(\sigma_1+\sigma_2)^{\,2}\,(2\sigma_2-1)^{\,2}\,(2\sigma_2+1)^{\,2}}
$$

We do not have at present a clear 4D gauge theory interpretation for this case.

2.2.6 $SU(2)^n$ linear quiver gauge theories

In this section we will be focusing on the case of linear $SU(2)^{\times n}$ quivers in the pure, non-conformal case. Our proposal is the following modification of the $SU(2)$ system

$$
\begin{aligned}
\tau_0^2\partial_{\log t_1}\partial_{\log t_n}\log\tau_0&=-t_1^{1/4}\cdots t_n^{1/4}\tau_1^2\\
\tau_1^2\partial_{\log t_1}\partial_{\log t_n}\log\tau_1&=-t_1^{1/4}\cdots t_n^{1/4}\tau_0^2
\end{aligned}
\tag{2.39}
$$

Notice that, in the case $n>2$, the system of equations explicitely describes only the dynamics associated to the irregular punctures moduli. As we will see in the following, the dependence on the moduli of the regular punctures is uniquely fixed by suitable asymptotic conditions on the solutions. These are obtained in the limiting cases of identity punctures leading to trivial monodromy or degenerating limits dividing the punctured Riemann sphere into disconnected components.

We solve to above equations in terms of the following generalised $SU(2)$ quiver Kiev Ansatz

$$
\tau_j(\{\sigma_k\},\{\eta_k\}|t_1,...,t_n)=\sum_{\substack{n_1,...,n_k\in\frac{j}{2}+\mathbb{Z}\\ i_1,...,i_n\in\mathbb{N}_0}}\prod_{l=1}^{n}\left(e^{2\pi i\eta_l\cdot n_l}t_l^{(\sigma_l+n_l)^2+i_l}\right)B_0(\{\sigma_k+n_k\})Z_{i_1,...,i_n}(\{\sigma_k+n_k\})
$$

where $Z_{0,...,0}\equiv 1$. Notice that the shift is simultaneous in all the lattices as there is no mixing. In the next subsection, by imposing appropriate asymptotic conditions, we will find that $B_0(\{\sigma_k+n_k\})=B^{\text{quiver}}(\{\sigma_k+n_k\})$ where

$$
B^{\text{quiver}}(\{\sigma_k\})=\frac{\prod_{i=1}^{n-1}G(1+m_{i,i+1}+\sigma_i\pm\sigma_{i+1})G(1+m_{i,i+1}-\sigma_i\pm\sigma_{i+1})}{\prod_{k=1}^{n}G(1\pm 2\sigma_k)}
$$

where $m_{i,j} \in \mathbb{C}$ are arbitrary bifundamental masses. We conjecture that these one-loop terms, along with the recursion relations arising from (2.39) and suitable additional constraints, lead to the identification

$$Z_{i_1,\dots,i_n}(\{\sigma_k\}) = \sum_{\substack{(\vec{Y}_1,\dots,\vec{Y}_n)\\ |Y_{k,1}|+|Y_{k,2}|=i_k}} \frac{\prod_{i=1}^{n-1} Z_{bifund.}(\sigma_i,\vec{Y}_i,\sigma_{i+1},\vec{Y}_{i+1},m_{i,i+1})}{\prod_{i=1}^{n} Z_{bifund.}(\sigma_i,\vec{Y}_i,\sigma_i,\vec{Y}_i,0)}$$

where $Z_{bifund.}$ is defined in 1.3.2.

2.2.6.1 One-loop normalisation

Examining the $\prod_l t_l^{1+2\sigma_l}$ term in (2.39) gives, for general n,

$$(1+2\sigma_1)(1+2\sigma_n) \sum_{p_1,\dots,p_n\in\{0,1\}} \frac{(-1)^{1+p_1+p_n}}{2} B_0(\{\sigma_k+p_k\})B_0(\{\sigma_k+1-p_k\})$$
$$= -B_0\left(\{\sigma_k+\frac{1}{2}\}\right)^2 \tag{2.40}$$

We prove this by induction. First, we need the auxiliary result that, for $k_1^+ \le k_1^- \le k_2^+ \le k_2^- \le \dots \le k_l^+ \le k_l^-$, if we denote

$$\boldsymbol{\sigma}_+ := (\sigma_1,\dots,\sigma_{k_1^+-1},\sigma_{k_1^+}+1,\dots,\sigma_{k_1^-}+1,\sigma_{k_1^-+1},\dots,\sigma_{k_2^+-1},\sigma_{k_2^+}+1,\dots,\sigma_{k_2^-}+1,\sigma_{k_2^-+1},\dots,\sigma_{k_l^+-1},\sigma_{k_l^+}+1,\dots,\sigma_{k_l^-}+1,\sigma_{k_l^-+1},\dots,\sigma_n)$$
$$\boldsymbol{\sigma}_- := (\sigma_1+1,\dots,\sigma_{k_1^+-1}+1,\sigma_{k_1^+},\dots,\sigma_{k_1^-},\sigma_{k_1^-+1}+1,\dots,\sigma_{k_2^+-1}+1,\sigma_{k_2^+},\dots,\sigma_{k_2^-},\sigma_{k_2^-+1}+1,\dots,\sigma_{k_l^+-1}+1,\sigma_{k_l^+},\dots,\sigma_{k_l^-},\sigma_{k_l^-+1}+1,\dots,\sigma_n+1)$$

then it follows that

$$\frac{B^{\text{quiver}}(\boldsymbol{\sigma}_+)B^{\text{quiver}}(\boldsymbol{\sigma}_-)}{-B^{\text{quiver}}\left(\{\sigma_k+\frac{1}{2}\}\right)^2} = \frac{\prod_{i=1}^{n-1}(1+\sigma_i+\sigma_{i+1}\pm m_{i,i+1})}{\prod_i (1+2\sigma_i)^2}$$
$$\times \prod_{q=1}^{l} \frac{(\sigma_{k_q^+-1}-\sigma_{k_q^+}\pm m_{k_q^+-1,k_q^+})(\sigma_{k_q^-}-\sigma_{k_q^-+1}\pm m_{k_q^-,k_q^-+1})}{(1+\sigma_{k_q^+-1}+\sigma_{k_q^+}\pm m_{k_q^+-1,k_1^+})(1+\sigma_{k_q^-}+\sigma_{k_q^-+1}\pm m_{k_q^-,k_q^-+1})}$$

Next we need another auxiliary result which we use to tame the summation in (2.40). Namely, for $n \ge 3$

$$\frac{-(\sigma_1-\sigma_2\pm m_{1,2})}{(1+\sigma_1+\sigma_2\pm m_{1,2})}\Bigg(1+\sum_{i_1=2}^{n-1}\frac{-(\sigma_{i_1}-\sigma_{i_1+1}\pm m_{i_1,i_1+1})}{(1+\sigma_{i_1}+\sigma_{i_1+1}\pm m_{i_1,i_1+1})}$$
$$+\sum_{i_1=2}^{n-1}\frac{-(\sigma_{i_1}-\sigma_{i_1+1}\pm m_{i_1,i_1+1})}{(1+\sigma_{i_1}+\sigma_{i_1+1}\pm m_{i_1,i_1+1})}\sum_{i_2=i_1+1}^{n-1}\frac{-(\sigma_{i_2}-\sigma_{i_2+1}\pm m_{i_2,i_2+1})}{(1+\sigma_{i_2}+\sigma_{i_2+1}\pm m_{i_1,i_2+1})}$$
$$+\sum_{i_1=2}^{n-1}\frac{-(\sigma_{i_1}-\sigma_{i_1+1}\pm m_{i_1,i_1+1})}{(1+\sigma_{i_1}+\sigma_{i_1+1}\pm m_{i_1,i_1+1})}\sum_{i_2=i_1+1}^{n-1}\frac{-(\sigma_{i_2}-\sigma_{i_2+1}\pm m_{i_2,i_2+1})}{(1+\sigma_{i_2}+\sigma_{i_2+1}\pm m_{i_1,i_2+1})}\sum_{i_3=i_2+1}^{n-1}\frac{-(\sigma_{i_3}-\sigma_{i_3+1}\pm m_{i_3,i_3+1})}{(1+\sigma_{i_3}+\sigma_{i_3+1}\pm m_{i_3,i_3+1})}$$
$$+\dots+\prod_{i=2}^{n-1}\frac{-(\sigma_i-\sigma_{i+1}\pm m_{i,i+1})}{(1+\sigma_i+\sigma_{i+1}\pm m_{i,i+1})}\Bigg)$$
$$=\frac{(\sigma_1-\sigma_2\pm m_{1,2})(1+2\sigma_2)(1+2\sigma_n)\prod_{i=3}^{n-1}(1+2\sigma_i)^2}{\prod_{i=1}^{n-1}(1+\sigma_i+\sigma_{i+1}\pm m_{i,i+1})}$$

This also follows from induction starting from $n = 3$, for which

$$-1 + \frac{(\sigma_2 - \sigma_3 \pm m_{2,3})}{(1 + \sigma_2 + \sigma_3 \pm m_{2,3})} = -\frac{(1 + 2\sigma_2)(1 + 2\sigma_3)}{(1 + \sigma_2 + \sigma_3 \pm m_{2,3})}$$

by iterating the identity we get

$$\begin{aligned}
&\frac{(\sigma_1 - \sigma_2 \pm m_{1,2})(1 + 2\sigma_2)(1 + 2\sigma_{n-1}) \prod_{i=3}^{n-2}(1 + 2\sigma_i)^2}{\prod_{i=1}^{n-2}(1 + \sigma_i + \sigma_{i+1} \pm m_{i,i+1})} \\
&+ \frac{-(\sigma_{n-1} - \sigma_n \pm m_{n-1,n})}{(1 + \sigma_{n-1} + \sigma_n \pm m_{n-1,n})}\left(1 + \sum_{i_1=2}^{n-2} \frac{-(\sigma_{i_1} - \sigma_{i_1+1} \pm m_{i_1,i_1+1})}{(1 + \sigma_{i_1} + \sigma_{i_1+1} \pm m_{i_1,i_1+1})} + \ldots\right) \\
&= \frac{(\sigma_1 - \sigma_2 \pm m_{1,2})(1 + 2\sigma_2)(1 + 2\sigma_{n-1}) \prod_{i=3}^{n-2}(1 + 2\sigma_i)^2}{\prod_{i=1}^{n-2}(1 + \sigma_i + \sigma_{i+1} \pm m_{i,i+1})}\left(1 + \frac{-(\sigma_{n-1} - \sigma_n \pm m_{n-1,n})}{(1 + \sigma_{n-1} + \sigma_n \pm m_{n-1,n})}\right) \\
&= \frac{(\sigma_1 - \sigma_2 \pm m_{1,2})(1 + 2\sigma_2)(1 + 2\sigma_n) \prod_{i=3}^{n-1}(1 + 2\sigma_i)^2}{\prod_{i=1}^{n-1}(1 + \sigma_i + \sigma_{i+1} \pm m_{i,i+1})}
\end{aligned}$$

which is what we wanted. Using both results, (2.40) becomes equivalent to the following identity after some reorganizing,

$$\begin{aligned}
&1 + \frac{(\sigma_1 - \sigma_2 \pm m_{1,2})(1 + 2\sigma_2)(1 + 2\sigma_n) \prod_{i=3}^{n-1}(1 + 2\sigma_i)^2}{\prod_{i=1}^{n-1}(1 + \sigma_i + \sigma_{i+1} \pm m_{i,i+1})} \\
&+ \frac{(\sigma_2 - \sigma_3 \pm m_{2,3})(1 + 2\sigma_3)(1 + 2\sigma_n) \prod_{i=4}^{n-1}(1 + 2\sigma_i)^2}{\prod_{i=2}^{n-1}(1 + \sigma_i + \sigma_{i+1} \pm m_{i,i+1})} \\
&+ \ldots + \frac{(\sigma_{n-2} - \sigma_{n-1} \pm m_{n-2,n-1})(1 + 2\sigma_{n-2})(1 + 2\sigma_n)}{(1 + \sigma_{n-2} + \sigma_{n-1} \pm m_{n-2,n-1})(1 + \sigma_n + \sigma_{n-1} \pm m_{n,n-1})} - \frac{(\sigma_n - \sigma_{n-1} \pm m_{n,n-1})}{(1 + \sigma_n + \sigma_{n-1} \pm m_{n,n-1})} \\
&= \frac{(1 + 2\sigma_1)(1 + 2\sigma_n) \prod_{i=2}^{n-1}(1 + 2\sigma_i)^2}{\prod_{i=1}^{n-1}(1 + \sigma_i + \sigma_{i+1} \pm m_{i,i+1})}
\end{aligned}$$

2.2.6.2 Instanton terms

The one instanton contributions can be obtained in a similar way. Let us focus on the detailed analysis of the simplest cases, starting with that of $n = 2$. To obtain the instanton terms, we need to impose correct boundary conditions.

- The first condition is the one corresponding to the identity puncture. This implies that when $m_{1,2} = 0$, by setting $\sigma_1 = \sigma_2$ kills off-diagonal terms in the expansion of the quiver tau function. Once we equate $t_1 = t_2$, the solution has to equal that of pure SU(2) as the equation it solves is the same. [5] A detailed proof of this is in the next section.

- The second condition is the one associated to the dividing degeneration limit. This is obtained by sending $m_{1,2} \to \infty$ while scaling $t_{1,2} \mapsto t_{1,2}/m_{1,2}^2$, inducing the factorization

$$Z_{k_1,k_2}(\sigma_1, \sigma_2)\left(\frac{t_1}{m_{1,2}}\right)^{k_1}\left(\frac{t_2}{m_{1,2}}\right)^{k_2} \to Z_{k_1}(\sigma_1)Z_{k_2}(\sigma_2)t_1^{k_1}t_2^{k_2}$$

[5]In the D-brane language, this condition can be seen most easily from the Hanany-Witten brane setup - this is the point at which the theory touches the Higgs branch and the gauge group gets broken down to the diagonal. There is no bifundamental, and the branes are fixed to move in unison, ignoring the intermediate NS5 brane.

This is consistent with (2.39). Indeed, under the scaling itself, the RHS goes to zero as $1/m_{1,2}$, while, the LHS automatically vanishes if the tau function factorizes.

2.2.6.3 Nekrasov function factorization

Let us show the claim from the previous section, that for the $n = 2$ quiver,

$$\begin{aligned} Z_{k_1,k_2}(\sigma,\sigma)|_{m_{1,2}=0} &= \sum_{\substack{|Y_1|+|Y_2|=k_1 \\ |W_1|+|W_2|=k_2}} \frac{Z_{bifund.}(\sigma,-\sigma,Y_1,Y_2,\sigma,-\sigma,W_1,W_2,0)}{Z_{adj.}(\sigma,-\sigma,Y_1,Y_2)Z_{adj.}(\sigma,-\sigma,W_1,W_2)} \\ &= \delta_{k_1,k_2} Z^{SU(2)}_{k_1}(\sigma) \end{aligned} \tag{2.41}$$

This can be seen to follow from

$$Z_{bifund.}(\sigma,-\sigma,Y_1,Y_2,\sigma,-\sigma,W_1,W_2,0) = \delta_{Y_1,W_1}\delta_{Y_2,W_2}Z_{adj.}(\sigma,-\sigma,Y_1,Y_2) \tag{2.42}$$

which we show to be true in the self-dual case. In the general Ω-background, the equality (2.42) is not true. Nevertheless, the full sum (2.41) seems to be true universally, although this follows from more complicated cancellations. In any case, we are interested only in the self-dual case $\epsilon_1 = 1$, $\epsilon_2 = -1$. With that in mind, we write

$$\begin{aligned} &Z_{bifund.}(\sigma,-\sigma,Y_1,Y_2,\sigma,-\sigma,W_1,W_2,0) = \\ &\prod_{i=1}^{2}\prod_{c\in Y_i}\xi((-1)^{1-i}2\sigma,Y_i,W_{1-i},c)\prod_{c\in W_i}\left(-\xi((-1)^{1-i}2\sigma,W_i,Y_{1-i},c)\right) \\ &\prod_{i=1}^{2}\prod_{c\in Y_i}\xi(0,Y_i,W_i,c)\prod_{c\in W_i}\left(-\xi(0,W_i,Y_i,c)\right) \end{aligned}$$

We prove the last line vanishes unless the Young diagrams are equal as $Y_1 = W_1$, $Y_2 = W_2$. Focusing on just one factor,

$$\xi(0,Y,W,c) = \mathrm{leg}(c,Y) + \mathrm{arm}(c,W) + 1$$

consider row diagrams $Y = (1^{l_1})$ and $W = (1^{l_2})$ with $l_1 \neq \tilde{l}_2$. WLOG, assume $l_1 > \tilde{l}_2$. In this case for $c = (l_1, 1)$ we have

$$\left.\begin{aligned} &\mathrm{leg}(c,Y) = (1^{l_1})^t_1 - l_1 = l_1 - l_1 = 0 \\ &\mathrm{arm}(c,W) = 0 - 1 = -1 \end{aligned}\right\} \Rightarrow \xi(0,Y,W,c) = 0 - 1 + 1 = 0$$

However, adding any amount of rows to any of the diagrams after the $j = 1$ one doesn't change this calculation. The other case is $l_1 = \tilde{l}_2$. Then, for the same cell, $\mathrm{arm}(c,W) = 1 - 1 = 0$, and $\xi(0,Y,W,c) \neq 0$. In fact, both the arm and the leg lengths have to be positive indefinite, since the cell c is contained within the diagrams Y and W, so $\xi(0,Y,W,c) \neq 0$ for the whole row, i.e. $i \in [1, l_1]$, $j = 1$. Next, we add rows to both diagrams, and we are in the same situation as before. If the rows are of equal length, the cell will be contained in both diagrams and the relative hook length will never vanish. Otherwise, if the j'th row is the first one of unequal length, with lengths $l_1 > \tilde{l}_2$, say, then the relative hook length of the cell $c = (l_1, j)$ vanishes by an analogous calculation. If all rows are equal, the diagrams are obviously the same, and there is no vanishing factor. Along with the trivial equality

$$Z_{bifund.}(\sigma, -\sigma, Y_1, Y_2, \sigma, -\sigma, Y_1, Y_2, 0) = Z_{adj.}(\sigma, -\sigma, Y_1, Y_2)$$

this proves our claim.

Direct calculations indicate this is true on the general Omega-background as well, but I do not have a proof available.

2.2.6.4 Instantons from the Kiev Ansatz

The equation for $n = 2$ yields two bilinear equations related one to the other by $\sigma_i \mapsto \sigma_i + 1/2$ symmetry:

$$\begin{aligned}
&\sum_{\substack{\mathbf{n},\mathbf{m}\in\mathbb{Z}^2\\ i_1,i_2\geq 0}} e^{2\pi\sqrt{-1}(\mathbf{n}+\mathbf{m})\cdot\boldsymbol{\eta}} t_1^{(\sigma_1+n_1)^2+i_1+(\sigma_2+n_2)^2+i_2} t_2^{(\sigma_1+m_1)^2+j_1+(\sigma_2+m_2)^2+j_2}\\
&\times (i_1 - i_2 + (n_1-n_2)(n_1+n_2+2\sigma_1))\,(j_1 - j_2 + (m_1-m_2)(m_1+m_2+2\sigma_2))\\
&\times B_0(\sigma_1+n_1, \sigma_2+m_1)B_0(\sigma_1+n_2, \sigma_2+m_2)Z_{i_1,j_1}(\sigma_1+n_1, \sigma_2+m_1)Z_{i_2,j_2}(\sigma_1+n_2, \sigma_2+m_2)\\
&= -\sum_{\substack{\mathbf{n},\mathbf{m}\in\mathbb{Z}^2\\ i_1,i_2\geq 0}} e^{2\pi\sqrt{-1}(\mathbf{n}+\mathbf{m})\cdot\boldsymbol{\eta}} t_1^{1+(\sigma_1+n_1)^2+\sigma_1+n_1+i_1+(\sigma_2+n_2)^2+\sigma_2+n_2+i_2}\\
&\times t_2^{1+(\sigma_1+m_1)^2+\sigma_1+m_1+j_1+(\sigma_2+m_2)^2+\sigma_2+m_2+j_2}\\
&\times B_0(\sigma_1+n_1+\frac{1}{2}, \sigma_2+m_1+\frac{1}{2})B_0(\sigma_1+n_2+\frac{1}{2}, \sigma_2+m_2+\frac{1}{2})\\
&\times Z_{i_1,j_1}(\sigma_1+n_1+\frac{1}{2}, \sigma_2+m_1+\frac{1}{2})Z_{i_2,j_2}(\sigma_1+n_2+\frac{1}{2}, \sigma_2+m_2+\frac{1}{2})
\end{aligned}$$

The $t_1 t_2^{1+2\sigma_2}$ term gives

$$Z_{1,0}(\sigma_1, \sigma_2+1) - Z_{1,0}(\sigma_1, \sigma_2) = -\frac{2B_0(\sigma_1+\frac{1}{2}, \sigma_2+\frac{1}{2})B_0(\sigma_1-\frac{1}{2}, \sigma_2+\frac{1}{2})}{(1+2\sigma_2)B_0(\sigma_1, \sigma_2)B_0(\sigma_1, \sigma_2+1)} = -\frac{1+2\sigma_2}{2\sigma_1^2}$$

once we put $B_0 = B^{\text{quiver}}$. The unique solution satisfying the boundary conditions is

$$Z_{1,0}(\sigma_1, \sigma_2) = \frac{m_{1,2}^2 + \sigma_1^2 - \sigma_2^2}{2\sigma_2^2}$$

Note also the obvious symmetry $Z_{i,j}(\sigma_1, \sigma_2) = Z_{j,i}(\sigma_2, \sigma_1)$ which leads to $Z_{0,1}$. Finally, the $t_1^{1+2\sigma_1} t_2^{1+2\sigma_2}$ term gets us

$$\begin{aligned}
&Z_{1,1}(\sigma_1, \sigma_2) - Z_{1,0}(\sigma_1, \sigma_2)Z_{0,1}(\sigma_1, \sigma_2)\\
&= 2\frac{B_0(\sigma_1+\frac{1}{2}, \sigma_2-\frac{1}{2})B_0(\sigma_1-\frac{1}{2}, \sigma_2+\frac{1}{2}) + B_0(\sigma_1+\frac{1}{2}, \sigma_2+\frac{1}{2})B_0(\sigma_1-\frac{1}{2}, \sigma_2-\frac{1}{2})}{B_0(\sigma_1, \sigma_2)^2}\\
&= -\frac{m_{1,2}^2 - \sigma_1^2 - \sigma_2^2}{4\sigma_1^2\sigma_2^2}
\end{aligned}$$

giving the correct mixed 2-instanton term

$$Z_{1,1}(\sigma_1, \sigma_2) = \frac{m_{1,2}^2(m_{1,2}^2 - 1) + \sigma_1^2 + \sigma_2^2 - (\sigma_1^2 - \sigma_2^2)^2}{4\sigma_1^2\sigma_2^2}$$

In general, we find that the coefficients of the form $Z_{k,0}(\boldsymbol{\sigma})$ satisfy simple recurrence relations, while mixed terms are determined uniquely. We checked agreement with instanton counting up to $Z_{3,3}(\boldsymbol{\sigma})$.
For the $SU(2)^3$ quiver, the boundary conditions are the logical extension of the above.

- When $m_{1,2} = 0$, setting $\sigma_1 = \sigma_2$ has to kill all terms Z_{k_1,k_2,k_3} with $k_1 \neq k_2$. Further, if we put $t_1 \mapsto t_1^{1/2}$ and $t_2 \mapsto t_1^{1/2}$, then
$$\left(Z_{k_1,k_2,k_3}(\sigma_1,\sigma_1,\sigma_3)|_{m_{1,2}=0}\right) t_1^{k_1} t_2^{k_2} t_3^{k_3} = Z_{k_1,k_3}(\sigma_1,\sigma_3) t^{k_1} t_3^{k_3}$$
where on the RHS we have the partition functions of the $SU(2)^2$ quiver. Similar considerations apply if $m_{2,3} = 0$ and $\sigma_2 = \sigma_3$. Clearly, at this point previous considerations apply, and we are free to set the remaining mass to zero and reach pure $SU(2)$ again.

- We can decouple bifundamental hypermultiplets individually. For instance, by sending $m_{1,2} \to \infty$ while scaling $t_{1,2} \mapsto t_{1,2}/m_{1,2}^2$, we obtain the factorization
$$Z_{k_1,k_2,k_3}(\sigma_1,\sigma_2,\sigma_3)\left(\frac{t_1}{m_{1,2}}\right)^{k_1}\left(\frac{t_2}{m_{1,2}}\right)^{k_2} t_3^{k_3} \to Z_{k_1}(\sigma_1) Z_{k_2,k_3}(\sigma_2,\sigma_3) t_1^{k_1} t_2^{k_2} t_3^{k_3}$$
Again, under any factorization, the LHS of (2.39) vanishes.

We present in brief the lowest order calculations for the $SU(2)^3$ quiver. The $t_1^{1+2\sigma_1} t_2^{1+2\sigma_2} t_3^{1+2\sigma_3}$ term leads to the one-loop term already discussed in general in the main text. Next, the $Z_{a,b,c}$ coefficients with positive integers $a^2+b^2+c^2 = 1$ are accessed by looking at $t_1^{1+2(1-a)\sigma_1} t_2^{1+2(1-b)\sigma_2} t_3^{1+2(1-c)\sigma_3}$ terms. This leads to

$$\begin{aligned}
&\left(-2m_{2,3}^2 + 2\sigma_2(\sigma_2+1) + 2\sigma_3(\sigma_3+1) + 1\right) Z_{1,0,0}(\sigma_1,\sigma_2,\sigma_3) \\
&= \left((\sigma_2-\sigma_3)^2 - m_{2,3}^2\right)\left(2Z_{1,0,0}(\sigma_1,\sigma_2,\sigma_3+1) - Z_{1,0,0}(\sigma_1,\sigma_2+1,\sigma_3)\right) \\
&+ \left((1+\sigma_2+\sigma_3)^2 - m_{2,3}^2\right) Z_{1,0,0}(\sigma_1,\sigma_2+1,\sigma_3+1) + \frac{(2\sigma_3+1)(2\sigma_2+1)^2}{2\sigma_1^2} \\
&\Rightarrow Z_{1,0,0}(\sigma_1,\sigma_2,\sigma_3) = \frac{m_{1,2}^2 + \sigma_1^2 - \sigma_2^2}{2\sigma_1^2},
\end{aligned}$$

an analogous equation for $Z_{0,0,1}$ which we omit, and finally

$$\begin{aligned}
&Z_{0,1,0}(\sigma_1,\sigma_2,\sigma_3) + Z_{0,1,0}(\sigma_1+1,\sigma_2,\sigma_3+1) \\
&= Z_{0,1,0}(\sigma_1,\sigma_2,\sigma_3+1) + Z_{0,1,0}(\sigma_1+1,\sigma_2,\sigma_3) + \frac{(2\sigma_1+1)(2\sigma_3+1)}{2\sigma_2^2} \\
&\Rightarrow Z_{0,1,0}(\sigma_1,\sigma_2,\sigma_3) = \frac{-4\sigma_2^2 m_{1,2} m_{2,3} + m_{2,3}^2\left(m_{1,2}^2 - \sigma_1^2 + \sigma_2^2\right) + (\sigma_2^2 - \sigma_3^2)\left(m_{1,2}^2 - \sigma_1^2 + \sigma_2^2\right)}{2\sigma_2^2}
\end{aligned}$$

The $Z_{a,b,c}$ coefficients with positive integers $a^2+b^2+c^2 = 2$ are likewise accessed by looking at $t_1^{1+2(1-a)\sigma_1} t_2^{1+2(1-b)\sigma_2} t_3^{1+2(1-c)\sigma_3}$ terms. They feature the terms we calculated in the previous step.

$$\begin{aligned}
&\left(Z_{0,1,0}\left(\sigma_1,\sigma_2,\sigma_3\right)+Z_{0,1,0}\left(\sigma_1,\sigma_2,\sigma_3+1\right)\right)Z_{1,0,0}\left(\sigma_1,\sigma_2,\sigma_3\right)+Z_{1,1,0}\left(\sigma_1,\sigma_2,\sigma_3\right)\\
&=\frac{\left(2\sigma_3+1\right)\left(-m_{1,2}^2+\sigma_1^2+\sigma_2^2\right)}{4\sigma_1^2\sigma_2^2}+2Z_{0,1,0}\left(\sigma_1,\sigma_2,\sigma_3\right)Z_{1,0,0}\left(\sigma_1,\sigma_2,\sigma_3+1\right)+Z_{1,1,0}\left(\sigma_1,\sigma_2,\sigma_3+1\right)\\
&\Rightarrow Z_{1,1,0}(\sigma_1,\sigma_2,\sigma_3)=-\frac{4\sigma_2^2m_{1,2}m_{2,3}\left(m_{1,2}^2+\sigma_1^2-\sigma_2^2\right)-m_{1,2}^2\left(m_{1,2}^2-1\right)\left(m_{2,3}^2+\sigma_2^2-\sigma_3^2\right)}{4\sigma_1^2\sigma_2^2}\\
&\qquad-\frac{\left(\sigma_1^4-\left(2\sigma_2^2+1\right)\sigma_1^2+\sigma_2^4-\sigma_2^2\right)\left(m_{2,3}^2+\sigma_2^2-\sigma_3^2\right)}{4\sigma_1^2\sigma_2^2}
\end{aligned}$$

as well as an analogous equation for $Z_{0,1,1}$, and

$$\begin{aligned}
&2Z_{1,0,1}\left(\sigma_1,\sigma_2,\sigma_3\right)+Z_{1,0,1}\left(\sigma_1,\sigma_2+1,\sigma_3\right)\\
&=Z_{0,0,1}\left(\sigma_1,\sigma_2+1,\sigma_3\right)Z_{1,0,0}\left(\sigma_1,\sigma_2,\sigma_3\right)+Z_{0,0,1}\left(\sigma_1,\sigma_2,\sigma_3\right)Z_{1,0,0}\left(\sigma_1,\sigma_2+1,\sigma_3\right)\\
&\quad+Z_{1,0,0}\left(\sigma_1,\sigma_2,\sigma_3\right)Z_{0,0,1}\left(\sigma_1,\sigma_2,\sigma_3\right)+\frac{\left(2\sigma_2+1\right)^2}{4\sigma_1^2\sigma_3^2}\\
&\Rightarrow Z_{1,0,1}(\sigma_1,\sigma_2,\sigma_3)=\frac{\left(m_{1,2}^2+\sigma_1^2-\sigma_2^2\right)\left(m_{2,3}^2-\sigma_2^2+\sigma_3^2\right)}{4\sigma_1^2\sigma_3^2}
\end{aligned}$$

Finally, $Z_{1,1,1}$ is found from the $t_1t_2t_3$ terms,

$$\begin{aligned}
&Z_{1,1,1}\left(\sigma_1,\sigma_2,\sigma_3\right)=\frac{\left(-m_{1,2}^2+\sigma_1^2+\sigma_2^2\right)\left(-m_{2,3}^2+\sigma_2^2+\sigma_3^2\right)}{8\sigma_1^2\sigma_2^2\sigma_3^2}\\
&=Z_{0,0,1}\left(\sigma_1,\sigma_2,\sigma_3\right)Z_{0,1,0}\left(\sigma_1,\sigma_2,\sigma_3\right)Z_{1,0,0}\left(\sigma_1,\sigma_2,\sigma_3\right)\\
&\quad+Z_{1,0,0}\left(\sigma_1,\sigma_2,\sigma_3\right)Z_{0,1,1}\left(\sigma_1,\sigma_2,\sigma_3\right)+Z_{1,1,0}\left(\sigma_1,\sigma_2,\sigma_3\right)Z_{0,0,1}\left(\sigma_1,\sigma_2,\sigma_3\right)\\
&\quad-2Z_{0,1,0}\left(\sigma_1,\sigma_2,\sigma_3\right)Z_{1,0,1}\left(\sigma_1,\sigma_2,\sigma_3\right)\\
&\Rightarrow Z_{1,1,1}(\sigma_1,\sigma_2,\sigma_3)=\frac{4\sigma_2^2\left(\sigma_2^2-\sigma_1^2\right)m_{1,2}m_{2,3}\left(m_{2,3}^2-\sigma_2^2+\sigma_3^2\right)-4\sigma_2^2m_{1,2}^3m_{2,3}\left(m_{2,3}^2-\sigma_2^2+\sigma_3^2\right)}{8\sigma_1^2\sigma_2^2\sigma_3^2}\\
&\quad+\frac{\left(\sigma_1^4-\left(2\sigma_2^2+1\right)\sigma_1^2+\sigma_2^4-\sigma_2^2\right)\left(-m_{2,3}^4+m_{2,3}^2+\sigma_2^4-\left(2\sigma_3^2+1\right)\sigma_2^2+\sigma_3^4-\sigma_3^2\right)}{8\sigma_1^2\sigma_2^2\sigma_3^2}\\
&\quad-\frac{m_{1,2}^2\left(m_{1,2}^2-1\right)\left(-m_{2,3}^4+m_{2,3}^2+\sigma_2^4-\left(2\sigma_3^2+1\right)\sigma_2^2+\sigma_3^4-\sigma_3^2\right)}{8\sigma_1^2\sigma_2^2\sigma_3^2}
\end{aligned}$$

2.2.7 Discussion

Let us comment on other interesting directions to investigate further.

- From the bulk four dimensional gauge theory perspective the τ-system we find and its possible generalizations are expected to describe chiral ring relations in presence of a surface operator. Schematically, we expect eq.(2.5) to derive from the following fusion rule among the chiral operator $\mathcal{O}=\operatorname{tr}\phi^2$ and the surface operator W_β

$$\langle:\mathcal{O}^2:W_\beta\rangle=-\frac{\boldsymbol{\beta}^\vee\cdot\boldsymbol{\beta}^\vee}{2}t^{1/h^\vee}\prod_{\boldsymbol{\alpha}\in\hat{\Delta},\boldsymbol{\alpha}\neq\boldsymbol{\beta}}\langle W_{\boldsymbol{\alpha}}\rangle^{-\boldsymbol{\alpha}\cdot\boldsymbol{\beta}^\vee},$$

  while higher chiral observables should generate the flows of the full *non-autonomous* Toda hierarchy.

- The τ-functions we compute in this work could be used to describe through their zeroes the spectrum of the quantum Toda chain integrable system along the lines of [25, 44].
- It should be possible to apply the approach proposed here to general class-$\mathcal{S}$ theories [96] by studying the related isomonodromic deformation problem, for example for circular quivers, generalising to other classical groups the results of [38, 39]. It would be also interesting to extend the analysis to non-self-dual Ω-background, which should amount to the quantization of the τ-systems, and its lift to five dimensional gauge theories on $\mathbb{R}^4 \times S^1$, which should correspond to quantum q-difference τ-systems [28, 40, 41, 46, 55, 220]. I address this last issue partly in the next section.
- The expansion in the large couplings regime should also be considered by extending the analysis of [47, 152]. Actually, the RG evolution at strong coupling can be analysed through late time expansion of the τ-functions. In particular, in [45] the solution in this regime for the A_n series has been given in terms of a matrix model describing the theory around the massless monopoles point which generalizes the $O(2)$ matrix model of [201]. As a related problem, it would be also interesting to priovide a Fredholm determinant/Pfaffian representation for the τ-functions presented here, see for example [33] for the case of orthogonal groups. It would also be interesting to study the extension to defects in supergroup gauge theories, see for example [173].

2.3 Other directions

Here I present other work related to extending the Painlevé/Gauge correspondence. First we look at $d = 5$ theories and $\mathfrak{q}$-difference equation lifts, then at $c = -2$ tau functions from blowup equations, and finally at a generalisation of the $G = U(2)$ tau system to an arbitrary amount of fundamental flavours.

2.3.1 5d Nekrasov functions

In the introduction and in the first section, we focused on $d = 4$ super Yang-Mills. Consider now pure 5 dimensional super Yang-Mills with 8 supercharges on $S^1_R \times \mathbb{R}^4_{\epsilon_1,\epsilon_2}$ with simple gauge group G on its Coulomb branch, parametrized by $\boldsymbol{\alpha} \in \mathfrak{g}^\vee = \mathrm{Lie}G^L$. Following [28], write $q_{1,2} = e^{R\epsilon_{1,2}}$, $\mathbf{u} = e^{R\boldsymbol{\alpha}}$, and let further $\boldsymbol{\omega} = \log \mathbf{u}/\sqrt{-\log q_1 \log q_2}$, where these are to be understood component-wise in the canonical vector space where the root system is realized. Then

$$\begin{aligned}
Z_{cl}(\mathbf{u}, q_1, q_2|z) &= \exp\left\{\log\left((q_1q_2)^{-h^\vee/2}R^{2h^\vee}z\right)\frac{\sum_i \log^2 u_i}{-2\log q_1 \log q_2}\right\} \\
&= \left((q_1q_2)^{-h^\vee/2}R^{2h^\vee}z\right)^{\frac{1}{2}\boldsymbol{\omega}^2} \\
Z_{1-loop}(\mathbf{u}, q_1, q_2) &= \prod_{\boldsymbol{\alpha}\in R}(\mathbf{u}^{\boldsymbol{\alpha}}; q_1, q_2)_\infty \\
Z_{inst}(\mathbf{u}, q_1, q_2|z) &= \sum_{k\geq 0}\left((q_1q_2)^{-h^\vee/2}R^{2h^\vee}z\right)^k Z_k(\mathbf{u}, q_1, q_2)
\end{aligned}$$

All information about the special functions will is collected in the appendix A. Invariance under $q_1 \leftrightarrow q_2$ and, separately, $\mathbf{u} \mapsto \mathbf{u}^{-1}$ is immediate for Z_{cl} and Z_{1-loop}, and therefore, for the full partition function, inasmuch as Z_{inst} follows from the classical and 1-loop asymptotics. For $SU(N)$, the instanton part is invariant under $q_1, q_2 \mapsto q_1^{-1}, q_2^{-1}$ [216, 217]. This is to be expected in the general case, as well. Recall that the localization in equivariant cohomology is done by a topologically twisted scalar supercharge Q which squares to a rotation up to exact terms,

$$Q^2 = \pounds_v + \ldots, \quad v = \epsilon_1(x_1\partial_{x_2} - x_2\partial_{x_1}) + \epsilon_2(x_3\partial_{x_4} - x_4\partial_{x_3})$$

Therefore, the signs of the ϵ parameters should be immaterial. On the other hand, the symmetry properties of the classical[6] and 1-loop parts are

$$\begin{aligned}
Z_{cl}(\mathbf{u}, q_1^{-1}, q_2^{-1}|z) &= (q_1q_2)^{h^\vee\cdot\frac{\sum_i \log^2 u_i}{-2\log q_1 \log q_2}} Z_{cl}(\mathbf{u}, q_1, q_2|z) \\
Z_{1-loop}(\mathbf{u}, q_1^{-1}, q_2^{-1}) &= \prod_{\boldsymbol{\alpha}\in R_+}(-\mathbf{u}^{\boldsymbol{\alpha}}\theta^{-1}(\mathbf{u}^{\boldsymbol{\alpha}}, q_1)\theta^{-1}(\mathbf{u}^{\boldsymbol{\alpha}}, q_2))Z_{1-loop}(\mathbf{u}, q_1, q_2) \\
&= (-1)^{|R_+|}\mathbf{u}^{2\boldsymbol{\rho}}\prod_{\boldsymbol{\alpha}\in R_+}(\theta^{-1}(\mathbf{u}^{\boldsymbol{\alpha}}, q_1)\theta^{-1}(\mathbf{u}^{\boldsymbol{\alpha}}, q_2))Z_{1-loop}(\mathbf{u}, q_1, q_2)
\end{aligned}$$

As is to be expected, for the self-dual $q_1 = \mathfrak{q}^{-1}$, $q_2 = \mathfrak{q}$ both asymmetry factors become unity. We then specialize to the $c = -2$ background, $q_1 = \mathfrak{q}^{-1}$, $q_2 = \mathfrak{q}^2$. In

[6] Sometimes, the instanton counting parameter is redefined to absorb the q_1q_2 factor, which is immaterial for the self-dual background.

this case, using

$$-z\theta^{-1}(z;\mathfrak{q}^{-1})\theta^{-1}(z;\mathfrak{q}^2) = \frac{\theta(z;\mathfrak{q})}{\theta(z;\mathfrak{q}^2)} = \theta(\mathfrak{q}z;\mathfrak{q}^2)$$

we find

$$Z_{cl}(\mathbf{u},\mathfrak{q},q^{-2}|z) = \mathfrak{q}^{h^\vee \frac{\sum_i \log^2 u_i}{4\log^2\mathfrak{q}}} Z_{cl}(\mathbf{u},\mathfrak{q}^{-1},\mathfrak{q}^2|z)$$

$$Z_{1-loop}(\mathbf{u},\mathfrak{q},\mathfrak{q}^{-2}) = \left(\prod_{\boldsymbol{\alpha}\in R_+}\theta(\mathfrak{q}\mathbf{u}^{\boldsymbol{\alpha}};\mathfrak{q}^2)\right) Z_{1-loop}(\mathbf{u},\mathfrak{q}^{-1},\mathfrak{q}^2)$$

Using the identity

$$\theta(\mathfrak{q}z;\mathfrak{q}^2) = \theta(\mathfrak{q}z^{-1};\mathfrak{q}^4)\theta(\mathfrak{q}z;\mathfrak{q}^4)$$

we rewrite the full expression more symmetrically,

$$A(\mathbf{u},\mathfrak{q}) := \frac{Z_{full}(\mathbf{u},\mathfrak{q},\mathfrak{q}^{-2}|z)}{Z_{full}(\mathbf{u},\mathfrak{q}^{-1},\mathfrak{q}^2|z)} = \mathfrak{q}^{h^\vee \frac{\sum_i \log^2 u_i}{4\log^2\mathfrak{q}}} \prod_{\boldsymbol{\alpha}\in R}\theta(\mathfrak{q}\mathbf{u}^{\boldsymbol{\alpha}};\mathfrak{q}^4)$$

defining an asymmetry factor which does not depend on z. Using properties of $\mathfrak{q}$-theta functions we find that, if $\boldsymbol{\lambda}$ is a miniscule coweight,

$$\frac{A(\mathbf{u}\mathfrak{q}^{2\boldsymbol{\lambda}},\mathfrak{q})}{A(\mathbf{u},\mathfrak{q})} = \mathbf{u}^{h^\vee\boldsymbol{\lambda}}\mathfrak{q}^{h^\vee\boldsymbol{\lambda}^2} \prod_{\substack{\boldsymbol{\alpha}\in R\\ \boldsymbol{\alpha}\cdot\boldsymbol{\lambda}=1}} \left(\frac{1}{-\mathfrak{q}\mathbf{u}^{\boldsymbol{\alpha}}}\right)$$

It is a fun exercise to show that

$$\sum_{\substack{\boldsymbol{\alpha}\in R\\ \boldsymbol{\alpha}\cdot\boldsymbol{\lambda}=1}} \boldsymbol{\alpha} = h^\vee\boldsymbol{\lambda}, \qquad \prod_{\substack{\boldsymbol{\alpha}\in R\\ \boldsymbol{\alpha}\cdot\boldsymbol{\lambda}=1}} z = z^{h^\vee\boldsymbol{\lambda}^2}$$

and, therefore,

$$\frac{A(\mathbf{u}\mathfrak{q}^{2\boldsymbol{\lambda}},\mathfrak{q})}{A(\mathbf{u},\mathfrak{q})} = (-1)^{h^\vee\boldsymbol{\lambda}^2}$$

If $\boldsymbol{\lambda}$ is not miniscule, but any coweight, the same holds, but it is harder to show[7]. We have

$$\frac{A(\mathbf{u}\mathfrak{q}^{2\boldsymbol{\lambda}},\mathfrak{q})}{A(\mathbf{u},\mathfrak{q})} = \mathbf{u}^{h^\vee\boldsymbol{\lambda}}\mathfrak{q}^{h^\vee\boldsymbol{\lambda}^2} \prod_{n=1}^{\infty}\prod_{\substack{\boldsymbol{\alpha}\in R\\ \boldsymbol{\alpha}\cdot\boldsymbol{\lambda}=n}} \left(\frac{1}{(-\mathfrak{q})^{n^2}\mathbf{u}^{n\boldsymbol{\alpha}}}\right) = (-1)^{h^\vee\boldsymbol{\lambda}^2} \tag{2.43}$$

2.3.1.1 $\mathfrak{q}$-Painlevé

Noting that if $\mathfrak{q} = e^R$, as $R \to 0$

$$D^2_{\mathfrak{q}}(f) := f(\mathfrak{q}z)f(\mathfrak{q}^{-1}z) - f(z)^2 = R^2 D^2(f) + \mathcal{O}(R)^4$$

[7] We can show it just for the simple coroots, since they along with the miniscule coweights generate the coweight lattice.

where $D^2(f) = f^2\partial^2_{\log z}\log f$, we expect bilinear relations similar to the ones in 4 dimensions. As before, if $\boldsymbol{\alpha}$ is a root corresponding to the a miniscule coweight $\boldsymbol{\lambda}$, we assign it the Kiev Ansatz

$$\tau_{\boldsymbol{\alpha}} = \sum_{\mathbf{n}\in\boldsymbol{\lambda}+Q^\vee} \underbrace{s^{\mathbf{m}}}_{\text{Defect}} \underbrace{z^{\frac{1}{2\log^2\mathfrak{q}}\sum_i \log^2(u_i\mathfrak{q}_i^m)}}_{Z_{cl.}} \underbrace{B_0(\mathbf{u}\mathfrak{q}^{\mathbf{m}})}_{Z_{1-loop}} \underbrace{\sum_i z^i Z_i(\mathbf{u}\mathfrak{q}^{\mathbf{m}})}_{Z_{inst}}$$

We can give the classical part a makeover by using, as before, $\boldsymbol{\omega} = \frac{1}{\log\mathfrak{q}}\log\mathbf{u}$, as then

$$Z_{cl}(\mathbf{u}\mathfrak{q}^{\mathbf{m}}) = z^{\frac{1}{2}\boldsymbol{\omega}^2+\boldsymbol{\omega}\cdot\mathbf{m}+\frac{1}{2}\mathbf{m}^2}$$

meaning

$$D^2_{\mathfrak{q}}(\tau_{\boldsymbol{\lambda}}) = \sum_{\substack{\mathbf{m}_1,\mathbf{m}_2\in Q^\vee_{\boldsymbol{\lambda}}\\ i_{1,2}\in\mathbb{Z}_{\geq 0}}} z^{\boldsymbol{\omega}^2+\frac{1}{2}\mathbf{m}_1^2+\frac{1}{2}\mathbf{m}_2^2+(\mathbf{m}+\mathbf{n})\cdot\boldsymbol{\omega}+i_1+i_2}\left(\mathfrak{q}^{\frac{1}{2}\mathbf{m}_1^2-\frac{1}{2}\mathbf{m}_2^2+(\mathbf{m}-\mathbf{n})\cdot\boldsymbol{\omega}+i_1-i_2}-1\right)$$
$$B_0(\mathbf{u}\mathfrak{q}^{\mathbf{m}_1})B_0(\mathbf{u}\mathfrak{q}^{\mathbf{m}_1})Z_{i_1}(\mathbf{u}\mathfrak{q}^{\mathbf{m}_1})Z_{i_2}(\mathbf{u}\mathfrak{q}^{\mathbf{m}_2})$$

Next I define the $\mathfrak{q}$-analogue of the 4d isomonodromic systems we already studied, which I lift in the minimal fashion:

$$D^2_{\mathfrak{q}}(\tau_{\boldsymbol{\alpha}}) = -\frac{\boldsymbol{\alpha}^2}{2}\prod_{\boldsymbol{\beta}\in\hat{\Delta}\setminus\{\boldsymbol{\alpha}\}}\tau_{\boldsymbol{\beta}}^{-\boldsymbol{\alpha}^\vee\cdot\boldsymbol{\beta}}$$

Here note that the Kiev Ansatz is defined only for the roots corresponding to miniscule coweights. The other tau functions can be obtained from the former.

2.3.1.2 1 loop term

Before considering the $\mathfrak{q}$-analogues of our isomonodromic equations, we begin the ritual of painstaking examination of properties of Z_{1-loop}. In the self-dual background we have

$$Z_{1-loop}(\mathbf{u}\mathfrak{q}^{\mathbf{m}},\mathfrak{q}) = \prod_{\boldsymbol{\alpha}\in R}(\mathbf{u}^{\boldsymbol{\alpha}}\mathfrak{q}^{\boldsymbol{\alpha}\cdot\mathbf{m}};\mathfrak{q}^{-1},\mathfrak{q})_\infty = \prod_{\boldsymbol{\alpha}\in R}(\mathbf{u}^{\boldsymbol{\alpha}}\mathfrak{q}^{1+\boldsymbol{\alpha}\cdot\mathbf{m}};\mathfrak{q},\mathfrak{q})_\infty^{-1} = \prod_{\boldsymbol{\alpha}\in R}\frac{(\mathbf{u}^{\boldsymbol{\alpha}}\mathfrak{q}^{\boldsymbol{\alpha}\cdot\mathbf{m}};\mathfrak{q})_\infty}{(\mathbf{u}^{\boldsymbol{\alpha}}\mathfrak{q}^{\boldsymbol{\alpha}\cdot\mathbf{m}};\mathfrak{q},\mathfrak{q})_\infty}$$

In particular we can write

$$\begin{aligned}\frac{Z_{1-loop}(\mathbf{u}\mathfrak{q}^{\mathbf{m}},\mathfrak{q})}{Z_{1-loop}(\mathbf{u},\mathfrak{q})} &= \prod_{\boldsymbol{\alpha}\in R}\frac{(\mathbf{u}^{\boldsymbol{\alpha}};\mathfrak{q},\mathfrak{q})_\infty}{(\mathbf{u}^{\boldsymbol{\alpha}}\mathfrak{q}^{\boldsymbol{\alpha}\cdot\mathbf{m}};\mathfrak{q},\mathfrak{q})_\infty}\frac{(\mathbf{u}^{\boldsymbol{\alpha}}\mathfrak{q}^{\boldsymbol{\alpha}\cdot\mathbf{m}};\mathfrak{q})_\infty}{(\mathbf{u}^{\boldsymbol{\alpha}};\mathfrak{q})_\infty}\\ &= \prod_{\substack{\boldsymbol{\alpha}\in R,n\in\mathbb{N}\\ \boldsymbol{\alpha}\cdot\mathbf{m}=n}}\frac{(\mathbf{u}^{\boldsymbol{\alpha}};\mathfrak{q},\mathfrak{q})_\infty}{(\mathbf{u}^{\boldsymbol{\alpha}}\mathfrak{q}^n;\mathfrak{q},\mathfrak{q})_\infty}\frac{(\mathbf{u}^{-\boldsymbol{\alpha}};\mathfrak{q},\mathfrak{q})_\infty}{(\mathbf{u}^{-\boldsymbol{\alpha}}\mathfrak{q}^{-n};\mathfrak{q},\mathfrak{q})_\infty}\frac{(\mathbf{u}^{\boldsymbol{\alpha}}\mathfrak{q}^n;\mathfrak{q})_\infty}{(\mathbf{u}^{\boldsymbol{\alpha}};\mathfrak{q})_\infty}\frac{(\mathbf{u}^{-\boldsymbol{\alpha}}\mathfrak{q}^{-n};\mathfrak{q})_\infty}{(\mathbf{u}^{-\boldsymbol{\alpha}}\mathfrak{q};\mathfrak{q})_\infty}\end{aligned}$$

Since,

$$\frac{(z;\mathfrak{q},\mathfrak{q})_\infty}{(z\mathfrak{q}^n;\mathfrak{q},\mathfrak{q})_\infty} = \prod_{k=0}^{n-1}\frac{(z\mathfrak{q}^k;\mathfrak{q},\mathfrak{q})_\infty}{(z\mathfrak{q}^{k+1};\mathfrak{q},\mathfrak{q})_\infty} = \prod_{k=0}^{n-1}(z\mathfrak{q}^k;\mathfrak{q})_\infty$$

$$\frac{(z^{-1};\mathfrak{q},\mathfrak{q})_\infty}{(z^{-1}\mathfrak{q}^{-n};\mathfrak{q},\mathfrak{q})_\infty} = \prod_{k=0}^{n-1}\frac{1}{(z^{-1}\mathfrak{q}^{k-n};\mathfrak{q})_\infty}$$

$$\frac{(z\mathfrak{q}^n;\mathfrak{q})_\infty}{(z;\mathfrak{q})_\infty} = \prod_{k=0}^{n-1}\frac{(z\mathfrak{q}^{k+1};\mathfrak{q})_\infty}{(z\mathfrak{q}^k;\mathfrak{q})_\infty} = \prod_{k=0}^{n-1}\frac{1}{1-z\mathfrak{q}^k}$$

$$\frac{(z^{-1}\mathfrak{q}^{-n};\mathfrak{q})_\infty}{(z^{-1};\mathfrak{q})_\infty} = \prod_{k=0}^{n-1}\left(1-z^{-1}\mathfrak{q}^{k-n}\right)$$

we have the end result

$$\frac{Z_{1-loop}(\mathbf{u}\mathfrak{q}^{\mathbf{m}},\mathfrak{q})}{Z_{1-loop}(\mathbf{u},\mathfrak{q})} = \prod_{\substack{\boldsymbol{\alpha}\in R, n\in\mathbb{N}\\ \boldsymbol{\alpha}\cdot\mathbf{m}=n}}\prod_{k=0}^{n-1}\frac{1-\mathbf{u}^{-\boldsymbol{\alpha}}\mathfrak{q}^{k-n}}{1-\mathbf{u}^{\boldsymbol{\alpha}}\mathfrak{q}^k}\frac{(\mathbf{u}^{\boldsymbol{\alpha}}\mathfrak{q}^k;\mathfrak{q})_\infty}{(\mathbf{u}^{-\boldsymbol{\alpha}}\mathfrak{q}^{k-n};\mathfrak{q})_\infty}$$

2.3.1.3 A_n

The $\mathfrak{q}$-analogue

$$D_{\mathfrak{q}}^2(\tau_0) = -z^{\frac{1}{n+1}}\tau_1\tau_n$$

leads to equations specifying the 1-loop term

$$(u_{p_1}u_{p_2})^{-1}(\mathfrak{q}u_{p_1}-u_{p_2})^2 B_0(\mathbf{u}\mathfrak{q}^{e_{p_1}-e_{p_2}})B_0(\mathbf{u}) = -2B_0(\mathbf{u}\mathfrak{q}^{e_{p_1}})B_0(\mathbf{u}\mathfrak{q}^{-e_{p_2}})$$

where $p_1\neq p_2$ and

$$Z_1(\mathbf{u}) = -2\frac{\mathfrak{q}}{(1-\mathfrak{q})^2}\sum_k\frac{B_0(\mathbf{u}\mathfrak{q}^{e_k})B_0(\mathbf{u}\mathfrak{q}^{-e_k})}{B_0(\mathbf{u})^2}$$

via the lowest order. Note that the term

$$-\frac{\mathfrak{q}}{(1-\mathfrak{q})^2} = \frac{1}{(1-\mathfrak{q}^{-1})(1-\mathfrak{q})}$$

corresponds to the character of $\mathbb{C}^2$ under the self-dual $U(1)^2$ rotation, corresponding to the center of the instanton.

2.3.1.4 B_n, D_n

The $\mathfrak{q}$-analogue

$$D_{\mathfrak{q}}^2(\tau_0) = D_{\mathfrak{q}}^2(\tau_1) \tag{2.44}$$

leads to the equations specifying the 1-loop term

$$(\mathfrak{q}u_{p_1}u_{p_2}-1)^2 B_0(\mathbf{u}\mathfrak{q}^{e_{p_1}+e_{p_2}})B_0(\mathbf{u}) = \mathfrak{q}(u_{p_1}-u_{p_2})^2 B_0(\mathbf{u}\mathfrak{q}^{e_{p_1}})B_0(\mathbf{u}\mathfrak{q}^{e_{p_2}})$$

where $p_1 \neq p_2$. Consider now

$$Z^{D_n}_{1-loop}(\mathbf{u}) = \prod_{i\neq j}(\frac{u_i}{u_j};\mathfrak{q}^{-1},\mathfrak{q})_\infty \prod_{i<j}(u_iu_j;\mathfrak{q}^{-1},\mathfrak{q})_\infty(\frac{1}{u_iu_j};\mathfrak{q}^{-1},\mathfrak{q})_\infty$$
$$Z^{B_n}_{1-loop}(\mathbf{u}) = Z^{D_n}_{1-loop}(\mathbf{u})\prod_i(u_i;\mathfrak{q}^{-1},\mathfrak{q})_\infty(\frac{1}{u_i};\mathfrak{q}^{-1},\mathfrak{q})_\infty$$

By the properties from a previous section, we see that the following holds for both functions ($X \in \{B, D\}$):

$$\frac{Z^{X_n}_{1-loop}(\mathbf{u}\mathfrak{q}^{e_{p_1}+e_{p_2}})Z^{X_n}_{1-loop}(\mathbf{u})}{Z^{X_n}_{1-loop}(\mathbf{u}\mathfrak{q}^{e_{p_1}})Z^{X_n}_{1-loop}(\mathbf{u}\mathfrak{q}^{e_{p_2}})} = \frac{\left(1-\frac{u_{p_1}}{u_{p_2}}\right)\left(1-\frac{u_{p_2}}{u_{p_1}}\right)(1-u_{p_1}u_{p_2})\left(1-\frac{1}{u_{p_1}u_{p_2}}\mathfrak{q}^{-2}\right)}{\left(1-\frac{u_{p_2}}{u_{p_1}}\mathfrak{q}^{-1}\right)\left(1-\frac{u_{p_1}}{u_{p_2}}\mathfrak{q}^{-1}\right)\left(1-\frac{1}{u_{p_1}u_{p_2}}\mathfrak{q}^{-1}\right)(1-u_{p_1}u_{p_2}\mathfrak{q})}$$
$$\times\frac{\left(\frac{1}{u_{p_1}u_{p_2}}\mathfrak{q}^{-1};\mathfrak{q}\right)_\infty\left(\frac{u_{p_1}}{u_{p_2}}\mathfrak{q}^{-1};\mathfrak{q}\right)_\infty\left(\frac{u_{p_2}}{u_{p_1}}\mathfrak{q}^{-1};\mathfrak{q}\right)_\infty(u_{p_1}u_{p_2}\mathfrak{q};\mathfrak{q})_\infty}{\left(\frac{1}{u_{p_1}u_{p_2}}\mathfrak{q}^{-2};\mathfrak{q}\right)_\infty\left(\frac{u_{p_1}}{u_{p_2}};\mathfrak{q}\right)_\infty\left(\frac{u_{p_2}}{u_{p_1}};\mathfrak{q}\right)_\infty(u_{p_1}u_{p_2};\mathfrak{q})_\infty}$$
$$=\frac{\left(1-\frac{u_{p_1}}{u_{p_2}}\right)\left(1-\frac{u_{p_2}}{u_{p_1}}\right)(1-u_{p_1}u_{p_2})\left(1-\frac{1}{u_{p_1}u_{p_2}}\mathfrak{q}^{-2}\right)}{\left(1-\frac{u_{p_2}}{u_{p_1}}\mathfrak{q}^{-1}\right)\left(1-\frac{u_{p_1}}{u_{p_2}}\mathfrak{q}^{-1}\right)\left(1-\frac{1}{u_{p_1}u_{p_2}}\mathfrak{q}^{-1}\right)(1-u_{p_1}u_{p_2}\mathfrak{q})}$$
$$\times\left(1-\frac{1}{u_{p_1}u_{p_2}}\mathfrak{q}^{-2}\right)^{-1}\left(1-\frac{u_{p_1}}{u_{p_2}}\mathfrak{q}^{-1}\right)\left(1-\frac{u_{p_2}}{u_{p_1}}\mathfrak{q}^{-1}\right)(1-u_{p_1}u_{p_2})^{-1}$$
$$=\frac{\left(1-\frac{u_{p_1}}{u_{p_2}}\right)\left(1-\frac{u_{p_2}}{u_{p_1}}\right)}{\left(1-\frac{1}{u_{p_1}u_{p_2}}\mathfrak{q}^{-1}\right)(1-u_{p_1}u_{p_2}\mathfrak{q})} = \frac{\mathfrak{q}(u_{p_1}-u_{p_2})^2}{(\mathfrak{q}u_{p_1}u_{p_2}-1)^2}$$

therefore we can set $B_0(\mathbf{u}) = f(\mathbf{u})Z_{1-loop}(\mathbf{u})$ and $f(\mathbf{u})$ will be a periodic on the coroot lattice. Next we have the 1-instanton

$$Z^{D_n}_1(\mathbf{u}) = \frac{\mathfrak{q}}{(1-\mathfrak{q})^2}\sum_k\frac{(1-u_k^2)^2}{u_k^2}\frac{B^{D_n}_0(\mathbf{u}\mathfrak{q}^{e_k})B^{D_n}_0(\mathbf{u}\mathfrak{q}^{-e_k})}{B^{D_n}_0(\mathbf{u})^2}$$
$$= \frac{\mathfrak{q}}{(1-\mathfrak{q})^2}\sum_k\frac{(1-u_k^2)^2}{u_k^2}\prod_{l\neq k}\left(1-\frac{u_k}{u_l}\right)^{-1}\left(1-\frac{u_l}{u_k}\right)^{-1}(1-u_ku_l)^{-1}\left(1-\frac{1}{u_ku_l}\right)^{-1}$$
$$= \frac{\mathfrak{q}}{(1-\mathfrak{q})^2}\sum_k(1-u_k^2)^2u_k^{2n-4}\prod_{l\neq k}\frac{u_l}{(u_k-u_l)^2(1-u_ku_l)^2}$$

For $n \geq 3$ we can compare these with universal expressions and we find agreement. The case $n = 2$ is special. Note that we have an isomorphism $D_2 \cong A_1 \times A_1$. In the 4 dimensional case, we found $\tau^{D_2} = (\tau^{A_1})^2$, but there we could use special properties of the Hirota derivative to find that

$$D^2((\tau_0^{A_1})^2) = 2(\tau_0^{A_1})^2D^2(\tau_0^{A_1}) = 2D^2(\tau_1^{A_1})(\tau_1^{A_1})^2 = D^2((\tau_1^{A_1})^2)$$

The $\mathfrak{q}$-analogue does not share the same properties. From the definition we get the following behaviour of the Hirota $\mathfrak{q}$-derivative:

$$D^2_\mathfrak{q}(f^n) = \left(D^2_\mathfrak{q}(f)+f^2\right)^n - f^{2n}$$
$$D^2_\mathfrak{q}(f\cdot g) = D^2_\mathfrak{q}(f)D^2_\mathfrak{q}(g) + f^2D^2_\mathfrak{q}(g) + g^2D^2_2(g)$$

Interestingly, we find that if we formulate the system (2.44) for D_2, then

$$\frac{(1-\mathfrak{q})^2}{\mathfrak{q}} Z^{D_2}_{1,\text{here}}(\mathbf{u}) = \frac{u_1^2+u_2^2-4u_1^2u_2^2+u_1^4u_2^2+u_1^2u_2^4}{(u_1-u_2)^2(u_1u_2-1)^2} = 1 + \frac{(1-\mathfrak{q})^2}{\mathfrak{q}} Z^{D_2}_{1,\text{elsewhere}}(\mathbf{u})$$

In particular, whereas in the 4d case we found that

$$Z^{D_2}_{1,4d}(\sigma_1+\sigma_2,\sigma_1-\sigma_2) = Z^{A_1}_{1,4d}(\sigma_1) + Z^{A_1}_{1,4d}(\sigma_2)$$

here we see that

$$Z^{D_2}_1(\mathbf{u}) = \frac{\mathfrak{q}}{(1-\mathfrak{q})^2} + Z^{A_1}_1(u_1u_2) + Z^{A_1}_1(u_1/u_2)$$

For the B_n gauge group, we find

$$Z^{B_n}_1(\mathbf{u}) = -\frac{\mathfrak{q}}{(1-\mathfrak{q})^2}\sum_k (1+u_k)^2 u_k^{2n-3}\prod_{l\neq k}\frac{u_l}{(u_k-u_l)^2(1-u_ku_l)^2}$$

which agrees with the literature for all $n \geq 2$.

