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Noncommutative Geometry, Symmetry and Real Division Algebras

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Abstract

We explore several aspects of algebraic noncommutative spaces described explicitly. In particular, we examine the construction of noncommutative Euclidean spaces by Dubois-Violette and Landi, and reinterpret it as a deformation via cocycle twists of a Hopf algebra action. We endow this class of algebras with the structure of a monoidal category to perform an explicit Tannaka-Krein reconstruction of the Hopf algebra of symmetries that is to be deformed. Next we explore products of real division algebras with a commutator determined by multiplication operators on these division algebras. We extend the theory to the case of the octonions in two different ways, and explicitly compute the symmetries of these specific families of noncommutative spheres and tori. Lastly, we will present some attempts at formalising the noncommutative octonionic Hopf fibration of $\mathbb{S}^{15}$ over $\mathbb{S}^{8}$. We show that an ansatz given by a family of differential operators discovered by Klim and Majid fails by exactly one associator, and we briefly describe some attempts to formalise the nonassociative Hopf-Galois extension corresponding to this fibration.

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Contents

Chapter 1

Introduction

Mathematics is often about duality: the interplay between the abstract and the concrete, between the analytic and synthetic. There is a very deep duality between two main areas of mathematics as well, between geometry and algebras. Starting with a space I can form an algebra by looking at the functions taking arguments in that space, and many algebras are in turn algebras of functions on a space that is sometimes hidden. Usually we consider these functions to take a point of our space and return a number. This creates a condition that these functions have to obey, as the values of these functions and hence the functions themselves *commute*. This means that their product does not depend on the order. Many interesting algebras, however, do not have this property. Physicists often model real processes via algebras, and in the real world the order in which one does things matters. Noncommutative geometry is based on the idea of taking this duality between algebras and spaces to its limit, to try to study *any* algebra as if it consists of functions on a (noncommutative) space.

There are many ways to translate classical results to noncommutative equivalents, and this is a field of active research. However, oftentimes this correspondence is not unique, as statements that are equivalent in the commutative setting become different when generalised. In order to find the "correct" generalisation, it is useful to check some simple examples. It is trivial to come up with examples of noncommutative spaces: take any noncommutative algebra, and consider it from a geometer's point of view. It is however quite a bit less trivial to come up with *good* examples: examples that preserve enough of a classical geometric flavour to see the remnants of geometric shapes and properties in the noncommutative algebra. This thesis attempts to construct and analyse some of these noncommutative spaces that are in some sense well-behaved. We construct these spaces out of three main ingredients: Simple classical spaces, their symmetries and the guidance of *number systems*.

That last component works as follows: We have all grown up with the number line, which happens to be both a space and an algebra. There is also the complex plane, which again happens to be a space made of numbers. This duality bestows them with many nice properties on the interplay of algebra and geometry. There are exactly two more spaces that behave somewhat like numbers; together these four are called the *real division algebras*. The remaining two are the quaternions and the octonions, and they are 4 and 8 dimensional respectively. The special properties of these spaces, their symmetries and their issues will be central in what follows.

This thesis is divided into four parts, the latter three of which are mostly individually readable. The first is a revision of the theory we will need for our constructions. This theory consists of the three ingredients mentioned before. First we very briefly review the classical spaces we will transform. Secondly we review a theory of symmetry; the theory of Hopf algebras. Hopf algebras are the perfect tool for us as they revolve around duality; they encapsulate both the duality between algebra and geometry as well as between abstraction and the concrete. We will use this in many ways, to start with abstract symmetries and end with concrete examples, to start with concrete examples and return with abstract symmetries. This deep connection between symmetries and objects bearing that symmetry gives us a powerful tool: when one deforms a Hopf algebra, so too will every space that has that symmetry be deformed. This is what we will use to create our noncommutative spaces. To round off the introduction we will revise the theory of real division algebras, interpreting also these as examples of very simple spaces with very special deformations along a symmetry.

The second part is about noncommutative products of flat spaces. The idea here is to confine the noncommutativity to the fault line between two normal spaces, with the goal of creating a minimally noncommutative space. We exhibit a new way of interpreting these noncommutative spaces via *monoidal categories*, which is a tool that is among other things capable of separating an algebra into an “external” and and “internal” part. This captures an essential point of this particular class of examples, where all the noncommutativity is caught in the interaction between the internal and the external. We reformulate the construction in this language, use a reconstruction theorem to reconstruct a Hopf algebra of symmetries, and indicate some ways of extending the theory to non-free algebras. After this reformulation, we have a short investigation of a very explicit nature as we try to formulate a way to exactly find how to deform the symmetries of a product of Euclidean spaces in order to end up at this noncommutative product.

The third part is very specifically about the case where the flat spaces are the generalised number lines mentioned before, the real division algebras. These spaces and their deformations give some concrete examples of the theory in the previous section. Furthermore, the numbers give us two properties that are very useful. The first is the existence of a multiplicative norm. By requiring that norm to be invariant we can pass from flat spaces to spheres and products of spheres, and thereby look at noncommutative variants of these. The second property is the homogeneity: every point on the number line looks the same. This results in particularly symmetric spaces, whose symmetries we will classify by explicitly using the symmetries of the division algebras.

The last part takes a closer look at these spheres. The existence of a multiplication on the spheres is a peculiarity that only exists for the four real division algebras, and they bestow the spheres with structures called *Hopf fibrations*. These geometric ways to make spheres out of spheres are very fruitful locations to insert Hopf algebraic deformations because of all the symmetry involved. There is however one case that is not particularly well understood: the case of the 15-sphere. This fibration is constructed using the octonions, which are non-associative, a property much more difficult to deal with than mere noncommutativity. One way to get around this classically is by looking at things locally. Things require more care in the dual function algebra setting, however. We use this as a testing ground for some ways to deal with nonassociativity in a noncommutative context, showing in particular that a method that seemed to circumvent the locality that is used classically by means of using differential operators turns out to be off by exactly one application of the associative law. After that, we talk about a proposed way to use the local approach and discuss an approach to gluing to serve as a starting point for further investigations.

Chapter 2

Preliminaries

In this work we touch on many different aspects of quantum groups, noncommutative geometry and the theory of real division algebras, so in this section I hope to contain much of the background necessary for the entire work. None of the material in this chapter is original, we repeat it for both completeness as well as to fix notation and conventions.

2.1 Algebraic Noncommutative Geometry

Usually noncommutative geometry à la Connes works with C*-algebras of operators [Con94]. We keep the functional analysis to a minimum and work with algebras[1], mostly polynomial rather than analytic, always over a field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We will stick to the notations of $\mathbb{S}^n$ for spheres and $\mathbb{T}^n(k)$ for tori. The algebra of polynomial functions on a space X is usually denoted by $\mathcal{A}(X)$.

As an incredibly brief review on how to look at these algebras: The classical spaces we look at are all affine varieties. This means they are of a kind that can be seen as embedded in $\mathbb{R}^n$, for some sufficiently large n, and then determined as the zero set of some polynomial(s) $\mathcal{P} : \mathbb{R}^n \to \mathbb{C}$. Consider polynomial functions on $\mathbb{R}^n$, that is, polynomials in n variables $\{x_i\}$, where each x_i is the function taking a point in $\mathbb{R}^n$ and sending it to the value of its i'th coordinate. We obtain the functions on our spaces by quotienting out by the polynomial(s) $\mathcal{P}$, as $\mathcal{P}$ ought to be zero by definition when evaluated on our space.

[1]Fittingly, one cannot commute the words "Algebraic" and "Noncommutative" in the title, as there is an approach to noncommutative geometry interpreting noncommutative spaces as categories of quasicoherent sheaves that goes by that name, recently championed by Kontsevich [KR00].

This notion of coordinate functions often leads to an abuse of notation where one states to be “multiplying coordinates", dropping the “functions", coordinates and functions actually being dual to one another. The point of view of noncommutative geometry is that this duality extends to noncommutative algebras: dual to many noncommutative algebras there exists a “noncommutative manifold” X. The algebra of “coordinate functions on X" no longer consists of functions taking actual values on points lying in this space, but rather abstract generators of a “function algebra". We will frequently still refer to these generators as coordinate functions. In this tract, algebras are presumed associative and unital unless mentioned, but never presumed commutative.

Quite often we have the extra data of a *real* or $*$-structure on our algebra. This structure consists of a $\mathbb{K}$-antilinear operator $*$ with the following properties:

$$(xy)^* = y^* x^*, (\lambda x)^* = \bar{\lambda} x^* \tag{2.1}$$

where $\lambda \in k$ and the bar is complex conjugation. We will sometimes use the bar also for the star structure in the algebra if the number of indices is too large. Note how every commutative algebra has a trivial $*$-structure, but this is no longer true in the noncommutative case.

There are some methods of passing back to the C*-algebraic realm, see for example [Rie93], or for quantum groups one can find some in any book on compact quantum groups like [Tim08]. We will refrain from doing so in this work.

2.2 Hopf algebras

Much in this thesis is written in the language of Hopf algebras, as they form the language of symmetry in noncommutative geometry. We will briefly review the parts of the theory that are important for following the arguments in this thesis. For a more complete overview, there are many books written on Hopf algebras and quantum groups, see for a reference [Kas95][CP95][Mon93][KS97]. We will for this short recap and notation mostly stick to [Maj95]. One important thing to mention is that in this area, not even the title *quantum group* is uniquely defined. The originator of the field, Drinfeld[2], in the founding paper [Dri86], defines it to be the Spec of a Hopf algebra, i.e. any Hopf algebra from a

[2]Drinfeld’s name often gets transliterated in a non-unique manner too, as Drinfel’d. We will stick to the spelling without apostrophe, as does he when writing his own name in the Latin alphabet.

geometric viewpoint, and so the terms Hopf algebra and quantum group are often used interchangeably. All of the references above use subtly different definitions. I myself will reserve the use of the term "quantum group" for Hopf algebras with a quasitriangular structure, to be defined in section 2.2.5.

2.2.1 Hopf algebras

There are many intuitions for what a Hopf algebra is, to give two: A Hopf algebra is an algebra with a good notion of a representation, and a Hopf algebra is an algebra whose structures are self-dual. As we will be dealing with symmetries of objects that are defined via duality, this language will be invaluable to us. Let us first begin with the axioms.

Definition 2.2.1. *A* unital associative algebra *is a vector space over $k = \mathbb{R}$ or $\mathbb{C}$ equipped with two linear maps, $m : \mathcal{A}\otimes\mathcal{A} \to \mathcal{A}$ and $\mathbb{1} : k \to \mathcal{A}$ such that the following commute:*

$$\begin{array}{ccc} \mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A} & \xrightarrow{\mathrm{id}\otimes m} & \mathcal{A}\otimes\mathcal{A} \\ \Big\downarrow{\scriptstyle m\otimes\mathrm{id}} & & \Big\downarrow{\scriptstyle m} \\ \mathcal{A}\otimes\mathcal{A} & \xrightarrow{\quad m \quad} & \mathcal{A} \end{array} \qquad \begin{array}{ccc} k\otimes\mathcal{A} & \xrightarrow{\mathbb{1}\otimes\mathrm{id}} & \mathcal{A}\otimes\mathcal{A} \\ & \searrow{\scriptstyle\cong} & \Big\downarrow{\scriptstyle m} \\ & & \mathcal{A} \end{array}$$

The diagrams are called associativity and unitarity. In the spirit of duality, one can ask what kind of algebraic structures one gets if one were to reverse all the arrows. The result is what is called a *coalgebra*:

Definition 2.2.2. *A* (counital coassociative) coalgebra *is a vector space over $k = \mathbb{R}$ or $\mathbb{C}$ equipped with two linear maps, $\Delta : \mathcal{C} \to \mathcal{C}\otimes\mathcal{C}$ and $\varepsilon : \mathcal{C} \to k$ such that the following commute:*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\quad\Delta\quad} & \mathcal{C}\otimes\mathcal{C} \\ \Big\downarrow{\scriptstyle\Delta} & & \Big\downarrow{\scriptstyle\mathrm{id}\otimes\Delta} \\ \mathcal{C}\otimes\mathcal{C} & \xrightarrow{\Delta\otimes\mathrm{id}} & \mathcal{C}\otimes\mathcal{C}\otimes\mathcal{C} \end{array} \qquad \begin{array}{ccc} \mathcal{C} & \xrightarrow{\Delta} & \mathcal{C}\otimes\mathcal{C} \\ & \searrow{\scriptstyle\cong} & \Big\downarrow{\scriptstyle\varepsilon\otimes\mathrm{id}} \\ & & k\otimes\mathcal{C} \end{array}$$

These relations are called coassociativity and counitarity, respectively. The intuition is quite frequently that a coproduct is a rule for "fair distribution". The counitarity relation means one does not lose information, when "subdividing" both halves remember in some sense what used to be the whole thing. The coassociativity means that after a subdivision one can further distribute starting from either half and end up with the same outcome. We will commonly use a piece of notation called *Sweedler notation*. In general, the coproduct of an element $c \in \mathcal{C}$ is given by $\Delta c = \sum_i c_{(1);i} \otimes c_{(2);i}$, some finite linear combination of elements $c_{(k);i} \in \mathcal{C}$. Sweedler notation leaves this summation implicit, hence this sum will be written as:

$$\Delta c := c_{(1)} \otimes c_{(2)} \tag{2.2}$$

Coassociativity implies that $c_{(1)_{(1)}} \otimes c_{(1)_{(2)}} \otimes c_{(2)} = c_{(1)} \otimes c_{(2)_{(1)}} \otimes c_{(2)_{(2)}}$, and hence either of these can be written as:

$$(\Delta \otimes \mathrm{id})\Delta c := c_{(1)} \otimes c_{(2)} \otimes c_{(3)} \tag{2.3}$$

While this notation allows for quite brief formulas in contexts that often become mired in notation, one should not forget that one is not dealing with simple tensors. To that end we will sometimes reintroduce an indexless summation sign as a reminder. If for all c we have $c_{(1)} \otimes c_{(2)} = c_{(2)} \otimes c_{(1)}$ the coalgebra is called *cocommutative*. The notion of a coalgebra itself is overshadowed by its dual, for a reference in the theory of coalgebras themselves see [Mic03]. We will unfortunately contribute to this neglect and only use the coalgebra structure in combination with the algebra structure to form a *bialgebra*.

Definition 2.2.3. *A* bialgebra *is an algebra* $(\mathcal{B}, m, \mathbb{1})$ *that is also a coalgebra* $(\mathcal{B}, \Delta, \varepsilon)$ *such that the structures commute in the following way:*

$$\begin{array}{ccccc}
\mathcal{B} \otimes \mathcal{B} & \xrightarrow{m} & \mathcal{B} & \xrightarrow{\Delta} & \mathcal{B} \otimes \mathcal{B} \\
\downarrow{\scriptstyle \Delta \otimes \Delta} & & & & \uparrow{\scriptstyle m \otimes m} \\
\mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B} & & \xrightarrow{\mathrm{id} \otimes \tau \otimes \mathrm{id}} & & \mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B}
\end{array}$$

$$\begin{array}{ccc}
\mathcal{B} \otimes \mathcal{B} & \xrightarrow{m} & \mathcal{B} \\
& {\scriptstyle \varepsilon \otimes \varepsilon} \searrow & \downarrow{\scriptstyle \varepsilon} \\
& & k
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{B} \otimes \mathcal{B} & \xleftarrow{\Delta} & \mathcal{B} \\
& \nwarrow {\scriptstyle \mathbb{1} \otimes \mathbb{1}} & \uparrow{\scriptstyle \mathbb{1}} \\
& & k
\end{array}$$

Where $\tau : \mathcal{B} \otimes \mathcal{B} \to \mathcal{B} \otimes \mathcal{B}$ is the transposition map flipping the two components. In words, the diagrams say, respectively, that the coproduct is an algebra map, the counit is a character, and the unit is what is called *grouplike*. This last notion is the following frequently useful[3] property. A grouplike element g of a coalgebra obeys:

$$\Delta(g) = g \otimes g \quad ; \quad \varepsilon(g) = 1 \tag{2.4}$$

Similarly there is a second notion of a *primitive* element. A primitive element $\mathfrak{g}$ of a bialgebra obeys:

$$\Delta\mathfrak{g} = 1 \otimes \mathfrak{g} + \mathfrak{g} \otimes 1 \tag{2.5}$$

Many naturally occurring bialgebras can in a unique way be equipped with the structure of a *Hopf algebra*, which has the additional datum of an *antipode map* S:

Definition 2.2.4. *A* Hopf algebra $\mathcal{H}$ *is a bialgebra equipped with a map* $S : \mathcal{H} \to \mathcal{H}$ *such that the following diagram commutes:*

$$\begin{array}{ccccc} \mathcal{H} & \xrightarrow{\varepsilon} & k & \xrightarrow{\mathbb{1}} & \mathcal{H} \\ \downarrow{\scriptstyle\Delta} & & & & \uparrow{\scriptstyle m} \\ \mathcal{H} \otimes \mathcal{H} & & \xrightarrow{S \otimes \mathrm{id}} & & \mathcal{H} \otimes \mathcal{H} \end{array}$$

This diagram says that S is an inverse to the identity map in the convolution algebra. The simplest intuition is to think of it as a linearised inverse, but one needs to take care as $S^2 \neq \mathrm{id}$ necessarily. This statement has as a consequence that S is an algebra anti-homomorphism as well as a coalgebra anti-homomorphism.

2.2.2 Hopf algebra examples

To round off the basic definitions I will briefly define three basic classes of Hopf algebras that will make their appearance:

Universal enveloping algebras

Given a Lie algebra $\mathfrak{g}$, one can form its *Universal Enveloping algebra* $\mathrm{UEA}(\mathfrak{g})$ as the universal associative algebra in which $\mathfrak{g}$ embeds, with the Lie bracket as commutator in the algebra. For technical constructions any textbook on Lie algebras will do, e.g. [Jac79].

[3] Being grouplike is a dual notion to being a function that is also an algebra homomorphism, which is of course a special yet useful class of functions.

Such an algebra is generated by polynomials in elements of $\mathfrak{g}$ (by the Poincaré-Birkhoff-Witt theorem), and hence by linearity of all structures involved and the fact that Δ, S and ε are (anti-)algebra maps, it is enough to specify the Hopf algebra structure on a basis of $\mathfrak{g}$. The following are the structure maps:

$$\begin{aligned} \Delta(x) &= x \otimes 1 + 1 \otimes x \\ \varepsilon(x) &= 0 \\ S(x) &= -x \end{aligned} \tag{2.6}$$

for $x \in \mathrm{UEA}(\mathfrak{g})$. This means all elements of the Lie algebra as it is embedded in $\mathrm{UEA}(\mathfrak{g})$ are primitive, giving us some intuition for the choice of that name.[4]

The group algebra $k[G]$

For a finite group G, using the group elements as a basis for an algebra over $\mathbb{K}$ gives an associative multiplication and an identity element. It forms a Hopf algebra if we in addition equip it with the following structure:

$$\begin{aligned} \Delta(g) &= g \otimes g \\ \varepsilon(g) &= 1 \\ S(g) &= g^{-1} \end{aligned} \tag{2.7}$$

This means the basis elements of the group in a group algebra are distinguished by being grouplike, explaining that name.[5] The counit is given by the trivial character on the group.

As a remark, these two examples are in some sense principal: there are theorems by Cartier-Gabriel-Milnor-Moore-Quillen-Sweedler[Krä19] that show that any cocommutative Hopf algebra $\mathcal{H}$ over $\mathbb{C}$ is in fact isomorphic to $\mathrm{UEA}(P(\mathcal{H})) \rtimes k[G(\mathcal{H})]$, for $P(\mathcal{H})$ the primitive elements and $G(\mathcal{H})$ the grouplike elements of H.

[4]As an example of the usefulness of Hopf algebra theory, the easiest proof of the existence of a Baker-Campbell-Hausdorff type formula goes via showing that all primitive elements lie in the Lie algebra.

[5]One does need to specify an embedding of the base group: $k[D_8]$ and $k[Q_8]$ are isomorphic as algebras but not as groups. Given the coproduct, one can recover the original non-linearised group by looking at the grouplike elements.

Functions on a group G

The last example of this section is dual to the previous example: functions $\mathcal{O}(G) \ni f : G \to k$. On finite groups we will look at polynomial functions, on infinite groups we will mostly restrict ourselves to matrix groups, which means we can take as functions polynomials in the projections onto any of the matrix components. In these cases, there is an isomorphism[Kas95]: $\mathcal{O}(G) \otimes \mathcal{O}(G) \cong \mathcal{O}(G \times G)$, which allows us to define for a function f:

$$\begin{aligned} \Delta(f)(g,h) &= f(gh) \\ \varepsilon(f) &= f(e) \\ (S(f))(g) &= f(g^{-1}) \end{aligned} \tag{2.8}$$

Where $e, g, h \in G$ and e is the identity. The first line is often a problem when defining duality; in general a function in $\mathcal{O}(G)\otimes\mathcal{O}(G)$ is not uniquely defined by its values on $G\otimes G$. Since these rules define the structure rather implicitly, let us state them explicitly as well for the basis δ_g on a finite group G for completeness:

$$\begin{aligned} \Delta(\delta_g) &= \sum_{h,k\in G} \delta_h\otimes\delta_{h^{-1}g} \\ \varepsilon(\delta_g) &= \begin{cases} 1 \text{ if } g = e \\ 0 \text{ otherwise} \end{cases} \\ S(\delta_g) &= \delta_{g^{-1}} \end{aligned} \tag{2.9}$$

2.2.3 Pairing

Given that the structures of a Hopf algebra are defined in a symmetric manner, it is to be expected that the dual to a Hopf algebra is itself also a Hopf algebra. However, as we remarked in the last section, this duality is not always that simple, as many Hopf algebras are infinite dimensional as vector spaces. There are a couple of notions of duality that are useful, for example see [Mon09]. The main notion we use is defined via pairing in [Maj95, p. 10]:

Definition 2.2.5. *Two Hopf algebras $\mathcal{A}, \mathcal{H}$ are paired if there is a bilinear map $\langle\ ,\ \rangle : \mathcal{A} \otimes \mathcal{H} \to k$ such that for all $\psi, \phi \in \mathcal{A}$ and $h, g \in \mathcal{H}$:*

$$\begin{aligned} \langle \phi\psi, h\rangle = \langle \phi \otimes \psi, \Delta h\rangle, &\quad \langle \mathbb{1}_{\mathcal{A}}, h\rangle = \varepsilon_{\mathcal{H}}(h), \\ \langle \Delta\phi, h\otimes g\rangle = \langle \phi, hg\rangle, &\quad \varepsilon_{\mathcal{A}}(\phi) = \langle \phi, \mathbb{1}_{\mathcal{H}}\rangle \\ \langle S(\phi), g\rangle &= \langle \phi, S(g)\rangle \end{aligned} \tag{2.10}$$

Like with any infinite-dimensional vector spaces it is not necessary that if $(\mathcal{A}, \mathcal{H})$ is a pairing, then so is $(\mathcal{H}, \mathcal{A})$. A pairing is *non-degenerate* if there are no nonzero elements in $\mathcal{A}$ and $\mathcal{H}$ that pair to zero with all elements of the other algebra. As a final remark, we may refer to the notion of *convolution algebra* of functions on a bialgebra $\mathcal{H}$. This is given by $(f \star g)(h) := (f \otimes g)(\Delta h)$.

2.2.4 Modules

The fact that our main examples are groups and Lie algebras reflects that a strong feature of Hopf algebras is their representation theory. In this chapter we introduce the notion of a (co)module and of a (co)module-(co)algebra. We will talk about the monoidal categories they form later in chapter 2.4, for a much more complete introduction to representation theory we refer to [Böh18] [Mon93] [Maj95]. Let us however begin by defining a module:

Definition 2.2.6. *An $\mathcal{H}$-*module *$(V, \triangleright)$ is a pair consisting of a vector space V and a map $\triangleright : \mathcal{H} \otimes V \to V$ such that the following commutes:*

$$\begin{array}{ccc} \mathcal{H} \otimes \mathcal{H} \otimes V & \xrightarrow{\mathrm{id}\otimes\triangleright} & \mathcal{H} \otimes V \\ {\scriptstyle m\otimes\mathrm{id}}\downarrow & & \downarrow{\scriptstyle\triangleright} \\ \mathcal{H} \otimes V & \xrightarrow{\quad\triangleright\quad} & V \end{array}$$

Modules appear all over mathematics, for example, representations of finite groups are modules over the group algebra and representations of a Lie algebra are modules over its universal enveloping algebra. The representation theoretic way of looking at the coalgebra structure is that given any two modules $(V, \triangleright), (W, \triangleright')$, we get a new module:

$$(V \otimes W, (\triangleright \otimes \triangleright') \circ \Delta) \tag{2.11}$$

by using the coproduct, and we get a module structure on $\mathbb{K}$ via ε. The set of all modules, together with maps $f : (V, \triangleright) \to (W, \triangleright')$ that obey the intertwining law:

$$f(h \triangleright v) = h \triangleright' f(v) \tag{2.12}$$

for all $h \in \mathcal{H}$ forms a category which we will denote by ${}_{\mathcal{H}}\mathcal{M}$.

