SISSA
Scuola Internazionale Superiore di Studi Avanzati

Mathematics Area - PhD course in
Mathematical Analysis, Modelling, and Applications

Hamiltonian PDEs in fluid dynamics: local and long time existence results

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Academic Year 2021-22

Contents

Chapter 1

Introduction

This Thesis contains new results concerning the Cauchy problem of the following dispersive partial differential equations arising in fluid dynamics under space periodic boundary conditions:

1. The compressible Euler-Korteweg system in (1.0.2) on $\mathbb{T}^d$;
2. The Quantum Hydrodynamics equations in (1.0.2) with $K(\rho) = \frac{\kappa}{\rho}$ on d-dimensional irrational tori;
3. The Gravity-Capillary water waves equations for bi-dimensional fluids with constant vorticity in (1.0.3).

As we shall describe later, these three systems of equations can be written as dispersive Hamiltonian quasi-linear PDEs of the form

$$\begin{cases} \partial_t u = \mathcal{L}u + \mathcal{P}(u), \quad u = u(t,x), \quad (t,x) \in [0,T] \times \mathbb{T}^d, \\ u(0,x) = u_0(x) \in H^s(\mathbb{T}^d), \end{cases} \qquad \mathbb{T}^d := \mathbb{R}^d / 2\pi\mathbb{Z}^d, \quad d \in \mathbb{N} \tag{1.0.1}$$

where $\mathcal{L}$ is an unbounded linear operator with purely imaginary spectrum, $\mathcal{P}$ is a nonlinear function of a complex unknown u and its derivatives (up to the same order of $\mathcal{L}$), vanishing quadratically at $u = 0$, and

$$H^s(\mathbb{T}^d) := \Big\{ u(x) = \sum_{j \in \mathbb{Z}^d} u_j e^{\mathrm{i} j \cdot x} \ : \ \|u\|_s^2 := \sum_{j \in \mathbb{Z}^d} |u_j|^2 \langle j \rangle^{2s} < +\infty \Big\}, \quad \langle j \rangle := \max\{1, |j|\}$$

is the Sobolev space of regularity s.

In this Thesis we shall consider the following dynamical questions:

- *Local well-posedness:* Given an initial datum $u_0 \in H^s(\mathbb{T}^d)$ (with s larger than some $\overline{s}$) determine if there exists a positive time $T_{\mathrm{loc}} > 0$ and a unique local, classical solution $u(t,x) \in C([0, T_{\mathrm{loc}}]; H^s(\mathbb{T}^d))$ of (1.0.1);
- *Long time existence:* For any initial datum $u_0(x)$ satisfying $\|u_0\|_{H^s} < \varepsilon \ll 1$ (with s larger than some $\overline{s}$) determine if the solution $u(t,x)$ of (1.0.1) exists and remains small for a long time $T_\varepsilon \gg T_{\mathrm{loc}}$, improving the local well-posedness time of existence which, for small $\varepsilon > 0$, is of size $T_{\mathrm{loc}} \sim \varepsilon^{-1}$.
- *Almost global existence:* For any $N \in \mathbb{N}$ prove that the time of existence is of order $T_\varepsilon \sim_N \varepsilon^{-N}$ taking s larger than some $\overline{s}$ depending on N.

The results of the present Thesis concern:

1. Local well posedness of the compressible Euler-Korteweg system (1.0.2) on $\mathbb{T}^d$;
2. Long time existence of the Quantum Hydrodynamics equations;
3. Almost global existence of Gravity-Capillary water waves equations with constant vorticity (1.0.3).

More precisely the main results of the present Thesis are the following:

1. **Local well posedness of the Euler-Korteweg equations on $\mathbb{T}^d$ ([32], Chapter 2):** We prove a local well-posedness result for the compressible, irrotational Euler-Korteweg (EK) system
$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho\vec{u}) = 0 \\ \partial_t\vec{u} + \vec{u}\cdot\nabla\vec{u} + \nabla g(\rho) = \nabla\big(K(\rho)\Delta\rho + \frac{1}{2}K'(\rho)|\nabla\rho|^2\big) \qquad \vec{u} = \nabla\phi, \\ \rho(0,x)\in H^s(\mathbb{T}^d),\, \phi(0,x)\in H^s(\mathbb{T}^d) \end{cases} \tag{1.0.2}$$
where $s > \frac{d}{2}+2$ (Theorem 1.1.1). This is the natural minimal regularity assumption for the quasi-linear PDE (1.0.2).

2. **Long time stability for Quantum Hydrodynamics (QHD) system ([74], Chapter 3):** In case
$$K(\rho) = \frac{\kappa}{\rho}, \quad \kappa\in\mathbb{R}^+,$$
system (1.0.2) is studied in the context of Quantum Hydrodynamics. We consider (1.0.2) on irrational d-dimensional torus $\mathbb{T}^d_\nu$ in (1.1.6) ($d = 2, 3$) and we prove, for almost all ν, the long time stability Theorem 1.1.2 which states that if the initial datum $\|\rho_0\|_{H^s} + \|\phi_0\|_{H^s} \le \varepsilon \ll 1$ is small for a large enough $s \gg 1$, then the solution remains of size ε up to a time
$$T_\varepsilon \ge \varepsilon^{-1-\frac{1}{d-1}}\log^{-d-2}\big(1+\varepsilon^{\frac{1}{1-d}}\big) \gg T_{\text{loc}} \sim \varepsilon^{-1}.$$

3. **Almost global existence of gravity-capillary water waves equations with constant vorticity ([33], Chapter 4):** We prove the almost global in time existence Theorem 1.1.3 of small amplitude space periodic solutions of the 1D gravity-capillary water waves equations with constant vorticity
$$\begin{cases} \partial_t\eta = G(\eta)\psi + \gamma\eta\eta_x \\ \partial_t\psi = -g\eta - \dfrac{1}{2}\psi_x^2 + \dfrac{1}{2}\dfrac{(\eta_x\psi_x + G(\eta)\psi)^2}{1+\eta_x^2} + \kappa\partial_x\Big[\dfrac{\eta_x}{(1+\eta_x^2)^{\frac{1}{2}}}\Big] + \gamma\eta\psi_x + \gamma\partial_x^{-1}G(\eta)\psi, \end{cases} \tag{1.0.3}$$
where $G(\eta) := G(\eta, \mathtt{h})$ is the Dirichlet-Neumann operator with depth $\mathtt{h}$ (see (4.1.3)).
In particular we prove that for any value of gravity $g > 0$, vorticity $\gamma\in\mathbb{R}$ and depth $\mathtt{h}\in(0,+\infty]$ and any surface tension $\kappa > 0$ belonging to a full measure set, for any $N\in\mathbb{N}$, any small initial data
$$\|\eta_0\|_{H_0^{s+\frac{1}{4}}} + \|\psi_0\|_{\dot{H}^{s-\frac{1}{4}}} < \varepsilon$$
with $s \gg 1$ give rise to solutions $\eta(t), \psi(t)$ which remain of size $\sim\varepsilon$ up to a time $T_\varepsilon \gtrsim \varepsilon^{-N-1}$, namely
$$\sup_{t\in[-T_\varepsilon, T_\varepsilon]} \|\eta(t)\|_{H_0^{s+\frac{1}{4}}} + \|\psi(t)\|_{\dot{H}^{s-\frac{1}{4}}} \lesssim \varepsilon \quad \text{with} \quad T_\varepsilon \gtrsim \varepsilon^{-N-1}.$$

The main focus of this Thesis is the proof of the almost global in time existence theorem, Theorem 1.1.3, which can be found in Chapter 4. In this introduction, we will place a greater emphasis on explaining this result, examining related literature, and discussing the key concepts and techniques used in its proof.

In the next section we will provide the precise statement of the results.

1.1 Main Results

In this section, we provide a detailed explanation of the results including comments on any new or noteworthy aspects.

1.1.1 Local well-posedness of the Euler-Korteweg equations on $\mathbb{T}^d$

In [32] we consider an initially irrotational velocity field that, under the evolution of (1.0.2), remains irrotational for all times. An irrotational vector field on $\mathbb{T}^d$ can be written as (Helmholtz decomposition)

$$\vec{u} = \vec{c}(t) + \nabla\phi\,, \quad \vec{c}(t) \in \mathbb{R}^d\,, \quad \vec{c}(t) = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} \vec{u}\,\mathrm{d}x\,, \tag{1.1.1}$$

where $\phi : \mathbb{T}^d \to \mathbb{R}$ is a scalar potential. By the second equation in (1.0.2) and $\operatorname{rot}\vec{u} = 0$, we get

$$\partial_t\vec{c}(t) = -\frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} \vec{u}\cdot\nabla\vec{u}\,\mathrm{d}x = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} -\frac{1}{2}\nabla(|\vec{u}|^2)\,\mathrm{d}x = 0 \quad \Longrightarrow \quad \vec{c}(t) = \vec{c}(0)$$

is independent of time. The (EK) system (1.0.2) is Galilean invariant: if $(\rho(t,x), \vec{u}(t,x))$ solves (1.0.2) then

$$\rho_{\vec{c}}(t,x) := \rho_{\vec{c}}(t, x+\vec{c}t)\,, \quad \vec{u}_{\vec{c}}(t,x) := \vec{u}(t, x+\vec{c}t) - \vec{c}$$

solve (1.0.2) as well. Thus, regarding the Euler-Korteweg system in a frame moving with a constant speed $\vec{c}(0)$, we may always consider in (1.1.1) that

$$\vec{u} = \nabla\phi\,, \quad \phi : \mathbb{T}^d \to \mathbb{R}\,.$$

The Euler-Korteweg equations (1.0.2) read, for irrotational fluids,

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho\nabla\phi) = 0 \\ \partial_t\phi + \frac{1}{2}|\nabla\phi|^2 + g(\rho) = K(\rho)\Delta\rho + \frac{1}{2}K'(\rho)|\nabla\rho|^2\,. \end{cases} \tag{1.1.2}$$

Our main contribution is the following local well posedness result for the solutions of (1.1.2) with initial data (ρ_0, ϕ_0) in Sobolev spaces $H^s(\mathbb{T}^d)$ under the natural mild regularity assumption $s > 2 + (d/2)$.

Theorem 1.1.1. (Local existence on $\mathbb{T}^d$) *Let $s > 2 + \frac{d}{2}$ and fix $s_0 \in (\frac{d}{2}, s-2]$. For any initial data*

$$(\rho_0, \phi_0) \in H^s(\mathbb{T}^d, \mathbb{R}) \times H^s(\mathbb{T}^d, \mathbb{R}) \quad \text{with} \quad \rho_0(x) > 0\,, \quad \forall x \in \mathbb{T}^d\,,$$

there exists $T := T(\|(\rho_0, \phi_0)\|_{s_0+2}, \min_x \rho_0(x)) > 0$ and a unique solution (ρ, ϕ) of (1.1.2) such that

$$(\rho, \phi) \in C^0\Big([-T,T], H^s(\mathbb{T}^d, \mathbb{R}) \times H^s(\mathbb{T}^d, \mathbb{R})\Big) \cap C^1\Big([-T,T], H^{s-2}(\mathbb{T}^d, \mathbb{R}) \times H^{s-2}(\mathbb{T}^d, \mathbb{R})\Big)$$

and $\rho(t,x) > 0$ for any $t \in [-T,T]$. Moreover, for $|t| \le T$, the solution map $(\rho_0, \phi_0) \mapsto (\rho(t,\cdot), \phi(t,\cdot))$ is locally defined and continuous in $H^s(\mathbb{T}^d, \mathbb{R}) \times H^s(\mathbb{T}^d, \mathbb{R})$.

The proof of Theorem 1.1.1 is the content of Chapter 2 and is summarized in Section 1.2.1. Here are some comments about it:

- **Regularity:** The velocity field $v = \nabla\phi$ belongs to the Sobolev space $H^{s-1}(\mathbb{T}^d;\mathbb{R}^d)$. In view of the Sobolev embedding $H^\sigma(\mathbb{T}^d;\mathbb{R}) \hookrightarrow L^\infty(\mathbb{T}^d;\mathbb{R})$, $\sigma > \frac{d}{2}$, the restriction $s > 2 + \frac{d}{2}$ is the minimal Sobolev regularity which guarantees that the velocity field v is Lipschitz. The Lipschitz regularity of v allows to define in classical way the fluid particles flow

$$\begin{cases} \dot{X}(t,x) = v(t, X(t,x)) \\ X(0,x) = x \in \mathbb{T}^d. \end{cases} \tag{1.1.3}$$

- **Hamiltonian structure:** Equations (1.1.2) are the Hamiltonian system generated by the Hamiltonian function

$$H(\rho,\phi) := \int_{\mathbb{T}^d} \frac{1}{2}\rho|\nabla\phi|^2 + \frac{1}{2}K(\rho)|\nabla\rho|^2 + G(\rho)\,\mathrm{d}x \tag{1.1.4}$$

where $G(\rho)$ is a primitive of $g(\rho)$, i.e. $G'(\rho) = g(\rho)$; this means that equations (1.1.2) can be written as

$$\partial_t \begin{pmatrix} \rho \\ \phi \end{pmatrix} = J \begin{pmatrix} \nabla_\rho H \\ \nabla_\phi H \end{pmatrix}, \quad J := \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix}, \tag{1.1.5}$$

where $(\nabla_\rho H, \nabla_\phi H)$ denote the L^2-gradients of $H(\rho,\phi)$. The Hamiltonian form, as given in equation (1.1.5), plays a significant role in the local and long time behavior of system (1.1.2). Even though it is not explicitly used, some crucial algebraic properties of the para-linearized system are consequences of the Hamiltonian structure.

- **Extended life span:** We do not know if the local solutions provided by Theorem 1.1.1 are global in time or not. In a forthcoming paper [103], we shall prove a set of long time existence results for the (EK)-system in 1-space dimension, in the same spirit of Theorem 1.1.3- [33].

1.1.2 Long- time stability of QHD system

In [74] we consider the quantum hydrodynamics system on an irrational torus of dimension 2 or 3

$$\begin{cases} \partial_t\rho = -\mathtt{m}\Delta\phi - \mathrm{div}(\rho\nabla\phi) \\ \partial_t\phi = -\frac{1}{2}|\nabla\phi|^2 - g(\mathtt{m}+\rho) + \dfrac{\kappa}{\mathtt{m}+\rho}\Delta\rho - \dfrac{\kappa}{2(\mathtt{m}+\rho)^2}|\nabla\rho|^2, \end{cases} \tag{QHD}$$

where $\mathtt{m} > 0$, $\kappa > 0$, the function g belongs to $C^\infty(\mathbb{R}_+;\mathbb{R})$ and $g(\mathtt{m}) = 0$. The function $\rho(t,x)$ is such that $\rho(t,x) + \mathtt{m} > 0$ and it has zero average in x. The space variable x belongs to the irrational torus

$$\mathbb{T}^d_\nu := (\mathbb{R}/2\pi\nu_1\mathbb{Z}) \times \cdots \times (\mathbb{R}/2\pi\nu_d\mathbb{Z}), \qquad d = 2,3, \tag{1.1.6}$$

with $\nu = (\nu_1, \ldots, \nu_d) \in [1,2]^d$. System (QHD) is a particular case of (1.0.2) when $K(\rho) = \frac{\kappa}{\rho}$ and $\rho \rightsquigarrow \rho + \mathtt{m}$. We assume the *strong* ellipticity condition

$$g'(\mathtt{m}) > 0. \tag{1.1.7}$$

We consider an initial condition (ρ_0, ϕ_0) having small size $\varepsilon \ll 1$ in the standard Sobolev space $H^s(\mathbb{T}^d_\nu)$ with $s \gg 1$. Since the equation has a quadratic nonlinear term, the local existence theory (which may be obtained in the spirit of [32, 72]) implies that the solution of (QHD) remains of size ε for times of magnitude $O(\varepsilon^{-1})$. The aim of our work [74] is to prove that, for *generic irrational tori*, the solution remains of size ε for longer times.

For $\phi \in H^s(\mathbb{T}^d_\nu)$ we define

$$\Pi_0\phi := \frac{1}{(2\pi)^d\nu_1\cdots\nu_d}\int_{\mathbb{T}^d_\nu}\phi(x)dx\,, \qquad \Pi_0^\perp := \mathrm{id} - \Pi_0\,. \tag{1.1.8}$$

Our main result is the following.

Theorem 1.1.2. *Let $d=2$ or $d=3$. There exists $s_0 \equiv s_0(d) > 0$ such that for almost all $\nu \in [1,2]^d$, for any $s \geq s_0$, $\mathtt{m} > 0$, $\kappa > 0$ there exist $C > 0$, $\varepsilon_0 > 0$ such that for any $0 < \varepsilon \leq \varepsilon_0$ we have the following. For any initial data $(\rho_0, \phi_0) \in H^s_0(\mathbb{T}^d_\nu) \times H^s(\mathbb{T}^d_\nu)$ such that*

$$\|\rho_0\|_{H^s(\mathbb{T}^d_\nu)} + \|\Pi_0^\perp\phi_0\|_{H^s(\mathbb{T}^d_\nu)} \leq \varepsilon\,, \tag{1.1.9}$$

there exists a unique solution of (QHD) with $(\rho(0),\phi(0)) = (\rho_0,\phi_0)$ such that

$$\begin{gathered}(\rho(t),\phi(t)) \in C^0\big([0,T_\varepsilon);H^s(\mathbb{T}^d_\nu)\times H^s(\mathbb{T}^d_\nu)\big)\bigcap C^1\big([0,T_\varepsilon);H^{s-2}(\mathbb{T}^d_\nu)\times H^{s-2}(\mathbb{T}^d_\nu)\big)\,,\\ \sup_{t\in[0,T_\varepsilon)}\Big(\|\rho(t,\cdot)\|_{H^s(\mathbb{T}^d_\nu)} + \|\Pi_0^\perp\phi(t,\cdot)\|_{H^s(\mathbb{T}^d_\nu)}\Big) \leq C\varepsilon\,, \qquad T_\varepsilon \geq \varepsilon^{-1-\frac{1}{d-1}}\log^{-d-2}\big(1+\varepsilon^{\frac{1}{1-d}}\big)\,.\end{gathered} \tag{1.1.10}$$

The proof of Theorem 1.1.2 is the content of Chapter 3 and is summarized in Section 1.2.2. Here are some comments about it:

- **Madelung transform:** System (QHD) is equivalent via the so-called Madelung transformation
$$\psi = \sqrt{\mathtt{m}+\rho}\,e^{\mathrm{i}\frac{\phi}{2\sqrt{\kappa}}} \tag{1.1.11}$$
to the following semi-linear Hamiltonian Schrödinger equation
$$\partial_t\psi = \mathrm{i}\Big(\frac{\hbar}{2}\Delta\psi - \frac{1}{\hbar}g(|\psi|^2)\psi\Big)\,, \quad \hbar := 2\sqrt{\kappa}.$$
- **Linear frequencies:** Thanks to (1.1.7), the linearized system near the equilibrium $(\rho,\phi) = (0,0)$ is a superposition of infinitely many harmonic oscillators with frequencies
$$\omega(j) := \sqrt{\frac{\hbar^2}{4}|j|^4_\nu + \mathtt{m}g'(\mathtt{m})|j|^2_\nu}\,, \quad |j|^2_\nu := \sum_{\ell=1}^d a_\ell|j_\ell|^2\,, \quad a_\ell := \nu_\ell^2\,, \quad \forall j \in \mathbb{Z}^d\setminus\{0\}. \tag{1.1.12}$$
- **Normal form and quasi-resonances:** The proof of Theorem 1.1.2 relies on Birkhoff normal form ideas and absence of three wave resonances
$$\omega(j_1) \pm \omega(j_2) \pm \omega(j_3) \neq 0 \quad \text{for almost every } \nu \in [1,2]^d. \tag{1.1.13}$$
The main obstacle to Birkhoff normal form are the presence of bad lower bounds of the three wave interactions, namely
$$|\omega(j_1) \pm \omega(j_2) \pm \omega(j_3)| \gtrsim \frac{1}{\mu_1^{d-1}\log^{d+1}\big(1+\mu_1^2\big)\mu_3^{M(d)}}\,, \tag{1.1.14}$$
where $\mu_1 := \max\{|j_1|,|j_2|,|j_3|\}$, $\mu_3 := \min\{|j_1|,|j_2|,|j_3|\}$ and some constant $M(d) > 0$. The lower bound in (1.1.14) allows a loss of derivatives with respect to the highest index μ_1. This loss of derivatives may be transformed in a loss of length of the lifespan following the ideas first introduced by Delort [56] and Ionescu-Pusateri [87]. For this reason we do not obtain the ε^{-2} life span which one expects from (1.1.13);

- **The square torus case:** If we consider the system (QHD) with x belonging to a square torus $\mathbb{T}^d$, then the corresponding linear frequencies $\omega(j)$ (in (1.1.12) with $\nu_1 = \cdots = \nu_d$) satisfy a good separation property and one can prove that an improved lower bound

$$|\omega(j_1) \pm \cdots \pm \omega(j_p)| \gtrsim \frac{1}{\mu_3^{M(d)}}, \tag{1.1.15}$$

holds for any $p \geq 3$, for almost every mass $\mathtt{m} > 0$ and $\mu_3 := \max_3\{|j_1|, \dots |j_p|\}$ is the third largest index among $\{|j_1|, \dots |j_p|\}$. Having (1.1.15) one can prove an almost-global stability result as in [15, 19] using also the Hamiltonian structure of the system.

- **Recent new developments:** We point out that our result has inspired recent new developments by Bambusi-Feola-Montalto [18]. They have achieved an almost global in time existence result, with a time of existence $T_\varepsilon \sim \varepsilon^{-r}$ for any $r \geq 1$, for several Schrödinger type equations, including equation (QHD).