2.3.2 (-1)-blowup relations

We can write the $d=5$ blowup relations for an arbitrary[8] simple gauge group G. The K-theoretic partition function on the blown up geometry $\hat{\mathbb{C}}^2$ is going to be given by[9] We call these blowup relations (-1)-blowup relations as they have to do with replacing the origin with the exceptional divisor which is a (-1)-curve. These relations relate $c=1$ to $c=-2$ tau functions, but cannot give us bilinear relations on the tau functions themselves.

$$\hat{Z}_{\boldsymbol{\lambda},d}(\mathbf{u},q_1,q_2|z) = (q_1q_2)^{-\frac{(4d-h^\vee)(h^\vee-1)}{48}} \sum_{\mathbf{n}\in\boldsymbol{\lambda}+Q^\vee} Z(\mathbf{u}q_1^{\mathbf{n}},q_1,q_2q_1^{-1}|q_1^{d-\frac{h^\vee}{2}}z) Z(\mathbf{u}q_2^{\mathbf{n}},q_1q_2^{-1},q_2|q_2^{d-\frac{h^\vee}{2}}z)$$

where d is the "exceptional divisor observable". The blowup formulas say that

$$\hat{Z}_{\boldsymbol{\lambda},d}(\mathbf{u},q_1,q_2|z) = \begin{cases} \left((q_1q_2)^{-h^\vee/2}R^{2h^\vee}z\right)^{\boldsymbol{\lambda}^2/2} Z(\mathbf{u},q_1,q_2|z) & d=0 \\ \chi_{Q^\vee}(\boldsymbol{\lambda}) Z(\mathbf{u},q_1,q_2|z) & 0<d<h^\vee \\ (-1)^{h^\vee\boldsymbol{\lambda}^2}\left((q_1q_2)^{+h^\vee/2}R^{2h^\vee}z\right)^{\boldsymbol{\lambda}^2/2} Z(\mathbf{u},q_1,q_2|z) & d=h^\vee \end{cases}$$

These we specialize to $q_2 = q_1^{-1} = \mathfrak{q}$. Note that, using (2.43), the $d=0$ and $d=h^\vee$ equations equate

$$\left(R^{2h^\vee}z\right)^{\boldsymbol{\lambda}^2/2} Z(\mathbf{u},\mathfrak{q}^{-1},\mathfrak{q}|z) = \sum_{\mathbf{n}\in\boldsymbol{\lambda}+Q^\vee} Z(\mathbf{u}\mathfrak{q}^{-\mathbf{n}},\mathfrak{q}^{-1},\mathfrak{q}^2|\mathfrak{q}^{\frac{h^\vee}{2}}z) Z(\mathbf{u}\mathfrak{q}^{\mathbf{n}},\mathfrak{q}^{-2},\mathfrak{q}|\mathfrak{q}^{-\frac{h^\vee}{2}}z) \tag{2.45}$$

while the others become

$$\chi_{Q^\vee}(\boldsymbol{\lambda}) Z(\mathbf{u},\mathfrak{q}^{-1},\mathfrak{q}|z) = \sum_{\mathbf{n}\in\boldsymbol{\lambda}+Q^\vee} Z(\mathbf{u}\mathfrak{q}^{-\mathbf{n}},\mathfrak{q}^{-1},\mathfrak{q}^2|\mathfrak{q}^{-d+\frac{h^\vee}{2}}z) Z(\mathbf{u}\mathfrak{q}^{\mathbf{n}},\mathfrak{q}^{-2},\mathfrak{q}|\mathfrak{q}^{d-\frac{h^\vee}{2}}z) \tag{2.46}$$

[8] Conjectural, except in case of $SU(N)$ by [216, 217], but with strong evidence.

[9] Comparing with [124], one sees that their "r" corresponds to our $h^\vee$. We would be in trouble if that were the rank, but what they consider is $SU(r)$ for which $r = h^\vee$.

2.3.2.1 Tau functions

Next we define $|P^\vee/Q^\vee|$ tau functions as the multiplicative Zak transforms of the hitherto considered partition functions. These we call

$$\tau_{\boldsymbol{\lambda}}(\mathbf{u},\boldsymbol{\sigma}|z) = \sum_{\mathbf{n}\in\boldsymbol{\lambda}+Q^\vee} \boldsymbol{\sigma}^{\mathbf{n}} Z(\mathbf{u}q^{\mathbf{n}}, q^{-1}, q|z)$$
$$\tau^{\pm}_{\boldsymbol{\lambda}}(\mathbf{u},\boldsymbol{\sigma}|z) = \sum_{\mathbf{n}\in\boldsymbol{\lambda}+Q^\vee} \boldsymbol{\sigma}^{\mathbf{n}/2} Z(\mathbf{u}q^{\mathbf{n}}, q^{\mp 1}, q^{\pm 2}|z)$$

Next we consider taking the same transform of the blowup equations (2.45), (2.46). As an example for $\boldsymbol{\lambda} = \vec{0}$, we can say in a unified fashion

$$\begin{aligned}\tau_0(\mathbf{u},\boldsymbol{\sigma}|z) =& \sum_{\mathbf{n},\mathbf{m}\in Q^\vee} \boldsymbol{\sigma}^{\mathbf{m}Z(\mathbf{u}\mathfrak{q}^{\mathbf{m}-\mathbf{n}},\mathfrak{q}^{-1},\mathfrak{q}^2|\mathfrak{q}^{-d+\frac{h^\vee}{2}}z)Z(\mathbf{u}\mathfrak{q}^{\mathbf{m}+\mathbf{n}},\mathfrak{q}^{-2},\mathfrak{q}|\mathfrak{q}^{d-\frac{h^\vee}{2}}z)}\\ =& \sum_{\mathbf{n}^+,\mathbf{n}^-\in Q^\vee} \boldsymbol{\sigma}^{\frac{\mathbf{n}^-}{2}} Z(\mathbf{u}\mathfrak{q}^{\mathbf{n}^-},\mathfrak{q}^{-1},\mathfrak{q}^2|\mathfrak{q}^{-d+\frac{h^\vee}{2}}z)\boldsymbol{\sigma}^{\frac{\mathbf{n}^+}{2}} Z(\mathbf{u}\mathfrak{q}^{\mathbf{n}^+},\mathfrak{q}^{-2},\mathfrak{q}|\mathfrak{q}^{d-\frac{h^\vee}{2}}z)\\ =&\tau_0^+(\mathbf{u},\boldsymbol{\sigma}|\mathfrak{q}^{-d+\frac{h^\vee}{2}}z)\tau_0^-(\mathbf{u},\boldsymbol{\sigma}|\mathfrak{q}^{d-\frac{h^\vee}{2}}z)\end{aligned}$$

for any $d \in 0, ..., h^\vee$. We can write this more symmetrically as

$$2\tau_0(\mathbf{u},\boldsymbol{\sigma}|z) = \tau_0^+(\mathbf{u},\boldsymbol{\sigma}|\mathfrak{q}^{-d+\frac{h^\vee}{2}}z)\tau_0^-(\mathbf{u},\boldsymbol{\sigma}|\mathfrak{q}^{d-\frac{h^\vee}{2}}z)+\tau_0^+(\mathbf{u},\boldsymbol{\sigma}|\mathfrak{q}^{d-\frac{h^\vee}{2}}z)\tau_0^-(\mathbf{u},\boldsymbol{\sigma}|\mathfrak{q}^{-d+\frac{h^\vee}{2}}z)$$

and restrict $d < h^\vee/2$, which gives $1 + \lfloor h^\vee/2 \rfloor$ equations. We can distinguish 2 cases, depending on the parity of $h^\vee$. In the case of an even Coxeter number we necessarily have the two equations

$$2\tau_0(z) = 2\tau_0^+(z)\tau_0^-(z) = \tau_0^+(\mathfrak{q}z)\tau_0^-(\mathfrak{q}^{-1}z) + \tau_0^+(\mathfrak{q}^{-1}z)\tau_0^-(\mathfrak{q}z)$$

Therefore, as in [27],

$$\tau_0(qz)\tau_0(q^{-1}z) - \tau_0(z)^2 = \tau_0^+(qz)\tau_0^-(qz)\tau_0^+(q^{-1}z)\tau_0^-(q^{-1}z) - \frac{1}{4}(\tau_0^+(qz)\tau_0^-(q^{-1}z) + \tau_0^+(q^{-1}z)\tau_0^-(qz))$$

$$= -\frac{1}{4}(\tau_0^+(qz)\tau_0^-(q^{-1}z) - \tau_0^+(q^{-1}z)\tau_0^-(qz))^2$$

In the case of an odd Coxeter number we necessarily have the two equations

$$2\tau_0(z) = \tau_0^+(q^{\frac12}z)\tau_0^-(q^{-\frac12}z)+\tau_0^+(q^{-\frac12}z)\tau_0^-(q^{\frac12}z) = \tau_0^+(q^{\frac32}z)\tau_0^-(q^{-\frac32}z)+\tau_0^+(q^{-\frac32}z)\tau_0^-(q^{\frac32}z)$$

Therefore,

$$\begin{aligned}&\tau_0(qz)\tau_0(q^{-1}z) - \tau_0(z)^2 = (\tau_0^+(q^{\frac32}z)\tau_0^-(q^{\frac12}z) + \tau_0^+(q^{\frac12}z)\tau_0^-(q^{\frac32}z))\\ &\times(\tau_0^+(q^{-\frac12}z)\tau_0^-(q^{-\frac32}z) + \tau_0^+(q^{-\frac32}z)\tau_0^-(q^{-\frac12}z))\\ &-(\tau_0^+(q^{\frac12}z)\tau_0^-(q^{-\frac12}z) + \tau_0^+(q^{-\frac12}z)\tau_0^-(q^{\frac12}z))(\tau_0^+(q^{\frac32}z)\tau_0^-(q^{-\frac32}z) + \tau_0^+(q^{-\frac32}z)\tau_0^-(q^{\frac32}z))\end{aligned}$$

$$= -(\tau_0^+(q^{-\frac32}z)\tau_0^-(q^{\frac12}z) - \tau_0^+(q^{\frac12}z)\tau_0^-(q^{-\frac32}z))(\tau_0^+(q^{-\frac12}z)\tau_0^-(q^{\frac32}z) - \tau_0^+(q^{\frac32}z)\tau_0^-(q^{-\frac12}z))$$

In [32], it was found that for $G = SU(2)$ the blowup equations imply the Toda-like tau form. For other groups, however, I find that I can rewrite the q-Painlevé Toda-like equations for the $c = 1$ tau functions in terms of the $c = -2$ tau functions using the above formulas. In simplifying the sum, differences of vectors in different lattice cosets can yield a vector in yet a third coset – attention must be payed to the root system in question. However, this ends up being a system of equations for the $c = -2$ tau functions. I was not able to reduce it further.

2.3.3 $U(2)$ with N_f fundamental flavors

Recall that the equation for the one-loop normalization obtained from inserting the Kiev Ansatz for $G = SU(2)$ in the appropriate tau form equation is (2.15). Consider writing it as follows,

$$\frac{B_0(\sigma_1+1,\sigma_2;\{m_k\})B_0(\sigma_1,\sigma_2-1;\{m_k\})}{B_0(\sigma_1+1,\sigma_2-1;\{m_k\})B_0(\sigma_1,\sigma_2;\{m_k\})} = -(1+\sigma_1-\sigma_2)^2 \tag{2.47}$$

where I have inserted some extra parameters, the explicit dependence on which is

$$B_0(\sigma_1,\sigma_2;\{m_k\}) = B_0^{fund.}(\sigma_1,\sigma_2;\{m_k\})B_0^{adj.}(\sigma_1,\sigma_2)$$

where

$$B_0^{fund.}(\sigma_1,\sigma_2;\{m_k\}) = \prod_{k=1}^{N_f} G(1+m_k+\sigma_1)G(1+m_k+\sigma_2)$$
$$B_0^{adj.}(\sigma_1,\sigma_2) = \frac{1}{G(1+\sigma_1-\sigma_2)G(1-\sigma_1+\sigma_2)}$$

as $\{e_1, e_2\}$ are weights of the fundamental ($\square$), and $\{e_1-e_2, e_2-e_1, 0\}$ of the adjoint representation ($\square\square$) of $U(2)$. We already know from before that

$$\frac{B_0^{adj}(\sigma_1+1,\sigma_2)B_0^{adj.}(\sigma_1,\sigma_2-1)}{B_0^{adj}(\sigma_1+1,\sigma_2-1)B_0^{adj}(\sigma_1,\sigma_2)} = -(1+\sigma_1-\sigma_2)^2$$

and we calculate directly that the same is true in this case, as

$$\begin{aligned}&\frac{B_0^{fund.}(\sigma_1+1,\sigma_2;\{m_k\})B_0^{fund.}(\sigma_1,\sigma_2-1;\{m_k\})}{B_0^{fund.}(\sigma_1+1,\sigma_2-1;\{m_k\})B_0^{fund.}(\sigma_1,\sigma_2;\{m_k\})}\\ &=\prod_{k=1}^{N_f}\frac{G(1+m_k+\sigma_1+1)G(1+m_k+\sigma_2)\cdot G(1+m_k+\sigma_1)G(1+m_k+\sigma_2-1)}{G(1+m_k+\sigma_1+1)G(1+m_k+\sigma_2-1)\cdot G(1+m_k+\sigma_1)G(1+m_k+\sigma_2)}\\ &=1\end{aligned}$$

Therefore, we have a solution of the one-loop normalization, with asymptotics

$$\log B_0 \sim \sum_i (\boldsymbol{\sigma}\cdot\boldsymbol{\lambda}_i^{adj})^2 \log\left(\boldsymbol{\sigma}\cdot\boldsymbol{\lambda}_i^{adj}\right)^2 - \sum_k\sum_i(\boldsymbol{\sigma}\cdot\boldsymbol{\lambda}_i^{fund}+m_k)^2\log\left(\boldsymbol{\sigma}\cdot\boldsymbol{\lambda}_i^{fund}+m_k\right)^2$$

Further, by inserting this solution into the recursion relation, which is unchanged, we obtain

$$Z_1(\boldsymbol{\sigma};m) = -\sum_{i=1}^{2}\frac{B_0(\boldsymbol{\sigma}\pm e_i);m)}{B_0(\boldsymbol{\sigma};m)^2} = \sum_{i=1}^{2}\frac{\prod_k(\sigma_i+m_k)}{\prod_{j\neq i}(\sigma_i-\sigma_j)^2}$$

and a more involved expression for higher instantons. These all agree with instanton counting (checked up to 4 instantons with $N_f = 1, 2$, up to 2 instantons with $N_f = 3, 4, 5$).

Next is the 2-instanton:

$$\begin{aligned} Z_2(\boldsymbol{\sigma}; m) = &- \frac{1}{4} \sum_{i=1}^{n+1} \frac{B_0(\boldsymbol{\sigma} \pm e_i; m)}{B_0(\boldsymbol{\sigma}; m)^2} [Z_1(\boldsymbol{\sigma} + e_i; m) + Z_1(\boldsymbol{\sigma} - e_i; m)] \\ &- \sum_{i<j}^{n+1} (\sigma_i - \sigma_j)^2 \frac{B_0(\boldsymbol{\sigma} \pm (e_i - e_j); m)}{B_0(\boldsymbol{\sigma}; m)^2} \end{aligned}$$

This is for $U(n)$, we specialize to $U(2)$ to get

$$\begin{aligned} Z_2(\sigma_1, \sigma_2; m) = &- \frac{1}{4} \frac{B_0(\sigma_1 \pm 1, \sigma_2; m)}{B_0(\boldsymbol{\sigma}; m)^2} [Z_1(\sigma_1 + 1, \sigma_2; m) + Z_1(\sigma_1 - 1, \sigma_2; m)] \\ &- \frac{1}{4} \frac{B_0(\sigma_1, \sigma_2 \pm 1; m)}{B_0(\boldsymbol{\sigma}; m)^2} [Z_1(\sigma_1, \sigma_2 + 1; m) + Z_1(\sigma_1, \sigma_2 - 1; m)] \\ &- (\sigma_1 - \sigma_2)^2 \frac{B_0(\sigma_1 + 1, \sigma_2 - 1; m) B_0(\sigma_1 - 1, \sigma_2 + 1; m)}{B_0(\boldsymbol{\sigma}; m)^2} \end{aligned}$$

The new thing to calculate is the last term. We consider just one flavor with mass m ($N_f = 1$).

$$\begin{aligned} \frac{B_0(\sigma_1 + 1, \sigma_2 - 1; m) B_0(\sigma_1 - 1, \sigma_2 + 1; m)}{B_0(\boldsymbol{\sigma}; m)^2} &= \left(\frac{G(1 + m + \sigma_1) G(1 + m + \sigma_2)}{G(1 + \sigma_{12}) G(1 - \sigma_{12})} \right)^{-2} \\ \frac{G(1 + m + \sigma_1 + 1) G(1 + m + \sigma_2 - 1)}{G(3 + \sigma_{12}) G(-1 - \sigma_{12})} &\cdot \frac{G(1 + m + \sigma_1 - 1) G(1 + m + \sigma_2 + 1)}{G(-1 + \sigma_{12}) G(3 - \sigma_{12})} \\ &= \frac{G(2 + m + \sigma_1) G(m + \sigma_1)}{G(1 + m + \sigma_2)^2} \cdot \frac{G(2 + m + \sigma_1) G(m + \sigma_2)}{G(1 + m + \sigma_2)^2} \\ &\times \frac{G(1 + \sigma_{12})^2 G(1 - \sigma_{12})^2}{G(3 + \sigma_{12}) G(-1 + \sigma_{12}) G(-1 - \sigma_{12}) G(3 - \sigma_{12})} \end{aligned}$$

Now use

$$\frac{G(2 + x) G(x)}{G(1 + x)^2} = \frac{\Gamma(1 + x) \Gamma(x) G(x)^2}{\Gamma(x)^2 G(x)^2} = x$$

for the first terms, and, treating independently σ_{12} and $-\sigma_{12}$,

$$\begin{aligned} \frac{G(1 + x)^2}{G(3 + x) G(-1 + x)} &= \frac{\Gamma(x)^2 \overbrace{\Gamma(-1 + x)^2 G(-1 + x)^2}^{G(x)^2}}{\Gamma(2 + x) \Gamma(1 + x) \Gamma(x) \Gamma(-1 + x) G(-1 + x)^2} \\ &= \frac{\Gamma(x) \Gamma(-1 + x)}{\Gamma(2 + x) \Gamma(1 + x)} = \frac{1}{(1 + x) x^2 (1 - x)} \end{aligned}$$

to get

$$\frac{B_0(\sigma_1 + 1, \sigma_2 - 1; m) B_0(\sigma_1 - 1, \sigma_2 + 1; m)}{B_0(\boldsymbol{\sigma}; m)^2} = \frac{(m + \sigma_1)(m + \sigma_2)}{(1 + \sigma_{12})^2 \sigma_{12}^4 (1 - \sigma_{12})^2}$$

Therefore we obtain the two-instanton term

$$
\begin{aligned}
Z_2(\sigma_1,\sigma_2;m) =& \frac{1}{4}\frac{m+\sigma_1}{\sigma_{12}^2}\left[\left(\frac{m+\sigma_1+1}{(1+\sigma_{12})^2}+\frac{m+\sigma_2}{(1+\sigma_{12})^2}\right)+\left(\frac{m+\sigma_1-1}{(-1+\sigma_{12})^2}+\frac{m+\sigma_2}{(-1+\sigma_{12})^2}\right)\right]\\
&+\frac{1}{4}\frac{m+\sigma_2}{\sigma_{12}^2}\left[\left(\frac{m+\sigma_1}{(-1+\sigma_{12})^2}+\frac{m+\sigma_2+1}{(-1+\sigma_{12})^2}\right)+\left(\frac{m+\sigma_1}{(1+\sigma_{12})^2}+\frac{m+\sigma_2-1}{(1+\sigma_{12})^2}\right)\right]\\
&-\frac{(m+\sigma_1)(m+\sigma_2)}{(1+\sigma_{12})^2\sigma_{12}^2(1-\sigma_{12})^2}
\end{aligned}
$$

Before we attempt to simplify, we have to look at what instanton counting tells us. We have the following components of the count:

$$
\begin{aligned}
Z_{\square\square,\emptyset} &= \frac{(m+\sigma_1)(m+\sigma_1-1)}{4\sigma_{12}^2(-1+\sigma_{12})^2}, \quad Z_{\emptyset,\square\square} = \frac{(m+\sigma_2)(m+\sigma_2-1)}{4\sigma_{12}^2(1+\sigma_{12})^2}\\
Z_{\substack{\square\\\square},\emptyset} &= \frac{(m+\sigma_1)(m+\sigma_1+1)}{4\sigma_{12}^2(1+\sigma_{12})^2}, \quad Z_{\emptyset,\substack{\square\\\square}} = \frac{(m+\sigma_2)(m+\sigma_2+1)}{4\sigma_{12}^2(-1+\sigma_{12})^2}\\
Z_{\square,\square} &= \frac{(m+\sigma_1)(m+\sigma_2)}{(-1+\sigma_{12})^2(1+\sigma_{12})^2}
\end{aligned}
$$

Therefore we can distribute and identify

$$
\begin{aligned}
Z_2(\sigma_1,\sigma_2;m) &= Z_{\substack{\square\\\square},\emptyset} + \frac{1}{4}\frac{m+\sigma_1}{\sigma_{12}^2}\frac{m+\sigma_2}{(1+\sigma_{12})^2}\\
+Z_{\square\square,\emptyset} + \frac{1}{4}\frac{m+\sigma_1}{\sigma_{12}^2}&\frac{m+\sigma_2}{(-1+\sigma_{12})^2} + \frac{1}{4}\frac{m+\sigma_2}{\sigma_{12}^2}\frac{m+\sigma_1}{(-1+\sigma_{12})^2} + Z_{\emptyset,\substack{\square\\\square}}\\
+\frac{1}{4}\frac{m+\sigma_2}{\sigma_{12}^2}&\frac{m+\sigma_1}{(1+\sigma_{12})^2} + Z_{\emptyset,\square\square} - \frac{(m+\sigma_1)(m+\sigma_2)}{(1+\sigma_{12})^2\sigma_{12}^2(1-\sigma_{12})^2}
\end{aligned}
$$

And finally we have that, collecting all the rest,

$$
\begin{aligned}
&\frac{1}{4}\frac{m+\sigma_1}{\sigma_{12}^2}\frac{m+\sigma_2}{(1+\sigma_{12})^2}+\frac{1}{4}\frac{m+\sigma_1}{\sigma_{12}^2}\frac{m+\sigma_2}{(-1+\sigma_{12})^2}+\frac{1}{4}\frac{m+\sigma_2}{\sigma_{12}^2}\frac{m+\sigma_1}{(-1+\sigma_{12})^2}\\
&+\frac{1}{4}\frac{m+\sigma_2}{\sigma_{12}^2}\frac{m+\sigma_1}{(1+\sigma_{12})^2}-\frac{(m+\sigma_1)(m+\sigma_2)}{(1+\sigma_{12})^2\sigma_{12}^2(1-\sigma_{12})^2}\\
&=\frac{(m+\sigma_1)(m+\sigma_2)}{2\sigma_{12}^2(1+\sigma_{12})^2(1-\sigma_{12})^2}\Bigg((1+\sigma_{12})^2+(-1+\sigma_{12})^2-2\Bigg)\\
&=\frac{(m+\sigma_1)(m+\sigma_2)}{(1+\sigma_{12})^2(1-\sigma_{12})^2}=Z_{\square,\square}
\end{aligned}
$$

Therefore,

$$
Z_2(\sigma_1,\sigma_2;m)^{\text{isomonodromy}} = Z_{\square\square,\emptyset}+Z_{\emptyset\square\square}+Z_{\substack{\square\\\square},\emptyset}+Z_{\emptyset,\substack{\square\\\square}}+Z_{\square,\square}=Z_2^{\text{counting}}
$$

2.3.3.1 $SU(2)$ vs $U(2)$

Note that in equation (2.47) we vary the vevs independently. This is the same situation as in instanton counting. Later, we can set $\sigma_1 = -\sigma_2 = \sigma$ and all our results reproduce massive $SU(2)$ theory. However, if we start from $SU(2)$ directly we cannot obtain anything other than pure theory. It is easy to check that

$$\frac{B_0(\sigma+1/2;\{m_k\})^2}{B_0(\sigma+1;\{m_k\})B_0(\sigma;\{m_k\})}=-(1+2\sigma)^2$$

is not satisfied by

$$B_0(\sigma)=\frac{\prod_{k=1}^{N_f}G(1+m_k\pm\sigma)}{G(1\pm 2\sigma)}$$

unless $N_f=0$. Here there is a mystery. Namely, the $U(2)$ Kiev Ansatz has to be the same as the $SU(2)$ one, in the sense that they have to share the same lattice $Q=\{(n,-n)|n\in\mathbb{Z}\}$. My manipulations in the previous section imply that

$$\tau(\boldsymbol{\sigma},\boldsymbol{\eta}|t):=\sum_{\mathbf{n}\in Q}e^{\boldsymbol{\eta}\cdot\mathbf{n}}t^{\frac{1}{2}(\boldsymbol{\sigma}+\mathbf{n})^2}B_0(\boldsymbol{\sigma}+\mathbf{n};m)Z(\boldsymbol{\sigma}+\mathbf{n};m|t)$$

satisfies

$$D^2_{\log t}[\tau(\boldsymbol{\sigma},\boldsymbol{\eta}|t)]=-\tau(\boldsymbol{\sigma}-(1,0),\boldsymbol{\eta}|t)\tau(\boldsymbol{\sigma}+(1,0),\boldsymbol{\eta}|t) \tag{2.48}$$

Consider, however, an alternative definition, with the lattice $Q=\mathbb{Z}^2$. In the following it will be convenient to write

$$B(\boldsymbol{\sigma},m|t)=B(\begin{bmatrix}\sigma_1\\ \sigma_2\end{bmatrix},m|t)$$

From the definition of B, it is clear that

$$B(\begin{bmatrix}\sigma_1+\omega\\ \sigma_2+\omega\end{bmatrix},m|t)=B(\begin{bmatrix}\sigma_1\\ \sigma_2\end{bmatrix},m+\omega|t)$$

holds. Consider defining $\sigma_\pm:=\sigma_1\pm\sigma_2$. Then

$$B(\begin{bmatrix}\sigma_1\\ \sigma_2\end{bmatrix},m|t)=B(\begin{bmatrix}\frac{\sigma_++\sigma_-}{2}\\ \frac{\sigma_+-\sigma_-}{2}\end{bmatrix},m|t)=B(\begin{bmatrix}\frac{\sigma_-}{2}\\ -\frac{\sigma_-}{2}\end{bmatrix},m+\frac{\sigma_+}{2}|t)$$

which is a consequence of the previous identity.

I now define the tau function with the lattice $\mathbb{Z}^2$,

$$\tilde{\tau}_{U(2)}(\begin{bmatrix}\sigma_1\\ \sigma_2\end{bmatrix},\begin{bmatrix}\eta_1\\ \eta_2\end{bmatrix},m|t)=\sum_{n_1,n_2\in\mathbb{Z}}e^{n_1\eta_1+n_2\eta_2}t^{\frac{1}{2}(\sigma_1+n_1)^2+\frac{1}{2}(\sigma_2+n_2)^2}B(\begin{bmatrix}\sigma_1+n_1\\ \sigma_2+n_2\end{bmatrix},m|t)$$

I want to compare this to

$$\tau_{U(2)}(\begin{bmatrix}\sigma_1\\ \sigma_2\end{bmatrix},\begin{bmatrix}\eta_1\\ \eta_2\end{bmatrix},m|t)=\sum_{\substack{n_1,n_2\in\mathbb{Z}\\ n_1+n_2=0}}e^{n_1\eta_1+n_2\eta_2}t^{\frac{1}{2}(\sigma_1+n_1)^2+\frac{1}{2}(\sigma_2+n_2)^2}B(\begin{bmatrix}\sigma_1+n_1\\ \sigma_2+n_2\end{bmatrix},m|t)$$

which I found satisfies (2.48). What equation does $\tilde{\tau}_{U(2)}$ satisfy? Let

$$n_\pm:=n_1\pm n_2,\quad \sigma_\pm:=\sigma_1\pm\sigma_2,\quad \eta_\pm:=\eta_1\pm\eta_2$$

and use it to rewrite the definition of $\tilde{\tau}_{U(2)}$,

$$\sum_{n_+,n_-\in\mathbb{Z}} e^{\frac{1}{2}n_+\eta_+}t^{(\frac{\sigma_+}{2}+\frac{n_+}{2})^2}e^{\frac{1}{2}n_-\eta_-}t^{(\frac{\sigma_-}{2}+\frac{n_-}{2})^2}B(\begin{bmatrix}\frac{\sigma_-+n_-}{2}+\frac{\sigma_++n_+}{2}\\ -\frac{\sigma_-+n_-}{2}+\frac{\sigma_++n_+}{2}\end{bmatrix},m|t)$$

and then use the property of the conformal block to write

$$\sum_{n_+,n_-\in\mathbb{Z}} e^{\frac{1}{2}n_+\eta_+}t^{(\frac{\sigma_+}{2}+\frac{n_+}{2})^2}e^{\frac{1}{2}n_-\eta_-}t^{(\frac{\sigma_-}{2}+\frac{n_-}{2})^2}B(\begin{bmatrix}\frac{\sigma_-+n_-}{2}\\ -\frac{\sigma_-+n_-}{2}\end{bmatrix},m+\frac{\sigma_++n_+}{2}|t)$$

Splitting the sum over n_- into even and odd parts,

$$\sum_{n_+\in\mathbb{Z}}\left(\sum_{n_-\text{ even}}+\sum_{n_-\text{ odd}}\right)e^{\frac{1}{2}n_+\eta_+}t^{(\frac{\sigma_+}{2}+\frac{n_+}{2})^2}e^{\frac{1}{2}n_-\eta_-}t^{(\frac{\sigma_-}{2}+\frac{n_-}{2})^2}B(\begin{bmatrix}\frac{\sigma_-+n_-}{2}\\ -\frac{\sigma_-+n_-}{2}\end{bmatrix},m+\frac{\sigma_++n_+}{2}|t)$$

I can rewrite

$$\tilde{\tau}_{U(2)}(\begin{bmatrix}\sigma_1\\ \sigma_2\end{bmatrix},\begin{bmatrix}\eta_1\\ \eta_2\end{bmatrix},m|t)=\sum_{n_+\in\mathbb{Z}}e^{\frac{1}{2}n_+\eta_+}t^{(\frac{\sigma_+}{2}+\frac{n_+}{2})^2}\Bigg(\tau_{U(2)}(\begin{bmatrix}\frac{\sigma_-}{2}\\ -\frac{\sigma_-}{2}\end{bmatrix},\begin{bmatrix}\frac{\eta_-}{2}\\ -\frac{\eta_-}{2}\end{bmatrix},m+\frac{\sigma_++n_+}{2}|t)$$

$$+e^{\frac{1}{2}\eta_-}\tau_{U(2)}(\begin{bmatrix}\frac{\sigma_-}{2}+\frac{1}{2}\\ -\frac{\sigma_-}{2}-\frac{1}{2}\end{bmatrix},\begin{bmatrix}\frac{\eta_-}{2}\\ -\frac{\eta_-}{2}\end{bmatrix},m+\frac{\sigma_++n_+}{2}|t)\Bigg)$$

Only in the special case where $N_f=0$ does the term in the brackets decouple, so that

$$\tilde{\tau}_{U(2)}(\begin{bmatrix}\sigma_1\\ \sigma_2\end{bmatrix},\begin{bmatrix}\eta_1\\ \eta_2\end{bmatrix}|t)=t^{\sigma_+^2/4}\theta_3\left(u=\frac{i}{4}(\eta_++\sigma_+\log t),q=t^{1/4}\right)$$
$$\times\left(\tau(\frac{\sigma_-}{2},\frac{\eta_-}{2}|t)+e^{\frac{1}{2}\eta_-}\tau(\frac{\sigma_-}{2}+\frac{1}{2},\frac{\eta_-}{2}|t)\right)$$

where $\tau(\sigma,\eta|t)$ is the $SU(2)$ PIII_3 tau function. Even in this case, however, it is unclear which equation is satisfied by $\tilde{\tau}_{U(2)}$. It seems, therefore, that $\tau_{U(2)}$ is a better definition.

2.3.3.2 Other groups

Similar considerations let us calculate $U(N)$, $SO(2N)$, $SO(2N+1)$ with fundamental matter and, via the accidental isomorphism $\mathfrak{so}(5)\cong\mathfrak{sp}(2)$ (this is complex), $Sp(4)$ (the 4 is real and corresponds to the compex 2) with *antisymmetric* multiplets, because the fundamental of $SO(5)$ corresponds to the antisymmetric representation of $Sp(4)$ via the isomorphism.

Again, this we check for quite high order in many ranks.

2.3.3.3 The tau chain

Recall 2.48,

$$D^2_{\log t}[\tau(\boldsymbol{\sigma},\boldsymbol{\eta}|t)]=-\tau(\boldsymbol{\sigma}-(1,0),\boldsymbol{\eta}|t)\tau(\boldsymbol{\sigma}+(1,0),\boldsymbol{\eta}|t)$$

Shifting $\boldsymbol{\sigma} \mapsto \boldsymbol{\sigma} + (1,0)$ gives us

$$D^2_{\log t}[\tau(\boldsymbol{\sigma} + (1,0), \boldsymbol{\eta}|t)] = -\tau(\boldsymbol{\sigma}, \boldsymbol{\eta}|t)\tau(\boldsymbol{\sigma} + (1,1), \boldsymbol{\eta}|t)$$

I can rewrite the last term in that equivalent way, since $(2,0) - (1,1) \in Q$. Notice that in the pure case, $\tau(\boldsymbol{\sigma}, \boldsymbol{\eta}|t) \propto \tau(\boldsymbol{\sigma} + (1,1), \boldsymbol{\eta}|t)$. Here, however,

$$\tau(\boldsymbol{\sigma}, \boldsymbol{\eta}|t) \not\propto \tau(\boldsymbol{\sigma} + (1,1), \boldsymbol{\eta}|t)$$

since neither the 1-loop term, nor the instanton volumes have that symmetry. So we obtain a **chain** of tau-functions which does not close.

The equation (2.48) should be compared with [234, Theorem 14] for PIII_2 and [236, eq. 0.17] in the general case[10]. It is not so simple to pass a verdict, label them exactly the same. In the case of PIII_2, in section 2.3.3.5 we do find that (2.48) reduces to a bilinear Toda equation. In the general case, the derivatives themselves are different, and aren't given in terms of $\partial_{\log t}$ but, for example, $t(1-t)\partial_t$ for PVI. Further, the Weyl group in these $U(2)$ tau functions acts on the initial condition $\boldsymbol{\sigma}$ only and not on the parameters of the equations – although in reductions to $SU(2)$ these get mixed with shifts of the masses, as seen in the next section.

This relationship should be explored more thoroughly.

2.3.3.4 Reducing to $N_f = 0$, PIII_3

In the pure gauge theory case, from the Kiev Ansatz we see that for $\omega \in \mathbb{C}$

$$\tau(\boldsymbol{\sigma} + \omega(1,1), \boldsymbol{\eta}|t) = t^{\omega^2} \sum_{\mathbf{n}\in Q} e^{\boldsymbol{\eta}\cdot\mathbf{n}} t^{\omega(\sigma_1+\sigma_2)+\frac{1}{2}(\boldsymbol{\sigma}+\mathbf{n})^2} B_0(\boldsymbol{\sigma}+\mathbf{n}) Z(\boldsymbol{\sigma}+\mathbf{n}|t)$$

In other words,

$$\tau\left((\sigma,-\sigma) + \omega(1,1), \boldsymbol{\eta}|t\right) = t^{\omega^2}\tau\left((\sigma,-\sigma), \boldsymbol{\eta}|t\right)$$

Note that to reduce to $SU(2)$, we need arguments equal but with opposite signs, i.e. both $\sigma_1 = -\sigma_2$ and $\eta_1 = -\eta_2$. We cannot do this for (σ_1+1, σ_2), but we can for $(\sigma_1 + 1/2, \sigma_2 - 1/2)$, obviously, just by setting $\sigma_1 = -\sigma_2$.
Using the previous relation, we can reduce $\tau(\boldsymbol{\sigma} + (1,0), \boldsymbol{\eta}|t)$ to $SU(2)$, since

$$\begin{aligned}\tau((\sigma,-\sigma) + (1,0), \boldsymbol{\eta}|t) &= \tau((\sigma,-\sigma) + (1/2,-1/2) + (1/2,1/2), \boldsymbol{\eta}|t) \\ &= t^{1/4}\tau((\sigma+1/2, -\sigma-1/2), \boldsymbol{\eta}|t)\end{aligned}$$

An analogous calculation gives $\tau((\sigma,-\sigma)-(1,0), \boldsymbol{\eta}|t) = t^{1/4}\tau((\sigma-1/2, -\sigma+1/2), \boldsymbol{\eta}|t)$. Therefore, the equations (2.48) tell us

$$D^2(\tau_0(\sigma,\eta|t)) = -t^{1/2}\tau_0(\sigma-1/2,\eta|t)\tau_0(\sigma+1/2,\eta|t)$$

Using $\mathbb{Z}$-periodicity of σ, we can write this as two equations

$$\begin{aligned}D^2(\tau_0(\sigma,\eta|t)) &= -t^{1/2}\tau_1(\sigma,\eta|t)^2 \\ D^2(\tau_1(\sigma,\eta|t)) &= -t^{1/2}\tau_0(\sigma,\eta|t)^2\end{aligned}$$

where we have defined

$$\tau_1(\sigma,\eta|t) := \tau_0(\sigma+1/2,\eta|t)$$

This is the Painlevé III_3 in tau form.

[10] I would like to thank M. Bershtein for pointing this out.

2.3.3.5 Reducing to $N_f = 1$, PIII$_2$

In the previous case, the main thing we did was close the chain, but now

$$\tau\left((\sigma,-\sigma)+\omega(1,1),\boldsymbol{\eta},m_1|t\right)\neq t^{\omega^2}\tau\left((\sigma,-\sigma),\boldsymbol{\eta},m_1|t\right)$$

so we cannot do that. However, a similar operation is possible. Consider the 1-loop term shifted by $\omega(1,1)$:

$$\begin{aligned}B_0\left(\boldsymbol{\sigma}+\omega(1,1),m_1\right)&=\frac{G(1+m_1+\sigma_1+\omega)G(1+m_1+\sigma_2+\omega)}{G(1+\sigma_1-\sigma_2)G(1-\sigma_1+\sigma_2)}\\&=\frac{G(1+(m_1+\omega)+\sigma_1)G(1+(m_1+\omega)+\sigma_2)}{G(1+\sigma_1-\sigma_2)G(1-\sigma_1+\sigma_2)}\\&=B_0(\boldsymbol{\sigma},m_1+\omega)\end{aligned}$$

Note that, at the level of instanton counting, we have the analogous result

$$Z(\boldsymbol{\sigma}+\omega(1,1),m_1|t)=Z(\boldsymbol{\sigma},m_1+\omega|t)$$

which is easy to see, because the fundamental contribution is equal to a prodcut of $m_1+\sigma_{1,2}+\epsilon_1(i_{1,2}-1)+\epsilon_2(j_{1,2}-1))$ terms for each of the Young diagrams. We see that we can 'transfer' shifts in both vev's to the mass. In particular, we get

$$\tau((\sigma,-\sigma)+\omega(1,1),\boldsymbol{\eta},m_1|t)=t^{\omega^2}\tau((\sigma,-\sigma),\boldsymbol{\eta},m_1+\omega|t)$$

Let us define $\tau_0(\sigma,\eta,m_1|t):=\tau((\sigma,-\sigma),(\eta,-\eta),m_1|t)$ as before. The equation (2.48) becomes after calculations analogous to the pure gauge case

$$D^2(\tau_0(\sigma,\eta,m_1|t))=-t^{1/2}\tau_0(\sigma-1/2,\eta,m_1-1/2|t)\tau_0(\sigma+1/2,\eta,m_1+1/2|t)$$

Note that in this case, we cannot use periodicity to simplify further to a square.

Indeed, define

$$\begin{aligned}\tau_1(\sigma,\eta,m_1|t)&=\tau_0(\sigma-1/2,\eta,m_1-1/2|t)\\\tau_2(\sigma,\eta,m_1|t)&=\tau_0(\sigma+1/2,\eta,m_1+1/2|t)\end{aligned}$$

Then the equation (2.48) reduces to

$$D^2(\tau_0)=-t^{1/2}\tau_1\tau_2$$

Is this Painlevé III$_2$? We can distinguish between two different tau forms of Painlevé equations, in general:

1. The first form is what I will refer to as the *full* form. It uses a single, unshifted tau function which we define starting from the equation written in terms of the transcendent. It is essentially a change of variables of the Painlevé equation

   For example, for PIII$_3$ we can take the Hirota derivative of one of the equations:

   $$D^2(D^2(\tau_0))=tD^2(\tau_1^2)$$

   and then simplify further the RHS using the other equation:

   $$D^2(D^2(\tau_0))=tD^2(\tau_1^2)=2t\tau_1^2D^2(\tau_1)=-2t^{1/2}D^2(\tau_0)(-t^{1/2}\tau_0^2)$$

   which, although it can be simplified a bit, is already enough, since it's written in terms of a single tau function.

2. The second form is what I will refer to as the *Bäcklund* form, because it uses many tau functions defined on a lattice of shifts in parameter space.

Returning to our equation $D^2(\tau_0) = -t^{1/2}\tau_1\tau_2$, we see that we are dealing, at this stage, with the Bäcklund form. We can use this equation to eliminate one of the tau functions, say τ_1. But then we need (at least) two more equations (since the problem is nonlinear), to eliminate another tau function, say τ_2, and to plug it all into a single equation which is just in terms of τ_0.

The Bäcklund form for PIII$_2$ is given in [234, Theorem 15], but explicit parameter identifications have not been made there. It is also available for $q-$Painlevé in [198] with an explicit dictionary. In that case we have

$$\begin{aligned}
\overline{\tau_1}\underline{\tau_2} - \tau_1\tau_2 &= q^{-\theta}(q-1)^{-1/2}t^{1/2}\tau_0^2 \\
\underline{\tau_1}\tau_2 - \tau_1\underline{\tau_2} &= q^{-1/4}(q-1)^{-1/2}t^{1/2}\tau_0^2 \\
D_q^2(\tau_0) &= -(q-1)^{1/2}t^{1/2}\tau_1\tau_2
\end{aligned}$$

This is equation (5.3) in [198], but I have written τ_0 instead of their τ_3, and I have written $D_q^2(f) := \overline{f}\underline{f} - f^2$.

If I write $q = e^R$ and $t \mapsto R^3 t$, and let $R \to 0$, at first order these equations reduce to

$$\begin{aligned}
\tau_2\partial_{\log t}\tau_1 - \tau_1\partial_{\log t}\tau_2 &= -t^{1/2}\tau_0^2 \\
D^2(\tau_0) &= -t^{1/2}\tau_1\tau_2
\end{aligned}$$

These are only 2 equations. I have checked that the first equation is true up to 4 instantons, but it is unclear where it comes from. Further, these equations are not the same as the already mentioned [234, Theorem 15], as they do not involve the mass/Painlevé parameter explicitly.

2.3.3.6 Reducing to $N_f = 2$, PIII$_1$

For two flavors, completely analogous calculations yield

$$\tau((\sigma,-\sigma)+\omega(1,1),\boldsymbol{\eta},m_1,m_2|t) = t^{\omega^2}\tau((\sigma,-\sigma),\boldsymbol{\eta},m_1+\omega,m_2+\omega|t)$$

Using that, the equation (2.48), which is true for all flavors, when specialized to $N_f = 2$, becomes

$$D^2(\tau_0(\sigma,\eta,m_1,m_2|t)) = -t^{1/2}\tau_0(\sigma-\frac{1}{2},\eta,m_1-\frac{1}{2},m_2-\frac{1}{2}|t)\tau_0(\sigma+\frac{1}{2},\eta,m_1+\frac{1}{2},m_2+\frac{1}{2}|t)$$

In [198], there are 4 tau functions which are considered. They all have shifts:

$$\begin{aligned}
\tau_1 = \tau_0(\sigma,\eta,m_1-\frac{1}{2},m_2|t), &\quad \tau_2 = \tau_0(\sigma,\eta,m_1+\frac{1}{2},m_2|t) \\
\tau_3 = \tau_0(\sigma-\frac{1}{2},\eta,m_1,m_2-\frac{1}{2}|t), &\quad \tau_4 = \tau_0(\sigma+\frac{1}{2},\eta,m_1,m_2+\frac{1}{2}|t)
\end{aligned}$$

In the last equation, we can therefore shift $m_1 \mapsto m_1 \pm \frac{1}{2}$ and get two equations,

$$\begin{aligned}
D^2(\tau_1) &= -t^{1/2}\tau_0(\sigma-\frac{1}{2},\eta,m_1-1,m_2-\frac{1}{2}|t)\tau_4 \\
D^2(\tau_2) &= -t^{1/2}\tau_0(\sigma+\frac{1}{2},\eta,m_1+1,m_2+\frac{1}{2}|t)\tau_3
\end{aligned}$$

Shifting instead, together, $\sigma \mapsto \sigma \pm \frac{1}{2}$, $m_2 \mapsto \frac{1}{2}$ yields

$$D^2(\tau_3) = -t^{1/2}\tau_0(\sigma - 1, \eta, m_1 - \frac{1}{2}, m_2 - 1|t)\tau_2$$
$$D^2(\tau_4) = -t^{1/2}\tau_0(\sigma + 1, \eta, m_1 + \frac{1}{2}, m_2 + 1|t)\tau_1$$

So in this case, all of the equations are qualitatively different than the ones in [198].

2.3.4 Discussion

The Toda-like equations for pure $d = 4$ $\mathcal{N} = 2$ gauge theories with arbitrary groups can be generalised to $d = 5$ $\mathcal{N} = 1$ theories on a circle by a straightforward lift of the ordinary Hirota derivative to a q-analogue. Due to the different behaviour of the q-version, some more complicated manipulations may become unavailable, for instance those involving the ordinary Y^n operators. It would be interesting, however, to obtain these equations using quiver mutations as in [26, 28, 40].

The link to blowup equations needs more work. In [254], (-2)-blowup relations were used – perhaps generalising this would work for $\mathfrak{g} = A_n$ theories with $n > 1$.

Finally, the $G = U(2)$ super Yang-Mills with an arbitrary number Nf of fundamental multiplets satisfying the same equation as the $Nf = 0$ case represents a mystery. From the SW/integrable systems 1.4 point of view, this should correspond to a deautonomised XXX spin chain. The fact that we cannot close it perhaps hints to some quasiperiodic boundary conditions. It would also be interesting if blowup relations can help derive this.

Chapter 3

Magical matrix models from three dimensions

3.1 Extending the ABJM/Painleve III_3 correspondence

In the introduction we have seen that ABJM computes the $\mathfrak{q}-$Painlevé III_3 tau function via the TS/ST/Tau correspondence. It is natural to ask for an extension of this correspondence to other $\mathfrak{q}$-difference equations.

In the first part, I present my work with my collaborators Naotaka Kubo and Tomoki Nosaka and my advisors Alessandro Tanzini and Giulio Bonelli [41]. We used recent progress on spectral curves to realise the tau function of $\mathfrak{q}$-PVI as a matrix model.

In the second part, I present my work with Tomoki Nosaka. There we computed the grand partition function of the D-type quiver exactly in terms of a Fredholm Pfaffian, for a simpler, degenerate case D_3, as well as for D_4, and we gave a rank-deformation which works. Both the exact computation and the rank-deformation are novel. We had hoped to use the ADE classification of superconformal Chern-Simons quivers to explore what kind of relations are satisfied by a quiver of D-type. With the data we have available, however, we were unable to find any bilinear relations. There are, however, other directions to go.

3.2 $\mathfrak{q}-$PVI matrix model

This section is based on joint work with Naotaka Kubo and Tomoki Nosaka [41]. We studied in detail the case of $\mathfrak{q}$-Painlevé VI, corresponding to five-dimensional $\mathcal{N}=1$ $\mathrm{SU}(2)$ gauge theory with four fundamental hypermultiplets $N_f=4$. Via geometric engineering this corresponds to topological strings on the local D_5 del Pezzo Calabi-Yau threefold.

This came about through a series of intensive studies of the four-nodes theory $\mathrm{U}(N)_k \times \mathrm{U}(N)_0 \times \mathrm{U}(N)_{-k} \times \mathrm{U}(N)_0$ [192, 209, 210] which describes N M2-branes placed on $\left(\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}^2/\mathbb{Z}_2\right)/\mathbb{Z}_k$ orbifold [146]. There it was found that the large $\mu=\log\kappa$ expansion of the modified grand potential $J(\mu)$ (related to the grand partition function as $\Xi(\kappa)=\sum_n e^{J(\mu+2\pi i n)}$) [183, 184, 208, 212] of this theory is consistent with the refined topological string free energy on local D_5 del Pezzo geometry at large radius. Later this result was generalized to the case of different ranks $\mathrm{U}(N_1)_k \times \mathrm{U}(N_2)_0 \times \mathrm{U}(N_3)_{-k} \times \mathrm{U}(N_4)_0$, finding that the three extra degrees of freedom of the rank differences realize a three dimensional sublattice of quantised values in the full Kähler moduli space of local D_5. This enables us to show that the five dimensional moduli space can be realized by turning on the Fayet-Iliopoulos parameters of the Chern-Simons matter theory. As it is the case for the TS/ST/tau correspondence between the ABJM theory and $\mathfrak{q}$-Painlevé III_3, by using the exact values of the partition function of $\mathrm{U}(N_1)_k \times \mathrm{U}(N_2)_0 \times \mathrm{U}(N_3)_{-k} \times \mathrm{U}(N_4)_0$ theory at fixed moduli we can check the τ-form of the $\mathfrak{q}$-Painlevé VI equation in the small κ expansion.

In section 3.2.1, we first recall some background material and fix our notations. In section 3.2.2, performed a detailed analysis of the matrix model of the quiver superconformal Chern-Simons theory and the related quantum curve. In section 3.2.4, we give a thorough check that the grand partition function of the above theory satisfies $\mathfrak{q}$-Painlevé equations, thus providing a conjectural Fredholm determinant representation for the corresponding τ-functions. In section 3.2.5 we describe the coalescence limits from the viewpoint of the analysis of matrix models and quantum curves, providing matrix model realizations of the $\mathfrak{q}$-Painlevé τ-functions. In section 3.2.6 we discuss the coalescence limit from the viewpoint of $\mathfrak{q}$-difference equations by considering both the perturbative gauge theory realisation of the τ-function and the magnetic matrix model one. Finally, in section 3.2.8 we discuss some open questions for further investigation. We collect in the appendices some relevant definitions and details of some computations.

3.2.1 Five dimensional gauge theory, $\mathfrak{q}$-Painlevé and TS/ST correspondence

3.2.1.1 Five dimensional gauge theory and $\mathfrak{q}$-deformed PVI equations in bilinear form

As mentioned in the introductory section 1.7, the relation between Painlevé VI differential equation and two dimensional Liouville CFT with $c = 1$ was first noticed in [100]. This connection arises from the formulation of Painlevé VI equation as the isomonodromic deformation problem of an auxiliary $sl(2, \mathbb{C})$ linear system on the Riemann sphere with four regular punctures. This linear system is solved in terms of the degenerate four point conformal block of $c = 1$ Liouville CFT [147]. By the AGT correspondence the isomonodromic τ function is given as the Fourier transform of the full Nekrasov partition function, the Nekrasov-Okounkov partition function introduced in [225]. In this sense the Painlevé equation can be viewed as a non-trivial identity among Nekrasov functions.

In the Introduction, there was some mention of $d = 5$ $\mathcal{N} = 1$ super Yang-Mills on the $d = 5$ Omega-background, which has a circle fibration and reduces to $\mathbb{R}^4 \times S^1$ in the Seiberg-Witten limit. More detail was given in the section on Painlevé-like equations.

By using the five dimensional uplift of AGT correspondence [11, 13] $\mathfrak{q}$-Virasoro four point conformal block can be computed in terms of the five dimensional Nekrasov-

Okounkov [225] (NO) partition function, as:

$$\begin{aligned}\tau\left(\theta_0,\theta_1,\theta_t,\theta_\infty;s,\sigma,t\right)&=\sum_{n\in\mathbb{Z}}s^n t^{(\sigma+n)^2-\theta_t^2-\theta_0^2}C\left(\theta_0,\theta_1,\theta_t,\theta_\infty;\sigma+n\right)Z\left(\theta_0,\theta_1,\theta_t,\theta_\infty;\sigma+n,t\right),\\ C\left(\theta_0,\theta_1,\theta_t,\theta_\infty;\sigma\right)&=\frac{\prod_{\epsilon,\epsilon'=\pm}G_{\mathfrak{q}}\left(1+\epsilon\theta_\infty-\theta_1+\epsilon'\sigma\right)G_{\mathfrak{q}}\left(1+\epsilon\sigma-\theta_t+\epsilon'\theta_0\right)}{G_{\mathfrak{q}}\left(1+2\sigma\right)G_{\mathfrak{q}}\left(1-2\sigma\right)},\\ Z\left(\theta_0,\theta_1,\theta_t,\theta_\infty;\sigma,t\right)&=\sum_{\lambda_+,\lambda_-}t^{|\lambda_+|+|\lambda_-|}\frac{\prod_{\epsilon,\epsilon'=\pm}N_{\phi,\lambda_{\epsilon'}}\left(\mathfrak{q}^{\epsilon\theta_\infty-\theta_1-\epsilon'\sigma}\right)N_{\lambda_\epsilon,\phi}\left(\mathfrak{q}^{\epsilon\sigma-\theta_t-\epsilon'\theta_0}\right)}{\prod_{\epsilon,\epsilon'}N_{\lambda_\epsilon,\lambda_{\epsilon'}}\left(\mathfrak{q}^{(\epsilon-\epsilon')\sigma}\right)}.\end{aligned}\tag{3.1}$$

Here C,Z are respectively the one-loop determinant and the instanton partition function of the $d=5$ $\mathcal{N}=1$ $SU(2)$ Yang-Mills with $N_f=4$, whose Seiberg-Witten curve reads

$$\begin{aligned}&\left(w-m_1'\right)\left(w-m_2'\right)t^2\\&+\left[-\left(\left(\frac{m_1'm_2'}{a_1a_2}\right)^{\frac12}+q'\left(\frac{a_1a_2}{m_3'm_4'}\right)^{\frac12}\right)w^2+Ew-m_1'm_2'\left(\left(\frac{a_1a_2}{m_1'm_2'}\right)^{\frac12}+q'\left(\frac{m_3'm_4'}{a_1a_2}\right)^{\frac12}\right)\right]t\\&+q'\left(\frac{m_1'm_2'}{m_3'm_4'}\right)^{\frac12}\left(w-m_3'\right)\left(w-m_4'\right)=0,\end{aligned}\tag{3.2}$$

where $a_1a_2=1$ and

$$\begin{aligned}\mathfrak{q}^{\theta_0}=\left(\frac{m_1'}{m_3'}\right)^{\frac12},\quad \mathfrak{q}^{\theta_1}=\left(m_2'm_4'\right)^{\frac12},\quad \mathfrak{q}^{\theta_t}=\left(m_1'm_3'\right)^{-\frac12},\\ \mathfrak{q}^{\theta_\infty}=\left(\frac{m_4'}{m_2'}\right)^{\frac12},\quad t=q'\left(\frac{m_2'm_4'}{m_1'm_3'}\right)^{\frac12}.\end{aligned}\tag{3.3}$$

The Omega-background is chosen to be self-dual $\epsilon_2=-\epsilon_1$, and $\mathfrak{q}=e^{-\beta\epsilon_1}$, where β is the radius of S^1 on which the five dimensional theory is compactified. This parameter identification can be obtained by comparing the results in [17] and [157]. The $\mathfrak{q}$-deformed Painlevé VI equation is defined through the $\mathfrak{q}$-difference version of the analogue isomonodromic deformation problem [158]. This suggests a connection between $\mathfrak{q}$-PVI system and $\mathfrak{q}$-Virasoro algebra – it was shown in [157] that the associated linear system can be solved in terms of $\mathfrak{q}$-Virasoro five point conformal blocks. Note that solving $\mathfrak{q}$-PVI equations does not fix uniquely the choice C in (3.1), as it was also the case for $\mathfrak{q}$-PIII$_3$ [31, 46]. We address this point further in section 3.2.8. This $\tau\left(\theta_0,\theta_1,\theta_t,\theta_\infty;s,\sigma,t\right)$ is a $\mathfrak{q}$-uplift of the isomonodromic τ function (1.10) of the Painlevé VI differential equation from the Introduction. The relation between the τ-function and the $\mathfrak{q}$-Painlevé transcendents is not obvious. Nevertheless, building on results in the differential case one can define them via some identities satisfied by the $\mathfrak{q}$-uplifted τ-function (3.1). Specifically, for $\mathfrak{q}$-PVI we define y,z as

$$y=\mathfrak{q}^{-2\theta_1-1}\cdot t\frac{\tau_3\tau_4}{\tau_1\tau_2},\quad z=-\mathfrak{q}^{\theta_t-\theta_1-1}\cdot t\frac{\underline{\tau}_7\tau_8}{\underline{\tau}_5\tau_6},$$

where $\tau_i, \underline{\tau}_i, \bar{\tau}_i$ are defined from a single τ-function $\tau\left(\theta_0, \theta_1, \theta_t, \theta_\infty; s, \sigma, t\right)$ as

$$\begin{aligned}
&\tau_1(t) = \tau\left(\theta_0, \theta_1, \theta_t, \theta_\infty + \frac{1}{2}; s, \sigma, t\right), && \tau_2(t) = \tau\left(\theta_0, \theta_1, \theta_t, \theta_\infty - \frac{1}{2}; s, \sigma, t\right),\\
&\tau_3(t) = \tau\left(\theta_0 + \frac{1}{2}, \theta_1, \theta_t, \theta_\infty; s, \sigma + \frac{1}{2}, t\right), && \tau_4(t) = \tau\left(\theta_0 - \frac{1}{2}, \theta_1, \theta_t, \theta_\infty; s, \sigma - \frac{1}{2}, t\right),\\
&\tau_5(t) = \tau\left(\theta_0, \theta_1 - \frac{1}{2}, \theta_t, \theta_\infty; s, \sigma, t\right), && \tau_6(t) = \tau\left(\theta_0, \theta_1 + \frac{1}{2}, \theta_t, \theta_\infty; s, \sigma, t\right),\\
&\tau_7(t) = \tau\left(\theta_0, \theta_1, \theta_t - \frac{1}{2}, \theta_\infty; s, \sigma + \frac{1}{2}, t\right), && \tau_8(t) = \tau\left(\theta_0, \theta_1, \theta_t + \frac{1}{2}, \theta_\infty; s, \sigma - \frac{1}{2}, t\right),\\
&\underline{\tau}_i(t) = \tau_i\left(\mathfrak{q}^{-1} t\right), \quad \bar{\tau}_i(t) = \tau_i(\mathfrak{q} t). && \qquad\qquad (3.4)
\end{aligned}$$

The $\mathfrak{q}$-PVI equations follow from the bilinear identities satisfied by the five dimensional Nekrasov-Okounkov partition function [157]:

$$\begin{aligned}
\tau_1\tau_2 - \mathfrak{q}^{-2\theta_1} \cdot t\tau_3\tau_4 - \left(1 - \mathfrak{q}^{-2\theta_1} \cdot t\right)\tau_5\tau_6 &= 0,\\
\tau_1\tau_2 - t\tau_3\tau_4 - \left(1 - \mathfrak{q}^{-2\theta_t} \cdot t\right)\underline{\tau}_5\bar{\tau}_6 &= 0,\\
\tau_1\tau_2 - \tau_3\tau_4 + \left(1 - \mathfrak{q}^{-2\theta_1} \cdot t\right)\mathfrak{q}^{2\theta_t}\underline{\tau}_7\bar{\tau}_8 &= 0,\\
\tau_1\tau_2 - \mathfrak{q}^{2\theta_t}\tau_3\tau_4 + \left(1 - \mathfrak{q}^{-2\theta_t} \cdot t\right)\mathfrak{q}^{2\theta_t}\tau_7\tau_8 &= 0,\\
\underline{\tau}_5\tau_6 + \mathfrak{q}^{-\theta_1 - \theta_\infty + \theta_t - \frac{1}{2}} \cdot t\underline{\tau}_7\tau_8 - \underline{\tau}_1\tau_2 &= 0,\\
\underline{\tau}_5\tau_6 + \mathfrak{q}^{-\theta_1 + \theta_\infty + \theta_t - \frac{1}{2}} \cdot t\underline{\tau}_7\tau_8 - \tau_1\underline{\tau}_2 &= 0,\\
\underline{\tau}_5\tau_6 + \mathfrak{q}^{\theta_0 + 2\theta_t}\underline{\tau}_7\tau_8 - \mathfrak{q}^{\theta_t}\underline{\tau}_3\tau_4 &= 0,\\
\underline{\tau}_5\tau_6 + \mathfrak{q}^{-\theta_0 + 2\theta_t}\underline{\tau}_7\tau_8 - \mathfrak{q}^{\theta_t}\tau_3\underline{\tau}_4 &= 0.
\end{aligned} \tag{3.5}$$

These eight equations provide the τ-form of $\mathfrak{q}$-Painlevé VI system.

3.2.1.2 TS/ST contribution

As detailed in the introductory section on TS/ST, it has been conjectured in [117] that the operator $\widehat{\mathcal{O}}^{-1}$, the inverse of the the operator $\widehat{\mathcal{O}} = \sum_{(m,n)\neq(0,0)} a_{m,n} e^{m\widehat{x} + n\widehat{p}}$, with $[\widehat{x}, \widehat{p}] = i\hbar$, which is quantum version of the mirror curve $W_X\left(e^x, e^p\right) = \sum_{m,n} a_{m,n} e^{mx+np} = 0$ in $\mathbb{C}^* \times \mathbb{C}^*$ to the topological string on a local toric Calabi-Yau threefold X, where W_X is the Newton polynomial of the CY3 X and the sum over m, n runs over a finite set in $\mathbb{Z}^2$ determined by the toric fun of X, known as the Newton polygon of the curve [19, 60, 143, 169], is trace class, that is $\mathrm{tr}\widehat{\mathcal{O}}^{-n}$ are finite for $n = 1, 2, \cdots$. This has been checked in some relevant cases in [167, 194] and our analysis in this work extends the verification to other cases. In particular, the following Fredholm operator admits a well defined spectral determinant:

$$\Xi(\kappa) = \det\left(1 + \kappa\widehat{\mathcal{O}}^{-1}\right). \tag{3.6}$$

This spectral determinant is conjectured to provide a non-perturbative completion of the free energy of the topological string on X. As mentioned in section 1.10.3, the perturbative expansion of the topological string free energy displays infinitely many poles, while the spectral determinant (3.6) is finite for an arbitrary value of the topological string coupling $\hbar$. Interestingly, from the viewpoint of the topological string free energy the right analytic properties of the spectral determinant are achieved by cancelling these poles by a proper combination of refined and unrefined topological string amplitudes [133].