By duality, where there is a module there is a *comodule*. We will define it in the by now hopefully familiar way of reversing arrows:

Definition 2.2.7. *An* $\mathcal{H}$-comodule (V, δ) *is a pair consisting of a vector space* V *and a map* $\delta : V \to V \otimes \mathcal{H}$ *such that the following commutes:*

$$\begin{array}{ccc} V & \xrightarrow{\quad \nabla \quad} & V \otimes \mathcal{H} \\ {\scriptstyle \nabla}\downarrow & & \downarrow{\scriptstyle \nabla \otimes \mathrm{id}} \\ V \otimes \mathcal{H} & \xrightarrow{\mathrm{id} \otimes \Delta} & V \otimes \mathcal{H} \otimes \mathcal{H} \end{array}$$

Also for coactions we will adopt a Sweedler-style notation:

$$\delta v = \sum_i v_i \otimes h_i := v_{(\underline{0})} \otimes h_{(\underline{1})} \tag{2.13}$$

The convention is that the index $(\underline{0})$ is reserved for elements of the comodule, whereas the positively numbered indices are for elements of the Hopf algebra. The set of comodules $(V, \delta), (W, \delta')$, together with the intertwining maps $f : V \to W$ that satisfy:

$$(f \otimes \mathrm{id})\delta v = \delta' f(v) \tag{2.14}$$

forms a category, which we denote $\mathcal{M}^{\mathcal{H}}$.

Similarly to how a coproduct gives a module structure to $V \otimes W$ for $V, W \in {}_{\mathcal{H}}\mathcal{M}$, we find that the product gives a comodule structure:

$$\delta(v \otimes w) = v_{(\underline{0})} \otimes w_{(\underline{0})} \otimes (v_{(\underline{1})} \cdot w_{(\underline{1})}) \tag{2.15}$$

2.2.5 Quasitriangular structure

A quasitriangular structure gives some structure to a Hopf algebra that is slightly weaker than it being cocommutative. From a representation theoretic point of view, the coproduct gives a way to divide an action $\triangleright$ over 2 different modules. Cocommutativity in that case implies that the coaction on $V \otimes W$ and $W \otimes V$ is the same. For many physical applications there could be more complicated statistics on a product of representations. The simplest possible requirement that is less restrictive than being cocommutative is to be cocommutative up to a conjugation by a symmetry. That is what a quasitriangular structure does. It is for this physical intuition that often the name quantum group is reserved to Hopf algebras with this structure. We will follow [Maj95] closely for this part.

Definition 2.2.8. *A* quasitriangular structure $\mathcal{R} := \sum \mathcal{R}_1 \otimes \mathcal{R}_2$ *on a Hopf algebra* $\mathcal{H}$ *is an element in* $\mathcal{H} \otimes \mathcal{H}$ *that is invertible, such that:*

$$\tau \circ \Delta h = \mathcal{R}(\Delta h)\mathcal{R}^{-1} \tag{2.16}$$

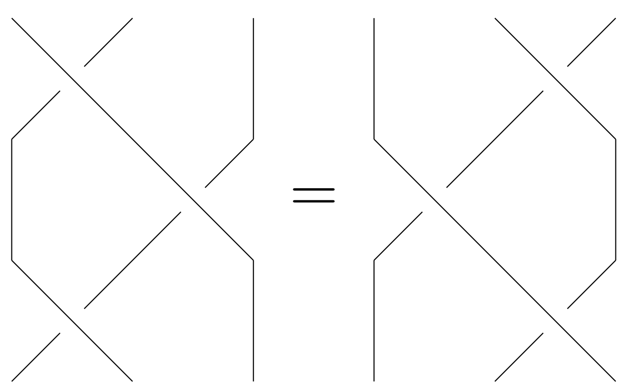
together with:

$$(\Delta \otimes \mathrm{id})\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23}; \quad (\mathrm{id} \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12} \tag{2.17}$$

Where in this case $\mathcal{R}_{12}$ means that $\mathcal{R}$ is acting on the first 2 slots of a tensor in $\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$, and similar for the others. Together the two equations imply that R obeys the *abstract Quantum Yang Baxter equation*:

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12} \tag{2.18}$$

This equation occurs frequently and is in fact one of the original reasons for developing quantum groups. Given such an $\mathcal{R}$ we get a solution in a vector space for every representation, and hence such an element $\mathcal{R}$ is often referred to as a *universal R-matrix*. The categorical interpretation of this equation is elucidating:



That is, the diagram corresponds to a variant of the third Reidemeister move for knots to be valid in Hopf algebraic string diagrams, making sure that the category of representations is what is called a *braided* monoidal category. We will briefly review these objects later in chapter 2.4.2.

Partially because it is somewhat accepted in the literature, partially because they occur a lot and the terms "quasitriangular Hopf algebra" and "qha" are unwieldy I will refer to a pair $(\mathcal{H}, \mathcal{R})$ as a *quantum group*. Of course there exists also a dual notion, called a dual quasitriangular structure. As we will not use this and the definition needs some care I will refer the interested reader to [Maj95, ch.2.2.].

2.2.6 Drinfeld Twists

One of the main tools we will use in this thesis to deform algebras is the notion of a *Drinfeld twist*. The idea is to make a new quantum group from an old one by introducing a conjugation around the coproduct. The equations we require for the new coproduct to be well-defined and coassociative turn out be strongly linked to cohomology of groups in the commutative examples mentioned before. Indeed, we can use this as an inspiration for a *Hopf cohomology*, however we will not talk about the coboundary maps. Although a definition exists and can be found in e.g. [Maj95], as mentioned there we do not in general have $\partial\partial = 1$ in degree > 2 for noncommutative Hopf algebras. One unfortunate further complication is that Majid refers to this theory as "cocycles" and the dual theory as "dual cocycles", whereas later literature such as [Asc+17] often refers to the dual theory as "cocycles" and the theory below as *twists*. As overused as the term "twist" is in noncommuative geometry, I will follow this use.

Definition 2.2.9. *A* Drinfeld twist *on a Hopf algebra* $\mathcal{H}$ *is an invertible element* $\chi \in \mathcal{H}\otimes\mathcal{H}$ *such that:*

$$(\mathbb{1}\otimes\chi)(\mathrm{id}\otimes\Delta)\chi = (\chi\otimes\mathbb{1})(\Delta\otimes\mathrm{id})\chi \tag{2.19}$$

It is counital if $(\varepsilon\otimes\mathrm{id})\chi = 1$.

We will presume all twists to be counital. The equation above will be referred to as the *twist condition*. Given a twist χ in a quantum group $(\mathcal{H}, \mathcal{R})$, the new, twisted quantum group is given by $(\mathcal{H}_\chi, \mathcal{R}_\chi)$, where $\mathcal{H}_\chi$ has the same algebra and counit structures, and the following coproduct, antipode and quasitriangular element:

$$\Delta_\chi h = \chi(\Delta h)\chi^{-1}, \quad \mathcal{R}_\chi = \tau(\chi)\mathcal{R}\chi^{-1}, \quad S_\chi h = ((\mathrm{id}\otimes S)\chi)(Sh)((\mathrm{id}\otimes S)\chi)^{-1} \tag{2.20}$$

Many twists yield isomorphic Hopf algebras. This often occurs when two twists are cohomologous, as defined below.

Definition 2.2.10. *Two twists χ, ϕ are* cohomologous *if $\chi = (\sigma\otimes\sigma)\phi(\Delta\sigma^{-1})$ for some invertible element $\sigma \in \mathcal{H}$.*

There are some special cases of note: A twist cohomologous to the identity equals $\chi = \Delta\sigma(\sigma\otimes\sigma)^{-1}$. The quasitriangular element $\mathcal{R}$ for an algebra that was formerly cocommutative is given by $\mathcal{R}_\chi = \tau(\chi)\chi^{-1}$.

The dual idea to a twist has a similar intuition. Since I have not talked much about coquasitriangular structures I will refrain from going into too much detail, but the dual picture for the twist itself is quite natural. Dual to a choice of an element, which can be written as a map $k \to \mathcal{H}\otimes\mathcal{H}$, we have a map $\gamma : \mathcal{H}\otimes\mathcal{H} \to k$.

Definition 2.2.11. *A* 2-cocycle *$\gamma : \mathcal{H}\otimes\mathcal{H} \to k$ on a Hopf algebra $\mathcal{H}$ is a convolution-invertible function such that:*

$$\sum \gamma(h_{(1)}\otimes k_{(1)})\gamma(l\otimes k_{(2)}h_{(2)}) = \sum \gamma(l_{(1)}\otimes h_{(1)})\gamma(l_{(2)}h_{(2)}\otimes k) \tag{2.21}$$

for all $h, k, l \in \mathcal{H}$. It is unital if $\gamma(h,1) = \gamma(1,h) = \varepsilon(h)\ \forall h \in \mathcal{H}$

The twisted product becomes:

$$h \cdot_\gamma k = \sum \chi(h_{(1)}\otimes k_{(1)})\chi^{-1}(h_{(3)}\otimes k_{(3)})h_{(2)}k_{(2)} \tag{2.22}$$

A remark here is that for grouplike elements the cocycle-deformed product is equal to the old product up to a scalar. We will see this later, where we will deform a group algebra.

2.2.7 (co)Module algebras

Quantum groups are made to (co-)act. The set of all objects with a particular quantum symmetry forms a world in itself, a category. Inside of this category one can define (co-)

algebras just the same. One reason to consider this is that a twist on a quantum group translates to a twist on all the objects that have this symmetry, including the algebras of functions of spaces with that symmetry, which is how we will form noncommutative spaces. An algebra with a quantum group action should have its own structures compatible with the Hopf algebra (co-)action. Let us begin there:

Definition 2.2.12. *An algebra and (left) $\mathcal{H}$-module $\mathcal{A}$ is a (left) $\mathcal{H}$-*module algebra *if the following diagram commutes*[6]*:*

$$\begin{array}{ccccc} \mathcal{H}\otimes\mathcal{A}\otimes\mathcal{A} & \xrightarrow{\mathrm{id}\otimes m_{\mathcal{A}}} & \mathcal{H}\otimes\mathcal{A} & \xrightarrow{\quad\triangleright\quad} & \mathcal{A} \\ \downarrow{\scriptstyle \Delta\otimes\mathrm{id}\otimes\mathrm{id}} & & & & \uparrow{\scriptstyle m_{\mathcal{A}}} \\ \mathcal{H}\otimes\mathcal{H}\otimes\mathcal{A}\otimes\mathcal{A} & \xrightarrow{\mathrm{id}\otimes\tau\otimes\mathrm{id}} & \mathcal{H}\otimes\mathcal{A}\otimes\mathcal{H}\otimes\mathcal{A} & \xrightarrow{\triangleright\otimes\triangleright} & \mathcal{A}\otimes\mathcal{A} \end{array}$$

The explanation for this diagram is that the bottom route is the action of $\mathcal{H}$ on $\mathcal{A}\otimes\mathcal{A}$ as defined by the coproduct, and so effectively we want the map $m_{\mathcal{A}} : \mathcal{A}\otimes\mathcal{A} \to \mathcal{A}$ to commute with the action of $\mathcal{H}$. If $\mathcal{H}$ is quasitriangular there is an element $\mathcal{R}$ generating a map between $V\otimes W$ and $W\otimes V$ in ${}_{\mathcal{H}}\mathcal{M}$, even when $V = W = \mathcal{A}$. This means that we have an inherent way to link $x \cdot y$ and $y \cdot x$ for $x, y \in \mathcal{A}$. More importantly, we have a new way to construct such noncommutative algebras, by means of twisting.

Theorem 2.2.13. *Let $\mathcal{A}$ be an associative $\mathcal{H}$-module algebra with multiplication m, and χ a twist on $\mathcal{H}$. Then $\mathcal{A}_\chi$ is an associative $\mathcal{H}_\chi$-module algebra with multiplication:*

$$m_\chi = m \circ (\chi^{-1} \triangleright \otimes \triangleright) \tag{2.23}$$

We will often use the fact that $\underline{\mathcal{H}}$ is itself a $\mathcal{H}$-module algebra by associativity to deform it. When viewing $\mathcal{H}$ as a module rather than an algebra we will underline it. This is of some importance, since when twisted as a $\mathcal{H}$-module algebra $\underline{\mathcal{H}}_\chi$ is no longer a Hopf algebra in general. This construction is very important for this thesis, almost all noncommutative spaces we will consider are made by variations on this theme.

Of course of all the above there exists a dualisation. In fact, there exist notions of comodule algebras, module coalgebras and comodule coalgebras. As we want to know

[6]The flip map in the bottom line is tricky. In the case where we move everything internal to the world of modules of another Hopf algebra, this flip map has to be the braiding in the appropriate category. As long as the Hopf algebra itself is not living in such an environment we stick to the regular flip.

about deformed geometric spaces, we will focus on algebras. All Hopf algebras we talk about can by some nondegenerate pairing be dualised to a Hopf algebra $\mathcal{H}^\circ$, which sends the actions to coactions and vice versa. Because of this, in theory the following is redundant. In practice however, it is useful to have a grasp on both sides of the duality as separate authors prefer different sides. Additionally, though often there is a good choice for the pairing this choice is in principle not unique. So let us define a *comodule algebra*:

Definition 2.2.14. *An algebra and (right) $\mathcal{H}$-comodule $\mathcal{A}$ is a (right) $\mathcal{H}$-*comodule algebra *if the following diagram commutes:*

$$\begin{array}{ccccc} \mathcal{A}\otimes\mathcal{A} & & \xrightarrow{\quad m_{\mathcal{A}}\quad} & & \mathcal{A} \\ \downarrow{\scriptstyle \delta\otimes\delta} & & & & \downarrow{\scriptstyle \delta} \\ \mathcal{A}\otimes\mathcal{H}\otimes\mathcal{A}\otimes\mathcal{H} & \xrightarrow{\mathrm{id}\otimes\tau\otimes\mathrm{id}} & \mathcal{A}\otimes\mathcal{A}\otimes\mathcal{H}\otimes\mathcal{H} & \xrightarrow{m_{\mathcal{A}}\otimes m} & \mathcal{A}\otimes\mathcal{H} \end{array}$$

Note that this diagram is not a strict dualisation of the previous one, as there is no coproduct on $\mathcal{A}$. We do however have a similar theorem to before:

Theorem 2.2.15. *Let $\mathcal{A}$ be an associative $\mathcal{H}$-comodule algebra with multiplication m, and γ a 2-cocycle on $\mathcal{H}$. Then $\mathcal{A}_\gamma$ is an associative $\mathcal{H}_\gamma$-comodule algebra with multiplication:*

$$m_\gamma = (m\otimes\gamma^{-1}) \circ (\delta\otimes\delta) \tag{2.24}$$

2.2.8 Quasi-Hopf algebras

In both theorems of the previous section as well as the section on twisting it was noted that the cocycle and twist conditions need to be obeyed in order to arrive at a (co)associative algebra. If we however do not impose this requirement on ourselves and accept that the algebras involved may become non-(co)associative, we can in fact use any invertible $\chi \in \mathcal{H}\otimes\mathcal{H}$ as a twist. This will require much more structure on the noncoassociative Hopf algebra, which is called a *quasitriangular quasi-Hopf algebra*. This notion is invented by Drinfeld [Dri89] and explained in [Maj95], and has several examples explained in [DAn15].

Definition 2.2.16. *A* quasitriangular quasi-Hopf algebra *is given by the decuplet:*

$$(\mathcal{H}, \Delta, \varepsilon, m, \mathbb{1}, \mathcal{R}, \phi, S, \alpha, \beta) \tag{2.25}$$

The elements $\alpha, \beta \in \mathcal{H}$ are only used to adapt the antipode to the nonassociative setting, which we will not go in to. The element $\phi \in \mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H}$ is the universal associator, whose purpose is given by:

$$(\mathrm{id} \otimes \Delta) \circ \Delta = \phi((\Delta \otimes \mathrm{id}) \circ \Delta)\phi^{-1} \tag{2.26}$$

All the coherence conditions between $\mathcal{R}$ and ϕ, and especially S will be left to the references. The main idea with regards to $\mathcal{R}$ is that the category of representations, in which $\mathcal{R}$ represents the braiding, is no longer a strict monoidal category and therefore has to obey pentagon and hexagon relations, which we leave in the appendix. This amount of nonassociativity is very mild, since one can in principle rebracket, it just comes with a cost of a multiplication.

We will use this theory mainly for the definition of the octonions by using a cocycle that does not obey the cocycle condition. This lies in the dual theory of a quasi-Hopf algebra, which we will not look at. The formula for the associator from using a noncocyclic invertible element $F : \mathcal{H}\otimes\mathcal{H} \to k$ is given by:

$$\phi(k, l, m) = \frac{F(k, l)F(k \cdot l, m)}{F(k, l \cdot m)F(l, m)} = F(k, l)F(k \cdot l, m)F(k, l \cdot m)F(l, m) \tag{2.27}$$

2.3 Hopf-Galois extensions

One of the fundamental building blocks of geometry is the notion of a *principal bundle*. The previous section was about one of the ways to dualise a group. If we want to study geometry in a dual setting, it is therefore a good idea to study the way Hopf algebras act on algebras and what conditions are necessary to get the dual of the notions of a free, transitive and locally trivial action. Very elegantly, this turns out to be a condition that is strongly related to the Galois condition of field extensions. We will look only at the Hopf-algebraic side, following in this section [Mon09], [Mon93], [BJM08] and [Asc+17]. For interest in the Galois theory side we refer to [BJ01] and [PG87].

We want to dualise the notion of a principal G-bundle $P \to GM$ with total space P, structure group G and base space M. We have 2 maps that are necessary to define this: the action $\rho : P \times G \to P$, and the projection $\pi : P \to M$. Dualising, we obtain an algebra $\mathcal{A}$ for the 'total space', a Hopf algebra $\mathcal{H}$ for the 'structure group' and an algebra $\mathcal{B}$ for the base space. We get a coaction $\delta : \mathcal{A} \to \mathcal{A}\otimes\mathcal{H}$, and an injection $i : \mathcal{B} \to \mathcal{A}$. The condition that the group acts fibrewise is on the geometric side equivalent to the request that the following commutes:

$$\begin{array}{ccc} P \times G & \xrightarrow{\pi_1} & P \\ \Big\downarrow{\scriptstyle \rho} & & \Big\downarrow{\scriptstyle \pi} \\ P & \xrightarrow{\pi} & M \end{array}$$

Where π_1 projects away the group. We can interpret that as similar to the action of a counit $\varepsilon \otimes \mathbb{1}$ on the group algebra, so reversing all the arrows we arrive at:

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{i} & \mathcal{A} \\ \Big\downarrow{\scriptstyle i} & & \Big\downarrow{\scriptstyle \mathrm{id} \otimes \mathbb{1}} \\ \mathcal{A} & \xrightarrow{\nabla} & \mathcal{A} \otimes \mathcal{H} \end{array}$$

That is to say, the coaction on $\mathcal{B}$ is trivial, or $\mathcal{B}$ is *coinvariant*. The fact that the group action has no fixed points dualises to the requirement that $\mathcal{B} = \mathcal{A}^{\mathrm{co}\,\mathcal{H}}$, i.e. $\mathcal{B} = \{a \in \mathcal{A} : \delta a = a \otimes \mathbb{1}\}$. Note that the set of coinvariants is always a subalgebra.

The remaining requirements are all captured in the *canonical map* given by:

$$\begin{aligned} \mathrm{can} &= (m_{\mathcal{A}} \otimes \mathrm{id})(\mathrm{id} \otimes \delta) : \mathcal{A} \otimes_{\mathcal{B}} \mathcal{A} \to \mathcal{A} \otimes \mathcal{H} \\ \mathrm{can}(a \otimes a') &= (a \cdot a'_{(\underline{0})}) \otimes a_{(\underline{1})} \end{aligned} \tag{2.28}$$

The bijectivity of this map is equivalent in the commutative case to stating that $\mathcal{A}$ is a principal $\mathcal{H}$ bundle with base space $\mathcal{B}$. If this is the case, we say that $\mathcal{A}$ is an *$\mathcal{H}$-Hopf Galois extension* over the algebra of coinvariants $\mathcal{B}$.

Some remarks: the map can is not in general an algebra map; indeed there is no algebraic structure defined in general for $\mathcal{A} \otimes_{\mathcal{B}} \mathcal{A}$ if $\mathcal{B}$ is not central in $\mathcal{A}$. Unlike for principal bundles, a Hopf-Galois extension over the ground field is not necessarily trivial, these objects are called *Galois objects*. Often it is required that $\mathcal{A}$ is faithfully flat as a $\mathcal{B}$-module, we do not use this requirement.

2.4 Monoidal categories

2.4.1 Introduction

The studies of Hopf algebras and monoidal categories are deeply linked. As briefly mentioned before, the set of (co-)modules of Hopf algebras has the structure of a monoidal category. The reverse is also true to some extent: if one has a monoidal category, then in some sense the maximal symmetry that all objects have in common forms a Hopf algebra, defining the symmetry via a relational complex rather than intrinsically. One great power of category theory is that it captures both the properties of large sets of objects, as well as describe some single objects themselves. We will later make use of both facets. Let us first define the necessary structures. We will follow [Eti+16], occasionally looking at [Maj95, ch.9], [MRV18], [Müg06] and [Böh18]. We will only use strict monoidal categories for this text. Let us first revise the definitions necessary:

Definition 2.4.1. *A* category $\mathcal{C}$ *consists of a collection of objects* $\mathrm{Ob}(\mathcal{C})$ *and morphisms (or arrows)* $\hom_{\mathcal{C}}(X, Y)$ *for all* $X, Y \in \mathrm{Ob}(\mathcal{C})$ *such that for every object* X *there exists an identity morphism* $\mathrm{id} : X \to X$*, and for every pair of morphisms* $f : X \to Y$ *and* $g : Y \to Z$ *there exists a composite* $g \circ f : X \to Z$ *such that the composition is associative, with* $f \circ \mathrm{id} = \mathrm{id} \circ f = f$*.*

We will quite often use the abuse of notation $X \in \mathcal{C}$ to mean that X is an object in $\mathcal{C}$. All categories we look at will be essentially small, so the $\hom$-sets are sets and the collection of objects can be assumed to be a set as well. We will presume some knowledge of functors and natural transformations, a standard reference is [Mac13].

Definition 2.4.2. *A* strict monoidal category $(\mathcal{C}, \otimes)$ *is a category equipped with a* product $\otimes$ *which is a bifunctor* $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ *such that* $(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$ *and a* unit object $\mathbb{1} \in \mathcal{C}$ *such that* $\mathbb{1} \otimes X = X = X \otimes \mathbb{1}$

There is some notational confusion between the object $X \otimes Y \in \mathcal{C}$ and the object $X \times Y \in \mathcal{C} \times \mathcal{C}$ to be wary of. As $\otimes$ is a bifunctor, it assigns product morphisms to pairs of morphisms. To pass to non-strict monoidal categories, every equality in the above definition is replaced by an isomorphism; for us this will become necessary when we consider modules over nonassociative algebras[7]. The extra axioms can be found in appendix

[7] In principle there exists Mac Lane's coherence theorem that claims that all monoidal categories are equivalent to strict categories, however some extra structure like a fiber functor does not follow along. One example is given in [EG01].

A.1. Let us give two quick examples of monoidal categories. Firstly, consider the category ${}_{\mathcal{H}}\mathcal{M}$ of $\mathcal{H}$-modules. We defined this category in 2.2.4, but to revise: the monoidal product is given by the tensor product of vector spaces, the module structure on the product of two modules $V \otimes W$ is given by the coproduct via $g \triangleright (V \otimes W) = (g_{(1)} \triangleright V) \otimes (g_{(2)} \triangleright W)$ and the unit object is guaranteed by the existence of a counit. The second example is on a much smaller scale: any algebra can itself be seen as a monoidal category, where every object is an element of the algebra and the category has only identity morphisms. The monoidal product is then equal to the algebra product.

2.4.2 Further structures

There are many other structures one would like to impose on a monoidal category. We will quickly revise some important ones.

Firstly, the existence of a fibre functor $F : \mathcal{C} \to \mathrm{Vect}$. This is an exact, faithful, functor $\mathcal{C} \to \mathbb{K}\,\mathrm{Vect}$ such that there exist isomorphisms $J_{X,Y} : F(X \otimes_{\mathcal{C}} Y) \to F(X) \otimes_{\mathrm{Vect}} F(Y)$ that obey an associativity requirement, see [Eti+16, ch. 5.1]. In the case we observe, all $J_{X,Y}$ are equalities and the associativity is trivial. One can interpret this as embedding the monoidal category into the category of $\mathbb{K}$-vector spaces. Quite often the functor is forgetful as the embedding is already implicitly there.