1.1.3 Almost global existence for Gravity-Capillary water waves equations with constant vorticity

We consider the Euler equations of hydrodynamics for a 2-dimensional perfect, incompressible, inviscid fluid *with constant vorticity* γ, under the action of *gravity* and *capillary* forces at the free surface. The fluid fills an ocean with depth $\mathtt{h} > 0$ (eventually infinite) and with space periodic boundary conditions, namely it occupies the time dependent region

$$\mathcal{D}_\eta := \big\{(x, y) \in \mathbb{T} \times \mathbb{R} \ : \ -\mathtt{h} < y < \eta(t, x)\big\}. \tag{1.1.16}$$

We refer to Appendix C for a rigorous derivation of the equation of motions and we describe here its main aspects. In case of a fluid with constant vorticity $v_x - u_y \equiv \gamma \in \mathbb{R}$, we express the velocity field, $\begin{pmatrix} u \\ v \end{pmatrix}$, as the sum of the Couette flow $\begin{pmatrix} \gamma y \\ 0 \end{pmatrix}$, which carries all the vorticity γ of the fluid, and an irrotational vector field, expressed as the gradient of a harmonic function Φ, called the *generalized velocity potential*:

$$\vec{v} = \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \gamma y \\ 0 \end{pmatrix} + \nabla\Phi.$$

Denoting by $\psi(t, x)$ the evaluation of the generalized velocity potential at the free interface $\psi(t, x) := \Phi(t, \eta(t, x))$, one recovers Φ by solving the elliptic problem

$$\Delta\Phi = 0 \ \text{ in } \ \mathcal{D}_\eta, \quad \Phi = \psi \ \text{ at } \ y = \eta(t, x), \quad \Phi_y = 0 \ \text{ as } \ y \to -\mathtt{h}. \tag{1.1.17}$$

The third condition in (1.1.17) is the impermeability property of the bottom and means that fluid particles can not cross the bottom. At the moving free surface $y = \eta(t, x)$ we impose the so-called kinematic boundary condition

$$v = \eta_t + u\eta_x \quad \text{ at } y = \eta(t, x). \tag{1.1.18}$$

Another boundary condition is a balance law for the pressure at the free surface $y = \eta(t, x)$ and means that the difference between the internal pressure of the fluid and the atmospheric pressure is compensate by the surface tension:

$$P = P_0 - \kappa \left(\frac{\eta_x}{\sqrt{1 + \eta_x^2}} \right)_x \quad \text{at } y = \eta(t, x).$$

The problem can be described by a closed system for the two time dependent functions $\eta(t,x), \psi(t,x)$:

$$\begin{cases} \partial_t \eta = G(\eta)\psi + \gamma\eta\eta_x \\ \partial_t \psi = -g\eta - \dfrac{1}{2}\psi_x^2 + \dfrac{1}{2}\dfrac{(\eta_x\psi_x + G(\eta)\psi)^2}{1+\eta_x^2} + \kappa\partial_x\Big[\dfrac{\eta_x}{(1+\eta_x^2)^{\frac{1}{2}}}\Big] + \gamma\eta\psi_x + \gamma\partial_x^{-1}G(\eta)\psi, \end{cases} \tag{1.1.19}$$

where $g > 0$ is the acceleration due to gravity, $\kappa > 0$ is the surface tension coefficient and $G(\eta)$ is the Dirichlet-Neumann operator

$$G(\eta)\psi := (-\Phi_x\eta_x + \Phi_y)|_{y=\eta(x)} \,. \tag{1.1.20}$$

The linearized Dirichlet-Neumann operator is given by the Fourier multiplier

$$G(0) = \begin{cases} |D| & \mathtt{h} = +\infty \\ D\tanh(\mathtt{h}D) & 0 < \mathtt{h} < +\infty \end{cases} \quad \text{where} \quad D := \frac{1}{\mathrm{i}}\partial_x. \tag{1.1.21}$$

Since the variable $\psi(t,x)$ is the trace at $y = \eta(t,x)$ of the potential $\Phi(x,y)$, it is defined up to a constant. As a consequence (1.1.19) depends only on $\psi - \frac{1}{2\pi}\int_{\mathbb{T}}\psi\mathrm{d}x$. For this reason we consider ψ belonging to the homogeneous Sobolev space

$$\dot{H}^s(\mathbb{T};\mathbb{R}) := {H^s(\mathbb{T};\mathbb{R})}\big/{\sim}, \quad \psi_1 \sim \psi_2 \iff \psi_1 - \psi_2 \equiv c \in \mathbb{R}.$$

Moreover the mass

$$\int_{\mathbb{T}} \eta(t,x)\,\mathrm{d}x \tag{1.1.22}$$

is constant along the evolution of (1.1.19) and is not restrictive to fix it to be zero. For this reason we consider the variable $\eta(t,x)$ in the Sobolev space of zero mean functions

$$H_0^s(\mathbb{T};\mathbb{R}) := \left\{ \eta \in H^s(\mathbb{T};\mathbb{R}) \colon \int_{\mathbb{T}} \eta(t,x)\,\mathrm{d}x = 0 \right\}. \tag{1.1.23}$$

Our main contribution is the following:

Theorem 1.1.3. (Almost global in time gravity-capillary water waves with constant vorticity) *For any value of the gravity $g > 0$, depth $\mathtt{h} \in (0,+\infty]$ and vorticity $\gamma \in \mathbb{R}$, there is a zero measure set $\mathcal{K} \subset (0,+\infty)$ such that, for any surface tension coefficient $\kappa \in (0,+\infty) \setminus \mathcal{K}$, for any N in $\mathbb{N}_0$, there is $s_0 > 0$ and, for any $s \geq s_0$, there are $\varepsilon_0 > 0, c > 0, C > 0$ such that, for any $0 < \varepsilon < \varepsilon_0$, any initial datum*

$$(\eta_0,\psi_0) \in H_0^{s+\frac{1}{4}}(\mathbb{T},\mathbb{R}) \times \dot{H}^{s-\frac{1}{4}}(\mathbb{T},\mathbb{R}) \quad \textit{with} \quad \|\eta_0\|_{H_0^{s+\frac{1}{4}}} + \|\psi_0\|_{\dot{H}^{s-\frac{1}{4}}} < \varepsilon\,,$$

system (1.1.19) has a unique classical solution (η,ψ) in

$$C^0\Big([-T_\varepsilon,T_\varepsilon], H_0^{s+\frac{1}{4}}(\mathbb{T},\mathbb{R}) \times \dot{H}^{s-\frac{1}{4}}(\mathbb{T},\mathbb{R})\Big) \quad \textit{with} \quad T_\varepsilon \geq c\varepsilon^{-N-1}\,, \tag{1.1.24}$$

satisfying the initial condition $\eta|_{t=0} = \eta_0, \psi|_{t=0} = \psi_0$. Moreover

$$\sup_{t\in[-T_\varepsilon,T_\varepsilon]} \big(\|\eta\|_{H_0^{s+\frac{1}{4}}} + \|\psi\|_{\dot{H}^{s-\frac{1}{4}}}\big) \leq C\varepsilon\,. \tag{1.1.25}$$

The proof of Theorem 1.1.3 is the content of Chapter 4 and is summarized in Section 1.2.3. Here are some comments about it:

1. HAMILTONIAN AND TRANSLATION INVARIANT STRUCTURE: By [124, 50, 45, 115] the equations (1.1.19) are the Hamiltonian system

$$\partial_t \begin{pmatrix} \eta \\ \psi \end{pmatrix} = J_\gamma \begin{pmatrix} \nabla_\eta H_\gamma(\eta,\psi) \\ \nabla_\psi H_\gamma(\eta,\psi) \end{pmatrix} \qquad \text{where} \qquad J_\gamma := \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & \gamma\partial_x^{-1} \end{pmatrix} \tag{1.1.26}$$

and

$$H_\gamma(\eta,\psi) := \frac{1}{2}\int_{\mathbb{T}} \big(\psi G(\eta)\psi + g\eta^2\big)\,\mathrm{d}x + \kappa \int_{\mathbb{T}} \sqrt{1+\eta_x^2}\,\mathrm{d}x + \frac{\gamma}{2}\int_{\mathbb{T}} \big(-\psi_x\eta^2 + \frac{\gamma}{3}\eta^3\big)\,\mathrm{d}x\,. \tag{1.1.27}$$

The L^2-gradients $(\nabla_\eta H_\gamma, \nabla_\psi H_\gamma)$ in (1.1.26) belong to (a dense subspace of) $\dot{L}^2(\mathbb{T}) \times L^2_0(\mathbb{T})$.

Since the bottom of $\mathcal{D}_\eta$ in (1.1.16) is flat, the Hamiltonian vector field X_γ, defined by the right hand side of (1.1.19), is translation invariant, namely

$$X_\gamma \circ \tau_\varsigma = \tau_\varsigma \circ X_\gamma\,, \quad \forall \varsigma \in \mathbb{R}\,, \qquad \text{where} \qquad \tau_\varsigma\colon f(x) \mapsto f(x+\varsigma) \tag{1.1.28}$$

is the translation operator. Equivalently the Hamiltonian H_γ in (1.1.27) satisfies $H_\gamma \circ \tau_\varsigma = H_\gamma$ for any $\varsigma \in \mathbb{R}$. By Noether theorem it induces the momentum $\int_{\mathbb{T}} \psi(x)\eta_x(x)\,\mathrm{d}x$ as a prime integral.

2. COMPARISON WITH [27]. We discuss the relation between Theorem 1.1.3 and the result in Berti-Delort [27]. Theorem 1.1.3 extends the result in [27] in two ways: (i) the equations (1.1.19) may have a *non zero vorticity*, whereas the water waves in [27] are irrotational, i.e. $\gamma = 0$. (ii) Also in the irrotational case Theorem 1.1.3 is new since the almost global existence result in [27] holds for initial data (η_0,ψ_0) even in x, whereas Theorem 1.1.3 applies to *any* (η_0,ψ_0). We remark that, in the irrotational case, the subspace of functions even in x -the so called standing waves- is invariant under evolution, whereas for $\gamma \neq 0$ it is not invariant under the flow of (1.1.19) and the approach of [27] can not be applied here.

3. PERIODIC SETTING VS $\mathbb{R}^d$. In a varieties of different scenarios, global in time results [76, 120, 85, 4, 77, 82, 86, 62] have been proved for irrotational water waves equations on $\mathbb{R}^d$ for sufficiently small, localized in space and regular enough initial data, exploiting the dispersive effects of the linear flow. So far no global existence is known for (1.1.19) in $\mathbb{R}^2$, not even for irrotational fluids. The breakthrough result [62] proves global existence in $\mathbb{R}^3$ if $\gamma = 0$. The periodic setting is deeply different, as the linear waves oscillate without decaying in time. The long time dynamics of the equations strongly depends on the presence of N*-wave resonant interactions* and the Hamiltonian and reversible nature of the equations.

4. DISPERSION RELATION AND NON-RESONANT PARAMETERS. The water waves equations (1.1.19) may be regarded as a *quasi-linear* complex PDE of the form

$$\partial_t u = -\mathrm{i}\Omega(D)u + \mathcal{N}(u,\overline{u}), \quad u(x) = \tfrac{1}{\sqrt{2\pi}}\sum\nolimits_{j\in\mathbb{Z}\setminus\{0\}} u_j\, e^{\mathrm{i}jx}\,,$$

where $\mathcal{N}$ is a quadratic non-linearity and $\Omega_j(\kappa)$ is the *dispersion relation*

$$\Omega_j(\kappa) := \omega_j(\kappa) + \frac{\gamma}{2}\frac{\mathtt{G}(j)}{j}\,, \qquad \omega_j(\kappa) := \sqrt{\mathtt{G}(j)\Big(g + \kappa j^2 + \frac{\gamma^2}{4}\frac{\mathtt{G}(j)}{j^2}\Big)}\,, \tag{1.1.29}$$

where $\mathtt{G}(\xi) = |\xi|\tanh(\mathtt{h}|\xi|)$ $(= |\xi|$ in infinite depth) is the symbol of the Dirichlet-Neumann operator $G(0)$. The linear frequencies $\Omega_j(\kappa)$ actually depend on $(\kappa, g, \mathtt{h}, \gamma)$. The restriction on such parameters in Theorem 1.1.3 arises to ensure the absence of N-wave resonant interactions

$$\Omega_{j_1}(\kappa) \pm \ldots \pm \Omega_{j_N}(\kappa) \neq 0 \tag{1.1.30}$$

(with quantitative lower bounds, see (4.1.13)) among integer indices $j_1, \ldots, j_N$ which are not super-action preserving, cfr. Definition 4.7.4. In Theorem 1.1.3 we fix $(g, \mathtt{h}, \gamma)$ and required $\kappa \notin \mathcal{K}$, but other choices are possible.

5. COMPARISON WITH THE APPROACH IN [58, 59] AND [27]. The Hamiltonian approach to para-differential calculus in [58, 59] is developed for quasi-linear Klein-Gordon equations and can not be applied to prove Theorem 1.1.3. Indeed for Klein-Gordon equations it is not required a reduction to x-independent para-differential operators up to smoothing remainders, since the dispersion relation is asymptotically linear. In this case, since the commutator between first order para-differential operators is still a first order para-differential operator, it is possible to implement a Hamiltonian Birkhoff normal form reduction in degrees of homogeneity, similarly to the semi-linear case. This approach can not be applied for (1.1.19) since its dispersion relation (1.1.29) is super-linear. It is for this reason that we shall reduce in Proposition 4.7.2 the para-linearized water waves equations to x-independent symbols up to smoothing remainders. This was done in [27] for $\gamma = 0$ (in a different way) but breaking the Hamiltonian structure. Incidentally we mention that the para-differential normal form in [27] is not a Birkhoff normal form: for standing waves it is not needed to reduce the x-independent symbols to Birkhoff normal form to deduce that the actions $|u_n|^2$ are prime integrals. Summarizing, the proof of Theorem 1.1.3 demands

- a reduction of the water waves equations (1.1.19) to para-differential x-independent symbols up to smoothing remainders, done in [27] for $\gamma = 0$ (in a different way) losing the Hamiltonian structure, and, additionally, reduce the x-independent symbols to super-action preserving Birkhoff normal form;
- preserve the Hamiltonian structure of the Birkhoff normal form, goal achieved in [58, 59] but only for Klein-Gordon equations.

The resolution of these requirements is a main achievement of our work.

1.1.4 Historical background of water waves

The study of the time evolution of water waves is a classical question, and one of the major problems in fluid dynamics. The very first attempt at a theory of water waves finds its origin in the *Principia* of Newton in the second half of the seventeenth century, but another century is needed for the foundational works by giants such as Euler, Laplace, Lagrange, Cauchy, Poisson, Bernoulli, and then by the British school with Russel, Robinson, Green, Airy, Stokes among others. We refer to the historical overview in [48]. We will now provide an overview of some recent and past findings.

Local Well-posedness. First results on the local existence of solutions to the initial value problem of the pure gravity water waves equations within a Sobolev class can be traced back to the pioneering works of Nalimov [104], Yosihara [122], and Craig [49], who studied the problem in one space dimension and under smallness assumptions on the initial data. For large Cauchy data, local existence in infinite depth has been proved by S. Wu in the breakthrough works [117] and [118] for 3D fluids. The similar question for a variable bottom in any dimension has been solved by Lannes [92], see also [93]. The case of local existence for the free surface incompressible Euler equation has been settled by Lindblad [95]. Arbitrary bottoms have been considered by Alazard, Burq and Zuily [3] for rough initial data. Concerning the case of positive κ, local existence of solutions with data in Sobolev spaces is due to Beyer and Gunther [35], Ming-Zhang [100], Coutand- Shkroller [46] and Shatah-Zeng [111, 112], for solutions of the incompressible free boundary Euler equation. Ifrim and Tataru [84] studied local existence when the pure gravity fluid has constant vorticity. The local existence problem with Cauchy data that are periodic in space, instead of lying in a Sobolev space on $\mathbb{R}^d$, has been established in Ambrose-Masmoudi [7] for $\kappa \geq 0$ in the case of infinite depth, and by Schweizer [109] for finite depth, even with a non zero vorticity. Non-localized Cauchy data lying in uniformly local spaces have been treated by Alazard, Burq and Zuily, in the case of arbitrary rough bottoms [2, 3]. Thanks to all the contributions mentioned above, the local well-posedness theory is presently well-understood in a variety of different scenarios.

Global in time existence results on $\mathbb{R}^d$. For initial data on the line which are sufficiently small, smooth and decaying at infinity, global existence results have been proved exploiting the dispersive properties of the flow. All the results we are aware of concern irrotational flows. The first contribution has been given by S. Wu [119] for a two dimensional fluid of infinite depth, proving that the solutions of the water waves equations with $\kappa = 0$ exist over a time interval of exponential length $e^{c/\varepsilon}$ when the size ε of the initial data goes to zero. For three dimensional fluids, global existence with small decaying data has been obtained independently by Germain, Masmoudi and Shatah [76] and by S. Wu [120]. Global existence for small data in one space dimension has been proved independently by Ionescu and Pusateri [85], Alazard and Delort [4] and by Ifrim and Tataru [82], for infinite depth fluids. For the capillary-gravity irrotational water waves equations global existence is known for three dimensional fluids in infinite depth by Deng, Ionescu, Pausader and Pusateri [62], the problem in 1D is still open. When the surface tension is positive, but the gravity g vanishes, global solutions in infinite depth fluids have been proved to exist by Germain, Masmoudi and Shatah [77] in dimension 2 and by Ionescu and Pusateri [86] in dimension 1.

Periodic and quasi-periodic solutions. For non-localized initial data, oscillation does not produce decay, resulting in a lack of global in time existence results. Despite this, several global space periodic and quasi-periodic solutions are known for water waves. One of the most well-known examples is the Stokes wave, which is a one dimensional traveling steady periodic wave. There is a huge literature about Stokes waves and we refer the interested reader to [44] for a more detailed explanation. Here we only mention that, after the pioneering work of Stokes [114], the first rigorous construction of small amplitude space periodic steady traveling waves goes back to the 1920's with the papers of Nekrasov [105], Levi-Civita [94] and Struik [113] for irrotational pure gravity waves. We also mention Zeidler [126] in which traveling waves are constructed in presence of the effects of capillarity and Wahlén [115], Martin [97] for constant vorticity fluids. Another class of solutions of water waves are known as periodic standing waves, which are periodic solutions that are even in both space and time. These solutions were constructed by Plotnikov-Toland in [106] and Iooss-Plotnikov-Toland in [88] for irrotational gravity waves and by Alazard-Baldi in [1] for irrotational gravity-capillary waves. Finally, we mention that quasi-periodic solutions have been constructed for water waves in [34, 14, 31, 69] using KAM techniques.

For the general Cauchy problem of water waves with periodic boundary conditions, as previously mentioned, there are no dispersive effects that can be used to control the solutions for all times through decay over time. Additionally, the quasi-linear nature of the equations prevents the use of semi-linear techniques. To study the long time dynamics of water waves we implement normal form methods. Before discussing the general principles of Birkhoff normal form method we briefly mention some recent findings about the long time existence for water waves.

Long time existence of water waves. For initial data of size ε we indicate below with T_ε the corresponding maximal time of existence and we outline different long time existence results proved in literature for *space* periodic water waves, with or without capillarity and vorticity.

(i) $T_\varepsilon \geq c\varepsilon^{-1}$. The local well posedness theory for free boundary Euler equations has been developed along several years in different scenarios in [104, 122, 49, 117, 118, 92, 2, 3, 35, 100, 109, 95, 46, 111, 112, 84, 7]. As a whole they prove the existence, for sufficiently nice initial data, of classical smooth solutions on a small time interval. When specialized to initial data of size ε in some Sobolev space, imply a time of existence larger than $c\varepsilon^{-1}$ (the non-linearity in (1.1.19) vanishes quadratically at zero). We remark that other large initial data can lead to breakdown in finite time, see for example the papers [43, 47] on "splash" singularities.

(ii) $T_\varepsilon \geq c\varepsilon^{-2}$. Wu [119], Ionescu-Pusateri [85], Alazard-Delort [4] for pure gravity waves, and Ifrim-Tataru [83], Ionescu-Pusateri [86] for $\kappa > 0$, $g = 0$ and $\mathtt{h} = +\infty$, proved that small data of size ε (periodic or on the line) give rise to irrotational solutions defined on a time interval at least $c\varepsilon^{-2}$. We quote [84] for $\kappa = 0$, $g > 0$, infinite depth and constant vorticity, [80] for irrotational fluids, and [79] in finite depth. All the previous results hold in absence of three wave interactions. Exploiting the Hamiltonian nature of the water waves equations, Berti-Feola-Franzoi [28] proved a $c\varepsilon^{-2}$ lower bound for the time of existence. The interesting fact is that in these cases three wave interactions may occur, giving rise to the well known Wilton ripples in fluid mechanics literature. We finally mention the $\varepsilon^{-\frac{5}{3}+}$ long time existence result [87] for periodic 2D gravity-capillary water waves.

(iii) $T_\varepsilon \geq c\varepsilon^{-3}$. A time of existence larger than $c\varepsilon^{-3}$ has been recently proved for the pure gravity water waves equations in deep water in Berti-Feola-Pusateri [29]. In this case four wave interactions may occur, but the Hamiltonian Birkhoff normal form turns out to be completely integrable by the formal computation in Zakharov-Dyachenko [125]. This result has been recently extended by S. Wu [121] for a larger class of initial data, developing a novel approach in configuration space, and, even more recently, by Deng-Ionescu-Pusateri [63] for waves with large period.