When X is the local D_5 del Pezzo, one obtains by geometric engineering $d=5$ $SU(2)$ super Yang-Mills gauge group minimally coupled to $N_f=4$ fundamental hypermultiplets. The quantum mirror curve is given by

$$\begin{aligned}\widehat{\mathcal{O}} = & \, q\left(\frac{m_1m_2}{m_3m_4}\right)^{\frac{1}{2}} e^{-\widehat{x}+\widehat{p}} - \left(\left(\frac{m_1m_2}{a_1a_2}\right)^{\frac{1}{2}} + q\left(\frac{a_1a_2}{m_3m_4}\right)^{\frac{1}{2}}\right) e^{\widehat{p}} + e^{\widehat{x}+\widehat{p}} \\ & - q\left(\frac{m_1m_2}{m_3m_4}\right)^{\frac{1}{2}} (m_3+m_4)\, e^{-\widehat{x}} + E - (m_1+m_2)\, e^{\widehat{x}} \\ & + q\,(m_1m_2m_3m_4)^{\frac{1}{2}}\, e^{-\widehat{x}-\widehat{p}} - m_1m_2\left(\left(\frac{a_1a_2}{m_1m_2}\right)^{\frac{1}{2}} + q\left(\frac{m_3m_4}{a_1a_2}\right)^{\frac{1}{2}}\right) e^{-\widehat{p}} \\ & + m_1m_2 e^{\widehat{x}-\widehat{p}}. \end{aligned} \tag{3.7}$$

Here m_i, q are related to m_i', q' in (3.2) through the following rescaling [44, 167]:

$$\log m_i' = \frac{2\pi}{\hbar}\log m_i, \quad \log q' = \frac{2\pi}{\hbar}\log q. \tag{3.8}$$

The Planck constant is related to the Omega deformation parameter $\mathfrak{q} = e^{-\beta\epsilon_1}$ as

$$\mathfrak{q} = e^{\frac{4\pi^2 i}{\hbar}}.$$

Together with (3.3), we obtain the following relation between Painlevé parameters $(\theta_0, \theta_1, \theta_t, \theta_\infty, t)$ and the coefficients of the quantum mirror curve (m_1, m_2, m_3, m_4, q)

$$\theta_0 = \frac{1}{4\pi i}\log\frac{m_1}{m_3}, \quad \theta_1 = \frac{1}{4\pi i}\log(m_2m_4), \quad \theta_t = \frac{1}{4\pi i}\log\frac{1}{m_1m_3},$$
$$\theta_\infty = \frac{1}{4\pi i}\log\frac{m_4}{m_2}, \quad \frac{\log t}{\log\mathfrak{q}} = \frac{1}{4\pi i}\log\left(\frac{q^2m_2m_4}{m_1m_3}\right).$$

The Coulomb vev σ is related to κ. Let us notice that in the correspondence the spectral determinant computes the $s=1$ NO partition function.

The TS/ST correspondence suggests that the spectral determinant $\Xi(\kappa)$ (3.6), with $\widehat{\mathcal{O}}$ given as (3.7), is equal to $\tau(\theta_0, \theta_1, \theta_t, \theta_\infty; 1, \sigma, t)$ (3.1) up to a κ-independent overall factor and upon an appropriate relation between κ and σ. Combining this with the five-dimensional uplift of the Painlevé/gauge theory correspondence [46] it follows that $\Xi(\kappa)$ should solve the $\mathfrak{q}$-Painlevé equations (3.4)-(3.5). Notice that while the Nekrasov-Okounkov partition function is written as a small t expansion, the spectral determinant is manifestly given as a small κ expansion, and hence solves $\mathfrak{q}$-Painlevé equations in a different regime. The difficult part of this program is to invert the operator $\widehat{\mathcal{O}}$ for a generic set of the coefficients m_1, m_2, m_3, m_4, q. It is here that the three dimensional Chern-Simons matter theory as described in 1.10.2 enters the story.

3.2.2 Chern-Simons matter matrix model and quantum curve

In [183, 184, 208, 212] it was found that $\mathcal{N}=4$ $\mathrm{U}(N_1)_k \times \mathrm{U}(N_2)_0 \times \mathrm{U}(N_3)_{-k} \times \mathrm{U}(N_4)_0$ superconformal Chern-Simons (CS) matter theory is related to the D_5 del

Pezzo geometry (3.2). More precisely, it was conjectured that the partition function of the CS matter theory computes the fermionic spectral traces of the inverse quantum mirror curve of D_5 del Pezzo geometry (3.7),

$$Z_k(N_1,N_2,N_3,N_4) = \frac{Z_k(N=0)}{N!}\int \frac{d^N x}{(2\pi)^N} \det_{i,j}^{N} \left\langle x_i \middle| \widehat{\mathcal{O}}^{-1} \middle| x_j \right\rangle ,$$

where we parametrize the four ranks as $N_1 = N + M_1$, $N_2 = N + M$, $N_3 = N + M_2$, $N_4 = N$, and assume $N, M_1, M_2, M \geq 0$. This is represented in Fig.3.1 where we also describe the IIB D-brane set-up. Therefore the grand partition function of this theory gives, after resummation, the spectral determinant (3.6):

$$\sum_{N=0}^{\infty} \kappa^N \frac{Z_k(N_1,N_2,N_3,N_4)}{Z_k(N=0)} = \det\left(1+\kappa\widehat{\mathcal{O}}^{-1}\right). \tag{3.9}$$

Here $\widehat{\mathcal{O}}$ is the quantum mirror curve (3.7) with $\hbar = 2\pi k$, where the three rank differences $N_i - N$ correspond to a three dimensional subspace of the five mass parameters of the curve, while the overall rank N is dual to the true* modulus. In order to turn on the remaining mass parameters we further introduce Fayet-Iliopoulos (FI) terms for each gauge node in the following way†

$$\begin{aligned} &\mathrm{U}\left(N_1\right)_k \times \mathrm{U}\left(N_2\right)_0 \times \mathrm{U}\left(N_3\right)_{-k} \times \mathrm{U}\left(N_4\right)_0 \\ \to\ &\mathrm{U}\left(N_1\right)_{k,\zeta_1} \times \mathrm{U}\left(N_2\right)_{0,-\zeta_1} \times \mathrm{U}\left(N_3\right)_{-k,\zeta_2} \times \mathrm{U}\left(N_4\right)_{0,-\zeta_2}. \end{aligned} \tag{3.10}$$

The partition function of the resulting gauge theory on S^3 is reduced by supersymmetric localisation to the following integral [164]

$$\begin{aligned} &Z_k\left(N;M_1,M_2,M,\zeta_1,\zeta_2\right) \\ &= \frac{i^{-\frac{N_1^2}{2}+\frac{N_3^2}{2}}}{N_1!N_2!N_3!N_4!}\int \prod_{i=1}^{N_1}\frac{d\lambda_i^{(1)}}{2\pi}\prod_{i=1}^{N_2}\frac{d\lambda_i^{(2)}}{2\pi}\prod_{i=1}^{N_3}\frac{d\lambda_i^{(3)}}{2\pi}\prod_{i=1}^{N_4}\frac{d\lambda_i^{(4)}}{2\pi} \\ &\quad\times e^{\frac{ik}{4\pi}\sum_{i=1}^{N_1}\left(\lambda_i^{(1)}\right)^2-\frac{ik}{4\pi}\sum_{i=1}^{N_3}\left(\lambda_i^{(3)}\right)^2} e^{-i\zeta_1\left(\sum_{i=1}^{N_1}\lambda_i^{(1)}-\sum_{i=1}^{N_2}\lambda_i^{(2)}\right)-i\zeta_2\left(\sum_{i=1}^{N_3}\lambda_i^{(3)}-\sum_{i=1}^{N_4}\lambda_i^{(4)}\right)} \\ &\quad\times\prod_{a=1}^{4}\frac{\prod_{i<j}^{N_a}\left(2\sinh\frac{\lambda_i^{(a)}-\lambda_j^{(a)}}{2}\right)^2}{\prod_{i=1}^{N_a}\prod_{j=1}^{N_{a+1}}2\cosh\frac{\lambda_i^{(a)}-\lambda_j^{(a+1)}}{2}}, \end{aligned} \tag{3.11}$$

where $N_5 = N_1$ and $\lambda_j^{(5)} = \lambda_j^{(1)}$.

3.2.3 Fermi gas formalism

We address the study of the matrix model (3.11) in the Fermi gas formalism. Our derivation extends the one in [183] to the present case with rank deformations

*Here we follow the standard terminology already used in [46] (see section 2.1) distinguishing the complex moduli of the mirror curve into a "true" modulus which we call κ and mass parameters.

†We could consider more general choices of FI parameters and also introduce hypermultiplet's masses. However, such extra deformations either do not preserve the Newton polygon of the quantum curve (3.2) or do not affect the quantum curve at all.

	012	3	4	5	6	7	8	9
D3	✓	✓						
NS5	✓		✓		✓		✓	
(1,k)5	✓		$\theta(k)$		$\theta(k)$		$\theta(k)$	

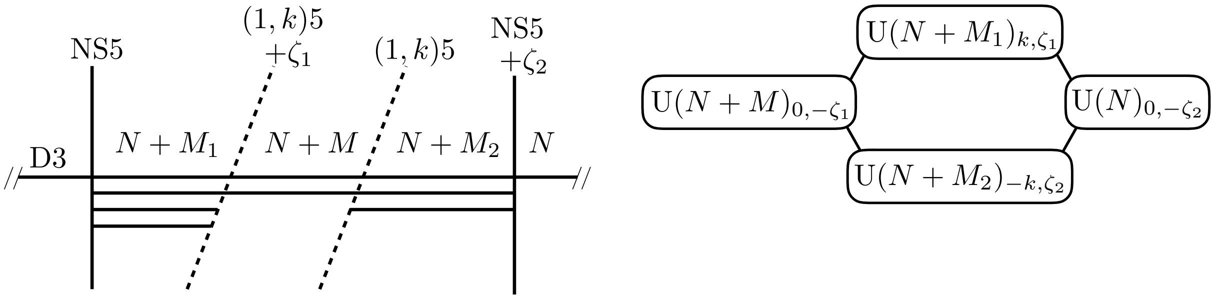


Figure 3.1: Left: Type IIB brane setup of the three dimensional superconformal Chern-Simons matter theory (3.10), where $\theta(k)$ in the row of $(1,k)$5-brane stands for the direction with an angle $\arctan(k)$ from the first axis in each of the pairs; Right: The quiver diagram of the three dimensional Chern-Simons matter theory realized by the brane setup.

and generic Fayet-Iliopoulos parameters. In the following we assume that the FI parameters ζ_i are real, we use the parametrization

$$N_1 = N + M_1, \quad N_2 = N + M, \quad N_3 = N + M_2, \quad N_4 = N,$$

and assume that N, M_1, M_2 and M are non-negative integers as in the main text. We also assume $M_1 \geq M$ and $M_2 \geq M$.

The integrand of the matrix model, after shifting all of the integration variables as $\mu \to \frac{\mu}{k}$, can be divided into two parts as

$$Z_k\left(N; M_1, M_2, M, \zeta_1, \zeta_2\right) = \frac{1}{N_2! N_4!} \int \prod_{n=1}^{N_4} \frac{d\mu_n}{\hbar} \prod_{n=1}^{N_2} \frac{d\nu_n}{\hbar} Y_{N_4,N_1,N_2}\left(0, \zeta_1; \mu, \nu\right) \left(Y_{N_4,N_3,N_2}\left(\zeta_2, 0; \mu, \nu\right)\right)^*, \tag{3.12}$$

where

$$\begin{aligned} &Y_{N_4,\tilde{N},N_2}\left(\zeta, \zeta'; \mu, \nu\right) \\ &= \frac{i^{-\frac{\tilde{N}^2}{2}}}{\tilde{N}!} \int \prod_{n=1}^{\tilde{N}} \frac{d\lambda_n}{\hbar} e^{\frac{i}{4\pi k} \sum_{n=1}^{\tilde{N}} \lambda_n^2} e^{-\frac{i\zeta}{k}\left(\sum_{n=1}^{N_4} \mu_n - \sum_{n=1}^{\tilde{N}} \lambda_n\right) - \frac{i\zeta'}{k}\left(\sum_{n=1}^{\tilde{N}} \lambda_n - \sum_{n=1}^{N_2} \nu_n\right)} \\ &\quad \times \frac{\prod_{m<m'}^{N_4} 2\sinh\frac{\mu_m - \mu_{m'}}{2k} \prod_{n<n'}^{\tilde{N}} 2\sinh\frac{\lambda_n - \lambda_{n'}}{2k}}{\prod_{m=1}^{N_4} \prod_{n=1}^{\tilde{N}} 2\cosh\frac{\mu_m - \lambda_n}{2k}} \frac{\prod_{m<m'}^{\tilde{N}} 2\sinh\frac{\lambda_m - \lambda_{m'}}{2k} \prod_{n<n'}^{N_2} 2\sinh\frac{\nu_n - \nu_{n'}}{2k}}{\prod_{m=1}^{\tilde{N}} \prod_{n=1}^{N_2} 2\cosh\frac{\lambda_m - \nu_n}{2k}}. \end{aligned} \tag{3.13}$$

We first focus on $Y_{N_4,\tilde{N},N_2}$ and rewrite it in the operator formalism. By combining the Cauchy determinant formula and the Vandermonde determinant formula [200]

$$\frac{\prod_{m<m'}^{K} 2\sinh\frac{\mu_m-\mu_{m'}}{2k}\prod_{n<n'}^{K+L} 2\sinh\frac{\lambda_n-\lambda_{n'}}{2k}}{\prod_{m=1}^{K}\prod_{n=1}^{K+L} 2\cosh\frac{\mu_m-\lambda_n}{2k}} = \det\begin{pmatrix} \left[(-1)^L\frac{e^{\frac{L}{2k}(\mu_m-\lambda_n)}}{2\cosh\frac{\mu_m-\lambda_n}{2k}}\right]_{m,n}^{K\times(K+L)} \\ \left[e^{\frac{1}{k}\left(\frac{L+1}{2}-r\right)\lambda_n}\right]_{r,n}^{L\times(K+L)} \end{pmatrix},$$

$$\frac{\prod_{m<m'}^{K+L} 2\sinh\frac{\lambda_m-\lambda_{m'}}{2k}\prod_{n<n'}^{K} 2\sinh\frac{\nu_n-\nu_{n'}}{2k}}{\prod_{m=1}^{K+L}\prod_{n=1}^{K} 2\cosh\frac{\lambda_m-\nu_n}{2k}}$$
$$= \det\begin{pmatrix} \left[(-1)^L\frac{e^{-\frac{L}{2k}(\lambda_m-\nu_n)}}{2\cosh\frac{\lambda_m-\nu_n}{2k}}\right]_{m,n}^{(K+L)\times K} & \left[e^{\frac{1}{k}\left(\frac{L+1}{2}-r\right)\lambda_m}\right]_{m,r}^{(K+L)\times L} \end{pmatrix}, \tag{3.14}$$

we can rewrite the third line of (3.13) as the determinant of the product of two matrices. The notation for the operator formalism is in (3.56). We will make use of the following identities

$$\begin{aligned}
(-1)^L\frac{e^{\frac{L}{2k}(\mu-\lambda)}}{2\cosh\frac{\mu-\lambda}{2k}} &= (-1)^L\int_{\mathbb{R}}\frac{dp}{2\pi}\frac{e^{\frac{i}{\hbar}(p-i\pi L)(\mu-\lambda)}}{2\cosh\frac{p}{2}} \\
&= k\left\langle\mu\left|\frac{1}{2\cosh\frac{\widehat{p}-i\pi L}{2}}\right|\lambda\right\rangle + \sum_{r}^{\left\lfloor\frac{L+1}{2}\right\rfloor}(-1)^{L+r+1}e^{\frac{1}{k}\left(\frac{L+1}{2}-r\right)(\mu-\lambda)}, \\
e^{\frac{1}{k}\sigma\lambda} &= \sqrt{k}\,\langle\!\langle 2\pi i\sigma\,|\,\lambda\rangle\,,
\end{aligned} \tag{3.15}$$

where in the second line, as we shifted the integration contour to $\mathbb{R}+i\pi L$, one obtains a summation over the resulting residues. Fortunately, the contribution to the determinant of the latter vanishes being a linear combination of rows. Indeed it is evident from (3.14) that the sum of the residues is a linear combination of the lower (or right) elements. To write all of factors in the operator formalism, we multiply the first matrix by a Fresnel factor. We also include the first FI factor, depending on ζ, in the first matrix and the second FI factor, depending on ζ', in the second matrix. After performing the similarity transformations

$$e^{-\frac{2\pi i\zeta}{\hbar}\widehat{x}}f\left(\widehat{p}\right)e^{\frac{2\pi i\zeta}{\hbar}\widehat{x}} = f\left(\widehat{p}+2\pi\zeta\right), \quad \langle\!\langle p|\,e^{\frac{2\pi i\zeta}{\hbar}\widehat{x}} = \langle\!\langle p-2\pi\zeta\,|\,,$$

we obtain

$$\begin{aligned}
&Y_{N_4,\tilde{N},N_2}\left(\zeta,\zeta';\mu,\nu\right) \\
&= \frac{i^{-\frac{\tilde{N}^2}{2}}}{\tilde{N}!}\int\prod_{n=1}^{\tilde{N}}\frac{d\lambda_n}{\hbar}\det\begin{pmatrix}\left[k\left\langle\mu_m\left|\frac{1}{2\cosh\frac{\widehat{p}+2\pi\zeta-i\pi\tilde{M}}{2}}e^{\frac{i}{2\hbar}\widehat{x}^2}\right|\lambda_n\right\rangle\right]_{m,n}^{N_4\times\tilde{N}} \\ \left[\sqrt{k}\left\langle\!\!\left\langle t_{-\zeta,\tilde{M},r}\left|e^{\frac{i}{2\hbar}\widehat{x}^2}\right|\lambda_n\right\rangle\right.\right]_{r,n}^{\tilde{M}\times\tilde{N}}\end{pmatrix} \\
&\quad\times\det\begin{pmatrix}\left[k\left\langle\lambda_m\left|\frac{1}{2\cosh\frac{\widehat{p}+2\pi\zeta'+i\pi(\tilde{N}-N_2)}{2}}\right|\nu_n\right\rangle\right]_{m,n}^{\tilde{N}\times N_2} & \left[\sqrt{k}\left\langle\lambda_m\left|-t_{\zeta',\tilde{N}-N_2,r}\right\rangle\!\!\right\rangle\right]_{m,r}^{\tilde{N}\times(\tilde{N}-N_2)}\end{pmatrix},
\end{aligned}$$

where $\tilde{M}=N_4-\tilde{N}$ and $t_{\zeta,n,r}$ is defined in (3.54).

We now return to the matrix model (3.12). Upon the similarity transformation

$$\int \prod_{n=1}^{N_4} d\mu_n \left|\mu_n\right\rangle \left\langle\mu_n\right| = \int \prod_{n=1}^{N_4} d\mu_n e^{\frac{i}{2\hbar}\widehat{x}^2} e^{\frac{i}{2\hbar}\widehat{p}^2} \left|\mu_n\right\rangle \left\langle\mu_n\right| e^{-\frac{i}{2\hbar}\widehat{p}^2} e^{-\frac{i}{2\hbar}\widehat{x}^2},$$
$$\int \prod_{n=1}^{\tilde{N}} d\lambda_n \left|\lambda_n\right\rangle \left\langle\lambda_n\right| = \int \prod_{n=1}^{\tilde{N}} d\lambda_n e^{\frac{i}{2\hbar}\widehat{p}^2} \left|\lambda_n\right\rangle \left\langle\lambda_n\right| e^{-\frac{i}{2\hbar}\widehat{p}^2},$$
$$\int \prod_{n=1}^{N_2} d\nu_n \left|\nu_n\right\rangle \left\langle\nu_n\right| = \int \prod_{n=1}^{N_2} d\nu_n e^{\frac{i}{2\hbar}\widehat{p}^2} \left|\nu_n\right\rangle \left\langle\nu_n\right| e^{-\frac{i}{2\hbar}\widehat{p}^2},$$

and by using the formulae

$$e^{-\frac{i}{2\hbar}\widehat{p}^2} e^{-\frac{i}{2\hbar}\widehat{x}^2} f\left(\widehat{p}\right) e^{\frac{i}{2\hbar}\widehat{x}^2} e^{\frac{i}{2\hbar}\widehat{p}^2} = f\left(\widehat{q}\right), \quad \left\langle\!\left\langle p\right|\right. e^{\frac{i}{2\hbar}\hat{x}^2} e^{\frac{i}{2\hbar}\widehat{p}^2} = \sqrt{i} e^{-\frac{i}{2\hbar}p^2} \left\langle p\right|,$$

we find that the matrix models can be written as

$$\begin{aligned} &Z_k\left(N; M_1, M_2, M, \zeta_1, \zeta_2\right) \\ &= \frac{1}{N_2! N_4!} e^{i\Theta_k\left(M_1, M_2, M, \zeta_1, \zeta_2\right)} \int \prod_{n=1}^{N_4} \frac{d\mu_n}{\hbar} \prod_{n=1}^{N_2} \frac{d\nu_n}{\hbar} \tilde{Y}_{N_4,N_1,N_2}\left(0, \zeta_1; \mu, \nu\right) \left(\tilde{Y}_{N_4,N_3,N_2}\left(\zeta_2, 0; \mu, \nu\right)\right)^*, \end{aligned} \tag{3.16}$$

where

$$\begin{aligned} &\tilde{Y}_{N_4,\tilde{N},N_2}\left(\zeta, \zeta'; \mu, \nu\right) \\ &= \frac{i^{-\frac{\tilde{N}^2}{2}+\frac{\tilde{M}}{2}}}{\tilde{N}!} \int \prod_{n=1}^{N} \frac{d\lambda_n}{\hbar} \det \begin{pmatrix} \left[k \left\langle \mu_m \middle| \frac{1}{2\cosh\frac{\widehat{x}+2\pi\zeta-i\pi\tilde{M}}{2}} \middle| \lambda_n \right\rangle\right]_{m,n}^{N_4\times\tilde{N}} \\ \left[\sqrt{k} \left\langle t_{-\zeta,\tilde{M},r} \middle| \lambda_n \right\rangle\right]_{r,n}^{\tilde{M}\times\tilde{N}} \end{pmatrix} \\ &\quad \times \det \begin{pmatrix} \left[k \left\langle \lambda_m \middle| \frac{1}{2\cosh\frac{\widehat{p}+2\pi\zeta'+i\pi\left(\tilde{N}-N_2\right)}{2}} \middle| \nu_n \right\rangle\right]_{m,n}^{\tilde{N}\times N_2} & \left[\sqrt{k} \left\langle \lambda_m \middle| -t_{\zeta',\tilde{N}-N_2,r} \right\rangle\!\right\rangle\right]_{m,r}^{\tilde{N}\times\left(\tilde{N}-N_2\right)} \end{pmatrix}, \end{aligned}$$

and

$$\begin{aligned} \Theta_k\left(M_1, M_2, M, \zeta_1, \zeta_2\right) &= \theta_k\left(M_1, 0\right) + \theta_k\left(M_1 - M, \zeta_1\right) - \theta_k\left(M_2, \zeta_2\right) - \theta_k\left(M_2 - M, 0\right), \\ \theta_k\left(M, \zeta\right) &= \frac{\pi}{k}\left[\frac{1}{12}\left(M^3 - M\right) - M\zeta^2\right]. \end{aligned}$$

$\tilde{Y}_{N_4,\tilde{N},N_2}$ can be computed as follows. Since the second determinant is an anti-symmetric function of λ_m, we can simplify the first determinant by using

$$\frac{1}{N!} \int d^N\lambda \det\left(\left[g_m\left(\lambda_n\right)\right]_{m,n}^{N\times N}\right) f\left(\lambda_1, \lambda_2, \ldots, \lambda_N\right) = \int d^N\lambda \prod_n^N g_n\left(\lambda_n\right) f\left(\lambda_1, \lambda_2, \ldots, \lambda_N\right),$$

which holds for any anti-symmetric function $f\left(\boldsymbol{\lambda}\right)$. We decompose the other determinant by using (3.15) and (3.14) backwards. Now we can perform the integration by using the delta functions coming form the inner products of the position operators. After a short computation, we obtain

$$
\begin{aligned}
\tilde{Y}_{N_4,\tilde{N},N_2}\left(\zeta,\zeta';\mu,\nu\right) =& i^{-\frac{\tilde{N}^2}{2}+\frac{\tilde{M}^2}{2}} e^{\frac{2\pi i\zeta\zeta'\tilde{M}}{k}} e^{-\frac{i\zeta'}{k}\left(\sum_{n=1}^{N_4}\mu_n-\sum_{n=1}^{N_2}\nu_n\right)} Z_k^{(\mathrm{CS})}\left(\tilde{M}\right)\\
&\times\prod_{n=1}^{N_4}\frac{\prod_{r=1}^{\tilde{M}}2\sinh\frac{\mu_n+t_{\zeta,\tilde{M},r}}{2k}}{2\cosh\frac{\mu_n+2\pi\zeta-i\pi\tilde{M}}{2}}\prod_{n=1}^{N_2}\frac{1}{\prod_{r=1}^{\tilde{M}}2\cosh\frac{\nu_n+t_{\zeta,\tilde{M},r}}{2k}}\\
&\times\frac{\prod_{m<m'}^{N_4}2\sinh\frac{\mu_m-\mu_{m'}}{2k}\prod_{n<n'}^{N_2}2\sinh\frac{\nu_n-\nu_{n'}}{2k}}{\prod_{m=1}^{N_4}\prod_{n=1}^{N_2}2\cosh\frac{\mu_m-\nu_n}{2k}},
\end{aligned}
$$

where

$$
Z_k^{(\mathrm{CS})}\left(L\right)=\frac{1}{k^{\frac{L}{2}}}\prod_{j<j'}^{L}2\sin\frac{\pi}{k}\left(j'-j\right),
$$

is the partition function of $\mathrm{U}\left(L\right)_k$ pure Chern-Simons theory. We again use the determinant formula (3.14) and the operator formula (3.15) for the factor at the third line, and we include the FI factors and the factors in the second line into the matrix. As a result, we obtain

$$
\begin{aligned}
&\tilde{Y}_{N_4,\tilde{N},N_2}\left(\zeta,\zeta';\mu,\nu\right)=i^{-\frac{N_4^2}{2}}e^{\frac{2\pi i\zeta\zeta'\tilde{M}}{k}}Z_k^{(\mathrm{CS})}\left(\tilde{M}\right) \qquad (3.17)\\
&\times\det\begin{pmatrix}\left[k\left\langle\mu_m\middle|S_{\tilde{M}}\left(\widehat{x}+2\pi\zeta\right)\frac{1}{2\cosh\frac{\widehat{p}+2\pi\zeta'-i\pi M}{2}}C_{\tilde{M}}\left(\widehat{x}+2\pi\zeta\right)\middle|\nu_n\right\rangle\right]_{m,n}^{N_4\times N_2}\\ \left[\sqrt{k}\left\langle\!\left\langle t_{-\zeta',M,r}\right|C_{\tilde{M}}\left(\widehat{x}+2\pi\zeta\right)\left|\nu_n\right\rangle\right]_{r,n}^{M\times N_2}\end{pmatrix},
\end{aligned}
$$

where

$$
S_L\left(x\right)=i^L\frac{\prod_{r=1}^{L}2\sinh\frac{x-2\pi i\left(\frac{L+1}{2}-r\right)}{2k}}{2\cosh\frac{x+i\pi L}{2}},\quad C_L\left(x\right)=\frac{1}{\prod_{r=1}^{L}2\cosh\frac{x-2\pi i\left(\frac{L+1}{2}-r\right)}{2k}}. \qquad (3.18)
$$

By using the recursive formula for the quantum dilogarithm functions (.84), (3.18) can be written in terms of the quantum dilogarithm as

$$
S_L\left(x\right)=e^{\frac{k-L}{2k}x}\frac{\Phi_b\left(\frac{x}{2\pi b}-\frac{iL}{2b}+\frac{i}{2}b\right)}{\Phi_b\left(\frac{x}{2\pi b}+\frac{iL}{2b}-\frac{i}{2}b\right)},\quad C_L\left(x\right)=e^{\frac{L}{2k}x}\frac{\Phi_b\left(\frac{x}{2\pi b}+\frac{iL}{2b}\right)}{\Phi_b\left(\frac{x}{2\pi b}-\frac{iL}{2b}\right)}.
$$

By substituting (3.17) into (3.16), we finally arrive at

$$
\begin{aligned}
&Z_k\left(N;M_1,M_2,M,\zeta_1,\zeta_2\right)\\
&=\frac{e^{i\Theta_k\left(M_1,M_2,M,\zeta_1,\zeta_2\right)}Z_k^{(\mathrm{CS})}\left(M_1\right)Z_k^{(\mathrm{CS})}\left(M_2\right)}{N!\left(N+M\right)!}\int\prod_{n=1}^{N}\frac{d\mu_n}{2\pi}\prod_{n=1}^{N+M}\frac{d\nu_n}{2\pi}\\
&\times\det\begin{pmatrix}\left[\left\langle\mu_m\middle|\widehat{D}_1^{\mathrm{VI}}\middle|\nu_n\right\rangle\right]_{m,n}^{N\times(N+M)}\\ \left[\left\langle\!\left\langle t_{0,M,r}\middle|\widehat{d}_1^{\mathrm{VI}}\middle|\nu_n\right\rangle\right]_{r,n}^{M\times(N+M)}\end{pmatrix}\\
&\times\det\begin{pmatrix}\left[\left\langle\nu_m\middle|\widehat{D}_2^{\mathrm{VI}}\middle|\mu_n\right\rangle\right]_{m,n}^{(N+M)\times N} & \left[\left\langle\nu_m\middle|\widehat{d}_2^{\mathrm{VI}}\middle|-t_{0,M,r}\right\rangle\!\right\rangle\right]_{m,r}^{(N+M)\times M}\end{pmatrix} \qquad (3.19)
\end{aligned}
$$

where

$$\widehat{D}_1^{\mathrm{VI}} = e^{-\frac{i\zeta_1}{k}\widehat{x}} S_{M_1}(\widehat{x}) \frac{1}{2\cosh\frac{\widehat{p}-i\pi M}{2}} e^{\frac{i\zeta_1}{k}\widehat{x}} C_{M_1}(\widehat{x}),$$
$$\widehat{d}_1^{\mathrm{VI}} = e^{\frac{i\zeta_1}{k}\widehat{x}} C_{M_1}(\widehat{x}),$$
$$\widehat{D}_2^{\mathrm{VI}} = C_{M_2}(\widehat{x}+2\pi\zeta_2) \frac{1}{2\cosh\frac{\widehat{p}+\pi i M}{2}} S_{M_2}(\widehat{x}+2\pi\zeta_2),$$
$$\widehat{d}_2^{\mathrm{VI}} = C_{M_2}(\widehat{x}+2\pi\zeta_2).$$

Let us perform a short digression on the formulas which can be used to rephrase our final result (3.19) in a more concise way, though we do not use them in the main text. Note that by using the formula

$$\frac{1}{N!}\int d^N\nu \det\left(\left[f_m(\nu_n)\right]_{m,n}^{N\times N}\right)\det\left(\left[g_n(\nu_m)\right]_{m,n}^{N\times N}\right) = \det\left(\left[\int d\nu f_m(\nu) g_n(\nu)\right]_{m,n}^{N\times N}\right), \tag{3.20}$$

the partition function for $M>0$ (3.19), which is written as a $(2N+M)$ dimensional integral, can be further reduced to a N dimensional integral:

$$\begin{aligned} &Z_k(N; M_1, M_2, M, \zeta_1, \zeta_2) \\ &= \frac{e^{i\Theta_k(M_1,M_2,M,\zeta_1,\zeta_2)} Z_k^{(\mathrm{CS})}(M_1) Z_k^{(\mathrm{CS})}(M_2)}{N!} \int \prod_{n=1}^{N} \frac{d\mu_n}{2\pi} \\ &\times \det\begin{pmatrix} \left[\left\langle \mu_m \middle| \widehat{D}_1^{\mathrm{VI}} \widehat{D}_2^{\mathrm{VI}} \middle| \mu_n \right\rangle\right]_{m,n}^{N\times N} & \left[\left\langle \mu_m \middle| \widehat{D}_1^{\mathrm{VI}} \widehat{d}_2^{\mathrm{VI}} \middle| -t_{0,M,s} \right\rangle\!\right\rangle\Big]_{m,s}^{N\times M} \\ \Big[\left\langle\!\left\langle t_{0,M,r} \middle| \widehat{d}_1^{\mathrm{VI}} \widehat{D}_2^{\mathrm{VI}} \middle| \mu_n \right\rangle\right]_{r,n}^{M\times N} & \Big[\left\langle\!\left\langle t_{0,M,r} \middle| \widehat{d}_1^{\mathrm{VI}} \widehat{d}_2^{\mathrm{VI}} \middle| -t_{0,M,s} \right\rangle\!\right\rangle\Big]_{r,s}^{M\times M} \end{pmatrix}, \end{aligned} \tag{3.21}$$

which implies that the grand partition function (3.9) can be written as [200]

$$\begin{aligned} \Xi_k(\kappa; M_1, M_2, M, \zeta_1, \zeta_2) &= \sum_{N=0}^{\infty} \kappa^N \frac{Z_k(N; M_1, M_2, M, \zeta_1, \zeta_2)}{Z_k(0; M_1, M_2, M, \zeta_1, \zeta_2)} \\ &= \mathrm{Det}\left(1+\kappa \widehat{D}_1^{\mathrm{VI}} \widehat{D}_2^{\mathrm{VI}}\right) \det_{r,s}\left[\left\langle\!\left\langle t_{0,M,r} \middle| \widehat{d}_1^{\mathrm{VI}} \frac{1}{1+\kappa \widehat{D}_2^{\mathrm{VI}} \widehat{D}_1^{\mathrm{VI}}} \widehat{d}_2^{\mathrm{VI}} \middle| -t_{0,M,s} \right\rangle\!\right\rangle\right]. \end{aligned}$$

3.2.3.1 $M = 0$ case

When $M=0$, the matrix model (3.21) simplifies to

$$\begin{aligned} Z_k(N; M_1, M_2, 0, \zeta_1, \zeta_2) &= e^{i\Theta_k(M_1,M_2,0,\zeta_1,\zeta_2)} Z_k^{(\mathrm{CS})}(M_1) Z_k^{(\mathrm{CS})}(M_2) \\ &\times \frac{1}{N!}\int \prod_{n=1}^{N} \frac{d\mu_n}{2\pi} \det\left(\left[\langle \mu_m | \widehat{\rho}_k(M_1, M_2, 0, \zeta_1, \zeta_2) | \mu_n \rangle\right]_{m,n}^{N\times N}\right), \end{aligned} \tag{3.22}$$

where

$$\begin{aligned} \widehat{\rho}_k(M_1, M_2, 0, \zeta_1, \zeta_2) &= \widehat{D}_1^{\mathrm{VI}} \widehat{D}_2^{\mathrm{VI}}\Big|_{M=0} \\ = S_{M_1}(\widehat{x}) \frac{1}{2\cosh\frac{\widehat{p}+2\pi\zeta_1}{2}} C_{M_1}(\widehat{x}) C_{M_2}(\widehat{x}+2\pi\zeta_2) &\frac{1}{2\cosh\frac{\widehat{p}}{2}} S_{M_2}(\widehat{x}+2\pi\zeta_2). \end{aligned} \tag{3.23}$$

This is the same as (3.29). For this expression, we can relate the density matrix to the quantum curve [167, 283]. The important relations are

$$
\begin{aligned}
C_L^{-1}(\widehat{x})\, e^{\pm\frac{1}{2}\widehat{p}} S_L^{-1}(\widehat{x}) &= e^{\mp\frac{1}{2}i\pi L} e^{\pm\frac{1}{2}\widehat{p}} \frac{\Phi_b\left(\frac{\widehat{x}}{2\pi b} - \frac{iL}{2b} \pm \frac{i}{2}b\right)}{\Phi_b\left(\frac{\widehat{x}}{2\pi b} + \frac{iL}{2b} \pm \frac{i}{2}b\right)} \frac{\Phi_b\left(\frac{\widehat{x}}{2\pi b} + \frac{iL}{2b} - \frac{i}{2}b\right)}{\Phi_b\left(\frac{\widehat{x}}{2\pi b} - \frac{iL}{2b} + \frac{i}{2}b\right)} e^{-\frac{1}{2}\widehat{x}} \\
&= e^{\pm\frac{1}{2}\widehat{p}} \left(e^{\pm\frac{1}{2}i\pi L} e^{\frac{1}{2}\widehat{x}} + e^{\mp\frac{1}{2}i\pi L} e^{-\frac{1}{2}\widehat{x}} \right), \\
S_L^{-1}(\widehat{x})\, e^{\pm\frac{1}{2}\widehat{p}} C_L^{-1}(\widehat{x}) &= e^{\pm\frac{1}{2}i\pi L} e^{-\frac{1}{2}\widehat{x}} \frac{\Phi_b\left(\frac{\widehat{x}}{2\pi b} + \frac{iL}{2b} - \frac{i}{2}b\right)}{\Phi_b\left(\frac{\widehat{x}}{2\pi b} - \frac{iL}{2b} + \frac{i}{2}b\right)} \frac{\Phi_b\left(\frac{\widehat{x}}{2\pi b} - \frac{iL}{2b} \mp \frac{i}{2}b\right)}{\Phi_b\left(\frac{\widehat{x}}{2\pi b} + \frac{iL}{2b} \mp \frac{i}{2}b\right)} e^{\pm\frac{1}{2}\widehat{p}} \\
&= \left(e^{\mp\frac{1}{2}i\pi L} e^{\frac{1}{2}\widehat{x}} + e^{\pm\frac{1}{2}i\pi L} e^{-\frac{1}{2}\widehat{x}} \right) e^{\pm\frac{1}{2}\widehat{p}},
\end{aligned}
\tag{3.24}
$$

where we used the Baker–Campbell–Hausdorff formula $e^{\alpha\widehat{x}} e^{\beta\widehat{p}} = e^{2\pi i\alpha\beta k} e^{\beta\widehat{p}} e^{\alpha\widehat{x}}$ and (.84). By using these relations, we obtain

$$
\begin{aligned}
& S_L(\widehat{x}) \frac{1}{2\cosh\frac{\widehat{p}}{2}} C_L(\widehat{x}) \\
&= \left[e^{\frac{1}{2}\widehat{p}} \left(e^{\frac{1}{2}i\pi L} e^{\frac{1}{2}\widehat{x}} + e^{-\frac{1}{2}i\pi L} e^{-\frac{1}{2}\widehat{x}} \right) + e^{-\frac{1}{2}\widehat{p}} \left(e^{-\frac{1}{2}i\pi L} e^{\frac{1}{2}\widehat{x}} + e^{\frac{1}{2}i\pi L} e^{-\frac{1}{2}\widehat{x}} \right) \right]^{-1}, \\
& C_L(\widehat{x}) \frac{1}{2\cosh\frac{\widehat{p}}{2}} S_L(\widehat{x}) \\
&= \left[\left(e^{-\frac{1}{2}i\pi L} e^{\frac{1}{2}\widehat{x}} + e^{\frac{1}{2}i\pi L} e^{-\frac{1}{2}\widehat{x}} \right) e^{\frac{1}{2}\widehat{p}} + \left(e^{\frac{1}{2}i\pi L} e^{\frac{1}{2}\widehat{x}} + e^{-\frac{1}{2}i\pi L} e^{-\frac{1}{2}\widehat{x}} \right) e^{-\frac{1}{2}\widehat{p}} \right]^{-1}.
\end{aligned}
\tag{3.25}
$$

The inverse of the density matrix is the product of the above two quantum curves. Therefore, we finally obtain

$$
\begin{aligned}
& \widehat{\rho_k}^{-1}(M_1, M_2, 0, \zeta_1, \zeta_2) \\
&= \left[\left(e^{-\frac{1}{2}i\pi L} e^{\frac{1}{2}\widehat{x}} + e^{\frac{1}{2}i\pi L} e^{-\frac{1}{2}\widehat{x}} \right) e^{\frac{1}{2}\widehat{p}} + \left(e^{\frac{1}{2}i\pi L} e^{\frac{1}{2}\widehat{x}} + e^{-\frac{1}{2}i\pi L} e^{-\frac{1}{2}\widehat{x}} \right) e^{-\frac{1}{2}\widehat{p}} \right] \\
&\quad \times \left[e^{\frac{1}{2}\widehat{p}} \left(e^{\frac{1}{2}i\pi L} e^{\frac{1}{2}\widehat{x}} + e^{-\frac{1}{2}i\pi L} e^{-\frac{1}{2}\widehat{x}} \right) + e^{-\frac{1}{2}\widehat{p}} \left(e^{-\frac{1}{2}i\pi L} e^{\frac{1}{2}\widehat{x}} + e^{\frac{1}{2}i\pi L} e^{-\frac{1}{2}\widehat{x}} \right) \right] \\
&= e^{\frac{\pi i(M_1-M_2)}{2} + \pi(\zeta_1+\zeta_2)} e^{\widehat{x}+\widehat{p}} + [e^{\frac{\pi i(-M_1-M_2)}{2} + \pi(\zeta_1+\zeta_2) + \pi ik} + e^{\frac{\pi i(M_1+M_2)}{2} + \pi(\zeta_1-\zeta_2) - \pi ik}] e^{\widehat{p}} \\
&\quad + e^{\frac{\pi i(-M_1+M_2)}{2} + \pi(\zeta_1-\zeta_2)} e^{-\widehat{x}+\widehat{p}} \\
&\quad + [e^{\frac{\pi i(-M_1-M_2)}{2} + \pi(-\zeta_1+\zeta_2)} + e^{\frac{\pi i(M_1+M_2)}{2} + \pi(\zeta_1+\zeta_2)}] e^{\widehat{x}} \\
&\quad + e^{\frac{\pi i(-M_1+M_2)}{2} + \pi(-\zeta_1-\zeta_2)} + e^{\frac{\pi i(-M_1+M_2)}{2} + \pi(\zeta_1+\zeta_2)} + e^{\frac{\pi i(M_1-M_2)}{2} + \pi(-\zeta_1+\zeta_2)} + e^{\frac{\pi i(M_1-M_2)}{2} + \pi(\zeta_1-\zeta_2)} \\
&\quad + [e^{\frac{\pi i(-M_1-M_2)}{2} + \pi(\zeta_1-\zeta_2)} + e^{\frac{\pi i(M_1+M_2)}{2} + \pi(-\zeta_1-\zeta_2)}] e^{-\widehat{x}} \\
&\quad + e^{\frac{\pi i(-M_1+M_2)}{2} + \pi(-\zeta_1+\zeta_2)} e^{\widehat{x}-\widehat{p}} + [e^{\frac{\pi i(-M_1-M_2)}{2} + \pi(-\zeta_1-\zeta_2) + \pi ik} + e^{\frac{\pi i(M_1+M_2)}{2} + \pi(-\zeta_1+\zeta_2) - \pi ik}] e^{-\widehat{p}} \\
&\quad + e^{\frac{\pi i(M_1-M_2)}{2} + \pi(-\zeta_1-\zeta_2)} e^{-\widehat{x}-\widehat{p}}.
\end{aligned}
\tag{3.26}
$$

This is the quantum curve associated to the (2,2) model for $M = 0$.

3.2.3.2 The quantum curve

The matrix model (3.11) can be written in the following form (see (3.21))

$$
\begin{aligned}
&Z_k\left(N;M_1,M_2,M,\zeta_1,\zeta_2\right)\\
&=\frac{e^{i\Theta_k(M_1,M_2,M,\zeta_1,\zeta_2)}Z_k^{(\mathrm{CS})}\left(M_1\right)Z_k^{(\mathrm{CS})}\left(M_2\right)}{N!}\int\prod_{n=1}^{N}\frac{d\mu_n}{2\pi}\\
&\times\det\begin{pmatrix}\left[\left\langle\mu_m\middle|\widehat{D}_1^{\mathrm{VI}}\widehat{D}_2^{\mathrm{VI}}\middle|\mu_n\right\rangle\right]_{m,n}^{N\times N} & \left[\left\langle\mu_m\middle|\widehat{D}_1^{\mathrm{VI}}\widehat{d}_2^{\mathrm{VI}}\middle|-t_{0,M,s}\right\rangle\!\right\rangle\right]_{m,s}^{N\times M}\\ \left[\left\langle\!\left\langle t_{0,M,r}\middle|\widehat{d}_1^{\mathrm{VI}}\widehat{D}_2^{\mathrm{VI}}\middle|\mu_n\right\rangle\right]_{r,n}^{M\times N} & \left[\left\langle\!\left\langle t_{0,M,r}\middle|\widehat{d}_1^{\mathrm{VI}}\widehat{d}_2^{\mathrm{VI}}\middle|-t_{0,M,s}\right\rangle\!\right\rangle\right]_{r,s}^{M\times M}\end{pmatrix}.
\end{aligned}
\tag{3.27}
$$

This is manifestly an ideal Fermi gas partition function only for $M=0$. Indeed, in this case (3.27) reduces to

$$
\begin{aligned}
&Z_k\left(N;M_1,M_2,0,\zeta_1,\zeta_2\right)\\
&=Z_k\left(0;M_1,M_2,0,\zeta_1,\zeta_2\right)\frac{1}{N!}\int\prod_{n=1}^{N}d\mu_n\det\left(\left[\left\langle\mu_m\left|\widehat{\rho}_k\left(M_1,M_2,0,\zeta_1,\zeta_2\right)\right|\mu_n\right\rangle\right]_{m,n}^{N\times N}\right),
\end{aligned}
\tag{3.28}
$$

with the density matrix $\widehat{\rho}_k(M_1,M_2,0,\zeta_1,\zeta_2)$ given as (3.23)

$$
\begin{aligned}
&\widehat{\rho}_k\left(M_1,M_2,0,\zeta_1,\zeta_2\right)\\
&=e^{\pi\zeta_2}e^{\left(-\frac{i\zeta_1}{k}+\frac{1}{2}-\frac{M_1}{2k}\right)\widehat{x}}\frac{\Phi_b\left(\frac{\widehat{x}}{2\pi b}-\frac{iM_1}{2b}+\frac{ib}{2}\right)}{\Phi_b\left(\frac{\widehat{x}}{2\pi b}+\frac{iM_1}{2b}-\frac{ib}{2}\right)}\frac{1}{2\cosh\frac{\widehat{p}}{2}}e^{\left(\frac{i\zeta_1}{k}+\frac{M_1+M_2}{2k}\right)\widehat{x}}\frac{\Phi_b\left(\frac{\widehat{x}}{2\pi b}+\frac{iM_1}{2b}\right)}{\Phi_b\left(\frac{\widehat{x}}{2\pi b}-\frac{iM_1}{2b}\right)}\\
&\times\frac{\Phi_b\left(\frac{\widehat{x}}{2\pi b}+\frac{iM_2}{2b}+\frac{\zeta_2}{b}\right)}{\Phi_b\left(\frac{\widehat{x}}{2\pi b}-\frac{iM_2}{2b}+\frac{\zeta_2}{b}\right)}\frac{1}{2\cosh\frac{\widehat{p}}{2}}e^{\left(\frac{1}{2}-\frac{M_2}{2k}\right)\widehat{x}}\frac{\Phi_b\left(\frac{\widehat{x}}{2\pi b}-\frac{iM_2}{2b}+\frac{ib}{2}+\frac{\zeta_2}{b}\right)}{\Phi_b\left(\frac{\widehat{x}}{2\pi b}+\frac{iM_2}{2b}-\frac{ib}{2}+\frac{\zeta_2}{b}\right)},
\end{aligned}
\tag{3.29}
$$

where $b=\sqrt{k}$.

Although an explicit Fermi gas formalism is presently not available in the case $M>0$, we conjecture a formula for the corresponding quantum curve. In [183] the following formula for the quantum curve at $M>0$ and vanishing FI parameters $\zeta_1=0$ and $\zeta_2=0$ was proposed[‡]

$$
\begin{aligned}
&\widehat{\rho}_k^{-1}\left(M_1,M_2,M,0,0\right)\\
&=e^{\frac{\pi i(-M_1+M_2)}{2}}e^{-\widehat{x}+\widehat{p}}+[e^{\frac{\pi i(-M_1-M_2)}{2}+\pi ik}+e^{\frac{\pi i(M_1+M_2)}{2}-\pi ik}]e^{\widehat{p}}+e^{\frac{\pi i(M_1-M_2)}{2}}e^{\widehat{x}+\widehat{p}}\\
&+[e^{\frac{\pi i(-M_1-M_2+2M)}{2}}+e^{\frac{\pi i(M_1+M_2-2M)}{2}}]e^{-\widehat{x}}+E+[e^{\frac{\pi i(-M_1-M_2+2M)}{2}}+e^{\frac{\pi i(M_1+M_2-2M)}{2}}]e^{\widehat{x}}\\
&+e^{\frac{\pi i(M_1-M_2)}{2}}e^{-\widehat{x}-\widehat{p}}+[e^{\frac{\pi i(-M_1-M_2)}{2}+\pi ik}+e^{\frac{\pi i(M_1+M_2)}{2}-\pi ik}]e^{-\widehat{p}}+e^{\frac{\pi i(-M_1+M_2)}{2}}e^{\widehat{x}-\widehat{p}}
\end{aligned}
\tag{3.30}
$$

Here E is a constant depending on k, M_1, M_2 and M.[§] Notice that the parameter dependence of the coefficients is multiplicative. In other words, the functions in

[‡]There are a couple of notational differences. First, the sign of the Chern-Simons level between an NS5-brane and a $(1,k)$5-brane in this work is opposite to that in [183]. Second, according to [183], we should care of the position of the node whose rank is the lowest. In this chapter, N_4 is always the smallest.

[§]The procedure in [183] cannot decide the overall phase, the constant term E and the constant shift of $\widehat{x}$ and $\widehat{p}$. We fixed the overall phase and the shift by comparing to the exact result (3.26). We will also fix the form of E in (3.32).

the exponents are linear combinations of the parameters. This is also the case for the exact result (3.26). Therefore, it is natural to assume that the full parameter dependence is also multiplicative. With this assumption, we can uniquely combine (3.30) and (3.26) into[¶]

$$\begin{aligned}
&\widehat{\rho}_k^{-1}\left(M_1, M_2, M, \zeta_1, \zeta_2\right)\\
&= e^{\frac{\pi i(-M_1+M_2)}{2}+\pi(\zeta_1-\zeta_2)}e^{-\widehat{x}+\widehat{p}} + [e^{\frac{\pi i(-M_1-M_2)}{2}+\pi(\zeta_1+\zeta_2)+\pi ik} + e^{\frac{\pi i(M_1+M_2)}{2}+\pi(\zeta_1-\zeta_2)-\pi ik}]e^{\widehat{p}}\\
&\quad + e^{\frac{\pi i(M_1-M_2)}{2}+\pi(\zeta_1+\zeta_2)}e^{\widehat{x}+\widehat{p}}\\
&\quad + [e^{\frac{\pi i(-M_1-M_2+2M)}{2}+\pi(\zeta_1-\zeta_2)} + e^{\frac{\pi i(M_1+M_2-2M)}{2}+\pi(-\zeta_1-\zeta_2)}]e^{-\widehat{x}} + E\\
&\quad + [e^{\frac{\pi i(-M_1-M_2+2M)}{2}+\pi(-\zeta_1+\zeta_2)} + e^{\frac{\pi i(M_1+M_2-2M)}{2}+\pi(\zeta_1+\zeta_2)}]e^{\widehat{x}}\\
&\quad + e^{\frac{\pi i(M_1-M_2)}{2}+\pi(-\zeta_1-\zeta_2)}e^{-\widehat{x}-\widehat{p}} + [e^{\frac{\pi i(-M_1-M_2)}{2}+\pi(-\zeta_1-\zeta_2)+\pi ik} + e^{\frac{\pi i(M_1+M_2)}{2}+\pi(-\zeta_1+\zeta_2)-\pi ik}]e^{-\widehat{p}}\\
&\quad + e^{\frac{\pi i(-M_1+M_2)}{2}+\pi(-\zeta_1+\zeta_2)}e^{\widehat{x}-\widehat{p}},
\end{aligned} \tag{3.31}$$

where

$$\begin{aligned}
E &= e^{\frac{\pi i(-M_1+M_2)}{2}+\pi(-\zeta_1-\zeta_2)+\pi ia_1M} + e^{\frac{\pi i(-M_1+M_2)}{2}+\pi(\zeta_1+\zeta_2)+\pi ia_2M}\\
&\quad + e^{\frac{\pi i(M_1-M_2)}{2}+\pi(-\zeta_1+\zeta_2)+\pi ia_3M} + e^{\frac{\pi i(M_1-M_2)}{2}+\pi(\zeta_1-\zeta_2)+\pi ia_4M},
\end{aligned} \tag{3.32}$$

with unknown constant parameters a_i. Notice that, when $M=0$, the quantum curve (3.31) – or equivalently, (3.26) – is factorized into the product of the two quantum curves appearing in (3.25). These quantum curves are the ones associated to the ABJM theory. Therefore for $M=0$ the quantum curve of the (2,2) model is factorized into a product of two ABJM quantum curves. On the other hand, when $M\neq 0$, this factorization does not occur, and this makes the inversion of (3.31) more difficult.

By comparing the coefficients of (3.31) with the coefficients of the quantum Seiberg-Witten curve (3.7) we can read off the parameters of the five dimensional gauge theory in the following way. First we compare the equation $\widehat{\rho}^{-1}|_{\widehat{x},\widehat{p}\to x,p}=0$ with the Seiberg-Witten curve at the four asymptotic regions $x=\pm\infty, p=\pm\infty$, see Fig.3.2:

$$\begin{aligned}
x\to\infty:&\quad e^p=\widetilde{m}_1,\widetilde{m}_2=-e^{\pi i(M_2-M)},\ -e^{\pi i(-M_1+M)-2\pi\zeta_1},\\
x\to-\infty:&\quad e^p=\widetilde{m}_3,\widetilde{m}_4=-e^{\pi i(M_1-M)-2\pi\zeta_1},\quad -e^{\pi i(-M_2+M)},\\
p\to\infty:&\quad e^x=\widetilde{t}_1,\widetilde{t}_3=-e^{\pi iM_2-2\pi\zeta_2-\pi ik},\quad -e^{-\pi iM_1+\pi ik},\\
p\to-\infty:&\quad e^x=\widetilde{t}_2,\widetilde{t}_4=-e^{\pi iM_1-\pi ik},\quad -e^{-\pi iM_2-2\pi\zeta_2+\pi ik}.
\end{aligned} \tag{3.33}$$

[¶]While this draft was in preparation, the quantum mirror curve for general values of $(M_1,M_2,M,\zeta_1,\zeta_2)$ also appeared in [94]. Our formula for the quantum mirror curve is their eq. (A.1) combined with eq. (3.5), where $m_i^{[\mathrm{FMMN}]}$ and $z_i^{[\mathrm{FMMN}]}$ in [94] being related to the rank differences M_1,M_2,M and the FI parameters ζ_1,ζ_2 as

$$\begin{aligned}
&m_1^{[\mathrm{FMMN}]}=e^{\pi i(-M_1+M_2)},\quad m_2^{[\mathrm{FMMN}]}=e^{\pi i(M-k)},\quad m_3^{[\mathrm{FMMN}]}=e^{\pi i(M_1+M_2-M-k)},\\
&z_1^{[\mathrm{FMMN}]}=e^{-2\pi\zeta_1},\quad z_3^{[\mathrm{FMMN}]}=e^{-2\pi\zeta_2}.
\end{aligned}$$

We thank Prof. Moriyama for pointing it out.

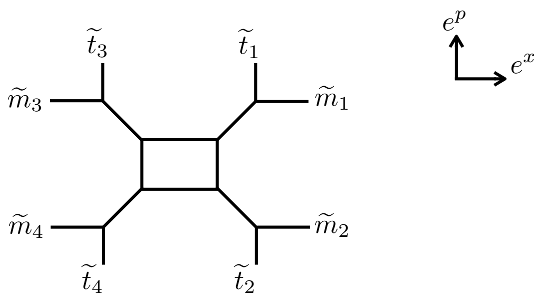


Figure 3.2: Five-brane web diagram corresponding to the classical limit of the quantum spectral curve (3.31). $\tilde{m}_i$ and $\tilde{t}_i$ are the asymptotic positions of the 5-branes in the limiting classical curve.

The quantities with tilde can be rescaled as

$$\left(\widetilde{m}_1, \widetilde{m}_2, \widetilde{m}_3, \widetilde{m}_4, \widetilde{t}_1, \widetilde{t}_2, \widetilde{t}_3, \widetilde{t}_4\right) \to \left(\alpha\widetilde{m}_1, \alpha\widetilde{m}_2, \alpha\widetilde{m}_3, \alpha\widetilde{m}_4, \beta\widetilde{t}_1, \beta\widetilde{t}_2, \beta\widetilde{t}_3, \beta\widetilde{t}_4\right),$$

with arbitrary non zero complex numbers α, β associated to the coordinates translation $(p, x) \to (p - \log\alpha, x - \log\beta)$. In (3.7) these ambiguities get removed by fixing the product $a_1 a_2$. The gauge coupling q is given by

$$q = \left(\frac{\widetilde{t}_3\widetilde{t}_4}{\widetilde{t}_1\widetilde{t}_2}\right)^{\frac{1}{2}},$$

which is by itself rescaling invariant. On the other hand the physical mass parameters are identified once $a_1 a_2 = 1$ is fixed. This amounts to set $\alpha = \left(\frac{\widetilde{t}_1}{\widetilde{m}_1\widetilde{m}_2\widetilde{t}_2}\right)^{\frac{1}{2}}$ so that

$$m_1 = \alpha\widetilde{m}_1 = \left(\frac{\widetilde{m}_1\widetilde{t}_1}{\widetilde{m}_2\widetilde{t}_2}\right)^{\frac{1}{2}}, \quad m_2 = \alpha\widetilde{m}_2 = \left(\frac{\widetilde{m}_2\widetilde{t}_1}{\widetilde{m}_1\widetilde{t}_2}\right)^{\frac{1}{2}},$$
$$m_3 = \alpha\widetilde{m}_3 = \left(\frac{\widetilde{m}_3\widetilde{t}_4}{\widetilde{m}_4\widetilde{t}_3}\right)^{\frac{1}{2}}, \quad m_4 = \alpha\widetilde{m}_4 = \left(\frac{\widetilde{m}_4\widetilde{t}_4}{\widetilde{m}_3\widetilde{t}_3}\right)^{\frac{1}{2}}.$$

By substituting the explicit expressions of $\widetilde{m}_i, \widetilde{t}_i$ we obtain

$$\begin{aligned} &m_1 = e^{\pi i(M_2 - M) + \pi(\zeta_1 - \zeta_2)}, \quad m_2 = e^{\pi i(-M_1 + M) + \pi(-\zeta_1 - \zeta_2)}, \quad m_3 = e^{\pi i(M_1 - M) + \pi(-\zeta_1 - \zeta_2)}, \\ &m_4 = e^{\pi i(-M_2 + M) + \pi(\zeta_1 - \zeta_2)}, \quad q = e^{\pi i(-M_1 - M_2) + 2\pi i k}. \end{aligned} \tag{3.34}$$

3.2.3.3 Parameter identification with q-PVI system

According to the discussion at the end of subsection 3.2.1.2, the relation between the q-PVI tau function and the grand partition function of the superconformal Chern-Simons quiver theory (3.10) follows by combining the above results (3.3), (3.6), (3.8), (3.34):

$$\begin{aligned} &\tau\left(\theta_0, \theta_1, \theta_t, \theta_\infty; 1, \kappa, t\right) \\ &= \frac{F\left(M_1, M_2, M, \zeta_1, \zeta_2\right)}{Z_k\left(0; M_1, M_2, M, \zeta_1, \zeta_2\right)} \sum_{N=0}^{\infty} \left(\Omega\left(M_1, M_2, M, \zeta_1, \zeta_2\right)\kappa\right)^N Z_k\left(N; M_1, M_2, M, \zeta_1, \zeta_2\right) \\ &= F\left(M_1, M_2, M, \zeta_1, \zeta_2\right) \det\left[1 + \Omega\left(M_1, M_2, M, \zeta_1, \zeta_2\right)\kappa\widehat{\rho}_k\left(M_1, M_2, M, \zeta_1, \zeta_2\right)\right], \end{aligned} \tag{3.35}$$

where $F(M_1, M_2, M, \zeta_1, \zeta_2)$ and $\Omega(M_1, M_2, M, \zeta_1, \zeta_2)$ are some functions independent of κ, with the following parameter identification:

$$\begin{pmatrix} \theta_0 \\ \theta_1 \\ \theta_t \\ \theta_\infty \\ \frac{\log t}{\log \mathfrak{q}} \end{pmatrix} = \frac{1}{4} \begin{pmatrix} -1 & 1 & 0 & 2 & 0 \\ -1 & -1 & 2 & 0 & -2 \\ -1 & -1 & 2 & 0 & 2 \\ 1 & -1 & 0 & 2 & 0 \\ -4 & -4 & 4 & 0 & 0 \end{pmatrix} \begin{pmatrix} M_1 - k \\ M_2 - k \\ M - k \\ -i\zeta_1 \\ -i\zeta_2 \end{pmatrix}. \tag{3.36}$$

As already discussed in the previous section, an explicit formula for the spectral density matrix $\widehat{\rho}_k$ is known to us only for $M = 0$. We therefore can calculate only the τ-functions generated by the action of the affine Weyl group transformations which fix the $M = 0$ condition. It is therefore useful to rewrite (3.36) after a suitable change of basis obtained by a Weyl group transformation $W(D_5)$ so that we can realize as many of the shifts in (3.4) as possible without varying M. Let us compute the relevant change of basis.

The full Weyl group is generated by the fundamental elements s_a, $a = 1, \ldots, 5$ associated to each node of D_5 Dynkin diagram which linearly realise the group action on the parameters $(M_1 - k, M_2 - k, M - k, -i\zeta_1, -i\zeta_2)$ as follows[‖]

$$s_1 = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & 0 & -1 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 & -1 \\ -\frac{1}{2} & -\frac{1}{2} & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \end{pmatrix}, \quad s_2 = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 1 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 1 \\ -\frac{1}{2} & -\frac{1}{2} & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \end{pmatrix}, \quad s_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$
$$s_4 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad s_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 1 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since each of these fundamental Weyl group transformation induces a similarity transformation $s : \widehat{\rho}^{-1} \to \widehat{U}\widehat{\rho}^{-1}\widehat{U}^{-1}$ on the quantum spectral curve, the spectral determinant (3.6) is invariant. We may therefore choose an arbitrary element $w \in W(D_5)$ to identify the $\mathfrak{q}$-PVI parameters as

$$\begin{pmatrix} \theta_0 \\ \theta_1 \\ \theta_t \\ \theta_\infty \\ \frac{\log t}{\log \mathfrak{q}} \end{pmatrix} = \frac{1}{4} \begin{pmatrix} -1 & 1 & 0 & 2 & 0 \\ -1 & -1 & 2 & 0 & -2 \\ -1 & -1 & 2 & 0 & 2 \\ 1 & -1 & 0 & 2 & 0 \\ -4 & -4 & 4 & 0 & 0 \end{pmatrix} w \begin{pmatrix} M_1 - k \\ M_2 - k \\ M - k \\ -i\zeta_1 \\ -i\zeta_2 \end{pmatrix}, \tag{3.37}$$

[‖]The explicit expressions for s_1, s_2, s_3, s_4, s_5 are written in [183] but in a different basis $(\log \bar{h}_1, \log \bar{h}_2, \log e_1, \log e_3, \log e_5)$, which is related to the current basis $(M_1 - k, M_2 - k, M - k, -i\zeta_1, -i\zeta_2)$ as

$$\begin{pmatrix} \log \bar{h}_1 \\ \log \bar{h}_2 \\ \log e_1 \\ \log e_3 \\ \log e_5 \end{pmatrix} = \pi i \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & -2 & 0 & 0 \\ 1 & 1 & -2 & 2 & 0 \\ 1 & 1 & 0 & 0 & -2 \\ 1 & 1 & -2 & -2 & 0 \end{pmatrix} \begin{pmatrix} M_1 - k \\ M_2 - k \\ M - k \\ -i\zeta_1 \\ -i\zeta_2 \end{pmatrix}.$$

instead of (3.36). In particular, if we choose

$$w = s_3 s_2 s_1 s_3 s_4 s_5 s_1 s_3 s_4 s_2 s_3 s_2 = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 0 & -1 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -\frac{1}{2} & -\frac{1}{2} & 1 & 0 & 0 \end{pmatrix},$$

we obtain

$$\begin{gathered} \theta_0 = \frac{-i\zeta_1 - i\zeta_2}{2}, \quad \theta_1 = \frac{M_1 + M_2 - M - k}{2}, \quad \theta_t = \frac{M - k}{2}, \\ \theta_\infty = \frac{i\zeta_1 - i\zeta_2}{2}, \quad t = \mathfrak{q}^{M_1 - k} = e^{\frac{2\pi i M_1}{k}}, \quad \mathfrak{q} = e^{\frac{2\pi i}{k}}. \end{gathered} \tag{3.38}$$

With this parameter identification, the allowed τ-functions are manifestly the ones not induced by a shift in θ_t, namely all of them but $\tau_{7,8}, \underline{\tau}_{7,8}, \bar{\tau}_{7,8}$ according to (3.4).