The second structure we would like to look into is the structure of a *braiding*. A braiding is a natural isomorphism $B_{X,Y} : X \otimes Y \to Y \otimes X$. In a strict monoidal category it has to obey:

$$\begin{array}{ccc} X \otimes Y \otimes Z & \xrightarrow{B_{X,Y} \otimes \mathrm{id}} & Y \otimes X \otimes Z \\ & \searrow^{B_{X,Y\otimes Z}} & \downarrow{\scriptstyle \mathrm{id} \otimes B_{X,Z}} \\ & & Y \otimes Z \otimes X \end{array} \qquad \begin{array}{ccc} X \otimes Y \otimes Z & \xrightarrow{\mathrm{id} \otimes B_{Y,Z}} & X \otimes Z \otimes Y \\ & \searrow^{B_{X\otimes Y,Z}} & \downarrow{\scriptstyle B_{X,Z} \otimes \mathrm{id}} \\ & & Z \otimes X \otimes Y \end{array}$$

which are the equivalents to the Yang-Baxter equation for the quasitriangular element. In the strict case one needs to include the appropriate associators.

The third structure we can equip our categories with are left and right duals, also called a *rigid* structure. An object V has a left dual V^* if there exists an evaluation morphism $\mathrm{ev} : V^* \otimes V \to \mathbb{1}$ and dually a coevaluation morphism $\mathrm{coev} : \mathbb{1} \to V \otimes V^*$ such that the following commute:

$$\begin{array}{ccc} V & \xrightarrow{\mathrm{coev}\otimes \mathrm{id}} & V\otimes V^*\otimes V \\ & \underset{\mathrm{id}}{\searrow} & \downarrow{\scriptstyle \mathrm{id}\otimes\mathrm{ev}} \\ & & V \end{array} \qquad \begin{array}{ccc} V^* & \xrightarrow{\mathrm{id}\otimes \mathrm{coev}} & V^*\otimes V\otimes V^* \\ & \underset{\mathrm{id}}{\searrow} & \downarrow{\scriptstyle \mathrm{ev}\otimes 1} \\ & & V^* \end{array}$$

The last structure we would like to mention is the structure of a *bar category* defined in [BM09]. We will not use this explicitly, although the omnipresence of complex structures in this work probably implies that this is a current shortcoming.

2.4.3 Tannaka-Krein reconstruction

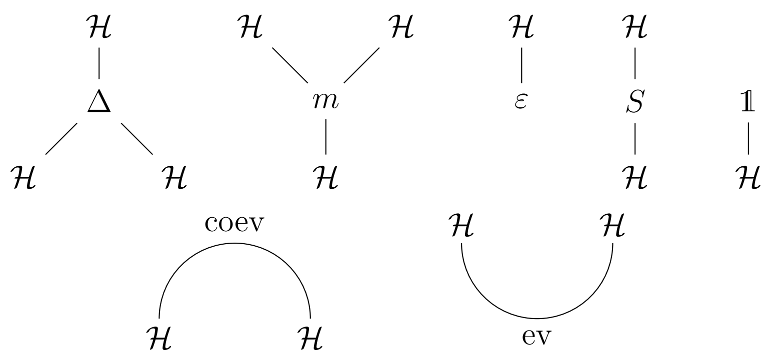
For all the above structures we can find some intuition or analogue for their meaning in the context of the monoidal category of modules of a Hopf algebra. Indeed, the fibre functor corresponds to a linear structure, the braiding to a quasitriangular element, the rigid structure to the antipode and the bar to a star structure. This intuitive correspondence is more than just an intuition, and in fact it should come as a justification for the correctness of the choice of axioms of a Hopf algebra that the set of symmetries of a braided rigid monoidal category with fiber functor to Vect always has the structure of a quantum group. There are many ways to find this Hopf algebra, ranging from technical coend constructions to a much more hands-on approach. Surprisingly, there are very few explicit examples in the literature of such a construction. As we will follow a very explicit example later, we will not repeat too much of the general theory here. We will instead provide a quick dictionary of terms, as many of the above structures have a direct connected structure in the reconstructed Hopf algebra:

Monoidal category	Hopf algebra
$\otimes$	Δ
$\mathbb{1}$	ε
Rigid structure	S
Braiding	$\mathcal{R}$
Bar structure	$*$-structure
Non-strict associator	Coassociator

2.4.4 String diagrams

A convenient way to interpret all the operations in a Hopf algebra in a categorical manner is via the pictorial language of string diagrams. The original reference here is [JS86], though most books above will discuss it at least somewhat.

The idea is as follows: we mentioned before that the Yang-Baxter equations are equivalent to the third Reidemeister move for knots, which indicates that we can do calculations using diagrams, by moving the lines defining operations around. The diagrams are supposed to be read from top to bottom, and contain the following components:



Any Hopf algebra property such as coassociativity or multiplicativity of the coproduct can be translated to a property of string diagrams rather than commutative diagrams as we did before. We will refrain from doing so right here, as we do not use many of these properties and the ones we do use are for quasi-Hopf algebras, which lack coassociativity. We will state any properties do use in situ.

Crossings are treacherous: they usually indicate a flip in the category one is working in. This means that to interpret them one needs to be aware if they are doing a calculation in a Hopf algebra, which usually lives in a category with the regular flip map τ, or if one is living in the category of representations of a Hopf algebra, where the flip map is given by the braiding $\mathcal{R}$.

The main point for which we will use these diagrams later is to aid in a computation, where otherwise the indices of the coproducts become unwieldy and unclear. One property of string diagrams that is of great help, which we will claim here without further proof, is the following: Any map between $\mathcal{H}^{\otimes n}$ and $\mathcal{H}^{\otimes m}$ defined by string diagrams has a dual map from $\mathcal{A}^{\otimes m}$ to $\mathcal{A}^{\otimes n}$ if there is a nondegenerate pairing between $\mathcal{H}$ and $\mathcal{A}$.

2.5 Real division algebras

2.5.1 Introduction

We will frequently run into the 4 real division algebras: the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions or Hamilton numbers $\mathbb{H}$ and the octonions or Cayley numbers $\mathbb{O}$. The first two are often fields of choice, whereas the latter two are less regularly used. Our reason to use them is to study spheres: the existence of a norm allows us to easily define a sphere of elements of constant norm, and the multiplicativity of the norm means the algebra structures descend down to this geometry. It is a well known theorem due to Hurwitz (proved in [Hur23], with a proof using string diagrams in [Str18]) that these four are the only real normed division algebras. General references on the subject are [Dix13][CS03][DM15][Bae02]. There are two different ways to define them, both of which will see use in this thesis.

2.5.2 Cayley-Dickson construction

The Cayley-Dickson construction works as follows: given any (not necessarily associative) algebra $\mathcal{A}$ with a $*$-structure, we can put a $*$-algebra structure on $\mathcal{A} \oplus \mathcal{A}$ as follows:

$$\begin{aligned} (a, b) \cdot (c, d) &= (ac - d^* b, da + bc^*) \\ (a, b)^* &= (a^*, -b) \end{aligned} \tag{2.29}$$

The principle behind this construction is explained in [Bae02] to be the following: the second copy of your algebra gets an extra copy of a square root of -1: $(0, 1) := \mathrm{i}_{\text{new}}$. This new complex unit implements the $*$-structure on $\mathcal{A}$ by means of conjugation, i.e. $\mathrm{i}_{\text{new}} \cdot ((a, 0) \cdot \mathrm{i}_{\text{new}}) = (a^*, 0)$. The above construction is exactly what is necessary to establish this.

It is well-known that this doubling comes with a drawback: starting with $\mathbb{R}$, you lose self-adjointness on your way to $\mathbb{C}$, commutativity at $\mathbb{H}$ and associativity at $\mathbb{O}$. After the octonions, applying another iteration of the Cayley-Dickson construction lands you outside the realm of the real division algebras in an algebra we call the *sedenions*[8] $\mathcal{S}$. This algebra

[8]Unfortunately we already use the symbol $\mathbb{S}$ for spheres, though there is something poetic in the fact that the sedenions had to hand in their symbol along with their claim to being a number system.

contains nonzero numbers[9] a, b such that $a \cdot b = 0$, which means that inverting can no longer work. This is a natural endpoint for many uses, as the norm $||x|| = \sqrt{x^* x}$ is no longer multiplicative, and hence the algebraic structure does no longer filter down to the unit ball of elements of norm one.

2.5.3 Twisted quasialgebra construction

The second way of constructing these algebras that we will look at is the one explored in [AM99] and its abridged version [AM19]. There, the authors use the Hopf algebra theory mentioned in chapter 2.2.6 to Drinfeld twist the algebra $\mathbb{R}\mathbb{Z}_2^n$. By this notation we mean the Hopf algebra of $\mathbb{R}$-linear multiples of $(\mathbb{Z}_2)^n$; we chose to avoid including brackets in the notation to avoid confusion with the group algebra $k[\mathbb{Z}_2]$. The basis of this algebra is given by elements e_k which have the product $e_k \cdot e_l = e_{k \oplus l}$. We choose to take $\oplus$ the operation in the group $(\mathbb{Z}_2)^n$, and to move the group operation to the index to have a multiplicative notation in the group algebra. Since this is confusing in words, let us do a quick example. For $n = 3$ we could for example write:

$$e_{\left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right)} \cdot e_{\left(\begin{smallmatrix}1\\1\\1\end{smallmatrix}\right)} = e_{\left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right) \oplus \left(\begin{smallmatrix}1\\1\\1\end{smallmatrix}\right)} = e_{\left(\begin{smallmatrix}0\\1\\1\end{smallmatrix}\right)}$$

Since this notation gets rather cumbersome, we will rather write the above as $e_{100} \cdot e_{111} = e_{011}$, or, interpreting these numbers as binary, as $e_4 \cdot e_7 = e_{4 \oplus 7} = e_3$. Whatever way we notate this, the basis elements are presumed grouplike in the Hopf algebra structure, i.e. $\Delta e_k = e_k \otimes e_k$, the unit is e_0 and the counit is given by $\epsilon(e_k) = 1$.

Let us use this as an example of how a Drinfeld twist procedure works, using a cocycle rather than a twist in this case. We equip the algebra $\mathbb{R}\mathbb{Z}_2^n$ with a cocycle $F : \mathbb{R}\mathbb{Z}_2^n \otimes \mathbb{R}\mathbb{Z}_2^n \to \mathbb{R}$ with the following properties:

$$\begin{gathered} F(e_k, e_l)^2 = 1 \\ F(e_0, e_l) = F(e_k, e_0) = 1 \end{gathered} \tag{2.30}$$

For all $k, l \in \mathbb{Z}_2^n$. This cocycle deforms the algebra structure on $\mathbb{R}\mathbb{Z}_2^n$ into an isomorphic algebra. Indeed, the convolution inverse of F is F and the basis on the algebra is grouplike,

[9] In fact, in [Mor98] it is claimed that the set of pairs in $\mathbb{S} \times \mathbb{S}$ that multiply to zero form a subspace isomorphic to the exceptional Lie algebra $G_2 = \mathrm{Aut}(\mathbb{O})$. A clearer explanation can be found in [BDI08].

so the deformed product on a basis is given by $e_k \cdot_F e_l = F(e_k, e_l)e_{k\oplus l}F(e_k, e_l) = e_k \cdot e_l$. The real deformation however can be found in a comodule algebra of the above algebra. Consider the canonical coaction δ of $\mathbb{R}\mathbb{Z}_2^n$ on itself, given by the coproduct. This turns $\underline{\mathbb{R}\mathbb{Z}_2^n}$ into a $\mathbb{R}\mathbb{Z}_2^n$-comodule algebra, as $\delta(\underline{e_k} \cdot \underline{e_l}) = (\underline{e_k \cdot e_l})\otimes(e_k \cdot e_l) = \delta(\underline{e_k}) \cdot \delta(\underline{e_l})$. The twist F on $\mathbb{R}\mathbb{Z}_2^n$, which effectively did nothing, now translates to a new product on the comodule algebra defined by $\underline{e_k} \cdot_F \underline{e_l} = F(e_k, e_l)\underline{e_{k\oplus l}}$. It is this product which we are after. Define the shorthand $F(k, l) := F(e_k, e_l)$. We recall from sections 2.2.6 and 2.2.8 the definitions of the *multiplicative commutator* and *associator* respectively by

$$\begin{aligned} \mathcal{R}(k, l) &: = F(k, l)F(l, k) \\ \phi(k, l, m) &: = \frac{F(k, l)F(k\oplus l, m)}{F(k, l\oplus m)F(l, m)} = F(k, l)F(k\oplus l, m)F(k, l\oplus m)F(l, m) \end{aligned} \tag{2.31}$$

where the last equality holds in our case where $F^2 = 1$. In the paper [AM19], Albuquerque and Majid show that we can view the entire Cayley-Dickson construction on an algebra $\mathcal{A}$ in this language. For the Cayley-Dickson construction the notion of a $*$-structure was central, and here we are given a distinguished grouplike basis, so we define the 1-cocycle $s : \mathcal{A} \to \mathbb{R}$ by $s(e_k)e_k = e_k^*$. In our example $s(e_k) = F(k, k)$. Then we have the following rules to establish a quasi-algebra structure $\hat{F}$ on $\mathcal{A}\otimes\mathbb{Z}_2$ with basis $\{e_k, \hat{i}e_l\} := \{k, \hat{i}\oplus l\}$:

$$\begin{gathered} \hat{F}(k, l) = F(k, l);\ \hat{F}(k, \hat{i}\oplus l) = s(k)F(k, l); \\ \hat{F}(\hat{i}\oplus k, l) = F(l, k);\ \hat{F}(\hat{i}\oplus k, \hat{i}\oplus l) = -s(k)F(l, k) \\ \hat{s}(k) = s(k);\ \hat{s}(\hat{i}\oplus k) = -1 \end{gathered} \tag{2.32}$$

Using these rules we can define the real division algebras iteratively. Let us talk in some more detail about the specifics.

2.5.4 Quaternions

The structure of the quaternions is defined as the numbers $\mathbb{H} \ni q := ae_0 + be_1 + ce_2 + de_3 = a + b\mathrm{i} + c\mathrm{j} + d\mathrm{ij}$ with multiplication as famously carved in Broom Bridge (paraphrased):

$$\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{ij}^2 = \mathrm{ijij} = -1 \tag{2.33}$$

As we will define the octonions later, which have an extra algebra generator, I prefer not to use k for the quaternions[10]. Taking the viewpoint of the quaternions as being a twisted copy of $\mathbb{R}\mathbb{Z}_2^2$, we can define the algebra by specifying just the table of minus signs:

F	e_{00}	$e_{01} = \mathrm{i}$	$e_{10} = \mathrm{j}$	$e_{11} = \mathrm{ij}$
e_{00}	1	1	1	1
e_{01}	1	-1	1	-1
e_{10}	1	-1	-1	1
e_{11}	1	1	-1	-1

As an example: $\mathrm{i} \cdot_F \mathrm{j} = F(e_{01}, e_{10}) e_{01} e_{10} = 1 \cdot e_{11} = \mathrm{ij}$. From here on out we will drop the subscript F as we presume the multiplication to be quaternionic.

To reiterate some basic facts: the quaternions admit a $*$-structure: for $q = a + b\mathrm{i} + c\mathrm{j} + d\mathrm{ij}$, we have $q^* = a - b\mathrm{i} - c\mathrm{j} - d\mathrm{ij}$. A routine computation shows that $qq^* = q^*q \in \mathbb{R}^{\geq 0}$, with $qq^* = 0$ if and only if $q = 0$, so we define $||q|| = \sqrt{q^*q}$. The quaternions form a division algebra, where $q^{-1} = \frac{q^*}{||q||^2}$. This norm is multiplicative, $||q \cdot p|| = ||q|| \cdot ||p||$, and hence the sphere $\mathbb{S}^3$ of quaternions of norm 1 is closed under multiplication and forms a group, isomorphic to $\mathrm{SU}(2)$. One can define many functions over the quaternions as one would with matrices, via power series; the existence of a multiplicative norm means convergence is defined as usual. As a final remark, the quaternions orthogonal to the identity are all purely imaginary, that is to say $q^* = -q$. There is no a priori distinguished basis in the quaternions, any imaginary quaternion q will by itself generate a copy of $\mathbb{C}$, and adjoining any quaternion p with an imaginary component orthogonal to q will generate the entirety of $\mathbb{H}$ algebraically [CS03].

In the rest of the text we will quite often use the left and right quaternion multiplication operators, acting on $\mathbb{R}^4$, which is isomorphic to $\mathbb{H}$ as a vector space. These operators will be called E_k^+ for left multiplication and E_k^- for right, following [DL18a] and [DL18a]. In fact, we almost exclusively use the slight alteration $J_k^\pm := \pm E_k^\pm$, so for convenience and reference let us print these:

[10] We instead opt to (sparingly) use the Dutch digraph ij; the combination of the letters i and j occurs often enough in that language for this combination to be regarded as one symbol.

(2.34)

$$J_1^+ = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad J_2^+ = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad J_3^+ = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

(2.35)

$$J_1^- = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad J_2^- = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad J_3^- = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

These operators can also be interpreted as twists of group actions via $(J_k^+)^\mu_{\ \nu} = F(k,\nu)\pi(e_k)^\mu_{\ \nu}$, where π is the left regular representation of $\mathbb{Z}_2^2$. Let us remark a couple more things about these sets of operators. Firstly, one can notice that each set forms a linearisation of the action of $\mathbb{H}$ on the sphere at the identity, and hence a basis for the Lie algebra $\mathfrak{su}(2)$. Secondly, from the associativity of $\mathbb{H}$ we get that the left and right multiplications commute, so any J^+ commutes with all J^- and vice versa. Furthermore, associativity gives:

(2.36)

$$J_k^+ J_l^+ e_m = F(k, l\oplus m)F(l,m)e_{k\oplus l\oplus m} = F(k\oplus l, m)F(k,l)e_{k\oplus l\oplus m} = F(k,l)J_{k\oplus l}^+ e_m$$

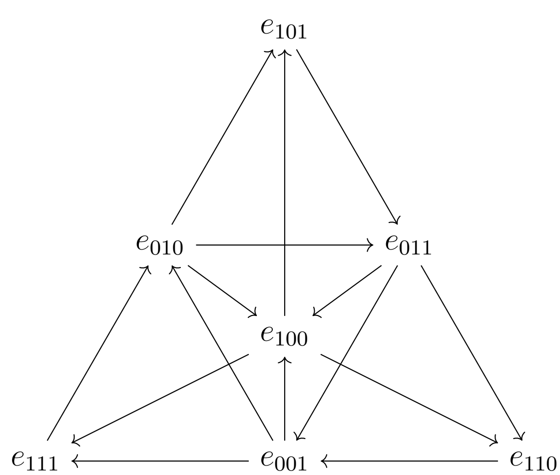
and similarly for the J^-.

2.5.5 Octonions

The Octonions are an 8-dimensional algebra generated by 3 algebraic generators, $\mathrm{i}, \mathrm{j}, \mathrm{k}$. As this algebra is no longer associative, while it is often convenient to consider the other vector space basis elements to be $\mathrm{ij}, \mathrm{ik}, \mathrm{jk}, \mathrm{i(jk)}$ we will prefer to just refer to them by the basis vectors e_k. Unlike the quaternions, whose multiplication table differs by at most a transpose, the octonions have 480 choices of F that yield isomorphic algebras. As a result there are different choices in the literature, in fact, many authors like the authoritative [Bae02] choose to work with a different basis in which the numbering is different. We choose to work with the numbering that is derived from $\mathbb{R}\mathbb{Z}_2^3$ as before. Our choice of multiplication table is the following, derived from the Cayley-Dickson procedure:

F	e_{000}	e_{001}	e_{010}	e_{011}	e_{100}	e_{101}	e_{110}	e_{111}
e_{000}	1	1	1	1	1	1	1	1
e_{001}	1	-1	1	-1	1	-1	-1	1
e_{010}	1	-1	-1	1	1	1	-1	-1
e_{011}	1	1	-1	-1	1	-1	1	-1
e_{100}	1	-1	-1	-1	-1	1	1	1
e_{101}	1	1	-1	1	-1	-1	-1	1
e_{110}	1	1	1	-1	-1	1	-1	-1
e_{111}	1	-1	1	1	-1	-1	1	-1

There is a mnemnonic for this table, given by the following diagram of the Fano plane:



This is to be interpreted as $F(a, b) = 1$ if the arrow goes from a to b. This encoding is a result of the interpretation of the Fano plane as nonidentity elements of $\mathbb{Z}_2^3$. Indeed, there is a variant of the same idea for the sedenions in [SHP15].[11]

The octonion twist and hence the octonions obey a few properties. Most importantly, it is the largest normed division algebra. It has a star structure where $e_k^* = -e_k$ for $k > 0$, and a multiplicative norm given algebraically by $||a|| = \sqrt{a^* a}$. The unit octonions form

[11]The nice cyclic structure of the arrows is equivalent to a structure of a cyclic triple system, which is equivalent to the algebra being a semisymmetric quasigroup. There is a rather sizeable amount of combinatorics done with this diagram which we shall not delve in to.

the 7-sphere $\mathbb{S}^7$, and the multiplicativity of the norm gives the structure of a Moufang algebraic quasigroup [KM10]. One important property which we cite from [AM99] is the fact that the multiplicative associator $\phi(x, y, z)$ is 1 if and only if x, y, z are linearly dependant over $\mathbb{Z}^2$, and -1 otherwise. This implies alternativity, which is the property that ϕ is symmetric in its arguments. It is related to the property of the octonions that they are, like the quaternions, homogeneous: any two imaginary orthogonal unit octonions generate a quaternion algebra, and any imaginary unit octonion orthogonal to that copy of the quaternions can be adjoined to generate the full octonion algebra.

The homogeneity of the octonions is reflected in its symmetry algebra G_2. This is a 14-dimensional Lie group that acts on the set of generating orthogonal unit triples, sending any such triple to any other such triple. A simple count: given any triple, the first coordinate has S^6 many options, the second is restrained to S^5 and the last one to S^3, yielding 14 dimensions in total.

We can use this property of ϕ also for a quick lemma about F we will use several times that holds for all real division algebras:

Lemma 2.5.1. *For F the octonion twist and $i, k \in \mathbb{Z}_2^3, k \neq 0$ we have $F(i, k)F(i \oplus k, k) = -1$.*

Proof. The proof is a simple one-line statement:

$$F(i, k)F(i \oplus k, k) = \phi(i, k, k)F(i, 0)F(k, k) = -1$$

□

For the computations later we will introduce operators $J^{\pm}_{k,\mathbb{O}}$ that are similar to the operators we defined on the quaternions, again given by the formulas $(J^+_k)^{\mu}_{\ \nu} = F(k, \nu)\pi(e_k)^{\mu}_{\ \nu}$ and $(J^-_k)^{\mu}_{\ \nu} = -F(\nu.k)\pi(e_k)^{\mu}_{\ \nu}$ for π the left regular representation of $\mathbb{Z}_2^3$. Since the octonions are not commutative, we notice that J^+ no longer commutes with J^-, and that there is in general no relation between the J^+_k for different k, unlike the multiplicative relationship for the quaternions. The J^--matrices used will be printed in the appendix for reference.

2.5.6 Beyond

Unfortunately, many things get lost after the doubling the octonions. The most important new failure in the sedenions is the fact that one gets zero divisors. This means that the algebra can no longer be multiplicatively normed, as the norm of a zero-divisor would have to be zero. Furthermore, a lesser-known problem is that the homogeneity we mentioned at the quaternions and octonions gets broken: the presence of zero divisors means that by rescaling there are such pairs of zero divisors on the unit ball. It is easily spotted that the elements of these pairs are orthogonal to one another, and they do not generate a quaternion algebra for the obvious reason that their product is zero. Similarly, not all elements of $\mathcal{S}$ are part of zero-divisor pairs. This all means that the automorphism group of the sedenions is nontrivial[12], and we can no longer rely on simplistic arguments to reduce a generic element to a special case. Lastly, we mentioned before that the octonions have the property of *alternativity*, implied by the special property of the associator to be -1 if and only if the entries are linearly independent. Both properties are lost here, leaving us with very little to hold on to.

[12]There is a series of idiosyncratic, unpublished arXiv preprints by R.P.C. de Marrais claiming some results in this direction.

Chapter 3

Noncommutative Products of Euclidean Spaces

3.1 Introduction

In this section we revise some of the results from the papers [DL18a] and [DL18b], which were the direct precursors to much of this thesis. In there, the authors define a class of noncommutative spaces, the noncommutative Euclidean spaces, via the very hands-on method of taking the algebras corresponding to two spaces and declaring the mixed products to be noncommutative. This has the very clear benefit of having the noncommutativity in some sense concentrated on the fault line between the component spaces. In this section we will do two things. To begin, we give a reinterpretation of these algebras as examples of categorical entities we call categorified algebras and show some ways to manipulate this class of objects. Afterwards, we return to the specifics and show some efforts in trying to understand the original construction in terms of twist deformations, and in finding nontrivial twists.