(iv) $T_\varepsilon \geq c_N\varepsilon^{-N}$ *for any* N. Berti-Delort [27] proved, for almost all the values of the surface tension $\kappa \in (0, +\infty)$, an almost global existence result as in Theorem 4.1.1 for the solutions of (1.1.19) in the case of *zero* vorticity $\gamma = 0$ *and* for initial data (η_0, ψ_0) even in x. The restriction on the capillary parameter arises to imply the absence of N-wave interactions, for any N. In [31] we prove Theorem 1.1.3 which extends the result of Berti-Delort to general periodic initial data (not only even in x) and for general constant vorticity $\gamma \in \mathbb{R}$ (not only $\gamma = 0$).

1.1.5 Birkhoff normal form

In the last years several authors investigated whether there is a stable behavior of solutions of small amplitude for several dispersive equations with periodic boundary conditions. Except for very specific PDEs, which are integrable, the most general and effective approach appears to be the Birkhoff normal form one. Here we outline the main concepts and challenges involved in implementing this technique.

Notation: Let $\begin{pmatrix} u \\ \overline{u} \end{pmatrix}$ be a couple of complex functions. We expand $u(x)$ using the Fourier basis $\frac{e^{\mathrm{i}jx}}{\sqrt{2\pi}}$ as

$$u(x) = \sum_{j\in\mathbb{Z}} u_j^+ \frac{e^{\mathrm{i}jx}}{\sqrt{2\pi}}, \quad \text{where} \quad u_j^+ := \widehat{u}_j = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}} u(x) e^{-\mathrm{i}jx}\,\mathrm{d}x \tag{1.1.31}$$

and the function $\overline{u}(x)$ using the Fourier basis $\frac{e^{-\mathrm{i}jx}}{\sqrt{2\pi}}$ as

$$\overline{u}(x) = \sum_{j\in\mathbb{Z}} u_j^- \frac{e^{-\mathrm{i}jx}}{\sqrt{2\pi}}, \quad \text{where} \quad u_j^- := \overline{\widehat{u}_j}. \tag{1.1.32}$$

The aim is to study the long time dynamics of the Hamiltonian PDE

$$\partial_t \begin{pmatrix} u \\ \overline{u} \end{pmatrix} = X_H(u) \quad \text{where} \quad X_H(u) := \begin{pmatrix} -\mathrm{i}\nabla_{\overline{u}} H(u,\overline{u}) \\ \mathrm{i}\nabla_u H(u,\overline{u}) \end{pmatrix}, \tag{1.1.33}$$

and, for simplicity's sake, we assume that H is a real valued polynomial Hamiltonian function of the form

$$H(u,\overline{u}) = H^{(2)}(u,\overline{u}) + H^{(3)}(u,\overline{u}) + \cdots + H^{(M)}(u,\overline{u}) \tag{1.1.34}$$

where:
• The quadratic Hamiltonian $H^{(2)}$ has the form

$$H^{(2)}(u,\overline{u}) = \sum_{j\in\mathbb{Z}^d} \Omega(j) u_j \overline{u}_j \tag{1.1.35}$$

with dispersion relation $\Omega : \mathbb{Z} \to \mathbb{R}$;
• For each $p = 3,\dots,M$, $H^{(p)}$ is a p-homogeneous Hamiltonian of the form

$$H^{(p)}(u,\overline{u}) = \sum_{\substack{j_1,\dots,j_p\in\mathbb{Z}^d \\ \sigma_1,\dots,\sigma_p\in\{\pm\}}} H^{\sigma_1,\dots,\sigma_p}_{j_1,\dots,j_p} u^{\sigma_1}_{j_1}\cdots u^{\sigma_p}_{j_p}, \quad p = 3,\dots,M \tag{1.1.36}$$

and $H^{\sigma_1,\dots,\sigma_p}_{j_1,\dots,j_p} \in \mathbb{C}$ are complex coefficients satisfying, since H is real valued,

$$\overline{H^{\sigma_1,\dots,\sigma_p}_{j_1,\dots,j_p}} = H^{-\sigma_1,\dots,-\sigma_p}_{j_1,\dots,j_p}. \tag{1.1.37}$$

To control the solutions of (1.1.34) for small initial data $\|u(0)\|_{H^s} \leq \varepsilon \ll 1$ for long time, we want to prove an *a priori* estimate of the form

$$\|u(t)\|^2_{H^s} \lesssim_{s,N} \|u(0)\|^2_{H^s} + \int_0^t \|u(\tau)\|^{N+2}_{H^s}\, \mathrm{d}\tau \tag{1.1.38}$$

which implies that

$$\|u(t)\|_{H^s} \lesssim_s \varepsilon, \quad \text{for any } |t| \leq T_\varepsilon \sim \varepsilon^{-N}.$$

As it will be clear in a moment, an estimate of the form (1.1.38) strongly depends on both the dispersion relation $\Omega(j)$ and on the perturbations $H^{(p)}$, $p = 3,\dots,M$.

Birkhoff normal form for semi-linear PDEs was first developed in the foundational papers by Bambusi [15], Bambusi-Grebért [19], Delort-Szeftel [60, 61] and Bambusi-Delort-Grebért-Szeftel [17]. Below we describe some aspects of that theory in a simplified setting.

Translation invariant Hamiltonian: We assume that the Hamiltonian H is invariant by translation, namely

$$H(u(\cdot+\tau),\overline{u}(\cdot+\tau)) = H(u,\overline{u}) \quad \text{for any} \quad \tau\in\mathbb{R}.$$

This property leads to the following Fourier restriction: for each homogeneous component $H^{(p)}$, one has

$$H^{\sigma_1,\dots,\sigma_p}_{j_1,\dots,j_p} \neq 0 \quad \Rightarrow \quad \sigma_1 j_1 + \cdots + \sigma_p j_p = 0. \tag{1.1.39}$$

An immediate consequence of (1.1.39) is that, on the support of each homogeneous components $H^{(p)}$, one has

$$\max\{|j_1|,\dots,|j_p|\} \sim \max{}_2\{|j_1|,\dots,|j_p|\} \tag{1.1.40}$$

where $\max_2\{|j_1|,\dots,|j_p|\}$ is the second largest number among $\{|j_1|,\dots,|j_p|\}$.

Semi-linear Hamiltonian: We assume that the p-homogeneous Hamiltonian $H^{(p)}$, $p = 3,\dots,M$, are semi-linear, namely we require that its coefficients satisfy, for some $\mu, C > 0$

$$|H^{\sigma_1,\dots,\sigma_p}_{j_1,\dots,j_p}| \leq C \max{}_3\{|j_1|,\dots,|j_p|\}^\mu, \quad \text{for any} \quad j_1,\dots,j_p\in\mathbb{Z}, \sigma_1,\dots,\sigma_p\in\{\pm\} \tag{1.1.41}$$

where $\max_3\{|j_1|,\ldots,|j_p|\}$ is the third largest number among $\{|j_1|,\ldots,|j_p|\}$. We denote the space of p-homogeneous semi-linear Hamiltonian functions as $\mathcal{P}_p^0$.

In view of the restrictions (1.1.39) and (1.1.40), one can prove that if $H^{(p)}$ is a semi-linear Hamiltonian then an important consequence of (1.1.41) is that the Hamiltonian vector field $X_{H^{(p)}}$ (defined as in (1.1.33)) is bounded on Sobolev spaces and there is $s_0 > 0$ such that for any $s \geq s_0$ one has

$$\|X_{H^{(p)}}(u,\overline{u})\|_{H^s} \lesssim_s \|u\|_{H^{s_0}}^{p-2}\|u\|_{H^s}. \tag{1.1.42}$$

Symplectic structure of the phase space: On the phase space we define the standard symplectic 2-form

$$\Omega_c\Big[\begin{pmatrix}u\\ \overline{u}\end{pmatrix},\begin{pmatrix}v\\ \overline{v}\end{pmatrix}\Big] := \int_{\mathbb{T}^d} \mathrm{i}(u(x)\overline{v}(x) - \overline{u}(x)v(x))\,\mathrm{d}x = \mathrm{i}\sum_{j\in\mathbb{Z}}(u_j\overline{v_j} - \overline{u_j}v_j). \tag{1.1.43}$$

The standard symplectic 2- form characterizes the Hamiltonian vector field X_H in (1.1.34), associated to the Hamiltonian H, as

$$\mathrm{d}H(u,\overline{u})\Big[\begin{pmatrix}v\\ \overline{v}\end{pmatrix}\Big] = \Omega_c\Big[X_H(u),\begin{pmatrix}v\\ \overline{v}\end{pmatrix}\Big], \quad X_H(u) = \begin{pmatrix}-\mathrm{i}\nabla_{\overline{u}}H(u,\overline{u})\\ \mathrm{i}\nabla_u H(u,\overline{u})\end{pmatrix}. \tag{1.1.44}$$

The symplectic 2-form induces the following Poisson bracket: given two Hamiltonian functions H and G we define

$$\{H,G\} := \mathrm{d}H[X_G] = \Omega_c\big[X_H,X_G\big] = \sum_{j\in\mathbb{Z}^d}\mathrm{i}\Big(\partial_{\overline{u_j}}H\partial_{u_j}G - \partial_{u_j}F\partial_{\overline{u_j}}H\Big). \tag{1.1.45}$$

A diffeomorphism $\phi : L^2 \to L^2$ is symplectic (or canonical) if

$$\phi^*\Omega_c := \Omega_c[\mathrm{d}\phi[\cdot],\mathrm{d}\phi[\cdot]] = \Omega_c. \tag{1.1.46}$$

If ϕ is symplectic and u solves the Hamiltonian system (1.1.33) then the variable $v = \phi^{-1}(u)$ solves the Hamiltonian system (see e.g. Lemma 4.3.15)

$$\dot{v} = X_{\widetilde{H}}(v), \quad \widetilde{H}(v) = H\circ\phi(v). \tag{1.1.47}$$

In other words, a symplectic change of variable preserves the Hamiltonian structure of the equations.

Hamiltonian structure in Fourier coordinates: The Hamiltonian system (1.1.33) can be written in Fourier coordinates as

$$\begin{aligned}\dot{u}_j &= -\mathrm{i}\partial_{\overline{u}_j}H(u,\overline{u})\\ &= -\mathrm{i}\Omega(j) - \mathrm{i}\partial_{\overline{u}_j}H^{(3)}(u,\overline{u}) - \cdots - \mathrm{i}\partial_{\overline{u}_j}H^{(M)}(u,\overline{u}).\end{aligned}$$

If $H^{(2)}$ is the quadratic Hamiltonian as in (1.1.36) and $G^{(p)}$ has the expansion

$$G^{(p)}(u,\overline{u}) = \sum_{\substack{j_1,\ldots,j_p\in\mathbb{Z}^d\\ \sigma_1,\ldots,\sigma_p\in\{\pm\}}} G_{j_1,\ldots,j_p}^{\sigma_1,\ldots,\sigma_p}u_{j_1}^{\sigma_1}\cdots u_{j_p}^{\sigma_p} \tag{1.1.48}$$

then the Poisson bracket reads

$$\{H^{(2)},G^{(p)}\} = \sum_{\substack{j_1,\ldots,j_p\in\mathbb{Z}^d\\ \sigma_1,\ldots,\sigma_p\in\{\pm\}}} \mathrm{i}[\sigma_1\Omega(j_1) + \cdots + \sigma_p\Omega(j_p)]G_{j_1,\ldots,j_p}^{\sigma_1,\ldots,\sigma_p}u_{j_1}^{\sigma_1}\cdots u_{j_p}^{\sigma_p}. \tag{1.1.49}$$

Moreover it is not difficult to prove that if $G^{(p)} \in \mathcal{P}_p^0$ and $F^{(q)} \in \mathcal{P}_q^0$ for some $p, q \geq 3$ then

$$\{G^{(p)}, F^{(q)}\} \in \mathcal{P}^0_{p+q-2}. \tag{1.1.50}$$

The Birkhoff normal form reduction: The unperturbed system generated by the quadratic Hamiltonian $H^{(2)}$ is made by infinitely many decoupled harmonic oscillators with frequencies given by the dispersion relation $j \mapsto \Omega(j)$. For this reason, since the quadratic Hamiltonian preserves all Sobolev norms and in view of the semi-linear estimate (1.1.42) for the perturbation $H^{(\geq 3)} := H^{(3)} + \cdots + H^{(M)}$, a small solution u of (1.1.34) satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{H^s} = \int_{\mathbb{T}} |D|^s X_{H^{(\geq 3)}}(u)\, |D|^s \overline{u}\, \mathrm{d}x \lesssim_s \|u\|^3_{H^s}, \tag{1.1.51}$$

which leads to the energy estimate (1.1.38) with $N = 1$ and a control for small solutions up to the local well-posedness time $T_\varepsilon \sim \varepsilon^{-1}$.

The idea of Birkhoff normal form is to look for a change of variables that removes iteratively, when it is possible, each term of homogeneity $p = 3, \ldots, M$ from the original Hamiltonian (1.1.34). At each step of the iterative reduction it is performed a change of variable of the form $v = \phi^{-1}(u)$ where $\phi = (\phi^\tau_G)_{|\tau=1}$ is the time one flow of the Hamiltonian vector field generated by a suitable Hamiltonian function G, namely

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}\tau}\phi^\tau_G(v) = X_G \circ \phi^\tau_G(v) \\ \phi^0_G(v) = v. \end{cases} \tag{1.1.52}$$

In this way, being ϕ a symplectic change of variable (see e.g. Lemma 4.3.14) and in view of (1.1.47), it is sufficient to compute the transformed Hamiltonian $\widetilde{H} = H \circ \phi$. We first note that

$$\frac{\mathrm{d}}{\mathrm{d}\tau} H \circ \phi^\tau_G(v) = \mathrm{d}H(\phi^\tau_G(v))[X_G\big(\phi^\tau_G(v)\big)] = \big\{H, G\big\} \circ \phi^\tau_G(v). \tag{1.1.53}$$

Then the transformed Hamiltonian has the Taylor expansion

$$H \circ \phi = H + \big\{H, G\big\} + \sum_{k \geq 2} \frac{1}{k!}\mathrm{Ad}^k_G[H], \quad \text{where} \quad \mathrm{Ad}_G[\cdot] := \{\cdot, G\}, \tag{1.1.54}$$

which is an expansion in increasing degree of homogeneity if the Hamiltonian G has at least degree of homogeneity 3 (see (1.1.50)).

We start from $p = 3$ and we generate the change of variable ϕ as above with a cubic Hamiltonian $G \equiv G^{(3)} \in \mathcal{P}_3^0$ to be determined. In view of (1.1.50) and (1.1.34) we deduce that

$$\widetilde{H}_1 := H \circ \phi = H^{(2)} + H^{(3)} + \{H^{(2)}, G\} + \widetilde{H}_1^{(\geq 4)}, \tag{1.1.55}$$

where $\widetilde{H}^{(\geq 4)}$ contains q-homogeneous Hamiltonian in $\mathcal{P}_q^0$ with $q \geq 4$. Then we remove the Hamiltonian of homogeneity 3 by solving the homological equation

$$H^{(3)} + \{H^{(2)}, G\} = 0 \tag{1.1.56}$$

which, written in Fourier coordinates, reads

$$-\mathrm{i}[\sigma_1\Omega(j_1) + \sigma_2\Omega(j_2) + \sigma_3\Omega(j_3)]G^{\sigma_1,\sigma_2,\sigma_3}_{j_1,j_2,j_3} = H^{\sigma_1,\sigma_2,\sigma_3}_{j_1,j_2,j_3}. \tag{1.1.57}$$

We can solve the above equation if the following non-resonance conditions hold:

$$\sigma_1\Omega(j_1) + \sigma_2\Omega(j_2) + \sigma_3\Omega(j_3) \neq 0. \tag{1.1.58}$$

In this case

$$G^{\sigma_1,\sigma_2,\sigma_3}_{j_1,j_2,j_3} := \frac{H^{\sigma_1,\sigma_2,\sigma_3}_{j_1,j_2,j_3}}{-\mathrm{i}[\sigma_1\Omega(j_1)+\sigma_2\Omega(j_2)+\sigma_3\Omega(j_3)]}. \tag{1.1.59}$$

Moreover, to guarantee that the coefficients in (1.1.59) define a semi-linear Hamiltonian $G^{(3)}$ in $\mathcal{P}^0_3$, it is necessary to require a strong lower bound of the form

$$|\sigma_1\Omega(j_1)+\sigma_2\Omega(j_2)+\sigma_3\Omega(j_3)| \gtrsim \frac{1}{\max_3\{|j_1|,|j_2|,|j_3|\}^\tau} \quad \text{for some } \tau>0. \tag{1.1.60}$$

Then the change of variable $v=\phi^{-1}(u)$ transforms the original Hamiltonian system into another Hamiltonian system for the variable v with Hamiltonian

$$\widetilde{H}_1(v) = H^{(2)}(v) + \widetilde{H}_1^{(\geq 4)}(v). \tag{1.1.61}$$

A consequence is that the variable v fulfills the improved energy estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2_{H^s} \lesssim_s \|v\|^4_{H^s}. \tag{1.1.62}$$

To analyze a resonant case, we iterate the above procedure to remove the quartic part of $\widetilde{H}_1^{(\geq 4)} = \widetilde{H}_1^{(4)} + \widetilde{H}_1^{(\geq 5)}$ in (1.1.61). We look for a change of variable ϕ_2 as the time one flow of the Hamiltonian vector field associated to a quartic Hamiltonian $G^{(4)}$ and the same computation as in (1.1.47) gives that the new transformed Hamiltonian has the form

$$\widetilde{H}_2 = \widetilde{H}_1\circ\phi_2 = H^{(2)} + \widetilde{H}_1^{(4)} + \{H^{(2)}, G^{(4)}\} + \widetilde{H}_2^{(\geq 5)}. \tag{1.1.63}$$

In this case, as it will be clear in a moment, it will never be possible to remove completely the quartic part of the Hamiltonian and the homological equation reads

$$\widetilde{H}_1^{(4)} + \{H^{(2)}, G^{(4)}\} = \mathcal{Z}^{(4)} \tag{1.1.64}$$

where $\mathcal{Z}^{(4)}$ is the quartic resonant Hamiltonian normal form, defined as

$$\mathcal{Z}^{\sigma_1,\sigma_2,\sigma_3,\sigma_4}_{j_1,j_2,j_3,j_4} := \begin{cases} [\widetilde{H}_1]^{\sigma_1,\sigma_2,\sigma_3,\sigma_4}_{j_1,j_2,j_3,j_4} & \text{if} \quad \sigma_1\Omega(j_1)+\sigma_2\Omega(j_2)+\sigma_3\Omega(j_3)+\sigma_4\Omega(j_4)=0. \\ 0 & \text{otherwise} \end{cases} \tag{1.1.65}$$

We solve (1.1.64) by defining the Fourier entries of $G^{(4)}$ as

$$G^{\sigma_1,\sigma_2,\sigma_3,\sigma_4}_{j_1,j_2,j_3,j_4} = \begin{cases} \frac{H^{\sigma_1,\sigma_2,\sigma_3,\sigma_4}_{j_1,j_2,j_3,j_4}}{-\mathrm{i}[\sigma_1\Omega(j_1)+\sigma_2\Omega(j_2)+\sigma_3\Omega(j_3)+\sigma_4\Omega(j_4)]} & \text{if} \quad \sigma_1\Omega(j_1)+\sigma_2\Omega(j_2)+\sigma_3\Omega(j_3)+\sigma_4\Omega(j_4)\neq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{1.1.66}$$

As for the cubic case, to prove that $G^{(4)}$ is a quartic semi-linear Hamiltonian in $\mathcal{P}^0_4$ we need the lower bound

$$\sigma_1\Omega(j_1)+\cdots+\sigma_4\Omega(j_4)\neq 0 \implies |\sigma_1\Omega(j_1)+\cdots+\sigma_4\Omega(j_4)| \gtrsim \frac{1}{\max_3\{|j_1|,\ldots,|j_4|\}^\tau}. \tag{1.1.67}$$

With this choice the Hamiltonian $\widetilde{H}_2$ reads

$$\widetilde{H}_2 = H^{(2)} + \mathcal{Z}^{(4)} + \widetilde{H}^{(\geq 5)} \tag{1.1.68}$$

and, in view of the quartic estimate (1.1.42) for the vector field $X_{\widetilde{H}^{(\geq 5)}}$, the dynamics of the Hamiltonian equation

$$\partial_t \begin{pmatrix} w \\ \overline{w} \end{pmatrix} = X_{\widetilde{H}_2}(w, \overline{w})$$

is dictated from the resonant Hamiltonian $H^{(2)} + \mathcal{Z}^{(4)}$ up to a time of order $\sim \varepsilon^{-3}$.

We shall deal with two different types of resonant Hamiltonians:

1. **Integrable Hamiltonians:** For any dispersion relations $j \to \Omega(j)$ the integrable monomials

$$|w_j|^2 |w_k|^2 = w_j \overline{w_j} w_k \overline{w_k} \tag{1.1.69}$$

are resonant. Indeed the corresponding divisors are

$$\Omega(j) - \Omega(j) + \Omega(k) - \Omega(k) = 0. \tag{1.1.70}$$

If the resonant normal form $\mathcal{Z}^{(4)}$ contains only integrable monomials then its dynamics can be completely integrated using that the actions

$$I_\ell(w) := |w_\ell|^2, \quad \ell \in \mathbb{Z} \tag{1.1.71}$$

are constant of motion. Indeed one verifies that

$$\{I_\ell(w), |w_j|^2 |w_k|^2\} = 0, \quad \text{for any} \quad \ell, j, k \in \mathbb{Z}. \tag{1.1.72}$$

As a consequence, if $\mathcal{Z}^{(4)}$ contains only integrable monomials, every Sobolev norm is constant along the flow of $H^{(2)} + \mathcal{Z}^{(4)}$.