By using (3.38) we find that $\tau_i, \underline{\tau}_i, \bar{\tau}_i$ for $M = 0$ are given by

$$\begin{aligned} \tau_{1,2} &= \tau\left(M_1, M_2, 0, \zeta_1 \mp \frac{i}{2}, \zeta_2 \pm \frac{i}{2}\right), \\ \tau_{3,4} &= \tau\left(M_1, M_2, 0, \zeta_1 \pm \frac{i}{2}, \zeta_2 \pm \frac{i}{2}\right), \\ \tau_{5,6} &= \tau\left(M_1, M_2 \mp 1, 0, \zeta_1, \zeta_2\right), \\ \underline{\tau}_{1,2} &= \tau\left(M_1 - 1, M_2 + 1, 0, \zeta_1 \mp \frac{i}{2}, \zeta_2 \pm \frac{i}{2}\right), \\ \underline{\tau}_{3,4} &= \tau\left(M_1 - 1, M_2 + 1, 0, \zeta_1 \pm \frac{i}{2}, \zeta_2 \pm \frac{i}{2}\right), \\ \underline{\tau}_5 &= \tau\left(M_1 - 1, M_2, 0, \zeta_1, \zeta_2\right), \\ \bar{\tau}_{1,2} &= \tau\left(M_1 + 1, M_2 - 1, 0, \zeta_1 \mp \frac{i}{2}, \zeta_2 \pm \frac{i}{2}\right), \\ \bar{\tau}_{3,4} &= \tau\left(M_1 + 1, M_2 - 1, 0, \zeta_1 \pm \frac{i}{2}, \zeta_2 \pm \frac{i}{2}\right), \\ \bar{\tau}_6 &= \tau\left(M_1 + 1, M_2, 0, \zeta_1, \zeta_2\right). \end{aligned} \tag{3.39}$$

Since the variables $\tau_{7,8}, \underline{\tau}_{7,8}, \bar{\tau}_{7,8}$ are obstructed to us, we can check the system (3.5) only after their elimination. This provides a subsystem of six equations in six variables out of (3.5) given by the first two equations, which explicitly do not involve $\tau_{7,8}, \underline{\tau}_{7,8}, \bar{\tau}_{7,8}$ and other four. Actually, by eliminating $\tau_{7,8}, \underline{\tau}_{7,8}, \bar{\tau}_{7,8}$ we can obtain from the remaining six equations in (3.5) three bilinear equations and a quartic equation. All in all, we then get the system

$$\tau_1\tau_2 - \mathfrak{q}^{-2\theta_1} \cdot t\tau_3\tau_4 - \left(1 - \mathfrak{q}^{-2\theta_1} \cdot t\right) \tau_5\tau_6 = 0, \tag{3.40}$$

$$\tau_1\tau_2 - t\tau_3\tau_4 - \left(1 - \mathfrak{q}^{-2\theta_t} \cdot t\right) \underline{\tau}_5\bar{\tau}_6 = 0, \tag{3.41}$$

$$\underline{\tau}_1\tau_2 - \mathfrak{q}^{-2\theta_\infty}\tau_1\underline{\tau}_2 - \left(1 - \mathfrak{q}^{-2\theta_\infty}\right) \underline{\tau}_5\tau_6 = 0, \tag{3.42}$$

$$\underline{\tau}_3\tau_4 - \mathfrak{q}^{2\theta_0}\tau_3\underline{\tau}_4 - \mathfrak{q}^{-\theta_t}\left(1 - \mathfrak{q}^{2\theta_0}\right) \underline{\tau}_5\tau_6 = 0, \tag{3.43}$$

$$\underline{\tau}_1\tau_2 - \mathfrak{q}^{-\theta_0-\theta_1-\theta_\infty-\frac{1}{2}} \cdot t\underline{\tau}_3\tau_4 - \left(1 - \mathfrak{q}^{-\theta_0-\theta_1-\theta_t-\theta_\infty-\frac{1}{2}} \cdot t\right) \underline{\tau}_5\tau_6 = 0, \tag{3.44}$$

$$\begin{aligned} &\left(\tau_1\tau_2 - \tau_3\tau_4\right)\left(\tau_1\tau_2 - \mathfrak{q}^{2\theta_t}\tau_3\tau_4\right) \\ &- \left(1 - \mathfrak{q}^{2\theta_1} \cdot t^{-1}\right)\left(1 - \mathfrak{q}^{2\theta_t} \cdot t^{-1}\right)\mathfrak{q}^{2\theta_\infty}\left(\underline{\tau}_1\tau_2 - \underline{\tau}_5\tau_6\right)\left(\tau_1\bar{\tau}_2 - \tau_5\bar{\tau}_6\right) = 0. \end{aligned} \tag{3.45}$$

We found that these hold provided the factors $F(M_1, M_2, 0, \zeta_1, \zeta_2)$ and $\Omega(M_1, M_2, 0, \zeta_1, \zeta_2)$ satisfy the following relations:

$$\begin{aligned}
&F_1F_2 = F_3F_4 = F_5F_6 = \underline{F}_5\bar{F}_6, \quad \underline{F}_1F_2 = F_1\underline{F}_2 = -\underline{F}_3F_4 = -F_3\underline{F}_4 = \underline{F}_5F_6, \\
&-i\Omega_1 = -i\Omega_2 = i\Omega_3 = i\Omega_4 = \Omega_5 = -\Omega_6 = i\underline{\Omega}_1 = i\underline{\Omega}_2 = -i\underline{\Omega}_3 = -i\underline{\Omega}_4 = -\underline{\Omega}_5 \\
&\quad = i\bar{\Omega}_1 = i\bar{\Omega}_2 = -i\bar{\Omega}_3 = -i\bar{\Omega}_4 = \bar{\Omega}_6.
\end{aligned} \tag{3.46}$$

Needless to say, here we adopt the very same notation as in (3.39) for the tau functions, both for the labels $1, 2, \cdots, 6$, and for the under/over-line as shifts of the parameters $(M_1, M_2, \zeta_1, \zeta_2)$. The following choice is consistent with all of the conditions (3.46):

$$F(M_1, M_2, 0, \zeta_1, \zeta_2) = e^{\frac{\pi(M_2-M_1)(\zeta_1+\zeta_2)}{2}}, \quad \Omega(M_1, M_2, 0, \zeta_1, \zeta_2) = e^{-\frac{\pi i(M_1-M_2)}{2}+\pi(\zeta_1+\zeta_2)},$$

and we adopt it from now on.

In the next section we explain how to check equations (3.40)-(3.45).

3.2.4 Checks of the q-Painlevé equations

In this subsection we provide non-trivial evidence that the τ-functions (3.39) defined as Fredholm determinants satisfy the bilinear and quartic equations (3.40)-(3.45). In order to do this, first we prove some symmetry properties of $\widehat{\rho}_k$ under a given set of linear transformations of the parameters $(M_1, M_2, \zeta_1, \zeta_2)$ which are a subset of the full $W(D_5)$ symmetry acting on the quantum mirror curve (3.31). Next, by using this symmetry property we provide two types of non-trivial checks of the bilinear/quartic equations: check around the symmetric points under transformations and the direct proof of the equations at the sub-leading order in κ with $(M_1, M_2, \zeta_1, \zeta_2)$ kept unfixed. Lastly, by reducing the problem using the above symmetries, we provide non-trivial checks of the equations at higher order in κ around a discrete set of points $(M_1, M_2, \zeta_1, \zeta_2)$.

3.2.4.1 Relation among bilinear equations under Weyl transformations

In this section we consider the discrete symmetry of the spectral density matrix $\widehat{\rho}_k(M_1, M_2, 0, \zeta_1, \zeta_2)$ given in (3.29) under similarity transformations which do not change the spectral determinant (3.6). In particular, under a suitable similarity transformation (cyclic permutation) we can rewrite the equivalent density matrix

$$\begin{aligned}
\widehat{\rho}_k(M_1, M_2, 0, \zeta_1, \zeta_2) = e^{\pi\zeta_2} e^{\left(-\frac{i\zeta_1}{k}+1-\frac{M_1+M_2}{2k}\right)\widehat{x}} &\frac{\Phi_b\left(\frac{\widehat{x}}{2\pi b}-\frac{iM_1}{2b}+\frac{ib}{2}\right)}{\Phi_b\left(\frac{\widehat{x}}{2\pi b}+\frac{iM_1}{2b}-\frac{ib}{2}\right)} \frac{\Phi_b\left(\frac{\widehat{x}}{2\pi b}-\frac{iM_2}{2b}+\frac{ib}{2}+\frac{\zeta_2}{b}\right)}{\Phi_b\left(\frac{\widehat{x}}{2\pi b}+\frac{iM_2}{2b}-\frac{ib}{2}+\frac{\zeta_2}{b}\right)} \frac{1}{2\cosh\frac{\widehat{p}}{2}} \\
\times e^{\left(\frac{i\zeta_1}{k}+\frac{M_1+M_2}{2k}\right)\widehat{x}} &\frac{\Phi_b\left(\frac{\widehat{x}}{2\pi b}+\frac{iM_1}{2b}\right)}{\Phi_b\left(\frac{\widehat{x}}{2\pi b}-\frac{iM_1}{2b}\right)} \frac{\Phi_b\left(\frac{\widehat{x}}{2\pi b}+\frac{iM_2}{2b}+\frac{\zeta_2}{b}\right)}{\Phi_b\left(\frac{\widehat{x}}{2\pi b}-\frac{iM_2}{2b}+\frac{\zeta_2}{b}\right)} \frac{1}{2\cosh\frac{\widehat{p}}{2}},
\end{aligned} \tag{3.47}$$

which for simplicity we still call $\widehat{\rho}_k$ (though they are not precisely equal to each other as operators).[**] We now consider the symmetry properties of the above density matrix among different values $(M_1, M_2, \zeta_1, \zeta_2)$ while $M = 0$ being preserved.

[**]Explicitly, $\widehat{\rho}_k$ in (3.29) and $\widehat{\rho}_k$ in (3.47) are related by the following similarity transformation:

$$\widehat{\rho}_k^{(3.47)} = \widehat{U}\widehat{\rho}_k^{(3.29)}\widehat{U}^{-1},$$

First, by reordering the quantum dilogarithms $\Phi_b\left(\frac{\widehat{x}}{2\pi b}+\cdots\right)$ within each group of four Φ_b separated by $\frac{1}{2\cosh\frac{\widehat{p}}{2}}$ we obtain

$$\widehat{\rho}_k\left(M_1,M_2,0,\zeta_1,\zeta_2\right)\sim\widehat{\rho}_k\left(\frac{M_1+M_2}{2}-i\zeta_2,\frac{M_1+M_2}{2}+i\zeta_2,0,\zeta_1,\frac{i\left(M_1-M_2\right)}{2}\right),$$
$$\widehat{\rho}_k\left(M_1,M_2,0,\zeta_1,\zeta_2\right)\sim\widehat{\rho}_k\left(\frac{M_1+M_2}{2}+i\zeta_2,\frac{M_1+M_2}{2}-i\zeta_2,0,\zeta_1,-\frac{i\left(M_1-M_2\right)}{2}\right). \tag{3.48}$$

Second, by exchanging the first line and the second line in (3.47), which is a cyclic permutation and can be realized by a similarity transformation, we obtain

$$\widehat{\rho}_k\left(M_1,M_2,0,\zeta_1,\zeta_2\right)\sim\widehat{\rho}_k\left(k-M_1,k-M_2,0,-\zeta_1,\zeta_2\right).$$

Lastly, we also find

$$\widehat{\rho}_k\left(M_1,M_2,0,\zeta_1,\zeta_2\right)\sim\widehat{\rho}_k\left(M_1,M_2,0,\zeta_2,\zeta_1\right). \tag{3.49}$$

It turns out that the above four transformations acting on $(M_1,M_2,\zeta_1,\zeta_2)$ are the four generators of the Weyl symmetry $W\left(A_3\times A_1\right)\subset W(D_5)$.

Notice that by using these symmetries one can generate the five bilinear equations (3.40)-(3.44) just from the two equations (3.40), (3.44). Let us denote the above Weyl reflections (3.48)-(3.49) as

$$\begin{aligned}
&r_1:\ (M_1,M_2,\zeta_1,\zeta_2)\to\left(\frac{M_1+M_2}{2}-i\zeta_2,\frac{M_1+M_2}{2}+i\zeta_2,\zeta_1,\frac{i(M_1-M_2)}{2}\right),\\
&r_2:\ (M_1,M_2,\zeta_1,\zeta_2)\to\left(\frac{M_1+M_2}{2}+i\zeta_2,\frac{M_1+M_2}{2}-i\zeta_2,\zeta_1,-\frac{i(M_1-M_2)}{2}\right),\\
&r_3:\ (M_1,M_2,\zeta_1,\zeta_2)\to\left(k-M_1,k-M_2,-\zeta_1,\zeta_2\right),\\
&r_4:\ (M_1,M_2,\zeta_1,\zeta_2)\to\left(M_1,M_2,\zeta_2,\zeta_1\right).
\end{aligned} \tag{3.50}$$

Then we find

$$\begin{aligned}
&(3.41)=-t\cdot[(3.40)]_{(r_2r_1)(M_1,M_2,\zeta_1,\zeta_2)},\qquad(3.42)=(3.44)-\mathfrak{q}^{-2\theta_\infty}\cdot[(3.44)]_{r_4(M_1,M_2,\zeta_1,\zeta_2)},\\
&(3.43)=\mathfrak{q}^{\theta_0+\theta_1-\theta_\infty+\frac{1}{2}}\left(\left[[(3.44)]_{r_3(M_1,M_2,\zeta_1,\zeta_2)}\right]_{t\to\mathfrak{q}^{-1}t}-[(3.44)]_{r_4(M_1,M_2,\zeta_1,\zeta_2)}\right),
\end{aligned} \tag{3.51}$$

where $[(\cdots)]_{r_i(M_1,M_2,\zeta_1,\zeta_2)}$ stands for the left-hand side of each equation with $(M_1,M_2,\zeta_1,\zeta_2)$ substituted by their images under r_i (3.50). We will use (3.51) in section 3.2.4.3 to check the bilinear equations at first order in κ.

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with

$$\widehat{U}=e^{\left(\frac{1}{2}-\frac{M_2}{2k}\right)\widehat{x}}\frac{\Phi_b\left(\frac{\widehat{x}}{2\pi b}+\frac{\zeta_2}{b}-\frac{iM_2}{2b}+\frac{ib}{2}\right)}{\Phi_b\left(\frac{\widehat{x}}{2\pi b}+\frac{\zeta_2}{b}+\frac{iM_2}{2b}-\frac{ib}{2}\right)}.$$

3.2.4.2 Special checks around symmetric points

Taking into account (3.50) and (3.51) we notice that (3.42),(3.43),(3.44) are trivially satisfied for some special choices of parameters $(M_1, M_2, \zeta_1, \zeta_2)$.

For example, by restricting on the subspace $\zeta_1 = \zeta_2$ which implies $\theta_\infty = 0$ by (3.38) and noticing that this is the fixed locus of the reflection r_4 in (3.50), we see that eq.(3.42) is satisfied.
In the same way we can also show the following by using Weyl transformations.

For $(M_1, M_2, \zeta_1, \zeta_2) = (M_1, M_2, \zeta_1, -\zeta_1)$, the coefficient of the third term in (3.43) vanishes, the coefficient of the second term is -1 and also $\tau_3(\kappa/\Omega_3)/F_3 = \tau_4(\kappa/\Omega_4)/F_4, \underline{\tau}_3(\kappa/\underline{\Omega}_3)/\underline{F}_3 = \underline{\tau}_4(\kappa/\underline{\Omega}_4)/\underline{F}_4$ (recall our definition of τ-functions in terms of the spectral determinant (3.35)) follow due to the Weyl symmetry $r_3r_4r_3$

$$r_3r_4r_3 : \,(M_1, M_2, \zeta_1, \zeta_2) \to (M_1, M_2, -\zeta_1, -\zeta_2).$$

Together with $\underline{F}_3F_4 = F_3\underline{F}_4$ and $\Omega_3 = \Omega_4 = -\underline{\Omega}_3 = -\underline{\Omega}_4$ (3.46) these implies $\underline{\tau}_3\tau_4 = \tau_3\underline{\tau}_4$, hence (3.43) is trivially satisfied.

For $(M_1, M_2, \zeta_1, \zeta_2) = \left(M_1, M_2, \zeta_1, \frac{i(M_1-M_2-1)}{2}\right)$, the coefficient of the third term in (3.44) vanishes, the coefficient of the second term is 1 and also $\underline{\tau}_1(\kappa/\underline{\Omega}_1)/\underline{F}_1 = \tau_4(\kappa/\Omega_4)/F_4, \tau_2(\kappa/\Omega_2)/F_2 = \underline{\tau}_3(\kappa/\underline{\Omega}_3)/\underline{F}_3$ follow due to the Weyl symmetry r_1. Hence, together with $\underline{\Omega}_1 = \Omega_4, \underline{\Omega}_3 = \Omega_2$ and $F_1\underline{F}_2 = -\underline{F}_3F_4$ (3.46), we find that (3.44) is trivially satisfied.

Moreover, we notice that (3.41) at $(M_1, M_2, \zeta_1, \zeta_2)$ is equivalent to (3.40) evaluated at a different point obtained by a Weyl transformation $(M_1', M_2', \zeta_1', \zeta_2') = (k - M_2, k - M_1, -\zeta_1, -\zeta_2)$, hence the equation (l.h.s. of (3.40))−(l.h.s. of (3.41))= 0 is trivially satisfied at the fixed points of this Weyl transformation, which are $(M_1, M_2, \zeta_1, \zeta_2) = (M_1, k - M_1, 0, 0)$.

3.2.4.3 Analysis to first order in κ

Due to the values of the overall coefficients of each term, the bilinear equations (3.40)-(3.45) are trivially satisfied at order κ^0, while the quartic equation (3.45) is trivial also at first order in κ. At higher order in κ these equations are non-trivial. For the bilinear equations (3.40)-(3.44), we can analytically check the equations at order κ for an arbitrary choice of the parameters $(M_1, M_2, \zeta_1, \zeta_2)$. First let us consider (3.40). At order κ the bilinear equation reduces to the following linear relation on $\widehat{\rho}_k(M_1, M_2, 0, \zeta_1, \zeta_2)$:

$$\begin{aligned} \mathrm{tr}\Bigg[& i\widehat{\rho}_k\left(M_1, M_2, 0, \zeta_1 - \frac{i}{2}, \zeta_2 + \frac{i}{2}\right) + i\widehat{\rho}_k\left(M_1, M_2, 0, \zeta_1 + \frac{i}{2}, \zeta_2 - \frac{i}{2}\right) \\ & + \mathfrak{q}^{-2\theta_1}\left(i\widehat{\rho}_k\left(M_1, M_2, 0, \zeta_1 + \frac{i}{2}, \zeta_2 + \frac{i}{2}\right) + i\widehat{\rho}_k\left(M_1, M_2, 0, \zeta_1 - \frac{i}{2}, \zeta_2 - \frac{i}{2}\right)\right) \\ & - \left(1 - \mathfrak{q}^{-2\theta_1}t\right)\left(\widehat{\rho}_k\left(M_1, M_2 - 1, 0, \zeta_1, \zeta_2\right) - \widehat{\rho}_k\left(M_1, M_2 + 1, 0, \zeta_1, \zeta_2\right)\right)\Bigg] = 0. \end{aligned} \tag{3.52}$$

Notice that for all the shifts of the parameters in (3.52), the arguments of the quantum dilogarithm in $\widehat{\rho}_k$ are different only by units of $\frac{i}{b}$. Hence by using the recursive relation (.84) we can express all the $\widehat{\rho}_k$s entering (3.52) by using only one of them up to rational factors:

$$\begin{aligned}
\widehat{\rho}_k\left(M_1, M_2, 0, \zeta_1-\frac{i}{2}, \zeta_2+\frac{i}{2}\right) &= ie^{-\frac{1}{2k}\widehat{x}}\left(1-e^{\frac{2\pi\zeta_2}{k}+\frac{\pi i M_2}{k}}e^{\frac{\widehat{x}}{k}}\right)\widehat{I}_1 e^{\frac{1}{2k}\widehat{x}}\left(1+e^{\frac{2\pi\zeta_2}{k}-\frac{\pi i M_2}{k}}e^{\frac{\widehat{x}}{k}}\right)\widehat{I}_2,\\
\widehat{\rho}_k\left(M_1, M_2, 0, \zeta_1+\frac{i}{2}, \zeta_2-\frac{i}{2}\right) &= -ie^{\frac{1}{2k}\widehat{x}}\left(1-e^{\frac{2\pi\zeta_2}{k}-\frac{\pi i M_2}{k}}e^{\frac{\widehat{x}}{k}}\right)\widehat{I}_1 e^{-\frac{1}{2k}\widehat{x}}\left(1+e^{\frac{2\pi\zeta_2}{k}+\frac{\pi i M_2}{k}}e^{\frac{\widehat{x}}{k}}\right)\widehat{I}_2,\\
\widehat{\rho}_k\left(M_1, M_2, 0, \zeta_1+\frac{i}{2}, \zeta_2+\frac{i}{2}\right) &= ie^{\frac{1}{2k}\widehat{x}}\left(1-e^{\frac{2\pi\zeta_2}{k}+\frac{\pi i M_2}{k}}e^{\frac{\widehat{x}}{k}}\right)\widehat{I}_1 e^{-\frac{1}{2k}\widehat{x}}\left(1+e^{\frac{2\pi\zeta_2}{k}-\frac{\pi i M_2}{k}}e^{\frac{\widehat{x}}{k}}\right)\widehat{I}_2,\\
\widehat{\rho}_k\left(M_1, M_2, 0, \zeta_1-\frac{i}{2}, \zeta_2-\frac{i}{2}\right) &= -ie^{-\frac{1}{2k}\widehat{x}}\left(1-e^{\frac{2\pi\zeta_2}{k}-\frac{\pi i M_2}{k}}e^{\frac{\widehat{x}}{k}}\right)\widehat{I}_1 e^{\frac{1}{2k}\widehat{x}}\left(1+e^{\frac{2\pi\zeta_2}{k}+\frac{\pi i M_2}{k}}e^{\frac{\widehat{x}}{k}}\right)\widehat{I}_2,\\
\widehat{\rho}_k\left(M_1, M_2-1, 0, \zeta_1, \zeta_2\right) &= e^{\frac{1}{2k}\widehat{x}}\widehat{I}_1 e^{-\frac{1}{2k}\widehat{x}}\left(1+e^{\frac{2\pi\zeta_2}{k}-\frac{\pi i M_2}{k}}e^{\frac{\widehat{x}}{k}}\right)\left(1+e^{\frac{2\pi\zeta_2}{k}+\frac{\pi i M_2}{k}}e^{\frac{\widehat{x}}{k}}\right)\widehat{I}_2,\\
\widehat{\rho}_k\left(M_1, M_2+1, 0, \zeta_1, \zeta_2\right) &= e^{-\frac{1}{2k}\widehat{x}}\left(1-e^{\frac{2\pi\zeta_2}{k}+\frac{\pi i M_2}{k}}e^{\frac{\widehat{x}}{k}}\right)\left(1-e^{\frac{2\pi\zeta_2}{k}-\frac{\pi i M_2}{k}}e^{\frac{\widehat{x}}{k}}\right)\widehat{I}_1 e^{-\frac{1}{2k}\widehat{x}}\widehat{I}_2,
\end{aligned}$$

where $\widehat{I}_1, \widehat{I}_2$ are

$$\begin{aligned}
\widehat{I}_1 &= e^{\pi\zeta_2}e^{\left(-\frac{i\zeta_1}{k}+1-\frac{M_1+M_2}{2k}\right)\widehat{x}}\frac{\Phi_b\left(\frac{\widehat{x}}{2\pi b}-\frac{iM_1}{2b}+\frac{ib}{2}\right)\Phi_b\left(\frac{\widehat{x}}{2\pi b}-\frac{iM_2}{2b}+\frac{ib}{2}+\frac{\zeta_2}{b}+\frac{i}{2b}\right)}{\Phi_b\left(\frac{\widehat{x}}{2\pi b}+\frac{iM_1}{2b}-\frac{ib}{2}\right)\Phi_b\left(\frac{\widehat{x}}{2\pi b}+\frac{iM_2}{2b}-\frac{ib}{2}+\frac{\zeta_2}{b}-\frac{i}{2b}\right)}\frac{1}{2\cosh\frac{\widehat{p}}{2}},\\
\widehat{I}_2 &= e^{\left(\frac{i\zeta_1}{k}+\frac{M_1+M_2}{2k}\right)\widehat{x}}\frac{\Phi_b\left(\frac{\widehat{x}}{2\pi b}+\frac{iM_1}{2b}\right)\Phi_b\left(\frac{\widehat{x}}{2\pi b}+\frac{iM_2}{2b}+\frac{\zeta_2}{b}+\frac{i}{2b}\right)}{\Phi_b\left(\frac{\widehat{x}}{2\pi b}-\frac{iM_1}{2b}-\frac{ib}{2}\right)\Phi_b\left(\frac{\widehat{x}}{2\pi b}-\frac{iM_2}{2b}+\frac{\zeta_2}{b}-\frac{i}{2b}\right)}\frac{1}{2\cosh\frac{\widehat{p}}{2}}.
\end{aligned} \tag{3.53}$$

Summing (3.53) with the coefficients in (3.52) we find

$$\begin{aligned}
&i\widehat{\rho}_k\left(M_1, M_2, 0, \zeta_1-\frac{i}{2}, \zeta_2+\frac{i}{2}\right)+i\widehat{\rho}_k\left(M_1, M_2, 0, \zeta_1+\frac{i}{2}, \zeta_2-\frac{i}{2}\right)\\
&\quad+\mathfrak{q}^{-2\theta_1}\left(i\widehat{\rho}_k\left(M_1, M_2, 0, \zeta_1+\frac{i}{2}, \zeta_2+\frac{i}{2}\right)+i\widehat{\rho}_k\left(M_1, M_2, 0, \zeta_1-\frac{i}{2}, \zeta_2-\frac{i}{2}\right)\right)\\
&\quad-\left(1-\mathfrak{q}^{-2\theta_1}t\right)\left(\widehat{\rho}_k\left(M_1, M_2-1, 0, \zeta_1, \zeta_2\right)-\widehat{\rho}_k\left(M_1, M_2+1, 0, \zeta_1, \zeta_2\right)\right)=0,
\end{aligned}$$

hence (3.40) is satisfied at order κ. We can also show (3.44) at order κ by a completely parallel calculation. So far we have proved that (3.40) and (3.44) hold at order κ for any values of $(M_1, M_2, \zeta_1, \zeta_2)$. Since the other three bilinear equations (3.41),(3.42),(3.43) can be written in some combinations of these two equations by using (3.51), our calculation also proves that (3.41),(3.42),(3.43) also hold at order κ for arbitrary $(M_1, M_2, \zeta_1, \zeta_2)$.

3.2.4.4 Exact values of $\mathrm{tr}\widehat{\rho}_k\left(M_1, M_2, 0, \zeta_1, \zeta_2\right)^n$ at integer points

In order to check the equations at higher order, we need to calculate the exact expressions for the higher traces of the spectral density matrix. In this subsection we explain that this can be done systematically when $k \in \mathbb{N}$ and $M_1, M_2, 2i\zeta_1 \in \mathbb{Z}$. For $M_1, M_2 \in \mathbb{Z}$, the ratios of quantum dilogarithm in the density matrix $\widehat{\rho}_k$ for

$M=0$ (3.29) reduce to products of hyperbolic functions:

$$\begin{aligned}\widehat{\rho}_k\left(M_1,M_2,0,\zeta_1,\zeta_2\right)=&\, i^{M_1+M_2}e^{-\frac{i\zeta_1\widehat{x}}{k}}\frac{1}{2\cosh\frac{\widehat{x}+\pi iM_1}{2}}\left(\prod_{r=1}^{M_1}2\sinh\frac{\widehat{x}-2\pi i\sigma_{1,r}}{2k}\right)\frac{1}{2\cosh\frac{\widehat{p}}{2}}\\ &\times\left(\prod_{r=1}^{M_1}\frac{1}{2\cosh\frac{\widehat{x}-2\pi i\sigma_{1,r}}{2k}}\right)e^{\frac{i\zeta_1\widehat{x}}{k}}e^{\frac{i\zeta_2\widehat{p}}{k}}\left(\prod_{r=1}^{M_2}\frac{1}{2\cosh\frac{\widehat{x}-2\pi i\sigma_{2,r}}{2k}}\right)\frac{1}{2\cosh\frac{\widehat{p}}{2}}\\ &\times\left(\prod_{r=1}^{M_2}2\sinh\frac{\widehat{x}-2\pi i\sigma_{2,r}}{2k}\right)\frac{1}{2\cosh\frac{\widehat{x}+\pi iM_2}{2}}e^{-\frac{i\zeta_2\widehat{p}}{k}},\end{aligned}$$

with $\sigma_{i,r}=\frac{M_i+1}{2}-r$. See the section 3.2.3 for details. For $k\in\mathbb{N}$, by using the following formula

$$\frac{\prod_{r=1}^{n}2\sinh\frac{x-2\pi i\left(\frac{n+1}{2}-r\right)}{2k}}{2\cosh\frac{x+\pi in}{2}}=\frac{i^{-n}}{\prod_{r=1}^{k-n}2\cosh\frac{x+2\pi i\left(\frac{k-n+1}{2}-r\right)}{2k}},\qquad(0\leq n\leq k)$$

which can be obtained by the standard formula $x^n-y^n=\prod_{j=1}^{n}\left(x-e^{\frac{2\pi ij}{n}}y\right)$, we can further rewrite $\widehat{\rho}_k\left(M_1,M_2,0,\zeta_1,\zeta_2\right)$ as

$$\begin{aligned}\widehat{\rho}_k\left(M_1,M_2,0,\zeta_1,\zeta_2\right)=&\, e^{-\frac{i\zeta_1\widehat{x}}{k}}\frac{1}{\prod_{r=1}^{k-M_1}2\cosh\frac{\widehat{x}+t_{0,k-M_1,r}}{2}}\frac{1}{2\cosh\frac{\widehat{p}}{2}}\frac{1}{\prod_{r=1}^{M_1}2\cosh\frac{\widehat{x}+t_{0,M_1,r}}{2}}e^{\frac{i\zeta_1\widehat{x}}{k}}e^{\frac{i\zeta_2\widehat{p}}{k}}\\ &\times\frac{1}{\prod_{r=1}^{M_2}2\cosh\frac{\widehat{x}+t_{\zeta_2,M_2,r}}{2}}\frac{1}{2\cosh\frac{\widehat{p}}{2}}\frac{1}{\prod_{r=1}^{k-M_2}2\cosh\frac{\widehat{x}+t_{\zeta_2,k-M_2,r}}{2}}e^{-\frac{i\zeta_2\widehat{p}}{k}},\end{aligned}$$

where we have defined $t_{\zeta,n,r}$ as

$$t_{\zeta,n,r}=2\pi\zeta+2\pi i\left(\frac{n+1}{2}-r\right).\tag{3.54}$$

By performing some similarity transformations we obtain

$$\widehat{\rho}_k\left(M_1,M_2,0,\zeta_1,\zeta_2\right)\sim\widehat{\rho}'=\widehat{\rho}_1'\widehat{\rho}_2',$$

with

$$\begin{aligned}\widehat{\rho}_1'&=\sqrt{A\left(\widehat{x}\right)}\frac{1}{2\cosh\frac{\widehat{p}}{2}}\sqrt{B\left(\widehat{x}\right)},\\ \widehat{\rho}_2'&=\sqrt{B\left(\widehat{x}\right)}\frac{1}{2\cosh\frac{\widehat{p}}{2}}\sqrt{A\left(\widehat{x}\right)},\\ A\left(x\right)&=\frac{e^{-\frac{i\zeta_1}{k}x}}{\left(\prod_{r=1}^{k-M_1}2\cosh\frac{x+t_{0,k-M_1,r}}{2k}\right)\left(\prod_{r=1}^{k-M_2}2\cosh\frac{x+t_{\zeta_2,k-M_2,r}}{2k}\right)},\\ B\left(x\right)&=\frac{e^{\frac{i\zeta_1}{k}x}}{\left(\prod_{r=1}^{M_1}2\cosh\frac{x+t_{0,M_1,r}}{2k}\right)\left(\prod_{r=1}^{M_2}2\cosh\frac{x+t_{\zeta_2,M_2,r}}{2k}\right)}.\end{aligned}$$

The operator $\widehat{\rho}'$ has the following property:

$$e^{\frac{\widehat{x}}{k}}\widehat{\rho}' - \widehat{\rho}' e^{\frac{\widehat{x}}{k}} = \sum_{a=1}^{2} \widehat{C}_a |0\rangle\!\rangle \langle\!\langle 0| \widehat{D}_a, \tag{3.55}$$

with

$$\widehat{C}_1 = e^{\frac{\widehat{x}}{2k}}\sqrt{A\left(\widehat{x}\right)}, \quad \widehat{C}_2 = -\widehat{\rho}_1' e^{\frac{\widehat{x}}{2k}}\sqrt{B\left(\widehat{x}\right)}, \quad \widehat{D}_1 = \sqrt{B\left(\widehat{x}\right)}e^{\frac{\widehat{x}}{2k}}\widehat{\rho}_2', \quad \widehat{D}_2 = \sqrt{A\left(\widehat{x}\right)}e^{\frac{\widehat{x}}{2k}}.$$

Here we have defined the position eigenstates $|x\rangle$ and the momentum eigenstates $|p\rangle\!\rangle$ with the following normalizations:

$$\begin{aligned} &\widehat{x}|x\rangle = x|x\rangle, \quad \langle x|y\rangle = 2\pi\delta\left(x-y\right), \\ &\widehat{p}|p\rangle\!\rangle = p|p\rangle\!\rangle, \quad \langle\!\langle p|p'\rangle\!\rangle = 2\pi\delta\left(p-p'\right), \quad \langle x|p\rangle\!\rangle = \frac{1}{\sqrt{k}}e^{\frac{ixp}{2\pi k}}. \end{aligned} \tag{3.56}$$

From (3.55) we can show

$$\begin{aligned} \operatorname{tr}\left(\widehat{\rho}'\right)^n = \frac{k}{2}\int \frac{dx}{2\pi}e^{-\frac{x}{k}}\sum_{\ell=0}^{n-1}\sum_{a=1}^{2}\Bigg(&\frac{d}{dx}\left(\langle x|\left(\widehat{\rho}'\right)^{\ell}\widehat{C}_a|0\rangle\!\rangle\right)\langle\!\langle 0|\widehat{D}_a\left(\widehat{\rho}'\right)^{n-1-\ell}|x\rangle \\ &-\langle x|\left(\widehat{\rho}'\right)^{\ell}\widehat{C}_a|0\rangle\!\rangle\frac{d}{dx}\left(\langle\!\langle 0|\widehat{D}_a\left(\widehat{\rho}'\right)^{n-1-\ell}|x\rangle\right)\Bigg). \end{aligned} \tag{3.57}$$

If we define $\phi_{a,\ell}\left(x\right)$ as

$$\phi_{1,\ell}\left(x\right) = \frac{1}{\sqrt{A\left(x\right)e^{\frac{x}{2k}}}}\langle x|\left(\widehat{\rho}'\right)^{\ell}\sqrt{A\left(\widehat{x}\right)}e^{\frac{\widehat{x}}{2k}}|0\rangle\!\rangle, \quad \phi_{2,\ell}\left(x\right) = \frac{1}{\sqrt{A\left(x\right)e^{\frac{x}{2k}}}}\langle x|\left(\widehat{\rho}'\right)^{\ell}\widehat{\rho}_1'\sqrt{B\left(\widehat{x}\right)}e^{\frac{\widehat{x}}{2k}}|0\rangle\!\rangle,$$

we can write the matrix elements with insertions of $\widehat{C}_a$ as

$$\langle x|\left(\widehat{\rho}'\right)^{\ell}\widehat{C}_1|0\rangle\!\rangle = \sqrt{A\left(x\right)e^{\frac{x}{2k}}}\phi_{1,\ell}\left(x\right), \quad \langle x|\left(\widehat{\rho}'\right)^{\ell}\widehat{C}_2|0\rangle\!\rangle = -\sqrt{A\left(x\right)e^{\frac{x}{2k}}}\phi_{2,\ell}\left(x\right). \tag{3.58}$$

By using the fact that $\langle x|\widehat{\rho}_1'|y\rangle = \langle y|\widehat{\rho}_2'|x\rangle$ and $\langle x|\widehat{\rho}'|y\rangle = \langle y|\widehat{\rho}'|x\rangle$ we can also write the matrix elements with $\widehat{D}_a$ in terms of $\phi_{a,\ell}\left(x\right)$, namely:

$$\langle\!\langle 0|\widehat{D}_1\left(\widehat{\rho}'\right)^{\ell}|x\rangle = \sqrt{A\left(x\right)e^{\frac{x}{2k}}}\phi_{2,\ell}\left(x\right), \quad \langle\!\langle 0|\widehat{D}_2\left(\widehat{\rho}'\right)^{\ell}|x\rangle = \sqrt{A\left(x\right)e^{\frac{x}{2k}}}\phi_{1,\ell}\left(x\right). \tag{3.59}$$

By using (3.58) and (3.59) we can rewrite (3.57) as

$$\operatorname{tr}\left(\widehat{\rho}'\right)^{\ell} = k\int\frac{dx}{2\pi}A\left(x\right)\sum_{\ell=0}^{n-1}\left(\frac{d\phi_{1,\ell}\left(x\right)}{dx}\phi_{2,n-1-\ell}\left(x\right) - \phi_{1,\ell}\left(x\right)\frac{d\phi_{2,n-1-\ell}\left(x\right)}{dx}\right).$$

Note that $\phi_{i,\ell}\left(x\right)$ can be calculated recursively as

$$\begin{aligned} \phi_{1,0}\left(x\right) &= \frac{1}{\sqrt{k}}, \quad \phi_{2,0}\left(x\right) = \frac{1}{k}\int\frac{dy}{2\pi}\frac{e^{\frac{y}{k}}}{e^{\frac{x}{k}}+e^{\frac{y}{k}}}B\left(y\right)\frac{1}{\sqrt{k}}, \\ \widetilde{\phi}_{i,\ell}\left(x\right) &= \frac{1}{k}\int\frac{dy}{2\pi}\frac{e^{\frac{y}{k}}}{e^{\frac{x}{k}}+e^{\frac{y}{k}}}A\left(y\right)\phi_{i,\ell}\left(y\right), \\ \phi_{i,\ell+1}\left(x\right) &= \frac{1}{k}\int\frac{dy}{2\pi}\frac{e^{\frac{y}{k}}}{e^{\frac{x}{k}}+e^{\frac{y}{k}}}B\left(y\right)\widetilde{\phi}_{i,\ell}\left(y\right). \end{aligned}$$

If we further assume $M_1 + M_2 + 2i\zeta_1 \in \mathbb{Z}$, we can apply the same technique as in [243] to evaluate these integrals. First we introduce a new variable $u = e^{\frac{x}{2k}}$, to rewrite $A(x)$ and $B(x)$ as

$$A(u) = e^{-\frac{\pi\zeta_2(k-M_2)}{k}} \frac{u^{2k-q}}{\prod_{r=1}^{k-M_1}\left(u^2 + e^{-\frac{1}{k}t_{0,k-M_1,r}}\right)\prod_{r=1}^{k-M_2}\left(u^2 + e^{-\frac{1}{k}t_{\zeta_2,k-M_2,r}}\right)},$$
$$B(u) = e^{-\frac{\pi\zeta_2 M_2}{k}} \frac{u^{q}}{\prod_{r=1}^{M_1}\left(u^2 + e^{-\frac{1}{k}t_{0,M_1,r}}\right)\prod_{r=1}^{M_2}\left(u^2 + e^{-\frac{1}{k}t_{\zeta_2,M_2,r}}\right)},$$

where $q = M_1 + M_2 + 2i\zeta_1$. The integration of $\phi_{2,0}(x)$ can be evaluated as [243]

$$\begin{aligned}\phi_{2,0}(u) &= \frac{1}{\pi}e^{-\frac{\pi\zeta_2 M_2}{k}}\int_0^\infty dv \frac{1}{u^2+v^2}\frac{v^{q+1}}{\prod_{r=1}^{M_-}\left(v^2 + e^{-\frac{1}{k}t_{0,M_-,r}}\right)\prod_{r=1}^{M_+}\left(v^2 + e^{-\frac{1}{k}t_{\zeta_2,M_+,r}}\right)}\frac{1}{\sqrt{k}} \\ &= \frac{1}{\pi}e^{-\frac{\pi\zeta_2 M_2}{k}}(-2\pi i)\sum_{w\in\mathbb{C}\backslash\mathbb{R}_{\geq 0}}\mathrm{Res}_{v=w}\left[\frac{1}{u^2+v^2}\right. \\ &\quad\left.\times\frac{v^{q+1}}{\prod_{r=1}^{M_-}\left(v^2 + e^{-\frac{1}{k}t_{0,M_-,r}}\right)\prod_{r=1}^{M_+}\left(v^2 + e^{-\frac{1}{k}t_{\zeta_2,M_+,r}}\right)}\frac{1}{\sqrt{k}}B_1\left(\frac{\log^{(+)} v}{2\pi i}\right)\right].\end{aligned}$$

Here $\log^{(+)} u$ is the logarithm with the branch cut chosen as $u \in \mathbb{R}_{\geq 0}$, and $B_j(x)$ are the Bernoulli polynomials.[††] The poles w contributing to $\phi_{2,0}(u)$ can be listed explicitly as

$$w = \begin{cases} \pm iu \\ \pm ie^{-\frac{1}{2k}t_{0,M_1,r}}, & (r = 1, 2, \cdots, M_1) \\ \pm ie^{-\frac{1}{2k}t_{\zeta_2,M_2,r}}, & (r = 1, 2, \cdots, M_2) \end{cases}.$$

[††] We can also chose $B_j(x)$ as any polynomials satisfying

$$B_{j+1}(x+1) - B_{j+1}(x) = (j+1)\,x^j.$$

In the same way the recursion relations of $\phi_{i,\ell}$ can be rewritten as

$$\begin{aligned}
\phi_{i,\ell}(u) &= \sum_{j\geq 0} \phi_{i,\ell}^{(j)}(u) (\log u)^j , \\
\widetilde{\phi}_{i,\ell}(u) &= \frac{1}{\pi} e^{-\frac{\pi\zeta_2(k-M_2)}{k}} \sum_{j\geq 0} \left(-\frac{(2\pi i)^{j+1}}{j+1} \right) \sum_{w\in\mathbb{C}\backslash\mathbb{R}_{\geq 0}} \mathrm{Res}_{v=w} \Bigg[\\
&\quad \frac{1}{u^2+v^2} \frac{v^{2k-q+1}\phi_{i,\ell}^{(j)}(v)}{\prod_{r=1}^{k-M_1}\left(v^2+e^{-\frac{1}{k}t_{0,k-M_1,r}}\right)\prod_{r=1}^{k-M_2}\left(v^2+e^{-\frac{1}{k}t_{\zeta_2,k-M_2,r}}\right)} \\
&\quad \times B_{j+1}\left(\frac{\log^{(+)} v}{2\pi i}\right)\Bigg] , \qquad (3.60) \\
\widetilde{\phi}_{i,\ell}(u) &= \sum_{j\geq 0} \widetilde{\phi}_{i,\ell}^{(j)}(u) (\log u)^j , \\
\phi_{i,\ell+1}(u) &= \frac{1}{\pi} e^{-\frac{\pi\zeta_2 M_2}{k}} \sum_{j\geq 0} \left(-\frac{(2\pi i)^{j+1}}{j+1} \right) \sum_{w\in\mathbb{C}\backslash\mathbb{R}_{\geq 0}} \mathrm{Res}_{v=w} \Bigg[\frac{1}{u^2+v^2} \qquad (3.61) \\
&\quad \frac{v^{q+1}\widetilde{\phi}_{i,\ell}^{(j)}(v)}{\prod_{r=1}^{M_1}\left(v^2+e^{-\frac{1}{k}t_{0,M_1,r}}\right)\prod_{r=1}^{M_2}\left(v^2+e^{-\frac{1}{k}t_{\zeta_2,M_2,r}}\right)} B_{j+1}\left(\frac{\log^{(+)} v}{2\pi i}\right)\Bigg] ,
\end{aligned}$$

where the poles contributing in (3.60) are

$$w = \begin{cases} \pm iu \\ \pm i e^{-\frac{1}{2k}t_{0,k-M_1,r}}, & (r=1,2,\cdots,k-M_1) \\ \pm i e^{-\frac{1}{2k}t_{\zeta_2,k-M_2,r}}, & (r=1,2,\cdots,k-M_2) \\ \text{poles of } \phi_{i,\ell}^{(j)}(w) \end{cases} ,$$

and the poles contributing in (3.61) are

$$w = \begin{cases} \pm iu \\ \pm i e^{-\frac{1}{2k}t_{0,M_1,r}}, & (r=1,2,\cdots,M_1) \\ \pm i e^{-\frac{1}{2k}t_{\zeta_2,M_2,r}}, & (r=1,2,\cdots,M_2) \\ \text{poles of } \widetilde{\phi}_{i,\ell}^{(j)}(w) \end{cases} .$$

We can show by induction that the poles of $\phi_{i,\ell}$ and $\widetilde{\phi}_{i,\ell}$ satisfy the following inclusions

$$\begin{aligned}
&\{\text{poles of } \phi_{i,\ell}(u)\} \\
&\quad \subset \left\{\pm e^{-\frac{1}{2k}t_{0,M_1,r}}\right\}_{r=1}^{M_1} \cup \left\{\pm e^{-\frac{1}{2k}t_{\zeta_2,M_2,r}}\right\}_{r=1}^{M_2} \cup \left\{\pm i e^{-\frac{1}{2k}t_{0,k-M_1,r}}\right\}_{r=1}^{k-M_1} \cup \left\{\pm i e^{-\frac{1}{2k}t_{\zeta_2,k-M_2,r}}\right\}_{r=1}^{k-M_2} , \\
&\left\{\text{poles of } \widetilde{\phi}_{i,\ell}(u)\right\} \\
&\quad \subset \left\{\pm i e^{-\frac{1}{2k}t_{0,M_1,r}}\right\}_{r=1}^{M_1} \cup \left\{\pm i e^{-\frac{1}{2k}t_{\zeta_2,M_2,r}}\right\}_{r=1}^{M_2} \cup \left\{\pm e^{-\frac{1}{2k}t_{0,k-M_1,r}}\right\}_{r=1}^{k-M_1} \cup \left\{\pm e^{-\frac{1}{2k}t_{\zeta_2,k-M_2,r}}\right\}_{r=1}^{k-M_2} ,
\end{aligned} \tag{3.62}$$

for both $i = 1, 2$ and any $\ell \geq 0$. Once we obtain $\{\phi_{i,\ell}\}_{\ell=0}^{n-1}$, we can calculate $\mathrm{tr}\,(\widehat{\rho}')^n$ as

$$\sum_{\ell=0}^{n-1}\left(\frac{d\phi_{1,\ell}}{du}\phi_{2,n-1-\ell} - \phi_{1,\ell}\frac{d\phi_{2,n-1-\ell}}{du}\right) = \sum_{j\geq 0}\Psi_n^{(j)}(u)(\log u)^j\,,$$

$$\mathrm{tr}\,(\widehat{\rho}')^n = \frac{k}{2\pi}e^{-\frac{\pi\zeta_2(k-M_2)}{k}}\sum_{j\geq 0}\left(-\frac{(2\pi i)^{j+1}}{j+1}\right)\sum_{w\in\mathbb{C}\setminus\mathbb{R}_{\geq 0}}\mathrm{Res}_{u=w}\Bigg[$$
$$\frac{u^{2k-q}}{\prod_{r=1}^{k-M_1}\left(u^2+e^{-\frac{1}{k}t_{0,k-M_1,r}}\right)\prod_{r=1}^{k-M_2}\left(u^2+e^{-\frac{1}{k}t_{\zeta_2,k-M_2,r}}\right)}\Psi_n^{(j)}(u)\,B_{j+1}\left(\frac{\log^{(+)}u}{2\pi i}\right)\Bigg].$$

Here the poles w contributing to $\mathrm{tr}\,(\widehat{\rho}')^n$ are the ones in the set $\{\text{poles of } \phi_{i,\ell}(w)\}$ listed in (3.62). By using these results, we are able to perform the checks of the bilinear and quartic equations for the τ-functions at higher order that we list in Table 3.1.

3.2.5 Coalescence limits: matrix models and quantum curves

The $\mathfrak{q}$-Painlevé equations were classified by their symmetry type in [246], where their coalescence patterns are also discussed. In particular we are interested in the following one concerning $\mathfrak{q}$-Painlevé VI equation

$$\mathrm{VI}\to\mathrm{V}\to\mathrm{III}_1\to\mathrm{III}_2\to\mathrm{III}_3. \tag{3.63}$$

The $\mathfrak{q}$-Painlevé III_3 equation, which is the end point of the above coalescence diagram, is related to the ABJM theory [46]. In this section, we study the above coalescence limits in terms of the matrix model and the quantum curve. The degeneration pattern of the quantum curves matches the one in (3.63). Since the coalescence limit is similar at all the steps, we study in detail the first one and omit the details for the others.

The coalescence can be seen as a degeneration of the tau function. In (3.35), we saw the relation between the $\mathfrak{q}$-Painlevé VI tau function and the grand partition function of the quiver superconformal Chern-Simons matter theory displayed in Fig.3.1. In (3.35) we normalized the grand partition function by $Z_k(0; M_1, M_2, M, \zeta_1, \zeta_2)$. In this section, we adopt a slightly different normalization factor, namely we define

$$Z_k^{\mathrm{VI}}(N; M_1, M_2, M, \zeta_1, \zeta_2) = \frac{Z_k(N; M_1, M_2, M, \zeta_1, \zeta_2)}{e^{i\Theta_k(M_1,M_2,M,\zeta_1,\zeta_2)}Z_k^{(\mathrm{CS})}(M_1)\,Z_k^{(\mathrm{CS})}(M_2)}$$
$$= \frac{1}{N!\,(N+M)!}\int\prod_{n=1}^{N}\frac{d\mu_n}{2\pi}\prod_{n=1}^{N+M}\frac{d\nu_n}{2\pi}\det\begin{pmatrix}\left[\left\langle\mu_m\middle|\widehat{D}_1^{\mathrm{VI}}\middle|\nu_n\right\rangle\right]_{m,n}^{N\times(N+M)}\\ \left[\left\langle\!\left\langle t_{0,M,r}\middle|\widehat{d}_1^{\mathrm{VI}}\middle|\nu_n\right\rangle\right]_{r,n}^{M\times(N+M)}\end{pmatrix}$$
$$\times\det\left(\left[\left\langle\nu_m\middle|\widehat{D}_2^{\mathrm{VI}}\middle|\mu_n\right\rangle\right]_{m,n}^{(N+M)\times N}\;\left[\left\langle\nu_m\middle|\widehat{d}_2^{\mathrm{VI}}\middle|-t_{0,M,r}\right\rangle\!\right\rangle\right]_{m,r}^{(N+M)\times M}\right), \tag{3.64}$$

(k, M_1, M_2, ζ_1)	(3.40)	(3.41)	(3.42)	(3.43)	(3.44)	(3.45)
$(2, 1, 0, 0)$		7				
$\left(2, 1, 1, -\frac{i}{2}\right)$	3	3	3	3	3	3
$(2, 1, 1, 0)$	2	2	2	2	2	2
$\left(2, 1, 1, \frac{i}{2}\right)$	3	3	3	3	3	3
$(3, 1, 1, -i)$			2	2	2	2
$\left(3, 1, 1, -\frac{i}{2}\right)$	2	2	2	2	2	2
$(3, 1, 1, 0)$	2	2	2	2	2	2
$\left(3, 1, 1, \frac{i}{2}\right)$	2	2	2	2	2	2
$(3, 1, 1, i)$			2	2	2	2
$\left(3, 1, 2, -\frac{i}{2}\right)$		2				2
$(3, 1, 2, 0)$	2	2	2	2	2	2
$\left(3, 1, 2, \frac{i}{2}\right)$		2				
$(3, 2, 1, 0)$	2	2	2	2	2	2
$\left(3, 2, 1, \frac{i}{2}\right)$						2
$(3, 2, 2, -i)$		2				2
$(3, 2, 2, 0)$	2	2				2
$\left(3, 2, 2, \frac{i}{2}\right)$	2	2	2	2	2	2
$(3, 2, 2, i)$						2
$\left(4, 1, 1, -\frac{i}{2}\right)$	2	2	2	2	2	2
$\left(4, 1, 1, \frac{i}{2}\right)$	2	2	2	2	2	
$(4, 1, 2, 0)$	2	2	2	2	2	2

Table 3.1: The list of (k, M_-, M_+, ζ_1) for which we have checked that bilinear equations (3.40), (3.41), (3.42), (3.43), (3.44), and the quartic equation (3.45) for the Fredholm determinant hold. Each number in the table means that we have confirmed the bilinear/quartic equation at least up to a possible $\mathcal{O}\left(\kappa^{\#+1}\right)$ correction. Blank cells stand for the cases where we could not check the equations beyond the first order in κ (we could check some of them with fixed values of ζ_2 to higher order in κ).

where we used (3.19). Notice that the integrand of Z_k^{VI} can also be written explicitly as

$$
\begin{aligned}
Z_k^{\mathrm{VI}}\left(N; M_1, M_2, M, \zeta_1, \zeta_2\right) = \frac{1}{N!\,(N+M)!} \int \prod_{n=1}^{N} \frac{d\mu_n}{2\pi k} \prod_{n=1}^{N+M} \frac{d\nu_n}{2\pi k} \\
\times \prod_{n=1}^{N} e^{\left(-\frac{i\zeta_1}{k}+\frac{2k-M-M_2}{2k}\right)\mu_n} \frac{\Phi_b\left(\frac{\mu_n}{2\pi b}-\frac{iM_1}{2b}+\frac{i}{2}b\right)\Phi_b\left(\frac{\mu_n}{2\pi b}-\frac{iM_2-2\zeta_2}{2b}+\frac{i}{2}b\right)}{\Phi_b\left(\frac{\mu_n}{2\pi b}+\frac{iM_1}{2b}-\frac{i}{2}b\right)\Phi_b\left(\frac{\mu_n}{2\pi b}+\frac{iM_2+2\zeta_2}{2b}-\frac{i}{2}b\right)} \\
\times \prod_{n=1}^{N+M} e^{\left(\frac{i\zeta_1}{k}+\frac{M_1+M_2}{2k}\right)\nu_n} \frac{\Phi_b\left(\frac{\nu_n}{2\pi b}+\frac{iM_1}{2b}\right)\Phi_b\left(\frac{\nu_n}{2\pi b}+\frac{iM_2+2\zeta_2}{2b}\right)}{\Phi_b\left(\frac{\nu_n}{2\pi b}-\frac{iM_1}{2b}\right)\Phi_b\left(\frac{\nu_n}{2\pi b}-\frac{iM_2-2\zeta_2}{2b}\right)} \\
\times \left(\frac{\prod_{m<m'}^{N} 2\sinh\frac{\mu_m-\mu_{m'}}{2k}\prod_{n<n'}^{N+M} 2\sinh\frac{\nu_n-\nu_{n'}}{2k}}{\prod_{m=1}^{N}\prod_{n=1}^{N+M} 2\cosh\frac{\mu_m-\nu_n}{2k}}\right)^2 .
\end{aligned}
\tag{3.65}
$$

The normalization factor in the first line of (3.64) is the prefactor appearing in (3.19) and is independent of N. This normalization factor provides a result consistent with the known result in [46] at the end of the coalescence, as we will comment later. Note that this definition does not contradict our previous analysis since for $M=0$ the two normalization factors coincide (see (3.22)).

For clarity, we will study the coalescence limit by treating at the same time the matrix model and the quantum curve. In such a way we can provide the relation between them while flowing along the coalescence. For this purpose, we first clarify the relation between the matrix model and the quantum curve for $\mathfrak{q}$-Painlevé VI. The conjecture (3.31) implies that

$$
\begin{aligned}
&Z_k^{\mathrm{VI}}\left(N; M_1, M_2, M, \zeta_1, \zeta_2\right) \\
&= Z_k^{\mathrm{VI}}\left(0; M_1, M_2, M, \zeta_1, \zeta_2\right) \frac{1}{N!} \int \prod_{n=1}^{N} \frac{d\mu_n}{2\pi} \det\left(\left[\left\langle \mu_m \middle| \widehat{\rho}_k^{\mathrm{VI}}\left(M_1, M_2, M, \zeta_1, \zeta_2\right) \middle| \mu_n \right\rangle\right]_{m,n}^{N\times N}\right),
\end{aligned}
\tag{3.66}
$$

where $\widehat{\rho}_k^{\mathrm{VI}}$ is the conjectural form of the inverse of $\widehat{\rho}_k^{-1}$ in (3.31). This is our starting point of the section.

The consistent ways to take the limit of the parameters in each step of the coalescence (3.63) can be pictorially described through the toric diagram. Indeed, by Painlevé/gauge theory correspondence [47] we know that each step of the coalescence corresponds to the decoupling limit of a fundamental hypermultiplet in the five dimensional $\mathcal{N}=1$ gauge theory. Since the mass parameters are encoded in the Seiberg-Witten curve, which we identify with $\widehat{\rho}_k^{-1}$, as the positions of the asymptotic regions at $x,p=\pm\infty$, i.e. the external legs in Fig.3.2 (see also Fig.3.3), the mass decoupling limit corresponds to sending one pair of horizontal and vertical legs to infinity (or to zero) while keeping the other legs fixed.

The first coalescence from $\mathfrak{q}$-Painlevé VI to $\mathfrak{q}$-Painlevé V is realized by sending the bottom-left pair of external legs to zero. This can be achieved by introducing the following new parameters

$$
iM_2 = iM_2' + 2\Lambda, \quad \zeta_1 = \zeta_1' - \Lambda, \quad \zeta_2 = \Lambda, \tag{3.67}
$$

and taking the limit $\Lambda\to\infty$ while keeping the other parameters fixed, and at the same time shifting $\widehat{p}$ as $\widehat{p}\to\widehat{p}+2\pi\zeta_2$ so that the locations of the other external legs

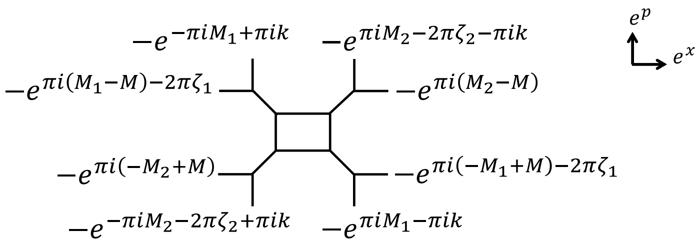

Figure 3.3: The asymptotic behavior of $\left(\widehat{\rho}_k^{\mathrm{VI}}\right)^{-1}$, or equivalently (3.31). This figure can be regarded as the five-brane web diagram of the 5d $\mathcal{N}=1$ SU(2) gauge theory with $N_f=4$ in Fig.3.2.

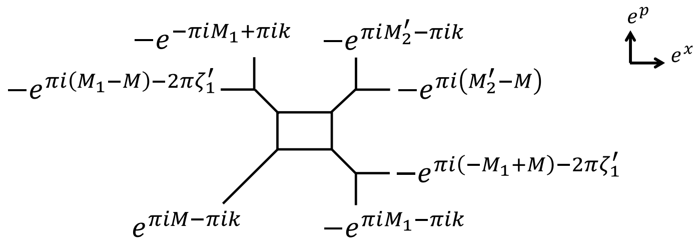

Figure 3.4: The asymptotic behavior of (3.68). This figure can be regarded as the five-brane web diagram of the 5d $\mathcal{N}=1$ SU(2) gauge theory with $N_f=3$. This diagram is the degeneration of the diagram in Fig.3.3.

(3.33) are kept finite:

$$
\begin{aligned}
\widetilde{m}_1^{\mathrm{VI}} &= -e^{\pi i(M_2-M)} && \to && \widetilde{m}_1^{\mathrm{V}} = e^{-2\pi\Lambda}\widetilde{m}_1^{\mathrm{VI}} = -e^{\pi i\left(M_2'-M\right)},\\
\widetilde{m}_2^{\mathrm{VI}} &= -e^{\pi i(-M_1+)-2\pi\zeta_1} && \to && \widetilde{m}_2^{\mathrm{V}} = e^{-2\pi\Lambda}\widetilde{m}_2^{\mathrm{VI}} = -e^{\pi i(-M_1+M)-2\pi\zeta_1'},\\
\widetilde{m}_3^{\mathrm{VI}} &= -e^{\pi i(M_1-M)-2\pi\zeta_1} && \to && \widetilde{m}_3^{\mathrm{V}} = e^{-2\pi\Lambda}\widetilde{m}_3^{\mathrm{VI}} = -e^{\pi i(M_1-M)-2\pi\zeta_1'},\\
\widetilde{m}_4^{\mathrm{VI}} &= -e^{\pi i(-M_2+M)} && \to && \widetilde{m}_4^{\mathrm{V}} = e^{-2\pi\Lambda}\widetilde{m}_4^{\mathrm{VI}} = -e^{\pi i\left(-M_2'+M\right)-4\pi\Lambda} = 0,\\
\widetilde{t}_1^{\mathrm{VI}} &= -e^{\pi iM_2-2\pi\zeta_2-\pi ik} && \to && \widetilde{t}_1^{\mathrm{V}} = \widetilde{t}_1^{\mathrm{VI}} = -e^{\pi iM_2'-\pi ik},\\
\widetilde{t}_2^{\mathrm{VI}} &= -e^{\pi iM_1-\pi ik} && \to && \widetilde{t}_2^{\mathrm{V}} = \widetilde{t}_2^{\mathrm{VI}} = -e^{\pi iM_1-\pi ik},\\
\widetilde{t}_3^{\mathrm{VI}} &= -e^{-\pi iM_1+\pi ik} && \to && \widetilde{t}_3^{\mathrm{V}} = \widetilde{t}_3^{\mathrm{VI}} = -e^{-\pi iM_1+\pi ik},\\
\widetilde{t}_4^{\mathrm{VI}} &= -e^{-\pi iM_2-2\pi\zeta_2+\pi ik} && \to && \widetilde{t}_4^{\mathrm{V}} = \widetilde{t}_4^{\mathrm{VI}} = -e^{-\pi iM_2'+\pi ik-4\pi\Lambda} = 0.
\end{aligned}
$$

See Fig.3.4.