First, to recap the previous results. Dubois-Violette and Landi define noncommutative products of Euclidean spaces as the spaces dual to the following algebra:

$$\mathcal{A}(\mathbb{R}^{N_1} \times_R \mathbb{R}^{N_2}) := \mathbb{R}\langle x^\mu, y^\alpha \rangle \big/ I$$

$$I = \langle x^\mu x^\nu - x^\nu x^\mu, y^\alpha y^\beta - y^\beta y^\alpha, x^\lambda y^\alpha - R^{\lambda\alpha}_{\beta\mu} y^\beta x^\mu, y^\alpha x^\lambda - (R^{\lambda\alpha}_{\beta\mu})^{-1} x^\mu y^\beta \rangle \tag{3.1}$$

For some matrix R governing the commutativity. In their papers, the authors establish many properties of these algebras, such as the fact that the matrix R obeys the Yang Baxter-equations if it is of the following form [DL18b, Prop. 5.3]:

Proposition 3.1.1. *Let A and C be the two commuting real $N_1 \times N_1$ matrices with A symmetric and C antisymmetric and let B and D be two commuting real $N_2 \times N_2$ matrices with B symmetric and D antisymmetric:*

$$[A, C] = 0 = [B, D], \tag{3.2}$$

$$^tA = A = \overline{A}, \qquad ^tB = B = \overline{B}, \qquad -\,^tC = C = \overline{C}, \qquad -\,^tD = D = \overline{D}, \tag{3.3}$$

Assume in addition that

$$A^2 \otimes B^2 + C^2 \otimes D^2 = \mathbb{I}_{N_1} \otimes \mathbb{I}_{N_2}. \tag{3.4}$$

Then the matrix $R^{\lambda\alpha}_{\beta\mu}$ of the form

$$R^{\lambda\alpha}_{\beta\mu} = A^\lambda_\mu\, B^\alpha_\beta + \mathrm{i}\, C^\lambda_\mu\, D^\alpha_\beta \tag{3.5}$$

Satisfies the Yang Baxter equations, and products in the algebra preserve the central element $\sum_{a,b} \|x_a\|^2 + \|y_b\|^2$.

The presence of the Yang-Baxter equation in this context, as well as the focus on braidings is very reminiscent of the cocycle twist of a module as defined in theorem 2.2.13. This is part of the motivation of this section: to understand the connection between the two languages.

3.2 Categorified algebras

In this section we will look at a particular way to look at these algebras that is specifically fit for these products of the algebras corresponding to Euclidean spaces. We will *categorify* the polynomial algebra to a monoidal category. There are two reasons for this approach which we outline below.

Firstly, if we believe that the above algebras are created by a twist deformation, then there ought to be a Hopf algebra behind it all. It is not difficult to find a Hopf algebra acting on the above by ansatz. However, this construction shows it to be in some sense universal. The R-matrix above is acting on a finite-dimensional space, so it is natural to think this is some representation of a universal $\mathcal{R}$-element in the Hopf algebra. One intuition for what we do is to take this single representation, generate a monoidal category almost as freely as possible, and use the powerful Tannaka-Krein reconstruction theorem to recover all the data from the single representation. This (re)construction is something we will demonstrate in detail as we believe there are very few explicit examples of a reconstruction of an algebra that is not clearly already an algebra of $\mathcal{H}$-modules in the literature[1].

The second reason for the categorification is one based in intuition: the idea of this approach is to split up the noncommutative algebra in a commutative main part, together with a noncommutative "internal" part. The theory of noncommutative products of Euclidean spaces is a perfect example: the two algebras commute up to an internal symmetry in each of the individual algebras. Inspired by the theory of algebraic stacks and representations of finite groups, we will relegate this internal part to data inside the target of a fibre functor to separate the internal twist from the external algebra.

3.2.1 Defining the category

Let us begin by describing this category which we simply call $\mathcal{C}$ in terms of objects and morphisms. Let $\mathcal{B}$ be an algebra of the form $\mathcal{A}(\mathbb{R}^n \times_R \mathbb{R}^m)$. To construct a monoidal category corresponding to $\mathcal{B}$, we begin with the objects. Consider the set of objects of $\mathcal{C}(\mathcal{A})$ to be the monomials X^aY^b. We will later equip this with a fibre functor to Vect, so the objects themselves can be thought of as vector spaces.

The difficult part is defining the morphisms. We want to define them keeping the Tannaka reconstruction in mind. For that reconstruction, one finds all the natural transformations of the fibre functor to itself. In concrete terms that comes down to all linear transformations that commute with the morphisms that we choose to put in. This is a delicate balance: In [MRV18] is described a version of this procedure to generate a universal Hopf algebra corresponding to a representation. To this end he defines a version of a *free* monoidal category, only putting in the morphisms strictly necessary to satisfy the axioms. This Hopf algebra is incredibly large and difficult to analyse, and for this reason we will require a few more morphisms.

Let us however first write down the morphisms necessary for the braided monoidal

[1]The book [MRV18] is the exception, though we will end up with quite different algebras.

structure. Our category will have the monoidal product:

$$\begin{aligned} \otimes : \mathcal{C}(\mathcal{A}) \times \mathcal{C}(\mathcal{A}) \to \mathcal{C}(\mathcal{A}) \\ X^aY^b \otimes X^cY^d \to X^{a+c}Y^{b+d} \end{aligned} \tag{3.6}$$

As mentioned before: all the nonassociative data lies inside the vector space and is therefore invisible on the categorical level. We are going to assume that the category structure is strict, so we need no associator or unitor maps. The only morphisms necessary are the braidings:

$$\mathcal{R}_{X^aY^b,X^cY^d} : X^aY^b \otimes X^cY^d \to X^cY^d \otimes X^aY^b \tag{3.7}$$

that obey the hexagon equations, which in a strict context is equal to:

$$\mathcal{R}_{X^aY^b,X^eY^f} \circ \mathcal{R}_{X^aY^b,X^cY^d} = \mathcal{R}_{X^aY^b,X^cY^d \otimes X^eY^f} \tag{3.8}$$

Together with the mirror image:

$$\mathcal{R}_{X^aY^b,X^eY^f} \circ \mathcal{R}_{X^cY^d,X^eY^f} = \mathcal{R}_{X^aY^b \otimes X^cY^d,X^eY^f} \tag{3.9}$$

By this equation and the lack of associators, the only data that is necessary is the braiding on X^2, Y^2 and XY.

We diverge from [MRV18] by requiring inclusion morphisms $i_{X,X^aY^b} : X \to X \otimes X^{a-1}Y^b$ and $i_{Y,X^aY^b} : Y \to X^aY^b \otimes Y$ for Y, given by the composition of the inclusion of $\mathcal{C}$ into $\mathcal{C} \times \mathcal{C}$ composed with the tensor product back to $\mathcal{C}$. The reason for this extra requirement is that this forces a coordinate change on X or Y to automatically affect any product of objects X and Y via a pushforward. It is this addition over Manin's approach that keeps the Hopf algebra one reconstructs small, as all the endomorphisms that will constitute the algebra have to arise from endomorphisms of the two simple objects.

We do want to check all the morphisms that this additional criterion generates to ensure consistency. The monoidal composition rule implies that:

$$
\begin{array}{ccccc}
X & & X & & X\otimes X \\
\Big\downarrow & \otimes & \Big\downarrow & \Longrightarrow & \Big\downarrow \\
X\otimes P & & X\otimes Q & & X\otimes P\otimes X\otimes Q
\end{array}
$$

For every $P, Q \in \mathcal{C}(\mathcal{A})$. By similar diagrams, using the braiding to reshuffle and induction, we get a morphism between every $X^\alpha Y^\beta$ and $X^a Y^b$ if $\alpha < a, \beta < b$. Now we require the following square to commute:

$$
\begin{array}{ccc}
X^\alpha Y^\beta \otimes X^\gamma Y^\delta & \xrightarrow{\mathcal{R}} & X^\gamma Y^\delta \otimes X^\alpha Y^\beta \\
\Big\downarrow i & & \Big\downarrow i \\
X^a Y^b \otimes X^c Y^d & \xrightarrow{\mathcal{R}} & X^c Y^d \otimes X^a Y^b
\end{array}
$$

For $\alpha < a, \beta < b, \gamma < c$ and $\delta < d$. This abstract equality is becoming hard to visualise, so let us pass to the concrete by defining the fibre functor. The fibre functor $F : \mathcal{C}(\mathcal{A}) \to \mathrm{Vect}$ sends X to $\mathbb{R}^n$, Y to $\mathbb{R}^m$, such that $F(P\otimes Q) = F(P)\otimes F(Q)$ for the standard vector space tensor product. This map sends $\mathcal{R}_{X,Y}$ to R, the concrete linear map, and the inclusion functors into the standard inclusions of vector spaces. For this functor to be a fibre functor, we need exactness and faithfulness. The latter just requires that morphisms that are different in $\mathcal{C}(\mathcal{A})$ map to different linear maps, which is a sensible requirement. Exactness requires us to talk about the abelian structure of $\mathcal{C}(\mathcal{A})$, which we have not defined yet. This requirement is stated in the book [Eti+16], however, in [Maj95] for example the abelian structure is not used for the reconstruction theorem. It is a possible further direction to see if the structure of monoidal category can be upgraded to the structure of a tensor category.

Having defined the category, there is a natural question if this formulation is equivalent. The answer is yes in this case, as can easily be seen by finding the generating objects X and Y, passing to the vector spaces $F(X)$ and $F(Y)$, and letting these vectors generate the algebra via products generated by the monoidal product. Another way of looking at it is by considering the direct sum of all the vector spaces $F(P)$, quotiented by all the relations obtained from the embeddings.

3.2.2 Tannaka-Krein

After all this setup, it is time to reconstruct the Hopf algebra of symmetries of $\mathcal{A}$. For now we skip the antipode, so we will keep it to a bialgebra. We will follow the guidance of [Maj95, ch.9].

First of all, to reconstruct the algebra itself, which is given by Nat(F, F). An element ν in Nat(F, F) is given by a matrix $\nu_{F(P)}$ on each object $F(P)$, for $P \in \mathcal{C}(\mathcal{A})$. These matrices need to obey:

$$\begin{array}{ccc} F(X) & \xrightarrow{i} & F(P) \\ \downarrow{\scriptstyle \nu_{F(X)}} & & \downarrow{\scriptstyle \nu_{F(P)}} \\ F(X) & \xrightarrow{F(i)} & F(P) \end{array}$$

Any object can be decomposed as follows:

$$\begin{aligned} F(X^nY^m) &= F(\underbrace{X\otimes\cdots\otimes X}_{n \text{ times}}\otimes\underbrace{Y\otimes\cdots\otimes Y}_{m \text{ times}}) \\ &= \underbrace{F(X)\otimes\cdots\otimes F(X)}_{n \text{ times}}\otimes\underbrace{F(Y)\otimes\cdots\otimes F(Y)}_{m \text{ times}} \end{aligned} \tag{3.10}$$

Hence any $\nu_{F(P)}$ is completely determined by ν_X and ν_Y. Note that the composition is well-defined: By diagrams like this we can see that $\nu \circ \nu'$ is still determined by $\nu_X\nu'_X$ and $\nu_Y\nu'_Y$, even though these matrices might not commute, as they are forced to act on different tensor factors and hence commute. There is one other naturality square:

$$\begin{array}{ccc} F(X\otimes Y) & \xrightarrow{\mathcal{R}} & F(Y\otimes X) \\ \downarrow{\scriptstyle \nu_{F(XY)}} & & \downarrow{\scriptstyle \nu_{F(XY)}} \\ F(X\otimes Y) & \xrightarrow{\mathcal{R}} & F(Y\otimes X) \end{array}$$

The meaning of this diagram becomes clearer if we put it into matrices. The tensor product maps $X\otimes Y \to XY$ and $Y\otimes X \to XY$ are linked by some linear map $R := F(\mathcal{R})$ on $F(XY)$. On elements:

$$x^i\otimes y^j = (xy)^{ij}, \quad y^j\otimes x^i = R^{ij}_{ab}(xy)^{ab} \tag{3.11}$$

This naturality square then says that for $A = \nu_X$ and $B = \nu_Y$, that:

$$(A\otimes B)^{\alpha\beta}_{ij} R^{ij}_{ab} (xy)^{ab} = R^{\alpha\beta}_{ij} (A\otimes B)^{ji}_{ab} (xy)^{ab} \tag{3.12}$$

Note that there is a transpose in the second $A\otimes B$. The algebra of functions $\mathcal{H} := \mathrm{Nat}(F, F)$ acting on the algebra $\mathcal{A}$ is isomorphic to the set of all the invertible matrices of the form $A\otimes B$ such that $A\otimes B$ commutes with R in this way. We will try to compute these algebras concretely in some special cases in the next chapter.

One may notice that these matrices are not a linear space. Indeed, the linear structure on this group is trivial, one can see it as a group algebra of finite linear combinations of group elements. This will be confirmed once we see the coalgebra structure.

The coproduct structure on $\mathcal{H}$ can be defined following [Maj95] by:

$$\Delta(h_{P,Q}) = h_{P\otimes Q} \tag{3.13}$$

For a natural transformation $h \in \mathcal{H}$. This equation requires some interpreting, as we like to think of $\mathcal{H}$ as an algebra without keeping track of all the objects it acts on. Luckily, the interpretation is quite clear: the coproduct $\Delta(h)$ shows how h would act on a tensor product of representations. In our case, given a matrix A acting on X, we know exactly how it would act on X^2, namely via $A\otimes A$. This means that every matrix A is grouplike. As this is true regardless of any of the scalar norms of the matrix, we find that the $\mathbb{K}$-linear structure of the Hopf algebra is different from the naive addition of matrices. We can see this happening as well with the counit, for which we can verify that the following square must commute:

$$\begin{array}{ccc} F(\mathbb{1})\otimes F(X) & \xrightarrow{\cong} & F(X) \\ \big\downarrow{\scriptstyle \nu_{\mathbb{1}}\otimes\nu_X} & & \big\downarrow{\scriptstyle \nu_X} \\ F(\mathbb{1})\otimes F(x) & \xrightarrow{\cong} & F(X) \end{array}$$

This implies that for every element we must have $\nu_{\mathbb{1}} = 1$. For a Lie group, this means that the Hopf algebra is uncountably infinitely dimensional as a vector space. We might want to remedy this by taking a double dual via pairing, using the fact that an infinite-dimensional pairing does not need to preserve the cardinality of the basis to send the Lie group to its Lie algebra. Before we take this double dual, we finish the reconstruction by reconstructing the quasitriangular element $\mathcal{R}$ as simply noticing that $R_{X,Y}$ definitely commutes with itself and is a finite linear combination of matrices $A\otimes B$, hence it can be found in $\mathcal{H}$.

There are many ways to define a pairing between this algebra and some choice of dual. Since it is fundamentally generated by the grouplike elements, which are themselves classified by matrices A, B, we will consider the algebra $\mathcal{H}^\circ$ to consist of the restriction of $C^\infty(\mathrm{SL}(n) \times \mathrm{SL}(m))$, linearised as follows:

$$f\left(\sum_i A_i \otimes B_i\right) = \sum_i f(A_i \otimes B_i) \tag{3.14}$$

This algebra has a clear product, and a coproduct defined by:

$$\Delta(f)\left(\sum_{i,j}(A_i\otimes B_i)\otimes(A_j\otimes B_j)\right) = f\left(\sum_{i,j}(A_iA_j)\otimes(B_iB_j)\right) \tag{3.15}$$

Noticing that (i, j) run over finitely many indices, the latter is a finite linear combination of elements of $C^\infty(\mathrm{SL}(n) \times \mathrm{SL}(m))\otimes C^\infty(\mathrm{SL}(n) \times \mathrm{SL}(m))$ and so lies in $\mathcal{H}^\circ\otimes\mathcal{H}^\circ$. The fact that the product on the algebra is dual to the grouplike coproduct is easy to verify. As a result these two algebras are paired with a pairing $\langle\ ,\ \rangle : \mathcal{H}^\circ\otimes\mathcal{H} \to k$, such that:

$$\langle f, x\rangle = f(x) \tag{3.16}$$

The reason we chose smooth functions is to pair it a second time, this time with the enveloping algebra $\mathrm{UEA}(\mathfrak{sl}(n)\oplus\mathfrak{sl}(m)) = \mathfrak{g}$, to finally arrive at the formalism of universal enveloping algebras. This way, we can follow the pairings to truly derive the action and the universal quasitriangular element $\mathcal{R}$ in that algebra from the representation.

Define this second pairing by $\langle\ ,\ \rangle_{\mathfrak{g}} : \mathfrak{g}\otimes\mathcal{H}^\circ \to k$, with the pairing:

$$\langle\mathfrak{h}, f\rangle_{\mathfrak{g}} = \mathfrak{h}(f)|_{\mathbb{1}} \tag{3.17}$$

We need to verify that this is a pairing. The equations

$$\langle\Delta(\mathfrak{h}), f\otimes g\rangle_{\mathfrak{g}} = \mathfrak{h}(f)|_{\mathbb{1}}g|_{\mathbb{1}} + f|_{\mathbb{1}}\mathfrak{h}(g)|_{\mathbb{1}} = \mathfrak{h}(fg)|_{\mathbb{1}} = \langle\mathfrak{h}, fg\rangle_{\mathfrak{g}} \tag{3.18}$$

and

$$\langle \mathfrak{h}\otimes\mathfrak{k}, \Delta(f)\rangle_{\mathfrak{g}} = \lim_{\varepsilon,\delta\to 0} \frac{f(\mathbb{1}+\varepsilon\mathfrak{h}+\delta\mathfrak{g}) - f(\mathbb{1})}{\varepsilon\delta} = \langle \mathfrak{h}\mathfrak{k}, f\rangle_{\mathfrak{g}} \tag{3.19}$$

together with the quick (co)unit check establish a pairing. We can take a quotient by any Lie algebra element that pairs to 0 with all functions to end up with the correct Lie algebra of the symmetry.

Given a pairing $\langle\ ,\ \rangle : \mathcal{H}^\circ\otimes\mathcal{H} \to k$ and a quasitriangular element $\mathcal{R} \in \mathcal{H}\otimes\mathcal{H}$ a coquasitriangular structure (as defined in chapter 2.2.5) $\Psi : \mathcal{H}^\circ\otimes\mathcal{H}^\circ \to k$ is given by:

$$\Psi(f,g) = \langle f\otimes g, \mathcal{R}\rangle \tag{3.20}$$

The dual is slightly harder to define, as the pairing does not necessarily define such an element, and the element need not be unique. Indeed, we will see later that the most common twists we consider are of the form $\exp(\hbar\mathfrak{h}\otimes\mathfrak{k})$, which do not actually lie in the universal enveloping algebra. The natural thing to consider here is $\langle\mathcal{R}_{\mathfrak{g}}, -\rangle_{\mathfrak{g}} = \Psi$, i.e. $\mathcal{R}_{\mathfrak{g}}(f\otimes g)|_{\mathbb{1}\otimes\mathbb{1}} = (f\otimes g)|_{\mathcal{R}}$ for all f, g. Finally to put it all together: Let $R = A_R\otimes B_R + \mathrm{i}C_R\otimes D_R$, we lift it to a $\mathcal{R}$ that is given by the natural transformation generated by the same object as an algebra element, and we find the "differential operator" that on every smooth function implements the evaluation on this algebra element. That last part can be reformulated as finding the infinitesimal differential operator that translates the function by the group element. The way to implement a translation by only using infinitesimal Lie algebra elements is exactly the exponential map, so the cocycle twist on the reconstructed Lie algebra will be given by an exponential of Lie algebra elements. We will look into this in the more concrete setting later.

3.2.3 Tori

Having defined these structures on products of Euclidean algebras, we will have a quick detour to show how this formalism deals with quotients. It turns out products of spheres have a very natural embedding, which we will demonstrate as follows.

So far, the object we chose have all been formally real, as the algebra generators are presumed Hermitean, and we have not dealt with the reconstruction of the antipode of the Hopf algebra. The antipode is related to the existence of dual objects in the category of representations of a Hopf algebra, and given that we work in Euclidean spaces we have access to an inner product via the norm. This intuitively means that we should be able to define the dual structure on some elements via looking at the adjoint maps of the inner product. The dual of some object X, X^* is then entirely defined by the evaluation map $\mathrm{ev}_X : X^* \otimes X \to \mathbb{1}$. Supposing we have a Euclidean norm, it might be interesting to check also the dual map, the coevaluation $\mathrm{coev}_X : \mathbb{1} \to X \otimes X^*$. Linearising, the coevaluation map sends a number to the tensor product of all elements that inner product to that number. Choosing a basis and a natural dual basis, this means

$$\mathrm{coev}_X(c) = c \sum_{k \in I} (e^k \otimes e_k) \tag{3.21}$$

If we choose X to be self-dual, this means that the coevaluation of 1 is equal to the sum of squares of the basis elements of X, which we used to generate the algebra. If we include this morphism in our construction of the category, we find for example that the reconstructed quasitriangular element $\mathcal{R}$ has to respect this bilinear form. Furthermore, we can localise at this element by declaring this morphism to be invertible, transporting us to the realm of spheres. In this formalism, we keep carrying the data of an isomorphism around, rather than viewing the objects as truly equal.

There are two interesting observations to make. Firstly, there is a slight extension one could easily consider: taking any other inner product passes from a noncommutative product of Euclidean spaces to a noncommutative product of Clifford algebras. This is still work in progress, but as is shown in [AM02] many of the constructions we will do later by considering the base algebra to be a copy of the octonions may have equivalents analysable by similar methods by considering the base algebra to be a Clifford algebra with the similar quasialgebra structure.

The second observation is that while it is no effort to take quotients internal to the fiber, taking quotients across objects is more tricky. Similar constructions have been studied before, as in [Day73], but we have yet to apply it to this setting.

3.3 Twist deformations of Euclidean algebras

In this section we take a quick look at the concrete way of framing the R-matrix as a Hopf algebraic twist. There has been quite some recent work in this direction from the viewpoint of deformation quantisation and star products, see for example the recent thesis [Web20].

We will be very brief on this subject, as we found few good results in this direction.

The approach is as follows. We consider a Hopf algebra of universal enveloping algebra type $\mathcal{H} = \mathrm{UEA}(\mathfrak{g})$, acting on our polynomial algebra $\mathcal{A} = \mathcal{A}(\mathbb{R}^n \times \mathbb{R}^m)$ corresponding to a product of Euclidean spaces. If $\mathfrak{g}$ acts by derivations, this is easily seen to be a module algebra, so we can deform the algebra $\mathcal{A}$ by considering a twist on $\mathcal{H}$. Let us begin there.

The twist condition is given by equation 2.19, to wit:

$$(\mathbb{1}\otimes\chi)(\mathrm{id}\otimes\Delta)\chi = (\chi\otimes\mathbb{1})(\Delta\otimes\mathrm{id})\chi \tag{3.22}$$

Finding nontrivial twists is a difficult task, even if you know they exist, see for example [DNS96]. We will therefore first begin with two examples that are almost trivial.

The first example is that of a coboundary, which as we mentioned is given by $\chi = \Delta\sigma(\sigma\otimes\sigma)^{-1}$ for some invertible $\sigma \in \mathcal{H}$. It is a simple check that this obeys the twist condition by virtue of (co)associativity.

The second example is much more central to the thesis, which is given by $\exp(\hbar\mathfrak{h}\otimes\mathfrak{k})$, for two primitive commuting elements $\mathfrak{h}$ and $\mathfrak{k}$ of $\mathcal{H}$. One might object to this twist, as a twist is supposed to be an element of $\mathcal{H}\otimes\mathcal{H}$, yet there is no way to implement the exponential map in the polynomially generated universal enveloping algebra. It is a quick check to show that if we do not go beyond this barrier, there are in fact no nontrivial twists. Let us take this quick diversion.

By the Poincaré-Birkhoff-Witt theorem there is a grading on elements of the universal enveloping algebra. Consider a generating basis $\mathfrak{h}_j$ of this algebra. The coproduct is an algebra morphism, and on a primitive element in this basis it acts as $\Delta(\mathfrak{h}_j) = 1\otimes\mathfrak{h}_j + \mathfrak{h}_j\otimes 1$, which means that any monomial $\prod_{j_k\in I}\mathfrak{h}_{j_k}$ has as a coproduct $\prod_{j_k\in I}(1\otimes\mathfrak{h}_{j_k} + \mathfrak{h}_{j_k}\otimes 1)$. Expanding this product, we see that in terms of grading this coproduct is given by a sum of terms in all the degrees between $|I|\otimes 0$ and $0\otimes|I|$, for I the index set of the monomial, with as total degree the constant $|I|$. Looking at the twist condition from the lens of this grading, we can consider the components of χ of maximal total degree N. Then the two sides of the equation of the twist condition will have terms of maximal total degree $2N$ as the product in the universal enveloping algebra is filtered, where the terms of this maximal total degree are subdivided as follows:

$$\begin{aligned} \mathrm{Deg}((\mathbb{1}\otimes\chi)(\mathrm{id}\otimes\Delta)\chi) &= (N_1, M_1 + N_{2_1}, M_1 + N_{2_1}) \\ \mathrm{Deg}((\chi\otimes\mathbb{1})(\Delta\otimes\mathrm{id})\chi) &= (M_1 + N_{1_1}, M_2 + N_{1_2}, N_2) \end{aligned} \tag{3.23}$$

for some $N_1 + N_2 = M_1 + M_2 = N$ and all N_{1_1}, N_{1_2} that sum to N_1. Consider the monomial in χ with maximal N_1. Then as $N_{1_1} \otimes N_{1_2}$ has a component that is identical to $N_1 \otimes 1$, this means that there is a nontrivial element on the right hand side of degree $(M_1 + N_1, M_2, N_2)$, whereas this is not possible to reach on the left-hand side. Therefore, no such twist can exist[2].