2. **Super-action preserving resonances:** This type of resonances arise, for example, from the interaction of two or more harmonic oscillators with the same frequencies of oscillation. Typical examples which arise in one dimensional PDEs is when the dispersion relation is even, namely $\Omega(j) = \Omega(-j)$. In this case, in addition to the integrable monomial (1.1.69), there are other two types of resonant monomials of the form

$$|w_j|^2 w_k \overline{w}_{-k}, \quad w_j \overline{w}_{-j} w_k \overline{w}_{-k}, \quad j, k \in \mathbb{Z} \tag{1.1.73}$$

with the corresponding divisors

$$\Omega(j) - \Omega(j) + \Omega(k) - \Omega(-k) = 0, \quad \Omega(j) - \Omega(-j) + \Omega(k) - \Omega(-k) = 0. \tag{1.1.74}$$

This type of monomials do not preserve the actions functionals $I_\ell(w)$ in (1.1.72) but they preserve the so-called super-actions, namely

$$J_n(w) := |w_n|^2 + |w_{-n}|^2, \quad \{J_n, |w_j|^2 w_k \overline{w}_{-k}\} = \{J_n, w_j \overline{w}_{-j} w_k \overline{w}_{-k}\} = 0, \tag{1.1.75}$$

for any $n \in \mathbb{N}$, $j, k \in \mathbb{Z}$. This means that the dynamics is not completely decoupled as in the integrable case but there can be an exchange of energy between the n-th Fourier mode and the $-n$-th Fourier mode. Nevertheless, by the conservation of the super-actions, the Sobolev norms

$$\|w\|_{H^s}^2 = \sum_{n \in \mathbb{N}} \langle n \rangle^{2s} J_n(w) \tag{1.1.76}$$

remains constant along the flow of the truncated Hamiltonian $H^{(2)} + \mathcal{Z}^{(4)}$ and, the solution w of the complete system

$$\partial_t \begin{pmatrix} w \\ \overline{w} \end{pmatrix} = X_{H^{(2)}+\mathcal{Z}^{(4)}}(w) + X_{H^{(\geq 5)}}(w), \tag{1.1.77}$$

satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2_{H^s} = \int_{\mathbb{T}} |D|^s X_{H^{(\geq 5)}}(w)\,|D|^s\overline{w}\,\mathrm{d}x \leq \|X_{H^{(\geq 5)}}(w)\|_{H^s}\|w\|_{H^s} \overset{(1.1.42)}{\lesssim_s} \|w\|^5_{H^s}. \tag{1.1.78}$$

In conclusion one can iterate the procedure if:

1. NON-RESONANCE CONDITION: For any $p \in \mathbb{N}$ one has
$$\sigma_1\Omega(j_1) + \cdots + \sigma_p\Omega(j_p) \neq 0 \tag{1.1.79}$$
except to the super-action preserving resonances $p = 2\ell$ and
$$\Omega(j_1) + \cdots + \Omega(j_\ell) - \Omega(j_{\ell+1}) - \cdots - \Omega(j_p) = 0, \quad \{|j_1|, \ldots, |j_\ell|\} = \{|j_{\ell+1}|, \ldots, |j_p|\}. \tag{1.1.80}$$

2. STRONG LOWER BOUND ON THE SMALL DENOMINATORS: In case (1.1.79) holds one has
$$|\sigma_1\Omega(j_1) + \cdots + \sigma_p\Omega(j_p)| \gtrsim \frac{1}{\max_3\{|j_1|, \ldots, |j_p|\}}. \tag{1.1.81}$$

3. SEMI-LINEAR PERTURBATION: If the Hamiltonian $H^{(p)}$ satisfies the semi-linear bound (1.1.41) and (1.1.81) holds then the generators $G^{(p)}$ defined as in (1.1.59), (1.1.66) satisfy (1.1.41) as well as each term in the expansion (1.1.54). Another important property is that the semi-linear estimate (1.1.42) for the generator X_G guarantees that the flow in (1.1.52) is well-defined in Sobolev spaces and one has
$$\|\phi^\tau_G(v)\|_{H^s} \sim_s \|v\|_{H^s} \tag{1.1.82}$$
which is the fundamental property which allows to deduce (1.1.38) by (1.1.78).

Normal form for semi-linear PDEs. Concerning semi-linear PDEs the long time existence problem has been extensively studied in literature. In addition to the previously mentioned foundational works of Bambusi [15], Bambusi-Grebért [19], and Delort-Szeftel [60, 61], we mention Faou-Grebért [69] regarding Birkhoff normal form theory for reversible PDEs and the paper [17] about long time existence of solutions for the semi-linear Klein-Gordon equation on Zoll manifolds which contains all the ideas of the preceding (and aforementioned) literature. The normal form for the completely resonant nonlinear Schrödinger equation on a torus $\mathbb{T}^d$ has been discussed by Procesi-Procesi in [107]. We quote also the paper [65] by Faou-Gauckler-Lubich about the long time stability of plane waves for the cubic Schrödinger equation $\mathbb{T}^d$, and the paper by Maspero-Procesi [98] about the stability of small finite gap solutions for the same equation on $\mathbb{T}^2$. In Faou-Grebért [67] is considered the case of analytic initial data and proved sub-exponential lower bounds for the stability time of the form $T_\varepsilon \gtrsim e^{\log(1/\varepsilon)^{1+b}}$ for $b > 0$ for a class of Schrödinger equations on $\mathbb{T}^d$. In [38], Biasco-Massetti-Procesi have improved, in the 1-dimensional case, the results in [67] using a different Diophantine non-resonance conditions on the linear frequencies. Recently, in [24, 25], Bernier-Grébert developed Birkhoff normal form techniques in low regularity.

All of the previously mentioned papers deal with non-resonant semi-linear problems with a strong lower bound for the small divisors (as in (1.1.81)). For problems with weak lower bounds for the small divisors (as in equation (1.1.14)), partial Birkhoff normal form results are obtained in [68, 23, 87, 56] using partition of energy ideas that will be outlined in Section 1.2.2.

In the recent paper [18], Bambusi-Feola-Montalto prove an almost global in time existence and stability for a large class of semi-linear Schrödinger type equations on irrational tori, even in the presence of a weak lower bound for the small divisors.

In the case that the non linearity $\mathcal{N}(u,\overline{u}) = X_{H(\geq 3)}$ contains derivatives of u, if one would follow the strategy used in the semi-linear case, one would end up with only formal results in the sense that the change of coordinates would be unbounded. In this direction we quote the early paper concerning the pure-gravity water waves equation by Craig-Worfolk [52].

Quasi-linear Hamiltonian PDEs: For quasi-linear Hamiltonian systems, as the water-waves system (1.1.19), the semi-linear estimate (1.1.41) does not hold and has to be replaced by an estimate of the form

$$|H^{\sigma_1,\ldots,\sigma_p}_{j_1,\ldots,j_p}| \leq C\max{}_3\{|j_1|,\ldots,|j_p|\}^{\mu}\max\{|j_1|,\ldots,|j_p|\}^{m} \tag{1.1.83}$$

for some loss of derivative $m > 0$. We shall say that $H^{(p)} \in \mathcal{P}^m_p$ meaning that is a p-homogeneous Hamiltonian with estimate (1.1.83). As a consequence the semi-linear approach leads only to a *formal* results for the following reasons:

1. LACK OF TRIVIAL LOCAL WELL-POSEDNESS: The starting point of the Birkhoff normal form reduction is the estimate (1.1.51) which allows to construct the solutions at least locally in time. For quasi-linear PDEs, and especially for water-waves, also the local well-posedness estimate (1.1.51) is far to be trivial and typically requires to write the water waves system using coordinates which are not standard Darboux coordinates;

2. FORMAL CHANGE OF VARIABLE: If the perturbation $H^{(p)}$ that we want to remove is in $\mathcal{P}^m_p$ for some $m > 0$ then, in general, the generator $G^{(p)}$ is in $\mathcal{P}^{m'}_p$ for some $m' \geq m$. Then the associated flow $\Phi^{\tau}_{G^{(p)}}$, which is in general not well-defined, is not a good change of variable in H^s;

3. FORMAL TRANSFORMED HAMILTONIAN: If G_1 is in $\mathcal{P}^{m_1}_p$ and G_1 is in $\mathcal{P}^{m_2}_q$ then in general
$$\{G_1, G_2\} \in \mathcal{P}^{m_1+m_2}_{p+q-2}. \tag{1.1.84}$$
As a consequence the expansion (1.1.54) is only formal because it increases the order of unboundedness.

Before discussing the literature regarding the Birkhoff normal form for quasi-linear PDEs, we quote the paper by Yuan-Zhang [123] and the paper by Feola-Montalto [75] in the case where the non-linearity contains derivatives of u of order strictly less than the order of the linearized operator $\Omega(D)$.

Normal form for quasi-linear PDEs. The first rigorous long time existence result concerning quasi-linear equations, has been obtained by Delort. In [57] Delort studied quasi-linear Hamiltonian perturbations of the Klein-Gordon equation on the circle, and in [58] the same equation on higher dimensional spheres. Here Delort introduces some classes of multi-linear maps which define para-differential operators (in the case of Klein-Gordon equation operators of order 1) similar to the abstract definition of spectrally localized map in Chapter 4 (Definition 4.2.71) but with the additional requirement to enjoy a symbolic calculus. We remark that in such papers Delort deeply uses the fact that the Klein-Gordon equation has a linear dispersion law (i.e. the operator $\Omega(D) \sim |D|$ and the non-linear term has order 1). A new different approach, in the case of super-linear dispersion law, is proposed in [27] for the irrotational water waves equations (1.1.19) with $\gamma = 0$ and then in [71] for quasi-linear Schrödinger equation. As we have already mentioned, in [27] is exploited only the time reversibility of the system which gives the long time existence result in the invariant subspace of standing waves. Finally in [33] we have developed a para-differential Hamiltonian Birkhoff normal form approach to water waves with constant vorticity which we shall explain in Section 1.2.3.

1.2 Strategy of the proofs

In this section we explain the main ideas of the proofs of Theorem 1.1.1, Theorem 1.1.2 and Theorem 1.1.3.

1.2.1 Theorem 1.1.1

The local solutions $(\rho(t), \phi(t))$ of Theorem 1.1.1 are constructed via a classical quasi-linear iterative scheme following the strategy first introduced by Kato [90]. We follow a para-differential approach which requires essentially two ingredients:

$i)$ The para-linearization of the system;

$ii)$ An energy estimate for the para-linearized system which controls the Sobolev norm $H^s(\mathbb{T}^d)$.

To understand the idea behind the quasi-linear iterative scheme we compare it with more simple iterative schemes.
The Picard scheme for Banach space ODEs: Consider first a Banach space ODE

$$\partial_t u = f(u), \quad u(0) \in Y \tag{1.2.1}$$

where Y is a Banach space and $f \in C^1(Y;Y)$. The solution of (1.2.1) is constructed by contraction principle which corresponds to a Picard-type iterative scheme of the form

$$\begin{cases} u_0 = u(0) \\ \partial_t u_n = f(u_{n-1}), \quad u_n(0) = u(0), \quad n \geq 1. \end{cases} \tag{1.2.2}$$

The convergence of the scheme is provided using that, since f is bounded on bounded sets, for small times, the norm of $\|u_n\|_Y$ remains bounded and, since f is Lipschitz, the norm of the differences $\|u_n - u_{n-1}\|_Y$ converges exponentially to zero. A different scheme has to be applied if the vector field is unbounded.
The Picard scheme for semi-linear PDEs: Consider then a semi-linear PDE:

$$\partial_t u = \mathrm{i}\Delta u + f(u), \quad u(0) \in H^s(\mathbb{T}^d;\mathbb{C}), \tag{1.2.3}$$

where $s > \frac{d}{2}$ and $f \in C^1(H^s;H^s)$. In this case it is possible to use the boundedness of the linear flow associated to $\partial_t u = \mathrm{i}\Delta u$ and find the solution u by a contraction argument on the Duhamel formula

$$u(t) = e^{\mathrm{i}t\Delta}u(0) + \int_0^t e^{\mathrm{i}(t-\tau)\Delta} f(u(\tau))\,\mathrm{d}\tau \tag{1.2.4}$$

which corresponds to the iterative scheme

$$\begin{cases} u_0 = u(0) \\ \partial_t u_n = \mathrm{i}\Delta u_n + f(u_{n-1}), \quad u_n(0) = u(0), \quad n \geq 1. \end{cases} \tag{1.2.5}$$

The new ingredient to achieve the convergence of (1.2.5) is the following energy equality for the linear flow

$$\|e^{\mathrm{i}t\Delta}u_0\|_{H^s} = \|u_0\|_{H^s}. \tag{1.2.6}$$

If the non-linearity $f(u)$ contains derivatives of u, this scheme does not converge and the non-linear term has to be treated in a non-perturbative way.

Iterative scheme for quasi-linear systems: To deal with the quasi-linear system of PDEs (1.1.2) we para-linearize it and we write it using complex coordinates (see Proposition 2.3.1) as

$$\partial_t U = \mathrm{i}A(U)U + f(U) \tag{1.2.7}$$

where:

- The operator $A(U)$ is a matrix of para-differential operators of the form

$$\begin{aligned} A(U) =&\mathrm{Op}^{\mathrm{BW}}\left(\begin{bmatrix} -\mathrm{i}(1+\mathtt{a}_+(U;x)) & -\mathrm{i}\mathtt{a}_-(U;x) \\ \mathrm{i}\mathtt{a}_-(U;x) & \mathrm{i}(1+\mathtt{a}_+(U;x)) \end{bmatrix}\sqrt{\mathtt{m}K(\mathtt{m})}|\xi|^2\right) \\ &+\mathrm{Op}^{\mathrm{BW}}\left(\begin{bmatrix} -\mathrm{i}\mathtt{b}(U;x)\cdot\xi & 0 \\ 0 & -\mathrm{i}\mathtt{b}(U;x)\cdot\xi \end{bmatrix}\right) \end{aligned} \tag{1.2.8}$$

  for some functions $\mathtt{a}_\pm$ and a vector $\mathtt{b}$ (see (2.3.6), (2.3.8)) and where the para-differential Weyl quantization is defined in Definition 2.2.2;

- $f(U)$ is a semi-linear term, satisfying, for any $s_0 > \frac{d}{2}$ and $s \geq s_0+2$,

$$\|f(U)\|_{H^s} \leq C(s, \|U\|_{s_0+2})\|U\|_{H^s}$$
$$\|f(U)-f(V)\|_s \leq \mathtt{C}(\|U\|_{s_0+2}, \|V\|_{s_0+2})\,\|U-V\|_s + \mathtt{C}(\|U\|_s, \|V\|_s)\,\|U-V\|_{s_0+2}.$$

We construct the solutions as the limit of an iterative scheme of the form

$$\begin{cases} U_0 = U(0) = \left[\begin{smallmatrix} u(0) \\ \overline{u}(0) \end{smallmatrix}\right] \\ \partial_t U_n = \mathrm{i}A(U_{n-1})U_n + f(U_{n-1}), \quad U_n(0) = U(0), \quad n \geq 1, \end{cases} \tag{1.2.9}$$

where $U_n = \begin{pmatrix} u_n \\ \overline{u}_n \end{pmatrix}$.

For fixed $s_0 > \frac{d}{2}$, in order to control the approximate solutions along the iteration, we prove that any solution of the linear problem

$$\partial_t V = \mathrm{i}A(U)V, \quad V(0) = U(0), \tag{1.2.10}$$

satisfies, for any $s \geq s_0+2$ and $U = \begin{pmatrix} u \\ \overline{u} \end{pmatrix} \in H^s(\mathbb{T}^d;\mathbb{C}^2)$ such that $\|U\|_{s_0} \leq r$ and $\|U\|_{s_0+2} \leq \Theta$, the *a priori* energy estimate

$$\|V(t)\|_s^2 \leq C_r\|V(0)\|_s^2 + C_\Theta\int_0^t \|V(\tau)\|_s^2 \mathrm{d}\tau. \tag{1.2.11}$$

The main difficulties which arise in proving (1.2.11) for (1.2.10) are not present in proving its semi-linear analogue (1.2.6), indeed:

$i)$ The semi-linear equation (1.2.5) is a scalar equation and the unbounded linear operator Δ is self-adjoint which implies that the L^2 norm is preserved by its flow whereas the operator $A(U)$ in (1.2.8) is not diagonal and there are no apparent reasons to have conservation of L^2 norm;

$ii)$ The operator Δ is constant coefficients, i.e. does not depend on x, which implies that its flow preserves any Sobolev norms while the unbounded linear operator $A(U)$ has variable coefficients and depends on the point U.

The energy estimate (1.2.11) is the main ingredient to prove the local well-posedness Theorem 1.1.1 and, since it deeply relies on the para-differential form (1.2.7) of the system, we outline below the main ideas behind para-differential calculus.

Para-differential calculus: Para-differential calculus was first introduced by J.M. Bony [40]. A key observation of Bony is that para-differential operators naturally arise when one performs a spectral analysis of nonlinear functionals. To explain this, consider the product of two functions with Fourier expansion $u(x) = \sum_{j\in\mathbb{Z}^d} \widehat{u}_j e^{\mathrm{i}j\cdot x}$ and $v(x) = \sum_{j\in\mathbb{Z}^d} \widehat{v}_j e^{\mathrm{i}j\cdot x}$ which can be written as

$$\begin{aligned} u(x)v(x) &= \sum_{j,k\in\mathbb{Z}^d} \widehat{u}_{j-k}\widehat{v}_k e^{\mathrm{i}j\cdot x} \\ &= \sum_{|j-k|\ll k} \widehat{u}_{j-k}\widehat{v}_k e^{\mathrm{i}j\cdot x} + \sum_{|j|\ll|j-k|} \widehat{u}_{j-k}\widehat{v}_k e^{\mathrm{i}j\cdot x} + \frac{1}{2\pi}\sum_{|j|\sim|j-k|} \widehat{u}_{j-k}\widehat{v}_k e^{\mathrm{i}j\cdot x} \\ &=: T_u v + T_v u + R(u,v). \end{aligned} \tag{1.2.12}$$

The advantage of the above splitting is that the operator $T_u v$ is linear with respect to v and is supported when the frequencies of v are much larger than the frequencies of u. As a consequence $T_u v$ has the same regularity of v with estimate (see Lemma 4.2.12)

$$\|T_u v\|_{H^s} \lesssim \|u\|_{H^{s_0}}\|v\|_{H^s}, \quad s_0 > \frac{d}{2}. \tag{1.2.13}$$

Moreover the remainder $R(u,v)$ is more regular than u and v with estimate

$$\|R(u,v)\|_{H^{s+\varrho-s_0}} \lesssim \|u\|_{H^s}\|v\|_{H^\varrho}, \quad s+\varrho \geq 0. \tag{1.2.14}$$

Inspired by formula (1.2.12) and to include also differential operators we define: given a symbol $a(x,\xi)$ its Weyl para-differential quantization as (see (2.2.12) for a more rigorous definition)

$$\mathrm{Op}^{\mathrm{BW}}(a(x,\xi)) := \sum_{|j-k|\ll|j+k|} \widehat{a}\Big(j-k, \frac{j+k}{2}\Big) u_k e^{\mathrm{i}j\cdot x}.$$

The main properties needed to prove (1.2.11) are:

- *B*oundedness (Theorem 2.2.12): If $a(x,\xi)$ is a symbol of order $m\in\mathbb{R}$ and $s_0>0$ then

$$\|\mathrm{Op}^{\mathrm{BW}}(a(x,\xi))u\|_{H^{s-m}} \lesssim |a|_{m,s_0,2(d+1)}\|u\|_{H^s}$$

  where the semi-norm $|a|_{m,s_0,2(d+1)}$ is defined in (2.2.3) and, similarly to (1.2.13), involves only the low norm H^{s_0} of the symbol a;

- *C*omposition (Theorem 2.2.13): The composition of two para-differential operators is still a para-differential operator (up to a smoothing remainder) whose symbol has the explicit asymptotic expansion (2.2.87);

- *C*ommutator: If $a(x,\xi)$ is a symbol of order $m\in\mathbb{R}$ and $b(x,\xi)$ is a symbol of order $m'\in\mathbb{R}$ then the commutator

$$[\mathrm{Op}^{\mathrm{BW}}(a), \mathrm{Op}^{\mathrm{BW}}(b)]$$

  is still a para-differential operator whose symbol has order $m+m'-1$.