Note that the ratio $\ell_4 = \widetilde{m}_4^{\mathrm{V}}\left(\widetilde{t}_4^{\mathrm{V}}\right)^{-1} = e^{\pi iM-\pi ik}$ is kept finite under the limit

$\Lambda \to \infty$. Under this procedure the quantum curve $\left(\widehat{\rho}_k^{\mathrm{VI}}\right)^{-1}$ transforms as

$$\begin{aligned}
&\left(\widehat{\rho}_k^{\mathrm{VI}}\left(M_1, M_2, M, \zeta_1, \zeta_2\right)\right)^{-1} \\
&\to \left(\widehat{\rho}_k^{\mathrm{V}}\left(M_1, M_2', M, \zeta_1'\right)\right)^{-1} \equiv \lim_{\Lambda\to\infty} e^{-\pi\zeta_2}\left(\widehat{\rho}_k^{\mathrm{VI}}\left(M_1, M_2, M, \zeta_1, \zeta_2\right)\right)^{-1}\Big|_{\widehat{p}\to\widehat{p}+2\pi\zeta_2} \\
&= \lim_{\Lambda\to\infty}\Bigg[e^{\frac{\pi i(-M_1+M_2')}{2}+\pi\zeta_1'}e^{-\widehat{x}+\widehat{p}} + [e^{\frac{\pi i(-M_1-M_2')}{2}+\pi\zeta_1'+\pi ik} + e^{\frac{\pi i(M_1+M_2')}{2}+\pi\zeta_1'-\pi ik}]e^{\widehat{p}} + e^{\frac{\pi i(M_1-M_2')}{2}+\pi\zeta_1'}e^{\widehat{x}+\widehat{p}} \\
&\quad + [e^{\frac{\pi i(-M_1-M_2'+2M)}{2}+\pi\zeta_1'-4\pi\Lambda} + e^{\frac{\pi i(M_1+M_2'-2M)}{2}-\pi\zeta_1'}]e^{-\widehat{x}} + E \\
&\quad + [e^{\frac{\pi i(-M_1-M_2'+2M)}{2}-\pi\zeta_1'} + e^{\frac{\pi i(M_1+M_2'-2M)}{2}+\pi\zeta_1'}]e^{\widehat{x}} \\
&\quad + e^{\frac{\pi i(M_1-M_2')}{2}-\pi\zeta_1'-4\pi\Lambda}e^{-\widehat{x}-\widehat{p}} + [e^{\frac{\pi i(-M_1-M_2')}{2}-\pi\zeta_1'-4\pi\Lambda+\pi ik} + e^{\frac{\pi i(M_1+M_2')}{2}-\pi\zeta_1'-\pi ik}]e^{-\widehat{p}} \\
&\quad + e^{\frac{\pi i(-M_1+M_2')}{2}-\pi\zeta_1'}e^{\widehat{x}-\widehat{p}}\Bigg] \\
&= e^{\frac{\pi i(-M_1+M_2')}{2}+\pi\zeta_1'}e^{-\widehat{x}+\widehat{p}} + [e^{\frac{\pi i(-M_1-M_2')}{2}+\pi\zeta_1'+\pi ik} + e^{\frac{\pi i(M_1+M_2')}{2}+\pi\zeta_1'-\pi ik}]e^{\widehat{p}} + e^{\frac{\pi i(M_1-M_2')}{2}+\pi\zeta_1'}e^{\widehat{x}+\widehat{p}} \\
&\quad + e^{\frac{\pi i(M_1+M_2'-2M)}{2}-\pi\zeta_1'}e^{-\widehat{x}} + E' + [e^{\frac{\pi i(-M_1-M_2'+2M)}{2}-\pi\zeta_1'} + e^{\frac{\pi i(M_1+M_2'-2M)}{2}+\pi\zeta_1'}]e^{\widehat{x}} \\
&\quad + e^{\frac{\pi i(M_1+M_2')}{2}-\pi\zeta_1'-\pi ik}e^{-\widehat{p}} + e^{\frac{\pi i(-M_1+M_2')}{2}-\pi\zeta_1'}e^{\widehat{x}-\widehat{p}},
\end{aligned} \tag{3.68}$$

where E is given by (3.32) and E' is given by

$$E' = e^{\frac{\pi i(-M_1+M_2')}{2}-\pi\zeta_1'+\pi ia_1M} + e^{\frac{\pi i(-M_1+M_2')}{2}+\pi\zeta_1'+\pi ia_2M} + e^{\frac{\pi i(M_1-M_2')}{2}-\pi\zeta_1'+\pi ia_3M}.$$

Here we have rescaled $\left(\widehat{\rho}_k^{\mathrm{VI}}\right)^{-1}$ by a factor $e^{-\pi\zeta_2}$ so that the coefficients of limiting curve remain finite.[‡‡] The resulting operator $\left(\widehat{\rho}_k^{\mathrm{V}}\right)^{-1}$ can be regarded as the quantum mirror curve of $\mathfrak{q}$-Painlevé V.

We now consider the same limit for the matrix model (3.64), or equivalently the left-hand side of the conjecture (3.66). First, corresponding to the shift $\widehat{p} \to \widehat{p}+2\pi\zeta_2$ in the quantum curve we perform the following similarity transformation

$$|\mu\rangle\langle\mu| \to e^{\frac{i\zeta_2}{k}\widehat{x}}|\mu\rangle\langle\mu|e^{-\frac{i\zeta_2}{k}\widehat{x}}, \quad |\nu\rangle\langle\nu| \to e^{\frac{i\zeta_2}{k}\widehat{x}}|\nu\rangle\langle\nu|e^{-\frac{i\zeta_2}{k}\widehat{x}}.$$

Corresponding to the overall rescaling of $\widehat{\rho}_k^{-1}$ by $e^{-\pi\zeta_2}$ we also multiply both sides of (3.66) by $e^{\pi\zeta_2 N}$. This changes $\widehat{D}_2^{\mathrm{VI}}$ to $e^{\pi\zeta_2}\widehat{D}_2^{\mathrm{VI}}$. All in all, $\widehat{D}_i^{\mathrm{V}}$ and $\widehat{d}_i^{\mathrm{V}}$ are changed

[‡‡] In terms of the grand partition function, which appears in the right hand side of (3.35), this operation corresponds to rescale κ as $e^{\pi\zeta_2}\kappa$.

Matching the similarity transformation and the shift guarantees the equality between the density matrix and the quantum curve at operator level. We explain this point at the end of this section.

into

$$e^{-\frac{i\zeta_2}{k}\widehat{x}}\widehat{D}_1^{\mathrm{VI}}e^{\frac{i\zeta_2}{k}\widehat{x}} = e^{-\frac{i\zeta_1'}{k}\widehat{x}}e^{\frac{k-M_1}{2k}\widehat{x}}\frac{\Phi_b\left(\frac{\widehat{x}}{2\pi b}-\frac{iM_1}{2b}+\frac{i}{2}b\right)}{\Phi_b\left(\frac{\widehat{x}}{2\pi b}+\frac{iM_1}{2b}-\frac{i}{2}b\right)}\frac{1}{2\cosh\frac{\widehat{p}-i\pi M}{2}}e^{\frac{i\zeta_1'}{k}\widehat{x}}e^{\frac{M_1}{2k}\widehat{x}}\frac{\Phi_b\left(\frac{\widehat{x}}{2\pi b}+\frac{iM_1}{2b}\right)}{\Phi_b\left(\frac{\widehat{x}}{2\pi b}-\frac{iM_1}{2b}\right)},$$

$$\widehat{d}_1^{\mathrm{VI}}e^{\frac{i\zeta_2}{k}\widehat{x}} = e^{\frac{i\zeta_1'}{k}\widehat{x}}e^{\frac{M_1}{2k}\widehat{x}}\frac{\Phi_b\left(\frac{\widehat{x}}{2\pi b}+\frac{iM_1}{2b}\right)}{\Phi_b\left(\frac{\widehat{x}}{2\pi b}-\frac{iM_1}{2b}\right)},$$

$$e^{\pi\zeta_2}e^{-\frac{i\zeta_2}{k}\widehat{x}}\widehat{D}_2^{\mathrm{VI}}e^{\frac{i\zeta_2}{k}\widehat{x}} = e^{2\pi\Lambda}e^{\frac{M_2'-4i\Lambda}{2k}\widehat{x}}\frac{\Phi_b\left(\frac{\widehat{x}}{2\pi b}+\frac{iM_2'+4\Lambda}{2b}\right)}{\Phi_b\left(\frac{\widehat{x}}{2\pi b}-\frac{iM_2'}{2b}\right)}\frac{1}{2\cosh\frac{\widehat{p}+i\pi M}{2}}e^{\frac{k-M_2'+4i\Lambda}{2k}\widehat{x}}$$
$$\times\frac{\Phi_b\left(\frac{\widehat{x}}{2\pi b}-\frac{iM_2'}{2b}+\frac{i}{2}b\right)}{\Phi_b\left(\frac{\widehat{x}}{2\pi b}+\frac{iM_2'+4\Lambda}{2b}-\frac{i}{2}b\right)},$$

$$e^{-\frac{i\zeta_2}{k}\widehat{x}}\widehat{d}_2^{\mathrm{VI}} = e^{\frac{\pi}{k}\Lambda\left(M_2'-2i\Lambda\right)}e^{\frac{M_2'-4i\Lambda}{2k}\widehat{x}}\frac{\Phi_b\left(\frac{\widehat{x}}{2\pi b}+\frac{iM_2'+4\Lambda}{2b}\right)}{\Phi_b\left(\frac{\widehat{x}}{2\pi b}-\frac{iM_2'}{2b}\right)},$$

where we used the new parameterization (3.67) for the right hand side. In this expression, we can take the $\Lambda\to\infty$ limit. In this limit, the quantum dilogarithm function behaves as (.85). In the third line, the factors depending on Λ and the divergent part of the asymptotic value of the quantum dilogarithm cancel out. On the other hand, in the fourth line, an overall factor

$$e^{\frac{i\pi M}{4k}\left(\left(iM_2'+2\Lambda\right)^2+4\Lambda^2\right)+\frac{i\pi M}{12}\left(k+k^{-1}\right)},$$

appears. However, the same factor also appears in the right hand side of (3.66) since this factor is independent of N. Therefore, we can get rid of it when we take the limit:

$$\begin{aligned}\lim_{\Lambda\to\infty}e^{-\frac{i\pi M}{4k}\left(\left(iM_2'+2\Lambda\right)^2+4\Lambda^2\right)-\frac{i\pi M}{12}\left(k+k^{-1}\right)}e^{\pi\zeta_2 N}Z_k^{\mathrm{VI}}\left(N;M_1,M_2,M,\zeta_1,\zeta_2\right)&\\ =Z_k^{\mathrm{V}}\left(N;M_1,M_2',M,\zeta_1'\right),&\end{aligned}\tag{3.69}$$

where we defined

$$\begin{aligned}Z_k^{\mathrm{V}}\left(N;M_1,M_2',M,\zeta_1'\right)&=\frac{1}{N!\left(N+M\right)!}\int\prod_{n=1}^{N}\frac{d\mu_n}{2\pi}\prod_{n=1}^{N+M}\frac{d\nu_n}{2\pi}\\&\times\det\begin{pmatrix}\left[\left\langle\mu_m\middle|\widehat{D}_1^{\mathrm{V}}\middle|\nu_n\right\rangle\right]_{m,n}^{N\times(N+M)}\\\left[\left\langle\left\langle t_{0,M,r}\middle|\widehat{d}_1^{\mathrm{V}}\middle|\nu_n\right\rangle\right]_{r,n}^{M\times(N+M)}\end{pmatrix}\\&\times\det\begin{pmatrix}\left[\left\langle\nu_m\middle|\widehat{D}_2^{\mathrm{V}}\middle|\mu_n\right\rangle\right]_{m,n}^{(N+M)\times N}&\left[\left\langle\nu_m\middle|\widehat{d}_2^{\mathrm{V}}\middle|-t_{0,M,r}\right\rangle\right\rangle\right]_{m,r}^{(N+M)\times M}\end{pmatrix},\end{aligned}$$

with $\widehat{D}_i^{\mathrm{V}}$ and $\widehat{d}_i^{\mathrm{V}}$ defined as

$$\widehat{D}_1^{\mathrm{V}} = e^{-\frac{i\zeta_1'}{k}\widehat{x}} e^{\frac{k-M_1}{2k}\widehat{x}} \frac{\Phi_b\left(\frac{\widehat{x}}{2\pi b} - \frac{iM_1}{2b} + \frac{i}{2}b\right)}{\Phi_b\left(\frac{\widehat{x}}{2\pi b} + \frac{iM_1}{2b} - \frac{i}{2}b\right)} \frac{1}{2\cosh\frac{\widehat{p}-i\pi M}{2}} e^{\frac{i\zeta_1'}{k}\widehat{x}} e^{\frac{M_1}{2k}\widehat{x}} \frac{\Phi_b\left(\frac{\widehat{x}}{2\pi b} + \frac{iM_1}{2b}\right)}{\Phi_b\left(\frac{\widehat{x}}{2\pi b} - \frac{iM_1}{2b}\right)},$$

$$\widehat{d}_1^{\mathrm{V}} = e^{\frac{i\zeta_1'}{k}\widehat{x}} e^{\frac{M_1}{2k}\widehat{x}} \frac{\Phi_b\left(\frac{\widehat{x}}{2\pi b} + \frac{iM_1}{2b}\right)}{\Phi_b\left(\frac{\widehat{x}}{2\pi b} - \frac{iM_1}{2b}\right)},, \qquad \widehat{d}_2^{\mathrm{V}} = \frac{e^{\frac{i}{2\hbar}\widehat{x}^2}}{\Phi_b\left(\frac{\widehat{x}}{2\pi b} - \frac{iM_2'}{2b}\right)}$$

$$\widehat{D}_2^{\mathrm{V}} = e^{\frac{i\pi}{4}k - \frac{i\pi}{2}M_2'} \frac{e^{\frac{i}{2\hbar}\widehat{x}^2}}{\Phi_b\left(\frac{\widehat{x}}{2\pi b} - \frac{iM_2'}{2b}\right)} \frac{1}{2\cosh\frac{\widehat{p}+i\pi M}{2}} \frac{\Phi_b\left(\frac{\widehat{x}}{2\pi b} - \frac{iM_2'}{2b} + \frac{i}{2}b\right)}{e^{\frac{i}{2\hbar}\widehat{x}^2}}.$$

The grand partition function associated to Z_k^{V} is expected to be the matrix model representation of the $\mathfrak{q}$-Painlevé V tau function. We remark that, as was the case for Z_k^{VI} (3.65), Z_k^{V} can also be expressed without using the operator formalism by using (3.15) and (3.14) as

$$\begin{aligned} Z_k^{\mathrm{V}}(N; M_1, M_2', M, \zeta_1') &= \frac{e^{\frac{i\pi}{4}kN - \frac{i\pi}{2}M_2'N}}{N!\,(N+M)!} \int \prod_{n=1}^{N} \frac{d\mu_n}{2\pi k} \prod_{n=1}^{N+M} \frac{d\nu_n}{2\pi k} \\ &\times \prod_{n=1}^{N} e^{\left(-\frac{i\zeta_1'}{k} + \frac{k-M_1}{2k}\right)\mu_n - \frac{i}{4\pi k}\mu_n^2} \frac{\Phi_b\left(\frac{\mu_n}{2\pi b} - \frac{iM_1}{2b} + \frac{i}{2}b\right)\Phi_b\left(\frac{\mu_n}{2\pi b} - \frac{iM_2'}{2b} + \frac{i}{2}b\right)}{\Phi_b\left(\frac{\mu_n}{2\pi b} + \frac{iM_1}{2b} - \frac{i}{2}b\right)} \\ &\times \prod_{n=1}^{N+M} e^{\left(\frac{i\zeta_1'}{k} + \frac{M_1}{2k}\right)\nu_n + \frac{i}{4\pi k}\nu_n^2} \frac{\Phi_b\left(\frac{\nu_n}{2\pi b} + \frac{iM_1}{2b}\right)}{\Phi_b\left(\frac{\nu_n}{2\pi b} - \frac{iM_1}{2b}\right)\Phi_b\left(\frac{\nu_n}{2\pi b} - \frac{iM_2'}{2b}\right)} \\ &\times \left(\frac{\prod_{m<m'}^{N} 2\sinh\frac{\mu_m-\mu_{m'}}{2k} \prod_{n<n'}^{N+M} 2\sinh\frac{\nu_n-\nu_{n'}}{2k}}{\prod_{m=1}^{N}\prod_{n=1}^{N+M} 2\cosh\frac{\mu_m-\nu_n}{2k}}\right)^2. \end{aligned}$$

Combining (3.66), (3.68) and (3.69), we find that the coalescence limit VI $\to$ V (3.67) reduces the conjecture (3.66) into the following

$$\begin{aligned} &Z_k^{\mathrm{V}}(N; M_1, M_2', M, \zeta_1') = \\ &= Z_k^{\mathrm{V}}(0; M_1, M_2', M, \zeta_1') \frac{1}{N!} \int \prod_{n=1}^{N} d\mu_n \det\left(\left[\left\langle \mu_m \,\middle|\, \widehat{\rho}_k^{\mathrm{V}}(M_1, M_2', M, \zeta_1') \,\middle|\, \mu_n \right\rangle\right]_{m,n}^{N\times N}\right). \end{aligned} \tag{3.70}$$

To implement the remaining steps of the coalescence, we can simply repeat the same procedure. First, we consider the coalescence from $\mathfrak{q}$-Painlevé V to $\mathfrak{q}$-Painlevé III$_1$, where we send the top-right legs to infinity. This is achieved by taking

$$iM_2' \to \infty,$$

under which the positions of the external legs become

$$\begin{aligned} &\widetilde{m}_1 = -e^{-\pi i M + \pi i M_2'} = \infty, \quad \widetilde{m}_2 = -e^{\pi i(-M_1+M) - 2\pi\zeta_1'}, \quad \widetilde{m}_3 = -e^{\pi i(M_1-M) - 2\pi\zeta_1'}, \\ &\widetilde{t}_1 = -e^{-\pi i k + \pi i M_2'} = \infty, \quad \widetilde{t}_2 = -e^{\pi i M_1 - \pi i k}, \quad \widetilde{t}_3 = -e^{-\pi i M_1 + \pi i k}, \\ &\ell_4 = \frac{\widetilde{m}_4}{t_4} = e^{\pi i M - \pi i k}, \end{aligned}$$

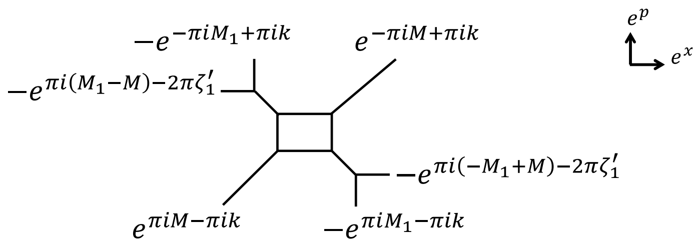

Figure 3.5: The asymptotic behavior of (3.71). This figure can be regarded as the five-brane web diagram of the 5d $\mathcal{N}=1$ SU (2) gauge theory with $N_f=2$. This diagram is the degeneration of the diagram in Fig.3.4.

with $\ell_1=\widetilde{m}_1\widetilde{t}_1^{-1}=e^{-\pi i M+\pi i k}$ finite; see Fig.3.5.

The quantum curve, with an appropriate overall rescaling, becomes

$$\lim_{iM_2'\to\infty} e^{-\frac{1}{2}i\pi M_2'}\left(\widehat{\rho}_k^{\mathrm{V}}\left(M_1,M_2',M,\zeta_1'\right)\right)^{-1}=\left(\widehat{\rho}_k^{\mathrm{III}_1}\left(M_1,M,\zeta_1'\right)\right)^{-1},$$

where we defined

$$\begin{aligned}
&\left(\widehat{\rho}_k^{\mathrm{III}_1}\left(M_1,M,\zeta_1'\right)\right)^{-1}\\
&=e^{-\frac{\pi i M_1}{2}+\pi\zeta_1'}e^{-\widehat{x}+\widehat{p}}+e^{\frac{\pi i M_1}{2}+\pi\zeta_1'-\pi i k}e^{\widehat{p}}\\
&\quad+e^{\frac{\pi i(M_1-2M)}{2}-\pi\zeta_1'}e^{-\widehat{x}}+\left[e^{-\frac{\pi i M_1}{2}-\pi\zeta_1'+\pi i a_1 M}+e^{-\frac{\pi i M_1}{2}+\pi\zeta_1'+\pi i a_2 M}\right]+e^{\frac{\pi i(M_1-2M)}{2}+\pi\zeta_1'}e^{\widehat{x}}\\
&\quad+e^{\frac{\pi i M_1}{2}-\pi\zeta_1'-\pi i k}e^{-\widehat{p}}+e^{-\frac{\pi i M_1}{2}-\pi\zeta_1'}e^{\widehat{x}-\widehat{p}}.
\end{aligned}\tag{3.71}$$

This operator can be regarded as the mirror curve corresponding to $\mathfrak{q}$-Painlevé III$_1$. Correspondingly, on the matrix model side we first multiply by an overall factor $e^{\frac{1}{2}i\pi M_2'N}$ and then take the $iM_2'\to\infty$ limit, to obtain

$$\lim_{iM_2'\to\infty} e^{\frac{1}{2}i\pi M_2'N}Z_k^{\mathrm{V}}\left(N;M_1,M_2',M,\zeta_1'\right)=Z_k^{\mathrm{III}_1}\left(N;M_1,M,\zeta_1'\right),$$

where we defined

$$\begin{aligned}
&Z_k^{\mathrm{III}_1}\left(N;M_1,M,\zeta_1'\right)\\
&=\frac{1}{N!\,(N+M)!}\int\prod_{n=1}^{N}\frac{d\mu_n}{2\pi}\prod_{n=1}^{N+M}\frac{d\nu_n}{2\pi}\det\begin{pmatrix}\left[\left\langle\mu_m\middle|\widehat{D}_1^{\mathrm{III}_1}\middle|\nu_n\right\rangle\right]_{m,n}^{N\times(N+M)}\\ \left[\left\langle\!\left\langle t_{0,M,r}\middle|\widehat{d}_1^{\mathrm{III}_1}\middle|\nu_n\right\rangle\right]_{r,n}^{M\times(N+M)}\end{pmatrix}\\
&\quad\times\det\left(\left[\left\langle\nu_m\middle|e^{\frac{i\pi}{4}k}e^{\frac{i}{2\hbar}\widehat{x}^2}\frac{1}{2\cosh\frac{\widehat{p}+i\pi M}{2}}e^{-\frac{i}{2\hbar}\widehat{x}^2}\middle|\mu_n\right\rangle\right]_{m,n}^{(N+M)\times N}\quad\left[\left\langle\nu_m\middle|e^{\frac{i}{2\hbar}\widehat{x}^2}\middle|-t_{0,M,r}\right\rangle\!\right\rangle\right]_{m,r}^{(N+M)\times M}\right).
\end{aligned}$$

In this limit, $\widehat{D}_1^{\mathrm{V}}$ and $\widehat{d}_1^{\mathrm{V}}$ are not affected:

$$\widehat{D}_1^{\mathrm{III}_1}=\widehat{D}_1^{\mathrm{V}},\quad\widehat{d}_1^{\mathrm{III}_1}=\widehat{d}_1^{\mathrm{V}}.$$

$Z_k^{\mathrm{III}_1}$ can be regarded as the matrix model corresponding to $\mathfrak{q}$-Painlevé III$_1$. Again

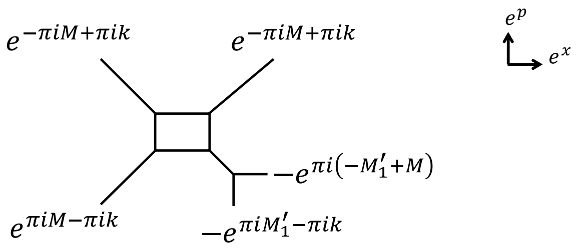

Figure 3.6: The asymptotic behavior of (3.74). This figure can be regarded as the five-brane web diagram of the 5d $\mathcal{N}=1$ SU(2) Yang-Mills theory with $N_f=1$. This diagram is degenerated one from the diagram in figure 3.5.

we can also write the integrand of $Z_k^{\mathrm{III}_1}$ in the product form:

$$\begin{aligned}
&Z_k^{\mathrm{III}_1}(N;M_1,M,\zeta_1') \\
&=\frac{e^{\frac{i\pi}{4}kN}}{N!\,(N+M)!}\int\prod_{n=1}^{N}\frac{d\mu_n}{2\pi k}\prod_{n=1}^{N+M}\frac{d\nu_n}{2\pi k}\prod_{n=1}^{N}e^{\left(-\frac{i\zeta_1'}{k}+\frac{k-M_1}{2k}\right)\mu_n-\frac{i}{4\pi k}\mu_n^2}\frac{\Phi_b\left(\frac{\mu_n}{2\pi b}-\frac{iM_1}{2b}+\frac{i}{2}b\right)}{\Phi_b\left(\frac{\mu_n}{2\pi b}+\frac{iM_1}{2b}-\frac{i}{2}b\right)} \\
&\quad\times\prod_{n=1}^{N+M}e^{\left(\frac{i\zeta_1'}{k}+\frac{M_1}{2k}\right)\nu_n+\frac{i}{4\pi k}\nu_n^2}\frac{\Phi_b\left(\frac{\nu_n}{2\pi b}+\frac{iM_1}{2b}\right)}{\Phi_b\left(\frac{\nu_n}{2\pi b}-\frac{iM_1}{2b}\right)}\left(\frac{\prod_{m<m'}^{N}2\sinh\frac{\mu_m-\mu_{m'}}{2k}\prod_{n<n'}^{N+M}2\sinh\frac{\nu_n-\nu_{n'}}{2k}}{\prod_{m=1}^{N}\prod_{n=1}^{N+M}2\cosh\frac{\mu_m-\nu_n}{2k}}\right)^2.
\end{aligned}$$

Combining the above results, the conjecture (3.66) is now reduced to the following

$$\begin{aligned}
&Z_k^{\mathrm{III}_1}(N;M_1,M,\zeta_1') \\
&=Z_k^{\mathrm{III}_1}(0;M_1,M,\zeta_1')\frac{1}{N!}\int\prod_{n=1}^{N}d\mu_n\det\left(\left[\langle\mu_m\,|\,\widehat{\rho}_k^{\mathrm{III}_1}(M_1,M,\zeta_1')\,|\,\mu_n\rangle\right]_{m,n}^{N\times N}\right)
\end{aligned}\tag{3.72}$$

We now consider the coalescence from $\mathfrak{q}$-Painlevé III_1 to $\mathfrak{q}$-Painlevé III_2. We send the top-left pair of external legs to $\widetilde{m}_3\to\infty$ and $\widetilde{t}_3\to 0$, by first shifting x,p as $\widehat{x}\to\widehat{x}-\pi\zeta_1'$, $\widehat{p}\to\widehat{p}-\pi\zeta_1'$ and then taking $\Lambda\to\infty$ with the following reparameterization

$$iM_1=iM_1'+\Lambda,\quad \zeta_1'=-\Lambda.\tag{3.73}$$

Under this procedure we are left with

$$\begin{aligned}
&\widetilde{m}_2=-e^{\pi i\left(-M_1'+M\right)}, \\
&\widetilde{t}_2=-e^{\pi iM_1'-\pi ik}, \\
&\ell_1=\frac{\widetilde{m}_1}{\widetilde{t}_1}=e^{-\pi iM+\pi ik},\quad \ell_4=\frac{\widetilde{m}_4}{\widetilde{t}_4}=e^{\pi iM-\pi ik},\quad \ell_3=\widetilde{m}_3\widetilde{t}_3=e^{-\pi iM+\pi ik},
\end{aligned}$$

see Fig.3.6. The quantum curve becomes

$$\lim_{\Lambda\to\infty}e^{\frac{1}{2}\pi\zeta_1'}\left(\widehat{\rho}_k^{\mathrm{III}_1}(M_1,M,\zeta_1')\right)^{-1}\Big|_{\widehat{x}\to\widehat{x}-\pi\zeta_1',\quad\widehat{p}\to\widehat{p}-\pi\zeta_1'}=\left(\widehat{\rho}_k^{\mathrm{III}_2}(M_1',M)\right)^{-1},$$

where

$$\begin{aligned}
\left(\widehat{\rho}_k^{\mathrm{III}_2}(M_1',M)\right)^{-1}=\;&e^{\frac{\pi iM_1'}{2}-\pi ik}e^{\widehat{p}} \\
&+e^{\frac{\pi i(M_1'-2M)}{2}}e^{-\widehat{x}}+e^{-\frac{\pi iM_1'}{2}+\pi ia_1M}+e^{\frac{\pi i(M_1'-2M)}{2}}e^{\widehat{x}} \\
&+e^{\frac{\pi iM_1'}{2}-\pi ik}e^{-\widehat{p}}+e^{-\frac{\pi iM_1'}{2}}e^{\widehat{x}-\widehat{p}}.
\end{aligned}\tag{3.74}$$

This operator can be regarded as the mirror curve corresponding to $\mathfrak{q}$-Painlevé III$_2$. On the matrix model side, we first multiply $e^{-\frac{1}{2}\pi\zeta_1' N}$, perform the similarity transformation

$$|\lambda\rangle\langle\lambda| \to e^{-\frac{i\zeta_1'}{2k}\widehat{x}}e^{\frac{i\zeta_1'}{2k}\widehat{p}}|\lambda\rangle\langle\lambda| e^{-\frac{i\zeta_1'}{2k}\widehat{p}}e^{\frac{i\zeta_1'}{2k}\widehat{x}},$$

both for $|\mu\rangle\langle\mu|$ and for $|\nu\rangle\langle\nu|$, and then take the limit $\Lambda\to\infty$ with the reparametrization (3.73). We end up with

$$\lim_{\Lambda\to\infty} e^{-\frac{i\pi M}{4k}\left(iM_1'+\Lambda\right)^2-\frac{i\pi M}{12}\left(k+k^{-1}\right)}e^{-\frac{1}{2}\pi\zeta_1' N}Z_k^{\mathrm{III}_1}\left(N;M_1,M,\zeta_1'\right) = Z_k^{\mathrm{III}_2}\left(N;M_1',M\right),$$

where

$$\begin{aligned}
&Z_k^{\mathrm{III}_2}\left(N;M_1',M\right)\\
&= \frac{1}{N!\,(N+M)!}\int\prod_{n=1}^{N}\frac{d\mu_n}{2\pi}\prod_{n=1}^{N+M}\frac{d\nu_n}{2\pi}\det\begin{pmatrix}\left[\left\langle\mu_m\middle|\widehat{D}_1^{\mathrm{III}_2}\middle|\nu_n\right\rangle\right]_{m,n}^{N\times(N+M)}\\ \left[\left\langle\left\langle t_{0,M,r}\middle|\widehat{d}_1^{\mathrm{III}_2}\middle|\nu_n\right\rangle\right]_{r,n}^{M\times(N+M)}\end{pmatrix}\\
&\quad\times\det\begin{pmatrix}\left[\left\langle\nu_m\middle|e^{\frac{i\pi}{4}k}e^{\frac{i}{2\hbar}\widehat{x}^2}\frac{1}{2\cosh\frac{\widehat{p}+i\pi M}{2}}e^{-\frac{i}{2\hbar}\widehat{x}^2}\middle|\mu_n\right\rangle\right]_{m,n}^{(N+M)\times N} & \left[\left\langle\nu_m\middle|e^{\frac{i}{2\hbar}\widehat{x}^2}\middle|-t_{0,M,r}\right\rangle\right\rangle\right]_{m,r}^{(N+M)\times M}\end{pmatrix}.
\end{aligned}$$

$\widehat{D}_1^{\mathrm{III}_1}$ and $\widehat{d}_1^{\mathrm{III}_1}$ are changed into

$$\widehat{D}_1^{\mathrm{III}_2} = e^{\frac{i\pi}{4}k}e^{-\frac{1}{2}i\pi M_1'}\frac{\Phi_b\left(\frac{\widehat{x}}{2\pi b}-\frac{iM_1'}{2b}+\frac{i}{2}b\right)}{e^{\frac{i}{2\hbar}\widehat{x}^2}}\frac{1}{2\cosh\frac{\widehat{p}-i\pi M}{2}}\frac{e^{\frac{i}{2\hbar}\widehat{x}^2}}{\Phi_b\left(\frac{\widehat{x}}{2\pi b}-\frac{iM_1'}{2b}\right)},$$

$$\widehat{d}_1^{\mathrm{III}_2} = \frac{e^{\frac{i}{2\hbar}\widehat{x}^2}}{\Phi_b\left(\frac{\widehat{x}}{2\pi b}-\frac{iM_1'}{2b}\right)}.$$

In the product form we have

$$\begin{aligned}
&Z_k^{\mathrm{III}_2}\left(N;M_1,M\right)\\
&= \frac{e^{\frac{i\pi}{2}kN-\frac{\pi iM_1'N}{2}}}{N!\,(N+M)!}\int\prod_{n=1}^{N}\frac{d\mu_n}{2\pi k}\prod_{n=1}^{N+M}\frac{d\nu_n}{2\pi k}\prod_{n=1}^{N}e^{-\frac{i}{2\pi k}\mu_n^2}\Phi_b\left(\frac{\mu_n}{2\pi b}-\frac{iM_1'}{2b}+\frac{i}{2}b\right)\\
&\quad\times\prod_{n=1}^{N+M}e^{\frac{i}{2\pi k}\nu_n^2}\frac{1}{\Phi_b\left(\frac{\nu_n}{2\pi b}-\frac{iM_1'}{2b}\right)}\left(\frac{\prod_{m<m'}^{N}2\sinh\frac{\mu_m-\mu_{m'}}{2k}\prod_{n<n'}^{N+M}2\sinh\frac{\nu_n-\nu_{n'}}{2k}}{\prod_{m=1}^{N}\prod_{n=1}^{N+M}2\cosh\frac{\mu_m-\nu_n}{2k}}\right)^2.
\end{aligned}$$

$Z_k^{\mathrm{III}_2}$ can be regarded as the matrix model corresponding to $\mathfrak{q}$-Painlevé III$_2$. Combining the above results, we obtain

$$Z_k^{\mathrm{III}_2}\left(N;M_1',M\right) = Z_k^{\mathrm{III}_2}\left(0;M_1',M\right)\frac{1}{N!}\int\prod_{n=1}^{N}d\mu_n\det\left(\left[\left\langle\mu_m\middle|\widehat{\rho}_k^{\mathrm{III}_2}\left(M_1',M\right)\middle|\mu_n\right\rangle\right]_{m,n}^{N\times N}\right). \tag{3.75}$$

Finally, we consider the coalescence from $\mathfrak{q}$-Painlevé III$_2$ to $\mathfrak{q}$-Painlevé III$_3$. We can send the bottom-right legs to $\widetilde{m}_2\to 0$ and $\widetilde{t}_2\to\infty$ by taking the $iM_1'\to\infty$ limit, which leaves $\ell_2=\widetilde{m}_2\widetilde{t}_2=e^{\pi iM-\pi ik}$ finite; see Fig.3.7.

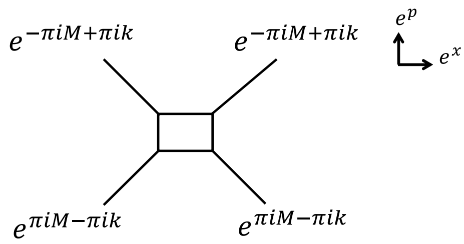

Figure 3.7: The asymptotic behavior of (3.76). This figure can be regarded as the five-brane web diagram of the 5d $\mathcal{N}=1$ SU (2) Yang-Mills theory. This diagram is the degeneration of the diagram in Fig.3.6.

The quantum curve becomes

$$\lim_{iM_1'\to\infty} e^{-\frac{1}{2}i\pi M_1'}\left(\widehat{\rho}_k^{\mathrm{III}_2}\left(M_1',M\right)\right)^{-1}=\left(\widehat{\rho}_k^{\mathrm{III}_3}\left(M\right)\right)^{-1},$$

where

$$\left(\widehat{\rho}_k^{\mathrm{III}_3}\left(M\right)\right)^{-1}=e^{-\pi i M}e^{\widehat{x}}+e^{-\pi i M}e^{-\widehat{x}}+e^{-\pi i k}e^{\widehat{p}}+e^{-\pi i k}e^{-\widehat{p}}. \tag{3.76}$$

This operator can be regarded as the mirror curve corresponding to $\mathfrak{q}$-Painlevé III_3. On the matrix model side we multiply the overall factor $e^{\frac{1}{2}i\pi M_1' N}$ and take the limit $iM_1'\to\infty$ to obtain

$$\lim_{iM_1'\to\infty} e^{\frac{1}{2}i\pi M_1' N} Z_k^{\mathrm{III}_2}\left(N;M_1',M\right)=Z_k^{\mathrm{III}_3}\left(N;M\right),$$

where

$$\begin{aligned}
&Z_k^{\mathrm{III}_3}\left(N;M\right)\\
&=\frac{1}{N!\left(N+M\right)!}\int\prod_{n=1}^{N}\frac{d\mu_n}{2\pi}\prod_{n=1}^{N+M}\frac{d\nu_n}{2\pi}\det\left(\begin{matrix}\left[\left\langle\mu_m\left|e^{\frac{i\pi}{4}k}e^{-\frac{i}{2\hbar}\widehat{x}^2}\frac{1}{2\cosh\frac{\widehat{p}-i\pi M}{2}}e^{\frac{i}{2\hbar}\widehat{x}^2}\right|\nu_n\right\rangle\right]_{m,n}^{N\times(N+M)}\\ \left[\left\langle\!\left\langle t_{0,M,r}\left|e^{\frac{i}{2\hbar}\widehat{x}^2}\right|\nu_n\right\rangle\right]_{r,n}^{M\times(N+M)}\end{matrix}\right)\\
&\times\det\left(\left[\left\langle\nu_m\left|e^{\frac{i\pi}{4}k}e^{\frac{i}{2\hbar}\widehat{x}^2}\frac{1}{2\cosh\frac{\widehat{p}+i\pi M}{2}}e^{-\frac{i}{2\hbar}\widehat{x}^2}\right|\mu_n\right\rangle\right]_{m,n}^{(N+M)\times N}\quad\left[\left\langle\nu_m\left|e^{\frac{i}{2\hbar}\widehat{x}^2}\right|-t_{0,M,r}\right\rangle\!\right\rangle\right]_{m,r}^{(N+M)\times M}\right).
\end{aligned}\tag{3.77}$$

As we mentioned in the beginning of this section, this matrix model coincides with the partition function of ABJM theory with the gauge group $\mathrm{U}(N)_{2k}\times\mathrm{U}(N+M)_{-2k}$. This integral can be regarded as the matrix model corresponding to $\mathfrak{q}$-Painlevé III_3. Combining the above results, we obtain

$$Z_k^{\mathrm{III}_3}\left(N;M\right)=Z_k^{\mathrm{III}_3}\left(0;M\right)\frac{1}{N!}\int\prod_{n=1}^{N}\frac{d\mu_n}{2\pi}\det\left(\left[\left\langle\mu_m\left|\widehat{\rho}_k^{\mathrm{III}_3}\left(M\right)\right|\mu_n\right\rangle\right]_{m,n}^{N\times N}\right) \tag{3.78}$$

In this section we derived the relations between the matrix models and the quantum curves in (3.66), (3.70), (3.72), (3.75) and (3.78). We expect that the grand canonical partition function of these matrix models provide a conjectural Fredholm determinant expression of the τ-functions for the relevant $\mathfrak{q}$-Painlevé equations, at least at the specific values of the moduli that we are analysing.

The result (3.78) we get is consistent with previous findings on $\mathfrak{q}$-PIII$_3$ equation. Indeed (3.76) is the quantum curve associated with local $\mathbb{P}^1\times\mathbb{P}^1$. The spectral determinant of this quantum curve was already known to satisfy the bilinear equation of the $\mathfrak{q}$-Painlevé III$_3$ [46]. On the other hand, the matrix model (3.77) we find $Z_k^{\mathrm{III}_3}(N;M)$ is the one associated with ABJM theory, as expected from the results in [46]. The identification of the matrix models goes along the same lines we followed in the beginning of the Section, eqs.(3.64) and (3.65). We get

$$Z_k^{\mathrm{III}_3}(N;M)=\frac{e^{\frac{i\pi}{2}kN}}{N!\,(N+M)!}\int\prod_{n=1}^{N}\frac{d\mu_n}{2\pi}\prod_{n=1}^{N+M}\frac{d\nu_n}{2\pi}e^{\frac{ik}{2\pi}\sum_{n=1}^{N}\mu_n^2-\frac{ik}{2\pi}\sum_{n=1}^{N+M}\nu_n^2}$$
$$\times\frac{\prod_{n<n'}^{N}\left(2\sinh\frac{\mu_n-\mu_{n'}}{2}\right)^2}{\prod_{m=1}^{N}\prod_{n=1}^{N+M}2\cosh\frac{\mu_m-\nu_n}{2}}\frac{\prod_{n<n'}^{N+M}\left(2\sinh\frac{\nu_n-\nu_{n'}}{2}\right)^2}{\prod_{m=1}^{N+M}\prod_{n=1}^{N}2\cosh\frac{\nu_m-\mu_n}{2}},\qquad(3.79)$$

which is the matrix model associated to the ABJM theory with $\mathrm{U}(N)_{2k}\times\mathrm{U}(N+M)_{-2k}$ gauge group. Notice that the Chern-Simons level gets renormalised along the flow to $k^{\mathrm{ABJM}}=2k$. Let us also observe that in the relation between ABJM theory and, $\mathfrak{q}$-Painlevé III$_3$, the rank parameter M corresponds to the time variable, while this is not the case for the $\mathfrak{q}$-Painlevé VI equation we start from.

For the particular case (3.78) we can actually prove our conjecture (3.66). Indeed in this case an explicit calculation of the inverse of the spectral density matrix $\left(\widehat{\rho}_k^{\mathrm{III}_3}\right)^{-1}$ can be performed by showing that this is indeed the quantum Seiberg-Witten curve for the local $\mathbb{P}^1\times\mathbb{P}^1$ geometry $\widehat{\mathcal{O}}_{\mathbb{P}^1\times\mathbb{P}^1}$. By using (3.25), we can show that [167]

$$\widehat{\rho}_k^{\mathrm{III}_3}(M)=\widehat{\mathcal{O}}_{\mathbb{P}^1\times\mathbb{P}^1}^{-1}=e^{\frac{i\pi}{2}k}\frac{\prod_j^M 2\sinh\frac{\widehat{x}+t_{0,M,j}}{4k}}{2\cosh\frac{\widehat{x}}{2}}\frac{1}{2\cosh\frac{\widehat{p}}{2}}\frac{1}{\prod_j^M 2\cosh\frac{\widehat{x}+t_{0,M,j}}{4k}},$$

so that (3.78) becomes

$$Z_k^{\mathrm{III}_3}(N;M)=\frac{i^{\frac{1}{2}M^2}e^{i\theta_{2k}(M,0)}}{N!}Z_{2k,M}^{(\mathrm{CS})}\int\prod_{n=1}^{N}\frac{d\mu_n}{2\pi}$$
$$\times\det\left(\left[\left\langle\mu_m\left|e^{\frac{i\pi}{2}k}\frac{1}{\prod_j^M 2\cosh\frac{\widehat{x}+t_{0,M,j}}{4k}}\frac{1}{2\cosh\frac{\widehat{p}}{2}}\frac{\prod_j^M 2\sinh\frac{\widehat{x}+t_{0,M,j}}{4k}}{2\cosh\frac{\widehat{x}}{2}}\right|\mu_n\right\rangle\right]_{m,n}^{N\times N}\right).$$

The above can be shown to coincide with (3.79) [142]. This is a non-trivial check of the conjecture (3.31). On the other hand, (3.70), (3.72) and (3.75) provide new conjectural relations between matrix models and quantum curves. Let us also stress that the normalisation we choose to define Z_k^{VI} in (3.64) is consistent with the one found in [46].

Finally, let us notice that the conjectured relations (3.70), (3.72) and (3.75) can actually be proved in the $M=0$ case by making use of the results of section 3.2.3. In these cases, the matrix models can be written in the form of (3.28) by using gluing formula (3.20), and the inverse of the density matrix coincides with the quantum Seiberg-Witten curve. Explicitly, (3.23) and (3.26) are equal as explained in section 3.2.3. For $M=0$, we can show that the other conjectured relations can be proved at the operator level in a similar way. For example, to show (3.70), it is enough to prove

$$\begin{aligned}
&\Bigg[e^{-\frac{i\zeta_1'}{k}\widehat{x}} e^{\frac{k-M_1}{2k}\widehat{x}} \frac{\Phi_b\left(\frac{\widehat{x}}{2\pi b} - \frac{iM_1}{2b} + \frac{i}{2}b\right)}{\Phi_b\left(\frac{\widehat{x}}{2\pi b} + \frac{iM_1}{2b} - \frac{i}{2}b\right)} \frac{1}{2\cosh\frac{\widehat{p}}{2}} e^{\frac{i\zeta_1'}{k}\widehat{x}} e^{\frac{M_1}{2k}\widehat{x}} \frac{\Phi_b\left(\frac{\widehat{x}}{2\pi b} + \frac{iM_1}{2b}\right)}{\Phi_b\left(\frac{\widehat{x}}{2\pi b} - \frac{iM_1}{2b}\right)} \\
&\quad \times e^{\frac{i\pi}{4}k} e^{-\frac{1}{2}i\pi M_2'} \frac{e^{\frac{i}{2\hbar}\widehat{x}^2}}{\Phi_b\left(\frac{\widehat{x}}{2\pi b} - \frac{iM_2'}{2b}\right)} \frac{1}{2\cosh\frac{\widehat{p}}{2}} \frac{\Phi_b\left(\frac{\widehat{x}}{2\pi b} - \frac{iM_2'}{2b} + \frac{i}{2}b\right)}{e^{\frac{i}{2\hbar}\widehat{x}^2}} \Bigg]^{-1} \\
&= \left[\left(e^{-\frac{1}{2}i\pi M_2'} e^{\frac{1}{2}\widehat{x}} + e^{\frac{1}{2}i\pi M_2'} e^{-\frac{1}{2}\widehat{x}} \right) e^{\frac{1}{2}\widehat{p}} + e^{\frac{1}{2}i\pi M_2'} e^{\frac{1}{2}\widehat{x}} e^{-\frac{1}{2}\widehat{p}} \right] \\
&\quad \times \left[e^{\pi\zeta_1'} e^{\frac{1}{2}\widehat{p}} \left(e^{\frac{1}{2}i\pi M_1} e^{\frac{1}{2}\widehat{x}} + e^{-\frac{1}{2}i\pi M_1} e^{-\frac{1}{2}\widehat{x}} \right) + e^{-\pi\zeta_1'} e^{-\frac{1}{2}\widehat{p}} \left(e^{-\frac{1}{2}i\pi M_1} e^{\frac{1}{2}\widehat{x}} + e^{\frac{1}{2}i\pi M_1} e^{-\frac{1}{2}\widehat{x}} \right) \right].
\end{aligned}$$

The right hand side is the quantum curve (3.68) with $M = 0$. This identity can be proved by using (3.24) and

$$\begin{aligned}
&e^{-\frac{i\pi}{4}k} e^{\frac{1}{2}i\pi M_2'} \frac{e^{\frac{i}{2\hbar}\widehat{x}^2}}{\Phi_b\left(\frac{\widehat{x}}{2\pi b} - \frac{iM_2'}{2b} + \frac{i}{2}b\right)} e^{\pm\frac{1}{2}\widehat{p}} \frac{\Phi_b\left(\frac{\widehat{x}}{2\pi b} - \frac{iM_2'}{2b}\right)}{e^{\frac{i}{2\hbar}\widehat{x}^2}} \\
&= e^{-\frac{i\pi}{4}k} e^{\frac{1}{2}i\pi M_2'} \frac{\Phi_b\left(\frac{\widehat{x}}{2\pi b} - \frac{iM_2'}{2b} \mp \frac{i}{2}b\right)}{\Phi_b\left(\frac{\widehat{x}}{2\pi b} - \frac{iM_2'}{2b} + \frac{i}{2}b\right)} e^{\pm\frac{1}{2}(\widehat{p}-\widehat{x})} = \begin{cases} \left(e^{-\frac{1}{2}i\pi M_2'} e^{\frac{1}{2}\widehat{x}} + e^{\frac{1}{2}i\pi M_2'} e^{-\frac{1}{2}\widehat{x}} \right) e^{\frac{1}{2}\widehat{p}} \\ e^{\frac{1}{2}i\pi M_2'} e^{\frac{1}{2}\widehat{x}} e^{-\frac{1}{2}\widehat{p}} \end{cases}.
\end{aligned}$$

We can also prove (3.72) and (3.75) at the operator level when $M = 0$.

3.2.6 Coalescence limits: $\mathfrak{q}$-difference equations

Having discussed the coalescence limits at the level of the matrix models, we now turn to the $\mathfrak{q}$-difference equations their grand canonical partition function conjecturally satisfy when identified with $\mathfrak{q}$-Painlevé τ-functions. In this section we consider the coalescence limit of the τ-functions in the short-time expansion as in (3.1). This is the large-radius expansion of the topological string whose gauge theory counterpart is the $d = 4$ $\mathcal{N} = 1$ SU(2) with four to zero flavors in the electric frame of its Coulomb branch. Although natural from the perspective of gauge theory, this limit is not ideal from the point of view of the matrix model, which we expect to be describing the dual magnetic frame [44, 45]. The main issue is the rescaling of the parameter s. Based on the TS/ST correspondence, we expect the spectral determinant to be calculating the τ-function (3.1) at $s = 1$ at any step in the coalescence series. This can be consistently implemented in a given set of identifications of parameters (3.37) by choosing a suitable set of Weyl transformations w so that the s parameter doesn't flow along the coalescence.

In the following we first briefly recall the standard flow as in [198], then we discuss the alternative flow in a concrete choice of parameterization.

3.2.6.1 The two types of flow

In 5 dimensions, we find there are two different ways to decouple a massive hypermultiplet. This section will serve as an introduction to details of both at the level of τ-functions. With that in mind, let us write a generic 5d Nekrasov-Okounkov τ-function, focusing on one particular hypermultiplet of mass θ whose decoupling will be described in the two schemes:

$$\begin{aligned}
\tau\left(\theta,\vec{\theta}_{\text{rest}};s,\sigma,t\right) &= \sum_{n\in\mathbb{Z}} s^n t^{(\sigma+n)^2} C\left(\theta,\vec{\theta}_{\text{rest}};\sigma+n\right) Z\left(\theta,\vec{\theta}_{\text{rest}};\sigma+n,t\right), \\
C\left(\theta,\vec{\theta}_{\text{rest}};\sigma\right) &= \left(\prod_{\epsilon=\pm} G_{\mathfrak{q}}(1-\theta+\epsilon\sigma)\right) C_{\text{rest}}\left(\vec{\theta}_{\text{rest}};\sigma\right), \\
Z\left(\theta,\vec{\theta}_{\text{rest}};\sigma,t\right) &= \sum_{\lambda_+,\lambda_-} t^{|\lambda_+|+|\lambda_-|} \prod_{\epsilon=\pm} N_{\phi,\lambda_\epsilon}\left(\mathfrak{q}^{-\theta+\epsilon\sigma}\right) z_{\text{rest};\lambda_+,\lambda_-}\left(\vec{\theta}_{\text{rest}};\sigma\right).
\end{aligned}$$

Here, C_{rest} and $z_{\text{rest};\lambda_+,\lambda_-}$ are respectively the one-loop and instanton terms describing the contributions of the rest of the theory.

Let us first recall the standard holomorphic decoupling limit of a massive hypermultiplet

$$\mathfrak{q}^{-\theta}\to\infty, \quad t\to 0, \quad t_1 = t\mathfrak{q}^{-\theta} \text{ finite.}$$

We will realize this limit by setting $\theta=\vartheta+i\Lambda$, $t=t_1\mathfrak{q}^{\vartheta+i\Lambda}$. In our case, $\mathfrak{q}=e^{2\pi i/k}$ with $k>0$, so $\mathfrak{q}^{i\Lambda}=e^{-\frac{2\pi i}{k}\Lambda}\to 0$ as $\Lambda\to\infty$. Up to some difference of imaginary units, this is the case worked out in [198], the main points of the calculation whereof we recall. To begin with, consider the instanton counting part $Z\left(\theta,\vec{\theta}_{\text{rest}};\sigma,t\right)$. Since

$$N_{\lambda,\mu}(u) = \prod_{c\in\lambda}\left(1-\mathfrak{q}^{-l_\lambda(c)-a_\mu(c)-1}u\right)\prod_{c\in\mu}\left(1-\mathfrak{q}^{l_\mu(c)+a_\lambda(c)+1}u\right),$$

we have

$$\mathfrak{q}^{|\lambda|(\vartheta+i\Lambda)} N_{\phi,\lambda}(u\mathfrak{q}^{-\vartheta-i\Lambda}) = \prod_{c\in\mu}\left(\mathfrak{q}^{\vartheta+i\Lambda}-\mathfrak{q}^{l_\mu(c)+a_\lambda(c)+1}u\right)\to f_\lambda(u^{-1}),$$

where $f_\lambda(u)=\prod_{c\in\mu}\left(-\mathfrak{q}^{l_\mu(c)+a_\lambda(c)+1}u^{-1}\right)$. This term is nothing but the five dimensional Chern-Simons coupling.

Slightly tougher is the calculation of the one-loop part, and here also the n-dependence of the summands has to be taken into account. The functional relations of Appendix A enable us to write for a shift $n\in\mathbb{Z}$

$$G_{\mathfrak{q}}\left(1+x\pm(\sigma+n)\right) = G_{\mathfrak{q}}\left(1+x\pm\sigma\right)\left[\frac{\Gamma_{\mathfrak{q}}(x+\sigma)}{\Gamma_{\mathfrak{q}}(x-\sigma)}\right]^n \prod_{j=0}^{|n|-1}[x+\operatorname{sgn}(n)\sigma]\prod_{k=1}^{j}[x\pm(\sigma+k)], \tag{3.80}$$

a relation the reader should make a mental note of, as we will return to it again from a different perspective. Presently though, we have $x=-\vartheta-i\Lambda$ and want the limit $\Lambda\to\infty$. Recalling the definition of $\mathfrak{q}$-numbers, we have for any $\alpha\in\mathbb{C}$

$$[\alpha-i\Lambda] = \frac{1-\mathfrak{q}^{\alpha-i\Lambda}}{1-\mathfrak{q}} \sim \frac{\mathfrak{q}^{\alpha-i\Lambda}}{\mathfrak{q}-1},$$

as $\Lambda \to \infty$. Before returning to (3.80), we find it useful to rewrite

$$\frac{\Gamma_{\mathfrak{q}}(x+\sigma)}{\Gamma_{\mathfrak{q}}(x-\sigma)} = \frac{\Gamma_{\mathfrak{q}}(x+\sigma)\Gamma_{\mathfrak{q}}(1+x-\sigma)}{\Gamma_{\mathfrak{q}}(1+x+\sigma)\Gamma_{\mathfrak{q}}(x-\sigma)} \cdot \frac{\Gamma_{\mathfrak{q}}(1+x+\sigma)}{\Gamma_{\mathfrak{q}}(1+x-\sigma)} = \frac{1-\mathfrak{q}^{x-\sigma}}{1-\mathfrak{q}^{x+\sigma}} \frac{\Gamma_{\mathfrak{q}}(1+x+\sigma)}{\Gamma_{\mathfrak{q}}(1+x-\sigma)}$$
$$\sim \mathfrak{q}^{-2\sigma} \frac{\Gamma_{\mathfrak{q}}(1+x+\sigma)}{\Gamma_{\mathfrak{q}}(1+x-\sigma)},$$

and define

$$\check{s} = s(\mathfrak{q}-1)^{-2\sigma} q^{2\sigma(\vartheta+i\Lambda-\frac{1}{2})} \frac{\Gamma_{\mathfrak{q}}(1-\vartheta-i\Lambda+\sigma)}{\Gamma_{\mathfrak{q}}(1-\vartheta-i\Lambda-\sigma)}. \tag{3.81}$$

Collecting it all and returning to (3.80) we have asymptotically

$$\begin{aligned} G_{\mathfrak{q}}\left(1-\vartheta-i\Lambda \pm (\sigma+n)\right) &\sim G_{\mathfrak{q}}\left(1-\vartheta-i\Lambda \pm \sigma\right) \left[\frac{\Gamma_{\mathfrak{q}}(-\vartheta-i\Lambda+\sigma)}{\Gamma_{\mathfrak{q}}(-\vartheta-i\Lambda-\sigma)}\right]^n (\mathfrak{q}-1)^{-n^2} q^{n^2(-\vartheta-i\Lambda)+n\sigma} \\ &= G_{\mathfrak{q}}\left(1-\vartheta-i\Lambda \pm \sigma\right) \left(\frac{\check{s}}{s}\right)^n (\mathfrak{q}-1)^{-(\sigma+n)^2+\sigma^2} \left[q^{-\vartheta-i\Lambda}\right]^{(\sigma+n)^2-\sigma^2} \\ &= X^{(1)}(\theta)^{-1} \left(\frac{\check{s}}{s}\right)^n (\mathfrak{q}-1)^{-(\sigma+n)^2} \left[\frac{t_1}{t}\right]^{(\sigma+n)^2}, \end{aligned}$$

where in the last line we defined the n-independent factor

$$X^{(1)}\left(\theta;\sigma\right)^{-1} = (\mathfrak{q}-1)^{\sigma^2} \mathfrak{q}^{\theta\sigma^2} G_{\mathfrak{q}}\left(1-\vartheta-i\Lambda \pm \sigma\right).$$

It's now clear that as $\Lambda \to \infty$,

$$\tau\left(\theta, \vec{\theta}_{\text{rest}}; s, \sigma, t\right) \to X^{(1)}\left(\theta;\sigma\right)^{-1} \tau^{\text{el}}\left(\vec{\theta}_{\text{rest}}; \check{s}, \sigma, t_1\right), \tag{3.82}$$

where

$$\begin{aligned} \tau^{\text{el}}\left(\vec{\theta}_{\text{rest}}; s, \sigma, t\right) &= \sum_{n\in\mathbb{Z}} s^n t^{(\sigma+n)^2} C^{\text{el}}\left(\vec{\theta}_{\text{rest}}; \sigma+n\right) Z^{\text{el}}\left(\vec{\theta}_{\text{rest}}; \sigma+n, t\right), \\ C^{\text{el}}\left(\vec{\theta}_{\text{rest}}; \sigma\right) &= (\mathfrak{q}-1)^{-\sigma^2} C_{\text{rest}}\left(\vec{\theta}_{\text{rest}}; \sigma\right), \\ Z^{\text{el}}\left(\vec{\theta}_{\text{rest}}; \sigma, t\right) &= \sum_{\lambda_+,\lambda_-} t^{|\lambda_+|+|\lambda_-|} \prod_{\epsilon=\pm} f_{\lambda_\epsilon}\left(\mathfrak{q}^{\epsilon\sigma}\right) z_{\text{rest};\lambda_+,\lambda_-}\left(\vec{\theta}_{\text{rest}}; \sigma\right), \end{aligned}$$

and the superscript "el" stands for the fact that the holomorphic decoupling limit is studied in the electric frame of the gauge theory in five dimensions. Being the relevant $\mathfrak{q}$-Painlevé equation homogeneous, the multiplicative redefinition in (3.82) preserves it. Notice that the holomorphic decoupling limit implies a redefinition of the s parameter as in (3.81).

A different kind of decoupling limit has to be defined if one studies the flow of solutions of the $\mathfrak{q}$-Painlevé equation with fixed initial condition $s = 1$ before and after the coalescence. We now turn to discuss a class of decoupling limits suitable in the sense above. Consider the limit

$$\mathfrak{q}^{-\theta} \to 0,$$

while keeping all other parameters fixed. In the following, we put $\theta = \vartheta - i\Lambda$ and consider the $\Lambda \to \infty$ limit. Consider again the instanton counting part $Z\left(\theta, \vec{\theta}_{\text{rest}}; \sigma, t\right)$. Since

$$N_{\lambda,\mu}(0) = \prod_{c\in\lambda} \left(1-\mathfrak{q}^{-l_\lambda(c)-a_\mu(c)-1}\cdot 0\right) \prod_{c\in\mu} \left(1-\mathfrak{q}^{l_\mu(c)+a_\lambda(c)+1}\cdot 0\right) = 1,$$

we are immediately left with just the $z_{\text{rest};\lambda_+,\lambda_-}$ part in each term of the sum. We now turn to the one-loop part, referring back to (3.80), this time with $x = -\vartheta + i\Lambda$. Consider first the $\mathfrak{q}$-numbers:

$$[\alpha + i\Lambda] = \frac{1-\mathfrak{q}^{\alpha+i\Lambda}}{1-\mathfrak{q}} \to \frac{1}{1-\mathfrak{q}}.$$

Therefore, the entire nested product in (3.80) gives $(1-\mathfrak{q})^{-n^2}$. Next consider the $\mathfrak{q}$-Gamma functions. Using their definition, we can write

$$\frac{\Gamma_{\mathfrak{q}}(x+\sigma)}{\Gamma_{\mathfrak{q}}(x-\sigma)} = (1-\mathfrak{q})^{-2\sigma}\frac{(q^{x-\sigma};q)_\infty}{(q^{x+\sigma};q)_\infty} \to (1-\mathfrak{q})^{-2\sigma},$$

since $\mathfrak{q}^x = \mathfrak{q}^{-\vartheta-i\Lambda} \to 0$ and $(0;\mathfrak{q})_\infty = 1$. All in all, we have

$$G_{\mathfrak{q}}\left(1-\vartheta+i\Lambda \pm (\sigma+n)\right) \sim G_{\mathfrak{q}}\left(1-\vartheta+i\Lambda\pm\sigma\right)(1-\mathfrak{q})^{-(\sigma+n)^2+\sigma^2}.$$

Therefore, letting

$$X^{(2)}\left(\theta;\sigma\right)^{-1} = (1-\mathfrak{q})^{\sigma^2} G_{\mathfrak{q}}\left(1-\theta\pm\sigma\right),$$

we have as $\Lambda\to\infty$,

$$\tau\left(\theta,\vec{\theta}_{\text{rest}};s,\sigma,t\right) \to X^{(2)}\left(\theta;\sigma\right)^{-1}\tau^{\text{magn}}\left(\vec{\theta}_{\text{rest}};s,\sigma,t\right)$$

where

$$\begin{aligned}
\tau^{\text{magn}}\left(\vec{\theta}_{\text{rest}};s,\sigma,t\right) &= \sum_{n\in\mathbb{Z}} s^n t^{(\sigma+n)^2} C^{\text{magn}}\left(\vec{\theta}_{\text{rest}};\sigma+n\right) Z^{\text{magn}}\left(\vec{\theta}_{\text{rest}};\sigma+n,t\right),\\
C^{\text{magn}}\left(\vec{\theta}_{\text{rest}};\sigma\right) &= (1-\mathfrak{q})^{-\sigma^2} C_{\text{rest}}\left(\vec{\theta}_{\text{rest}};\sigma\right),\\
Z^{\text{magn}}\left(\vec{\theta}_{\text{rest}};\sigma,t\right) &= \sum_{\lambda_+,\lambda_-} t^{|\lambda_+|+|\lambda_-|} z_{\text{rest};\lambda_+,\lambda_-}\left(\vec{\theta}_{\text{rest}};\sigma\right).
\end{aligned}$$

The superscript "magn" indicates that the limit is suitable to preserve the matrix model realisation of the τ-function, which conjecturally describes the gauge theory in the magnetic frame.

3.2.6.2 Coalescence limits in the electric frame

Having illustrated the calculation of coalescence limits at a generic step, we turn to concrete realizations of our dictionary. We first show that the dictionary (3.36) reproduces the coalescence limits in the electric frame as described in [198]. Formula (3.36) reads in components as

$$\theta_0 = \frac{M_2 - M_1 - 2i\zeta_1}{4}, \quad \theta_1 = \frac{2M - M_1 - M_2 + 2i\zeta_2}{4}, \quad \theta_t = \frac{2M - M_1 - M_2 - 2i\zeta_2}{4},$$
$$\theta_\infty = \frac{M_1 - M_2 - 2i\zeta_1}{4}, \quad t = \mathfrak{q}^{k+M-M_1-M_2}, \quad \mathfrak{q} = e^{\frac{2\pi i}{k}}.$$

Let us now examine the holomorphic decoupling limits in the matrix model parameters. First, the redefinition $M_2 = M_2' - 2i\Lambda$, $\zeta_1 = \zeta_1' - \Lambda$, $\zeta_2 = \Lambda$ leads to

$$\mathfrak{q}^{-\theta_1-\theta_\infty} \to \infty, \quad t\to 0, \quad t_1 = t\mathfrak{q}^{-\theta_1-\theta_\infty} = \mathfrak{q}^{\frac{M-2M_1-M_2'+i\zeta_1'}{2}}, \quad \Lambda\to\infty,$$

which corresponds to §3 of [198]. Next, it can be checked that setting $M_2' = -i\Lambda$ leads to

$$\mathfrak{q}^{-\theta_t+\theta_0} \to \infty, \quad t_1 \to 0, \quad t_2 = t_1 \mathfrak{q}^{-\theta_t+\theta_0} = \mathfrak{q}^{-M_1}, \quad \Lambda \to \infty,$$

which corresponds to §4 of [198], up to a redefinition $\theta_0 \leftrightarrow -\theta_0$, which is a symmetry of the original $\mathfrak{q}-$PVI tau function. Next, it can be checked that $M_1 = M_1' - i\Lambda$, $\zeta_1' = -\Lambda$ leads to

$$\mathfrak{q}^{-\theta_t-\theta_0} \to \infty, \quad t_2 \to 0, \quad t_3 = t_2 \mathfrak{q}^{-\theta_t-\theta_0} = \mathfrak{q}^{-\frac{M+M_1'}{2}}, \quad \Lambda \to \infty,$$

which corresponds to §5 of [198]. Finally, one can check that $M_1' = -i\Lambda$ leads to

$$\mathfrak{q}^{-\theta_1+\theta_\infty} \to \infty, \quad t_3 \to 0, \quad t_4 = t_3 \mathfrak{q}^{-\theta_1+\theta_\infty} = \mathfrak{q}^{-M}, \infty,$$

which corresponds to §6 of [198].