One solution to this conundrum is by considering the twist to lie in some wider algebra $\mathcal{H}[[\hbar]]$. This comes with another caveat, however, which is that it is too easy to find twists here, as there are no restrictions on convergence. Indeed, while in the old algebra there are no nontrivial invertible elements, in this algebra any element with nonzero constant part is invertible, as one can Taylor-expand $\dfrac{1}{c_0 + \hbar\alpha_{>0}}$ for any $\alpha \in \mathcal{H}$. Indeed, one can imagine solving the twist condition with a very cumbersome interminable process of adding infinite higher-degree terms to cancel any anomalies. Ultimately, we are interested in the deformation down in the representation. This can push us down the path of requiring the matrix norm of the twist to exist when in this representation, which is a more analytical direction. We decided to take the simpler way first, and consider twists of *exponential type*, that is, $\chi = \exp(\sum \alpha \otimes \beta)$, for $\alpha, \beta \in \mathcal{H}$, as motivated by the earlier results in the reconstruction of the Hopf algebra from the monoidal category.

Plugging this ansatz into the twist condition, we find:

$$\exp(\mathbb{1} \otimes \alpha \otimes \beta) \exp(\alpha \otimes \Delta(\beta)) = \exp(\alpha \otimes \beta \otimes \mathbb{1}) \exp(\Delta\alpha \otimes \beta) \tag{3.24}$$

where we once again used that Δ is an algebra map to take it inside the exponential. Suppose that α, β and both components of $\Delta(\alpha)$ and $\Delta(\beta)$ all commuted among each other. In that case we can merge the exponentials without incurring any Baker-Campbell-Hausdorff type terms, and end up with the equation:

$$\mathbb{1} \otimes \alpha \otimes \beta + \alpha \otimes \Delta(\beta) = \alpha \otimes \beta \otimes \mathbb{1} + \Delta\alpha \otimes \beta \tag{3.25}$$

There are two things to notice: First of all, this is linear, which greatly simplifies things. Secondly, by another degree argument, we can see that there are no nontrivial solutions with bounded degree > 1. For example if $|\alpha| = A > 1$ and $|\beta| = B > 1$ and the left hand side will contain a component in degree $(A, 1, B - 1)$, which is impossible to create on

[2]This argument is possibly more convoluted than an argument that there are no invertible elements whatsoever, which is another requirement for the twist to exist. As we have other degree arguments later we use this as a warmup.

the right hand side by the same monomial. By linearity, without loss of generality there is only one monomial in each degree. Therefore any such term will need to be cancelled off by a second term inside of the cocycle with the same total degree. The only way one could end up with another element in $(A, 1, B)$ is by having α' of degree $A+1$ and β' of degree $B-1$. This argument can then be repeated to see that all the cancellations together imply that every element in the cocycle with total degree $A+B$ can have its tensor factors multiplied together to exactly a constant element. Since this one element determines the entire twist in this degree, and we know that there exists a coboundary determined by any γ, we can identify them as the same. This argument does not hold in degree 1, as there the degrees in the equation are $(1,1,0), (1,0,1), (0,1,1)$ on both sides, and therefore they cancel perfectly fine and do in fact do so for every primitive commuting α and β.

This means that the only way to have a nontrivial twist is to have either α not commute with β, or to have $\alpha_{(1)}$ not commute with α, or both. The case of $\alpha_{(1)}$ not commuting with α is more common than it seems at first hand, as the simple example of $\Delta(\mathfrak{h}_1\mathfrak{h}_2) = 1\otimes\mathfrak{h}_1\mathfrak{h}_2 + \mathfrak{h}_1\otimes\mathfrak{h}_2 + \mathfrak{h}_2\otimes\mathfrak{h}_1 + \mathfrak{h}_1\mathfrak{h}_2\otimes 1$ demonstrates[3]. There is no a priori reason for $\mathfrak{h}_1\otimes\mathfrak{h}_2$ to commute with $\mathfrak{h}_1\mathfrak{h}_2\otimes 1$. We spent some time investigating this class of degree-type arguments, complicated by the existence of Baker-Campbell-Hausdorff-type extra terms. If α and β do not commute but both do commute with their own coproducts, then the BCH terms in the exponential are constrained to only the middle tensor factor, which gives some hope of an argument of this type working. If α does not commute with its own coproduct, all bets are off as there are no degrees necessarily preserved. We have not found any bounded twists of the above type; the fact that the extra degrees of freedom are constrained to the middle tensor factor in the first case of α and β commuting with their coproducts leads us to conjecture that this setting is too constrained for nontrivial twists to exist, whereas we do not feel so confident in the other setting.

We end up equipped only with twists of the form $\exp(\mathrm{i}\hbar\mathfrak{h}\otimes\mathfrak{k})$. In the setting we are interested in, there is a very fruitful way to guarantee that $\mathfrak{h}$ and $\mathfrak{k}$ commute: consider a Lie algebra $\mathfrak{g}$ that decomposes as $\mathfrak{g}_1\oplus\mathfrak{g}_2$, each half acting on a different tensor factor. Picking $\mathfrak{h}\in\mathfrak{g}_1$ and $\mathfrak{k}\in\mathfrak{g}_2$, $\mathfrak{h}$ and $\mathfrak{k}$ are sure to commute. The factor of i in the exponent is necessary for the unitarity, to work with the $*$-structure. Given such a twist, as mentioned in section

[3]Interestingly, this particular type of noncommutation is also the reason why this cocycle condition for Hopf algebras in degrees higher than 2 does not necessarily annihilate coboundaries.

2.2.6, the universal quasitriangular element $\mathcal{R}$ in the twisted Hopf algebra is given by:

$$\mathcal{R}_\chi = \tau(\chi)\chi^{-1} = \exp(\mathrm{i}\hbar(\mathfrak{h}\otimes\mathfrak{k} - \mathfrak{k}\otimes\mathfrak{h})) \tag{3.26}$$

Acting with this element on a module algebra is completely determined by its action on basis elements, which by invertibility is necessarily of a form given by an invertible matrix. Given any pair of Lie algebra elements, one can represent them by a matrix, take the matrix exponential, and end up with an R-matrix. Given an R-matrix, one needs to find the corresponding Lie algebra elements, which are likely not unique by virtue of the periodicity of the exponential map. Locally, one can take a matrix logarithm.

There are two last things to comment on: One is that in the next chapter we will be looking at a class of twist which would correspond to a case in which the two factors $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are almost but not entirely independent, as we will find more interesting examples when these two are in fact connected in some sense. The idea there is that we do not want to have complete freedom in being able to without loss of generality pick a pure tensor product of operators. The only way to describe this lack of freedom in this context is by considering the Hopf algebra itself to not entirely factorise in two parts, each acting on different algebras, as the two components would be independent. Instead, one needs to have a split action of a single component across the two factors. In this case, finding a nontrivial twist is profoundly more difficult, as for the case of $\mathfrak{sl}(2)$ already we do not have two independent commuting elements to create a twist of the form $\exp(\hbar\mathfrak{h}\otimes\mathfrak{k})$.

The second comment is that we looked for a while into what it would even mean to be a nontrivial twist, as there are a number of nontrivial twists known. For example, there exist twists like the Jordanian twist, and also there are unconstructed twists connecting the $\mathfrak{sl}_q(2)$ algebras, whose existence is proved nonconstructively. We did not find what this would look like in the representation, but it is hard to imagine it looking particularly different from a commutation given by an arbitrary matrix acting on the generators, as demonstrated by the reconstruction procedure. The construction of the noncommutative algebra as a twisted module algebra means that the twist has to act via the generators, which limits the options quite severely.

3.4 Further directions

The approach of categorifying a noncommutative algebra has a number of parallels. For this part, the lack of quotients in our algebra worked very nicely in the construction for the monoidal category, more care would need to be taken to extend it to a more general noncommutative variety. There are parallels we would like to draw to finite group theory: In a noncommutative group the category of representations has a reconstruction theorem using the same Tannaka-Krein theorem. There, there is a correspondence between the noncommutative algebra $\mathbb{C}G$ and the set of conjugacy classes in the group. One could think of each conjugacy class as a point with an internal structure, as the commutative limit is equal to requiring that the conjugacy classes are exactly normal points. Doing so we see that there is a correspondence between noncommutative algebras, points with an internal symmetry and categories of representations of these algebras. Unlike this intuition from finite groups however, in our work above we ran into infinities due to the fact that the algebra is not finite-dimensional. We would like to continue the investigation into different algebras interpreted as monoidal categories, extend the formalism to finite groups and other affine varieties than tori.

As for the explicit reconstruction of the cocycle, there are many interesting directions being researched at this moment. The most relevant open question to the rest of this paper that we already touched upon would be to understand better how to deform a diagonal action of a single algebra, to better mimic the case we will look at later where we do not want to be able to lose all information without loss of generality.

Chapter 4

Noncommutative Products of Normed Division Algebras and their Symmetries

4.1 Introduction

This chapter is based on [LW22], in preparation as of writing. In this chapter, we will look at a different direction to extend the work in [DL18a] and [DL18b]. This direction is the special case where the R-matrix and the Euclidean spaces it acts on are given by real normed division algebras $\{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$. The reason for this interest is twofold. Firstly, deforming the normed division algebras by means of an action generated by these algebras themselves will also deform the sphere of unit elements in these algebras. As a result, the deformation passes from a product of Euclidean flat spaces to the spheres, giving us a construction of noncommutative spheres. Furthermore, by passing to the unit sphere in both Euclidean algebras we end up getting a noncommutative product of spheres, or a generalised noncommutative torus.

Secondly, this case turned out to be of particular interest when in [DL18a] it was found that the set of parameters $\mathbf{u}$ for the deformation was shown to have the particularly nice structure of $\mathbb{P}^1(\mathbb{V}^{(n)})$ for the cases of $\mathbb{V}^{(n)} \in \{\mathbb{R}, \mathbb{C}\}$. Furthermore, in [DL18a] it was remarked that the quaternionic case admitted a particularly nice symmetry by exploiting the fact that right and left multiplication commute in this case. In what follows we investigate the extent to which both of these properties generalise to the further Cayley-Dickson algebras.

There turn out to be two approaches to this theory in higher dimensions. One is to keep things real and close to a Hopf-algebraic nature, which is useful when dealing with the matter abstractly. We will be only living in the representations, but sticking to the theme of the previous chapter many intuitions and notations do come from the Hopf algebraic world. The second way is to embrace the explicit nature and rewrite the equations in a form using complex variables that gives a greatly simplified explicit matrix for the commutation. This form is elegant but has some issues when trying to recover the abstract form. We will split this chapter principally in two parts corresponding to these two approaches, and secondarily into parts corresponding to the degree of the algebras. Before we split off, let us first take a look at the base cases of $\mathbb{R}$ and $\mathbb{C}$.

4.1.1 Products of $\mathbb{R}$

Noncommutative products of two copies of $\mathbb{R}$ are given by by what is called the *quantum plane*[Kas95][KS97][Maj95][1] and serve as a base case of our interests:

Definition 4.1.1. *The* quantum plane $\mathcal{A}(\mathbb{R}^2_q)$ *is given by the algebra freely generated by two hermitian coordinate functions* x, y *subject to the commutation relation:*

$$xy = q\, yx \tag{4.1}$$

for some nonzero $q \in \mathbb{R}$.

This quantum plane has been studied extensively, as in the above references. We will however look at some different aspects, inspired by the idea to extend to further normed division algebras. Without the requirement to preserve the norm, it is a simple exercise in linear algebra to show that no coordinate transformation can reduce the parameter space of q any more than an isomorphism between $\mathcal{A}(\mathbb{R}^2_q)$ and $\mathcal{A}(\mathbb{R}^2_{q^{-1}})$. We specify to a choice of q that preserves the norm, that is to say, that keeps the element $x^*x + y^*y = x^2 + y^2$ central[2] In order to accomplish this, we need that form to commute with x and y; taking a commutator with x reveals that in order to keep the norm central we in fact need $q^2 = 1$. Imposing this, the commutator with y is similarly satisfied. The parameter space of choices

[1] Be warned, different sources place the q on different sides of the equals sign. Both Kassel and Majid use the other convention, we stick to the one consistent with Dubois-Violette and Landi.

[2] This terminology is unfortunately a bit confusing, as the norm is an element of our algebra, not an operation on the elements. The elements x and y are classically coordinate functions, so the function $x^2 + y^2$ is in fact the Euclidean norm function on the underlying space.

of q that preserve the norm is $\mathbb{Z}_2$. The quotient space setting the central element $x^2 + y^2$ to 1 in the case of $q = +1$ is of course $\mathcal{A}(\mathbb{S}^1)$. For $q = -1$, one ends up with quite a different space: a noncommutative algebra $\mathcal{A}(\mathbb{S}^1_{-1})$, some sort of fermionic circle. As a circle is 1-dimensional, and a single generator is hard to make commute with itself, this object is a bit of an aberration, but as we have not seen any mention of it in the references we will make a quick few observations before moving on to the complex setting.

For both $\mathbb{S}^1$ and $\mathbb{S}^1_{-1}$ we have an exact sequence:

$$0 \to \mathbb{R}[x] \to \mathcal{A}(\mathbb{S}^1_*) \to \mathbb{R}\mathbb{Z}_2 \to 0 \tag{4.2}$$

This point of view is taking the algebra $\mathbb{R}[x, y] / \langle x^2 + y^2 - 1 \rangle$, reducing away all the factors of y^2 into polynomials in x, and interpreting the resulting algebra as $\mathbb{R}[x] \oplus y\mathbb{R}[x]$ with a product $y^2 \equiv 1 \mod \mathbb{R}[x]$. The difference of the two algebras is the action that $\mathbb{Z}_2$ has on $\mathbb{R}[x]$. For the classical circle the action is trivial, for the fermionic one this action is sending x^n to $(-1)^n x^n$.

This way of looking at the circle is not a very fruitful one classically: projecting out the y's is akin to looking at the circle as a $\mathbb{Z}_2$-bundle over a line with a singular action at the endpoints. The better way to look at the circle algebra is by factorising $x^2 + y^2$ into $z = x + iy$ and $w = x - iy$, having $zw = 1 \implies w = z^{-1}$, and noticing that the remaining algebra is $\mathbb{R}[x, x^{-1}]$. For the noncommutative algebra this substitution does not immediately yield recognisable results. We first resolve the algebra by considering it matrix-valued. Writing the generators as:

$$x = \begin{pmatrix} u & 0 \\ 0 & -u \end{pmatrix} \quad y = \begin{pmatrix} 0 & v \\ -v & 0 \end{pmatrix} \tag{4.3}$$

we notice that all the noncommutativity is captured by the matrix part, and the components themselves lie in commutative algebras. It is straightforward to check that the matrix splits as:

$$\begin{pmatrix} A & B \\ B^\alpha & A^\alpha \end{pmatrix} \tag{4.4}$$

where α denotes the action sending each u^n to $(-u)^n$, and A lies in the algebra defined by

$\mathbb{R}[u,v^2]/\langle u^2+v^2-1\rangle$, B lies in $v\mathbb{R}[u,v^2]/\langle u^2+v^2-1\rangle$. To wrap this cursory analysis up, the algebra $\mathbb{R}[u,v^2]/\langle u^2+v^2-1\rangle$ admits the coordinate substitution we used for the circle, being equivalent to $u = \cos\theta$ and $v = \sin\theta$. The absence of the odd terms in v is then equivalent to the algebra being the algebra of invariants under the action $\theta \to -\theta$. This is a GIT quotient, again sending the circle to a piece of line with two singular endpoints. One is not to expect a more circularly symmetric interpretation, given that the choice of x and y is integral to the definition of this space.

4.1.2 Products of $\mathbb{C}$

Passing to the complex case we end up with a one-parameter family $\mathbb{R}^4_u$, which was introduced in [CL01]. We will describe it first by passing to real generators, using an isomorphism $\mathbb{C}^2_u \simeq \mathbb{R}^4_u$. Let the latter be generated by Hermitian coordinate functions (x^0, x^1) and (y^0, y^1). We generate the algebra $\mathcal{A}(\mathbb{R}^4_u)$ by imposing the following algebra relations:

$$x^0x^1 = x^1x^0, \quad y^0y^1 = y^1y^0, \quad x^\lambda y^\alpha = R^{\lambda\alpha}_{\beta\mu}\, y^\beta x^\mu, \quad y^\alpha x^\lambda = (R^{\lambda\alpha}_{\beta\mu})^{-1} x^\mu y^\beta \tag{4.5}$$

where all indices lie in $\{1,2\}$, with matrix $R^{\lambda\mu}_{\nu\rho}$ given explicitly by

$$R^{\lambda\alpha}_{\beta\rho} = u^0\, \delta^\lambda_\rho \otimes \delta^\alpha_\beta + \mathrm{i}u^1\, J^\lambda_\rho \otimes J^\alpha_\beta \qquad \text{and} \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{4.6}$$

The tensor products are mostly for clarity, as one can see which components are acted on by the indices. We will need a couple of remarks. Firstly, let us clarify why we are interested in this particular form of the R-matrix. We take the symmetric part to be in some sense undeformed, and the antisymmetric part of R to be a tensor product of Lie algebra elements corresponding to the action of $\mathrm{U}(1) \subset \mathbb{C}^*$ on $\mathbb{C}$ by left multiplication, the corresponding real Lie algebra being proportional to an action by multiplication by i, which is implemented by the matrix J.

Secondly, we will take a look at the $*$-structure of R, as later we will see that this becomes less clear. The fundamental thing we would like is that our algebra itself has a good $*$-structure. This means that the following diagram has to commute:

$$\begin{array}{ccc} (x^\lambda y^\alpha)^* & \longrightarrow & (R^{\lambda\alpha}_{\beta\mu}\, y^\beta x^\mu)^* \\ \big\downarrow & & \big\downarrow \\ y^\alpha x^\lambda & \longrightarrow & \overline{R}^{\lambda\alpha}_{\beta\mu} x^\mu y^\beta \end{array}$$

That is, the star of R is its complex conjugate, taking care to have it act on the correct indices. We did not have to transpose the matrix.

As a related remark, the coordinate functions are taken to be Hermitian. This means that it is natural to take the coefficients of the algebra in $\mathbb{R}$. One might however quickly spot that the R-matrix has a factor of i present. This factor is necessary for the matrix to be unitary. It does however mean that the values this algebra takes are necessarily complex. Later we will try to view this factor of i as really changing our coefficients to a complex algebra. For now we will consider this complexity to be constrained to the deformation parameter and we will carry on working over the real numbers.

The relations (4.5) can be combined into the block matrix:

$$\begin{pmatrix} x^\lambda x^\alpha \\ x^\lambda y^\alpha \\ y^\lambda x^\alpha \\ y^\lambda y^\alpha \end{pmatrix} = \mathcal{R} \begin{pmatrix} x^\beta x^\mu \\ x^\beta y^\mu \\ y^\beta x^\mu \\ y^\beta y^\mu \end{pmatrix} := \begin{pmatrix} \mathbb{1}_2 & 0 & 0 & 0 \\ 0 & 0 & R^{\lambda\alpha}_{\beta\mu} & 0 \\ 0 & \overline{R}^{\lambda\alpha}_{\beta\mu} & 0 & 0 \\ 0 & 0 & 0 & \mathbb{1}_2 \end{pmatrix} \begin{pmatrix} x^\beta x^\mu \\ x^\beta y^\mu \\ y^\beta x^\mu \\ y^\beta y^\mu \end{pmatrix} \tag{4.7}$$

Lastly, the parameter u should be investigated. To conform with the $*$-structure we would like u^0 and u^1 to be real. Further restrictions on u come once again from the requirement that the algebra preserves the norm $\|x\|^2 = (x^0)^2 + (x^1)^2$ and $\|y\|^2 = (y^0)^2 + (y^1)^2$. To generalise this easily later, let us compute the commutator of x^n with $\sum_k (y^k)^2$. Using summation convention for only indices appearing both higher and lower, we get for each fixed k:

$$\begin{aligned} x^n (y^k)^2 &= R^{nk}_{lm}\, y^l x^m y^k \\ &= R^{nk}_{lm}\, R^{mk}_{pq}\, y^l y^p x^q \\ &= (u^0\, \delta^n_m\, \delta^k_l + \mathrm{i} u^1 J^n_m\, J^k_l)(u^0\, \delta^m_q\, \delta^k_p + \mathrm{i} u^1 J^m_q\, J^k_p) y^l y^p x^q \\ &= (u^0)^2\, (y^k)^2\, x^n + 2\mathrm{i} u^0 u^1 (y^k J^k_q\, y^q J^n_p\, x^p) + (u^1)^2\, (J^k_q\, y^q)^2\, x^n \end{aligned} \tag{4.8}$$

Finally summing over k we find that the antisymmetry of J^k_q implies that the second term in the final equation always appears in cancelling pairs, whereas the final term is just a permutation because of the square. Hence, for the norm to be central all we require is $(u^0)^2 + (u^1)^2 = 1$. Combined with the reality condition, that forces $u^0 = \cos\theta$ and $u^1 = \sin\theta$. There is one more symmetry of the parameter sending θ to $-\theta$, showing that the moduli space of parameters is ${\mathbb{S}^1}/{\mathbb{Z}_2} \cong \mathbb{P}^1(\mathbb{R})$.

Using the centrality of the two norms, we can quotient the deformed algebras $\mathbb{R}^4_\theta$ by $\langle \|x\|^2 + \|y\|^2 - 1\rangle$ to obtain the noncommutative sphere $\mathbb{S}^3_\theta$, or quotient it even further by $\langle \|x\|^2 - 1, \|y\|^2 - 1\rangle$ to arrive at the celebrated noncommutative torus $\mathbb{T}^2_\theta$. These algebras have been studied extensively elsewhere.

Moving on, we will investigate what happens if we take this i present to be the seed for a complexification of the algebra. This process is not in general possible, as one can see from the following argument. We act on each 2-dimensional real vector space with a 2×2-dimensional real matrix J. We want to translate this to an action on a 1-dimensional complex vector space by a 1×1 complex matrix. The former has four degrees of freedom, where we are left with only two at the end. Therefore this complexification, if it is possible, loses a hidden redundancy, but it is only possible if the matrix has this redundancy to begin with. Reasoning from the other side, we find that the eigenvalues of a complex number z acting by multiplication as a matrix on a real 2-dimensional vector space come in complex conjugate pairs, and by a diagonalisation argument we find that the converse holds as well, making this the requirement for a real matrix to be translated into a complex matrix.

In this particular case it turns out to be quite simple: Consider the complex algebra generators $z_1 = x^0 + \mathrm{i}x^1$, $z_2 = y^0 + \mathrm{i}y^1$. On these generators, the relations 4.5 read:

$$\begin{aligned} [z_1, z_1^*] &= 0 = [z_2, z_2^*] \\ z_1 z_2 = e^{-\mathrm{i}\theta} z_2 z_1, &\quad z_1 z_2^* = e^{\mathrm{i}\theta} z_2^* z_1 \\ z_1^* z_2 = e^{\mathrm{i}\theta} z_2 z_1^*, &\quad z_1^* z_2^* = e^{-\mathrm{i}\theta} z_2^* z_1^* \end{aligned} \tag{4.9}$$

Where the last two equations are made redundant by the star structure. In this form, there is another symmetry that becomes quite clear: rescaling the z_i by a complex number does not change the commutation relations. Consequently, there is a $\mathbb{T}^2 = \mathbb{C}^* \times \mathbb{C}^*$ action on the algebra which factors down to $\mathbb{S}^3_\theta$ and $\mathbb{T}^2_\theta$.

There is one pitfall when working with the algebra in this complex form: it requires care to turn back. We have introduced an element i in the coordinates to work well with the pre-existing factor of i in the matrix. However, this means that when taking a real or imaginary part one would simply erase that unmissable factor of i. Let us demonstrate what happens when we add and expand the second line in equation 4.9:

$$(z_1 z_2 + z_1 z_2^*)/2 = x^0 y^0 + \mathrm{i}\, x^1 y^0 = \cos\theta\, y^0 x^0 + \sin\theta\, y^1 x^0 + \mathrm{i}\left(\cos\theta\, y^0 x^1 + \sin\theta\, y^1 x^1\right)$$
$$(z_1 z_2^* - z_1 z_2)/2 = x^1 y^1 - \mathrm{i}\, x^0 y^1 = \cos\theta\, y^1 x^1 - \sin\theta\, y^0 x^1 + \mathrm{i}\left(-\cos\theta\, y^1 x^0 + \sin\theta\, y^0 x^0\right)$$

Taking the real and imaginary parts of this equation gives us different commutation relations from the original 4.5, and these different relations do not yield a consistent $*$-algebra. However, if one splits this equation in the following way:

$$\begin{aligned} x^0 y^0 &= \cos\theta\, y^0 x^0 + \mathrm{i}\sin\theta\, y^1 x^1 \\ x^1 y^1 &= \cos\theta\, y^1 x^1 + \mathrm{i}\sin\theta\, y^0 x^0 \\ x^0 y^1 &= -\mathrm{i}\sin\theta\, y^0 x^1 + \cos\theta\, y^1 x^0 \\ x^1 y^0 &= -\mathrm{i}\sin\theta\, y^1 x^0 + \cos\theta\, y^0 x^1 \end{aligned}$$

one gets back the relations in (4.5), (4.6).