Symmetrization of (1.2.10)**:** In view of $i)$ below (1.2.11), we first reduce (1.2.10) to a scalar hyperbolic system. To do so we consider the matrix of functions associated to the principal order

$$\begin{bmatrix} -\mathrm{i}(1+\mathtt{a}_+(U;x)) & -\mathrm{i}\mathtt{a}_-(U;x) \\ \mathrm{i}\mathtt{a}_-(U;x) & \mathrm{i}(1+\mathtt{a}_+(U;x)) \end{bmatrix} \tag{1.2.15}$$

and we note that its eigenvalues are $\pm \mathrm{i}\lambda(U;x)$ where

$$\lambda(U;x) := \sqrt{(1+\mathtt{a}_+(U;x))^2 - \mathtt{a}_-(U;x)^2}.$$

We diagonalize the matrix of functions in (1.2.15) with a symmetric matrix of the form

$$F(U;x) = \begin{bmatrix} f(U;x) & g(U;x) \\ g(U;x) & f(U;x) \end{bmatrix}, \quad f^2 - g^2 = 1, \quad F^{-1}(U;x) = \begin{bmatrix} f(U;x) & -g(U;x) \\ -g(U;x) & f(U;x) \end{bmatrix}.$$

and

$$F^{-1} \begin{bmatrix} \mathrm{i}(1+\mathtt{a}_+) & -\mathrm{i}\mathtt{a}_- \\ \mathrm{i}\mathtt{a}_- & \mathrm{i}(1+\mathtt{a}_+) \end{bmatrix} F = \begin{bmatrix} -\mathrm{i}\lambda & 0 \\ 0 & \mathrm{i}\lambda \end{bmatrix}. \tag{1.2.16}$$

As a consequence the system for the new variable

$$W = \mathrm{Op}^{\mathrm{BW}}\left(F^{-1}\right)V \tag{1.2.17}$$

reads

$$\partial_t W = \mathrm{Op}^{\mathrm{BW}}\left(\begin{bmatrix} -\mathrm{i}\lambda\sqrt{\mathtt{m}K(\mathtt{m})}|\xi|^2 - \mathrm{i}\mathtt{b}\cdot\xi & 0 \\ 0 & \mathrm{i}\lambda\sqrt{\mathtt{m}K(\mathtt{m})}|\xi|^2 + \mathrm{i}\mathtt{b}\cdot\xi \end{bmatrix} \right) W \tag{1.2.18}$$

up to semi-linear terms. Moreover one has

$$\|W\|_{H^s} \sim \|V\|_{H^s}. \tag{1.2.19}$$

System (1.2.18) for W has the advantage of being diagonal (up to bounded perturbation) but, since its principal order has not constant coefficients, it is necessary to define a *modified energy* to get (1.2.11).
The modified energy: In order to get (1.2.11) we define the modified energy

$$\begin{aligned} \|V\|^2_{s;U} :=& \langle \mathrm{Op}^{\mathrm{BW}}\left(\lambda^s(U;x)|\xi|^{2s}\right)W, W\rangle \\ =& \langle \mathrm{Op}^{\mathrm{BW}}\left(\lambda^s(U;x)|\xi|^{2s}\right)\mathrm{Op}^{\mathrm{BW}}\left(F^{-1}(U;x)\right)V, \mathrm{Op}^{\mathrm{BW}}\left(F^{-1}(U;x)\right)V\rangle \end{aligned} \tag{1.2.20}$$

where we introduce the scalar product

$$\langle V, W\rangle := 2\mathrm{Re}\int_{\mathbb{T}^d} v(x)\overline{w}(x)\,\mathrm{d}x, \qquad V = \begin{bmatrix} v \\ \overline{v} \end{bmatrix}, \quad W = \begin{bmatrix} w \\ \overline{w} \end{bmatrix}.$$

The modified energy is equivalent to the Sobolev norm of V and W, i.e.

$$\|V\|^2_{s;U} \sim_r \|W\|^2_{H^s} \sim_r \|V\|^2_{H^s} \tag{1.2.21}$$

and it satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\|V\|^2_{s;U} \le C_\Theta \|V\|^2_{s;U}. \tag{1.2.22}$$

Remark 1.2.1. We consider solutions which are not small and we do not invert the operator $\mathrm{Op}^{\mathrm{BW}}\left(F^{-1}\right)$ in (1.2.18) and the operator $\mathrm{Op}^{\mathrm{BW}}\left(\lambda^\sigma(U;x)|\xi|^{2\sigma}\right)$ in (1.2.20). As a consequence the equivalence of the norms in (1.2.19) and (1.2.21) has to be intended as in (2.4.21) and it is obtained by means of a parametrix and a Gårding-type argument (see Lemma 2.4.5).

Remark 1.2.2 (Weyl quantization)**.** We use the para-differential calculus with Weyl quantization because, thanks to its symmetries, it reveals easily some cancellations due to self-adjointness. The main algebraic property of the Weyl quantization is:

$$\mathrm{Op}^{\mathrm{BW}}(a)^* = \mathrm{Op}^{\mathrm{BW}}(\overline{a}) \tag{1.2.23}$$

for any symbol a (see Definition 2.2.1) and where $\mathrm{Op}^{\mathrm{BW}}(a)^*$ is the adjoint of $\mathrm{Op}^{\mathrm{BW}}(a)$. Another remarkable algebraic property which we exploit is formula (2.2.90) which states the following: given a symbol three symbols a, b and c of order respectively m, m' and m'' then

$$\mathrm{Op}^{\mathrm{BW}}(a)\mathrm{Op}^{\mathrm{BW}}(b)\mathrm{Op}^{\mathrm{BW}}(c) = \mathrm{Op}^{\mathrm{BW}}(a\,b\,c + \sigma(a,b,c)) + R_{m+m'+m''-2}$$

where $\sigma(a,b,c)$ is a symbol of order $m+m'+m''-1$ and is anti-symmetric with respect to a and c, namely $\sigma(a,b,c) = -\sigma(c,b,a)$ whereas $R_{m+m'+m''-2}$ is an operator of order $m+m'+m''-2$.

1.2.2 Theorem 1.1.2

In general (1.1.2) is a system of quasi-linear equations. The case (QHD), i.e., system (1.1.2) with the particular choice $K(\rho) = \frac{\kappa}{\rho}$, reduces, for small solutions, to a semi-linear Schrödinger equation. This is a consequence of the fact that the Madelung transform (introduced for the first time in the seminal work by Madelung [96]) is well defined for small solutions. In other words one can introduce the new variable

$$\psi := \sqrt{\mathtt{m}+\rho}\,e^{\mathrm{i}\phi/\hbar} \tag{1.2.24}$$

(see Section 3.2 for details), where $\hbar = 2\sqrt{k}$, one obtains the equation

$$\partial_t \psi = \mathrm{i}\Big(\frac{\hbar}{2}\Delta\psi - \frac{1}{\hbar}g(|\psi|^2)\psi\Big). \tag{1.2.25}$$

Since $g(\mathtt{m}) = 0$, such equation has an equilibrium point at $\psi = \sqrt{\mathtt{m}}$. The study of the stability of small solutions for (1.1.2) is equivalent to the study of the stability of the variable $z := \psi - \sqrt{\mathtt{m}}$. The equation for z reads

$$\partial_t z = -\mathrm{i}\Big(\frac{\hbar|D|_\nu^2}{2} + \frac{\mathtt{m}g'(\mathtt{m})}{\hbar}\Big)z - \mathrm{i}\frac{\mathtt{m}g'(\mathtt{m})}{\hbar}\overline{z} + f(z),$$

where f is a smooth function having a zero of order 2 at $z=0$, i.e., $|f(z)| \lesssim |z|^2$, and $|D|_\nu^2$ is the Fourier multiplier with symbol

$$|\xi|_\nu^2 := \sum_{i=1}^d a_i|\xi_i|^2, \quad a_i := \nu_i^2, \quad \forall \xi \in \mathbb{Z}^d. \tag{1.2.26}$$

The aim is to use a Birkhoff normal form/modified energy technique in order to reduce the size of the nonlinearity $f(z)$. To do that, it is convenient to perform some preliminary reductions. First of all we want to eliminate the term $-\mathrm{i}\frac{\mathtt{m}g'(\mathtt{m})}{\hbar}\overline{z}$. In other words we want to diagonalize the operator

$$\mathcal{L} = \begin{pmatrix} \frac{\hbar}{2}|D|_\nu^2 + \frac{1}{\hbar}\mathtt{m}g'(\mathtt{m}) & \frac{1}{\hbar}\mathtt{m}g'(\mathtt{m}) \\ \frac{1}{\hbar}\mathtt{m}g'(\mathtt{m}) & \frac{\hbar}{2}|D|_\nu^2 + \frac{1}{\hbar}\mathtt{m}g'(\mathtt{m}) \end{pmatrix} \tag{1.2.27}$$

which is a matrix of Fourier multipliers with symbol

$$\mathcal{L}(j) := \begin{pmatrix} \frac{\hbar}{2}|j|_\nu^2 + \frac{1}{\hbar}\mathtt{m}g'(\mathtt{m}) & \frac{1}{\hbar}\mathtt{m}g'(\mathtt{m}) \\ \frac{1}{\hbar}\mathtt{m}g'(\mathtt{m}) & \frac{\hbar}{2}|j|_\nu^2 + \frac{1}{\hbar}\mathtt{m}g'(\mathtt{m}) \end{pmatrix}, \quad j \in \mathbb{Z}^d. \tag{1.2.28}$$

The matrix in (1.2.28) is diagonalizable for any $j \neq 0$ whereas for $j = 0$ it is not. For this reason it is necessary to rewrite the equation in a system of coordinates which does not involve the zero mode.

Elimination of the zero mode. The degeneracy of the matrix $\mathcal{L}(0)$ in (1.2.28) is due to the symmetries of the system. To clarify this point we note that, linearizing system (1.1.2) in (ρ, ϕ) variable, we get a Fourier multiplier with symbol

$$\mathcal{A}(j) := \begin{pmatrix} 0 & \mathtt{m}|j|_\nu^2 \\ -\frac{\kappa}{\mathtt{m}}|j|_\nu^2 - g'(m) & 0 \end{pmatrix} \tag{1.2.29}$$

which, for the zero mode, gives the non-diagonalizable matrix

$$\mathcal{A}(0) := \begin{pmatrix} 0 & 0 \\ -g'(\mathtt{m}) & 0 \end{pmatrix}. \tag{1.2.30}$$

On the other hand, thanks to conservation of the mass

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^d_\nu} (\mathtt{m} + \rho)\,\mathrm{d}x = 0, \tag{1.2.31}$$

we can consider the variable ρ to have zero average. Moreover system (QHD) is invariant by the one parameter family of transformations

$$\begin{pmatrix} \rho \\ \phi \end{pmatrix} \to \begin{pmatrix} \rho \\ \phi + c \end{pmatrix}, \quad c \in \mathbb{R}. \tag{1.2.32}$$

As a consequence we can consider (QHD) as a closed system for the variables $(\rho, \Pi_0^\perp \phi)$ which do not include the zero mode.

Back to the Madelung variable $\psi = \sqrt{\mathtt{m} + \rho}\, e^{\mathrm{i}\phi/\hbar}$ in (1.2.24) we note that (1.2.31) induces the conservation of L^2 norm of ψ

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\psi\|_{L^2}^2 = 0, \quad \|\psi\|_{L^2}^2 \equiv \mathtt{m} \tag{1.2.33}$$

and (1.2.32) implies that equation (1.2.25) for ψ is invariant under phase rotations

$$\psi \to \psi e^{\mathrm{i}c}, \quad c \in \mathbb{R}. \tag{1.2.34}$$

In Section 3.2.2 we shall use the invariance (1.2.34) as well as the L^2 norm preservation (1.2.33) to eliminate the dynamics of the zero mode. In particular we find a new variable z in (3.2.19) whose Fourier mode $\neq 0$ describes the complete dynamics of (1.2.25). Since the linearized equation at $z = 0$ remains unchanged, we diagonalize the matrix in (1.2.27).

Diagonalized system and dispersion law: After the diagonalization of the matrix in (1.2.27) we end up with the diagonal, quadratic, Hamiltonian, semi-linear equation

$$\partial_t w = -\mathrm{i}\omega(D)w - \mathrm{i}\partial_{\overline{w}}\widetilde{\mathcal{K}}^{(3)}_{\mathtt{m}}(w, \overline{w}) - \mathrm{i}\partial_{\overline{w}}\widetilde{\mathcal{K}}^{(\geq 4)}_{\mathtt{m}}(w, \overline{w}) \tag{1.2.35}$$

where:

• $\omega(D)$ is the Fourier multiplier associated to the dispersion relation

$$\omega(j) := \sqrt{\frac{\hbar^2}{4}|j|_\nu^4 + \mathtt{m}g'(\mathtt{m})|j|_\nu^2}, \quad j \in \mathbb{Z}^d \setminus \{0\}; \tag{1.2.36}$$

• $\widetilde{\mathcal{K}}^{(3)}_{\mathtt{m}}(w,\overline{w})$ is a cubic real Hamiltonian of the form

$$\widetilde{\mathcal{K}}^{(3)}_{\mathtt{m}}(w,\overline{w}) = \sum_{\substack{\vec{\sigma}\in\{-1,1\}^3,\ \vec{\jmath}\in(\mathbb{Z}^d\setminus\{0\})^3\\ \sigma_1 j_1+\sigma_2 j_2+\sigma_3 j_3=0}} \mathcal{K}^{\vec{\sigma}}_{\vec{\jmath}}\, w^{\sigma_1}_{j_1} w^{\sigma_2}_{j_2} w^{\sigma_3}_{j_3}\,, \qquad |\mathcal{K}^{\vec{\sigma}}_{\vec{\jmath}}| \lesssim 1; \tag{1.2.37}$$

• $\widetilde{\mathcal{K}}^{(\geq 4)}_{\mathtt{m}}(w,\overline{w})$ is a real Hamiltonian such that

$$\|\partial_{\overline{w}}\widetilde{\mathcal{K}}^{(\geq 4)}_{\mathtt{m}}(w,\overline{w})\|_{H^s} \lesssim \|w\|^3_{H^s}, \tag{1.2.38}$$

for any $w \in H^s$ sufficiently small. At this point we are ready to define a suitable modified energy. Our primary aim is to control the derivative of the H^s-norm of the solution

$$\frac{d}{dt}N_s(w)\,, \quad N_s(w) := \sum_{j\in\mathbb{Z}^d\setminus\{0\}} |j|^{2s}|w_j|^2 \tag{1.2.39}$$

for the longest time possible. Using the Hamiltonian structure of the equation, we write (1.2.39) as

$$\frac{d}{dt}N_s(w) = \{N_s, \widetilde{\mathcal{K}}^{(3)}_{\mathtt{m}}\}(w). \tag{1.2.40}$$

We perturb the Sobolev energy by homogeneous functionals of degree 3 such that their time derivatives cancel out the main contribution (i.e., the one coming from cubic terms) in (1.2.40), up to remainders of higher order. Following normal form ideas, we define, given a tri-linear Hamiltonian

$$H(w) = \sum_{\substack{\vec{\sigma}\in\{-1,1\}^3,\ \vec{\jmath}\in(\mathbb{Z}^d\setminus\{0\})^3\\ \sigma_1 j_1+\sigma_2 j_2+\sigma_3 j_3=0}} H^{\vec{\sigma}}_{\vec{\jmath}}\, w^{\sigma_1}_{j_1} w^{\sigma_2}_{j_2} w^{\sigma_3}_{j_3},$$

the adjoint action associated to $\mathcal{K}^{(2)}(w) := \sum_{j\in\mathbb{Z}^d\setminus\{0\}} \omega(j)|w_j|^2$ and its (formal) inverse as

$$\begin{aligned} \mathrm{ad}_{\mathcal{K}^{(2)}} H(w) :=& \{\mathcal{K}^{(2)}, H\}(w) \\ =& \sum_{\substack{\vec{\sigma}\in\{-1,1\}^3,\ \vec{\jmath}\in(\mathbb{Z}^d\setminus\{0\})^3\\ \sigma_1 j_1+\sigma_2 j_2+\sigma_3 j_3=0}} \mathrm{i}(\sigma_1\omega(j_1)+\sigma_2\omega(j_2)+\sigma_3\omega(j_3)) H^{\vec{\sigma}}_{\vec{\jmath}}\, w^{\sigma_1}_{j_1} w^{\sigma_2}_{j_2} w^{\sigma_3}_{j_3} \\ \mathrm{ad}^{-1}_{\mathcal{K}^{(2)}} H(w) :=& \sum_{\substack{\vec{\sigma}\in\{-1,1\}^3,\ \vec{\jmath}\in(\mathbb{Z}^d\setminus\{0\})^3\\ \sigma_1 j_1+\sigma_2 j_2+\sigma_3 j_3=0}} \frac{1}{\mathrm{i}(\sigma_1\omega(j_1)+\sigma_2\omega(j_2)+\sigma_3\omega(j_3))} H^{\vec{\sigma}}_{\vec{\jmath}}\, w^{\sigma_1}_{j_1} w^{\sigma_2}_{j_2} w^{\sigma_3}_{j_3}. \end{aligned} \tag{1.2.41}$$

In this way, defining

$$E_3(w) := \mathrm{ad}^{-1}_{\mathcal{K}^{(2)}}\{N_s, \widetilde{\mathcal{K}}^{(3)}_{\mathtt{m}}\}(w), \tag{1.2.42}$$

we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(N_s(w)+E_3(w)) = \{E_3, \widetilde{\mathcal{K}}^{(3)}_{\mathtt{m}}\}(w). \tag{1.2.43}$$

The right hand side of the above equality is a quartic Hamiltonian but there are small divisors problem which leads to a loss of derivatives.

Quasi-resonances: The denominator

$$\pm\omega(j_1) \pm \omega(j_2) \pm \omega(j_3)$$

in (1.2.41) can accumulate to zero and leads to a loss of derivatives. Therefore we need to impose some lower bounds on the small divisors. Here we exploit the irrationality of the torus $\mathbb{T}^d_\nu$. We prove indeed that for almost any $\nu \in [1,2]^d$, there exists $\gamma > 0$ such that

$$| \pm \omega(j_1) \pm \omega(j_2) \pm \omega(j_3)| \geq \frac{\gamma}{\mu_1^{d-1} \log^{d+1}\left(1+\mu_1^2\right) \mu_3^{M(d)}}, \tag{1.2.44}$$

if $\pm j_1 \pm j_2 \pm j_3 = 0$, we denoted by $M(d)$ a positive constant depending on the dimension d and μ_i the i-st largest integer among $|j_1|, |j_2|$ and $|j_3|$. It is nowadays well known, see for instance [15, 19], that the power of μ_3 is not dangerous if we work in H^s with s big enough. Unfortunately we have also a power of the highest frequency μ_1 which represents, in principle, a loss of derivatives. However, this loss of derivatives may be transformed in a loss of length of the lifespan through partition of frequencies, as done for instance in [56, 87, 68, 23], that we shall now explain.

Frequencies partition: First of all we note that, thanks to (1.2.38), the non trivial contribution to the dynamics for times $T \lesssim \varepsilon^{-2}$ comes only from the truncated system

$$\partial_t w = -\mathrm{i}\omega(D) w - \mathrm{i}\partial_{\overline{w}} \widetilde{\mathcal{K}}^{(3)}(w, \overline{w}). \tag{1.2.45}$$

Then to fix ideas suppose that the sum in (1.2.37) is supported in $|j_1| \geq |j_2| \geq |j_3|$ and note that, if the corresponding signs satisfy $\sigma_1\sigma_2 = +$, then the corresponding small divisor satisfies

$$|\omega(j_1) + \omega(j_2) \pm \omega(j_3)| \gtrsim 1. \tag{1.2.46}$$

This suggest the splitting

$$\widetilde{\mathcal{K}}^{(3)}_{\mathtt{m}}(w, \overline{w}) = \widetilde{\mathcal{K}}^{(3,+)}(w, \overline{w}) + \widetilde{\mathcal{K}}^{(3,-)}(w, \overline{w}), \tag{1.2.47}$$

where $\widetilde{\mathcal{K}}^{(3,+)}(w, \overline{w})$ is the Hamiltonian obtained restricting the sum in (1.2.37) to the signs such that $\sigma_1\sigma_2 = +$ and similarly $\widetilde{\mathcal{K}}^{(3,-)}(w, \overline{w})$ is restricted when $\sigma_1\sigma_2 = -$. Thanks to the strong non-resonance condition (1.2.46), the Hamiltonian $\widetilde{\mathcal{K}}^{(3,+)}(w, \overline{w})$ can be easily removed by including in the modified energy E_3 the term

$$E_3^+(w) := \mathrm{ad}^{-1}_{\mathcal{K}^{(2)}} \{N_s, \widetilde{\mathcal{K}}^{(3,+)}\}.$$

On the other hand the contribution to the H^s energy estimate coming from $\widetilde{\mathcal{K}}^{(3,-)}_{\mathtt{m}}(w, \overline{w})$ can be easily bounded by

$$\{N_s, \widetilde{\mathcal{K}}^{(3,-)}_{\mathtt{m}}\} \lesssim \|w\|_{H^{s_0}} \|w\|_{H^{s-1}} \|w\|_{H^s} \tag{1.2.48}$$

for some $s_0 > 0$. To transform the gain of derivatives in the above estimate in an improved smallness, we do a high-low frequencies decomposition of the Hamiltonian

$$\widetilde{\mathcal{K}}^{(3,-)} = \widetilde{\mathcal{K}}^{(3,-)}_{\leq N} + \widetilde{\mathcal{K}}^{(3,-)}_{>N}, \qquad \begin{cases} (\widetilde{\mathcal{K}}^{(3,-)}_{\leq N})^{\vec{\sigma}}_{\vec{\jmath}} = (\widetilde{\mathcal{K}}^{(3,-)})^{\vec{\sigma}}_{\vec{\jmath}} & |j_2| \leq N \\ 0 & \text{oterwise.} \end{cases}$$

In this way one has, for the high frequencies part, the improved bound

$$|\{N_s, \widetilde{\mathcal{K}}^{(3,-)}_{>N}\}| \lesssim N^{-1} \|w\|^3_{H^s}. \tag{1.2.49}$$

Thanks to (1.2.44) we reduce the low frequencies part by adding, in the modified energy, a low frequencies term whose unboundedness is controlled in terms of N

$$E_3^- := \mathrm{ad}^{-1}_{\mathcal{K}^{(2)}} \{N_s, \widetilde{\mathcal{K}}^{(3,-)}_{\leq N}\}, \quad |(E_3)^{\vec{\sigma}}_{\vec{\jmath}}| \lesssim_s N^{d-2} \log^{d+1}(1+N) |j_3|^{M+1} |j_1|^{2s}.$$

Finally defining $E_3 = E_3^+ + E_3^-$ we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(N_s(w) + E_3(w)) \lesssim N^{d-2}\log^{d+1}(1+N)\|w\|_{H^s}^4 + N^{-1}\|w\|_{H^s}^3. \tag{1.2.50}$$

The long time result Theorem 1.1.2 is obtained by a classical bootstrap argument for the small Sobolev norm $\|w\|_{H^s} \lesssim \varepsilon$ and optimizing the above estimate taking $N \sim \varepsilon^{-\frac{1}{d-1}}$.