3.2.6.3 Coalescence limits in the magnetic frame

In the following subsections we focus on the coalescence limits suitable for the magnetic frame, where the τ-functions at $s = 1$ are identified with the grand partition functions of the suitable quiver Chern-Simons matrix models. This can be achieved by modifying the dictionary (3.36) as in (3.37) by a specific class of Weyl transformations. By inspection, one finds that there are $48 = 4! \times 2$ possibilities corresponding to the permutations of four masses and the inversion $t \to t^{-1}$.

We now describe in detail a specific parameter identification among them and describe the corresponding coalescence limits. At each step, we give the sets of shifted τ functions and their bilinear relations. These depend on the specific choice of coalescence. All the choices, however, end on the identical $\mathfrak{q}$-PIII$_3$ equation. For the other choices, see Appendix C.

3.2.6.4 $\mathfrak{q}$-PVI $\to$ $\mathfrak{q}$-PV

We consider the dictionary:

$$\begin{pmatrix} \theta_0 \\ \theta_1 \\ \theta_t \\ \theta_\infty \\ \frac{\log t}{\log \mathfrak{q}} \end{pmatrix} = \frac{1}{4} \begin{pmatrix} -1 & 1 & 0 & 2 & 0 \\ -1 & -1 & 2 & 0 & -2 \\ -1 & -1 & 2 & 0 & 2 \\ 1 & -1 & 0 & 2 & 0 \\ -4 & -4 & 4 & 0 & 0 \end{pmatrix} \tilde{w} \begin{pmatrix} M_1 - k \\ M_2 - k \\ M - k \\ -i\zeta_1 \\ -i\zeta_2 \end{pmatrix},$$

with the Weyl group element

$$\tilde{w} = s_2 s_1 s_3 s_4 s_2 s_3 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -1 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & -1 & 0 & -1 \\ 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \end{pmatrix},$$

which leads to the simple identification

$$\theta_0 = -\frac{i}{2}\left(\zeta_1 - \zeta_2\right), \quad \theta_1 = \frac{M_1 - k}{2}, \quad \theta_t = \frac{M_2 - k}{2}, \quad \theta_\infty = -\frac{i}{2}\left(\zeta_1 + \zeta_2\right), \quad t = \mathfrak{q}^{M-k} = e^{\frac{2\pi i M}{k}}, \quad \mathfrak{q} = e^{\frac{2\pi i}{k}}.$$

We proceed with the limit by setting $M_2 = M_2' - 2i\Lambda$, $\zeta_1 = \zeta_1' - \Lambda$, $\zeta_2 = \Lambda$. We have

$$q^{-\theta_t+\theta_0} = e^{\frac{\pi i(k-M_2'-i\zeta_1')}{k}} e^{\frac{4\pi\Lambda}{k}} \to 0, \quad \Lambda \to \infty.$$

Further, let $\theta_\star = \theta_0 + \theta_t = \frac{M_2'-i\zeta_1'-k}{2}$ and define the $\mathfrak{q}$-PV tau function as

$$\tau^{\mathrm{V}}\left(\theta_1, \theta_\star, \theta_\infty; s, \sigma, t\right) = \sum_{n\in\mathbb{Z}} s^n t^{(\sigma+n)^2} C^{\mathrm{V}}\left(\theta_1, \theta_\star, \theta_\infty; \sigma+n\right) Z^{\mathrm{V}}\left(\theta_1, \theta_\star, \theta_\infty; \sigma+n, t\right),$$

$$C^{\mathrm{V}}\left(\theta_1, \theta_\star, \theta_\infty; \sigma\right) = (1-\mathfrak{q})^{-\sigma^2} \prod_{\epsilon,\epsilon'=\pm} G_{\mathfrak{q}}\left(1+\epsilon\theta_\infty - \theta_1 + \epsilon'\sigma\right) \prod_{\epsilon=\pm} \frac{G_{\mathfrak{q}}\left(1-\theta_\star+\epsilon\sigma\right)}{G_{\mathfrak{q}}\left(1+2\epsilon\sigma\right)},$$

$$Z^{\mathrm{V}}\left(\theta_1, \theta_\star, \theta_\infty; \sigma, t\right) = \sum_{\lambda_+,\lambda_-} t^{|\lambda_+|+|\lambda_-|} \frac{\prod_{\epsilon,\epsilon'=\pm} N_{\phi,\lambda_{\epsilon'}}\left(\mathfrak{q}^{\epsilon\theta_\infty-\theta_1-\epsilon'\sigma}\right) \prod_{\epsilon=\pm} N_{\lambda_\epsilon,\phi}\left(\mathfrak{q}^{\epsilon\sigma-\theta_\star}\right)}{\prod_{\epsilon,\epsilon'} N_{\lambda_\epsilon,\lambda_{\epsilon'}}\left(\mathfrak{q}^{(\epsilon-\epsilon')\sigma}\right)}.$$

Let us consider the limit for the τ functions. First, define the scaling prefactors

$$X_{1,2} = X\left(\theta_0, \theta_1, \theta_t, \theta_\infty \pm \frac{1}{2}, \sigma, t\right), \quad X_{3,4} = X\left(\theta_0 \pm \frac{1}{2}, \theta_1, \theta_t, \theta_\infty, \sigma \pm \frac{1}{2}, t\right),$$
$$X_{5,6} = X\left(\theta_0, \theta_1 \mp \frac{1}{2}, \theta_t, \theta_\infty, \sigma, t\right), \quad X_{7,8} = X\left(\theta_0, \theta_1, \theta_t \mp \frac{1}{2}, \theta_\infty, \sigma \pm \frac{1}{2}, t\right),$$

with

$$X(\theta_0, \theta_1, \theta_t, \theta_\infty, \sigma, t) = t^{-\theta_t^2-\theta_0^2}(1-\mathfrak{q})^{\sigma^2} \prod_{\epsilon=\pm} G_{\mathfrak{q}}\left(1-\theta_t+\theta_0+\epsilon\sigma\right)^{-1}.$$

Here, the first or second index will correspond to the choice of the upper or lower sign, respectively. Next, define the shifted $\mathfrak{q}-$PV tau functions

$$\tau^{\mathrm{V}}_{1,2} = \tau^{\mathrm{V}}\left(\theta_1, \theta_\star, \theta_\infty \pm \frac{1}{2}; s, \sigma, t\right), \quad \tau^{\mathrm{V}}_{3,4} = \tau^{\mathrm{V}}\left(\theta_1, \theta_\star \pm \frac{1}{2}, \theta_\infty; s, \sigma \pm \frac{1}{2}, t\right),$$
$$\tau^{\mathrm{V}}_{5,6} = \tau^{\mathrm{V}}\left(\theta_1 \mp \frac{1}{2}, \theta_\star, \theta_\infty; s, \sigma, t\right).$$

Then, our discussion in subsection 3.2.6.1 implies that

$$X_i\tau_i^{\mathrm{VI}}(\theta_0, \theta_1, \theta_t, \theta_\infty; s, \sigma, t) \to \tau_i^{\mathrm{V}}(\theta_1, \theta_\star, \theta_\infty; s, \sigma, t), \quad i = 1, 2, 3, 4, 5, 6$$
$$X_7\tau_7^{\mathrm{VI}}(\theta_0, \theta_1, \theta_t, \theta_\infty; s, \sigma, t) \to s^{-1}\cdot\tau_4^{\mathrm{V}}(\theta_0, \theta_*, \theta_\infty; s, \sigma, t),$$
$$X_8\tau_8^{\mathrm{VI}}(\theta_0, \theta_1, \theta_t, \theta_\infty; s, \sigma, t) \to s\cdot\tau_3^{\mathrm{V}}(\theta_0, \theta_*, \theta_\infty; s, \sigma, t).$$

For the tau functions as we have defined them, w We then obtain the following nontrivial bilinear identities for the $\mathfrak{q}$-PV τ-functions – omitting the "V" superscript for readability:

$$\tau_1\tau_2 - (1-\mathfrak{q})^{-\frac{1}{2}}t^{\frac{1}{2}}q^{-2\theta_1}\tau_3\tau_4 - \left(1-\mathfrak{q}^{-2\theta_1}t\right)\tau_5\tau_6 = 0,$$
$$\tau_1\tau_2 - (1-\mathfrak{q})^{-\frac{1}{2}}t^{\frac{1}{2}}\tau_3\tau_4 - \underline{\tau}_5\overline{\tau}_6 = 0,$$
$$\tau_1\tau_2 - (1-\mathfrak{q})^{-\frac{1}{2}}t^{-\frac{1}{2}}\tau_3\tau_4 + (1-\mathfrak{q})^{-\frac{1}{2}}t^{-\frac{1}{2}}\left(1-\mathfrak{q}^{-2\theta_1}t\right)\overline{\tau}_3\underline{\tau}_4 = 0,$$
$$\underline{\tau}_5\tau_6 + \mathfrak{q}^{-\frac{1}{4}-\theta_1-\theta_\infty}(1-\mathfrak{q})^{-\frac{1}{2}}t^{\frac{1}{2}}\tau_3\underline{\tau}_4 - \underline{\tau}_1\tau_2 = 0,$$
$$\underline{\tau}_5\tau_6 + \mathfrak{q}^{-\frac{1}{4}-\theta_1+\theta_\infty}(1-\mathfrak{q})^{-\frac{1}{2}}t^{\frac{1}{2}}\underline{\tau}_3\tau_4 - \tau_1\underline{\tau}_2 = 0,$$
$$\underline{\tau}_5\tau_6 + \mathfrak{q}^{\frac{1}{2}+\theta_\star}(1-\mathfrak{q})^{-\frac{1}{2}}t^{-\frac{1}{2}}\left(\tau_3\underline{\tau}_4 - \underline{\tau}_3\tau_4\right) = 0.$$

Note that the fourth and eighth equation from (3.5) became trivial.

3.2.6.5 $\mathfrak{q}$-PV $\to$ $\mathfrak{q}$-PIII$_1$

The next limit amounts to $iM_2' \to \infty$ which means

$$\mathfrak{q}^{-\theta_\star} = \mathfrak{q}^{\frac{k+i\zeta_1'}{2}} e^{-\pi\Lambda/k} \to 0, \quad \Lambda \to \infty.$$

Let us define the appropriate τ function as

$$\begin{aligned}
\tau^{\mathrm{III}_1}\left(\theta_1,\theta_\infty;s,\sigma,t\right) &= \sum_{n\in\mathbb{Z}} s^n t^{(\sigma+n)^2} C^{\mathrm{III}_1}\left(\theta_1,\theta_\infty;\sigma+n\right) Z^{\mathrm{III}_1}\left(\theta_1,\theta_\infty;\sigma+n,t\right),\\
C^{\mathrm{III}_1}\left(\theta_1,\theta_\infty;\sigma\right) &= (1-\mathfrak{q})^{-2\sigma^2} \frac{\prod_{\epsilon,\epsilon'=\pm} G_{\mathfrak{q}}\left(1-\theta_1+\epsilon\theta_\infty+\epsilon'\sigma\right)}{\prod_{\epsilon=\pm} G_{\mathfrak{q}}\left(1+2\epsilon\sigma\right)},\\
Z^{\mathrm{III}_1}\left(\theta_1,\theta_\infty;\sigma,t\right) &= \sum_{\lambda_+,\lambda_-} t^{|\lambda_+|+|\lambda_-|} \prod_{\epsilon,\epsilon'=\pm} \frac{N_{\phi,\lambda_\epsilon}\left(\mathfrak{q}^{-\theta_1+\epsilon\theta_\infty-\epsilon'\sigma}\right)}{N_{\lambda_\epsilon,\lambda_{\epsilon'}}\left(\mathfrak{q}^{(\epsilon-\epsilon')\sigma}\right)}.
\end{aligned}$$

Calculations are completely analogous to the previous section. Because of $\mathfrak{q}^{-\theta_*} \to 0$, it's easy to see that $Z^V\left(\theta_0,\theta_*,\theta_t;\sigma,t\right) \to Z^{\mathrm{III}_1}\left(\theta_0,\theta_t;\sigma,t\right)$. We define the scaling prefactors

$$\begin{aligned}
X_{1,2}(\theta_1,\theta_\star,\theta_\infty,\sigma) &= X\left(\theta_1,\theta_\star,\theta_\infty \pm \frac{1}{2},\sigma\right),\\
X_{3,4}(\theta_1,\theta_\star,\theta_\infty,\sigma) &= X\left(\theta_1,\theta_\star \pm \frac{1}{2},\theta_\infty,\sigma \pm \frac{1}{2}\right),\\
X_{5,6}(\theta_1,\theta_\star,\theta_\infty,\sigma) &= X\left(\theta_1 \mp \frac{1}{2},\theta_\star,\theta_\infty,\sigma\right),
\end{aligned}$$

with

$$X(\theta_1,\theta_\star,\theta_\infty,\sigma) = (1-\mathfrak{q})^{-\sigma^2} \prod_{\epsilon=\pm} G_{\mathfrak{q}}\left(1-\theta_\star+\epsilon\sigma\right)^{-1},$$

and the shifted $\mathfrak{q}$-PIII$_1$ tau functions

$$\begin{aligned}
&\tau_{1,2}^{\mathrm{III}_1} = \tau^{\mathrm{III}_1}\left(\theta_1,\theta_\infty \pm \frac{1}{2};s,\sigma,t\right),\\
&\tau_3^{\mathrm{III}_1} = s^{\frac{1}{2}}\tau^{\mathrm{III}_1}\left(\theta_1,\theta_\infty;s,\sigma+\frac{1}{2},t\right), \qquad \tau_{4,5}^{\mathrm{III}_1} = \tau^{\mathrm{III}_1}\left(\theta_1 \mp \frac{1}{2},\theta_\infty;s,\sigma,t\right),
\end{aligned}$$

so that in the limit we find

$$\begin{aligned}
X_{1,2}\tau_{1,2}^{\mathrm{V}}(\theta_1,\theta_\star,\theta_\infty;s,\sigma,t) &\to \tau_{1,2}^{\mathrm{III}_1}(\theta_1,\theta_\infty;s,\sigma,t),\\
X_{3,4}\tau_{3,4}^{\mathrm{V}}(\theta_1,\theta_\star,\theta_\infty;s,\sigma,t) &\to s^{\mp\frac{1}{2}}\tau_3^{\mathrm{III}_1}(\theta_1,\theta_\infty;s,\sigma,t),\\
X_{5,6}\tau_{5,6}^{\mathrm{V}}(\theta_1,\theta_\star,\theta_\infty;s,\sigma,t) &\to \tau_{4,5}^{\mathrm{III}_1}(\theta_1,\theta_\infty;s,\sigma,t).
\end{aligned}$$

The resulting bilinear equations are the following:

$$\begin{aligned}
\tau_1\tau_2 + (\mathfrak{q}-1)^{-1}\mathfrak{q}^{-2\theta_1}t^{\frac{1}{2}}\tau_3^2 - \left(1-\mathfrak{q}^{-2\theta_1}t\right)\tau_4\tau_5 &= 0,\\
\tau_1\tau_2 + (\mathfrak{q}-1)^{-1}t^{\frac{1}{2}}\tau_3^2 - \underline{\tau}_4\bar{\tau}_5 &= 0,\\
\tau_1\tau_2 + (\mathfrak{q}-1)^{-1}t^{-\frac{1}{2}}\tau_3^2 - (\mathfrak{q}-1)^{-1}t^{-\frac{1}{2}}\left(1-\mathfrak{q}^{-2\theta_1}t\right)\underline{\tau}_3\bar{\tau}_3 &= 0,\\
\underline{\tau}_4\tau_5 - (\mathfrak{q}-1)^{-1}\mathfrak{q}^{-\frac{1}{4}-\theta_1-\theta_\infty}t^{\frac{1}{2}}\underline{\tau}_3\tau_3 - \underline{\tau}_1\tau_2 &= 0,\\
\underline{\tau}_4\tau_5 - (\mathfrak{q}-1)^{-1}\mathfrak{q}^{-\frac{1}{4}-\theta_1+\theta_\infty}t^{\frac{1}{2}}\underline{\tau}_3\tau_3 - \underline{\tau}_2\tau_1 &= 0.
\end{aligned}$$

3.2.6.6 $\mathfrak{q}$-PIII$_1$ $\to$ $\mathfrak{q}$-PIII$_2$

We now consider the limit $M_1 \to M_1' - i\Lambda, \zeta_1' = -\Lambda$ which in our dictionary implies

$$\mathfrak{q}^{-\theta_1+\theta_\infty} = \mathfrak{q}^{\frac{k-M_1'}{2}} e^{-2\pi\Lambda/k} \to \infty.$$

Let us denote by $\theta_* = \theta_1 + \theta_\infty$ and define the appropriate τ function as

$$\begin{aligned}
\tau^{\mathrm{III}_2}\left(\theta_*; s, \sigma, t\right) &= \sum_{n\in\mathbb{Z}} s^n t^{(\sigma+n)^2} C^{\mathrm{III}_2}\left(\theta_*; \sigma+n\right) Z^{\mathrm{III}_2}\left(\theta_*; \sigma+n, t\right), \\
C^{\mathrm{III}_2}\left(\theta_*; \sigma\right) &= (1-\mathfrak{q})^{-3\sigma^2} \prod_{\epsilon=\pm} \frac{G_{\mathfrak{q}}\left(1-\theta_*+\epsilon\sigma\right)}{G_{\mathfrak{q}}\left(1+2\epsilon\sigma\right)}, \\
Z^{\mathrm{III}_2}\left(\theta_*; \sigma, t\right) &= \sum_{\lambda_+,\lambda_-} t^{|\lambda_+|+|\lambda_-|} \frac{\prod_{\epsilon=\pm} N_{\phi,\lambda_\epsilon}\left(\mathfrak{q}^{-\theta_*-\epsilon\sigma}\right)}{\prod_{\epsilon,\epsilon'=\pm} N_{\lambda_\epsilon,\lambda_{\epsilon'}}\left(\mathfrak{q}^{(\epsilon-\epsilon')\sigma}\right)}.
\end{aligned}$$

We define the scaling prefactors

$$\begin{aligned}
X_{1,2}(\theta_1,\theta_\infty,\sigma) &= X\left(\theta_1, \theta_\infty \pm \frac{1}{2}, \sigma\right), \\
X_3(\theta_1,\theta_\infty,\sigma) &= X\left(\theta_1, \theta_\infty, \sigma + \frac{1}{2}\right), \\
X_{4,5}(\theta_1,\theta_\infty,\sigma) &= X\left(\theta_1 \mp \frac{1}{2}, \theta_\infty, \sigma\right),
\end{aligned}$$

with

$$X(\theta_1,\theta_\infty,\sigma) = (1-\mathfrak{q})^{-\sigma^2} \prod_{\epsilon=\pm} G_{\mathfrak{q}}\left(1-\theta_1+\theta_\infty+\epsilon\sigma\right)^{-1},$$

and the shifted τ functions:

$$\begin{aligned}
\tau_{1,2}^{\mathrm{III}_2} &= \tau^{\mathrm{III}_2}\left(\theta_* \pm \frac{1}{2}; s, \sigma, t\right), \\
\tau_3^{\mathrm{III}_2} &= s^{\frac{1}{2}} \cdot \tau^{\mathrm{III}_2}\left(\theta_*; s, \sigma + \frac{1}{2}, t\right).
\end{aligned}$$

Then we have as $\Lambda \to \infty$,

$$\begin{aligned}
X_{1,5}\tau_{1,5}^{\mathrm{III}_1}(\theta_1,\theta_\infty; s,\sigma,t) &\to \tau_1^{\mathrm{III}_2}(\theta_*; s,\sigma,t), \\
X_{2,4}\tau_{2,4}^{\mathrm{III}_1}(\theta_1,\theta_\infty; s,\sigma,t) &\to \tau_2^{\mathrm{III}_2}(\theta_*; s,\sigma,t), \\
X_3\tau_3^{\mathrm{III}_1}(\theta_1,\theta_\infty; s,\sigma,t) &\to \tau_3^{\mathrm{III}_2}(\theta_*; s,\sigma,t).
\end{aligned}$$

The resulting bilinear equations are

$$\begin{aligned}
\tau_1\tau_2 - (1-\mathfrak{q})^{-\frac{1}{2}}\, t^{\frac{1}{2}} \tau_3^2 - \overline{\tau}_1\underline{\tau}_2 &= 0, \\
\tau_1\tau_2 - (1-\mathfrak{q})^{-\frac{1}{2}}\, t^{-\frac{1}{2}} \left(\tau_3^2 - \underline{\tau}_3\overline{\tau}_3\right) &= 0, \\
\underline{\tau}_1\tau_2 - \tau_1\underline{\tau}_2 - (1-\mathfrak{q})^{-\frac{1}{2}}\, t^{\frac{1}{2}} \mathfrak{q}^{-\frac{1}{4}-\theta_*} \underline{\tau}_3\tau_3 &= 0.
\end{aligned}$$

3.2.6.7 $\mathfrak{q}$-PIII$_2$ $\to$ $\mathfrak{q}$-PIII$_3$

Finally we have $M_1' = -i\Lambda, \Lambda \to \infty$, which corresponds to

$$\mathfrak{q}^{-\theta_*} = -e^{-\pi\Lambda/k} \to 0.$$

Let us define the appropriate τ function

$$\begin{aligned}
\tau^{\mathrm{III}_3}(s,\sigma,t) &= \sum_{n\in\mathbb{Z}} s^n t^{(\sigma+n)^2} C^{\mathrm{III}_3}(\sigma+n)\, Z^{\mathrm{III}_3}(\sigma+n,t)\,,\\
C^{\mathrm{III}_3}(\sigma) &= (1-\mathfrak{q})^{-4\sigma^2}\prod_{\epsilon=\pm}\frac{1}{G_{\mathfrak{q}}(1+2\epsilon\sigma)},\\
Z^{\mathrm{III}_3}(\theta_0,\theta_t;\sigma,t) &= \sum_{\lambda_+,\lambda_-} t^{|\lambda_+|+|\lambda_-|}\frac{1}{\prod_{\epsilon,\epsilon'=\pm} N_{\lambda_\epsilon,\lambda_{\epsilon'}}\left(\mathfrak{q}^{(\epsilon-\epsilon')\sigma}\right)}.
\end{aligned}$$

To perform the limit, we first define the scaling prefactors

$$X_{1,2}(\theta_*,\sigma) = X\left(\theta_*\pm\frac{1}{2},\sigma\right),\quad X_3(\theta_*,\sigma) = X\left(\theta_\star,\sigma+\frac{1}{2}\right)$$

with

$$X(\theta_*,\sigma) = (1-\mathfrak{q})^{-\sigma^2}\prod_{\epsilon=\pm} G_{\mathfrak{q}}(1-\theta_\star+\epsilon\sigma)^{-1}\,,$$

and the shifted τ functions

$$\tau_1^{\mathrm{III}_3} = \tau^{\mathrm{III}_3}(s,\sigma,t),\quad \tau_2^{\mathrm{III}_3} = s^{\frac{1}{2}}\tau^{\mathrm{III}_3}\left(s,\sigma+\frac{1}{2},t\right).$$

Then we have as $\Lambda\to\infty$,

$$\begin{aligned}
X_{1,2}\tau_{1,2}^{\mathrm{III}_2}(\theta_*;s,\sigma,t) &\to \tau_1^{\mathrm{III}_3}(s,\sigma,t),\\
X_3\tau_3^{\mathrm{III}_2}(\theta_*;s,\sigma,t) &\to \tau_2^{\mathrm{III}_2}(s,\sigma,t),
\end{aligned}$$

and, as expected, we find these to satisfy the bilinear equations of $\mathfrak{q}$-Painlevé III$_3$,

$$\begin{aligned}
\tau_1^2 - \underline{\tau}_1\bar{\tau}_1 - t^{\frac{1}{2}}\tau_2^2 &= 0,\\
\tau_1^2 - t^{-\frac{1}{2}}\left(\tau_2^2 - \underline{\tau}_2\bar{\tau}_2\right) &= 0.
\end{aligned}$$

3.2.7 Different coalsecence limit

The identification in the previous section is not the only possible limit. An analysis of the scaling parameters identifies $4!\times 2 = 48$ different elements of the Weyl group which lead to a well-defined endpoint in $\mathfrak{q}-$PIII$_3$. We consider another choice. We pick

$$\begin{pmatrix}\theta_0\\ \theta_1\\ \theta_t\\ \theta_\infty\\ \frac{\log t}{\log\mathfrak{q}}\end{pmatrix} = \frac{1}{4}\begin{pmatrix}-1 & 1 & 0 & 2 & 0\\ -1 & -1 & 2 & 0 & -2\\ -1 & -1 & 2 & 0 & 2\\ 1 & -1 & 0 & 2 & 0\\ -4 & -4 & 4 & 0 & 0\end{pmatrix}\tilde{w}\begin{pmatrix}M_1-k\\ M_2-k\\ M-k\\ -i\zeta_1\\ -i\zeta_2\end{pmatrix}.$$

with the Coxeter group element

$$\tilde{w} = s_3s_4s_5s_2s_3s_4s_5s_1s_3s_4s_5s_2s_3 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & -1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -\frac{1}{2} & -\frac{1}{2} & 1 & 0 & 0 \end{pmatrix},$$

which leads to the identification

$$\theta_0 = \frac{-i\zeta_1 - i\zeta_2}{2}, \quad \theta_1 = \frac{M_2 - M}{2}, \quad \theta_t = \frac{M - M_1}{2}, \quad \theta_\infty = \frac{i\zeta_1 - i\zeta_2}{2},$$
$$t = \mathfrak{q}^{-M_1 - k} = e^{-\frac{2\pi i M_1}{k}}, \quad \mathfrak{q} = e^{\frac{2\pi i}{k}}. \tag{3.83}$$

3.2.7.1 $\mathfrak{q}$-PVI $\to$ $\mathfrak{q}$-PV

Let $\theta_* = \frac{1}{2}(M_2' - M - i\zeta_1')$, in terms of which (3.83) under $M_2 = M_2' - 2i\Lambda, \quad \zeta_1 = \zeta_1' - \Lambda, \quad \zeta_2 = \Lambda$ becomes

$$-\theta_1 + \theta_\infty = -\theta_*, \quad -\theta_1 - \theta_\infty = -\theta_* - i\zeta_1' + 2i\Lambda, \quad \Lambda \to \infty$$

Noting that as $\mathfrak{q} = e^{2\pi i/k}$ and $k > 0$, we have $\mathfrak{q}^{-\theta_1 - \theta_\infty} = \mathfrak{q}^{-\theta_* - i\zeta_1'} e^{-4\pi\Lambda/k} \to 0$. To perform the limit, first define the scaling prefactors

$$X(\theta_0, \theta_1, \theta_t, \theta_\infty, \sigma) = (1 - \mathfrak{q})^{-\sigma^2} \prod_{\epsilon = \pm} G_{\mathfrak{q}} \left(1 - \theta_1 - \theta_\infty + \epsilon\sigma\right)^{-1}$$

$$X_{1,2} = X\left(\theta_0, \theta_1, \theta_t, \theta_\infty \pm \frac{1}{2}, \sigma\right) \qquad X_{3,4} = X\left(\theta_0 \pm \frac{1}{2}, \theta_1, \theta_t, \theta_\infty, \sigma \pm \frac{1}{2}\right)$$
$$X_{5,6} = X\left(\theta_0, \theta_1 \mp \frac{1}{2}, \theta_t, \theta_\infty, \sigma\right) \qquad X_{7,8} = X\left(\theta_0, \theta_1, \theta_t \mp \frac{1}{2}, \theta_\infty, \sigma \pm \frac{1}{2}\right)$$

Here, the first or second index will correspond to the choice of the upper or $\pm$ sign, respectively. Note that the prefactor is not exactly the same as the one found from the grand canonical partition function of the 3 dimensional matrix model. This is not surprising, however, as we expect these functions to differ by the arbitrary functions Z_i. Next, define the shifted $\mathfrak{q}-$PV tau functions

$$\tau_1^{\mathrm{V}} = \tau^{\mathrm{V}}\left(\theta_0, \theta_* - \frac{1}{2}, \theta_t; s, \sigma, t\right) \qquad \tau_2^{\mathrm{V}} = \tau^{\mathrm{V}}\left(\theta_0, \theta_* + \frac{1}{2}, \theta_t; s, \sigma, t\right)$$
$$\tau_3^{\mathrm{V}} = \tau^{\mathrm{V}}\left(\theta_0 + \frac{1}{2}, \theta_*, \theta_t; s, \sigma + \frac{1}{2}, t\right) \qquad \tau_4^{\mathrm{V}} = \tau^{\mathrm{V}}\left(\theta_0 - \frac{1}{2}, \theta_*, \theta_t; s, \sigma - \frac{1}{2}, t\right)$$
$$\tau_5^{\mathrm{V}} = \tau^{\mathrm{V}}\left(\theta_0, \theta_*, \theta_t - \frac{1}{2}; s, \sigma + \frac{1}{2}, t\right) \qquad \tau_6^{\mathrm{V}} = \tau^{\mathrm{V}}\left(\theta_0, \theta_*, \theta_t + \frac{1}{2}; s, \sigma - \frac{1}{2}, t\right)$$

Then, our discussion implies that

$$X_i\tau_i^{\mathrm{VI}}(\theta_0, \theta_1, \theta_t, \theta_\infty; s, \sigma, t) \to \tau_i^{\mathrm{V}}(\theta_0, \theta_*, \theta_\infty; s, \sigma, t), \quad i = 1, 2, 3, 4$$
$$X_5\tau_5^{\mathrm{VI}}(\theta_0, \theta_1, \theta_t, \theta_\infty; s, \sigma, t) \to \tau_1^{\mathrm{V}}(\theta_0, \theta_*, \theta_\infty; s, \sigma, t),$$
$$X_6\tau_6^{\mathrm{VI}}(\theta_0, \theta_1, \theta_t, \theta_\infty; s, \sigma, t) \to \tau_2^{\mathrm{V}}(\theta_0, \theta_*, \theta_\infty; s, \sigma, t),$$
$$X_i\tau_i^{\mathrm{VI}}(\theta_0, \theta_1, \theta_t, \theta_\infty; s, \sigma, t) \to \tau_{i-2}^{\mathrm{V}}(\theta_0, \theta_*, \theta_\infty; s, \sigma, t), \quad i = 7, 8$$

We remark that these are again not the same limits as in [198], where the authors find that, up to the prefactors, τ_5 and τ_6 degenerate to $\mathfrak{q}$-shifted τ_1 and τ_2, respectively. For the tau functions as we have defined them, we obtain the following nontrivial bilinear identities, omitting the "V" superscript for readability:

$$\begin{aligned}
\tau_1\tau_2 - (1-\mathfrak{q})^{\frac{1}{2}} \cdot t\tau_3\tau_4 - \left(1-\mathfrak{q}^{-2\theta_t} \cdot t\right) \underline{\tau}_1\bar{\tau}_2 &= 0, \\
(1-\mathfrak{q})^{-\frac{1}{2}}\tau_1\tau_2 - \tau_3\tau_4 + \mathfrak{q}^{2\theta_t}\underline{\tau}_5\bar{\tau}_6 &= 0, \\
(1-\mathfrak{q})^{-\frac{1}{2}}\tau_1\tau_2 - \mathfrak{q}^{2\theta_t}\tau_3\tau_4 + \left(1-\mathfrak{q}^{-2\theta_t} \cdot t\right)\mathfrak{q}^{2\theta_t}\tau_5\tau_6 &= 0, \\
\underline{\tau}_1\tau_2 + (1-\mathfrak{q})^{\frac{1}{2}}\mathfrak{q}^{-\theta_*+\theta_t-\frac{1}{2}} \cdot t\underline{\tau}_5\tau_6 - \tau_1\underline{\tau}_2 &= 0, \\
(1-\mathfrak{q})^{-\frac{1}{2}}\underline{\tau}_1\tau_2 + \mathfrak{q}^{\theta_0+2\theta_t}\underline{\tau}_5\tau_6 - \mathfrak{q}^{\theta_t}\underline{\tau}_3\tau_4 &= 0, \\
(1-\mathfrak{q})^{-\frac{1}{2}}\underline{\tau}_1\tau_2 + \mathfrak{q}^{-\theta_0+2\theta_t}\underline{\tau}_5\tau_6 - \mathfrak{q}^{\theta_t}\tau_3\underline{\tau}_4 &= 0.
\end{aligned}$$

Note that the first and fifth equation from (3.5) became trivial.

3.2.7.2 $\mathfrak{q}$−PV → $\mathfrak{q}$−PV$_{\mathbf{deg}}$/$\mathfrak{q}$−PIII$_1$

The next limit is $iM_2' \to \infty$ which means

$$\mathfrak{q}^{-\theta_*} = \mathfrak{q}^{\frac{M+i\zeta_1'}{2}} e^{-\pi\Lambda/k} \to 0, \quad \Lambda \to \infty$$

Technically speaking, this limit lands us in the degenerate case of $\mathfrak{q}$-Painlevé V, $\mathfrak{q}$-PV_{deg}. This, however, becomes $\mathfrak{q}$−PIII$_1$ after a change of variables. Therefore, we shall not distinguish them, and write $\mathfrak{q}$−PIII$_1$. Let us define the appropriate tau function as

$$\begin{aligned}
\tau^{\mathrm{III}_1}\left(\theta_0,\theta_t;s,\sigma,t\right) &= \sum_{n\in\mathbb{Z}} s^n t^{(\sigma+n)^2} C^{\mathrm{III}_1}\left(\theta_0,\theta_t;\sigma+n\right) Z^{\mathrm{III}_1}\left(\theta_0,\theta_t;\sigma+n,t\right), \\
C^{\mathrm{III}_1}\left(\theta_0,\theta_t;\sigma\right) &= (1-\mathfrak{q})^{-2\sigma^2} \frac{\prod_{\epsilon,\epsilon'=\pm} G_{\mathfrak{q}}\left(1+\epsilon\sigma-\theta_t+\epsilon'\theta_0\right)}{\prod_{\epsilon=\pm} G_{\mathfrak{q}}\left(1+2\epsilon\sigma\right)}, \\
Z^{\mathrm{III}_1}\left(\theta_0,\theta_t;\sigma,t\right) &= \sum_{\lambda_+,\lambda_-} t^{|\lambda_+|+|\lambda_-|} \prod_{\epsilon,\epsilon'=\pm} \frac{N_{\lambda_\epsilon,\phi}\left(\mathfrak{q}^{\epsilon\sigma-\theta_t-\epsilon'\theta_0}\right)}{N_{\lambda_\epsilon,\lambda_{\epsilon'}}\left(\mathfrak{q}^{(\epsilon-\epsilon')\sigma}\right)}
\end{aligned}$$

Calculations are completely analogous to the previous section. Because of $\mathfrak{q}^{-\theta_*} \to 0$, it's easy to see that $Z^V\left(\theta_0,\theta_*,\theta_t;\sigma,t\right) \to Z^{\mathrm{III}_1}\left(\theta_0,\theta_t;\sigma,t\right)$. To deal with the 1-loop terms, using the function

$$X(\theta_0,\theta_*,\theta_t,\sigma) = t^{\theta_0^2+\theta_t^2}(1-\mathfrak{q})^{-\sigma^2}\prod_{\epsilon=\pm} G_{\mathfrak{q}}\left(1-\theta_*+\epsilon\sigma\right)^{-1}$$

define the scaling prefactors

$$\begin{aligned}
X_{1,2}(\theta_0,\theta_*,\theta_t,\sigma) &= X\left(\theta_0,\theta_*\mp\frac{1}{2},\theta_t,\sigma\right), \\
X_{3,4}(\theta_0,\theta_*,\theta_t,\sigma) &= X\left(\theta_0\pm\frac{1}{2},\theta_*,\theta_t,\sigma\pm\frac{1}{2}\right), \\
X_{5,6}(\theta_0,\theta_*,\theta_t,\sigma) &= X\left(\theta_0,\theta_*,\theta_t\mp\frac{1}{2},\sigma\pm\frac{1}{2}\right)
\end{aligned}$$

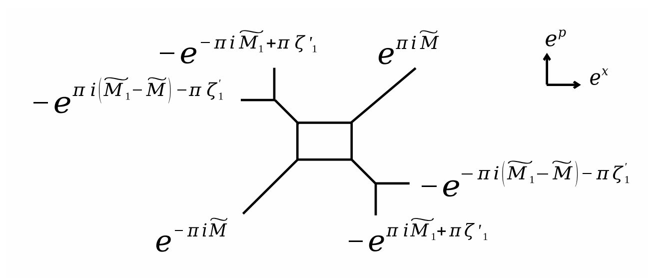

Figure 3.8: A rewriting of 3.5. The $E_3^{(1)}$ symmetry is generated by exchanging the asymptotics purely in the e^p directions, likewise for purely e^x, and finally by exchanging the asymptotic in the e^{x+p} direction with the asymptotics in the e^{-x} and e^{-p} direction simultaneously.

We define the shifted $\mathfrak{q}$-PIII$_1$ tau functions

$$
\begin{aligned}
\tau_1^{\mathrm{III}_1} &= \tau^{\mathrm{III}_1}\left(\theta_0,\theta_t;s,\sigma,t\right) \\
\tau_2^{\mathrm{III}_1} &= \tau^{\mathrm{III}_1}\left(\theta_0+\frac{1}{2},\theta_t;s,\sigma+\frac{1}{2},t\right) \qquad \tau_3^{\mathrm{III}_1} = \tau^{\mathrm{III}_1}\left(\theta_0-\frac{1}{2},\theta_t;s,\sigma-\frac{1}{2},t\right) \\
\tau_4^{\mathrm{III}_1} &= \tau^{\mathrm{III}_1}\left(\theta_0,\theta_t-\frac{1}{2};s,\sigma+\frac{1}{2},t\right) \qquad \tau_5^{\mathrm{III}_1} = \tau^{\mathrm{III}_1}\left(\theta_0,\theta_t+\frac{1}{2};s,\sigma-\frac{1}{2},t\right)
\end{aligned}
$$

Then in the limit we find

$$
\begin{aligned}
X_i\tau_i^{\mathrm{V}}(\theta_0,\theta_*,\theta_t,\sigma;s,\sigma,t) &\to \tau_1^{\mathrm{III}_1}(\theta_0,\theta_t,\sigma;s,\sigma,t), \quad i=1,2 \\
X_i\tau_i^{\mathrm{V}}(\theta_0,\theta_*,\theta_t,\sigma;s,\sigma,t) &\to \tau_{i-1}^{\mathrm{III}_1}(\theta_0,\theta_t,\sigma;s,\sigma,t), \quad i=3,4,5,6
\end{aligned}
$$

The bilinear equations obtained from this limit are the following:

$$
\begin{aligned}
\tau_1^2-\left(1-\mathfrak{q}^{-2\theta_t}\cdot t\right)\underline{\tau}_1\overline{\tau}_1-(1-\mathfrak{q})\cdot t^{\frac{1}{2}}\tau_2\tau_3 &= 0 \\
\tau_1^2-(1-\mathfrak{q})\cdot t^{-\frac{1}{2}}\left(\tau_2\tau_3-\underline{\tau}_4\overline{\tau}_5\right) &= 0, \\
\tau_1^2-(1-\mathfrak{q})\mathfrak{q}^{2\theta_t}\cdot t^{-\frac{1}{2}}\tau_2\tau_3+(1-\mathfrak{q})\left(1-\mathfrak{q}^{-2\theta_t}\cdot t\right)\mathfrak{q}^{2\theta_t}\cdot t^{-\frac{1}{2}}\tau_4\tau_5 &= 0, \\
\underline{\tau}_1\tau_1-(1-\mathfrak{q})\mathfrak{q}^{1/4+\theta_0+\theta_t}\cdot t^{-\frac{1}{2}}\left(\underline{\tau}_2\tau_3-\underline{\tau}_4\tau_5\right) &= 0, \\
\underline{\tau}_1\tau_1-(1-\mathfrak{q})\mathfrak{q}^{1/4-\theta_0+\theta_t}\cdot t^{-\frac{1}{2}}\left(\tau_2\underline{\tau}_3-\underline{\tau}_4\tau_5\right) &= 0.
\end{aligned}
$$

3.2.7.3 $\mathfrak{q}-$PIII$_1$ $\to$ $\mathfrak{q}-$PIII$_2$

Before taking the limit, we comment on the $E_3^{(1)}$ symmetry of the setup and our choice of $t=\mathfrak{q}^{-M_1}$. Once we have rewritten $(\hat{x},\hat{p})\to(\hat{x}-\pi\zeta_1',\hat{p}-\pi\zeta_1')$, the curve will have asymptotic values as shown in 3.8, where $\tilde{M}=M+k,\tilde{M}_1=M_1+k$. There is a Coxeter group acting on the curve by exchanging asymptotic values. In this particular case we have the very simple action

$$
\begin{aligned}
&\tilde{s}_1: e^{-\pi i\tilde{M}_1+\pi\zeta_1'} \to e^{\pi iM_1+\pi\zeta_1'} \\
&\tilde{s}_2: e^{\pi i(\tilde{M}-\tilde{M}_1)-\pi\zeta_1'} \to e^{-\pi i(M-M_1)-\pi\zeta_1'} \\
&\tilde{s}_3: e^{\pi i(\tilde{M}-2\tilde{M}_1)} \to e^{-\pi i\tilde{M}}
\end{aligned}
$$

and, further $s_4 = s_3^{-1}$, which means s_1 sends $\tilde{M}_1 \to -\tilde{M}_1$. Therefore, at this point a redefinition could have been made, had we opted for a different dictionary at the beginning.

Here we are in a slightly different position as the limit $M_1 \to M_1' - i\Lambda, \zeta_1' = -\Lambda$ means

$$
\begin{aligned}
t &= \mathfrak{q}^{-M_1'} e^{-2\pi\Lambda/k} \to 0 \\
\mathfrak{q}^{-\theta_t - \theta_0} &= \mathfrak{q}^{\frac{M_1' - M}{2}} e^{2\pi\Lambda/k} \to \infty
\end{aligned}
$$

with $t_1 = t\mathfrak{q}^{-\theta_t-\theta_0} = \mathfrak{q}^{-\frac{M+M_1'}{2}}$ fixed. Let us denote by $\theta_\star = \theta_t - \theta_0$ and define the appropriate tau function as

$$
\begin{aligned}
\tau^{\mathrm{III}_2}\left(\theta_\star; s, \sigma, t\right) &= \sum_{n\in\mathbb{Z}} s^n t^{(\sigma+n)^2} C^{\mathrm{III}_2}\left(\theta_\star; \sigma+n\right) Z^{\mathrm{III}_2}\left(\theta_\star; \sigma+n, t\right), \\
C^{\mathrm{III}_2}\left(\theta_\star; \sigma\right) &= (1-\mathfrak{q})^{-2\sigma^2}(\mathfrak{q}-1)^{-\sigma^2} \frac{\prod_{\epsilon=\pm} G_{\mathfrak{q}}\left(1+\epsilon\sigma-\theta_\star\right)}{G_{\mathfrak{q}}\left(1+2\epsilon\sigma\right)}, \\
Z^{\mathrm{III}_2}\left(\theta_\star; \sigma, t\right) &= \sum_{\lambda_+,\lambda_-} t^{|\lambda_+|+|\lambda_-|} \frac{\prod_{\epsilon=\pm} N_{\lambda_\epsilon,\phi}\left(\mathfrak{q}^{\epsilon\sigma-\theta_\star}\right) f_{\lambda_\epsilon}(\mathfrak{q}^{-\epsilon\sigma})}{\prod_{\epsilon,\epsilon'=\pm} N_{\lambda_\epsilon,\lambda_{\epsilon'}}\left(\mathfrak{q}^{(\epsilon-\epsilon')\sigma}\right)}
\end{aligned}
$$

where $f_\lambda(u) = \prod_{c\in\lambda}\left(-\mathfrak{q}^{l_\lambda(c)+a_\phi(c)+1}u^{-1}\right)$. We define the scaling prefactors

$$
X_1(\theta_0,\theta_t,\sigma) = (\mathfrak{q}-1)^{-\sigma^2}\mathfrak{q}^{-(\theta_t+\theta_0)\sigma^2}\prod_{\epsilon=\pm} G_{\mathfrak{q}}\left(1-\theta_0-\theta_t+\epsilon\sigma\right)^{-1},
$$

$$
\begin{aligned}
X_2(\theta_0,\theta_t,\sigma) &= X_1\left(\theta_0+\frac{1}{2},\theta_t,\sigma+\frac{1}{2}\right), \quad X_3(\theta_0,\theta_t,\sigma) = X_1\left(\theta_0-\frac{1}{2},\theta_t,\sigma-\frac{1}{2}\right), \\
X_4(\theta_0,\theta_t,\sigma) &= X_1\left(\theta_0,\theta_t-\frac{1}{2},\sigma+\frac{1}{2}\right), \quad X_5(\theta_0,\theta_t,\sigma) = X_1\left(\theta_0,\theta_t+\frac{1}{2},\sigma-\frac{1}{2}\right),
\end{aligned}
$$

and the shifted tau functions:

$$
\begin{aligned}
\tau_1^{\mathrm{III}_2} &= \tau^{\mathrm{III}_2}\left(\theta_\star; s,\sigma,t\right) \\
\tau_2^{\mathrm{III}_2} &= \tau^{\mathrm{III}_2}\left(\theta_\star-\frac{1}{2}; s,\sigma+\frac{1}{2},\mathfrak{q}^{-\frac{1}{2}}t\right) \\
\tau_3^{\mathrm{III}_2} &= \tau^{\mathrm{III}_2}\left(\theta_\star+\frac{1}{2}; s,\sigma-\frac{1}{2},\mathfrak{q}^{\frac{1}{2}}t\right).
\end{aligned}
$$

To perform the limit, we further put

$$
s = \check{s}(\mathfrak{q}-1)^{-2\sigma}\mathfrak{q}^{2(1-\theta_0-\theta_t)\sigma}\prod_{\epsilon=\pm}\Gamma_{\mathfrak{q}}\left(1-\theta_0-\theta_t+\epsilon\sigma\right)^{-\epsilon}
$$

Then we have as $\Lambda \to \infty$,

$$
\begin{aligned}
&X_i\tau_i^{\mathrm{III}_1}(\theta_0,\theta_t;s,\sigma,t) \to \tau_i^{\mathrm{III}_2}(\theta_\star;\check{s},\sigma,t), \quad i=1,2,3 \\
&X_4\tau_4^{\mathrm{III}_1}(\theta_0,\theta_t;s,\sigma,t) \to \tau_2^{\mathrm{III}_2}(\theta_\star;\check{s},\sigma,\mathfrak{q}t) \\
&X_5\tau_5^{\mathrm{III}_1}(\theta_0,\theta_t;s,\sigma,t) \to \tau_3^{\mathrm{III}_2}(\theta_\star;\check{s},\sigma,\mathfrak{q}^{-1}t)
\end{aligned}
$$

for the details of the calculation see the similar calculation of Proposition 3.1 of [198]. The bilinear equations of $\mathfrak{q}$-PIII$_1$ degenerate to

$$\begin{aligned}\tau_1^2-(1-\mathfrak{q}^{-\theta_\star}\cdot t)\underline{\tau}_1\bar{\tau}_1+(\mathfrak{q}-1)^{\frac{1}{2}}t^{\frac{1}{2}}\tau_2\tau_3=0\\ \tau_1^2-(\mathfrak{q}-1)^{\frac{1}{2}}(1-\mathfrak{q}^{-\theta_\star}t)\mathfrak{q}^{\theta_\star}t^{-\frac{1}{2}}\left(\tau_2\tau_3+\bar{\tau}_2\underline{\tau}_3\right)=0\\ \underline{\tau}_1\tau_1-(\mathfrak{q}-1)^{\frac{1}{2}}\mathfrak{q}^{1/4}t^{-\frac{1}{2}}\left(\underline{\tau}_2\tau_3-\underline{\tau}_3\tau_2\right)=0\end{aligned}$$

3.2.7.4 $\mathfrak{q}$−PIII$_2$ → $\mathfrak{q}$−PIII$_3$

In the final limit we have $M_1'=-i\Lambda,\Lambda\to\infty$. Similarly as in the last step,

$$\begin{aligned}t=\mathfrak{q}^{-M/2}e^{-\pi\Lambda/k}\to 0\\ \mathfrak{q}^{-\theta_\star}=\mathfrak{q}^{-M/2}e^{+\pi\Lambda/k}\to\infty\end{aligned}$$

with $t_1=t\mathfrak{q}^{-\theta_\star}=\mathfrak{q}^{-M}$ fixed. Let us define the appropriate tau function

$$\begin{aligned}\tau^{\mathrm{III}_3}\left(s,\sigma,t\right)=\sum_{n\in\mathbb{Z}}s^nt^{(\sigma+n)^2}C^{\mathrm{III}_3}\left(\sigma+n\right)Z^{\mathrm{III}_3}\left(\sigma+n,t\right),\\ C^{\mathrm{III}_3}\left(\sigma\right)=(1-\mathfrak{q})^{-2\sigma^2}(\mathfrak{q}-1)^{-2\sigma^2}\prod_{\epsilon=\pm}\frac{1}{G_\mathfrak{q}\left(1+2\epsilon\sigma\right)},\\ Z^{\mathrm{III}_3}\left(\theta_0,\theta_t;\sigma,t\right)=\sum_{\lambda_+,\lambda_-}t^{|\lambda_+|+|\lambda_-|}\frac{\prod_{\epsilon=\pm}f_{\lambda_\epsilon}(\mathfrak{q}^{-\epsilon\sigma})^2}{\prod_{\epsilon,\epsilon'=\pm}N_{\lambda_\epsilon,\lambda_{\epsilon'}}\left(\mathfrak{q}^{(\epsilon-\epsilon')\sigma}\right)}\end{aligned}$$

To perform the limit, we first define the scaling prefactors

$$X_1(\theta_\star,\sigma)=(\mathfrak{q}-1)^{-\sigma^2}\mathfrak{q}^{-\theta_\star\sigma^2}\prod_{\epsilon=\pm}G_\mathfrak{q}\left(1-\theta_\star+\epsilon\sigma\right)^{-1},$$

$$X_2(\theta_\star,\sigma)=C_1\left(\theta_\star-\frac{1}{2},\sigma+\frac{1}{2}\right),\qquad X_3(\theta_\star,\sigma)=C_1\left(\theta_\star+\frac{1}{2},,\sigma-\frac{1}{2}\right)$$

We define the shifted tau functions

$$\tau_1^{\mathrm{III}_3}=\tau^{\mathrm{III}_3}(s,\sigma,t),\quad \tau_2^{\mathrm{III}_3}=s^{\frac{1}{2}}\tau^{\mathrm{III}_3}\left(s,\sigma+\frac{1}{2},t\right).$$

To perform the limit, we define

$$s=\check{s}(\mathfrak{q}-1)^{-2\sigma}\mathfrak{q}^{(1-2\theta_\star)\sigma}\prod_{\epsilon=\pm}\Gamma_\mathfrak{q}\left(1-\theta_\star+\epsilon\sigma\right)^{-\epsilon}$$

Then we have as $\Lambda\to\infty$,

$$\begin{aligned}&X_1\tau_1^{\mathrm{III}_2}(\theta_\star;s,\sigma,t)\to\tau_1^{\mathrm{III}_3}(\check{s},\sigma,t)\\ &X_2\tau_2^{\mathrm{III}_2}(\theta_\star;s,\sigma,t)\to\tau_2^{\mathrm{III}_2}(\check{s},\sigma,t)\\ &X_3\tau_3^{\mathrm{III}_2}(\theta_\star;s,\sigma,t)\to\tau_1^{\mathrm{III}_3}\left(\check{s},\sigma-\frac{1}{2},t\right)=s^{\frac{1}{2}}\tau_2^{\mathrm{III}_2}(\check{s},\sigma,t)\end{aligned}$$

We find these satisfy the bilinear equations

$$\begin{aligned}\tau_1^2-\underline{\tau}_1\bar{\tau}_1+t^{\frac{1}{2}}\underline{\tau}_2\bar{\tau}_2=0\\ \tau_2^2-\underline{\tau}_2\bar{\tau}_2+t^{\frac{1}{2}}\underline{\tau}_1\bar{\tau}_1=0\end{aligned}$$

The second equation is the degeneration of a linear combination of the first two $\mathfrak{q}$-PIII$_2$ equations. By Proposition 4.2 in [32] the functions $\check{\tau}_i=(\mathfrak{q}t;\mathfrak{q},\mathfrak{q})_\infty\tau_i$, $i=1,2$ satisfy the usual $\mathfrak{q}$-PIII$_3$ equations in tau form.

3.2.8 Discussion and open questions

In this work we proposed that the grand partition function of the four node circular quiver superconformal Chern-Simons theory, see Fig.3.1, solves the $\mathfrak{q}$-deformed Painlevé VI equation in τ-form [246, 268]. We showed that this theory describes the full moduli space of $SU(2)$ gauge theory on $\mathbb{R}^4 \times S^1$ with $N_f = 4$ by extending the previous findings in [183, 184, 208, 212]. Let us notice that a solution to the above $\mathfrak{q}$-difference system was previously proposed in terms of the Nekrasov-Okounkov partition function of the gauge theory in [157]. While this solution is valid in a short time expansion, which is perturbative in the gauge coupling, the Fredholm determinant realisation arising from the quiver Chern-Simons theory can be naturally expanded in the different regime of small κ, corresponding to the magnetic phase of the gauge theory. We provide several explicit checks of this proposal at low orders in κ. Our result therefore allows to study the five dimensional gauge theory in terms of a matrix model in a regime which is otherwise difficult to access. Moreover, the explicit results obtained in this work, motivated by TS/ST correspondence, give a stronger check of the latter and enlarge the set of examples where its rigorous realisation is verified.

Let us list in the following several questions left open by our analysis that it would be interesting to further investigate.

- The matrix models discussed in this work can be used to study systematically the dual prepotential of five dimensional SU(2) gauge theories with $N_f \leq 4$. The four dimensional limit can be also studied by introducing a suitable dual scaling along the lines of [44].
- Generalise our matrix model to the case $M \neq 0$ in a representation which can be analytically continued in M as well as in M_1, M_2.
- We provided analytic evidence of our conjecture by explicit checks at low orders in κ and numerical checks at fixed moduli. It would be great to be able to provide an analytic proof, either by induction in the power of κ or by suitable Ward identities on the matrix model itself.
- The $\mathfrak{q}$-Painlevé VI τ-functions given as the Nekrasov-Okounkov partition function (3.1) and the one proposed in this work as the grand partition function of Chern-Simons quiver (3.35) should be matched by fixing the ambiguity of the C coefficients and relating the parameters κ and σ. The latter are linked through the quantum mirror map whose explicit expression is proposed in [95], which one could check by comparing the two expressions for the τ-function.
- The identification of the τ-function with the spectral determinant of the quantum operator implies that the analysis of the zeroes of the first solves the quantum spectrum of the latter [25, 44]. Therefore, the results obtained in this work provide a method to quantize the integrable spin chain systems associated to 5d gauge theories with $N_f \leq 4$. It would be interesting to pursue this direction and compare the results with the ones that can be derived from the Nekrasov-Shatashvili quantization method [228].
- Conversely, by extending the relevant $\mathfrak{q}$-difference equations to higher rank simple gauge groups, it would be possible to use them as a tool to compute

the multi-instantons expansions of matter gauge theories in five dimensions, by extending the approach elaborated in [43] for the four dimensional case.

- $\mathfrak{q}$-PIII$_3$ equation has been shown to be related to five dimensional Nakajima-Yoshioka blowup equation [217] for pure Super Yang-Mills [31, 32]. It would be interesting to extend such an analysis to the $\mathfrak{q}$-Painlevé equations corresponding to the gauge theory in five dimensions coupled to massive hypermultiplets.

- One can also consider the mass deformed quiver Chern-Simons matter theories [108, 144]. It was found that the grand partition function of the ABJM theory with $\mathcal{N} = 6$ preserving mass deformation satisfies a modified version of $\mathfrak{q}$-PIII$_3$ [231]. It would be worth investigating whether the grand partition function of the four node quiver theory with mass deformation also obey a modified version of $\mathfrak{q}$-PVI. Conversely, one could also investigate the grand partition function of the mass deformed CS quiver theories by exploiting the $\mathfrak{q}$-difference bilinear equations, this in particular concerning the novel phase transition which was discovered for the mass deformed ABJM theory in [141, 232, 233] and which is expected to exist also for more general quiver Chern-Simons matter theories with mass deformation [141, 230].

- Generalise the Chern-Simons matter quiver theory and identify its grand canonical partition function with a Fredholm determinant of a suitable quantum operator and a related integrable system.

- Investigate the insertion of observables and their rôle in these correspondences. It would be particularly interesting to find the relevant observable of the three-dimensional Chern-Simons quiver theory allowing to describe the full set of initial conditions of $\mathfrak{q}$-Painlevé equations, or from the five dimensional gauge theory viewpoint, the insertion of real co-dimension two defects. We expect this to provide a description of the wave functions of the associated quantum integrable systems.

- The relation between the S^3 quiver matter Chern Simons partition functions and the NO partition function of 5d gauge theories on $\mathbb{R}^4 \times S^1$ based on bilinear $\mathfrak{q}$-Painlevé is suitable to be dimensionally lift to a relation between a quiver supersymmetric gauge theory on $S^3 \times S^1$ and $\mathcal{N} = 1$ gauge theory on $\mathbb{R}^4 \times T^2$. Let us notice in this perspective that the Fermi gas formalism applied to the 3d partition function was crucial to the study of $\mathfrak{q}$-Painlevé system we performed. It is indeed known that the Fermi gas formalism extends to the Schur index of a certain class of four dimensional gauge theories [53, 54, 79] where the relevant integrand is as elliptic lift of our (3.11). It is therefore natural to expect that these Schur indices could be related to some gauge theories on $\mathbb{R}^4 \times T^2$ and to a related cluster integrable systems.

- A direct link between the three dimensional Chern-Simons quiver theories on S^3 and the gauge theories on $\mathbb{R}^4 \times S^1$ involved in this game is to our knowledge still missing. It is expected to arise from a chain of string theory dualities and geometric transitions, that it would be worth exploring also with the aim of a deeper understanding of the TS/ST correspondence.

3.3 D-type quiver

The Fermi gas formalism of superconformal Chern-Simons theories on S^3 reveals a quantum curve to which the TS/ST/tau formalism can be applied. However, by themselves, quivers of $\mathcal{N}=3$ $d=3$ Chern-Simons theories with bifundamental matter have an ADE classification [122]. ABJM, corresponding to local $\mathbb{P}^1\times\mathbb{P}^1$, can be seen to fit in the A_n type. A natural question is whether the D or E type quivers lead to interesting relations. This was the question my coauthor Nosaka-san and I set out to answer, so we considered the simplest D type quiver, the extended D_4 quiver. We only considered its rank deformation, without turning on any mass deformations or FI terms, in hope to connect with the $q-$Painlevé-like theory corresponding to massless $\mathfrak{g}=D_4=\mathfrak{so}(8)$ $d=5$ $\mathcal{N}=1$ gauge theory. In section 3.3.2 we also formally extended (or shrunk) the quiver to a $\hat{D}_3$ quiver, which is computationally much more tractable. The starting point was the following rank-deformation, which was found to be exactly computable:

$$\begin{tikzcd}
U(N)_0 \arrow[dr, shift left] & & U(N)_0 \arrow[dl, shift left] \\
& U(2N)_0 \arrow[ul, shift left] \arrow[ur, shift left] \arrow[dl, shift left] \arrow[dr, shift left] & \\
U(N+M)_{-2k} \arrow[ur, shift left] & & U(N+M)_{2k} \arrow[ul, shift left]
\end{tikzcd}$$

Figure 3.9: The rank-deformed $\hat{D}_4$ quiver.