We will conclude this section by expanding the tensor product in (4.5) to collect all the deformation in a single matrix, which we will use as a reference for generalisations later:

$$\begin{pmatrix} x^0 y^0 \\ x^1 y^1 \\ x^0 y^1 \\ x^1 y^0 \end{pmatrix} = \begin{pmatrix} t & \mathrm{i}u & 0 & 0 \\ \mathrm{i}u & t & 0 & 0 \\ 0 & 0 & t & -\mathrm{i}u \\ 0 & 0 & -\mathrm{i}u & t \end{pmatrix} \begin{pmatrix} y^0 x^0 \\ y^1 x^1 \\ y^1 x^0 \\ y^0 x^1 \end{pmatrix} \tag{4.10}$$

with $t = \cos\theta$, $u = \sin\theta$. We will later also expand these parameters, viewing them as lying in a matrix:

$$t^2 + u^2 = 1 = \det\begin{pmatrix} t & \mathrm{i}u \\ \mathrm{i}u & t \end{pmatrix} = \det\begin{pmatrix} t & -\mathrm{i}u \\ -\mathrm{i}u & t \end{pmatrix} \tag{4.11}$$

4.2 Abstract form: Quaternions

4.2.1 Spaces

Let us begin with the first analogue of the R-matrix given in 4.6:

$$R^{\lambda\alpha}_{\beta\rho} = u^0\ \delta^\lambda_\rho \otimes \delta^\alpha_\beta + \mathrm{i}\sum_{k,l} v^k (J^-_k)^\lambda_\rho \otimes u^l (J^-_l)^\alpha_\beta \tag{4.12}$$

The analogue of the matrix J in the complex case is now a linear combination of the matrices J^-_k defined back in 2.35. The choice for J^- rather than J^+ is arbitrary, made for consistency with [DL18a]. As in that paper, we will reduce the degrees of freedom significantly, by noticing that the homogeneity of the quaternions allows us without loss of generality take the first J^- to be proportional to J^-_1. This does not exhaust the symmetry: as mentioned in chapter 2.5.4 one can generate the quaternion algebra with any two orthogonal imaginary quaternions. Hence we can use the further symmetry to reduce $\sum_l u_l J^-_l$ to $u^1 J^-_1 + u^2 J^-_2$. We can once again check that the quadratic norms are central; this time we will do it in a more abstract setting to immediately also prove it for the octonions. Starting off with 4.8, we will suppress most indices for clarity, replacing their role of keeping track on what we act with tensor products, and replace deltas with $\mathbb{1}$. Since we expect it to be true for any value of $\mathbf{u}$, we expect the commutator to be 0 for each $u^k J^-_k$ individually. We will however interpret $u^k J^-_k$ as a summation in order to be sure that there are no interactions between the different terms, and to make it generalisable for the cases later where there are more J-matrices.

Theorem 4.2.1. *For $n = 0, 1, 2, 3$, the norms $\sum_a (x^a)^2$ and $\sum_a (y^a)^2$ are central in $\mathcal{A}(\mathbb{R}^{2^n} \times_R \mathbb{R}^{2^n})$ if $\sum_k (u^k)^2 = 1$.*

Proof. The action of J^-_k on x^n is given as in 2.5.5 by $J^-_k x^n = -F(n,k)x^{n\oplus k}$. Using this, let us commute x^a with a term $(y^b)^2$:

$$\begin{aligned}
x^a y^b y^b =& m(m\otimes\mathbb{1})(\mathbb{1}\otimes\tau)(\mathbb{1}\otimes R)(\tau\otimes\mathbb{1})(R\otimes\mathbb{1})x^a\otimes y^b\otimes y^b \\
=& m(m\otimes\mathbb{1})(\mathbb{1}\otimes\tau)(u^0\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1} + \mathrm{i}\,\mathbb{1}\otimes J_1^-\otimes(u^l J_l^-)) \\
& \cdot(\tau\otimes\mathbb{1})(u^0\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1} + \mathrm{i}\,J_1^-\otimes(u^k J_k^-)\otimes\mathbb{1})x^a\otimes y^b\otimes y^b \\
=& m(m\otimes\mathbb{1})(\mathbb{1}\otimes\tau)(u^0\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1} + \mathrm{i}\,\mathbb{1}\otimes J_1^-\otimes(u^l J_l^-)) \\
& \cdot(u^0 y^b\otimes x^a\otimes y^b + u^k F(b,k)F(a,1)y^{b\oplus k}\otimes x^{a\oplus 1}\otimes y^b) \\
=& (u^0)^2 y^b y^b x^a + u^0 u^k F(b,k)F(a,1)y^{b\oplus k}y^b x^{a\oplus 1} \\
& + u^0 u^l F(b,l)F(a,1)y^b y^{b\oplus l}x^{a\oplus 1} + u^k u^l F(b,k)F(a,1)F(a\oplus 1,1)F(b,l)y^{b\oplus k}y^{b\oplus l}x^a
\end{aligned}$$

This last line gets summed over all values for a and b. Using lemma 2.5.1 we find that $F(a,1)F(a\oplus 1,1) = -1$. Using the same lemma, we have:

$$\sum_b F(b,k)y^{b\oplus k}y^b = \sum_{b'=b\oplus k} F(b'\oplus k,k)y^{b'}y^{b'\oplus k} = -\sum_{b'} F(b',k)y^{b'\oplus k}y^{b'} = 0 \tag{4.13}$$

Where we can use the lemma as $k \neq 0$. The only remaining piece is:

$$\sum_b u^k u^l F(b,k)F(b,l)y^{b\oplus k}y^{b\oplus l}x^a \tag{4.14}$$

For $k = l$ this is clearly equal to $(u^k)^2\sum_b (y^b)^2 x^a$, which would show what we want. For $k \neq l$, we want to show that there is a cancellation between the b-term and the $b\oplus l\oplus k$-term as only they carry the same polynomial. This is equivalent to saying that $F(b,k)F(b\oplus k\oplus l,k)F(b,l)F(b\oplus k\oplus l,k) = -1$ for all b and $k, l \neq 0$. Using lemma 2.5.1 to send $F(b\oplus k\oplus l,k)$ to $-F(b\oplus l,k)$ we arrive at $(-1)^2 F(b,l)F(b\oplus k,l)F(b,k)F(b+l,k)$. Here we can use the definition of ϕ twice to end up at:

$$\begin{aligned}
F(b,k)F(b\oplus k,l)F(b,l)F(b\oplus l,k) &= \phi(b,k,l)F(b,k\oplus l)F(k,l)\phi(b,l,k)F(b,l\oplus k)F(l,k) \\
&= F(k,l)F(l,k) = -1
\end{aligned} \tag{4.15}$$

Which proves the claim. □

As a tangential remark, the very last computation of equation 4.14 has an interesting corollary: setting q and y equal to 1 we find that $\sum F(b,k)F(b,l) \propto \delta_l^k$. This is an algebraic way to show that the matrices F are *Hadamard*, that is, matrices whose only entries are ± 1 with orthogonal rows and columns. These matrices have a lot of uses in information theory and have already been investigated in this context in [AM99].

To return to the matter at hand, we find that the real parameters $\mathbf{u} = (u^0, u^1, u^2)$ need to obey $(u^0)^2 + (u^1)^2 + (u^2)^2 = 1$, placing $\mathbf{u}$ in $\mathbb{S}^2 = \mathbb{P}^1(\mathbb{C})$. Using this we can define a quaternionic noncommutative torus $\mathbb{T}^{\mathbb{H}}_{\mathbf{u}}$ as before, by duality:

$$\mathcal{A}(\mathbb{T}^{\mathbb{H}}_{\mathbf{u}}) = {\mathcal{A}(\mathbb{R}^4 \times_{\mathbf{u}} \mathbb{R}^4)}\big/{\langle \|x\|^2 - 1, \|y\|^2 - 1\rangle} \tag{4.16}$$

and similarly for a noncommutative 7-sphere $\mathbb{S}^7_{\mathbf{u}} \simeq (\mathbb{H}^2_1)_{\mathbf{u}}$

$$\mathcal{A}(\mathbb{T}^{\mathbb{H}}_{\mathbf{u}}) = {\mathcal{A}(\mathbb{R}^4 \times_{\mathbf{u}} \mathbb{R}^4)}\big/{\langle \|x\|^2 + \|y\|^2 - 1\rangle} \tag{4.17}$$

As for the algebras defined by the complex numbers, much has been written about the quaternionic sphere and torus, which we shall not seek to repeat.

4.2.2 Symmetries

In [DL18a], the authors find a useful source of symmetry of the algebras $\mathcal{A}(\mathbb{R}^4 \times_{\mathbf{u}} \mathbb{R}^4)$. The idea is that the quaternions by virtue of their associativity have commuting left and right multiplication operators. As we only use the right multiplication to define our R-matrix, we find that using the J^+ gives a source of maps on x and y preserving the commutation relations.

It follows that the multiplicative group $\mathbb{H}^* \times \mathbb{H}^*$ acts by automorphisms of the $*$-algebra $\mathcal{A}_{\mathbf{u}}$. This induces an action of $\mathrm{U}_1(\mathbb{H}) \times \mathrm{U}_1(\mathbb{H})$ on $\mathcal{A}_{\mathbf{u}}$ by restriction to $\mathrm{U}_1(\mathbb{H})$. This action passes to the quotients and hence defines an action of the classical quaternionic torus $\mathbb{T}^2_{\mathbb{H}} = \mathrm{U}_1(\mathbb{H}) \times \mathrm{U}_1(\mathbb{H})$ on the noncommutative quaternionic torus $(\mathbb{T}^2_{\mathbb{H}})_{\mathbf{u}}$. This situation is of course parallel to the complex case we studied before, which lead to an action of the classical torus $\mathbb{T}^2$ on the noncommutative torus $\mathbb{T}^2_\theta$.

We will further investigate the set of symmetries of these algebras to see if there are any others. In fact, there is a single further continuous symmetry, which we will find by looking at an infinitesimal deformation of the coordinates.

Consider the equivalent of equation (4.7), written as a tensor product:

$$\begin{pmatrix} x^{\cdot} \otimes x^{\cdot} \\ x^{\cdot} \otimes y^{\cdot} \\ y^{\cdot} \otimes x^{\cdot} \\ y^{\cdot} \otimes y^{\cdot} \end{pmatrix} = \begin{pmatrix} \mathbb{1}_{4\otimes 4} & 0 & 0 & 0 \\ 0 & 0 & R & 0 \\ 0 & \overline{R} & 0 & 0 \\ 0 & 0 & 0 & \mathbb{1}_{4\otimes 4} \end{pmatrix} \begin{pmatrix} x^{\cdot} \otimes x^{\cdot} \\ x^{\cdot} \otimes y^{\cdot} \\ y^{\cdot} \otimes x^{\cdot} \\ y^{\cdot} \otimes y^{\cdot} \end{pmatrix} \tag{4.18}$$

Where we suppressed the indices, but keep a dot to remind ourselves that we are dealing with vectors of variables. We are interested in symmetries of this algebra that respect the splitting between the algebras generated by the x and y, so consider the deformation $x^{\cdot} \to x^{\cdot} + Mx^{\cdot}$ and $y^{\cdot} \to y^{\cdot} + Ny^{\cdot}$. Using the fact that the linearised inverse of the deformation $\mathbb{1}_4 + M$ is given by $\mathbb{1}_4 - M$, we find that the first-order terms reduce to:

$$R(\mathbb{1} \otimes M) + R(N \otimes \mathbb{1}) - (M \otimes \mathbb{1})R - (\mathbb{1} \otimes N)R = 0 \tag{4.19}$$

The equation for $\overline{R}$ reduces to the same as M, N are real. Using our explicit form for R, we derive:

$$(J_1^- N - M J_1^-) \otimes (u^1 J_1^- + u^2 J_2^-) + J_1^- \otimes (M(u^1 J_1^- + u^2 J_2^-) - (u^1 J_1^- + u^2 J_2^-)N) = 0 \tag{4.20}$$

Two simple tensor products are equal if and only if the components are proportional, so for some constant κ:

$$(J_1^- N - M J_1^-) = \kappa J_1^- \tag{4.21}$$

$$M(u^1 J_1^- + u^2 J_2^-) - (u^1 J_1^- + u^2 J_2^-)N = \kappa(u^1 J_1^- + u^2 J_2^-) \tag{4.22}$$

We can rescale $M' = M + \kappa\mathbb{1}$ to get rid of the right hand side. In fact, in order for the symmetry to descend to the sphere we would need $M \in \mathfrak{so}(4)$, so without loss of generality $\kappa = 0$.

Solving for N in the two equations we find:

$$\begin{aligned}
N =& - J_1^- M' J_1^- \\
=& - \frac{1}{|u|^2}(u^1 J_1^- + u^2 J_2^-) M' (u^1 J_1^- + u^2 J_2^-) \\
=& \frac{1}{|u|^2}(u^1 J_1^- + u^2 J_2^-) J_1^- N J_1^- (u^1 J_1^- + u^2 J_2^-) \\
=& \frac{1}{|u|^2}((u^1)^2 N - (u^2)^2 J_3^- N J_3^-)
\end{aligned}$$

Where we used the fact that $J_1^- J_2^- = J_3^-$, which, as we remarked in chapter 2.5.4, fails a priori for the octonions. Here we get that the infinitesimal symmetries are classified by exactly the matrices N that commute with J_3^-. This includes the aforementioned copy of $\mathrm{SU}(2)$ acting "on the right" via J_i^+, and J_3^- itself as a basis of antisymmetric operators. For the case of symmetries that do not descend down to the sphere, the above argument still holds, however there are other matrices that commute with J_3^-. The simplest way to find them is to pass to a complex basis in which J_3^- is diagonal, and notice that it is diagonalisable with eigenvalues $\{\mathrm{i}, \mathrm{i}, -\mathrm{i}, -\mathrm{i}\}$. The commutant therefore consists of exactly 2 copies of $\mathrm{SU}(2)$ and one identity scaling matrix for each. The remaining generators can be found to be:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{4.23}$$

To find the symmetries of this deformation in the end we need to exponentiate the Lie algebra generators. As the generators all square to a scalar, this exponential is easy to take.

4.3 Abstract form: Octonions

4.3.1 Spaces

We once again start off by looking at the general form as in 4.6 and 4.12:

$$R^{\lambda\alpha}_{\beta\rho} = u^0\, \delta^\lambda_\rho \otimes \delta^\alpha_\beta + \mathrm{i} \sum_{k,l} v^k\, (J^-_{k,\mathbb{O}})^\lambda_\rho \otimes u^l\, (J^-_{l,\mathbb{O}})^\alpha_\beta \tag{4.24}$$

Where this time the sum runs from 1 to 7. One remark to begin with is that this algebra is itself associative even though the octonions are involved: the only thing we change is a commutator. As with the quaternions, the octonions have a homogeneity: their automorphism group G_2 is given by any map that sends three orthogonal generators to three orthogonal generators. In this case, we can again without loss of generality take $v^k = \delta^k_1$. On the right hand side, we have the freedom of a rotation by G_2 keeping the vector e_1 fixed[3]. This rotation can be shown to in the most general case reduce the operator R to:

$$R^{\lambda\alpha}_{\beta\mu} = t\, \delta^\lambda_\mu \delta^\alpha_\beta + \mathrm{i}\, (J^-_{1,\mathbb{O}})^\lambda_\mu\, (u^1 J^-_{1,\mathbb{O}} + u^2 J^-_{2,\mathbb{O}} + u^3 J^-_{3,\mathbb{O}} + u^4 J^-_{4,\mathbb{O}})^\alpha_\beta \tag{4.25}$$

The $J_{\mathbb{O}}$ matrices used can be found in the appendix. The matrices represent a multiplication operation that is not associative, yet they are still associative themselves. For the centrality of the norm, we would like to refer back to theorem 4.2.1. In this last case, we find that the parameter space is equal to $\mathbb{S}^4 = \mathbb{P}^1(\mathbb{H})$, following the pattern that the parameter spaces are given by projective spaces. Using these invariant norms we once again obtain two sets of noncommutative spaces: the noncommutative octonionic torus with coordinate algebra:

$$\mathcal{A}(\mathbb{T}^{\mathbb{O}}_{\mathbf{u}}) = \mathcal{A}(\mathbb{C}^4 \times_{\mathbf{u}} \mathbb{C}^4) \big/ \langle \|x\|^2 - 1, \|y\|^2 - 1 \rangle \tag{4.26}$$

And noncommutative 15-dimensional spheres defined by

$$\mathcal{A}(\mathbb{S}^{15}_{\mathbf{u}}) = \mathcal{A}(\mathbb{C}^4 \times_{\mathbf{u}} \mathbb{C}^4) \big/ \langle \|x\|^2 + \|y\|^2 - 1 \rangle \tag{4.27}$$

[3] Technically one would change a basis in the algebra to show that the algebras are isomorphic. Using the fact that G_2 is an algebra automorphism one can see that a change of basis by some automorphism ϕ has the same effect as an automorphism ϕ^{-1} acting on the operator R.

4.3.2 Symmetries

In the octonionic case the symmetries are perhaps of greater interest, as we can no longer write down a simple ansatz as we did for the quaternions. The previous ansatz of right multiplication operators no longer works, as due to the nonassociativity the left multiplication operators no longer commute with the right multiplication operators. We can however do the same computation of infinitesimal symmetries as for the quaternionic case. In this case the computations are mostly the same, except for some points which we noticed before. Let us pick up there.

The equivalents of (4.21) for the octonions are as follows:

$$M' J^-_{1,\mathbb{O}} - J^-_{1,\mathbb{O}} N = 0 \tag{4.28}$$

$$M'(u^1 J^-_{1,\mathbb{O}} + u^2 J^-_{2,\mathbb{O}} + u^3 J^-_{3,\mathbb{O}} + u^4 J^-_{4,\mathbb{O}}) - (u^1 J^-_{1,\mathbb{O}} + u^2 J^-_{2,\mathbb{O}} + u^3 J^-_{3,\mathbb{O}} + u^4 J^-_{4,\mathbb{O}}) N = 0 \tag{4.29}$$

from which it can readily be derived that any infinitesimal symmetry N of the coordinate algebra of the octonions obeys the following relation for $\mathbf{u} \neq 0$:

$$((u^2)^2 + (u^3)^2 + (u^4)^2) N = J^-_{1,\mathbb{O}} (u^2 J^-_{2,\mathbb{O}} + u^3 J^-_{3,\mathbb{O}} + u^4 J^-_{4,\mathbb{O}}) N (u^2 J^-_{2,\mathbb{O}} + u^3 J^-_{3,\mathbb{O}} + u^4 J^-_{4,\mathbb{O}}) J^-_{1,\mathbb{O}} \tag{4.30}$$

For the octonions this product does not simplify appreciably, as the nonassociativity of the algebra means that the left multiplication operators do not form a closed set under composition. We will split up the symmetries in two cases.

Firstly we have a look at the symmetries that are independent of $\mathbf{u}$. In this case, we require N to commute with both $J^-_{1,\mathbb{O}} J^-_{2,\mathbb{O}}$, $J^-_{1,\mathbb{O}} J^-_{3,\mathbb{O}}$ and $J^-_{1,\mathbb{O}} J^-_{4,\mathbb{O}}$ independently. We checked the possible N by computer calculation, which yielded an 8-dimensional algebra. This is to be expected, as can be seen by naively diagonalising. The matrix has a $+\mathrm{i}$ and a $-\mathrm{i}$ eigenspace which are respected by the J-matrices, and every independent commutator imposes a simple equality among pairs of diagonal elements. The algebra consists of matrices of the form:

$$\begin{pmatrix} a & -b & -c & -d & -e & -f & -g & -h \\ b & a & d & -c & -f & e & h & -g \\ c & -d & a & b & -g & -h & e & f \\ d & c & -b & a & -h & g & -f & e \\ -e & f & g & h & a & b & c & d \\ f & e & h & -g & -b & a & d & -c \\ g & -h & e & f & -c & -d & a & b \\ h & g & -f & e & -d & c & -b & a \end{pmatrix} \tag{4.31}$$

Where one can see explicitly that the symmetry generators also carry the structure of group multiplication matrices together with a set of signs. To make the algebra respect the Euclidean structure and able to be passed down to the sphere, we require the algebra generators to be antisymmetric matrices, as then the Lie algebra elements lie in the Lie algebra $\mathfrak{o}(8)$. One can see from the explicit form above that all but two of the generators are antisymmetric[4].

We can use the formalism of quasialgebras and cocycles to construct these symmetries by hand as well. Let g_k be the shift operator on $(\mathbb{Z}_2)^3$, sending e_l to $e_{k\oplus l}$. We can write N as a sum of operators of the following form:

$$N =: \sum_k N_k : e_l \to f(k,l)e_{k\oplus l} = f(k,l)g_k e_l \tag{4.32}$$

where f takes values in $\mathbb{C}$. This is possible as operators of this type form a highly degenerate basis of the space of matrices. In this peculiar basis, we notice that the commutation relations fix every g_n-component. Indeed, the operators with which N ought to commute act via g_3 and g_5 respectively, and as the g_n form an abelian group of shifts we see that commutation only changes minus signs, not group elements. Hence we can ignore the summation and look to any specific choice of k. Acting on a basis octonion e_l and remembering that $J^-_{k,\mathbb{O}}$ acts by sending it to $J^-_{k,\mathbb{O}}e_l = -F(l,k)e_{l\oplus k}$, we get that (4.30) becomes:

$$F(l\oplus k\oplus 1,1)F(l\oplus k\oplus 3,2)f_a(k,l\oplus 3)F(l\oplus 1,2)F(l,1) = f_a(k,l) \tag{4.33}$$

[4]Indeed, the one symmetric off-diagonal generator looks quite out of place, and deserves a better explanation than we have currently available.

Similarly for the components corresponding to $J^-_{3,\mathbb{O}}$ and $J^-_{4,\mathbb{O}}$ respectively:

$$F(l\oplus k\oplus 1,1)F(l\oplus k\oplus 2,3)f_a(k,l\oplus 2)F(l\oplus 1,3)F(l,1)=f_a(k,l) \tag{4.34}$$

$$F(l\oplus k\oplus 1,1)F(l\oplus k\oplus 5,4)f_a(k,l\oplus 5)F(l\oplus 1,4)F(l,1)=f_a(k,l) \tag{4.35}$$

Together with the requirement that $f(k,0)=1$. If we make the ansatz that $f_a(x,y)=F(x,y)p(x,y)$ for some fudge factor p, we can use the associativity condition to simplify. The choice $F(x,y)$ instead of $F(y,x)\propto J^-_{x,\mathbb{O}}$ will become useful later. It can be justified a priori by the fact that the symmetry in the quaternionic case is given by the opposite, i.e. *left* quaternion multiplication, where the matrices $J^-_{x,\mathbb{O}}$ act from the right. Using cocycle gymnastics, we find for 4.33:

(4.36)

$$\begin{aligned}
p(k,l)=&F(l\oplus k\oplus 1,1)F(l\oplus k\oplus 3,2)F(k,l\oplus 3)F(l\oplus 1,2)F(l,1)F(k,l)p(k,l\oplus 3)\\
&\qquad\Downarrow F(l\oplus 1,2)F(l,1)=\phi(l,1,2)F(1,2)F(l,1\oplus 2)\\
=&F(l\oplus k\oplus 1,1)F(l\oplus k\oplus 3,2)F(k,l\oplus 3)F(l,3)F(1,2)\phi(l,1,2)F(k,l)p(k,l\oplus 3)\\
&\qquad\Downarrow F(k,l\oplus 3)F(l,3)F(k,l)=\phi(k,l,3)F(l\oplus k,3)\\
=&F(l\oplus k\oplus 1,1)F(l\oplus k\oplus 3,2)F(l\oplus k,3)F(1,2)\phi(k,l,3)\phi(l,1,2)p(k,l\oplus 3)\\
&\qquad\Downarrow F(l\oplus k\oplus 3,2)F(l\oplus k,3)=\phi(l\oplus k,3,2)F(3,2)F(l\oplus k,1)\\
=&F(l\oplus k\oplus 1,1)F(l\oplus k,1)F(3,2)F(1,2)\phi(l\oplus k,3,2)\phi(k,l,3)\phi(l,1,2)p(k,l\oplus 3)\\
&\qquad\Downarrow F(l\oplus k\oplus 1,1)F(l\oplus k,1)=\phi(l\oplus k,1,1)F(1,1)F(l\oplus k,0)=-1\\
=&-F(3,2)F(1,2)\phi(l\oplus k,3,2)\phi(k,l,3)\phi(l,1,2)p(k,l\oplus 3)\\
&\qquad\Downarrow F(3,2)F(1,2)=-1\\
=&\phi(l\oplus k,3,2)\phi(k,l,3)\phi(l,1,2)p(k,l\oplus 3)
\end{aligned}$$

Similarly for the $J^-_{3,\mathbb{O}}$ and $J^-_{4,\mathbb{O}}$-components:

$$p(k,l) = \phi(l \oplus k, 2, 3)\phi(k, l, 2)\phi(l, 1, 3)p(k, l \oplus 2) \tag{4.37}$$

$$p(k,l) = \phi(l \oplus k, 5, 4)\phi(k, l, 5)\phi(l, 1, 4)p(k, l \oplus 5) \tag{4.38}$$

These equations imply that the only obstructions to a constant p and hence a symmetry given by right octonion multiplication are exactly the associator products above.