1.2.3 Theorem 1.1.3

The life span estimate (1.1.24) and the bound (1.1.25) for the solutions of (1.1.19) follow by an energy estimate for $\|(\eta,\psi)\|_{X^s} := \|\eta\|_{H_0^{s+\frac14}} + \|\psi\|_{\dot H^{s-\frac14}}$ of the form

$$\|(\eta,\psi)(t)\|_{X^s}^2 \lesssim_{s,N} \|(\eta,\psi)(0)\|_{X^s}^2 + \int_0^t \|(\eta,\psi)(\tau)\|_{X^s}^{N+3}\,\mathrm{d}\tau\,. \tag{1.2.51}$$

The fact that the right hand side of the above estimate contains the same norm $\|\ \|_{X^s}$ of the left hand side is non trivial at all because the equations (1.1.19) are quasi-linear. Also the presence of the exponent N is highly not trivial because the non-linearity in (1.1.19) vanishes only quadratically at $(\eta,\psi) = (0,0)$. This will be a consequence of the Hamiltonian Birkhoff normal form reduction.

Wahlén coordinates and complex Hamiltonian form: As we already pointed out in (1.1.27), system (1.1.19) is Hamiltonian but (η,ψ) are not standard Darboux variables. However, following Wahlén [115], we introduce new coordinates

$$\begin{pmatrix}\eta\\ \zeta\end{pmatrix} = \begin{pmatrix}\eta\\ \psi - \frac{\gamma}{2}\partial_x^{-1}\eta\end{pmatrix}, \tag{1.2.52}$$

which are standard Darboux variables, in the sense that, in these coordinates, the water waves system has the form

$$\partial_t\begin{pmatrix}\eta\\ \zeta\end{pmatrix} = J\begin{pmatrix}\nabla_\eta H_\gamma(\eta,\zeta)\\ \nabla_\zeta H_\gamma(\eta,\zeta)\end{pmatrix} \qquad \text{where} \qquad J := \begin{pmatrix}0 & \mathrm{Id}\\ -\mathrm{Id} & 0\end{pmatrix}. \tag{1.2.53}$$

Next, passing to the complex variable u, we get the system

$$\partial_t u = -\mathrm{i}\Omega(D)u - \mathrm{i}\nabla_{\overline{u}} H^{(\geq 3)}(U), \quad J_c := \begin{pmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{pmatrix}, \quad U := \begin{pmatrix}u\\ \overline{u}\end{pmatrix} \tag{1.2.54}$$

where $\Omega(D)$ is the Fourier multiplier whose symbol is the water waves dispersion relation in (1.1.29) and $H^{(\geq 3)}(U) = H^{(3)}(U) + \cdots + H^{(N+3)}(U) + \ldots$ is the part of the real Hamiltonian $H_\gamma(\eta,\zeta)$ (written in complex coordinate U) with homogeneity ≥ 3. The energy estimate (1.2.51) will follow from the equivalent complex energy estimate

$$\|u(t)\|_{\dot H^s}^2 \lesssim_{s,N} \|u(0)\|_{\dot H^s}^2 + \int_0^t \|u(\tau)\|_{\dot H^s}^{N+3}\,\mathrm{d}\tau\,, \quad N \geq 0. \tag{1.2.55}$$

The fact that the right hand side in (1.2.55) contains the same norm $\|\ \|_{\dot H^s}$ of the left hand side is non trivial at all because the non-linear term $J_c\nabla_U H^{(\geq 3)}(U)$ contains derivatives being (1.2.54) a quasi-linear system.

Symmetrization of the system: The first step is to prove estimate (1.2.55) for $N = 0$, which is already not trivial. We follow a scheme close to Alazard-Burq-Zuily [2]:

(i) We para-linearize the water-waves system (Lemma 4.5.1 and Lemma 4.5.3);

(ii) We write it in complex variable;

(iii) We write the system using a complex version of the so-called good unknown of Alinhac (Lemma 4.6.4)

$$\omega = \psi - \mathrm{Op}^{BW}(B)\eta; \tag{1.2.56}$$

(iv) We symmetrize the para-differential vector field of positive order (Lemma 4.6.5).

This procedure slightly differs from the one in [2] (who first perform the good unknown of Alinhac change of variables (1.2.56)) but we prefer to follow this order because after step (ii) we obtain a system for the complex variable u which is a complex Darboux variable (see (1.2.54)).

Summarizing Lemma 4.6.4 and Lemma 4.6.5 we find a transformation

$$V = \begin{pmatrix} v \\ \overline{v} \end{pmatrix} = \mathcal{G}_1(U)U, \quad \text{with} \quad \|v\|_{\dot{H}^s} \sim \|u\|_{\dot{H}^s},$$

such that the variable V solves

$$\partial_t V = -\mathrm{i}\boldsymbol{\Omega}(D)V + \mathrm{Op}^{BW}_{\mathtt{vec}}\Big(\mathrm{i}a_{\frac{3}{2}}(V;t,x,\xi)\Big)V + \mathrm{Op}^{BW}(A_0(V;t,x,\xi))V + R(V;t)V, \tag{1.2.57}$$

where $a_{\frac{3}{2}}$ is a real symbol of order $\frac{3}{2}$ and $A_0(V;t,x,\xi)$ is a matrix of symbols of order 0, $R(V;t)$ are smoothing operators, we used the notation in (4.2.24) for diagonal para-differential matrices and $\boldsymbol{\Omega}(D)$ is the diagonal matrix associated to the Fourier multiplier $\Omega(D)$ (see (4.5.10)). Actually due to the fact that we do not completely invert our transformations, the high homogeneity part of the symbols and the smoothing remainder in the above equation remain expressed in terms of U but we skip to discuss this technical detail as does not play an important role in the proof.

Proceeding in the same way as in Section 1.2.1 at this level of the proof one can prove an energy estimate of the form (1.2.55) with $N = 0$. To improve the estimate (1.2.55) for arbitrary $N \geq 1$ we develop a Hamiltonian normal form approach adapted to quasi-linear PDEs. As we will explain this requires several additional ideas with respect to the classical normal form approach explained in Section 1.1.5.

A formal Birkhoff normal form approach: As we pointed out at the end of Section 1.1.5, for quasi-linear systems the perturbative approach of the reduction in degree of homogeneity leads only to formal results. We explain here again, using a different formalism with respect to Section 1.1.5, the difficulty that one immediately encounters when using a direct Birkhoff normal form approach. We aim to eliminate the quadratic terms present in the right-hand side of equation (1.2.57). Then expanding (1.2.57) in homogeneity we write it as

$$\partial_t V = -\mathrm{i}\boldsymbol{\Omega}(D)V + X^{(2)}(V) + \dots \tag{1.2.58}$$

where $X^{(2)}(V)$ is a quadratic, unbounded vector field which we want to remove by means of a change of variable. The general idea to do so is to look for a change of variable of the form

$$W = \mathcal{F}(V) = \mathcal{F}^{\tau}(V)_{|\tau=1}, \quad \begin{cases} \partial_\tau \mathcal{F}^{\tau}(V) = G^{(2)}(\mathcal{F}^{\tau}(V)) \\ \mathcal{F}^0(V) = V \end{cases} \tag{1.2.59}$$

where $G^{(2)}(V)$ is an appropriate quadratic vector field selected to cancel $X^{(2)}(V)$ up to higher homogeneity remainders. The new equation for W reads

$$\partial_t W = \mathrm{d}_V\mathcal{F}(V)[-\mathrm{i}\boldsymbol{\Omega}(D)V + X^{(2)}(V) + \dots] \tag{1.2.60}$$

$$= -\mathrm{i}\boldsymbol{\Omega}(D)W \tag{1.2.61}$$

$$+ X^{(2)}(V) + \mathrm{i}\boldsymbol{\Omega}(D)G^{(2)}(V) + \mathrm{d}_V G^{(2)}(V)[-\mathrm{i}\boldsymbol{\Omega}(D)] \tag{1.2.62}$$

$$+ \mathrm{d}_V G^{(2)}(V)[X^{(2)}(V)] + \dots \tag{1.2.63}$$

where in (1.2.62) we collect the quadratic parts and in (1.2.63) the terms in the vector field which are at least cubic. To remove the quadratic part of the equation one imposes the homological equation

$$X^{(2)}(V) + \mathrm{i}\boldsymbol{\Omega}(D)G^{(2)}(V) + \mathrm{d}_V G^{(2)}(V)[-\mathrm{i}\boldsymbol{\Omega}(D)] = 0 \tag{1.2.64}$$

from which one finds $G^{(2)}(V)$ if three wave interactions are absent for the frequencies $\Omega_j(\kappa)$. We immediately note that

1. In general, if $X^{(2)}(V)$ is unbounded, the solution $G^{(2)}(V)$ of (1.2.64) is unbounded and it is not clear if the change of variable is well-defined;
2. The higher homogeneity terms, as $\mathrm{d}_V G^{(2)}(V)[X^{(2)}(V)]$, accumulate derivatives.

In conclusion a direct approach to Birkhoff normal form for water waves does not seem to be possible and a preliminary reduction in decreasing para-differential degree has to be performed before the reduction in degree of homogeneity.

Linearly Hamiltonian para-differential normal form: To overcome to the unboundedness of water-waves, starting from the system (1.2.57) for the variable V, we perform in Section 4.6 several para-differential change of variable which reduce iteratively the system in decreasing para-differential order. The system will be reduced to an unbounded, non-linear Fourier multiplier up to a smoothing remainder. In particular the final outcome of Section 4.6 is Proposition 4.6.1 where is provided a para-differential transformation $W = \mathbf{B}(U;t)U$ such that W solves

$$\partial_t W = \mathrm{Op}^{BW}_{\mathtt{vec}}\Big(\mathrm{i}\mathtt{m}_{\frac{3}{2}}(U;t,\xi)\Big)W + R(U;t)W \tag{1.2.65}$$

where $\mathtt{m}_{\frac{3}{2}}(U;t,\xi)$ is a Fourier multiplier (i.e. independent of x) whose imaginary part has order 0 and vanishes at order $O(\|U\|^{N+1})$ and $R(U;t)$ is a smoothing remainder.

Let us make some comment:

1. The linear map $\mathbf{B}(U;t)$, as para-differential operators, is a spectrally localized map see Definition 4.2.16.
2. A pseudo-differential version of our para-differential reduction has been developed for irrotational gravity-capillary water waves to overcome small divisors problems. In particular in Alazard-Baldi [1] it is used to prove the existence of periodic standing waves and in Berti-Montalto [34] for the existence of quasi-periodic standing waves. For the first time, similar reduction is performed in a non-linear context by Berti-Delort [27] for water-waves and then by Feola- Iandoli [70] for reversible quasi-linear Schrödinger equations. The main difference is that in [27] and [70], the linear Hamiltonian structure on the symbols (as defined in Definition 4.3.7) is not preserved, while our reduction method maintains it. Even though it is only the starting point to recover the full non-linear Hamiltonian structure, a more accurate analysis is required to preserve the linear Hamiltonian one, as compared to [27], indeed:
   - (i) The para-linearization formula of the Dirichlet-Neumann operator in [27] contains non-explicit symbols of negative order (see (4.5.17)) which *a priori* do not satisfy the linear Hamiltonian structure in (4.3.10). Despite this, we recover the linear Hamiltonian structure in complex coordinates at every degree of homogeneity in Lemma 4.5.5 thanks to the abstract Lemma 4.3.20;
   - (ii) The map $\mathbf{B}(U;t)$ is obtained by composing iteratively several para-differential transformations. To preserve the linear Hamiltonian structure, each transformation has to be linearly symplectic, for this reason we have to realize it as a U dependent linear flow $\mathcal{G}(U)$ of some linearly Hamiltonian operator as we shall explain in the next paragraph;

3. The liner map $\mathbf{B}(U;t)$ preserves only the linear Hamiltonian structure and the full non-linear Hamiltonian one is not preserved; as a consequence the equation (1.2.65) is not Hamiltonian;

To recover the full non-linear Hamiltonian structure we prove the abstract Theorem 4.4.1 and we apply it to the linearly symplectic map $\mathbf{B}(U;t)$.

To fix ideas we illustrate in the next paragraph the way we proceed to preserve the Hamiltonian structure, up to homogeneity N, in a generic transformation step along the proof of Theorem 1.1.3.

Symplectic conjugation step up to homogeneity N. Consider a real-to-real system in para-differential form

$$\partial_t U = X(U) = \mathrm{Op}^{\mathrm{BW}}(A(U;t,x,\xi))[U] + R(U;t)[U]\,, \quad U = \begin{bmatrix} u \\ \overline{u} \end{bmatrix}\,, \tag{1.2.66}$$

where $A(U;t,x,\xi)$ is a matrix of symbols and $R(U;t)$ are ϱ-smoothing operators, which admit a homogeneous expansion up to homogeneity N, whereas the terms with homogeneity $> N$ are dealt, as in [27], as time dependent symbols and remainders, see Section 4.2.1. This is quite convenient from a technical point of view because it does not demand much information about the higher degree terms. Moreover this enables to directly use the para-linearization of the Dirichlet-Neumann operator proved in [27]. System (1.2.66) is Hamiltonian up to homogeneity N, namely the homogeneous components of the vector field $X(U)$ of degree $\leq N+1$ have the Hamiltonian form

$$J_c \nabla H(U) \qquad \text{where} \qquad J_c = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix} \tag{1.2.67}$$

is the Poisson tensor and $H(U)$ is a real valued pluri-homogeneous Hamiltonian of degree $\leq N+2$. Moreover the para-differential operator $\mathrm{Op}^{\mathrm{BW}}(A(U))$ in (1.2.66) is a linear Hamiltonian operator, up to homogeneity N, namely of the form $\mathrm{Op}^{\mathrm{BW}}(A(U)) = J_c \mathrm{Op}^{\mathrm{BW}}(B(U))$ where $B(U)$ is a symmetric operator up to homogeneity N, see Definition 4.3.6.

In order to prove energy estimates for (1.2.66) we transform it under several changes of variables. Actually we do not really perform changes of variables of the phase space, but we proceed in the time dependent setting due to the high homogeneity terms. Let us discuss a typical transformation step. Let $\mathcal{G}(U;t) := \mathcal{G}^\tau(U;t)|_{\tau=1}$ be the time 1-flow

$$\partial_\tau \mathcal{G}^\tau(U;t) = J_c \mathrm{Op}^{\mathrm{BW}}\big(B(U;\tau,t,x,\xi)\big)\mathcal{G}^\tau(U;t)\,, \quad \mathcal{G}^0(U;t) = \mathrm{Id}\,, \tag{1.2.68}$$

generated by a linearly Hamiltonian operator $J_c \mathrm{Op}^{\mathrm{BW}}\big(B(U;\tau,t,x,\xi)\big)$ up to homogeneity N. The transformation $\mathcal{G}(U;t)$ is invertible and bounded on $\dot{H}^s(\mathbb{T}) \times \dot{H}^s(\mathbb{T})$ for any $s \in \mathbb{R}$ and it admits a pluri-homogeneous expansion $\mathcal{G}_{\leq N}(U)$, which is an unbounded operator if the generator $J_c \mathrm{Op}^{\mathrm{BW}}\big(B\big)$ is unbounded, see Section 4.3.3. If U solves (1.2.66) then the variable

$$W := \mathcal{G}(U;t)U \tag{1.2.69}$$

solves a new system in para-differential form

$$\partial_t W = X_+(W) = \mathrm{Op}^{\mathrm{BW}}(A_+(W;t,x,\xi))[W] + R_+(W;t)[W] \tag{1.2.70}$$

(actually the symbols and remainders of homogeneity $> N$ in (1.2.70) are still expressed in terms of U, but for simplicity we skip to discuss this issue here). In Section 4.6 we perform several transformations of this kind, choosing suitable generators $J_c \mathrm{Op}^{BW}\big(B\big)$ (either bounded or unbounded) in order to obtain a diagonal matrix A_+ with x-independent symbols.

We remark that, with this procedure, since the time one flow map $\mathcal{G}(U;t)$ of the linear Hamiltonian system (1.2.68) is only *linearly* symplectic up to homogeneity N, namely

$$\mathcal{G}(U;t)^\top E_c \mathcal{G}(U;t) = E_c + E_{>N}(U;t)\,, \quad E_{>N}(U;t) = O(\|U\|^{N+1})\,,$$

the new system (1.2.70) *is not* Hamiltonian anymore, not even its pluri-homogeneous components of degree $\leq N+1$. The new system (1.2.70) is only linearly symplectic, up to homogeneity N, see Lemma 4.3.15. In order to obtain a new Hamiltonian system up to homogeneity N, we use the Darboux results of Section 4.4 to construct perturbatively a "symplectic corrector" of the transformation (1.2.69).

Let us say some words about the construction of the symplectic corrector. We remark that the perturbed symplectic tensor $E_{\leq N}(V)$ induced by the non-symplectic transformation $\mathcal{G}_{\leq N}(U)$ is *not* a smoothing perturbation of the standard Poisson tensor E_c. However, Lemmata 4.4.4 and 4.4.5 prove that, for any pluri-homogeneous vector field $X(V)$, we have

$$E_{\leq N}(V)[X(V)] = E_c X(V) + \nabla \mathcal{W}(V) + \text{ smoothing vector fields} + \text{high homogeneity terms}$$

where $\mathcal{W}(V)$ is a scalar function. This algebraic structural property enables us to prove the Darboux Proposition 4.4.7, thus Theorem 4.4.1, via a deformation argument à la Moser. We also remark that the operators $R_{\leq N}(\cdot)$ of Theorem 4.4.1 are smoothing for arbitrary $\varrho \geq 0$, since they have 2 equivalent frequencies, namely $\max_2(n_1,\ldots,n_{p+1}) \sim \max(n_1,\ldots,n_{p+1})$ in (4.2.38), arising by applications of Lemma 4.2.21. This property compensates the presence of unbounded operators in $\mathcal{G}_{\leq N}(U)$.

In conclusion, Theorem 4.4.1 provides a nonlinear map $W + R_{\leq N}(W)W$, where $R_{\leq N}(W)$ are pluri-homogeneous ϱ-smoothing operators (for arbitrary $\varrho > 0$) such that the pluri-homogeneous map

$$\mathcal{D}_N(U) := (\mathrm{Id} + R_{\leq N}(\cdot)) \circ \mathcal{G}_{\leq N}(U)U$$

is symplectic up to homogeneity N, i.e.

$$\left[\mathrm{d}_U \mathcal{D}_N(U)\right]^\top E_c \left[\mathrm{d}_U \mathcal{D}_N(U)\right] = E_c + E_{>N}(U) \tag{1.2.71}$$

where $E_c := J_c^{-1}$ is the standard symplectic tensor and $E_{>N}(U)$ is an operator of homogeneity degree $\geq N+1$. As a consequence, since (1.2.66) is Hamiltonian up to homogeneity N, the variable

$$Z(t) := \mathcal{D}(U(t);t) := W(t) + R_{\leq N}(W(t)) = (\mathrm{Id} + R_{\leq N}(\cdot)) \circ \mathcal{G}(U(t);t)U(t)$$

satisfies a system which is *Hamiltonian up to homogeneity* N as well, and which has, since $R_{\leq N}(\cdot)$ are smoothing operators, the same para-differential form as in (1.2.70),

$$\partial_t Z = X_{++}(Z) = \mathrm{Op}^{\mathrm{BW}}\left(A_{++}(Z;t,x,\xi)\right)[Z] + R_{++}(Z;t)[Z]\,. \tag{1.2.72}$$

This is the content of Theorem 4.7.1. Note that the matrix of symbols $A_{++}(Z;t,x,\xi)$ in (1.2.72) is obtained by substituting in $A_+(W;t,x,\xi)$ the relation $W = Z - R_{\leq N}(Z) + \ldots$ obtained inverting $Z = W + R_{\leq N}(W)$ approximately up to homogeneity N. This procedure is rigorously justified in Lemmata A.0.4 and A.0.5.