The matrix model corresponding to this quiver Chern-Simons theory on S^3 is, by rules of supersymmetric localisation given in CITE,

$$Z=\frac{1}{(N!)^2((N+M)!)^2(2N)!}\int\left(\frac{\mathrm{d}\xi}{2\pi}\right)^N\left(\frac{\mathrm{d}\xi'}{2\pi}\right)^{N+M}\left(\frac{\mathrm{d}z}{2\pi}\right)^{2N}\left(\frac{\mathrm{d}\eta}{2\pi}\right)^N\left(\frac{\mathrm{d}\eta'}{2\pi}\right)^{N+M}e^{-\frac{ik}{2\pi}\sum\limits_{i=1}^{N+M}\xi_i'^2-\eta_i'^2}$$

$$\frac{\prod_{i<j}^N\left(2\sinh\frac{\xi_i-\xi_j}{2}\right)^2\left(2\sinh\frac{\eta_i-\eta_j}{2}\right)^2\prod_{i<j}^{N+M}\left(2\sinh\frac{\xi_i'-\xi_j'}{2}\right)^2\left(2\sinh\frac{\eta_i'-\eta_j'}{2}\right)^2\prod_{i<j}^{2N}\left(2\sinh\frac{z_i-z_j}{2}\right)^2}{\prod_{i=1}^{2N}\prod_{p=1}^N 2\cosh\frac{\xi_p-z_i}{2}2\cosh\frac{\eta_p-z_i}{2}\prod_{q=1}^{N+M}2\cosh\frac{\xi_q'-z_i}{2}2\cosh\frac{\eta_q'-z_i}{2}}$$

There are several Cauchy-van der Monde identities we use:

$$\begin{aligned}
\frac{\prod_{i<j}^{N} 2\sinh\frac{x_i-x_j}{2}\prod_{i<j}^{N+M} 2\sinh\frac{y_i-y_j}{2}}{\prod_{i=1}^{N}\prod_{j=1}^{N+M} 2\cosh\frac{x_i-y_j}{2}} &= (-1)^{NM}\prod_i e^{M\frac{x_i-y_i}{2}}\det\begin{pmatrix} \frac{1}{2\cosh\frac{x_i-y_j}{2}} \\ e^{\ell_r y_j}\end{pmatrix} \\
&= (-1)^{\binom{M}{2}}\prod_i e^{M\frac{y_i-x_i}{2}}\det\begin{pmatrix} \frac{1}{2\cosh\frac{x_i-y_j}{2}} \\ e^{-\ell_r y_j}\end{pmatrix} \\
\frac{\prod_{i<j}^{N} 2\sinh\frac{x_i-x_j}{2}\prod_{i<j}^{N+M} 2\sinh\frac{y_i-y_j}{2}}{\prod_{i=1}^{N}\prod_{j=1}^{N+M} 2\sinh\frac{x_i-y_j}{2}} &= (-1)^{\binom{N}{2}}\prod_i e^{M\frac{x_i-y_i}{2}}\det\begin{pmatrix} \frac{1}{2\sinh\frac{x_i-y_j}{2}} \\ e^{\ell_r y_j}\end{pmatrix} \\
&= (-1)^{\binom{N}{2}+\binom{M}{2}}\prod_i e^{M\frac{y_i-x_i}{2}}\det\begin{pmatrix} \frac{1}{2\sinh\frac{x_i-y_j}{2}} \\ e^{-\ell_r y_j}\end{pmatrix}
\end{aligned}$$

where $\ell_r = M + 1/2 - r$. We use all of them as to cancel all factors of $e^{\pm Mx/2}$ with $x = \xi, z, \eta$, that is, those with CS level zero. The result is

$$Z = \frac{1}{(N!)^2((N+M)!)^2(2N)!}\int\left(\frac{\mathrm{d}\xi}{2\pi}\right)^N\left(\frac{\mathrm{d}\xi'}{2\pi}\right)^{N+M}\left(\frac{\mathrm{d}z}{2\pi}\right)^{2N}\left(\frac{\mathrm{d}\eta}{2\pi}\right)^{N}\left(\frac{\mathrm{d}\eta'}{2\pi}\right)^{N+M}$$

$$\left(\prod_{i=1}^{N+M} e^{-\frac{ik}{2\pi}\xi_i'^2 - M\xi_i'}e^{\frac{ik}{2\pi}\eta_i'^2+M\eta_i'}\right)\det\begin{pmatrix}\left[\frac{1}{2\sinh\frac{\xi_i-\xi_j'}{2}}\right]_{i,j:N\times(N+M)} \\ \left[e^{\ell_r\xi_j'}\right]_{r,j:M\times(N+M)}\end{pmatrix}$$

$$\det\begin{pmatrix}\left[\frac{1}{2\cosh\frac{z_i-x_j}{2}}\right]_{i,j:2N\times(2N+M)} \\ \left[e^{\ell_r x_j}\right]_{r,j:M\times(2N+M)}\end{pmatrix}\det\begin{pmatrix}\left[\frac{1}{2\cosh\frac{z_i-y_j}{2}}\right]_{i,j:2N\times(2N+M)} \\ \left[e^{-\ell_r y_j}\right]_{r,j:M\times(2N+M)}\end{pmatrix}$$

$$\det\begin{pmatrix}\left[\frac{1}{2\sinh\frac{\eta_i-\eta_j'}{2}}\right]_{i,j:N\times(N+M)} \\ \left[e^{-\ell_r x_j}\right]_{r,j:M\times(N+M)}\end{pmatrix}$$

where $x_i = (\xi_i, \xi_i')$, $y_i = (\eta_i, \eta_i')$. Next we use the Matsumoto-Moriyama-Andréief formula [208, B.4],

$$\int \mathrm{d}^N x\det\begin{pmatrix}[f_i(x_j)]_{i,j:(N+M)\times N} & [v_{is}]_{i,s:(N+M)\times M}\end{pmatrix}\det\begin{pmatrix}[g_i(x_j)]_{i,j:(N+M')\times N} & [w_{is}]_{i,s:(N+M')\times M'}\end{pmatrix}$$

$$= N!(-1)^{MM'}\det\begin{pmatrix}\left[\int\mathrm{d}x f_i(x)g_j(x)\right]_{i,j:(N+M)\times(N+M')} & [v_{is}]_{i,s:(N+M)\times M} \\ [w_{is}]_{i,s:(N+M')\times M'} & [0]_{M\times M'}\end{pmatrix}$$

so that defining the convolution kernel

$$L(x,y) = \frac{1}{2\cosh\frac{x}{2}}\circ\frac{1}{2\cosh\frac{y}{2}} = \int\frac{\mathrm{d}z}{2\pi}\frac{1}{2\cosh\frac{x-z}{2}}\frac{1}{2\cosh\frac{y-z}{2}}$$

we get

$$Z = \frac{(-1)^M}{(N!)^2((N+M)!)^2} \int \left(\frac{\mathrm{d}\xi}{2\pi}\right)^N \left(\frac{\mathrm{d}\xi'}{2\pi}\right)^{N+M} \left(\frac{\mathrm{d}\eta}{2\pi}\right)^N \left(\frac{\mathrm{d}\eta'}{2\pi}\right)^{N+M}$$

$$\left(\prod_{i=1}^{N+M} e^{-\frac{ik}{2\pi}\xi_i'^2 - M\xi_i'} e^{\frac{ik}{2\pi}\eta_i'^2 + M\eta_i'}\right) \det \begin{pmatrix} \left[\frac{1}{2\sinh\frac{\xi_i - \xi_j'}{2}}\right]_{i,j:N\times(N+M)} \\ \left[e^{\ell_r \xi_j'}\right]_{r,j:M\times(N+M)} \end{pmatrix}$$

$$\det \begin{pmatrix} L(\xi_i,\eta_j) & L(\xi_i,\eta_j') & e^{\ell_r\xi_i} \\ L(\xi_i',\eta_j) & L(\xi_i',\eta_j') & e^{\ell_r\xi_i'} \\ e^{\ell_s\eta_j} & e^{\ell_s\eta_j'} & 0 \end{pmatrix} \det \begin{pmatrix} \left[\frac{1}{2\sinh\frac{\eta_i - \eta_j'}{2}}\right]_{i,j:N\times(N+M)} \\ \left[e^{-\ell_r x_j}\right]_{r,j:M\times(N+M)} \end{pmatrix}$$

Defining further

$$Y(x,y) = \frac{e^{\frac{ik}{2\pi}x^2 + Mx}}{2\sin\frac{x-y}{2}}, \quad V_r(x) = e^{\ell_r x}, \quad W_r(x) = e^{-\ell_r x}, \quad W_r^+(x) = e^{\frac{ik}{2\pi}x^2 + Mx} W_r(x),$$

$$X(x,y) = \frac{e^{-\frac{ik}{2\pi}y^2 - My}}{2\sin\frac{x-y}{2}}, V_r^-(x) = e^{-\frac{ik}{2\pi}x^2 - Mx} V_r(x)$$

lets us write further applications of the formula for the η' and then for the ξ' integrations as

$$Z = \frac{(-1)^M}{(N!)^2(N+M)!} \int \left(\frac{\mathrm{d}\xi}{2\pi}\right)^N \left(\frac{\mathrm{d}\xi'}{2\pi}\right)^{N+M} \left(\frac{\mathrm{d}\eta}{2\pi}\right)^N \left(\prod_{i=1}^{N+M} e^{-\frac{ik}{2\pi}\xi_i'^2 - M\xi_i'}\right)$$

$$\det \begin{pmatrix} \left[\frac{1}{2\sinh\frac{\xi_i - \xi_j'}{2}}\right]_{i,j:N\times(N+M)} \\ \left[e^{\ell_r \xi_j'}\right]_{r,j:M\times(N+M)} \end{pmatrix} \det \begin{pmatrix} L(\xi_i,\eta_j) & -(L\circ Y)(\xi_i,\eta_j) & (L\circ W_s^+)(\xi_i) & V_s(\xi_i) \\ L(\xi_i',\eta_j) & -(L\circ Y)(\xi_i',\eta_j) & (L\circ W_s^+)(\xi_i') & V_s(\xi_i') \\ W_r^t(\eta_j) & -(W_r^t\circ Y)(\eta_j) & W_r^t\circ W_s^+ & 0 \end{pmatrix}$$

$$= \frac{(-1)^M}{(N!)^2} \int \left(\frac{\mathrm{d}\xi}{2\pi}\right)^N \left(\frac{\mathrm{d}\eta}{2\pi}\right)^N \det \begin{pmatrix} L(\xi_i,\eta_j) & -(L\circ Y)(\xi_i,\eta_j) & (L\circ W_s^+)(\xi_i) & V_s(\xi_i) \\ (X\circ L)(\xi_i,\eta_j) & -(X\circ L\circ Y)(\xi_i,\eta_j) & (X\circ L\circ W_s^+)(\xi_i) & (X\circ V_s)(\xi_i) \\ (V_r^{-t}\circ L)(\eta_j) & -(V_r^{-t}\circ L\circ Y)(\eta_j) & V_r^{-t}\circ L\circ W_s^+ & V_r^{-t}\circ V_s \\ W_r^t(\eta_j) & -(W_r^t\circ Y)(\eta_j) & W_r^t\circ W_s^+ & 0 \end{pmatrix}$$

Next we consider a generalisation of the de Brujin identity,

$$\int \mathrm{d}^N x \det \begin{pmatrix} [f_i(x_j)]_{i,j:(2N+M)\times N} & [g_i(x_j)]_{i,j:(2N+M)\times N} & [v_{is}]_{i,s:(2N+M)\times M} \end{pmatrix}$$

$$= N!(-1)^{\binom{N}{2}+\binom{M}{2}} \mathrm{Pf} \begin{pmatrix} \left[\int \mathrm{d}x(f_i g_j - f_j g_i)\right]_{i,j:(2N+M)\times(2N+M)} & [v_{is}]_{i,s:(2N+M)\times M} \\ [-v_{si}]_{s,i:M\times(2N+M)} & [0]_{M\times M} \end{pmatrix}$$

to get rid of the η integration and get the final expression, where $\tilde{Y} = Y - Y^t$,

$$Z = \frac{(-1)^{\binom{N}{2}+\frac{M(M+1)}{2}}}{N!} \int \left(\frac{\mathrm{d}\xi}{2\pi}\right)^N$$

$$\mathrm{Pf} \begin{pmatrix} (L\circ\tilde{Y}\circ L^t)(\xi_i,\xi_j) & (L\circ\tilde{Y}\circ L^t\circ X^t)(\xi_i,\xi_j) & (L\circ\tilde{Y}\circ L^t\circ V_s^-)(\xi_i) & (L\circ\tilde{Y}\circ W_s)(\xi_i) & (L\circ W_s^+)(\xi_i) & V_s(\xi_i) \\ (X\circ L\circ\tilde{Y}\circ L^t)(\xi_i,\xi_j) & (X\circ L\circ\tilde{Y}\circ L^t\circ X^t)(\xi_i,\xi_j) & (X\circ L\circ\tilde{Y}\circ L^t\circ V_s^-)(\xi_i) & (X\circ L\circ\tilde{Y}\circ W_s)(\xi_i) & (X\circ L\circ W_s^+)(\xi_i) & (X\circ V_s)(\xi_i) \\ (V_r^{-t}\circ L\circ\tilde{Y}\circ L^t)(\xi_j) & (V_r^{-t}\circ L\circ\tilde{Y}\circ L^t\circ X^t)(\xi_j) & V_r^{-t}\circ L\circ\tilde{Y}\circ L^t\circ V_s^- & V_r^{-t}\circ L\circ\tilde{Y}\circ W_s & V_r^{-t}\circ L\circ W_s^+ & V_r^{-t}\circ V_s \\ (W_r^t\circ\tilde{Y}\circ L^t)(\xi_j) & (W_r^t\circ\tilde{Y}\circ L^t\circ X^t)(\xi_j) & W_r^t\circ\tilde{Y}\circ L^t\circ V_s^- & W_r^t\circ\tilde{Y}\circ W_s & W_r^t\circ W_s^+ & 0 \\ -(W_r^{+t}\circ L^t)(\xi_j) & -(W_r^{+t}\circ L^t\circ X^t)(\xi_j) & -W_r^{+t}\circ L^t\circ V_s^- & -W_r^{+t}\circ W_s & 0 & 0 \\ -V_r^t(\xi_j) & -(V_r^t\circ X^t)(\xi_j) & -V_r^t\circ V_s^- & 0 & 0 & 0 \end{pmatrix}$$

To calculate the grand canonical potential, we use the following conjectural formula for the rank-deformed extension of the Fredholm Pfaffian formula, found for $M = 0$ in [199, Proposition 2.1],

$$\sqrt{\det\left[\begin{pmatrix}\Omega & 0\\ 0 & 0\end{pmatrix} + \begin{pmatrix} zP & zV \\ -V^t & \alpha\end{pmatrix}\right]} = \sum_{k=0}^{N_\infty}(-1)^{\binom{k}{2}} z^k \sum_{\substack{S\subset\{1,2,\dots,N_\infty\}\\ |S|=k}} \mathrm{Pf}\begin{pmatrix} P_S & V_S \\ -V_S^t & \alpha\end{pmatrix}$$

where

$$\Omega = \begin{pmatrix} [0]_{N_\infty\times N_\infty} & [\delta_{ij}]_{i,j:N_\infty\times N_\infty} \\ [-\delta_{ij}]_{i,j:N_\infty\times N_\infty} & [0]_{N_\infty\times N_\infty}\end{pmatrix}, P = \begin{pmatrix} [A_{ij}]_{i,j:N_\infty\times N_\infty} & [B_{ij}]_{i,j:N_\infty\times N_\infty} \\ [C_{ij}]_{i,j:N_\infty\times N_\infty} & [D_{ij}]_{i,j:N_\infty\times N_\infty}\end{pmatrix},$$
$$V = \begin{pmatrix} [v_{is}]_{i,s:N_\infty\times 2M} \\ [w_{is}]_{i,s:N_\infty\times 2M}\end{pmatrix}, \alpha = \left([\alpha_{rs}]_{r,s:2M\times 2M}\right),$$

the sum is over ordered pairs $S = (n_1, n_2, ..., n_k)$ with $1 \neq n_1 \neq ... \neq n_k \neq N_\infty$ which determine

$$P_S = \begin{pmatrix} [A_{a_i,a_j}]_{i,j:k\times k} & [B_{a_i,a_j}]_{i,j:k\times k} \\ [C_{a_i,a_j}]_{i,j:k\times k} & [D_{a_i,a_j}]_{i,j:k\times k}\end{pmatrix}, V_S = \begin{pmatrix} [v_{a_i,s}]_{i,s:k\times 2M} \\ [w_{a_i,s}]_{i,s:k\times 2M}\end{pmatrix}.$$

The $N_\infty \to \infty$ limit implies that if we define the grand partition function as

$$\Xi_{k,M}(z) = (-1)^{\frac{M(M+1)}{2}} \sum_{N\geq 0} z^N Z(N)$$

then it becomes the Fredholm Pfaffian

$$\Xi_{k,M}(z) =$$
$$\sqrt{\det\begin{pmatrix}
zL\circ\tilde{Y}\circ L^t & \mathbb{1} + zL\circ\tilde{Y}\circ L^t\circ X^t & zL\circ\tilde{Y}\circ L^t\circ V_s^- & zL\circ\tilde{Y}\circ W_s & zL\circ W_s^+ & zV_s \\
-\mathbb{1} + zX\circ L\circ\tilde{Y}\circ L^t & zX\circ L\circ\tilde{Y}\circ L^t\circ X^t & zX\circ L\circ\tilde{Y}\circ L^t\circ V_s^- & zX\circ L\circ\tilde{Y}\circ W_s & zX\circ L\circ W_s^+ & zX\circ V_s \\
V_r^{-t}\circ L\circ\tilde{Y}\circ L^t & V_r^{-t}\circ L\circ\tilde{Y}\circ L^t\circ X^t & V_r^{-t}\circ L\circ\tilde{Y}\circ L^t\circ V_s^- & V_r^{-t}\circ L\circ\tilde{Y}\circ W_s & V_r^{-t}\circ L\circ W_s^+ & V_r^{-t}\circ V_s \\
W_r^t\circ\tilde{Y}\circ L^t & W_r^t\circ\tilde{Y}\circ L^t\circ X^t & W_r^t\circ\tilde{Y}\circ L^t\circ V_s^- & W_r^t\circ\tilde{Y}\circ W_s & W_r^t\circ W_s^+ & 0 \\
-W_r^{+t}\circ L^t & -W_r^{+t}\circ L^t\circ X^t & -W_r^{+t}\circ L^t\circ V_s^- & -W_r^{+t}\circ W_s & 0 & 0 \\
-V_r^t & -V_r^t\circ X^t & -V_r^t\circ V_s^- & 0 & 0 & 0
\end{pmatrix}}$$

Standard row and column operations we use to simplify this determinant are as follows, with $\tilde{X} = X - X^t$:

1. (3rd column) $\mapsto$ (3rd column) $-$ (1st column)$\circ V_s^-$
2. (2nd row) $\mapsto$ (2nd row) $-$ $X\circ$(1st row)
3. (2nd column) $\mapsto$ (2nd column) $-$ (1st column)$\circ X^t$
4. (1st row) $\mapsto$ (1st row) $+$ $zL\circ\tilde{Y}\circ L^t\circ$(2nd row)
5. (3rd column) $\mapsto$ (3rd column) $-$ (2nd column)$\circ(1 - zL\circ\tilde{Y}\circ L^t\circ\tilde{X})^{-1}\circ V_s^-$
6. (4th column) $\mapsto$ (4th column) $-$ (2nd column)$\circ(1 - zL\circ\tilde{Y}\circ L^t\circ\tilde{X})^{-1}\circ W_s$
7. (5th column) $\mapsto$ (5th column) $-$ (2nd column)$\circ(1 - zL\circ\tilde{Y}\circ L^t\circ\tilde{X})^{-1}\circ W_s^+$
8. (6th column) $\mapsto$ (6th column) $-$ (2nd column)$\circ(1 - zL\circ\tilde{Y}\circ L^t\circ\tilde{X})^{-1}\circ V_s$

They lead to the open-string factorised expression

$$\Xi_{k,M}(z) = \sqrt{\det(\mathbb{1} + z\rho)}\sqrt{\det(H)}$$

where $\rho = -L \circ \tilde{Y} \circ L^t \circ \tilde{X}$ is the M-independent inverse spectral curve and H is a $4M \times 4M$ matrix composed of 16 $M \times M$ blocks $[H_{ij}]_{i,j:4\times 4}$. If we define

$$L_1(z) = (1 - zL \circ \tilde{Y} \circ L^t \circ \tilde{X})^{-1}, L_2(z) = (1 - z\tilde{Y} \circ L^t \circ \tilde{X} \circ L)^{-1}$$
$$L_3(z) = (1 - zL \circ \tilde{X} \circ L^t \circ \tilde{Y})^{-1}, L_4(z) = (1 - z\tilde{X} \circ L \circ \tilde{y} \circ L^t)^{-1}$$

then we can write

$$\begin{pmatrix} H_{11} & H_{12} & H_{13} & H_{14} \\ H_{21} & H_{22} & H_{23} & H_{24} \\ H_{31} & H_{32} & H_{33} & H_{34} \\ H_{41} & H_{42} & H_{43} & H_{44} \end{pmatrix} =$$

$$\begin{pmatrix} V_r^{-t} \circ L_1(z) \circ L \circ \tilde{Y} \circ L^t \circ V_s^- & V_r^{-t} \circ L_1(z) \circ L \circ \tilde{Y} \circ W_s & V_r^{-t} \circ L_1(z) \circ L \circ W_s^+ & V_r^{-t} \circ L_1(z) \circ V_s \\ W_r^t \circ L_2(z) \circ \tilde{Y} \circ L^t \circ V_s^- & W_r^t \circ L_2(z) \circ \tilde{Y} \circ W_s & W_r^t \circ L_2(z) \circ W_s^+ & zW_r^t \circ L_2(z) \circ \tilde{Y} \circ L^t \circ \tilde{X} \circ V_s \\ -W_r^{+t} \circ L_3(z) \circ L^t \circ V_s^- & -W_r^{+t} \circ L_3(z) \circ W_s & -zW_r^{+t} \circ L_3(z) \circ L^t \circ \tilde{X} \circ L \circ W_s^+ & -zW_r^{+t} \circ L_3(z) \circ L^t \circ \tilde{X} \circ V_s \\ -V_r^t \circ L_4(z) \circ V_s^- & -zV_r^t \circ L_4(z) \circ \tilde{X} \circ L \circ \tilde{Y} \circ W_s & -zV_r^t \circ L_4(z) \circ \tilde{X} \circ L \circ W_s^+ & -zV_r^t \circ L_4(z) \circ \tilde{X} \circ V_s \end{pmatrix}$$

3.3.1 Quantum mechanics and divergences

It's convenient at this point to rescale the integration variables $x \mapsto x/k$, which rescales $L, \tilde{X}, \tilde{Y}$ by $1/k$ and V_r, V_r^-, W_s, W_s^+ by $1/\sqrt{k}$. We introduce quantum mechanical notation with canonically conjugate $[\hat{q}, \hat{p}] = 2\pi i k$ and label position eigenstates as ordinary kets $\hat{q}\,|x\rangle = x\,|x\rangle$ and momentum eigenstates as double bracketed kets $\hat{p}|x\rangle\!\rangle = x|x\rangle\!\rangle$ as in (3.56). Then

$$L(x, y) = \langle x|\,(2\cosh\frac{\hat{p}}{2})^{-2}\,|y\rangle\,,$$

$$\tilde{X}(x, y) = \langle x| \left\{\frac{\tanh\frac{\hat{p}}{2}}{2i}, e^{-\frac{i}{2\pi k}\hat{q}^2 - \frac{M}{k}\hat{q}}\right\} |y\rangle\,, \tilde{Y}(x, y) = \langle x| \left\{\frac{\tanh\frac{\hat{p}}{2}}{2i}, e^{\frac{i}{2\pi k}\hat{q}^2 + \frac{M}{k}\hat{q}}\right\} |y\rangle$$

which use

$$\lim_{\epsilon\to 0}\int \frac{\mathrm{d}p}{2\pi i}\tanh\frac{p}{2}e^{ixp - \epsilon p} = \frac{1}{\sinh \pi x}$$

and

$$\begin{aligned} V_r(x) &= \langle x| - 2\pi i\ell_r\rangle\!\rangle & V_r^t(x) &= \langle\!\langle 2\pi i\ell_r|x\rangle \\ V_r^-(x) &= \langle x|e^{-\frac{i}{2\pi k}\hat{q}^2 - \frac{M}{k}\hat{q}}| - 2\pi i\ell_r\rangle\!\rangle & V_r^{-t}(x) &= \langle\!\langle 2\pi i\ell_r|e^{-\frac{i}{2\pi k}\hat{q}^2 - \frac{M}{k}\hat{q}}|x\rangle \\ W_s(x) &= \langle x|2\pi i\ell_s\rangle\!\rangle & W_s^t(x) &= \langle\!\langle -2\pi i\ell_s|x\rangle \\ W_s^+(x) &= \langle x|e^{\frac{i}{2\pi k}\hat{q}^2 + \frac{M}{k}\hat{q}}|2\pi i\ell_s\rangle\!\rangle & W_s^{+t}(x) &= \langle\!\langle -2\pi i\ell_s|e^{\frac{i}{2\pi k}\hat{q}^2 + \frac{M}{k}\hat{q}}|x\rangle \end{aligned}$$

We can reformulate the block sub-matrices H_{ij} in terms of 1d quantum mechanics. However, we find that, as written, some of the elements diverge. Consider

$$H_{12} = \langle\!\langle 2\pi i\ell_r|e^{-\frac{i}{2\pi k}\hat{q}^2 - \frac{M}{k}\hat{q}}\left(1 - z\left(2\cosh\frac{\hat{p}}{2}\right)^{-2}\left\{\frac{\tanh\frac{\hat{p}}{2}}{2i}, e^{\frac{i}{2\pi k}\hat{q}^2 + \frac{M}{k}\hat{q}}\right\}\right.$$
$$\left.\left(2\cosh\frac{\hat{p}}{2}\right)^{-2}\left\{\frac{\tanh\frac{\hat{p}}{2}}{2i}, e^{-\frac{i}{2\pi k}\hat{q}^2 - \frac{M}{k}\hat{q}}\right\}\right)^{-1}\left(2\cosh\frac{\hat{p}}{2}\right)^{-2}\left\{\frac{\tanh\frac{\hat{p}}{2}}{2i}, e^{\frac{i}{2\pi k}\hat{q}^2 + \frac{M}{k}\hat{q}}\right\}|2\pi i\ell_s\rangle\!\rangle$$

At the very end of this expression, from the anticommutator we have

$$\tanh\frac{\hat{p}}{2}|2\pi i\ell_s\rangle\rangle = \tanh\pi i\left(M+\frac{1}{2}-s\right)|2\pi i\ell_s\rangle\rangle$$

which diverges. However, we have found that the divergences of the second and fourth columns respectively are proportional the first and third column, and verbatim with the rows. This is a case-by-case check. Therefore, the following elementary operations remove the divergences:

1. (2nd column) $\mapsto$ (2nd column) + $\frac{i}{2}\tanh\pi i\ell_s$ (3rd column)
2. (4nd column) $\mapsto$ (4nd column) $-$ $\frac{iz}{2}\tanh\pi i\ell_s$ (1st column)
3. (2nd row) $\mapsto$ (2nd row) + $\frac{i}{2}\tanh\pi i\ell_r$ (3rd row)
4. (4nd row) $\mapsto$ (4nd row) $-$ $\frac{iz}{2}\tanh\pi i\ell_r$ (1st row)

The blocks $H_{11}, H_{13}, H_{31}, H_{33}$ stayed the same. Let us display some of new the blocks of the transformed matrix H.

$$\begin{aligned}
H_{12} &= -V_r^{-t}\circ L_1(z)\circ L\circ Y^t\circ W_s,\\
H_{14} &= -V_r^{-t}\circ V_s + zV_r^{-t}\circ L_1(z)\circ L\circ\tilde{Y}\circ L^t\circ X\circ V_s\\
H_{22} &= -zW_r^t\circ Y\circ L_2(z)\circ L^t\tilde{X}\circ L\circ Y^t\circ W_s\\
H_{24} &= zW_r^t\circ Y\circ L_2(z)\circ L^t\circ X\circ V_s
\end{aligned}$$

In fact, these are enough to determine all the others. Using the antisymmetry of $\tilde{X}$ and $\tilde{Y}$, we can show that in the new matrix,

$$(H_{ij})_{rs} = -(H_{ji})_{sr}$$

Similarly, using $V_r(-x)^* = W_r(x)$, $V_r^-(-x)^* = W_r^+(x)$, $X(-x,-y)^* = Y^t(x,y)$, $Y(-x,-y)^* = X^t(x,y)$ and $L(-x,-y)^* = L(x,y)$ we have elementwise

$$H_{33} = zH_{11}^*, H_{34} = zH_{12}^*, H_{31} = -H_{13}^*, H_{32} = -H_{14}*$$
$$H_{42} = -H_{24}^*, H_{44} = zH_{22}^*$$

This means $H_{11}, H_{12}, H_{13}, H_{14}, H_{22}, H_{24}$ are enough to determine the others.

3.3.2 $\hat{D}_3$ quiver exact calculation

In fact, we can observe that everything up to now holds for the $\hat{D}_r$ quivers

$$\begin{tikzcd}
U(N)_0 \arrow[dr, shift left] & & & & & & U(N)_0 \arrow[dl, shift left] \\
& U(2N)_0 \arrow[ul, shift left] \arrow[dl, shift left] \arrow[r, shift left] & U(2N)_0 \arrow[l, shift left] \arrow[r, shift left] & \cdots \arrow[l, shift left] \arrow[r, shift left] & U(2N)_0 \arrow[l, shift left] \arrow[r, shift left] & U(2N)_0 \arrow[l, shift left] \arrow[ur, shift left] \arrow[dr, shift left] & \\
U(N+M)_{-2k} \arrow[ur, shift left] & & & & & & U(N+M)_{-k} \arrow[ul, shift left]
\end{tikzcd}$$

with $r-3$ $U(2N)_0$ nodes in the middle. The difference is that

$$L(x,y) = \int \frac{\mathrm{d}^{r-3}w}{(4\pi k)^{r-3}} \frac{1}{\cosh \frac{x-w_1}{2k}} \frac{1}{\cosh \frac{w_1-w_2}{2k}} \cdots \frac{1}{\cosh \frac{w_{r-3}-y}{2k}}$$

The property $L(-x,-y)^* = L(x,y)$ still holds. In the case of higher rank q-Painlevé equations, D_3 satisfied the same equation as $D_{r\geq 4}$. Therefore, we have tried computing the $r=3$, $\hat{D}_3$ model. We will comment on its $3d$ interpretation later. At the level of the exact computation, the difference is only that now

$$L(x,y) = \frac{1}{2k\cosh\frac{x-y}{2k}} = \langle x|\, \frac{1}{2\cosh\frac{\hat{p}}{2}}\, |y\rangle$$

This makes the $\hat{D}_3$ model vastly simpler to compute.

3.3.2.1 $M = 0$

The one particle density matrix for the $\hat{D}_r$ quiver is $\rho = -L \circ \tilde{Y} \circ L^t \circ \tilde{X}$, which becomes for $r=3$ in the 1d quantum mechanics notation the operator

$$\hat{\rho} = \frac{1}{2\cosh\frac{\hat{p}}{2}} \frac{\tanh\frac{\hat{p}}{2} + \tanh\frac{\hat{p}+2\hat{q}}{2}}{2} \frac{1}{2\cosh\frac{\hat{p}+2\hat{q}}{2}} \frac{\tanh\frac{\hat{p}}{2} + \tanh\frac{\hat{p}+2\hat{q}}{2}}{2}$$

Given the structure of $\hat{\rho}$, it makes sense to introduce a different basis of canonically conjugate position and momentum operators,

$$\hat{Q} = \hat{p} + 2\hat{q}, \quad \hat{P} = \hat{p}, \quad [\hat{Q}, \hat{P}] = 2\pi i\kappa, \quad \kappa = 2k$$

in terms of which $\hat{\rho}$ is similar to

$$\hat{\rho} \sim \sqrt{\frac{1}{2\cosh\frac{\hat{Q}}{2}}} \frac{\tanh\frac{\hat{Q}}{2} + \tanh\frac{\hat{P}}{2}}{2} \frac{1}{2\cosh\frac{\hat{P}}{2}} \frac{\tanh\frac{\hat{Q}}{2} + \tanh\frac{\hat{Q}}{2}}{2} \sqrt{\frac{1}{2\cosh\frac{\hat{Q}}{2}}}$$

Using the following Fourier transform formulas for $r = 0, 1, 2$

$$_Q\langle x|\, \frac{1}{2\cosh\frac{\hat{P}}{2}} \left(\frac{\tanh\frac{\hat{P}}{2}}{2}\right)^r |y\rangle_Q = \frac{1}{2\kappa\cosh\frac{x-y}{2\kappa}} \left(\delta_{r0} + \delta_{r1}\frac{i(x-y)}{2\pi\kappa} + \delta_{r3}\left[\frac{1}{8} - \frac{1}{2}\left(\frac{x-y}{2\pi k}\right)^2\right]\right)$$

we can express the matrix elements as

$$_Q\langle x|\,\hat{\rho}\,|y\rangle_Q = \frac{E(x)E(y)}{e^{\frac{x}{\kappa}}+e^{\frac{y}{\kappa}}}\frac{1}{\kappa}\sum_{i=1}^{r} f_i(x)g_i(y)$$

where

$$\begin{aligned}
E(x) &= \frac{e^{\frac{x}{2\kappa}}}{\sqrt{2\cosh\frac{x}{2}}} \\
f_1 &= \frac{1}{16}+\frac{ix\tanh\frac{x}{2}}{4\pi\kappa}-\frac{1}{2}\left(\frac{x}{2\pi\kappa}\right)^2, \qquad g_1(x)=1 \\
f_2(x) &= 1, \qquad\qquad g_2(x)=\frac{1}{16}-\frac{ix\tanh\frac{x}{2}}{4\pi\kappa}-\frac{1}{2}\left(\frac{x}{2\pi\kappa}\right)^2 \\
f_3(x) &= \frac{\tanh\frac{x}{2}}{2}+\frac{ix}{2\pi\kappa}, \qquad\qquad g_3(x)=\frac{\tanh\frac{x}{2}}{2}-\frac{ix}{2\pi\kappa},
\end{aligned}$$

This enables us to use the TWYP formalism mentioned in the introduction. Namely, the previous expression is equivalent to

$$e^{\frac{\hat{Q}}{\kappa}}\hat{\rho}+\hat{\rho}e^{\frac{\hat{Q}}{\kappa}} = \sum_{i=1}^{3} f_i(\hat{Q})E(\hat{Q})|0\rangle\!\rangle_{PP}\langle\!\langle 0|E(\hat{Q})g_i(\hat{Q})$$

and can be seen to lead to a recursive expression for $\hat{\rho}^n$, whose traces are needed to compute the spectral determinant,

$$e^{\frac{\hat{Q}}{\kappa}}\hat{\rho}^n+(-1)^n\hat{\rho}^n e^{\frac{\hat{Q}}{\kappa}} = \sum_{l=0}^{n-1}(-1)^l\hat{\rho}^l\left[\sum_{i=1}^{3} f_i(\hat{Q})E(\hat{Q})|0\rangle\!\rangle_{PP}\langle\!\langle 0|E(\hat{Q})g_i(\hat{Q})\right]\hat{\rho}^{n-1-l}$$

which lets us express the matrix element in the $\hat{Q}$ basis as

$$\begin{aligned}
_Q\langle x|\,\hat{\rho}^n\,|y\rangle_Q &= \frac{E(x)E(y)}{e^{\frac{x}{\kappa}}-(-1)^n e^{\frac{y}{k}}}\sum_{l=0}^{n-1}\sum_{i=1}^{3}(-1)^l\phi_l^{(i)}(x)\psi_{n-1-l}^{(i)}(y), \\
\phi_l^{(i)}(x) = \frac{1}{E(x)}{}_Q\langle x|\,\hat{\rho}^l f_i(\hat{Q})E(\hat{Q})|0\rangle\!\rangle_P, &\quad \phi_l^{(i)}(x)=\frac{1}{E(x)}{}_P\langle\!\langle 0|E(\hat{Q})g_i(\hat{Q})\hat{\rho}^l\,|x\rangle_Q
\end{aligned}$$

It's obvious we have the recursion formula

$$\phi_l^{(i)}(x) = \int\frac{\mathrm{d}y}{2\pi}\frac{E(y)}{E(x)}{}_Q\langle x|\,\hat{\rho}\,|y\rangle_Q\,\phi_{l-1}^{(i)}(y)$$

The substitution $u=e^{\frac{x}{\kappa}}=e^{\frac{x}{2k}}$, $v=e^{\frac{y}{2k}}$ lets us rewrite this in terms of rational functions. Explicitly,

$$\phi_l^{(i)}(u) = \int_0^\infty\frac{\mathrm{d}y}{2\pi}\frac{v^k}{(u+v)(v^{2k}+1)}\left[\frac{1}{8}+\frac{(u^{2k}-1)(v^{2k}-1)}{4(u^{2k}+1)(v^{2k}+1}+\frac{i}{4\pi}\left(\frac{u^{2k}-1}{u^{2k}+1}+\frac{v^{2k}-1}{v^{2k}+1}\right)\log u\right.$$

$$\left.-\frac{1}{8\pi^2}(\log u)^2+\left[-\frac{i}{4\pi}\left(\frac{u^{2k}-1}{u^{2k}+1}+\frac{v^{2k}-1}{v^{2k}+1}\right)+\frac{\log u}{4\pi^2}\right]\log v-\frac{1}{8\pi^2}(\log v)^2\right]\phi_{l-1}^{(i)}(v)$$

Expanding $\phi_l^{(i)}(u)$ as

$$\phi_l^{(i)}(u) = \sum_{j\geq 0} \phi_l^{(i),j}(u)(\log u)^j$$

where $\phi_l^{(i),j}(u)$ are rational functions of their argument lets us use the following formula valid for a rational function $f(v)$ and $j \geq 0$,

$$\begin{aligned}\int_0^\infty \mathrm{d}v f(v)(\log v)^j &= -\frac{(2\pi i)^j}{j+1}\oint_\gamma \mathrm{d}v f(v) B_{j+1}\left(\frac{\log^+ v}{2\pi i}\right)\\ &= -\frac{(2\pi i)^{j+1}}{j+1}\sum_{w:\text{poles in }\mathbb{C}\backslash\mathbb{R}_{\geq 0}} \mathrm{Res}_{v=w}\, f(v)B_{j+1}\left(\frac{\log^+ v}{2\pi i}\right)\end{aligned}$$

where $\gamma \in \mathbb{C}$ is a keyhole contour avoiding the positive reals and $B_j(x)$ are Bernoulli polynomials. Then

$$\phi_l^{(i)}(u) = \sum_{j\geq 0}\sum_{w:\text{poles in }\mathbb{C}\backslash\mathbb{R}_{\geq 0}} \mathrm{Res}_{v=w}\frac{1}{2\pi}\frac{v^k}{(u+v)(v^{2k}+1)}\phi_{l-1}^{(i),j}(v)$$

$$\left[-\frac{(2\pi i)^{j+1}}{j+1}B_{j+1}\left(\frac{\log^+ v}{2\pi i}\right)\left(\frac{1}{8}+\frac{(u^{2k}-1)(v^{2k}-1)}{4(u^{2k}+1)(v^{2k}+1}+\frac{i}{4\pi}\left(\frac{u^{2k}-1}{u^{2k}+1}+\frac{v^{2k}-1}{v^{2k}+1}\right)\log u-\frac{1}{8\pi^2}(\log u)^2\right)\right.$$

$$-\frac{(2\pi i)^{j+2}}{j+2}B_{j+2}\left(\frac{\log^+ v}{2\pi i}\right)\left[-\frac{i}{4\pi}\left(\frac{u^{2k}-1}{u^{2k}+1}+\frac{v^{2k}-1}{v^{2k}+1}\right)+\frac{\log u}{4\pi^2}\right]$$
$$\left.-\frac{(2\pi i)^{j+3}}{j+3}B_{j+3}\left(\frac{\log^+ v}{2\pi i}\right)\left(-\frac{1}{8\pi^2}\right)\right]$$

The poles themselves are $v=-u$ and $v=e^{\frac{\pi i}{k}(m-1/2)}$ for $m=1,2,...,2k$. $\psi_l^i(u)$ are then

$$\psi_l^{(1)}(u) = (\phi_l^{(2)})^*, \quad \psi_l^{(2)}(u) = (\phi_l^{(1)})^*, \quad \psi_l^{(3)}(u) = (\phi_l^{(3)})^*$$

Then the traces may be calculated as

$$\begin{aligned}\mathrm{tr}\,\hat\rho^n &= \begin{cases}\frac{k}{2\pi}\int_0^\infty \mathrm{d}u\frac{u^{k-1}}{u^{2k}+1}\sum\limits_{l=0}^{n-1}(-1)^l\sum\limits_{i=1}^{3}\phi_l^{(i)}(u)\psi_{n-1-l}^{(i)}(u), & n \text{ odd}\\ \frac{k}{2\pi}\int_0^\infty \mathrm{d}u\frac{u^{k}}{u^{2k}+1}\sum\limits_{l=0}^{n-1}(-1)^l\sum\limits_{i=1}^{3}\frac{\mathrm{d}\phi_l^{(i)}(u)}{\mathrm{d}u}\psi_{n-1-l}^{(i)}(u), & n \text{ even}\end{cases}\\ &= \begin{cases}\sum\limits_{j\geq 0}\sum\limits_{w:\text{poles in }\mathbb{C}\backslash\mathbb{R}_{\geq 0}}\mathop{\mathrm{Res}}\limits_{u=w}\frac{k}{2\pi}\frac{u^{k-1}}{u^{2k}+1}\Phi_n^j(u)\left(-\frac{(2\pi i)^{j+1}}{j+1}\right)B_{j+1}\left(\frac{\log^+ u}{2\pi i}\right), & n \text{ odd}\\ \sum\limits_{j\geq 0}\sum\limits_{w:\text{poles in }\mathbb{C}\backslash\mathbb{R}_{\geq 0}}\mathop{\mathrm{Res}}\limits_{u=w}\frac{k}{2\pi}\frac{u^{k}}{u^{2k}+1}\tilde\Phi_n^j(u)\left(-\frac{(2\pi i)^{j+1}}{j+1}\right)B_{j+1}\left(\frac{\log^+ u}{2\pi i}\right), & n \text{ even}\end{cases}\end{aligned}$$

where $\Phi_n^j(u)$, $\tilde\Phi_n^j(u)$ are the rational functions defined from the expansions

$$\sum_{l=0}^{n-1}\sum_{i=1}^{3}\phi_l^{(i)}(u)\psi_{n-1-l}^{(i)}(u) = \sum_{j\geq 0}\Phi_n^j(u)(\log u)^j$$

$$\sum_{l=0}^{n-1}(-1)^l\sum_{i=1}^{3}\frac{\mathrm{d}\phi_l^{(i)}(u)}{\mathrm{d}u}\psi_{n-1-l}^{(i)}(u) = \sum_{j\geq 0}\tilde\Phi_n^j(u)(\log u)^j$$

For $k = 1$ we obtain the following spectral traces:

$$\begin{aligned}
&\operatorname{tr}\hat{\rho} = \frac{1}{32}, \quad \operatorname{tr}\hat{\rho}^2 = \frac{3}{2048} - \frac{1}{96\pi^2}, \quad \operatorname{tr}\hat{\rho}^3 = -\frac{35}{524288} + \frac{11}{15360\pi^2}, \\
&\operatorname{tr}\hat{\rho}^4 = -\frac{105}{16777216} + \frac{323}{5160960\pi^2}, \quad \operatorname{tr}\hat{\rho}^5 = \frac{3063808059358336\pi^2 + 5733585\pi^4}{28701118955520\pi^4}, \\
&\operatorname{tr}\hat{\rho}^6 = \frac{13971}{549755813888} - \frac{341497159}{1269625887129600\pi^2} + \frac{1}{5529600\pi^4} \\
&\operatorname{tr}\hat{\rho}^7 = \frac{2298257971609600 + 16378220515098624\pi^2 + 30139135715681280\pi^4 3223083922183125\pi^6}{4910354586227158548480000\pi^6}
\end{aligned}$$

We can calculate for higher k as well. This lets us read off the partition functions $Z_{k=1,M=0}(N)$. For the $k = 1, 2, 3$, the first three read,

$$\begin{aligned}
Z_{1,0}(1) &= \frac{1}{64}, \quad Z_{1,0}(2) = -\frac{1}{4096} + \frac{1}{384\pi^2}, \\
Z_{1,0}(3) &= -\frac{17}{1048576} + \frac{59}{368640\pi^2} \\
Z_{2,0}(1) &= \frac{1}{128}, \quad Z_{2,0}(2) = \frac{35}{262144} - \frac{1}{768\pi^2}, \\
Z_{2,0}(3) &= \frac{313}{33554432} - \frac{41}{1572864\pi} - \frac{1}{98304\pi^2} \\
Z_{3,0}(1) &= \frac{1}{196}, \quad Z_{3,0}(2) = -\frac{883}{10077696} + \frac{1}{1152\pi^2}, \\
Z_{3,0}(3) &= -\frac{7172861}{1671768834048} + \frac{29}{1574640\sqrt{3}\pi} + \frac{89}{9953280\pi^2}
\end{aligned}$$

Already for small N, plotting these against the semiclassical expansion

$$\begin{gathered}
Z^{\text{pert.}}_{k,0}(N) = e^{A_k} C_k^{-1/3} \operatorname{Ai}\left[C_k^{-1/3}(N - B_k) \right], \\
C_k = \frac{1}{3\pi^2 k}, \quad B_k = \frac{\pi^2 C_k}{3} - \frac{5}{18k} + \frac{k}{4}, \quad A_k = \frac{A^{\text{ABJM}}_{6k} + 9A^{\text{ABJM}}_{2k}}{2}
\end{gathered}$$

already shows similar behaviour. This is a good check of correctness of our calculation.

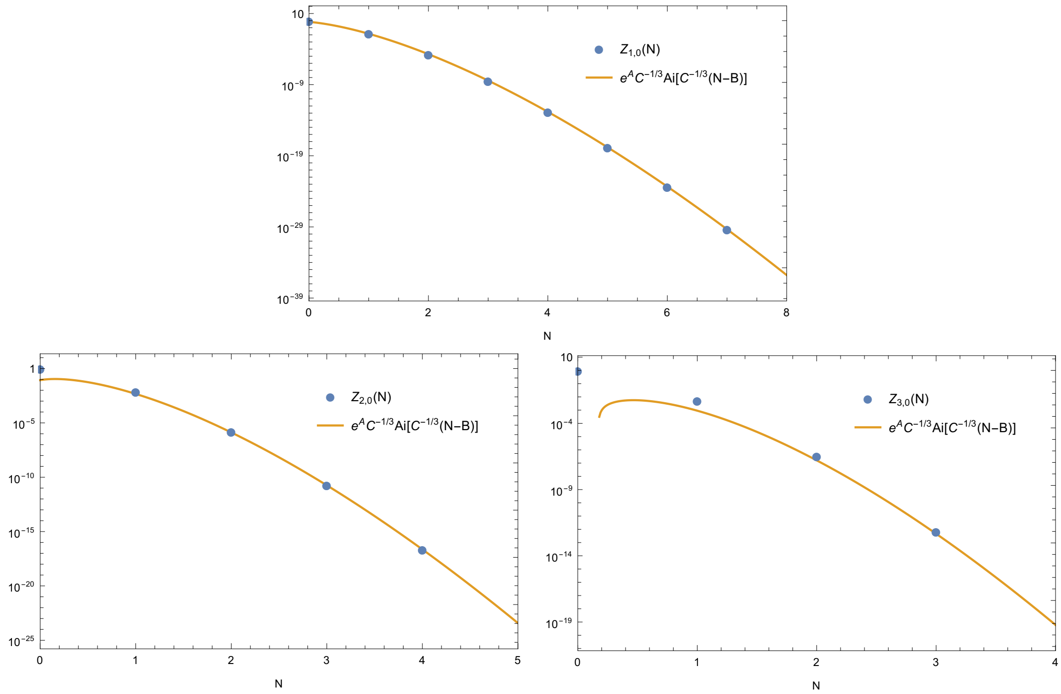

Figure 3.10: Plots of the $\hat{D}_3$-quiver model against the semiclassical expansions for $k = 1, 2, 3$.

3.3.2.2 Equivalence to $(3,1)_{2k}$ model

Consider again the partition function for the $M = 0$ $\hat{D}_3$ quiver, which we can get by substituting back $L(x, y) = (2\cosh\frac{x-y}{2})^{-1}$,

$$
\begin{aligned}
Z_{k,M=0}(N) = &\frac{1}{(N!)^K} \int \left(\frac{\mathrm{d}\xi}{2\pi}\right)^N \int \left(\frac{\mathrm{d}\xi'}{2\pi}\right)^N \int \left(\frac{\mathrm{d}\eta}{2\pi}\right)^N \int \left(\frac{\mathrm{d}\eta'}{2\pi}\right)^N \prod_i e^{\frac{ik}{2\pi}(\eta_i'^2 - \xi_i'^2)} \\
&\det_{i,j}^{N} \left(\frac{1}{2\sinh\frac{\eta_i - \eta_j'}{2}}\right) \det_{i,j}^{2N} \left(\frac{1}{2\cosh\frac{x_i - y_j}{2}}\right) \det_{i,j}^{N} \left(\frac{1}{2\sinh\frac{\xi_i - \xi_j'}{2}}\right) \\
= &\frac{1}{(N!)^K} \int \left(\frac{\mathrm{d}\xi}{2\pi}\right)^N \int \left(\frac{\mathrm{d}\xi'}{2\pi}\right)^N \int \left(\frac{\mathrm{d}\eta}{2\pi}\right)^N \int \left(\frac{\mathrm{d}\eta'}{2\pi}\right)^N \prod_i e^{\frac{ik}{2\pi}(\eta_i'^2 - \xi_i'^2)} \\
&\frac{\prod_{i<j} \left(2\sinh\frac{\xi_i - \xi_j}{2}\right)^2 \left(2\sinh\frac{\xi_i' - \xi_j'}{2}\right)^2 \left(2\sinh\frac{\eta_i - \eta_j}{2}\right)^2 \left(2\sinh\frac{\eta_i' - \eta_j'}{2}\right)^2}{\prod_{i,j} 2\cosh\frac{\xi_i - \eta_j}{2} 2\cosh\frac{\xi_i - \eta_j'}{2} 2\cosh\frac{\xi_i' - \eta_j}{2} 2\cosh\frac{\xi_i' - \eta_j'}{2}} \\
= &\frac{1}{N!} \int \left(\frac{\mathrm{d}\xi}{2\pi}\right)^N \det_{i,j} \langle \xi_i | \frac{1}{2\cosh\frac{\hat{p}}{2}} \frac{1}{2\cosh\frac{\hat{p}}{2}} e^{-\frac{i}{2\pi k}\hat{q}^2} \frac{1}{2\cosh\frac{\hat{p}}{2}} e^{\frac{i}{2\pi k}\hat{q}^2} \frac{1}{2\cosh\frac{\hat{p}}{2}} |\xi_j\rangle
\end{aligned}
$$

where $|\xi\rangle$ are $\hat{q}$ eigenstates. This expression coincides with the partition function of the $(3,1)$ model of $\mathcal{N} = 4$ circular quiver superconformal Chern-Simons theories with level $2k$. The exact values of $Z_k(N)$ were already computed for this model in [129]. They agree with our results.

This leads to an interesting conjecture, as the equivalence of partition functions means

$$\sum_{N\geq 0} z^N Z_{k,0}^{\hat{D}_3}(N) = \sqrt{\det\left(1+z\frac{1}{2\cosh\frac{\hat{Q}}{2}}\frac{\tanh\frac{\hat{Q}}{2}+\tanh\frac{\hat{P}}{2}}{2}\frac{1}{2\cosh\frac{\hat{P}}{2}}\frac{\tanh\frac{\hat{Q}}{2}+\tanh\frac{\hat{P}}{2}}{2}\right)}$$

$$\sum_{N\geq 0} z^N Z_{2k}^{(3,1)}(N) = \det\left(1+z\frac{1}{(2\cosh\frac{\hat{Q}}{2})^3}\frac{1}{2\cosh\frac{\hat{P}}{2}}\right)$$

are the same, which seems nontrivial not only from the point of view of operator theory but also from a geometric standpoint, as the latter operator can be easily inverted and be expressed in terms of a Newton polygon. For the former, this geometric side is unclear.

3.3.2.3 $M > 0$

First, for $N = 0$ we can easily obtain

$$\begin{aligned}
Z_{k,M}(0) &= (-1)^M \det\begin{pmatrix} V_r^{-t}\circ L\circ W_s^+ & V_r^{-t}\circ V_s \\ W_r^t\circ W_s^+ & 0\end{pmatrix} \\
&= (-1)^M \det\begin{pmatrix} \frac{(-1)^M e^{-\frac{\pi i}{2k}(\ell_r^2-\ell_s^2)}}{4k\cos\frac{\pi(\ell_r-\ell_s)}{2k}} & (2ki)^{-1/2}e^{-\frac{\pi i}{2k}(\ell_r+\ell_s-M)^2} \\ (2k/i)^{-1/2}e^{\frac{\pi i}{2k}(\ell_r+\ell_s-M)^2} & 0\end{pmatrix}
\end{aligned}$$

Interestingly, this matrix satisfies $Z_{k,M}(0) = Z_{k,2k-M}(0)$, and it vanishes for $M > 2k$. It would be interesting to see whether this "Hanany-Witten duality" and "s-rule" hold for the full theory. For both $M, N > 0$, we have to calculate both the spectral traces $\operatorname{tr}\hat{\rho}^n$ as well as the block-matrix corrections. Luckily, it turns out we are firmly in the open-string formalism, as $\hat{\rho}(M) \sim \hat{\rho}(0)$. To see this, write

$$\begin{aligned}
\hat{\rho}(M) &= -\frac{1}{2\cosh\frac{\hat{P}}{2}}\left\{\frac{\tanh\frac{\hat{P}}{2}}{2i}, e^{\frac{i}{4\pi\kappa}\hat{Q}^2+\frac{M}{\kappa}\hat{Q}}\right\}\frac{1}{2\cosh\frac{\hat{P}}{2}}\left\{\frac{\tanh\frac{\hat{P}}{2}}{2i}, e^{-\frac{i}{4\pi\kappa}\hat{Q}^2-\frac{M}{\kappa}\hat{Q}}\right\} \\
&= \frac{1}{2\cosh\frac{\hat{P}}{2}}\left[\frac{\tanh\frac{\hat{P}}{2}}{2}e^{\frac{i}{4\pi\kappa}\hat{Q}^2+\frac{M}{\kappa}\hat{Q}}\frac{1}{2\cosh\frac{\hat{P}}{2}}e^{-\frac{i}{4\pi\kappa}\hat{Q}^2-\frac{M}{\kappa}\hat{Q}}\frac{\tanh\frac{\hat{P}}{2}}{i}\right. \\
&\quad+\frac{\tanh\frac{\hat{P}}{2}}{2}e^{\frac{i}{4\pi\kappa}\hat{Q}^2+\frac{M}{\kappa}\hat{Q}}\frac{1}{2\cosh\frac{\hat{P}}{2}}\frac{\tanh\frac{\hat{P}}{2}}{i}e^{-\frac{i}{4\pi\kappa}\hat{Q}^2-\frac{M}{\kappa}\hat{Q}} \\
&\quad+e^{\frac{i}{4\pi\kappa}\hat{Q}^2+\frac{M}{\kappa}\hat{Q}}\frac{1}{2\cosh\frac{\hat{P}}{2}}\frac{\tanh\frac{\hat{P}}{2}}{2}e^{-\frac{i}{4\pi\kappa}\hat{Q}^2-\frac{M}{\kappa}\hat{Q}}\frac{\tanh\frac{\hat{P}}{2}}{i} \\
&\quad\left.+e^{\frac{i}{4\pi\kappa}\hat{Q}^2+\frac{M}{\kappa}\hat{Q}}\frac{1}{2\cosh\frac{\hat{P}}{2}}\left(\frac{\tanh\frac{\hat{P}}{2}}{i}\right)^2 e^{-\frac{i}{4\pi\kappa}\hat{Q}^2-\frac{M}{\kappa}\hat{Q}}\right]
\end{aligned}$$

Then the similarity transformation $\hat{\rho}(M) \mapsto e^{-\frac{i}{4\pi\kappa}\hat{P}^2}e^{-\frac{i}{4\pi\kappa}\hat{Q}^2}\hat{\rho}(M)e^{\frac{i}{4\pi\kappa}\hat{Q}^2}e^{\frac{i}{4\pi\kappa}\hat{P}^2}$ gets rid of some of the $e^{\pm\frac{i}{4\pi\kappa}\hat{Q}^2}$ factors, and this along with the identity

$$e^{-\frac{i}{4\pi\kappa}\hat{P}^2}e^{-\frac{i}{4\pi\kappa}\hat{Q}^2}f(\hat{P})e^{\frac{i}{4\pi\kappa}\hat{Q}^2}e^{\frac{i}{4\pi\kappa}\hat{P}^2} = f(\hat{Q})$$

yields

$$\hat{\rho}(M) \sim \frac{1}{2\cosh\frac{\hat{Q}}{2}}\left[\frac{\tanh\frac{\hat{Q}}{2}}{2}e^{\frac{M}{\kappa}\hat{Q}}\frac{1}{2\cosh\frac{\hat{P}}{2}}e^{-\frac{M}{\kappa}\hat{Q}}\frac{\tanh\frac{\hat{Q}}{2}}{2} + \frac{\tanh\frac{\hat{Q}}{2}}{2}e^{\frac{M}{\kappa}\hat{Q}}\frac{1}{2\cosh\frac{\hat{P}}{2}}\frac{\tanh\frac{\hat{P}}{2}}{2}e^{-\frac{M}{\kappa}\hat{Q}}\right.$$

$$\left. + e^{\frac{M}{\kappa}\hat{Q}}\frac{1}{2\cosh\frac{\hat{P}}{2}}\frac{\tanh\frac{\hat{P}}{2}}{2}e^{-\frac{M}{\kappa}\hat{Q}}\frac{\tanh\frac{\hat{Q}}{2}}{2} + e^{\frac{M}{\kappa}\hat{Q}}\frac{1}{2\cosh\frac{\hat{P}}{2}}\left(\frac{\tanh\frac{\hat{P}}{2}}{2}\right)^2 e^{-\frac{M}{\kappa}\hat{Q}}\right]$$

$$= e^{\frac{M}{\kappa}\hat{Q}}\frac{1}{2\cosh\frac{\hat{Q}}{2}}\frac{\tanh\frac{\hat{Q}}{2}+\tanh\frac{\hat{P}}{2}}{2}\frac{1}{2\cosh\frac{\hat{P}}{2}}\frac{\tanh\frac{\hat{Q}}{2}+\tanh\frac{\hat{P}}{2}}{2}e^{-\frac{M}{\kappa}\hat{Q}}$$

where the i-periodicity of the hyperbolic functions was used. Therefore, we do not need to calculate the traces again, as $\operatorname{tr}\hat{\rho}(M)^n = \operatorname{tr}\hat{\rho}(M=0)^n$.