To ensure that we do in fact get a symmetry we require for example the following commutativity conditions on the associators:

$$\begin{aligned} p(k, l \oplus 1) &= p(k, (l \oplus 3) \oplus 2) \\ &= p(k, (l \oplus 2) \oplus 3) \end{aligned}$$

Which is equivalent to:

$$\begin{aligned} &\phi(2, 3, l \oplus k \oplus 3)\phi(2, l \oplus 3, k)\phi(3, 1, l \oplus 3)\phi(2, 3, l \oplus k)\phi(3, l, k)\phi(2, 1, l) \\ =&\phi(2, 3, l \oplus k)\phi(2, l, k)\phi(3, 1, l)\phi(2, 3, l \oplus k \oplus 2)\phi(3, l \oplus 2, k)\phi(2, 1, l \oplus 2) \end{aligned} \tag{4.39}$$

With similar equations holding for all the other values, as $2, 3, 5$ span $\mathbb{Z}_2^3$. Using the property that ϕ only depends on the span of the numbers it operates on, this particular equation like some others simplifies to:

$$\phi(2, l+1, k)\phi(3, l+1, k)\phi(2, l, k)\phi(3, l, k) = 1 \tag{4.40}$$

More generally, the following was found to be true in the octonions:

$$\phi(a, l, k)\phi(a+b, l, k)\phi(a, l+b, k)\phi(a+b, l+b, k) = 1 \tag{4.41}$$

The remaining equations that do not simplify were checked to hold by computer for all l, k, although a more concrete explanation remains elusive.

Using these equations we can consider a symmetry of the whole algebra; that is, one not necessarily descending to the sphere. A priori this is 64-dimensional since this relies only on a choice of $p(k, l)$. The coefficients $p(k, l)$ needs to satisfy 3 equations, and by the above computations these equations are independent, which means that given $p(k, l)$ we have $p(k, l + a)$ for any a. This leaves an 8-dimensional algebra of options.

In order to descend to a symmetry on the sphere one needs to obey the further condition $f(k, l) = -f(l, k)$, as the Lie algebra element N is antisymmetric in that case. This means that for k nonzero we require $p(k, l) = p(l, k)$. This gives us in principle further commutativity conditions on the associators to verify.

For this 6-dimensional algebra we computed the Cartan matrix to see if the complexified Lie algebra decomposed into anything known, and it does: the complexified algebra is isomorphic to $\mathfrak{su}(2, \mathbb{C}) \otimes \mathfrak{su}(2, \mathbb{C})$.

Lastly, it is worth checking if there are more symmetries if we allow the symmetries to depend on $\mathbf{u}$. In that case we can notice by octonion homogeneity that $u^2 J^-_{2,\mathbb{O}} + u^3 J^-_{3,\mathbb{O}} + u^4 J^-_{4,\mathbb{O}}$ for generic $\mathbf{u}$ behaves like a generic unit octonion orthogonal to $J^-_{1,\mathbb{O}}$, and so, for the purposes of this calculation, can be rotated to any basis element. This rotation matrix will depend on $\mathbf{u}$, but once that is finished, we end up in the same case as previously analysed, but with only one equation on the $p(k, l)$ instead of 3. As a result, for a fixed $\mathbf{u}$ the number of dimensions of our space of symmetries goes up to 32.

4.4 Beyond

Given this setup and the fact that there are no a priori uses of the existence of inverses, it is quite tempting to see how much of this theory extends beyond the octonions. Unfortunately it is not to be. The most important two failures that mark this as an endpoint for us are the following. The lack of homogeneity, as mentioned in chapter 2.5.6, means we can no longer reduce the parameter space of deformations to a simple enough form to be analysed. Even if one were to just postulate an R-matrix of a similar form as in equation 4.24, in order to pass to noncommutative tori or spheres we require that the norm is central. We ran computer-based calculations to check if x^1 commutes with $\sum_0^{15} (y^k)^2$ in the algebra with R-matrix:

$$R^{\lambda\alpha}_{\beta\mu} = \mathrm{i}\, (J^-_{1,\mathcal{S}})^{\lambda}_{\mu}\, (J^-_{8,\mathcal{S}})^{\alpha}_{\beta} \tag{4.42}$$

Where $J^-_{8,\mathcal{S}}$ is once again defined by $J^-_{8,\mathcal{S}}\, y^n = -F(n, 8) y^{n \oplus 8}$. This specific R-matrix gives a perfect failure on lemma 2.5.1, which we use to prove that the terms not proportional to x^a disappear, and the computer verifies that these terms do not disappear. The matrix still obeys the Hadamard property, and the tests we ran seem to indicate that equation 4.14 is still satisfied for the sedenions, although we have not attempted to prove this.

4.5 Concrete form

In this section, we will go back to the explicit (complex) matrix form of the equations. This section is split off for a reason, namely that this method invites a different way to generalise, which turns out to not be equivalent. We have not been able as of yet to find a satisfying unification. For this reason, the two approaches are split off to avoid confusing results of one with the other. This explicit method also fixes the norm and can therefore generate all the corresponding spaces, we will not repeat these constructions. The most explicit difference however will be visible in the parameter space classifying the tori and spheres.

We start as with the complex equations 4.9 and the matrix 4.10, of which we will repeat the reduced form:

$$R^{\lambda\alpha}_{\beta\rho} = u^0\ \delta^\lambda_\rho \otimes \delta^\alpha_\beta + \mathrm{i}(J^-_1)^\lambda_\rho \otimes (u_1 J^-_1 + u_2 J^-_2)^\alpha_\beta \tag{4.43}$$

The idea of using complex coordinates to simplify the complex action seems natural enough, so the most inviting next step is to try quaternionic combinations of the coordinate functions. However, there are pitfalls. Our matrix R is acting on the algebra generators. If our algebra of generators becomes $\mathbb{H}$-valued, then even if the deformation parameter is 0 the generated algebra will be noncommutative and looking at it naively we generate a fundamentally different algebra. Indeed, we already need to take a lot of care to ensure we preserve the $*$-structure when we pass to the complex case, as we mentioned before in section 4.1.2. We choose at this point to not consider taking $\mathbb{H}$-linear combinations.

Instead, we will stick to complex coordinates, absorbing the naturally occurring parameter i into the complexified algebra. Let us begin with the quaternions. The commutation relations 4.5 and 4.12 can be written by acting on the explicit complex generators:

$$z^0 = x^0 + \mathrm{i}x^3, \quad z^1 = x^1 + \mathrm{i}x^2 \qquad \text{and} \qquad w^0 = y^0 + \mathrm{i}y^3, \quad w^1 = y^1 + \mathrm{i}y^2 \tag{4.44}$$

As mentioned in section 4.1.2, it is not trivial to be able to reduce a real matrix to a complex matrix of lower dimension; the pairs of variables chosen is one of the choices that allows for this decomposition. As an example of how to derive the matrix, the explicit action of

J_2^- on w^0 is given by:

$$J_2^- w^0 = J_2^- y^0 + \mathrm{i} J_2^- y^3 = -y^2 + \mathrm{i} y^1 = \mathrm{i} w^1 \tag{4.45}$$

Combining all the terms like this, and including the factor of i in R in the w-term we end up at:

$$z^\lambda w^\alpha = t\, w^\alpha z^\lambda + \begin{pmatrix} 0 & u \\ \bar{u} & 0 \end{pmatrix}^{\alpha}_{\beta} w^\beta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{\lambda}_{\mu} z^\mu \tag{4.46}$$

together with

$$z^{*\lambda} w^\alpha = t\, w^\alpha z^{*\lambda} + \begin{pmatrix} 0 & u \\ \bar{u} & 0 \end{pmatrix}^{\alpha}_{\beta} w^\beta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{\lambda}_{\mu} z^{*\mu} \tag{4.47}$$

These two equations have the same form, and indeed it is a quick check that J_1^- acts on z^0 and z^1 in a way that is agnostic about the minus sign within the definition of z. It is possible to derive the other two equations from this by enforcing the $*$-structure, again taking care that the conjugate transpose of the matrix is not the correct $*$-structure in this case. Rather, the $*$-structure is defined by requiring that the resulting polynomial algebra itself is a $*$-algebra, i.e. by inverting the relations for specific values of λ and α to obtain $\bar{R}$, for example:

$$\begin{aligned} z^0 w^0 &= t w^0 z^0 + u w^1 z^1 \to w^{*0} z^{*0} = t z^{*0} w^{*0} + \bar{u} z^{*1} w^{*,1} \\ z^1 w^1 &= t w^1 z^1 - \bar{u} w^0 z^0 \to w^{*1} z^{*1} = t z^{*1} w^{*1} - u z^{*0} w^{*,0} \end{aligned} \tag{4.48}$$

And inverting this together with the $w^{*0} z^{*1}$ systems of equations yields:

$$z^{*\lambda} w^{*\alpha} = t\, w^{*\alpha} z^{*\lambda} - \begin{pmatrix} 0 & \bar{u} \\ u & 0 \end{pmatrix}^{\alpha}_{\beta} w^{*\beta} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{\lambda}_{\mu} z^{*\mu} \tag{4.49}$$

And similarly:

$$z^\lambda w^{*\alpha} = t\, w^{*\alpha} z^\lambda - \begin{pmatrix} 0 & \bar{u} \\ u & 0 \end{pmatrix}^\alpha_\beta w^{*\beta} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^\lambda_\mu z^\mu \tag{4.50}$$

or equivalently one can derive these equations in a similar way as the first two.

In this case, as mentioned before, the parameters t and u obey $t^2 + \|u\|^2 = 1$. We can arrange them in a matrix:

$$U = \begin{pmatrix} t & u \\ -\bar{u} & t \end{pmatrix} \tag{4.51}$$

which is unitary with unit determinant.

Passing on to the octonions, we will rather than start from 4.25 work by analogy, using the doubling to our advantage. Let $\mathrm{i}, \mathrm{j}, \mathrm{ij}$ be the quaternion units once again. Then we will define the complex coordinate functions to be:

$$\begin{aligned} z^0 &= x^0 + x^3\mathrm{ij}, \quad z^3 = x^1 + x^2\mathrm{ij}, \quad z^1 = x^4 + x^7\mathrm{ij}, \quad z^2 = x^5 + x^6\mathrm{ij} \\ w^0 &= y^0 + y^3\mathrm{ij}, \quad w^3 = y^1 + y^2\mathrm{ij}, \quad w^1 = y^4 + y^7\mathrm{ij}, \quad w^2 = y^5 + y^6\mathrm{ij} \end{aligned} \tag{4.52}$$

and their conjugates. We upgrade the R- matrix to:

$$\begin{aligned} z^\lambda w^\alpha = t\, w^\alpha z^\lambda &+ \begin{pmatrix} 0 & u^0 & 0 & 0 \\ \bar{u}^0 & 0 & 0 & 0 \\ 0 & 0 & 0 & u^0 \\ 0 & 0 & \bar{u}^0 & 0 \end{pmatrix}^\alpha_\beta w^\beta \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}^\lambda_\mu z^\mu \\ &+ \begin{pmatrix} 0 & 0 & \bar{u}^1 & 0 \\ 0 & 0 & 0 & \bar{u}^1 \\ u^1 & 0 & 0 & 0 \\ 0 & u^1 & 0 & 0 \end{pmatrix}^\alpha_\beta w^\beta \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}^\lambda_\mu z^\mu, \end{aligned} \tag{4.53}$$

This choice of R can be motivated by noticing that we need an extra matrix as in the abstract case, where we upgraded the J_1^--matrices to $J_{1,\mathbb{O}}^-$ as well as requiring a second term with $J_{2,\mathbb{O}}^-$. As before, the relations for z^* and w^* can be found in similar manners; the simplest way to find them is to show what transformations the parameters $\mathbf{u}$ undergo when performing a conjugation on a single element. We can then test explicitly by hand and computer that this R-matrix is unitary and preserves the norm. The matrix of parameters also doubles to a matrix:

$$U = t\,\mathbb{I}_4 + \begin{pmatrix} 0 & u^0 & \bar{u}^1 & 0 \\ -\bar{u}^0 & 0 & 0 & \bar{u}^1 \\ -u^1 & 0 & 0 & -u^0 \\ 0 & -u^1 & \bar{u}^0 & 0 \end{pmatrix} \tag{4.54}$$

Which is again unitary and has unit determinant, implying $t^2 + \|x\|^2 = 1$.

We can compute the commutant of this R-matrix quite easily if we care about the symmetries independant of the parameter u, as that involves computing the matrices that commute with both parts of the sum simultaneously. The symmetries that may depend on the parameter u are a lot more difficult to find; in this form it is not clear that a symmetry argument holds and the two components might have cancellations. The parallel to equation 4.30 is:

$$N = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & u^0 & 0 & 0 \\ \bar{u}^0 & 0 & 0 & 0 \\ 0 & 0 & 0 & u^0 \\ 0 & 0 & \bar{u}^0 & 0 \end{pmatrix} N \begin{pmatrix} 0 & u^0 & 0 & 0 \\ \bar{u}^0 & 0 & 0 & 0 \\ 0 & 0 & 0 & u^0 \\ 0 & 0 & \bar{u}^0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{4.55}$$

and

$$N = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & \bar{u}^1 & 0 \\ 0 & 0 & 0 & \bar{u}^1 \\ u^1 & 0 & 0 & 0 \\ 0 & u^1 & 0 & 0 \end{pmatrix} N \begin{pmatrix} 0 & 0 & \bar{u}^1 & 0 \\ 0 & 0 & 0 & \bar{u}^1 \\ u^1 & 0 & 0 & 0 \\ 0 & u^1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \tag{4.56}$$

Where because of the independence of the parameter we can split the complex u matrices into their real and imaginary parts, and N will have to commute with both. All the matrices involved turn out to be diagonal, so their commutant is easily seen to be complex diagonal matrices.

The exact connection with the abstract case remains under investigation.

4.6 Further outlook

There are a few points which we view as potential continuations. Firstly, the 6-dimensional symmetry of the octonion algebra is not necessarily small, but has little octonionic flavour to it. We believe this is natural, as any algebra generated by two elements and their commutation relations cannot be nonassociative. As a result, we would be interested in finding if there is any clear way to go with triple products of the form:

$$\begin{array}{c} \mathbb{R}^8 \\ \times_S \;\; \times_T \\ \mathbb{R}^8 \;\; \times_R \;\; \mathbb{R}^8 \end{array}$$

where we expect there to be three sets of commutation relations that obey some compatibility, and perhaps one needs to even consider an additional associator on products of three elements. This data allows for too much freedom, so for the problem to be tractable we will need a natural structure to present itself. Luckily there are some candidates; there exists a *triality* on three 8-dimensional representations of $\mathrm{Spin}(8)$ which has strong connections to the octonions [DM15][CS03]. We hope that this trio of representations could somehow serve as a good model for the three copies of $\mathbb{R}^8$. There is another well-known algebra that is strongly linked to a trio of octonion algebras: the off-diagonal elements of the exceptional Jordan algebra. As of yet we do not have any idea on how to construct the natural representations necessary to form the R-matrices, though it might be worth investigating. Lastly, we would like to check any further connections with the monoidal categorical setting discussed in the previous chapter.

Secondly, there are a few unfinished pieces that are worth considering further. For one, I would like to extend the octonion cocycle calculus. It feels slightly redundant, as any computer can easily prove large numbers of identities this particular 8 by 8 matrix obeys. However, we believe that there is some merit in developing this calculus further anyway, precisely because a priori there is a large redundancy of identities. By developing a calculus one can make clear which properties are "essential" and which ones are "accidental". Take for example the computed Lie algebra of symmetries. There are many ways of interpreting exactly what the meaning is of the sequences of ± 1's that make up the nontrivial parts, but ultimately there are only $2^8/2 = 128$ of these sequences up to sign. Having an explicit formula in terms of associators would not necessarily speed up computations, but it would provide a reason for why the symmetries are what they are.

Another unfinished thread is the relation between the abstract and concrete forms. We believe the abstract form to be in some sense more fundamental, but there is something to be said for the simplicity of doubling that brings one to the concrete forms. Exploring the relation between the two would be a natural task to finish.

Lastly, there is the obvious matter of wanting to learn more about the noncommutative spaces involved. The concreteness of this formalism means the created spaces are in theory good examples to test noncommutative geometric notions on. In particular, more than most other spaces this way of creating spaces has a connection with quaternionic and complex elements. Multiplication by complex or quaternionic scalars is a good way to construct classical Hopf fibrations, so it is a natural thing to attempt similar constructions in this setting. We will in fact look more at Hopf fibrations in specific in the next chapter.

Chapter 5

Noncommutative Octonionic Hopf Fibration

5.1 Introduction

The *Hopf fibration*[1] has been a very early example of a nontrivial source of principal bundles, and has been studied extensively for its role in homotopy among many other things. For a good exploration of interpretations in low dimensions we refer to [Lyo03].

As principal bundles are a good source of noncommutative manifolds via twist deformation, this has been a rich source of noncommutative spheres, for an example see [BDZ04]. However, the octonionic Hopf fibration seems to resist: in the non-dual setting one has to choose charts. In this section we record some attempts at dualising this construction.

Let us first record some basics: I will follow [DM15, ch.12] here. The Hopf fibrations are the following fibrations:

$$\begin{array}{cccc} \mathbb{S}^1 & \mathbb{S}^3 & \mathbb{S}^7 & \mathbb{S}^{15} \\ \big\downarrow{\scriptstyle \mathbb{S}^0} & \big\downarrow{\scriptstyle \mathbb{S}^1} & \big\downarrow{\scriptstyle \mathbb{S}^3} & \big\downarrow{\scriptstyle \mathbb{S}^7} \\ \mathbb{S}^1 & \mathbb{S}^2 & \mathbb{S}^4 & \mathbb{S}^8 \end{array}$$

Where the first three are principal with groups $\mathbb{S}^0 \cong \mathbb{Z}_2$, $\mathbb{S}^1 \cong \mathrm{U}(1)$, $\mathbb{S}^3 \cong \mathrm{SU}(2)$ and the

[1] It seems to be either a coincidence or a matter of large numbers that Heinz Hopf has the algebra, the algebraic extension and fibration named after him, and that they come together in this way. It does not seem he ever thought of Hopf algebras as objects of symmetries.

last one is in fact not a group as it is isomorphic to the unit octonions, which are of course non-associative.

One way to describe the fibration is as follows. Consider an element

$$v = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{K}^2 \tag{5.1}$$

for $\mathbb{K}$ a real division algebra. We can see v as a vector of spherical coordinates if we require that the norm $v^*v = \|x\|^2 + \|y\|^2 = 1$. We can also look at the opposite product given by:

$$vv^* = \begin{pmatrix} \|x\|^2 & xy^* \\ yx^* & \|y\|^2 \end{pmatrix} \tag{5.2}$$

This matrix is Hermitian by definition, and has trace is given by $t = \|x\|^2 + \|y\|^2$, which is equal to 1 by assumption. The reduced diagonal $u = \frac{1}{2}\|x\|^2 - \|y\|^2$ together with the hypercomplex coordinate $z = xy^* \in \mathbb{K}$ together obey

$$|z|^2 + u^2 = \|x\|^2\|y\|^2 + \frac{1}{4}(\|x\|^4 - 2\|x\|^2\|y\|^2 + \|y\|^4) = 1 \tag{5.3}$$

and therefore form a parametrisation of $\mathbb{S}^{n+1}$. Now, the information lost is clear: vv^* is invariant if we change v by a phase to vq, for $q \in \mathbb{K}, \|q\| = 1$. However, there is one problem with this statement: we need to be able to rebracket $(vq)(q^*v^*)$. The trick is to use the octonionic homogeneity. For any v, there is a unique z containing all the octonionic information. Any algebra generated by two octonions is at most quaternionic, and therefore has no issues with associativity. Therefore we can pick a "square root" of z, called w, such that w is a complex vector embedded in $\mathbb{O}^2$, and we find $ww^* = vv^*$. Then the argument goes by transitivity: as any v can be reached in this way we can do the computation in the associative realm. The explicit choice of w is given by:

$$w = \begin{pmatrix} \frac{xy^*}{\|y\|} \\ \|y\| \end{pmatrix} \tag{5.4}$$

unless $y = 0$. This is where a deviation from the quaternions lies: the fact that this computation depends on the choice of a chart.

We will try to dualise this setup. The natural language to work in is the language of

Hopf-Galois extensions, although there is one natural problem: there is no Hopf algebra structure on $\mathbb{S}^7$. We will next repeat some of the results in [KM10], where they define a *Hopf quasigroup* and a *Hopf coquasigroup* to deal with this specific case.

Definition 5.1.1. *A* Hopf quasigroup *is a (possibly nonassociative) unital algebra $\mathcal{H}$ equipped with algebra homomorphisms $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$, $\epsilon : \mathcal{H} \to k$ forming a coassociative coalgebra and a map $S : \mathcal{H} \to \mathcal{H}$ such that*

$$m(\mathrm{id} \otimes m)(S \otimes \mathrm{id} \otimes \mathrm{id})(\Delta \otimes \mathrm{id}) = \epsilon \otimes \mathrm{id} = m(\mathrm{id} \otimes m)(\mathrm{id} \otimes S \otimes \mathrm{id})(\Delta \otimes \mathrm{id}) \tag{5.5}$$

$$m(m \otimes \mathrm{id})(\mathrm{id} \otimes S \otimes \mathrm{id})(\mathrm{id} \otimes \Delta) = \mathrm{id} \otimes \epsilon = m(m \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{id} \otimes S)(\mathrm{id} \otimes \Delta) \tag{5.6}$$

Definition 5.1.2. *A* Hopf coquasigroup *is a unital associative algebra $\mathcal{H}$ equipped with counital algebra homomorphisms $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$, $\epsilon : k \to \mathcal{H}$, and linear map $S : \mathcal{H} \to \mathcal{H}$ such that*

$$(m \otimes \mathrm{id})(S \otimes \mathrm{id} \otimes \mathrm{id})(\mathrm{id} \otimes \Delta)\Delta = 1 \otimes \mathrm{id} = (m \otimes \mathrm{id})(\mathrm{id} \otimes S \otimes \mathrm{id})(\mathrm{id} \otimes \Delta)\Delta \tag{5.7}$$

$$(\mathrm{id} \otimes m)(\mathrm{id} \otimes S \otimes \mathrm{id})(\Delta \otimes \mathrm{id})\Delta = \mathrm{id} \otimes 1 = (\mathrm{id} \otimes m)(\mathrm{id} \otimes \mathrm{id} \otimes S)(\Delta \otimes \mathrm{id})\Delta \tag{5.8}$$

The intuition here is as follows: In the quasigroup case, we yield associativity, except in the special case where we need a term of the form $x^{-1}(xy)$, in which case it is permitted to "let x in". Without that permission the notion of inverse becomes murky, so this is necessary, and it captures the most powerful aspect of the octonions. For the coquasigroup definition all we do here is reverse the arrows. We would like to remind the reader that Sweedler notation becomes tricky in non-coassociative settings, as there is no way in general to simplify subscripts of subscripts. The point of these definitions comes from [KM10, Prop 5.7], where they show that $k[\mathbb{S}^{2^n-1}]$ is a Hopf coquasigroup for all n, and dually $\mathbb{S}^7$ is a quasigroup, $\mathbb{C}\mathbb{S}^7$ is a Hopf quasigroup.

5.2 Vector fields on $\mathbb{S}^7$

The first attempt to establish a Hopf-Galois extension on $\mathbb{S}^{15}$ was done by instead considering the associative algebra of vector fields defined in [KM10, ch 6.1]. The vector fields are defined by

$$\partial^i = \sum_a F(a,i) x_{a+i} \frac{\partial}{\partial x_a} \tag{5.9}$$

Here we take both a and i to lie in $(\mathbb{Z}_2)^3$, and i is nonzero.

We will take these vector fields to be acting diagonally on S^{15} parametrised by x_a, y_a, again with a in $(\mathbb{Z}_2)^3$, such that of course $\sum_a x_a^2 + y_a^2 = 1$. This means that we effectively double the vector fields to:

$$\sum_a F(a,i)(x_{a+i} \frac{\partial}{\partial x_a} + y_{a+i} \frac{\partial}{\partial y_a}) \tag{5.10}$$

We are interested in the invariant polynomials under this action, hoping that they square to 1, forming another sphere. It is easy to verify that $\sum_a x_a^2 - y_a^2$ is in fact invariant under all vector fields. The only way for this set of polynomials to square to 1 is to use the one relation we have, which is $\sum_a x_a^2 + y_a^2 = 1$. The first possible solution is to find polynomials p_a such that:

$$(\sum_a x_a^2 - y_a^2)^2 + \sum_b p_b^2 = (\sum_a x_a^2 + y_a^2)^2 = 1 \tag{5.11}$$

From this we can deduce that our polynomials are all quadratic, and by the difference of squares we get that $\sum_b p_b^2 = 4(\sum_a x_a^2)(\sum_b y_b^2)$. This relation is the Degen eight-square identity. Hence we would like to see if we can get invariant polynomials that solve this equation.