Hamiltonian Birkhoff normal form. We perform the Hamiltonian Birkhoff normal form reduction in Section 4.7 for any value of the surface tension κ outside the set $\mathcal{K}$ defined in Theorem B.0.1. We start from the Hamiltonian (up to homogeneity N) equation of the form

$$\begin{aligned}\partial_t Z_0 =& \mathrm{Op}^{BW}_{\mathtt{vec}}\left(\mathrm{i}\mathtt{m}_{\frac{3}{2}}(Z_0;t,\xi)\right)Z_0 + R(Z_0;t)Z_0 \\ =& -\mathrm{i}\boldsymbol{\Omega}(D)Z_0 + J_c\nabla H^{(\geq 3)}(Z_0) + \mathrm{Op}^{BW}_{\mathtt{vec}}\left(\mathrm{i}(\mathtt{m}_{\frac{3}{2}})_{>N}(U;t,\xi)\right)Z_0 + R_{>N}(U;t)Z_0\end{aligned}$$

for the variable Z_0 obtained from $W = \mathbf{B}(U;t)U$ by applying the Darboux corrector (see Proposition 4.7.2). We first iteratively reduce the p–homogeneous x-independent para-differential symbol to its super-action-preserving component, via the linear flow generated by an unbounded Fourier multiplier, see (4.7.42). Since such transformation is only linearly symplectic, we apply again Theorem 4.7.1 to recover a Hamiltonian system up to homogeneity N, see system (4.7.56). Finally we reduce the $(p+1)$-homogeneous component of the Hamiltonian smoothing vector field to its super-action preserving part, see (4.7.70). The key property is that, a super-action preserving Hamiltonian, Poisson commutes with the super-actions functionals (see e.g (1.1.75)). After N iterations, the final outcome is the Hamiltonian Birkhoff normal form system (4.7.21), which has the form

$$\partial_t Z = J_c \nabla H^{(\mathtt{SAP})}(Z) + \mathrm{Op}^{BW}_{\mathtt{vec}}\Big(-\mathrm{i}(\mathtt{m}_{\frac{3}{2}})_{>N}(U;t,\xi)\Big)Z + R_{>N}(U;t)U \tag{1.2.73}$$

where $H^{(\mathtt{SAP})}(Z)$ is a super-action preserving Hamiltonian (Definition 4.7.8) and the higher order homogeneity para-differential and smoothing terms admit energy estimates in Sobolev spaces (the imaginary part of the symbol $(\mathtt{m}_{\frac{3}{2}})_{>N}$ has order zero). Here are some comments:

- The super action preserving Hamiltonian contains only p-homogeneous monomials of even degree $p = 2\ell$ of the form

$$u_{j_1}\cdots u_{j_\ell}\overline{u}_{j_{\ell+1}}\cdots\overline{u}_{j_p}, \quad \{|j_1|,\ldots,|j_\ell|\} = \{|j_{\ell+1}|,\ldots,|j_p|\} \tag{1.2.74}$$

- The monomials in (1.2.74) can be either of the form $|u_{j_1}|^2\cdots|u_{j_\ell}|^2$ (integrable monomials) or of the more general form

$$u_{j_1}\cdots u_{j_{\ell'}}\overline{u}_{-j_1}\cdots\overline{u}_{-j_{\ell'}} \times \text{integrable monomials}. \tag{1.2.75}$$

  A non-integrable vector field correspondent to the monomial in (1.2.75) in general does not preserve the Sobolev norm even if it is reversible, nevertheless if it is Hamiltonian, as we explained in Section 1.1.5, it preserves the super-actions

$$J_n(z) = |z_{-n}|^2 + |z_n|^2, \quad \text{for any} \quad n\in\mathbb{N}$$

  and all Sobolev norms are constant along its flow. For this reason the approach in [27, 70] allows to deal only with standing waves (which are not invariant under the flow of (1.1.19) with $\gamma \neq 0$);

- For irrotational fluids ($\gamma = 0$), super-action preserving monomials are always resonant because the dispersion relation reduce to $\Omega_j(\kappa) = \omega_j(\kappa)$ which is even with respect to j. On the other hand, when $\gamma \neq 0$, the monomials in (1.2.75) can be either resonant or non-resonant: indeed the odd part of the dispersion relation $\gamma\frac{\mathtt{G}(j)}{j}$ allows (using also the momentum restriction induced by the invariance by translation) to exclude non-integrable 4 and 6 waves interactions, see Remark 4.7.16. However there are several super-action preserving monomials which are resonant for any value of $\gamma \in \mathbb{R}$ and for infinite depth $h = \infty$:

$$z_{n_1}\overline{z_{-n_1}}z_{n_2}\overline{z_{-n_2}}z_{-n_3}\overline{z_{n_3}}z_{-n_4}\overline{z_{n_4}}, \quad n_1+n_2 = n_3+n_4. \tag{1.2.76}$$

  For this reason a Hamiltonian approach is necessary also in case $\gamma \neq 0$.

Energy estimates: The Hamiltonian Birkhoff normal form equation $\partial_t Z = J_c\nabla H^{(\mathtt{SAP})}(Z)$ obtained neglecting the terms of homogeneity larger than N in (1.2.73) possesses the super–actions $|z_{-n}|^2 + |z_n|^2$, for any $n \in \mathbb{N}$, as prime integrals. Thus it preserves the Sobolev norms and the solutions of (1.2.73) with initial data of size ε have energy estimates up to times of order ε^{-N-1}. In conclusion, since the Sobolev norms of U in (1.2.54) and Z in (1.2.73) are equivalent, we deduce energy estimates for (1.2.54),

$$\|U(t)\|_{\dot{H}^s}^2 \lesssim_{s,N} \|U(0)\|_{\dot{H}^s}^2 + \int_0^t \|U(\tau)\|_{\dot{H}^s}^{N+3} \mathrm{d}\tau$$

valid up to times of order ε^{-N-1}. A standard bootstrap argument concludes the proof of Theorem 1.1.3.

Chapter 2

Local well posedness of the Euler-Korteweg system

In this chapter we prove the local well-posedness result outlined in Section 1.1.1. The chapter is self-contained. Also Section 2.1 provides a self-contained introduction to the problem, including the statement of the main result, and a review of relevant literature on the topic.

2.1 Introduction to Chapter 2

We consider the compressible Euler-Korteweg (EK) system

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \vec{u}) = 0 \\ \partial_t \vec{u} + \vec{u} \cdot \nabla \vec{u} + \nabla g(\rho) = \nabla\big(K(\rho)\Delta\rho + \frac{1}{2}K'(\rho)|\nabla\rho|^2\big)\,, \end{cases} \tag{2.1.1}$$

which is a modification of the Euler equations for compressible fluids to include capillary effects, under space periodic boundary conditions $x \in \mathbb{T}^d := (\mathbb{R}/2\pi\mathbb{Z})^d$. The scalar variable $\rho(t,x) > 0$ is the density of the fluid and $\vec{u}(t,x) \in \mathbb{R}^d$ is the time dependent velocity field. The functions $K(\rho)$, $g(\rho)$ are defined on $\mathbb{R}^+$, smooth, and $K(\rho)$ is positive.

The quasi-linear equations (2.1.1) appear in a variety of physical contexts modeling phase transitions [64], water waves [41], quantum hydrodynamics where $K(\rho) = \kappa/\rho$ [10], see also [42].

Local well posedness results for the (EK)-system have been obtained in Benzoni-Gavage, Danchin and Descombes [22] for initial data sufficiently localized in the space variable $x \in \mathbb{R}^d$. Then, thanks to dispersive estimates, global in time existence results have been obtained for small irrotational data by Audiard-Haspot [13], assuming the sign condition $g'(\rho) > 0$. The case of quantum hydrodynamics corresponds to $K(\rho) = \kappa/\rho$ and, in this case, the (EK)-system is formally equivalent, via Madelung transform, to a semi-linear Schrödinger equation on $\mathbb{R}^d$. Exploiting this fact, global in time weak solutions have been obtained by Antonelli-Marcati [10, 11] also allowing $\rho(t,x)$ to become zero (see also the recent paper [12]).

In [32] we prove a local in time existence result for the solutions of (2.1.1), with space periodic boundary conditions, under natural minimal regularity assumptions on the initial datum in Sobolev spaces, see Theorem 2.1.1. Relying on this result, in a forthcoming paper [103], we shall prove a set of long time existence results for the (EK)-system in 1-space dimension, in the same spirit of [27], [28].

We consider an initially irrotational velocity field that, under the evolution of (2.1.1), remains irrotational for all times. An irrotational vector field on $\mathbb{T}^d$ reads (Helmholtz decomposition)

$$\vec{u} = \vec{c}(t) + \nabla\phi\,, \quad \vec{c}(t) \in \mathbb{R}^d\,, \quad \vec{c}(t) = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} \vec{u}\,\mathrm{d}x\,, \tag{2.1.2}$$

where $\phi : \mathbb{T}^d \to \mathbb{R}$ is a scalar potential. By the second equation in (2.1.1) and $\operatorname{rot} \vec{u} = 0$, we get

$$\partial_t \vec{c}(t) = -\frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \vec{u} \cdot \nabla \vec{u} \, \mathrm{d}x = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} -\frac{1}{2} \nabla (|\vec{u}|^2) \, \mathrm{d}x = 0 \quad \Longrightarrow \quad \vec{c}(t) = \vec{c}(0)$$

is independent of time. Note that if the dimension $d = 1$, the average $\frac{1}{2\pi} \int_{\mathbb{T}} u(t,x) \mathrm{d}x$ is an integral of motion for (2.1.1), and thus any solution $u(t,x)$, $x \in \mathbb{T}$, of the (EK)-system (2.1.1) has the form (2.1.2) with $c(t) = c(0)$ independent of time, that is $u(t,x) = c(0) + \phi_x(t,x)$.

The (EK) system (2.1.1) is Galilean invariant: if $(\rho(t,x), \vec{u}(t,x))$ solves (2.1.1) then

$$\rho_{\vec{c}}(t,x) := \rho_{\vec{c}}(t, x + \vec{c}t), \quad \vec{u}_{\vec{c}}(t,x) := \vec{u}(t, x + \vec{c}t) - \vec{c}$$

solve (2.1.1) as well. Thus, regarding the Euler-Korteweg system in a frame moving with a constant speed $\vec{c}(0)$, we may always consider in (2.1.2) that

$$\vec{u} = \nabla \phi, \quad \phi : \mathbb{T}^d \to \mathbb{R}.$$

The Euler-Korteweg equations (2.1.1) read, for irrotational fluids,

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \nabla \phi) = 0 \\ \partial_t \phi + \frac{1}{2} |\nabla \phi|^2 + g(\rho) = K(\rho) \Delta \rho + \frac{1}{2} K'(\rho) |\nabla \rho|^2 . \end{cases} \tag{2.1.3}$$

The main result of the present chapter proves local well posedness for the solutions of (2.1.3) with initial data (ρ_0, ϕ_0) in Sobolev spaces

$$H^s(\mathbb{T}^d) := \Big\{ u(x) = \sum_{j \in \mathbb{Z}^d} u_j e^{\mathrm{i} j \cdot x} \ : \ \|u\|_s^2 := \sum_{j \in \mathbb{Z}^d} |u_j|^2 \langle j \rangle^{2s} < +\infty \Big\}$$

where $\langle j \rangle := \max\{1, |j|\}$, under the natural mild regularity assumption $s > 2 + (d/2)$. Along the chapter, $H^s(\mathbb{T}^d)$ may denote either the Sobolev space of real valued functions $H^s(\mathbb{T}^d, \mathbb{R})$ or the complex valued ones $H^s(\mathbb{T}^d, \mathbb{C})$.

Theorem 2.1.1. (Local existence on $\mathbb{T}^d$) *Let $s > 2 + \frac{d}{2}$ and fix $s_0 \in (\frac{d}{2}, s-2]$. For any initial data*

$$(\rho_0, \phi_0) \in H^s(\mathbb{T}^d, \mathbb{R}) \times H^s(\mathbb{T}^d, \mathbb{R}) \quad \text{with} \quad \rho_0(x) > 0, \quad \forall x \in \mathbb{T}^d,$$

there exists $T := T(\|(\rho_0, \phi_0)\|_{s_0+2}, \min_x \rho_0(x)) > 0$ and a unique solution (ρ, ϕ) of (2.1.3) such that

$$(\rho, \phi) \in C^0\Big([-T,T], H^s(\mathbb{T}^d, \mathbb{R}) \times H^s(\mathbb{T}^d, \mathbb{R})\Big) \cap C^1\Big([-T,T], H^{s-2}(\mathbb{T}^d, \mathbb{R}) \times H^{s-2}(\mathbb{T}^d, \mathbb{R})\Big)$$

and $\rho(t,x) > 0$ for any $t \in [-T,T]$. Moreover, for $|t| \le T$, the solution map $(\rho_0, \phi_0) \mapsto (\rho(t,\cdot), \phi(t,\cdot))$ is locally defined and continuous in $H^s(\mathbb{T}^d, \mathbb{R}) \times H^s(\mathbb{T}^d, \mathbb{R})$.

We remark that it is sufficient to prove the existence of a solution of (2.1.3) on $[0,T]$ because system (2.1.3) is reversible: the Euler-Korteweg vector field X defined by (2.1.3) satisfies $X \circ \mathcal{S} = -\mathcal{S} \circ X$, where $\mathcal{S}$ is the involution

$$\mathcal{S} \begin{pmatrix} \rho \\ \phi \end{pmatrix} := \begin{pmatrix} \rho^\vee \\ -\phi^\vee \end{pmatrix}, \quad \rho^\vee(x) := \rho(-x). \tag{2.1.4}$$

Thus, denoting by $(\rho, \phi)(t,x) = \Omega^t(\rho_0, \phi_0)$ the solution of (2.1.3) with initial datum (ρ_0, ϕ_0) in the time interval $[0,T]$, we have that $\mathcal{S}\Omega^{-t}(\mathcal{S}(\rho_0, \phi_0))$ solves (2.1.3) with the same initial datum but in the time interval $[-T, 0]$.

Let us make some comments about the phase space of system (2.1.3). Note that the average $\frac{1}{(2\pi)^d}\int_{\mathbb{T}^d}\rho(x)\,\mathrm{d}x$ is a prime integral of (2.1.3) (conservation of the mass), namely

$$\frac{1}{(2\pi)^d}\int_{\mathbb{T}^d}\rho(x)\,\mathrm{d}x = \mathtt{m}\,, \quad \mathtt{m}\in\mathbb{R}\,, \tag{2.1.5}$$

remains constant along the solutions of (2.1.3). Note also that the vector field of (2.1.3) depends only on $\phi - \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d}\phi\,\mathrm{d}x$. As a consequence, the variables $(\rho-\mathtt{m},\phi)$ belong naturally to some Sobolev space $H^s_0(\mathbb{T}^d)\times\dot{H}^s(\mathbb{T}^d)$, where $H^s_0(\mathbb{T}^d)$ denotes the Sobolev space of functions with zero average

$$H^s_0(\mathbb{T}^d) := \Big\{u\in H^s(\mathbb{T}^d)\,:\,\int_{\mathbb{T}^d}u(x)\mathrm{d}x = 0\Big\}$$

and $\dot{H}^s(\mathbb{T}^d)$, $s\in\mathbb{R}$, the corresponding homogeneous Sobolev space, namely the quotient space obtained by identifying all the $H^s(\mathbb{T}^d)$ functions which differ only by a constant. For simplicity of notation we denote the equivalent class $[u] := \{u+c, c\in\mathbb{R}\}$, just by u. The homogeneous norm of $u\in\dot{H}^s(\mathbb{T}^d)$ is $\|u\|_s^2 := \sum_{j\in\mathbb{Z}^d\setminus\{0\}}|u_j|^2|j|^{2s}$. We shall denote by $\|\ \|_s$ either the Sobolev norm in H^s or that one in the homogeneous space $\dot{H}^s$, according to the context.

Let us make some comments about the proof. First, in view of (2.1.5), we rewrite system (2.1.3) in terms of $\rho\rightsquigarrow\mathtt{m}+\rho$ with $\rho\in H^s_0(\mathbb{T}^d)$, obtaining

$$\begin{cases}\partial_t\rho = -\mathtt{m}\Delta\phi - \mathrm{div}\,(\rho\nabla\phi)\\ \partial_t\phi = -\frac{1}{2}|\nabla\phi|^2 - g(\mathtt{m}+\rho) + K(\mathtt{m}+\rho)\Delta\rho + \frac{1}{2}K'(\mathtt{m}+\rho)|\nabla\rho|^2\,.\end{cases} \tag{2.1.6}$$

Then Theorem 2.1.1 follows by the following result, that we are going to prove

Theorem 2.1.2. *Let $s>2+\frac{d}{2}$, $0<\mathtt{m}_1<\mathtt{m}_2$ and fix $s_0\in(\frac{d}{2},s-2]$. For any initial data of the form $(\mathtt{m}+\rho_0,\phi_0)$ with $(\rho_0,\phi_0)\in H^s_0(\mathbb{T}^d)\times\dot{H}^s(\mathbb{T}^d)$ and $\mathtt{m}_1<\mathtt{m}+\rho_0(x)<\mathtt{m}_2$, $\forall x\in\mathbb{T}^d$, there exists $T=T\big(\|(\rho_0,\phi_0)\|_{s_0+2},\min_x(\mathtt{m}+\rho_0(x))\big)>0$ and a unique solution $(\mathtt{m}+\rho,\phi)$ of (2.1.6) such that*

$$(\rho,\phi)\in C^0\Big([0,T],H^s_0(\mathbb{T}^d,\mathbb{R})\times\dot{H}^s(\mathbb{T}^d,\mathbb{R})\Big)\cap C^1\Big([0,T],H^{s-2}_0(\mathbb{T}^d,\mathbb{R})\times\dot{H}^{s-2}(\mathbb{T}^d,\mathbb{R})\Big)$$

and $\mathtt{m}_1<\mathtt{m}+\rho(t,x)<\mathtt{m}_2$ holds for any $t\in[0,T]$. Moreover, for $|t|\le T$, the solution map $(\rho_0,\phi_0)\mapsto(\rho(t,\cdot),\phi(t,\cdot))$ is locally defined and continuous in $H^s_0(\mathbb{T}^d)\times\dot{H}^s(\mathbb{T}^d)$.

We consider system (2.1.6) as a system on the homogeneous space $\dot{H}^s\times\dot{H}^s$, that is we study

$$\begin{cases}\partial_t\rho = -\mathtt{m}\Delta\phi - \mathrm{div}\,((\Pi_0^\perp\rho)\nabla\phi)\\ \partial_t\phi = -\frac{1}{2}|\nabla\phi|^2 - g(\mathtt{m}+\Pi_0^\perp\rho) + K(\mathtt{m}+\Pi_0^\perp\rho)\Delta\rho + \frac{1}{2}K'(\mathtt{m}+\Pi_0^\perp\rho)|\nabla\rho|^2\end{cases} \tag{2.1.7}$$

where $\Pi_0^\perp$ is the projector onto the Fourier modes of index $\neq 0$. For simplicity of notation we shall not distinguish between systems (2.1.7) and (2.1.6). In Section 2.3, we para-linearize (2.1.6), i.e. (2.1.7), up to bounded semi-linear terms (for which we do not need Bony para-linearization formula). Then, introducing a suitable complex variable, we transform it into a quasi-linear type Schrödinger equation, see system (2.3.4), defined in the phase space

$$\dot{\mathbf{H}}^s := \Big\{U=\begin{pmatrix}u\\ \overline{u}\end{pmatrix}:\quad u\in\dot{H}^s(\mathbb{T}^d,\mathbb{C})\Big\}\,,\quad \|U\|_s^2 := \|U\|^2_{\dot{\mathbf{H}}^s} = \|u\|_s^2 + \|\overline{u}\|_s^2\,. \tag{2.1.8}$$

We use para-differential calculus in the Weyl quantization, because it is quite convenient to prove energy estimates for this system. Since (2.3.4) is a quasi-linear system, in order to prove local well posedness (Proposition 2.4.1) we follow the strategy, initiated by Kato [90], of constructing inductively a sequence of linear problems whose solutions converge to the solution of the quasi-linear equation. Such a scheme has been widely used, see e.g. [99, 2, 22, 72] and reference therein.

The equation (2.1.3) is a Hamiltonian PDE. We do not exploit explicitly this fact, but it is indeed responsible for the energy estimate of Proposition 2.4.4. The method of proof of Theorem 2.1.1 is similar to the one in Feola-Iandoli [70] for Hamiltonian quasi-linear Schrödinger equations on $\mathbb{T}^d$ (and Alazard-Burq-Zuily [2] in the case of gravity-capillary water waves in $\mathbb{R}^d$). The main difference is that we aim to obtain the minimal smoothness assumption $s > 2 + (d/2)$. This requires to optimize several arguments, and, in particular, to develop a sharp para-differential calculus for periodic functions that we report in the Appendix in a self-contained way. Some other technical differences are in the use of the modified energy (Section 2.4.2), the mollifiers (2.4.17) which enables to prove energy estimates independent of ε for the regularized system, the argument for the continuity of the flow in H^s.

We now set some notation that will be used throughout the chapter. Since $K : \mathbb{R}_+ \to \mathbb{R}$ is positive, given $0 < \mathtt{m}_1 < \mathtt{m}_2$, there exist constants $c_K, C_K > 0$ such that

$$c_K \le K(\rho) \le C_K\,, \qquad \forall \rho \in (\mathtt{m}_1, \mathtt{m}_2)\,. \tag{2.1.9}$$

Since the velocity potential ϕ is defined up to a constant, we may assume in (2.1.6) that

$$g(\mathtt{m}) = 0\,. \tag{2.1.10}$$

From now on *we fix* s_0 so that

$$\frac{d}{2} < s_0 < s - 2\,. \tag{2.1.11}$$

The initial datum $\rho_0(x)$ belongs to the open subset of $H_0^{s_0}(\mathbb{T}^d)$ defined by

$$\mathcal{Q} := \left\{\rho \in H_0^{s_0}(\mathbb{T}^d) \ \ : \ \ \mathtt{m}_1 < \mathtt{m} + \rho(x) < \mathtt{m}_2\right\} \tag{2.1.12}$$

and we shall prove that, locally in time, the solution of (2.1.6) stays in this set.

We write $a \lesssim b$ with the meaning $a \le Cb$ for some constant $C > 0$ which does not depend on relevant quantities.

2.2 Para-differential calculus

We introduce the notions of para-differential calculus that we shall use for the proof of Theorem 2.1.1. As in Theorem 2.1.1 we want to reach the minimal regularity assumption given by the energy method, in Section 2.2.1, we demonstrate the classic results of para-differential calculus with a specific focus on their optimality in terms of regularity with respect to the variable x of the symbols. We shall prove only the results needed for the proof of Theorem 2.1.1. For a more detailed understanding of para-differential calculus, we refer readers to the book [99], which mostly inspired this present section.