To calculate the block-matrix correction, recursive methods can be used. First, note that due to the presence of poles at $\pi i m$, $m \in \mathbb{Z}$, the following formulas hold for $M \geq 0$, as can be seen by using a rectangular contour,

$$e^{-\frac{M}{\kappa}x}\langle x|\frac{1}{2\cosh\frac{\hat{p}}{2}}\left(\frac{\tanh\frac{\hat{p}}{2}}{2}\right)^i|y\rangle e^{\frac{M}{\kappa}y} = (-1)^M\langle x|\frac{1}{2\cosh\frac{\hat{p}}{2}}|y\rangle + \Delta_i(x,y)$$

where we need only

$$\Delta_0(x,y) = \frac{1}{\kappa}\sum_{m=1}^{M}(-1)^{m-1}e^{-\frac{\ell_m}{\kappa}(x-y)}, \quad \Delta_1(x,y) = \frac{i(x-y)}{2\pi\kappa}\Delta_0(x,y)$$

$$\Delta_2(x,y) = \left(\frac{1}{8} - \frac{1}{2}\left(\frac{x-y}{2\pi\kappa}\right)^2\right)\Delta_0(x,y)$$

Whereas for the opposite sign, writing also $M \leq 0$,

$$e^{\frac{M}{\kappa}x}\langle x|\frac{1}{2\cosh\frac{\hat{p}}{2}}\left(\frac{\tanh\frac{\hat{p}}{2}}{2}\right)^i|y\rangle e^{-\frac{M}{\kappa}y} = (-1)^M\langle x|\frac{1}{2\cosh\frac{\hat{p}}{2}}|y\rangle + \tilde{\Delta}_i(x,y)$$

with

$$\tilde{\Delta}_0(x,y) = \frac{1}{\kappa}\sum_{m=1}^{M}(-1)^{m-1}e^{\frac{\ell_m}{\kappa}(x-y)}, \quad \tilde{\Delta}_1(x,y) = \frac{i(x-y)}{2\pi\kappa}\tilde{\Delta}_0(x,y)$$

$$\tilde{\Delta}_2(x,y) = \left(\frac{1}{8} - \frac{1}{2}\left(\frac{x-y}{2\pi\kappa}\right)^2\right)\tilde{\Delta}_0(x,y)$$

Then, expanding the first column,

$$(H_{11})_{rs} = V_r^{-t} \circ L \circ \tilde{Y} \circ \left(1 - zL^t \circ \tilde{X} \circ L \circ \tilde{Y}\right)^{-1} \circ L^t \circ V_s^- = \sum_{n\geq 0} z^n (H_{11})_{rs,n}$$

$$(H_{21})_{rs} = W_r^t \circ Y \circ \left(1 - zL^t \circ \tilde{X} \circ L \circ \tilde{Y}\right)^{-1} \circ L^t \circ V_s^- = \sum_{n\geq 0} z^n (H_{21})_{rs,n}$$

$$(H_{31})_{rs} = -W_r^{+t} \circ Y \circ \left(1 - zL^t \circ \tilde{X} \circ L \circ \tilde{Y}\right)^{-1} \circ L^t \circ V_s^- = \sum_{n\geq 0} z^n (H_{31})_{rs,n}$$

$$\begin{aligned}(H_{41})_{rs} &= -V_r^t \circ V_s^- + zV_r^t \circ X^t \circ L \circ \tilde{Y} \circ \left(1 - zL^t \circ \tilde{X} \circ L \circ \tilde{Y}\right)^{-1} \circ L^t \circ V_s^- \\ &= -V_r^t \circ V_s^- + \sum_{n\geq 0} z^{n+1} (H_{41})_{rs,n}\end{aligned}$$

we find we can write

$$\begin{aligned}(H_{11})_{rs,n} &= -\frac{\sqrt{i}}{2\pi} e^{-\frac{\pi i}{\kappa}\ell_r^2 - \frac{\pi i}{\kappa}\ell_r} \int \mathrm{d}u \Bigg[\frac{(-1)^M}{v_r + u}\frac{i(\log v_r - \log u)}{2\pi} + \kappa\Delta_1(-2\pi i\ell_r, x) e^{\frac{\pi i}{\kappa}\ell_r - \frac{x}{2\kappa}} \\ &\quad + \left(\frac{(-1)^M}{v_r + u} + \kappa\Delta_0(-2\pi i\ell_r, x) e^{\frac{\pi i}{\kappa}\ell_r - \frac{x}{2\kappa}}\right)\left(-\frac{1}{2}\frac{u^{2k}-1}{u^{2k}+1}\right)\Bigg] \lambda_{s,n}^{(1)}(u) \\ (H_{21})_{rs,n} &= \frac{i\sqrt{\kappa}}{2\pi} e^{\frac{\pi i}{\kappa}(M-\ell_r)^2} \int \mathrm{d}u \frac{1}{2}\frac{u^{2k}-1}{u^{2k}+1} u^{M-\ell_r-\frac{1}{2}} \lambda_{s,n}^{(1)}(u) \\ (H_{31})_{rs,n} &= -\frac{\sqrt{\kappa}}{2\pi} e^{\frac{\pi i}{\kappa}(M-\ell_r)^2} \int \mathrm{d}u u^{M-\ell_r-\frac{1}{2}} \lambda_{s,n}^{(1)}(u) \\ (H_{41})_{rs,n} &= \frac{1}{2\pi\sqrt{i}} e^{-\frac{\pi i}{\kappa}\ell_r^2 - \frac{\pi i}{\kappa}\ell_r} \int \mathrm{d}u \Bigg[\frac{(-1)^M}{v_r + u}\left(\frac{1}{8} - \frac{(\log v_r - \log u)^2}{8\pi^2}\right) + \kappa\Delta_2(-2\pi i\ell_r, x) e^{\frac{\pi i}{\kappa}\ell_r - \frac{x}{2\kappa}} \\ &\quad + \left(\frac{(-1)^M}{v_r + u}\frac{i(\log v_r - \log u)}{2\pi} + \kappa\Delta_1(-2\pi i\ell_r, x) e^{\frac{\pi i}{\kappa}\ell_r - \frac{x}{2\kappa}}\right)\left(-\frac{1}{2}\frac{u^{2k}-1}{u^{2k}+1}\right)\Bigg] \lambda_{s,n}^{(1)}(u)\end{aligned}$$

where $u = e^{\frac{x}{\kappa}}$, $v_r = e^{-\frac{2\pi i}{\kappa}\ell_r}$, and the coefficients $\lambda_{s,n}^{(1)}(u)$ are given by the following recursion relation:

$$\begin{aligned}\lambda_{s,0}^{(1)}(u) &= \frac{1}{\sqrt{\kappa}} e^{-\frac{\pi i}{\kappa}(M-\ell_s)^2} \frac{u^k}{u^{2k}+1} u^{\ell_s - M - \frac{1}{2}} \\ \lambda_{s,n}^{(1)}(u) &= \frac{1}{2\pi}\int \mathrm{d}v \frac{u^k}{u^{2k}+1}\Bigg[-\frac{1}{2}\frac{u^{2k}-1}{u^{2k}+1}\frac{1}{2}\frac{v^{2k}-1}{v^{2k}+1}\left(\frac{(-1)^M}{u+v} + \kappa\Delta_0(x,y) e^{-\frac{x+y}{2\kappa}}\right) \\ &\quad + \frac{1}{2}\left(\frac{u^{2k}-1}{u^{2k}+1} + \frac{v^{2k}-1}{v^{2k}+1}\right)\left(\frac{(-1)^M}{u+v}\frac{i(\log u - \log v)}{2\pi} + \kappa\Delta_1(x,y) e^{-\frac{x+y}{2\kappa}}\right) \\ &\quad - \frac{(-1)^M}{u+v}\left(\frac{1}{8} - \frac{(\log u - \log v)^2}{8\pi^2}\right) + \kappa\Delta_2(x,y) e^{-\frac{x+y}{2\kappa}}\Bigg] \lambda_{s,n-1}^{(1)}(v)\end{aligned}$$

where $v = e^{\frac{y}{\kappa}}$. The second column is

$$(H_{12})_{rs} = -V_r^{-t} \circ \left(1 - zL \circ \tilde{Y} \circ L^t \circ \tilde{X}\right)^{-1} \circ L \circ Y^t \circ W_s = \sum_{n\geq 0} z^n (H_{12})_{rs,n}$$

$$(H_{22})_{rs} = -zW_r \circ Y \circ L^t \circ \left(1 - zL \circ Y \circ L^t \circ \tilde{X}\right)^{-1} \circ L \circ Y^t \circ W_s = \sum_{n\geq 0} z^{n+1} (H_{22})_{rs,n}$$

$$\begin{aligned}(H_{32})_{rs} &= -W_r^{+t} \circ W_s + zW_r^{+t} \circ L^t \circ \tilde{X} \circ \left(1 - zL \circ Y \circ L^t \circ \tilde{X}\right)^{-1} \circ L \circ Y^t \circ W_s \\ &= -W_r^{+t} \circ W_s + \sum_{n\geq 0} z^{n+1} (H_{32})_{rs,n}\end{aligned}$$

$$(H_{42})_{rs} = -zV_r^t \circ X^t \circ \left(1 - zL \circ Y \circ L^t \circ \tilde{X}\right)^{-1} \circ L \circ Y^t \circ W_s = \sum_{n\geq 0} z^{n+1} (H_{42})_{rs,n}$$

and yields the coefficients

$$(H_{12})_{rs,n} = -\frac{\sqrt{\kappa}}{2\pi} e^{-\frac{\pi i}{\kappa}(\ell_r - M)^2} \int \mathrm{d}u u^{\ell_r - M - \frac{1}{2}} \lambda^{(2)}_{s,n}(u)$$

$$\begin{aligned}(H_{22})_{rs,n} = \frac{1}{2\pi\sqrt{-i}} e^{\frac{\pi i}{\kappa}\ell_r^2 - \frac{\pi i}{\kappa}\ell_r} \int \mathrm{d}u \Bigg[&\frac{(-1)^M}{v_r + u} \left(\frac{1}{8} - \frac{(\log v_r - \log u)^2}{8\pi^2}\right) + \kappa\tilde{\Delta}_2(-2\pi i \ell_r, x) e^{\frac{\pi i}{\kappa}\ell_r - \frac{x}{2\kappa}} \\ &+ \left(\frac{(-1)^M}{v_r + u} \frac{i(\log v_r - \log u)}{2\pi} + \kappa\tilde{\Delta}_1(-2\pi i \ell_r, x) e^{\frac{\pi i}{\kappa}\ell_r - \frac{x}{2\kappa}}\right) \left(\frac{1}{2}\frac{u^{2k}-1}{u^{2k}+1}\right) \Bigg] \lambda^{(2)}_{s,n}(u)\end{aligned}$$

$$\begin{aligned}(H_{32})_{rs,n} = \frac{\sqrt{-i}}{2\pi} e^{\frac{\pi i}{\kappa}\ell_r^2 - \frac{\pi i}{\kappa}\ell_r} \int \mathrm{d}u \Bigg[&\frac{(-1)^M}{v_r + u} \frac{i(\log v_r - \log u)}{2\pi} + \kappa\tilde{\Delta}_1(-2\pi i \ell_r, x) e^{\frac{\pi i}{\kappa}\ell_r - \frac{x}{2\kappa}} \\ &+ \left(\frac{(-1)^M}{v_r + u} + \kappa\tilde{\Delta}_0(-2\pi i \ell_r, x) e^{\frac{\pi i}{\kappa}\ell_r - \frac{x}{2\kappa}}\right) \left(\frac{1}{2}\frac{u^{2k}-1}{u^{2k}+1}\right) \Bigg] \lambda^{(2)}_{s,n}(u)\end{aligned}$$

$$(H_{42})_{rs,n} = -\frac{i\sqrt{\kappa}}{2\pi} e^{-\frac{\pi i}{\kappa}(\ell_r - M)^2} \int \mathrm{d}u \frac{1}{2}\frac{u^{2k}-1}{u^{2k}+1} u^{\ell_r - M - \frac{1}{2}} \lambda^{(2)}_{s,n}(u)$$

where $\lambda^{(2)}_{s,n}(u)$ satisfy the recursion relation

$$\lambda^{(2)}_{s,0}(u) = \frac{1}{\sqrt{\kappa}} e^{-\frac{\pi i}{\kappa}(\ell_s - M)^2} \frac{u^k}{u^{2k}+1} \left(\frac{i}{2}\frac{u^{2k}-1}{u^{2k}+1}\right) u^{-\ell_s + M - \frac{1}{2}}$$

$$\begin{aligned}\lambda^{(2)}_{s,n}(u) = \frac{1}{2\pi} \int \mathrm{d}v \left(-\frac{u^k}{u^{2k}+1}\right) \Bigg[&\frac{1}{2}\frac{u^{2k}-1}{u^{2k}+1}\frac{1}{2}\frac{v^{2k}-1}{v^{2k}+1} \left(\frac{(-1)^M}{u+v} + \kappa\tilde{\Delta}_0(x,y) e^{-\frac{x+y}{2\kappa}}\right) \\ &+ \frac{1}{2}\left(\frac{u^{2k}-1}{u^{2k}+1} + \frac{v^{2k}-1}{v^{2k}+1}\right) \left(\frac{(-1)^M}{u+v}\frac{i(\log u - \log v)}{2\pi} + \kappa\tilde{\Delta}_1(x,y) e^{-\frac{x+y}{2\kappa}}\right) \\ &+ \frac{(-1)^M}{u+v}\left(\frac{1}{8} - \frac{(\log u - \log v)^2}{8\pi^2}\right) + \kappa\tilde{\Delta}_2(x,y) e^{-\frac{x+y}{2\kappa}} \Bigg] \lambda^{(2)}_{s,n-1}(v)\end{aligned}$$

The third column is given by

$$
\begin{aligned}
(H_{13})_{rs} &= -V_r^{-t} \circ \left(1 - zL \circ \tilde{Y} \circ L^t \circ \tilde{X}\right)^{-1} \circ L \circ W_s^+ = \sum_{n\geq 0} z^n (H_{13})_{rs,n} \\
(H_{23})_{rs} &= W_r^t \circ W_s^+ + zW_r^t \circ Y \circ \left(1 - zL \circ \tilde{Y} \circ L^t \circ \tilde{X}\right)^{-1} \circ L \circ W_s^t \\
&= W_r^t \circ W_s^+ + \sum_{n\geq 0} z^{n+1} (H_{23})_{rs,n} \\
(H_{33})_{rs} &= -zW_r^{+t} \circ \left(1 - zL \circ \tilde{Y} \circ L^t \circ \tilde{X}\right)^{-1} \circ L \circ W_s^+ = \sum_{n\geq 0} z^{n+1} (H_{33})_{rs,n} \\
(H_{43})_{rs} &= zV_r^t \circ X^t \circ \left(1 - zL \circ \tilde{Y} \circ L^t \circ \tilde{X}\right)^{-1} \circ L \circ W_s^+ = \sum_{n\geq 0} z^{n+1} (H_{43})_{rs,n}
\end{aligned}
$$

We find the coefficients

$$
\begin{aligned}
(H_{13})_{rs,n} &= \frac{\sqrt{\kappa}}{2\pi} e^{-\frac{\pi i}{\kappa}(\ell_r - M)^2} \int \mathrm{d}u u^{\ell_r - M - \frac{1}{2}} \lambda^{(3)}_{s,n}(u) \\
(H_{23})_{rs,n} &= -\frac{1}{2\pi\sqrt{-i}} e^{\frac{\pi i}{\kappa}\ell_r^2 - \frac{\pi i}{\kappa}\ell_r} \int \mathrm{d}u \Bigg[\frac{(-1)^M}{v_r + u} \left(\frac{1}{8} - \frac{(\log v_r - \log u)^2}{8\pi^2} \right) + \kappa\tilde{\Delta}_2(-2\pi i \ell_r, x) e^{\frac{\pi i}{\kappa}\ell_r - \frac{x}{2\kappa}} \\
&\quad + \left(\frac{(-1)^M}{v_r + u} \frac{i(\log v_r - \log u)}{2\pi} + \kappa\tilde{\Delta}_1(-2\pi i \ell_r, x) e^{\frac{\pi i}{\kappa}\ell_r - \frac{x}{2\kappa}} \right) \left(\frac{1}{2} \frac{u^{2k} - 1}{u^{2k} + 1} \right) \Bigg] \lambda^{(3)}_{s,n}(u) \\
(H_{33})_{rs,n} &= \frac{i}{2\pi\sqrt{-i}} e^{\frac{\pi i}{\kappa}\ell_r^2 - \frac{\pi i}{\kappa}\ell_r} \int \mathrm{d}u \Bigg[\frac{(-1)^M}{v_r + u} \frac{i(\log v_r - \log u)}{2\pi} + \kappa\tilde{\Delta}_1(-2\pi i \ell_r, x) e^{\frac{\pi i}{\kappa}\ell_r - \frac{x}{2\kappa}} \\
&\quad + \left(\frac{(-1)^M}{v_r + u} + \kappa\tilde{\Delta}_0(-2\pi i \ell_r, x) e^{\frac{\pi i}{\kappa}\ell_r - \frac{x}{2\kappa}} \right) \left(\frac{1}{2} \frac{u^{2k} - 1}{u^{2k} + 1} \right) \Bigg] \lambda^{(3)}_{s,n}(u) \\
(H_{43})_{rs,n} &= \frac{i\sqrt{\kappa}}{2\pi} e^{-\frac{\pi i}{\kappa}(\ell_r - M)^2} \int \mathrm{d}u \frac{1}{2} \frac{u^{2k} - 1}{u^{2k} + 1} u^{\ell_r - M - \frac{1}{2}} \lambda^{(3)}_{s,n}(u)
\end{aligned}
$$

formally identical up to signs to the previous column. The coefficients $\lambda^{(3)}_{s,n}(u)$ are given by the same recursion relation as $\lambda^{(2)}_{s,n}(u)$ but with a different initial condition,

$$
\begin{aligned}
\lambda^{(3)}_{s,0}(u) &= \frac{1}{\sqrt{\kappa}} e^{\frac{\pi i}{\kappa}(\ell_s - M)^2} \frac{u^k}{u^{2k} + 1} u^{-\ell_s + M - \frac{1}{2}} \\
\lambda^{(3)}_{s,n}(u) &= \frac{1}{2\pi} \int \mathrm{d}v \left(-\frac{u^k}{u^{2k} + 1} \right) \Bigg[\frac{1}{2} \frac{u^{2k} - 1}{u^{2k} + 1} \frac{1}{2} \frac{v^{2k} - 1}{v^{2k} + 1} \left(\frac{(-1)^M}{u + v} + \kappa\tilde{\Delta}_0(x, y) e^{-\frac{x+y}{2\kappa}} \right) \\
&\quad + \frac{1}{2} \left(\frac{u^{2k} - 1}{u^{2k} + 1} + \frac{v^{2k} - 1}{v^{2k} + 1} \right) \left(\frac{(-1)^M}{u + v} \frac{i(\log u - \log v)}{2\pi} + \kappa\tilde{\Delta}_1(x, y) e^{-\frac{x+y}{2\kappa}} \right) \\
&\quad + \frac{(-1)^M}{u + v} \left(\frac{1}{8} - \frac{(\log u - \log v)^2}{8\pi^2} \right) + \kappa\tilde{\Delta}_2(x, y) e^{-\frac{x+y}{2\kappa}} \Bigg] \lambda^{(3)}_{s,n-1}(v)
\end{aligned}
$$

The last column is

$$(H_{14})_{rs} = V_r^{-t} \circ V_s + zV_r^{-t} \circ L \circ \tilde{Y} \circ \left(1 - zL^t \circ \tilde{X} \circ L \circ \tilde{Y}\right)^{-1} \circ L^t \circ X \circ V_s = \sum_{n\geq 0} z^n (H_{11})_{rs,n}$$

$$= V_r^{-t} \circ V_s + \sum_{n\geq 0} z^{n+1} (H_{41})_{rs,n}$$

$$(H_{24})_{rs} = zW_r^t \circ Y \circ \left(1 - zL^t \circ \tilde{X} \circ L \circ \tilde{Y}\right)^{-1} \circ L^t \circ x \circ V_s = \sum_{n\geq 0} z^{n+1} (H_{24})_{rs,n}$$

$$(H_{34})_{rs} = -zW_r^{+t} \circ \left(1 - zL^t \circ \tilde{X} \circ L \circ \tilde{Y}\right)^{-1} \circ L^t \circ X \circ V_s = \sum_{n\geq 0} z^{n+1} (H_{34})_{rs,n}$$

$$(H_{44})_{rs} = z^2 V_r^t \circ X^t \circ L \circ \tilde{Y} \circ \left(1 - zL^t \circ \tilde{X} \circ L \circ \tilde{Y}\right)^{-1} \circ L^t \circ X \circ V_s = \sum_{n\geq 0} z^{n+2} (H_{44})_{rs,n}$$

where the expansion coefficients can be written as

$$(H_{14})_{rs,n} = \frac{\sqrt{i}}{2\pi} e^{-\frac{\pi i}{\kappa}\ell_r^2 - \frac{\pi i}{\kappa}\ell_r} \int \mathrm{d}u \Bigg[\left(\frac{(-1)^M}{v_r + u} + \kappa \Delta_0(-2\pi i \ell_r, x) e^{\frac{\pi i}{\kappa}\ell_r - \frac{x}{2\kappa}} \right) \left(\frac{1}{2} \frac{u^{2k}-1}{u^{2k}+1} \right)$$

$$- \left(\frac{(-1)^M}{v_r + u} \frac{i(\log v_r - \log u)}{2\pi} + \kappa \Delta_1(-2\pi i \ell_r, x) e^{\frac{\pi i}{\kappa}\ell_r - \frac{x}{2\kappa}} \right) \Bigg] \lambda_{s,n}^{(4)}(u)$$

$$(H_{24})_{rs,n} = \frac{i\sqrt{\kappa}}{2\pi} e^{\frac{\pi i}{\kappa}(M-\ell_r)^2} \int \mathrm{d}u \frac{1}{2} \frac{u^{2k}-1}{u^{2k}+1} u^{M-\ell_r-\frac{1}{2}} \lambda_{s,n}^{(4)}(u)$$

$$(H_{34})_{rs,n} = -\frac{\sqrt{\kappa}}{2\pi} e^{\frac{\pi i}{\kappa}(M-\ell_r)^2} \int \mathrm{d}u u^{M-\ell_r-\frac{1}{2}} \lambda_{s,n}^{(4)}(u)$$

$$(H_{44})_{rs,n} = \frac{1}{2\pi\sqrt{i}} e^{-\frac{\pi i}{\kappa}\ell_r^2 - \frac{\pi i}{\kappa}\ell_r} \int \mathrm{d}u \Bigg[\frac{(-1)^M}{v_r + u} \left(\frac{1}{8} - \frac{(\log v_r - \log u)^2}{8\pi^2} \right) + \kappa \Delta_2(-2\pi i \ell_r, x) e^{\frac{\pi i}{\kappa}\ell_r - \frac{x}{2\kappa}}$$

$$- \left(\frac{(-1)^M}{v_r + u} \frac{i(\log v_r - \log u)}{2\pi} + \kappa \Delta_1(-2\pi i \ell_r, x) e^{\frac{\pi i}{\kappa}\ell_r - \frac{x}{2\kappa}} \right) \left(\frac{1}{2} \frac{u^{2k}-1}{u^{2k}+1} \right) \Bigg] \lambda_{s,n}^{(4)}(u)$$

and the coefficients $\lambda_{s,n}^{(4)}(u)$ are given by the following recursion relation:

$$\lambda_{s,0}^{(4)}(u) = \frac{i}{\sqrt{\kappa}} e^{-\frac{\pi i}{\kappa}(M-\ell_s)^2} \frac{u^k}{u^{2k}+1} \frac{1}{2} \frac{u^{2k}-1}{u^{2k}+1} u^{\ell_s - M - \frac{1}{2}}$$

$$\lambda_{s,n}^{(4)}(u) = \frac{1}{2\pi} \int \mathrm{d}v \frac{u^k}{u^{2k}+1} \Bigg[-\frac{1}{2} \frac{u^{2k}-1}{u^{2k}+1} \frac{1}{2} \frac{v^{2k}-1}{v^{2k}+1} \left(\frac{(-1)^M}{u+v} + \kappa \Delta_0(x,y) e^{-\frac{x+y}{2\kappa}} \right)$$

$$+ \frac{1}{2} \left(\frac{u^{2k}-1}{u^{2k}+1} + \frac{v^{2k}-1}{v^{2k}+1} \right) \left(\frac{(-1)^M}{u+v} \frac{i(\log u - \log v)}{2\pi} + \kappa \Delta_1(x,y) e^{-\frac{x+y}{2\kappa}} \right)$$

$$- \frac{(-1)^M}{u+v} \left(\frac{1}{8} - \frac{(\log u - \log v)^2}{8\pi^2} \right) + \kappa \Delta_2(x,y) e^{-\frac{x+y}{2\kappa}} \Bigg] \lambda_{s,n-1}^{(4)}(v)$$

Having calculated these blocks, we can use the spectral traces we already calculated to obtain the rank-deformed grand partition function $\Xi_{k,M}(z)$. We find the following result, where we omit the $(-1)^{M(M+1)/2}$ sign:

$$
\begin{aligned}
&\Xi_{1,0}(z) = 1 + \frac{z}{64} + \left(-\frac{1}{4096} + \frac{1}{384\pi^2}\right) z^2 + \left(-\frac{17}{1048576} + \frac{59}{368640\pi^2}\right) z^3 \\
&+ \left(\frac{85}{134217728} - \frac{121}{18350080\pi^2} + \frac{1}{294912\pi^4}\right) z^4 \\
&+ \left(\frac{397}{8589934592} - \frac{8379787}{20927899238400\pi^2} + \frac{1199}{2548039680\pi^4}\right) z^5 \\
&+ \left(-\frac{1033}{54975581388} + \frac{1014424093}{48753634065776640\pi^2} - \frac{5165}{228304355328\pi^4} + \frac{1}{339738624\pi^6}\right) z^6 + \mathcal{O}(z^7) \\
&\Xi_{1,1}(z) = \frac{1}{2} + \left(\frac{1}{128} + \frac{1}{32\pi^2}\right) z + \left(-\frac{1}{8192} + \frac{25}{18432\pi^2} + \frac{1}{3072\pi^4}\right) z^2 \\
&+ \left(-\frac{1}{131072} + \frac{121981}{1857945600\pi^2} + \frac{169}{1769472\pi^4} + \frac{1}{737280\pi^6}\right) z^3 \\
&+ \left(\frac{87}{268435456} + \frac{804697}{178362777600\pi^4} + \frac{49}{47185920\pi^6} + \frac{1}{330301440\pi^8} - \frac{14129}{3853516800\pi^2}\right) z^4 \\
&+ \Big(\frac{287}{17179869184} + \frac{10002037}{179789679820800\pi^4} + \frac{2380391}{14269022208000\pi^6} \\
&+ \frac{841}{190253629440\pi^8} + \frac{1}{237817036800\pi^{10}} - \frac{7951622183}{46168214077440000\pi^2}\Big) z^5 \\
&+ \Big(-\frac{4235}{4398046511104} - \frac{260802634510919}{1184713307798372352000\pi^4} + \frac{29158861}{2696845197312000\pi^6} + \frac{32481809}{19177565847552000\pi^8} \\
&+ \frac{1369}{136982613196800\pi^{10}} + \frac{1}{251134790860800\pi^{12}} + \frac{785289917597813}{6757253681516642304000\pi^2}\Big) z^6 + \mathcal{O}(z^7) \\
&\Xi_{2,0}(z) = 1 + \frac{z}{128} + \Big(\frac{35}{262144} - \frac{1}{768\pi^2}\Big) z^2 + \Big(\frac{313}{33554432} - \frac{1}{98304\pi^2} - \frac{41}{1572864\pi}\Big) z^3 \\
&+ \Big(\frac{124679}{274877906944} - \frac{212743}{105696460800\pi^2} + \frac{1}{1179648\pi^4} - \frac{817}{1006632960\pi}\Big) z^4 + \mathcal{O}(z^5) \\
&\Xi_{2,1}(z) = \frac{1}{4} + \Big(\frac{1}{256\pi} - \frac{1}{256\pi^2}\Big) z + \Big(\frac{123}{1048576} - \frac{1}{4608\pi^2} - \frac{1}{49152\pi^3} + \frac{1}{49152\pi^4} - \frac{439}{1474560\pi}\Big) z^2 \\
&+ \Big(\frac{81}{536870912} - \frac{451471}{158544691200\pi^2} - \frac{203}{18874368\pi^3} + \frac{521}{75497472\pi^4} - \frac{1}{31457280\pi^5} + \frac{1}{15728640\pi^6} + \frac{137377}{105696460800\pi}\Big) z^3 \\
&+ \Big(\frac{317087}{1099511627776} - \frac{1649}{72477573120\pi^6} + \frac{1}{42278584320\pi^7} + \frac{1}{21139292160\pi^8} + \frac{4031}{72477573120\pi^5} \\
&- \frac{1601313263}{3409345039564800\pi^2} + \frac{24706351}{30440580710400\pi^3} + \frac{6278849}{60881161420800\pi^4} - \frac{10532702423}{12500931811737600\pi}\Big) z^4 + \mathcal{O}(z^5) \\
&\Xi_{2,2}(z) = \frac{1}{8} + \Big(\frac{3}{1024} - \frac{1}{192\pi}\Big) z + \Big(\frac{73}{2097152} - \frac{7}{294912\pi^2} + \frac{5}{98304\pi^4} - \frac{5}{49152\pi}\Big) z^2 \\
&+ \Big(\frac{1113}{268435456} - \frac{7559}{1509949440\pi^2} + \frac{30433}{6794772480\pi^3} + \frac{7}{12582912\pi^4} + \frac{1}{2359296\pi^5} - \frac{188801}{15854469120\pi}\Big) z^3 + \mathcal{O}(z^4)
\end{aligned}
$$

The coefficients $Z_{k,M}(N)$ show agreement with the perturbative Airy formula, with a shift in B_k.

However, we were unable to find a bilinear formula satisfied by $\Xi_{k,M}(z)$. It does not satisfy the expected q-Painlevé-like D-type formula.

3.3.3 $\hat{D}_4$ quiver exact calculation

The $\hat{D}_4$ quiver is the one we started with. In this case, we have calculated $\Xi_{k,M}(z)$ for the special case $k = 1/2, 1$ and $M = 0$, that is without the rank deformation. Due to how involved the calculations are, there is no use in calculating further until we can invert the spectral density operator and find the mirror curve.

The one-particle density is in this case written up to a similarity transformation as

$$
\hat{\rho} = \frac{1}{2\cosh\frac{\hat{Q}}{2}} \frac{\tanh\frac{\hat{Q}}{2} + \tanh\frac{\hat{P}}{2}}{2} \frac{1}{\left(2\cosh\frac{\hat{P}}{2}\right)^2} \frac{\tanh\frac{\hat{Q}}{2} + \tanh\frac{\hat{P}}{2}}{2} \frac{1}{2\cosh\frac{\hat{Q}}{2}}
$$

Using the Fourier transforms for $r = 0, 1, 2$

$$
{}_Q\langle x| \frac{1}{\left(2\cosh\frac{\hat{P}}{2}\right)^2} \left(\frac{\tanh\frac{\hat{P}}{2}}{2}\right)^r |y\rangle_Q = \frac{1}{2\kappa\sinh\frac{x-y}{2}} \frac{x-y}{2\pi\kappa}
$$

$$
\left[\delta_{r0} + \frac{i}{2}\frac{x-y}{2\pi\kappa}\delta_{r1} + \frac{1}{12}\left(1 - \left(\frac{x-y}{2\pi\kappa}\right)^2\right)\delta_{r2}\right]
$$

the matrix element of $\hat{\rho}$ the $\hat{Q}$ eigenbasis is

$$
{}_Q\langle x|\,\hat{\rho}\,|y\rangle_Q = \frac{1}{2\cosh\frac{x}{2}} \frac{1}{2\cosh\frac{y}{2}} \frac{1}{2\kappa\sinh\frac{x-y}{2\kappa}} \frac{x-y}{2\pi\kappa} \Big[\frac{\tanh\frac{x}{2}}{2}\frac{\tanh\frac{y}{2}}{2}
$$

$$
+ \left(\frac{\tanh\frac{x}{2}}{2} + \frac{\tanh\frac{y}{2}}{2}\right)\frac{i}{2}\frac{x-y}{2\pi\kappa} + \frac{1}{12}\left(1 - 2\left(\frac{x-y}{2\pi\kappa}\right)^2\right)\Big]
$$

which we can rewrite in the form suitable for TWYP calculations,

$$
{}_Q\langle x|\,\hat{\rho}\,|y\rangle_Q = \frac{E(x)E(y)}{e^{\frac{x}{\kappa}} - e^{\frac{y}{\kappa}}} \frac{1}{\kappa} \sum_{i=1}^{4} f_i(x) g_i(x)
$$

with

$$
E(x) = \frac{e^{\frac{x}{2\kappa}}}{2\cosh\frac{x}{2}}
$$

$$
f_1(x) = \frac{x}{2\pi\kappa}\left(\frac{1}{12} + \frac{i}{4}\frac{x}{2\pi\kappa} - \frac{1}{6}\left(\frac{x}{2\pi\kappa}\right)^2\right) \qquad g_1(x) = 1
$$

$$
f_2(x) = 1, \qquad g_2(x) = \frac{x}{2\pi\kappa}\left(-\frac{1}{12} + \frac{i}{4}\frac{x}{2\pi\kappa} + \frac{1}{6}\left(\frac{x}{2\pi\kappa}\right)^2\right)
$$

$$
f_3(x) = \frac{\tanh\frac{x}{2}}{2}\frac{x}{2\pi\kappa} + \frac{i}{2}\left(\frac{x}{2\pi\kappa}\right)^2, \qquad g_3(x) = \frac{\tanh\frac{x}{2}}{2} - i\frac{x}{2\pi\kappa}
$$

$$
f_4(x) = \frac{\tanh\frac{x}{2}}{2} + i\frac{x}{2\pi\kappa} \qquad g_4(x) = -\frac{\tanh\frac{x}{2}}{2}\frac{x}{2\pi\kappa} + \frac{i}{2}\left(\frac{x}{2\pi\kappa}\right)^2
$$

The same form allows us the same manipulations to go through and yield

$$
{}_Q\langle x|\,\hat{\rho}^n\,|y\rangle_Q = \frac{E(x)E(y)}{e^{\frac{x}{\kappa}} - e^{\frac{y}{\kappa}}} \sum_{l=0}^{n-1}\sum_{i=1}^{4} \phi_l^{(i)}(x)\psi_{n-1-l}^{(i)}(y),
$$

$$
\phi_l^{(i)}(x) = \frac{1}{E(x)}{}_Q\langle x|\,\hat{\rho}^n E(\hat{Q}) f_i(\hat{Q})|0\rangle\!\rangle, \psi_l^{(i)}(x) = \langle\!\langle 0| g_i(\hat{Q}) E(\hat{Q})\hat{\rho}^n\,|x\rangle_Q \frac{1}{E(x)}
$$

Recursion relations can be obtained from this expression, namely

$$
\phi_0^{(i)} = \frac{1}{\sqrt{\kappa}} f_i(x)
$$

$$
\phi_l^{(i)}(x) = \int \frac{\mathrm{d}y}{2\pi} \frac{E(y)^2}{e^{\frac{x}{\kappa}} - e^{\frac{y}{\kappa}}} \frac{1}{\kappa} \frac{x-y}{2\pi\kappa} \Big[\frac{\tanh\frac{x}{2}}{2}\frac{\tanh\frac{y}{2}}{2}
$$

$$
+ \left(\frac{\tanh\frac{x}{2}}{2} + \frac{\tanh\frac{y}{2}}{2}\right)\frac{i}{2}\frac{x-y}{2\pi\kappa} + \frac{1}{12}\left(1 - 2\left(\frac{x-y}{2\pi\kappa}\right)^2\right)\Big]\phi_{l-1}^{(i)}(y)
$$

The coefficients $\psi_n^{(i)}(x)$ can be obtained from $\phi_n^{(i)}(x)$ by

$$\psi_l^{(j)} = \phi_l^{(j+1)}|_{i\mapsto -i}, i = 1, 3$$
$$\psi_l^{(j)} = -\phi_l^{(j-1)}|_{i\mapsto -i}, i = 2, 4$$

where $i \mapsto -i$ means changing only the imaginary units explicitly appearing in f_i's and the matrix element ${}_Q\langle x|\,\hat\rho\,|y\rangle_Q$, not fully complex-conjugating. Once again new variables $u = e^{\frac{x}{\kappa}}$, $v = e^{\frac{y}{\kappa}}$ turn this into a rational-with-logs expression amenable to the Bernoulli residue trick,

$$\begin{aligned}
\phi_l^i(u) = \frac{1}{2\pi}\int_0^\infty \mathrm{d}v \frac{v^\kappa}{(u-v)(v^\kappa+1)^2}\Big\{ \\
\Big[\frac{1}{8\pi}\frac{u^\kappa-1}{u^\kappa+1}\frac{v^\kappa-1}{v^\kappa+1} + \frac{i}{16\pi^2}\left(\frac{u^\kappa-1}{u^\kappa+1}+\frac{v^\kappa-1}{v^\kappa+1}\right)\log u + \frac{1}{24\pi}\left(1-\frac{\log^2 u}{2\pi^2}\right)\Big]\log u \\
-\Big[\frac{1}{8\pi}\frac{u^\kappa-1}{u^\kappa+1}\frac{v^\kappa-1}{v^\kappa+1} + \frac{i}{8\pi^2}\left(\frac{u^\kappa-1}{u^\kappa+1}+\frac{v^\kappa-1}{v^\kappa+1}\right)\log u + \frac{1}{24\pi}\left(1-\frac{3\log^2 u}{2\pi^2}\right)\Big]\log v \\
+\Big[\frac{i}{16\pi^2}\left(\frac{u^\kappa-1}{u^\kappa+1}+\frac{v^\kappa-1}{v^\kappa+1}\right) - \frac{\log u}{16\pi^3}\Big]\log^2 v + \frac{1}{48\pi^3}\log^3 v\Big\}\phi_{l-1}^{(i)}(v) \\
= \sum_{j\geq 0}\sum_{w:\text{poles in }\mathbb{C}\setminus\mathbb{R}_{\geq 0}} \mathrm{Res}_{v=w}\frac{1}{2\pi}\frac{v^\kappa}{(u-v)(v^\kappa+1)^2}\phi_{l-1}^{(i),j}(v)\Big[\\
-\frac{(2\pi i)^{j+1}}{j+1}B_{j+1}\left(\frac{\log^+ v}{2\pi i}\right)\Big(\frac{1}{8\pi}\frac{u^\kappa-1}{u^\kappa+1}\frac{v^\kappa-1}{v^\kappa+1} + \frac{i}{16\pi^2}\left(\frac{u^\kappa-1}{u^\kappa+1}+\frac{v^\kappa-1}{v^\kappa+1}\right)\log u \\
+\frac{1}{24\pi}\left(1-\frac{\log^2 u}{2\pi^2}\right)\Big)\log u \\
+\frac{(2\pi i)^{j+2}}{j+2}B_{j+2}\left(\frac{\log^+ v}{2\pi i}\right)\Big(\frac{1}{8\pi}\frac{u^\kappa-1}{u^\kappa+1}\frac{v^\kappa-1}{v^\kappa+1} + \frac{i}{8\pi^2}\left(\frac{u^\kappa-1}{u^\kappa+1}+\frac{v^\kappa-1}{v^\kappa+1}\right)\log u \\
+\frac{1}{24\pi}\left(1-\frac{3\log^2 u}{2\pi^2}\right)\Big) \\
-\frac{(2\pi i)^{j+3}}{j+3}B_{j+3}\left(\frac{\log^+ v}{2\pi i}\right)\left(\frac{i}{16\pi^2}\left(\frac{u^\kappa-1}{u^\kappa+1}+\frac{v^\kappa-1}{v^\kappa+1}\right) - \frac{1}{16\pi^3}\log u\right) \\
-\frac{(2\pi i)^{j+4}}{j+4}B_{j+1}\left(\frac{\log^+ v}{2\pi i}\right)\frac{1}{48\pi^3}\Big]
\end{aligned}$$

where as before $\phi_l^{(i),j}(u)$ are rational functions coming from the expansion

$$\phi_l^{(i)}(u) = \sum_{j\geq 0}\phi_l^{(i),j}(u)\log^j u$$

and the poles are located at $v = u$ and $v = e^{\frac{2\pi i}{\kappa}}(m-\frac{1}{2})$ for $m = 1, 2, ..., \kappa$. The spectral traces can then be recovered from

$$\mathrm{tr}\,\hat\rho^n = \sum_{j\geq 0}\sum_{w:\text{poles in }\mathbb{C}\setminus\mathbb{R}_{\geq 0}} \mathrm{Res}_{u=w}\frac{\kappa}{4\pi}\frac{u^\kappa}{(u^\kappa+1)^2}\left(-\frac{(2\pi i)^{j+1}}{j+1}\right)B_{j+1}\left(\frac{\log^+ v}{2\pi i}\right)\Phi_n^{(j)}(u)$$

where the rational functions $\Phi_n^{(j)}(u)$ come from the expansion of

$$\sum_{l=0}^{n-1}\sum_{i=1}^{4}\left(\frac{\mathrm{d}\phi_l^{(i)}}{\mathrm{d}u}\psi_{n-1-l}^{(i)}(u) - \phi_l^{(i)}(u)\frac{\mathrm{d}\psi_{n-1-l}^{(i)}}{\mathrm{d}u}\right) = \sum_{j\geq 0}\Phi_n^{(j)}(u)\log^j u$$

and the poles are at $w = e^{\frac{2\pi i}{\kappa}(m-\frac{1}{2})}$, $m = 1, 2, ..., \kappa$. Interestingly, unlike the $\hat{D}_3$ case, there are no terms like $u^{\frac{\kappa}{2}}$. Therefore, we can set $\kappa = 1$, which is Chern-Simons level of $k = \frac{1}{2}$. The first several partition functions in this case are

$$\begin{aligned}
Z_{\frac{1}{2},0}(1) &= \frac{1}{48\pi^2} \\
Z_{\frac{1}{2},0}(2) &= \frac{1}{302400\pi^2} - \frac{1}{7680\pi^6} \\
Z_{\frac{1}{2},0}(3) &= \frac{337}{99532800\pi^4} - \frac{13}{5529600\pi^6} - \frac{17}{5160960\pi^8} - \frac{1763}{5588352000\pi^2} \\
Z_{\frac{1}{2},0}(4) &= -\frac{19}{412876800\pi^{10}} - \frac{1}{9083289600\pi^{12}} - \frac{151}{65028096000\pi^8} \\
&\quad - \frac{10879}{243855360000\pi^4} + \frac{110671}{1170505728000\pi^6} + \frac{33161}{9323233920000\pi^2} \\
Z_{\frac{1}{2},0}(5) &= \frac{225860389}{4833164464128000000\pi^4} - \frac{17004137}{12051526975488000\pi^6} \\
&\quad - \frac{100577927}{29596083977940480000\pi^2} + \frac{731}{1498247331840\pi^{10}} - \frac{1583}{3269984256000\pi^{12}} \\
&\quad - \frac{1}{991895224320\pi^{14}} + \frac{348757}{337983528960000\pi^8}
\end{aligned}$$

We have results up to $N = 11$. For $k = 1$ the first several partition functions are

$$\begin{aligned}
Z_{1,0}(1) &= \frac{1}{96\pi^2} \\
Z_{1,0}(2) &= \frac{3}{1048576} + \frac{7}{589824\pi^4} - \frac{1}{245760\pi^6} - \frac{36277}{1238630400\pi^2} \\
Z_{1,0}(3) &= -\frac{7}{268435456} + \frac{191887}{1426902220800\pi^4} - \frac{1541}{5662310400\pi^6} \\
&\quad - \frac{31}{660602880\pi^8} + \frac{180619357}{732476473344000\pi^2} \\
Z_{1,0}(4) &= \frac{828980993}{1687625794584576000\pi^4} + \frac{59868161}{127850438983680000\pi^6} \\
&\quad - \frac{8380522497631}{3754029823064801280000\pi^2} + \frac{243}{1099511627776} - \frac{293}{1268357529600\pi^{10}} \\
&\quad + \frac{19}{4650644275200\pi^{12}} - \frac{3834259}{2130840649728000\pi^8} \\
Z_{1,0}(5) &= -\frac{484825215577667}{3243481767127988305920000\pi^4} - \frac{226751681927}{76587526968783667200000\pi^6} \\
&\quad + \frac{205353144313178193 73}{101691370219939157627633664 0000\pi^2} - \frac{525}{281474976710656} \\
&\quad - \frac{16850329}{2454728428486656000\pi^{10}} - \frac{3779}{59528246722560000\pi^{12}} \\
&\quad + \frac{8443}{1625121135525888000\pi^{14}} + \frac{92575189433}{28352113349020876800000\pi^8}
\end{aligned}$$

We have results up to $N = 8$. Results are available for $k = 3/2$ and $k = 2$, as well.

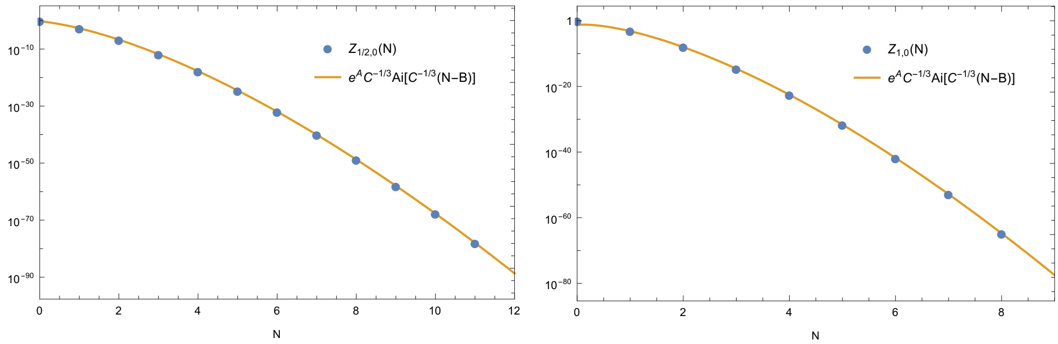

Figure 3.11: Plots of the $\hat{D}_4$-quiver model against the semiclassical expansions for $k = 1/2, 1$.

3.3.4 Discussion

We were unable to find bilinear relations among the functions considered. However, there is a clear way forward – by turning on FI parameters and mass deformations, the symmetries of the matrix model may give a clue as to what the spectral curve is, and thereby tell us what the model is computing.

There are other possible directions:

- We can further investigate the rank-deformed coefficient $B_k(M)$ and see if it satisfies a Seiberg-like duality, and see whether this duality is consistent with the type IIB realisation.

- Identify the worldsheet instanton exponents and see if it is consistent with topologically nontrivial cycles on the orbifold $Y_7 = S^7/G$ of the dual space-time. This can be compared with [211], where membrane instantons were obtained.

- Likewise, the shift of B_k can be compared with the prescription in [34].

- If a mass deformation is turned on, we can find a phase transition with respect to a real mass. We can investigate whether the critical mass coincides with the point where the real part of the worldsheet instanton exponent vanishes.

Appendices

A Quantum dilogarithm and other special functions

In the following we assume $|\mathfrak{q}| \neq 1$. We recall the definition of $\mathfrak{q}-$numbers,

$$[u] = \frac{1-\mathfrak{q}^u}{1-\mathfrak{q}},$$

and the infinite multiple $\mathfrak{q}$-Pochhammer symbol

$$(z;\mathfrak{q}_1,\ldots,\mathfrak{q}_k)_\infty = \exp\left\{\left(-\sum_{p=1}^{\infty}\frac{z^p}{p}\frac{1}{1-\mathfrak{q}_1^p}\cdots\frac{1}{1-\mathfrak{q}_k^p}\right)\right\} \overset{|\mathfrak{q}|<1}{=} \prod_{l_1,\ldots,l_k=0}^{\infty}\left(1-z\mathfrak{q}_1^{l_1}\cdots\mathfrak{q}_k^{l_k}\right).$$

This is defined for $|z|<1$, but can be analytically continued to $z\in\mathbb{C}$ using the latter equality. We extend the definition to an empty symbol $(z;)_\infty := 1-z$. For any $k\geq 1$ we then have the relations

$$\frac{(z;\mathfrak{q}_1,\ldots,\mathfrak{q}_k)_\infty}{(\mathfrak{q}_1 z;\mathfrak{q}_1,\ldots,\mathfrak{q}_k)_\infty} = (z;\mathfrak{q}_2,\ldots,\mathfrak{q}_k)_\infty.$$

Let us introduce the $\mathfrak{q}$-Gamma and $\mathfrak{q}$-Barnes G functions

$$\Gamma_{\mathfrak{q}}(u) = \frac{(\mathfrak{q};\mathfrak{q})_\infty}{(\mathfrak{q}^u;\mathfrak{q})_\infty}(1-\mathfrak{q})^{u-1},\quad G_{\mathfrak{q}}(u) = \frac{(\mathfrak{q}^u;\mathfrak{q},\mathfrak{q})_\infty}{(\mathfrak{q};\mathfrak{q},\mathfrak{q})_\infty}(\mathfrak{q};\mathfrak{q})_\infty^{u-1}(1-\mathfrak{q})^{-\frac{(u-1)(u-2)}{2}},$$

where $|\mathfrak{q}|<1$, which satisfy the $\mathfrak{q}$-analogues of the usual properties of Gamma and Barnes G functions,

$$\Gamma_{\mathfrak{q}}(u+1) = [u]\Gamma_{\mathfrak{q}}(u),\quad G_{\mathfrak{q}}(u+1) = \Gamma_{\mathfrak{q}}(u)G_{\mathfrak{q}}(u)$$

and are both equal to one at $u=1$ and log-convex [229]. These can also be easily analytically continued to $|\mathfrak{q}|>1$ using

$$\Gamma_{\mathfrak{q}}(u) = \mathfrak{q}^{\frac{(u-1)(u-2)}{2}}\Gamma_{\mathfrak{q}^{-1}}(u),\quad G_{\mathfrak{q}}(u) = \mathfrak{q}^{\frac{(u-1)(u-2)(u-3)}{6}}G_{\mathfrak{q}^{-1}}(u).$$

Let us finally introduce the quantum dilogarithm function $\Phi_b(z)$ as [167]

$$\Phi_b(z) = \frac{\left(e^{2\pi b\left(z+\frac{i}{2}\left(b+\frac{1}{b}\right)\right)};e^{2\pi i b^2}\right)_\infty}{\left(e^{\frac{2\pi}{b}\left(z-\frac{i}{2}\left(b+\frac{1}{b}\right)\right)};e^{-\frac{2\pi i}{b^2}}\right)_\infty}.$$

Note that $\Phi_b(z)$ satisfies the following recursive relations

$$\frac{\Phi_b(z+ib)}{\Phi_b(z)} = \frac{1}{1+e^{\pi i b^2}e^{2\pi b z}}, \quad \frac{\Phi_b\left(z+\frac{i}{b}\right)}{\Phi_b(z)} = \frac{1}{1+e^{\frac{\pi i}{b^2}}e^{\frac{2\pi z}{b}}}, \tag{.84}$$

and that its asymptotic behavior is

$$\Phi_b(z) \sim \begin{cases} \exp\left(i\pi z^2 + \frac{\pi i}{12}\left(b^2+b^{-2}\right)\right) & (\mathrm{Re}[z] \to \infty) \\ 1 & (\mathrm{Re}[z] \to -\infty) \end{cases}. \tag{.85}$$

The K-theoretic analogue of the 1-loop term uses

$$\begin{aligned} \gamma_{\epsilon_1,\epsilon_2}(x) &= \sum_{n=1}^{\infty} \frac{1}{d} \frac{e^{-Rdx}}{(e^{Rd\epsilon_1}-1)(e^{Rd\epsilon_2}-1)} = \sum_{n=1}^{\infty} \frac{e^{Rdx}}{d} \sum_{k_1,k_2=0}^{\infty} q_1^{dk_1} q_2^{dk_2} \\ &= -\sum_{k_1,k_2=0}^{\infty} \log\left(1-e^{Rx}q_1^{k_1}q_2^{k_2}\right) \end{aligned}$$

So that

$$\begin{aligned} \exp\{-\gamma_{\epsilon_1,\epsilon_2}(x)\} &= \prod_{k_1,k_2=0}^{\infty} \left(1-e^{Rdx}q_1^{k_1}q_2^{k_2}\right) = (e^{Rx}; q_1, q_2)_\infty \\ Z_{1-loop}(\mathbf{u}, q_1, q_2) &= \prod_{\boldsymbol{\alpha}\in R} \exp\{-\gamma_{\epsilon_1,\epsilon_2}(\boldsymbol{\alpha}\cdot\boldsymbol{a})\} = \prod_{\boldsymbol{\alpha}\in R} (\mathbf{u}^{\boldsymbol{\alpha}}; q_1, q_2)_\infty \end{aligned}$$

B The deep Coulomb approximation

Many times in the main text it was useful to check equivariant volume calculations by performing a large $\boldsymbol{\sigma}$ limit and studying the leading term, especially when we had either nothing to compare with at all or the blowup technique turned out to be too computationally costly. To describe this, we first define the auxiliary function

$$Z_n^{dC}(\epsilon_1,\epsilon_2,\boldsymbol{\sigma}) = \frac{1}{n!}\left(\frac{-1}{\epsilon_1\epsilon_2}Z_1(1,-1,\boldsymbol{\sigma})\right)^n$$

which we call the deep Coulomb instanton function. This is going to be the large $\boldsymbol{\sigma}$ limit, in which the only contribution to the equivariant volume given by the n-fold symmetric product of one instantons terms and more complicated configurations of Young diagrams are not involved. We claim that

$$Z_n^{dC}(\epsilon_1,\epsilon_2,\boldsymbol{\sigma}) = \lim_{\gamma\to 0}\gamma^{-(2h^\vee-2)n}Z_n(\epsilon_1,\epsilon_2,\boldsymbol{\sigma}/\gamma) \tag{.86}$$

In other words, this is the leading part under the scaling

$$Z_n(\epsilon_1,\epsilon_2,\boldsymbol{\sigma}/\gamma) = \gamma^{(2h^\vee-2)n}Z_n^{dC}(\epsilon_1,\epsilon_2,\boldsymbol{\sigma}) + \mathcal{O}(\gamma^{(2h^\vee-2)n+1})$$

Clearly, $Z_1^{dC}(1,-1,\boldsymbol{\sigma}) = Z_1(1,-1,\boldsymbol{\sigma})$. In this case, the subleading terms are absent, and that the scaling is correct can be seen from the universal 1-instanton term, since there are $2h^\vee - 2$ terms in the denominator of (1.6). In the refined case $\epsilon_1+\epsilon_2\neq 0$, this is no longer true, but it's immediate that (.86) is true.

We prove the rest by induction, using (1.7). In the following we split the sum into one with $\mathbf{m}=0$ and the rest. For $n>1$ write

$$\begin{aligned}
&\lim_{\gamma\to 0}\frac{Z_n(\epsilon_1,\epsilon_2,\boldsymbol{\sigma}/\gamma)}{Z_n^{dC}(\epsilon_1,\epsilon_2,\boldsymbol{\sigma}/\gamma)}\\
&=\lim_{\gamma\to 0}\frac{1}{n^2\epsilon_1\epsilon_2}\Bigg(\sum_{\substack{i_1+i_2=n\\ i_{1,2}<n}}(\epsilon_1 i_1+(\epsilon_1+\epsilon_2)i_2)(-\epsilon_2 i_2)\frac{Z_{i_1}(\epsilon_1,\epsilon_2,\boldsymbol{\sigma}/\gamma)Z_{i_2}(-\epsilon_2,\epsilon_1+\epsilon_2,\boldsymbol{\sigma}/\gamma)}{Z_n^{dC}(\epsilon_1,\epsilon_2,\boldsymbol{\sigma}/\gamma)}\\
&+\sum_{\substack{\frac{1}{2}\mathbf{m}^2+i_1+i_2=n\\ \mathbf{0}\neq\mathbf{m}\in Q^\vee,\, i_{1,2}<n}}\frac{\left(\epsilon_1 i_1+(\epsilon_1+\epsilon_2)i_2+\mathbf{m}\cdot\boldsymbol{\sigma}/\gamma+\frac{1}{2}\mathbf{m}^2(2\epsilon_1+\epsilon_2)\right)}{L(\epsilon_1,\epsilon_1+\epsilon_2,\boldsymbol{\sigma}/\gamma,\mathbf{m})}\\
&\left(\epsilon_1 i_1+(\epsilon_1+\epsilon_2)(i_2-n)+\mathbf{m}\cdot\boldsymbol{\sigma}/\gamma+\frac{1}{2}\mathbf{m}^2(2\epsilon_1+\epsilon_2)\right)\\
&\frac{Z_{i_1}(\epsilon_1,\epsilon_2,\boldsymbol{\sigma}/\gamma+\epsilon_1\mathbf{m})Z_{i_2}(-\epsilon_2,\epsilon_1+\epsilon_2,\boldsymbol{\sigma}/\gamma+(\epsilon_1+\epsilon_2)\mathbf{m})}{Z_n^{dC}(\epsilon_1,\epsilon_2,\boldsymbol{\sigma}/\gamma)}\Bigg)
\end{aligned}$$

Assume (.86) is true for all $n'<n$. Then the first sum becomes

$$\begin{aligned}
&\frac{1}{n^2\epsilon_1\epsilon_2}\sum_{\substack{i_1+i_2=n\\ i_{1,2}<n}}(\epsilon_1 i_1+(\epsilon_1+\epsilon_2)i_2)(-\epsilon_2 i_2)\frac{(i_1+i_2)!}{i_1!i_2!}\left(-\frac{\epsilon_1}{\epsilon_1+\epsilon_2}\right)^{i_2}\\
&=\frac{1}{n^2\epsilon_1\epsilon_2}\sum_{i=1}^{n-1}\left(-(n-i)^2\epsilon_1\epsilon_2+n(n-i)(\epsilon_1+\epsilon_2)\right)\binom{n}{i}\left(-\frac{\epsilon_1}{\epsilon_1+\epsilon_2}\right)^{i}\\
&=1-\frac{\epsilon_2^{n-2}\left((n-1)\epsilon_1+n\epsilon_2\right)}{n(\epsilon_1+\epsilon_2)^{n-1}}+\frac{(n-1)\epsilon_2^{n-2}}{n(\epsilon_1+\epsilon_2)^{n-2}}=1-\frac{\epsilon_2^{n-1}}{n(\epsilon_1+\epsilon_2)^{n-1}}
\end{aligned}$$

In the second sum, all terms except the ones proportional to $\boldsymbol{\sigma}$ can be ignored, as well as all the shifts. It becomes

$$\frac{1}{n^2\epsilon_1\epsilon_2}\sum_{\substack{\frac{1}{2}\mathbf{m}^2+i_1+i_2=n\\ \mathbf{0}\neq\mathbf{m}\in Q^\vee,\, i_{1,2}<n}}(\mathbf{m}\cdot\boldsymbol{\sigma})^2\lim_{\gamma\to 0}\frac{Z^{dC}_{i_1}(\epsilon_1,\epsilon_2,\boldsymbol{\sigma}/\gamma)Z^{dC}_{i_2}(-\epsilon_2,\epsilon_1+\epsilon_2,\boldsymbol{\sigma}/\gamma)}{\gamma^2 L(\epsilon_1,\epsilon_1+\epsilon_2,\boldsymbol{\sigma}/\gamma,\mathbf{m})Z^{dC}_n(\epsilon_1,\epsilon_2,\boldsymbol{\sigma}/\gamma)}$$
$$=\frac{1}{n^2\epsilon_1\epsilon_2(-Z_1(1,-1,\boldsymbol{\sigma}))}\sum_{\substack{\frac{1}{2}\mathbf{m}^2+i_1+i_2=n\\ \mathbf{0}\neq\mathbf{m}\in Q^\vee,\, i_{1,2}<n}}(\mathbf{m}\cdot\boldsymbol{\sigma})^2\,\epsilon_1\epsilon_2\frac{(i_1+i_2+1)!}{i_1!i_2!}\left(-\frac{\epsilon_1}{\epsilon_1+\epsilon_2}\right)^{i_2}$$
$$\lim_{\gamma\to 0}\frac{1}{\gamma^{2h^\vee}L(\epsilon_1,\epsilon_1+\epsilon_2,\boldsymbol{\sigma}/\gamma,\mathbf{m})}$$

The limit is dependent on the incidence properties of the colattice vector with respect to the roots. Note that if $\mathbf{m}^2=2$, $|\{\boldsymbol{\alpha}\in R|\boldsymbol{\alpha}\cdot\mathbf{m}=-1\}|=2h^\vee-4$ which contribute 1 term each in (1.8) and $|\{\boldsymbol{\alpha}\in R|\boldsymbol{\alpha}\cdot\mathbf{m}=\pm 2\}|=2$ which contribute 2 terms each. These vectors are the short coroots. For any other nonzero vector, the number of terms is greater than $2h^\vee$. Explicitly,

$$\lim_{\gamma\to 0}\gamma^{2h^\vee}L(\epsilon_1,\epsilon_1+\epsilon_2,\boldsymbol{\sigma}/\gamma,\mathbf{m})=\begin{cases}0, & \text{if } \mathbf{m}=\mathbf{0},\\ (\mathbf{m}\cdot\boldsymbol{\sigma})^4\prod\limits_{\boldsymbol{\alpha}\cdot\mathbf{m}=1}(\boldsymbol{\alpha}\cdot\boldsymbol{\sigma}), & \text{if } \mathbf{m}\in R^\vee_{\text{short}},\\ \infty & \text{otherwise.}\end{cases}$$

Therefore, the sum becomes

$$\frac{1}{n^2(-Z_1(1,-1,\boldsymbol{\sigma}))}\sum_{\mathbf{m}\in R^\vee_{\text{short}}}\frac{1}{(\mathbf{m}\cdot\boldsymbol{\sigma})^2\prod\limits_{\boldsymbol{\alpha}\cdot\mathbf{m}=1}(\boldsymbol{\alpha}\cdot\boldsymbol{\sigma})}\sum_{\substack{i_1+i_2=n-1\\ i_{1,2}<n}}\frac{(i_1+i_2+1)!}{i_1!i_2!}\left(-\frac{\epsilon_1}{\epsilon_1+\epsilon_2}\right)^{i_2}$$
$$=\frac{\epsilon_2^{n-1}}{n(\epsilon_1+\epsilon_2)^{n-1}}$$

using (1.6). This proves the claim. Moreover, we can prove quite easily that the full expansion around $\gamma=0$ has to be even,

$$\frac{Z_n(\epsilon_1,\epsilon_2,\boldsymbol{\sigma}/\gamma)}{Z^{dC}_n(\epsilon_1,\epsilon_2,\boldsymbol{\sigma}/\gamma)}=1+\sum_{k>1}z_k(\boldsymbol{\sigma})\gamma^{2k},$$

since this expression is holomorphic and Weyl invariant, and the Weyl group is generated by reflections. This can also be inferred from the blowup formula by a more involved, but straightforward calculation.

C Weyl transformations

We give the list of $2 \cdot 4!$ Weyl transformations discussed in 3.2.5 in terms of the generators, realized also as matrices, arranged by length. A positive integer $i_1 \ldots i_k$ represents the product $s_{i_1} \cdots s_{i_k}$:

$$21343 = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}, 213413 = \begin{pmatrix} \tfrac12 & \tfrac12 & -1 & 0 & -1 \\ \tfrac12 & \tfrac12 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \tfrac12 & -\tfrac12 & 0 & 0 & 0 \end{pmatrix}, 213423 = \begin{pmatrix} \tfrac12 & \tfrac12 & -1 & 0 & 1 \\ \tfrac12 & \tfrac12 & -1 & 0 & -1 \\ 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -\tfrac12 & \tfrac12 & 0 & 0 & 0 \end{pmatrix},$$

$$432134 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}, 452134 = \begin{pmatrix} \tfrac12 & \tfrac12 & -1 & 1 & 0 \\ \tfrac12 & \tfrac12 & -1 & -1 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ \tfrac12 & -\tfrac12 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}, 2134213 = \begin{pmatrix} 0 & 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$4532134 = \begin{pmatrix} \tfrac12 & \tfrac12 & 0 & 1 & 0 \\ \tfrac12 & \tfrac12 & 0 & -1 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ \tfrac12 & -\tfrac12 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}, 2345134 = \begin{pmatrix} \tfrac12 & \tfrac12 & -1 & 0 & -1 \\ \tfrac12 & \tfrac12 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 & 0 \\ \tfrac12 & -\tfrac12 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, 1345234 = \begin{pmatrix} \tfrac12 & \tfrac12 & -1 & 0 & 1 \\ \tfrac12 & \tfrac12 & -1 & 0 & -1 \\ 1 & 1 & -1 & 0 & 0 \\ \tfrac12 & -\tfrac12 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix},$$

$$432134131 = \begin{pmatrix} \tfrac12 & \tfrac12 & 0 & 0 & -1 \\ \tfrac12 & \tfrac12 & 0 & 0 & 1 \\ 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \tfrac12 & -\tfrac12 & 0 & 0 & 0 \end{pmatrix}, 452134131 = \begin{pmatrix} \tfrac12 & \tfrac12 & -1 & 1 & 0 \\ \tfrac12 & \tfrac12 & -1 & -1 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ \tfrac12 & -\tfrac12 & 0 & 0 & 0 \end{pmatrix}, 432134232 = \begin{pmatrix} \tfrac12 & \tfrac12 & 0 & 0 & 1 \\ \tfrac12 & \tfrac12 & 0 & 0 & -1 \\ 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -\tfrac12 & \tfrac12 & 0 & 0 & 0 \end{pmatrix},$$

$$452134232 = \begin{pmatrix} \tfrac12 & \tfrac12 & -1 & 1 & 0 \\ \tfrac12 & \tfrac12 & -1 & -1 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -\tfrac12 & \tfrac12 & 0 & 0 & 0 \end{pmatrix}, 4532134131 = \begin{pmatrix} \tfrac12 & \tfrac12 & 0 & 1 & 0 \\ \tfrac12 & \tfrac12 & 0 & -1 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ \tfrac12 & -\tfrac12 & 0 & 0 & 0 \end{pmatrix}, 2345134131 = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$1345234131 = \begin{pmatrix} 0 & 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}, 2345321341 = \begin{pmatrix} \tfrac12 & \tfrac12 & 0 & 0 & -1 \\ \tfrac12 & \tfrac12 & 0 & 0 & 1 \\ 1 & 1 & -1 & 0 & 0 \\ \tfrac12 & -\tfrac12 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, 4532134232 = \begin{pmatrix} \tfrac12 & \tfrac12 & 0 & 1 & 0 \\ \tfrac12 & \tfrac12 & 0 & -1 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -\tfrac12 & \tfrac12 & 0 & 0 & 0 \end{pmatrix},$$

$$2345134232 = \begin{pmatrix} 0 & 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, 1345234232 = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}, 1345321342 = \begin{pmatrix} \tfrac12 & \tfrac12 & 0 & 0 & 1 \\ \tfrac12 & \tfrac12 & 0 & 0 & -1 \\ 1 & 1 & -1 & 0 & 0 \\ \tfrac12 & -\tfrac12 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix},$$

$$4321343213 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, 4521343213 = \begin{pmatrix} \tfrac12 & \tfrac12 & -1 & 1 & 0 \\ \tfrac12 & \tfrac12 & -1 & -1 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ -\tfrac12 & \tfrac12 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, 3213452134 = \begin{pmatrix} \tfrac12 & \tfrac12 & -1 & -1 & 0 \\ \tfrac12 & \tfrac12 & -1 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ \tfrac12 & -\tfrac12 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$23453213431 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, 13453213432 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}, 45321343213 = \begin{pmatrix} \tfrac12 & \tfrac12 & 0 & 1 & 0 \\ \tfrac12 & \tfrac12 & 0 & -1 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ -\tfrac12 & \tfrac12 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$23451343213 = \begin{pmatrix} \tfrac12 & \tfrac12 & -1 & 0 & 1 \\ \tfrac12 & \tfrac12 & -1 & 0 & -1 \\ 1 & 1 & -1 & 0 & 0 \\ -\tfrac12 & \tfrac12 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, 13452343213 = \begin{pmatrix} \tfrac12 & \tfrac12 & -1 & 0 & -1 \\ \tfrac12 & \tfrac12 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 & 0 \\ -\tfrac12 & \tfrac12 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}, 32134532134 = \begin{pmatrix} \tfrac12 & \tfrac12 & 0 & -1 & 0 \\ \tfrac12 & \tfrac12 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ \tfrac12 & -\tfrac12 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$1345321342131 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}, 3213452134131 = \begin{pmatrix} \tfrac12 & \tfrac12 & -1 & -1 & 0 \\ \tfrac12 & \tfrac12 & -1 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ -\tfrac12 & \tfrac12 & 0 & 0 & 0 \end{pmatrix}, 2345321342132 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$3213452134232 = \begin{pmatrix} \tfrac12 & \tfrac12 & -1 & -1 & 0 \\ \tfrac12 & \tfrac12 & -1 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \tfrac12 & -\tfrac12 & 0 & 0 & 0 \end{pmatrix}, 5432134521343 = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, 13453213432131 = \begin{pmatrix} \tfrac12 & \tfrac12 & 0 & 0 & -1 \\ \tfrac12 & \tfrac12 & 0 & 0 & 1 \\ 1 & 1 & -1 & 0 & 0 \\ -\tfrac12 & \tfrac12 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix},$$

$$32134532134131 = \begin{pmatrix} \tfrac12 & \tfrac12 & 0 & -1 & 0 \\ \tfrac12 & \tfrac12 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ -\tfrac12 & \tfrac12 & 0 & 0 & 0 \end{pmatrix}, 23453213432132 = \begin{pmatrix} \tfrac12 & \tfrac12 & 0 & 0 & 1 \\ \tfrac12 & \tfrac12 & 0 & 0 & -1 \\ 1 & 1 & -1 & 0 & 0 \\ -\tfrac12 & \tfrac12 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, 32134532134232 = \begin{pmatrix} \tfrac12 & \tfrac12 & 0 & -1 & 0 \\ \tfrac12 & \tfrac12 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \tfrac12 & -\tfrac12 & 0 & 0 & 0 \end{pmatrix},$$

$$32134521343213 = \begin{pmatrix} \tfrac12 & \tfrac12 & -1 & -1 & 0 \\ \tfrac12 & \tfrac12 & -1 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ -\tfrac12 & \tfrac12 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}, 54321345213413 = \begin{pmatrix} \tfrac12 & \tfrac12 & -1 & 0 & -1 \\ \tfrac12 & \tfrac12 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ -\tfrac12 & \tfrac12 & 0 & 0 & 0 \end{pmatrix}, 54321345213423 = \begin{pmatrix} \tfrac12 & \tfrac12 & -1 & 0 & 1 \\ \tfrac12 & \tfrac12 & -1 & 0 & -1 \\ 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ \tfrac12 & -\tfrac12 & 0 & 0 & 0 \end{pmatrix},$$

$$54321345432134 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, 321345321343213 = \begin{pmatrix} \tfrac12 & \tfrac12 & 0 & -1 & 0 \\ \tfrac12 & \tfrac12 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ -\tfrac12 & \tfrac12 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}, 543213452134213 = \begin{pmatrix} 0 & 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

$$54321345432134131 = \begin{pmatrix} \tfrac12 & \tfrac12 & 0 & 0 & -1 \\ \tfrac12 & \tfrac12 & 0 & 0 & 1 \\ 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ -\tfrac12 & \tfrac12 & 0 & 0 & 0 \end{pmatrix}, 54321345432134232 = \begin{pmatrix} \tfrac12 & \tfrac12 & 0 & 0 & 1 \\ \tfrac12 & \tfrac12 & 0 & 0 & -1 \\ 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ \tfrac12 & -\tfrac12 & 0 & 0 & 0 \end{pmatrix}, 543213454321343213 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

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