By Hurwitz's theorem there is no 16-square identity for polynomials[2], which tells us that if there is a way to find these invariant polynomials, it must break down after the octonions.

The Degen polynomials, when the variables are indexed by $(\mathbb{Z}_2)^3$ rather than $\{0...7\}$, have the property that the indices of the terms of a given p_b all add up to the same element. The reason for this is their connection to the octonions: the polynomials are given by a twisted coproduct on the basis elements of $\mathbb{C}\mathbb{Z}_2^3$, which will become relevant later. It is

[2]As mentioned in chapter 2.5.6, there are identities for rational functions. I have not been able to find a use for them here.

possible to write all the polynomials as follows:

$$p_b = \sum_{\alpha+\beta=b} F(b_\alpha, b_\beta)\, x_{b_\alpha} y_{b_\beta} = \sum F(b_{(1)}, b_{(2)}) x_{b_{(1)}} \otimes x_{b_{(2)}} \tag{5.12}$$

The property of being like a coproduct turns out to be necessary for canceling out under the action by the vector fields as we see later. The fact that the coefficients are the same F as the cocycle for octonions is important for the sum squaring to one.

Each individual term in the derivations has the effect of sending x_a to x_{a+i}, effectively performing a translation on the index. Therefore when we act with ∂^i on a term $x_a y_b$, the result will be sent to $x_{a+i}y_b \pm x_a y_{b+i}$, up to a sign. We want $\partial^i p_c$ to be zero for all i, so we need our terms to cancel. The preimage of $x_{a+i}y_b$ under ∂^i is given by $x_a y_b$ as well as $x_{a+i}y_{b+i}$, as we live in $(\mathbb{Z}_2)^3$ where the characteristic is 2 so $i+i=0$. The same preimages also occur for $x_a y_{b+i}$. This means that in order for any polynomial containing the term $x_a y_b$ to be sent to zero, that polynomial must contain $x_{a+i}y_{b+i}$ for all i. This is a complete closed orbit. Note that $a+i+b+i = a+b$ for all i, which means that the sum of the indices must be constant in each polynomial, which forces the coproduct-like behaviour noted above.

For cancellation to actually happen, we need the following:

$$\partial^i \sigma(a,b) x_a y_b = -\partial^i \sigma(a+i, b+i) x_{a+i} x_{b+i} \tag{5.13}$$

Where $\sigma(a,b)$ is the coefficient of the term $x_a y_b$ in the polynomial. We make the following ansatz:

$$\sigma(a,b) = F(a,b)\theta(a,b) \tag{5.14}$$

In the quaternionic case the polynomials have coefficients almost proportional to F, so that is what the ansatz is based on. Note also the parallel with section 4.3.2. The quaternion case shows one problem, which is that F is not totally antisymmetric off the diagonal whereas the polynomials are, as $F(0,i) = F(i,0) = 1$. The solution there turns out to have $\theta(a,0) = -1, \theta(a,i) = 1$.

Plugging the ansatz into the cancellation equation we get:

$$F(a,i)F(a,b)\theta(a,b)x_{a+i}y_b = -F(b+i,i)F(a+i,b+i)\theta(a+i,b+i)x_{a+i}y_b$$
$$F(b,i)F(a,b)\theta(a,b)x_a y_{b+i} = -F(a+i,i)F(a+i,b+i)\theta(a+i,b+i)x_a y_{b+i} \tag{5.15}$$

Using the fact that $F^2 = 1$, we can rearrange the F's in order to get an associativity equation in both cases. Remember that the associator is defined by:

$$\phi(x,y,z) = \frac{F(x,y)F(x+y,z)}{F(x,y+z)F(y,z)} \tag{5.16}$$

Rearranging:

$$F(a,i)F(a+i,b+i)\theta(a,b) = -F(b+i,i)F(a,b)\theta(a+i,b+i) \tag{5.17}$$

$$\phi(a,i,b+i)\frac{F(i,b+i)}{F(b+i,i)} = -\frac{\theta(a+i,b+i)}{\theta(a,b)} \tag{5.18}$$

Similarly for the other equation:

$$F(b,i)F(a+i,b+i)\theta(a,b) = -F(a+i,i)F(a,b)\theta(a+i,b+i)$$
$$\phi(a+i,i,b)\frac{F(b,i)}{F(i,b)} = -\frac{\theta(a,b)}{\theta(a+i,b+i)} \tag{5.19}$$

F is almost antisymmetric. A little care needs to be taken with $F(a,a) = F(a,a)$ and $F(a,0) = F(0,a) = 1$. Also for the octonions, setting $\hat{\theta}(a,0) = -1, \hat{\theta}(a,i) = 1$ solves this problem. The special cases are $b = 0$ and $b = i$, as $i = 0$ is not allowed by assumption. If $b = 0$, then:

$$\frac{\hat{\theta}(a+i,b+i)}{\hat{\theta}(a,b)} = \frac{1}{-1} \tag{5.20}$$

And if $b = i$:

$$\frac{\hat{\theta}(a+i,b+i)}{\hat{\theta}(a,b)} = \frac{-1}{1} \tag{5.21}$$

The converse is also true, if $\hat{\theta} = -1$ then $b = 0$ or $b = i$. Hence if we make the new ansatz

$\theta(a,b) = \chi(a,b)\hat{\theta}(a,b)$, then the equations above simplify to:

$$\phi(a,i,b+i) = \frac{\chi(a+i,b+i)}{\chi(a,b)}$$
$$\phi(a+i,i,b) = \frac{\chi(a,b)}{\chi(a+i,b+i)} \tag{5.22}$$

As mentioned in chapter 2.5.5 the octonion associator $\phi(x,y,z)$ is 1 if x,y,z are linearly dependent, and -1 otherwise. Hence here both associators are actually equal to $\phi(a,i,b)$.

To finish off the impossibility of the existence of quadratic polynomials annihilated by this set of vector fields, we just need to show that:

$$\phi(a,i,b) \neq \frac{\chi(a,b)}{\chi(a+i,b+i)} \tag{5.23}$$

for some χ. Consider the case $a = i$. Then the LHS is 1, so $\chi(i,b) = \chi(0,i+b)$ for all b,i, so χ only depends on the sum of its arguments. If χ only depends on the sum of its arguments, then $\chi(a,b) = \chi(a+i,b+i)$ and the associator ϕ must be trivial. Contradiction follows if ϕ is nontrivial, and hence for the octonions this vector field does not annihilate a polynomial algebra of spheres.

5.3 Quasi-Hopf-Galois extensions on tori

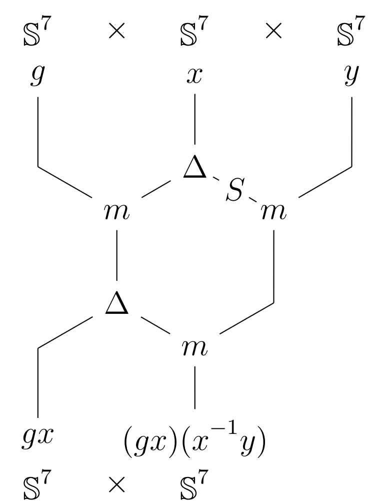
The next approach is by attempting to work with a weak notion of action. There exists a notion of Hopf quasimodule, defined in [BJ12], where they only require an associativity with the inverse element. In our case, the octonions are power-associative so this should not be a problem and the requirement is very lenient. The ultimate goal is to find a notion of Hopf-Galois extension for this context that is strong enough to work for the purposes of [Asc+17], so we can define noncommutative spheres by putting a cocycle on $\mathcal{A}(\mathbb{S}^7)$. The outline of how to do this is as follows: we view $\mathbb{S}^{15}$ in the context of the *Hopf construction*. We will detail this more later, but the gist is that we decompose the sphere $\mathbb{S}^{15}$ into an interval of higher tori $\mathbb{S}^7 \times [0,1] \times \mathbb{S}^7$, and quotient this at the endpoints. The first step is then to understand this quasi-action on the tori, and later we can patch it up.

For our algebra, we would like to act on $\mathbb{O}\oplus\mathbb{O}$, which creates the problem that a priori, by acting with g on $v = (x,y)$ as (gx, gy), there are two different associators involved when checking the action condition $g \triangleright (h \triangleright (x, y)) \cong (gh) \triangleright (x, y)$. This is what prevented us from showing (vv^*) was invariant. The idea to fix this is to use the division algebra structure of $\mathbb{O}$ and rewrite this action in a way that is equivalent for all algebras that are not $\mathbb{O}$, namely as:

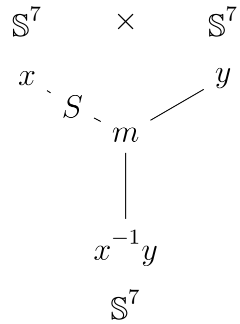
$$g \triangleright (x, y) = (gx, (gx)(x^{-1}y)) \tag{5.24}$$

This way of defining the action yields the same associator on both elements when repeated, and hence intuitively it should cancel out when squaring. There is one downside to this approach: it is not defined at $x = 0$. Let us first work in a chart where $x \neq 0$, and address that possibility later when we return to the Hopf construction. To aid us in dualising things later and to remove any dependency on Sweedler notation in noncoassociative contexts, we will convert this calculation into the language of string diagrams, as we defined in 2.4.4.

To draw the string diagram, we need to have a way of splitting x into three parts, as the same x appears three times in the equation. This is naturally done with a coproduct, but we have no natural coproduct on a general nonzero octonion; we have an already defined coproduct on $\mathbb{S}^7$ via the quasigroup structure. We will therefore proceed as follows: we formulate the action of $\mathbb{S}^7$ on $\mathbb{S}^7 \times \mathbb{S}^7$, and later glue all these copies of $\mathbb{S}^7 \times \mathbb{S}^7$ together with a technique called the *Hopf construction*. First, let us write equation 5.24 as a string diagram:



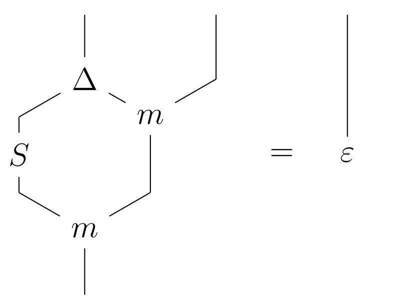
Where we used that the comultiplication is grouplike. We would like to adjoin the diagram:



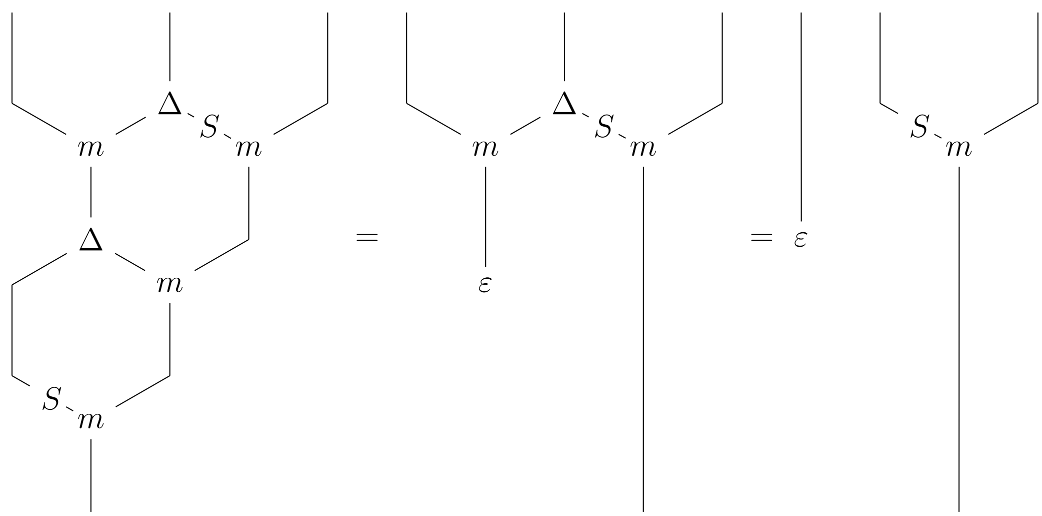
If we call the former diagram $\triangleright$ and the latter ϕ, then the composition of the diagrams is given by the following diagram:

$$\mathbb{S}^7 \times \mathbb{S}^7 \times \mathbb{S}^7 \underset{\varepsilon \otimes \mathrm{id} \otimes \mathrm{id}}{\overset{\triangleright}{\rightrightarrows}} \mathbb{S}^7 \times \mathbb{S}^7 \xrightarrow{\phi} \mathbb{S}^7$$

If this diagram is a coequaliser, that is, the universal diagram such that all paths commute, then $\mathbb{S}^7$ is the quotient of $\mathbb{S}^7 \times \mathbb{S}^7$. Hence we would like to show using string diagram methods that this holds. We need to use the first of the Hopf quasigroup axioms in equation 5.5, which in diagrammatic form reads:

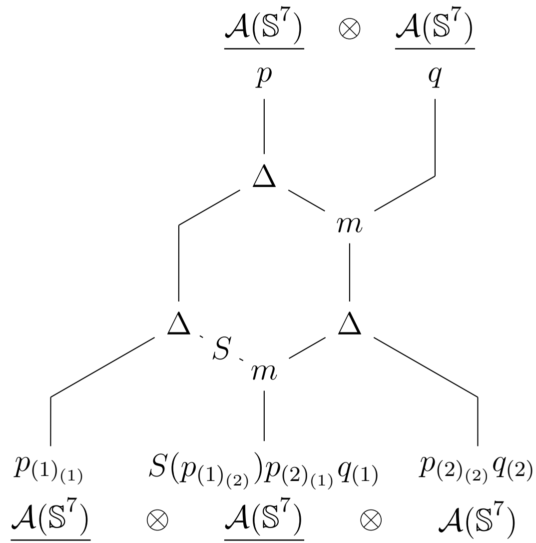


Combining the three diagrams into one calculation, we get:

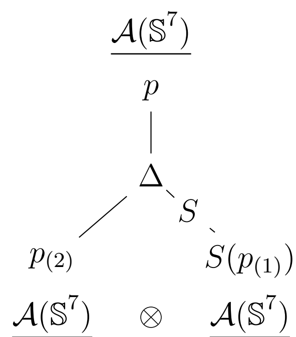


For the last equality we used that ε is an algebra morphism. This establishes the fact that this “action” has a candidate coequaliser map ϕ going to $\mathbb{S}^7$, and hence these maps give a fibration. To show that this map truly defines the coequaliser one needs to show it is universal. The naturality of its definition gives an indication of this. We will omit a proof as in the dual setting this requirement is implied by the Hopf-Galois condition, which we will talk about later.

The reason why we went through the effort of phrasing these maps into all in these diagrams is because if we want to do the same in the dual setting of algebras, we can now just invert all the diagrams. As everything is defined dually, these computations look exactly the same. The same maps define a quasicoaction by the Hopf coquasigroup $k[\mathbb{S}^7]$, the coaction composed with antipode on $k[\mathbb{S}^7]$ defines the candidate equaliser of this map. This explains why in the previous section the invariant polynomials looked like a coproduct: the algebra of coproducts is coinvariant under this quasicoaction. The dual diagrams are given by:



and for the candidate equaliser:



Here we explicitly see a reason for the existence of Degen polynomials solving the eight-square identity: the coinvariants of the coaction given by this slightly more complicated comultiplication are given exactly by the polynomials generated by $\sum_k F(j,j)F(k\oplus j,j)x_{k\oplus j}x_j$, which are the polynomials that appear in that identity.

We want to establish a notion of Hopf-Galoisness, to show that this construction is equivalent to some notion of principal bundle for quasigroups. Ultimately, the goal is to lift this to the entire sphere using a Hopf construction, and then to use the results of [Asc+17] to establish that any deformation of the Hopf quasigroup $\mathcal{A}(\mathbb{S}^7)$ can be used to obtain deformed noncommutative 15-spheres. We are going to start by requiring the standard condition for Hopf-Galoisness, requiring the canonical map to be bijective. The canonical map is still well-defined in the Hopf quasigroup case, and so the question of whether it is bijective is well-posed. Let us recall that the canonical map is given by:

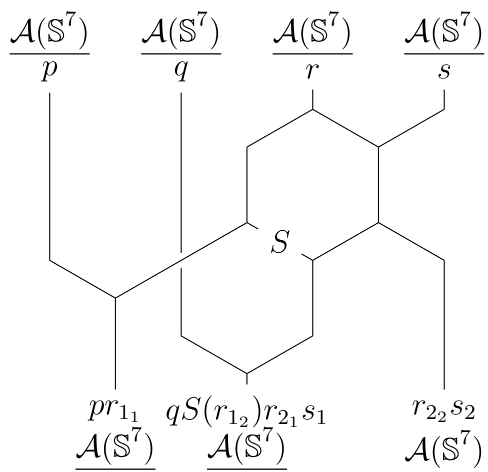
(5.25)

$$\mathrm{can} = (m_{\underline{\mathcal{A}(\mathbb{S}^7\otimes\mathbb{S}^7)}}\otimes \mathrm{id})(\mathrm{id}\otimes\delta) : \underline{\mathcal{A}(\mathbb{S}^7\otimes\mathbb{S}^7)} \underset{\underline{\mathcal{A}(\mathbb{S}^7)}}{\otimes} \underline{\mathcal{A}(\mathbb{S}^7\otimes\mathbb{S}^7)} \to \underline{\mathcal{A}(\mathbb{S}^7\otimes\mathbb{S}^7)}\otimes\mathcal{A}(\mathbb{S}^7)$$

$$\mathrm{can}(a\otimes a') = (a\cdot a'_{(\underline{0})})\otimes a_{(\underline{1})}$$

Where the underline once again indicates that the algebra is thought of as a module rather than a Hopf algebra. The tensor over $\underline{\mathcal{A}(\mathbb{S}^7)}$ means that the specific embedding of $\mathcal{A}(\mathbb{S}^7)$ given by the equaliser in both algebra halves can be mapped over the tensor product; that is, for a polynomial of the form $\sum p\otimes q \underset{\underline{\mathcal{A}(\mathbb{S}^7)}}{\otimes} r\otimes s$, any factor of $\sum p\otimes q$ that contains a Degen polynomial can be transferred over the tensor product to multiply $r\otimes s$ instead.

Let us fill our choice of coaction into this equation. Diagrammatically, the canonical map is given by the following:



Where we left out the symbols indicating multiplication and comultiplication as they are clear. We need to show that this map is in fact bijective, reminding ourselves that this map is not necessarily an algebra map, which means we have to check on any polynomial rather than just generators. As a quick remark, notice how we are working in a polynomial algebra and hence our multiplication is commutative and associative for now.

The proof of Hopf-Galoisness is unfortunately not quite complete with all the details, so we will provide just a sketch below.

We notice that by acting with the counit on any pair of the generators, we can recover the third: this way we obtain the products pr, $qS(r_1)r_2s = qs$ and rs. For surjectivity, we want to show that one can invert this map to obtain any individual p, q, r, s up to factors of $a_1 \otimes S(a_2)$ which may be transferred between $p \otimes q$ and $r \otimes s$. Intuitively, this is correct: in the pairs pr, qs and rs we see that if we can scale p by a factor a, q by a^{-1}, r by a^{-1} and s by a, then these products are invariant, and this is exactly the porting around of a factor of $a \otimes a^{-1}$, and additionally, it should be clear that this is the only degree of freedom available as providing for example p/r fixes everything else. However, there is a caveat: we need to have access to any a to port around, and it is not true that all these polynomials a can be written as a component b_1 of a coproduct of some polynomial b. Take for a counterexample the polynomial x; there is no way to multiply it to a multiple of $x^2 + y^2$. The idea to establish surjectivity is then as follows: If we could factor polynomials, we are done, as we can find the greatest common divisor between any two products and so retrieve the polynomials. In other words, the obstruction to our factorisation comes from these invertible polynomial tensor products. Hence the only thing left to prove is that a tensor product of polynomials multiplies to one if and only if it appears in a coproduct of the Degen kind. The if is clear, the only if not so much.

As for injectivity, for Hopf-Galois extensions it was shown in [Sch90] that it is enough to show that $\underline{\mathcal{A}(\mathbb{S}^7 \times \mathbb{S}^7)}$ is an injective $\mathcal{A}(\mathbb{S}^7)$-module to establish that it is Hopf-Galois. Now, we need to add the word 'quasi' to Hopf-Galois, which makes it a little less obvious, and this is another gap we need to fill in.

5.4 Hopf construction

The remaining piece before we can claim we have fixed the Hopf fibration on $\mathbb{S}^{15}$ is to assemble this fibration of products of spheres $\mathbb{S}^7 \times \mathbb{S}^7$ into the 15-sphere. There is a topological construction, named the *Hopf construction* that classically does this. This construction says that given a map from topological spaces:

$$f : X \times Y \to Z \tag{5.26}$$

there exists a lift to

$$\hat{f} : X * Y \to \Sigma Z \tag{5.27}$$

where $A * B$ is the *join*, given by:

$$X * Y = {}^{(X \times I \times Y)}\!\diagup_{\sim} \tag{5.28}$$

where I is the unit interval $[0,1]$ and $\sim$ is an equivalence relation given by:

$$(x,0,y_1) \sim (x,0,y_2) \ \ , \ \ (x_1,1,y) \sim (x_2,1,y) \tag{5.29}$$

Signifying a contraction of the spheres at the respective endpoints. The suspension ΣZ is similarly given by a quotient:

$$\Sigma Z = {}^{(Z \times I)}\!\diagup_{\sim} \ \text{ for } \ (x_1,0) \sim (x_2,0) \ \ , \ \ (x_1,1) \sim (x_2,1) \tag{5.30}$$

Similarly pinching at the endpoints. The reason we are interested in this construction is because we have the exact case of $f : \mathbb{S}^7 \times \mathbb{S}^7 \to \mathbb{S}^7$, and we would like to lift it to the sphere. We also have the problem of only being defined at one of the endpoints, which this construction singles out for us. We can define similar diagrams for the action defined by:

$$g \triangleright' (x,y) = (((x^*)(y^*)^{-1})((y^*)g^*), y^*g^*)^* = ((gy)(y^{-1}x), gy) \tag{5.31}$$

Then in order to do the Hopf construction, one needs to glue these two functions together into one function defined everywhere. This gluing procedure is well-defined classically, but to write it down in a way that is dualisable is not trivial. This is the current state of affairs. There is a paper [DHH14] in which the authors detail a noncommutative join construction. This is done in the C^*-algebraic realm, and it is not trivial to find a good algebraic equivalent to $C^\infty([0,1])$. Similarly, the classical join construction is usually done in the category of topological spaces, not the category of algebraic varieties. There are methods we are considering, like using functions that pinch away the endpoints in a similar way to how one can define a circle by the algebra:

$$\langle X^n, Y : Y^2 = 1 - X^2 \rangle \tag{5.32}$$

Viewing the X-coordinate as the important one and the Y-coordinate as order 2 with an extra cocycle.

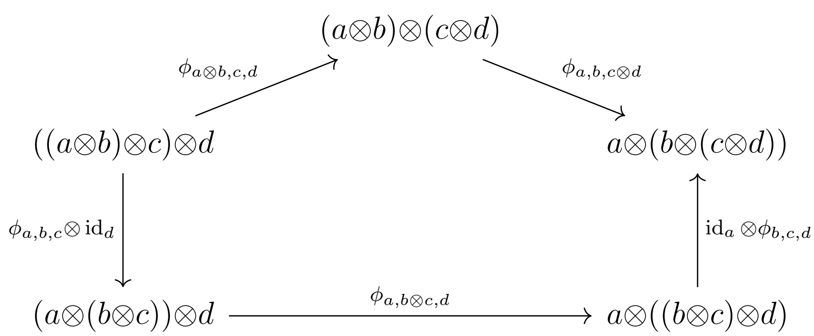
The most natural attempt is by using the extra variable as a parameter rather than a coordinate function. We include the parameter t as parametrising the radius squared of the sphere X and $1-t$ for Y, and this stays that way when dualising. While this creates a rather natural-looking construction, the break between coordinate functions and parameters for the dual of t is a bit troubling. One could replace t by $\|X\|^2 - \|Y\|^2$, but then the matter returns of how to use this element to parametrise the coinvariants.

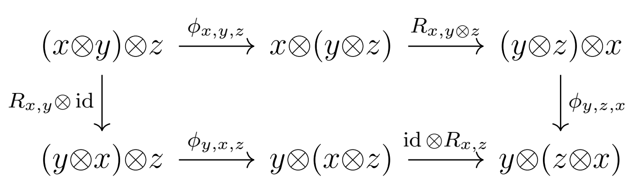
Appendix A

Appendices

A.1 Non-strict monoidal category relations

The pentagon and hexagon relations for monoidal categories with associator ϕ and braiding R are given by the following diagrams:





A.2 Explicit matrices

The octonionic J-matrices used explicitly are given by:

$$J^-_{1,\mathbb{O}} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$J^-_{2,\mathbb{O}} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}$$

$$J^-_{3,\mathbb{O}} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}$$

$$J_{4,\mathbb{O}}^{-} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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