It is worth noting that in Section 4.2, we will introduce a slightly modified version of para-differential calculus specifically tailored for the purpose of proving Theorem 1.1.3. Indeed, Theorem 1.1.3 necessitates careful monitoring of the multilinear expansion of the symbols with respect to the solution U, but it does not require any particular attention to the threshold of regularity.

2.2.1 Para-differential calculus in low regularity

The main results of this section are the continuity Theorem 2.2.12 and the composition Theorem 2.2.13, which require mild regularity assumptions of the symbols in the space variable (they are deduced by the sharper results proved in Theorems 2.2.10 and 2.2.11 in the Appendix). This is needed in order to prove the local existence Theorem 2.1.1 with the natural minimal regularity on the initial datum $(\rho_0, \phi_0) \in H^s \times H^s$ with $s > 2 + \frac{d}{2}$.

Along the section $\mathscr{W}$ may denote either the Banach space $L^\infty(\mathbb{T}^d)$, or the Sobolev spaces $H^s(\mathbb{T}^d)$, or the Hölder spaces $W^{\varrho,\infty}(\mathbb{T}^d)$, introduced in Definition 2.2.6. Given a multi-index $\beta \in \mathbb{N}_0^d$ we define $|\beta| := \beta_1 + \ldots + \beta_d$.

Definition 2.2.1. (Symbols with finite regularity)
Given $m \in \mathbb{R}$ and a Banach space $\mathscr{W} \in \{L^\infty(\mathbb{T}^d), H^s(\mathbb{T}^d), W^{\varrho,\infty}(\mathbb{T}^d)\}$, we denote by $\Gamma^m_{\mathscr{W}}$ the space of functions $a : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{C}$, $a(x,\xi)$, which are C^∞ with respect to ξ and such that, for any $\beta \in \mathbb{N}_0^d$, there exists a constant $C_\beta > 0$ such that

$$\left\|\partial_\xi^\beta a(\cdot,\xi)\right\|_{\mathscr{W}} \le C_\beta \langle\xi\rangle^{m-|\beta|}, \quad \forall \xi \in \mathbb{R}^d. \tag{2.2.1}$$

We denote by $\Sigma^m_{\mathscr{W}}$ the subclass of symbols $a \in \Gamma^m_{\mathscr{W}}$ which are spectrally localized*, that is*

$$\exists \delta \in (0,1): \qquad \widehat{a}(j,\xi) = 0, \quad \forall |j| \ge \delta\langle\xi\rangle, \tag{2.2.2}$$

where $\widehat{a}(j,\xi) := (2\pi)^{-d}\int_{\mathbb{T}^d} a(x,\xi)e^{-\mathrm{i}j\cdot x}\mathrm{d}x$, $j \in \mathbb{Z}^d$, are the Fourier coefficients of the function $x \mapsto a(x,\xi)$.

We endow $\Gamma^m_{\mathscr{W}}$ with the family of norms defined, for any $n \in \mathbb{N}_0$, by

$$|a|_{m,\mathscr{W},n} := \max_{|\beta|\le n} \sup_{\xi\in\mathbb{R}^d} \left\|\langle\xi\rangle^{-m+|\beta|}\,\partial_\xi^\beta a(\cdot,\xi)\right\|_{\mathscr{W}}. \tag{2.2.3}$$

When $\mathscr{W} = H^s$, we also denote $\Gamma^m_s \equiv \Gamma^m_{H^s}$ and $|a|_{m,s,n} \equiv |a|_{m,H^s,n}$. We denote by $\Gamma^m_s \otimes \mathcal{M}_2(\mathbb{C})$ the 2×2 matrices $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ of symbols in Γ^m_s and $|A|_{m,\mathscr{W},n} := \max_{i=1,\ldots,4}\{|a_i|_{m,\mathscr{W},n}\}$. Similarly we denote by $\Gamma^m_s \otimes \mathbb{R}^d$ the d-dimensional vectors of symbols in Γ^m_s.

Let us make some simple remarks:

- (i) given a function $a(x) \in \mathscr{W}$ then $a(x) \in \Gamma^0_{\mathscr{W}}$ and

$$|u|_{0,\mathscr{W},n} = \|u\|_{\mathscr{W}}, \forall n \in \mathbb{N}_0. \tag{2.2.4}$$

- (ii) For any $s_0 > \frac{d}{2}$ and $0 \le \varrho' \le \varrho$, we have that

$$|a|_{m,L^\infty,n} \lesssim |a|_{m,W^{\varrho',\infty},n} \lesssim |a|_{m,W^{\varrho,\infty},n} \lesssim |a|_{m,H^{s_0+\varrho},n}, \quad \forall n \in \mathbb{N}_0. \tag{2.2.5}$$

- (iii) If $a \in \Gamma^m_{\mathscr{W}}$, then, for any $\alpha \in \mathbb{N}_0^d$, we have $\partial_\xi^\alpha a \in \Gamma^{m-|\alpha|}_{\mathscr{W}}$ and

$$|\partial_\xi^\alpha a|_{m-|\alpha|,\mathscr{W},n} \lesssim |a|_{m,\mathscr{W},n+|\alpha|}, \quad \forall n \in \mathbb{N}_0. \tag{2.2.6}$$

- (iv) If $a \in \Gamma^m_{H^s}$, resp. $a \in \Gamma^m_{W^{\varrho,\infty}}$, then $\partial_x^\alpha a \in \Gamma^m_{H^{s-|\alpha|}}$, resp. $\partial_x^\alpha a \in \Gamma^m_{W^{\varrho-|\alpha|,\infty}}$ for any multi-indices α with $|\alpha| \le \varrho$, and

$$|\partial_x^\alpha a|_{m,s-|\alpha|,n} \lesssim |a|_{m,s,n}, \quad \text{resp. } |\partial_x^\alpha a|_{m,W^{\varrho-|\alpha|,\infty},n} \lesssim |a|_{m,W^{\varrho,\infty},n}, \quad \forall n \in \mathbb{N}_0. \tag{2.2.7}$$

• (v) If $a, b \in \Gamma^m_{\mathscr{W}}$ then $ab \in \Gamma^m_{\mathscr{W}}$ with $|ab|_{m+m',\mathscr{W},n} \lesssim |a|_{m,\mathscr{W},N}|b|_{m',\mathscr{W},n}$ for any $n \in \mathbb{N}_0$. In particular, if $a, b \in \Gamma^m_s$ with $s > d/2$ then $ab \in \Gamma^{m+m'}_s$ and

$$|ab|_{m+m',s,n} \lesssim |a|_{m,s,n}|b|_{m',s_0,n} + |a|_{m,s_0,n}|b|_{m',s,n}\,, \quad \forall n \in \mathbb{N}_0\,. \tag{2.2.8}$$

Let $\epsilon \in (0,1)$ and consider a C^∞, even cut-off function $\chi\colon \mathbb{R}^d \to [0,1]$ such that

$$\chi(\xi) = \begin{cases} 1 & \text{if } |\xi| \le 1.1 \\ 0 & \text{if } |\xi| \ge 1.9\,, \end{cases} \qquad \chi_\epsilon(\xi) := \chi\left(\frac{\xi}{\epsilon}\right). \tag{2.2.9}$$

Given a symbol a in $\Gamma^m_{\mathscr{W}}$ we define the *regularized* symbol

$$a_\chi(x,\xi) := \chi_{\epsilon\langle\xi\rangle}(D)a(x,\xi) = \sum_{j\in\mathbb{Z}^d} \chi_\epsilon\Big(\frac{j}{\langle\xi\rangle}\Big)\,\widehat{a}(j,\xi)\,e^{\mathrm{i}j\cdot x}\,. \tag{2.2.10}$$

Note that a_χ is analytic in x (it is a trigonometric polynomial) and it is spectrally localized.

In order to define the Bony-Weyl quantization of a symbol $a(x,\xi)$ we first remind the Weyl quantization formula

$$\mathrm{Op}^W(a)[u] := \sum_{j\in\mathbb{Z}^d}\Big(\sum_{k\in\mathbb{Z}^d} \widehat{a}\Big(j-k, \frac{k+j}{2}\Big)\,u_k\Big)e^{\mathrm{i}j\cdot x}\,. \tag{2.2.11}$$

Definition 2.2.2. (Bony-Weyl quantization) *Given a symbol $a \in \Gamma^m_{\mathscr{W}}$, we define the* Bony-Weyl para-differential operator $\mathrm{Op}^{BW}(a) = \mathrm{Op}^W(a_\chi)$ *that acts on a periodic function u as*

$$\begin{aligned}\big(\mathrm{Op}^{BW}(a)[u]\big)(x) &:= \sum_{j\in\mathbb{Z}^d}\Big(\sum_{k\in\mathbb{Z}^d} \widehat{a}_\chi\Big(j-k, \frac{j+k}{2}\Big)\,u_k\Big)e^{\mathrm{i}j\cdot x} \\ &= \sum_{j\in\mathbb{Z}^d}\Big(\sum_{k\in\mathbb{Z}^d} \widehat{a}\Big(j-k, \frac{j+k}{2}\Big)\,\chi_\epsilon\Big(\frac{j-k}{\langle j+k\rangle}\Big)\,u_k\Big)e^{\mathrm{i}j\cdot x}\,.\end{aligned} \tag{2.2.12}$$

If $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ is a matrix of symbols in Γ^m_s, then $\mathrm{Op}^{BW}(A)$ is defined as the matrix valued operator $\begin{pmatrix} \mathrm{Op}^{BW}(a_1) & \mathrm{Op}^{BW}(a_2) \\ \mathrm{Op}^{BW}(a_3) & \mathrm{Op}^{BW}(a_4) \end{pmatrix}$.

Given a symbol $a(\xi)$ independent of x, then $\mathrm{Op}^{BW}(a)$ is the Fourier multiplier operator

$$\mathrm{Op}^{BW}(a)u = a(D)u = \sum_{j\in\mathbb{Z}^d} a(j)\,u_j\,e^{\mathrm{i}j\cdot x}\,.$$

Note that if $\chi_\epsilon\Big(\frac{k-j}{\langle k+j\rangle}\Big) \neq 0$ then $|k-j| \le \epsilon\langle j+k\rangle$ and therefore, for $\epsilon \in (0,1)$,

$$\frac{1-\epsilon}{1+\epsilon}|k| \le |j| \le \frac{1+\epsilon}{1-\epsilon}|k|\,, \quad \forall j,k \in \mathbb{Z}^d\,. \tag{2.2.13}$$

This relation shows that the action of a para-differential operator does not spread much the Fourier support of functions. In particular $\mathrm{Op}^{\mathrm{BW}}(a)$ sends a constant function into a constant function and therefore $\mathrm{Op}^{\mathrm{BW}}(a)$ sends homogenous spaces into homogenous spaces.

Remark 2.2.3. Actually, if $\chi_\epsilon\big(\frac{k-j}{\langle k+j\rangle}\big) \neq 0$, $\epsilon \in (0,1/4)$, then $|j| \le |j+k| \le 3|j|$, for all $j,k \in \mathbb{Z}^d$.

2.2.2 Bony-Weyl calculus in periodic Hölder spaces

In this section we develop in a self-contained manner para-differential calculus for space periodic symbols $a(x,\xi)$ which belong to the Banach scale of Hölder spaces $W^{\varrho,\infty}(\mathbb{T}^d)$. The main results are the continuity Theorem 2.2.10 and the composition Theorem 2.2.11, which require mild regularity assumptions of the symbols in the space variable, and imply Theorems 2.2.12 and 2.2.13. We first provide some preliminary technical results.

Technical lemmas. In the following we denote by ∂_m, $m = 1,\dots,d$ the *discrete derivative*, defined for functions $f\colon \mathbb{Z}^d \to \mathbb{C}$ as

$$(\partial_m f)(n) := f(n) - f(n - \vec{e}_m), \quad n \in \mathbb{Z}^d, \tag{2.2.14}$$

where $\vec{e}_m$ denotes the usual unit basis vector of $\mathbb{N}_0^d$ with 0 components expect the m-th one. Given a multi-index $\beta \in \mathbb{N}_0^d$, we set $\partial^\beta f := \partial_1^{\beta_1} \cdots \partial_d^{\beta_d} f$.

We shall use the Leibniz rule for finite differences in the following form: given $k \in \mathbb{N}$, $m = 1,\dots,d$, there exist constants C_{k_1,k_2} (binomial coefficients) such that

$$(\partial_m^k)(fg)(n) = \sum_{k_1+k_2=k} C_{k_1,k_2} (\partial_m^{k_1} f)(n - k_2)(\partial_m^{k_2} g)(n). \tag{2.2.15}$$

Moreover, when using discrete derivatives, the analogous of the integration by parts formula is given by the *Abel resummation formula*:

$$\sum_{n\in\mathbb{Z}^d} e^{\mathrm{i}n\cdot z} \beta(x,n) = -\frac{1}{e^{\mathrm{i}\vec{e}_m\cdot z} - 1} \sum_{n\in\mathbb{Z}^d} e^{\mathrm{i}n\cdot z} (\partial_m \beta)(x,n), \quad \forall m = 1,\dots,d. \tag{2.2.16}$$

Lemma 2.2.4. *Let $K\colon \mathbb{T}^d \to \mathbb{C}$ be a function satisfying, for constants A and B, the estimate*

$$|K(y)| \lesssim A^d B \min\Big(1, \min_{1\le m\le d} \frac{1}{|A\, 2\sin\frac{y_m}{2}|^{(d+1)}}\Big), \quad \forall y \in \mathbb{T}^d. \tag{2.2.17}$$

Then

$$\int_{\mathbb{T}^d} |K(y)|\,\mathrm{d}y \lesssim B. \tag{2.2.18}$$

Proof. If $A \le 1$ the bound (2.2.18) follows trivially integrating the first inequality in (2.2.17). Then we suppose $A > 1$. We split the integral in (2.2.18) as

$$\int_{\mathbb{T}^d} |K(y)|\,\mathrm{d}y = \int_{\mathbb{T}^d \cap \{|y|\le \frac{1}{A}\}} |K(y)|\,\mathrm{d}y + \int_{\mathbb{T}^d \cap \{|y| > \frac{1}{A}\}} |K(y)|\,\mathrm{d}y. \tag{2.2.19}$$

We bound the first integral using the first inequality in (2.2.17), getting

$$\int_{\mathbb{T}^d \cap \{|y|\le \frac{1}{A}\}} |K(y)|\,\mathrm{d}y \lesssim A^d B\, \mathrm{meas}\Big(y \in [-\pi,\pi]^d\colon\ |y| \le \frac{1}{A}\Big) \lesssim B. \tag{2.2.20}$$

To bound the second integral in (2.2.19) we use that, for some $c > 0$, $\max_{1\le m\le d} \big|\sin\big(\frac{y_m}{2}\big)\big| \ge c|y|$, $\forall y \in [-\pi,\pi]^d$, and therefore the second inequality in (2.2.17) implies

$$\int_{\mathbb{T}^d \cap \{|y| > \frac{1}{A}\}} |K(y)|\,\mathrm{d}y \lesssim A^d B \int_{\{y\in\mathbb{R}^d\colon |y| > \frac{1}{A}\}} \frac{\mathrm{d}y}{|Ay|^{d+1}} \overset{z=Ay}{\lesssim} B \int_{\{|z|>1\}} \frac{\mathrm{d}z}{|z|^{d+1}} \lesssim B. \tag{2.2.21}$$

The bounds (2.2.20)-(2.2.21) and (2.2.19) imply (2.2.18). $\square$

The next lemma represents a Fourier multiplier operator acting on periodic functions as a convolution integral on $\mathbb{R}^d$. The key step is the use of Poisson summation formula.

Lemma 2.2.5. *Let $\chi \in \mathcal{S}(\mathbb{R}^d)$. Then the Fourier multiplier $\chi_\theta(D) := \chi(\theta^{-1}D)$, $\theta \geq 1$, acting on a periodic function $u \in L^1(\mathbb{T}^d, \mathbb{C})$ can represented by*

$$\chi_\theta(D)u = \int_{\mathbb{R}^d} u(y)\psi_\theta(x-y)\mathrm{d}y = \int_{\mathbb{R}^d} u(x-y)\psi_\theta(y)\mathrm{d}y \tag{2.2.22}$$

where $\psi_\theta(z) := \theta^d \psi(\theta z)$ and ψ denotes the anti-Fourier transform of χ on $\mathbb{R}^d$.

Proof. For $\theta \geq 1$ we write

$$\chi\left(\frac{D}{\theta}\right)u = \int_{\mathbb{T}^d} u(y)\, h_\theta(x-y)\mathrm{d}y \qquad \text{where} \qquad h_\theta(z) := \frac{1}{(2\pi)^d}\sum_{j\in\mathbb{Z}^d} \chi\left(\frac{j}{\theta}\right) e^{\mathrm{i}j\cdot z}. \tag{2.2.23}$$

Then the Fourier transform $\widehat{\psi_\theta}(\xi) = \int_{\mathbb{R}^d} \theta^d \psi(\theta z)\, e^{-\mathrm{i}z\cdot\xi}\mathrm{d}z = \int_{\mathbb{R}^d} \psi(y)\, e^{-\mathrm{i}y\cdot\frac{\xi}{\theta}}\mathrm{d}y = \widehat{\psi}\left(\frac{\xi}{\theta}\right) = \chi\left(\frac{\xi}{\theta}\right)$, and, using *Poisson summation formula*, we write the periodic function $h_\theta(z)$ in (2.2.23) as

$$h_\theta(z) = \frac{1}{(2\pi)^d}\sum_{j\in\mathbb{Z}^d} \widehat{\psi_\theta}(j) e^{\mathrm{i}j\cdot z} = \sum_{j\in\mathbb{Z}^d} \psi_\theta(z + 2\pi j).$$

Therefore the integral (2.2.23) is

$$\begin{aligned}
\chi(\theta^{-1}D)u &= \sum_{j\in\mathbb{Z}^d}\int_{\mathbb{T}^d} u(y)\,\psi_\theta(x-y+2\pi j)\mathrm{d}y = \sum_{j\in\mathbb{Z}^d}\int_{[0,2\pi]^d+2\pi j} u(y)\psi_\theta(x-y)\mathrm{d}y \\
&= \int_{\mathbb{R}^d} u(y)\psi_\theta(x-y)\mathrm{d}y = \int_{\mathbb{R}^d} u(x-y)\psi_\theta(y)\mathrm{d}y
\end{aligned}$$

proving (2.2.22). □

We now give the definition and basic properties of the Hölder spaces $W^{\varrho,\infty}(\mathbb{T}^d)$.

Definition 2.2.6. (Periodic Hölder spaces) *Given $\varrho \in \mathbb{N}_0$, we denote by $W^{\varrho,\infty}(\mathbb{T}^d)$ the space of continuous functions $u : \mathbb{T}^d \to \mathbb{C}$, 2π-periodic in each variable $(x_1, \dots, x_d)$, whose derivatives of order ϱ are in L^∞, equipped with the norm $\|u\|_{W^{\varrho,\infty}} := \sum_{|\alpha|\leq\varrho} \|\partial_x^\alpha u\|_{L^\infty}$, $\alpha \in \mathbb{N}_0^d$. In case $\varrho > 0$, $\varrho \notin \mathbb{N}$, we denote $\lfloor\varrho\rfloor$ the integer part of ϱ, and we define $W^{\varrho,\infty}(\mathbb{T}^d)$ as the space of functions u in $C^{\lfloor\varrho\rfloor}(\mathbb{T}^d, \mathbb{C})$ whose derivatives of order $\lfloor\varrho\rfloor$ are $(\varrho - \lfloor\varrho\rfloor)$-Hölder-continuous, that is*

$$[\partial_x^\alpha u]_\varrho := \sup_{x\neq y} \frac{|\partial_x^\alpha u(x) - \partial_x^\alpha u(y)|}{|x-y|^{\varrho-\lfloor\varrho\rfloor}} < +\infty, \quad \forall |\alpha| = \lfloor\varrho\rfloor,$$

equipped with the norm

$$\|u\|_{W^{\varrho,\infty}} := \sum_{|\alpha|\leq\lfloor\varrho\rfloor} \|\partial_x^\alpha u\|_{L^\infty} + \sum_{|\alpha|=\lfloor\varrho\rfloor} [\partial_x^\alpha u]_\varrho.$$

For $\varrho = 0$ the norm $\| \; \|_{W^{\varrho,\infty}} = \| \; \|_{L^\infty}$.

The Hölder spaces $W^{\varrho,\infty}(\mathbb{T}^d)$ can be described by the Paley-Littlewood decomposition of a function. Consider the locally finite partition on unity

$$1 = \chi(\xi) + \sum_{k\geq 1} \varphi(2^{-k}\xi) , \quad \varphi(z) := \chi(z) - \chi(2z) , \tag{2.2.24}$$

where $\chi : \mathbb{R}^d \to \mathbb{R}$ is the cut-off function defined in (2.2.9). It induces the decomposition of a distribution $u \in \mathcal{S}'(\mathbb{T}^d)$ as

$$u = \sum_{k\geq 0} \Delta_k u \quad \text{where} \quad \Delta_0 := \chi(D) , \, \Delta_k := \varphi(2^{-k}D) = \chi_{2^k}(D) - \chi_{2^{k-1}}(D) , \, k \geq 1 . \tag{2.2.25}$$

We also set

$$S_k := \sum_{0\leq j\leq k} \Delta_j = \chi_{2^k}(D) . \tag{2.2.26}$$

The Paley-Littlewood theory of the Hölder spaces $W^{\varrho,\infty}(\mathbb{T}^d)$ follows as in $\mathbb{R}^d$, see e.g. [99], once we represent the Fourier multipliers Δ_k as integral convolution operators on $\mathbb{R}^d$, by Lemma 2.2.5. In particular the following smoothing estimates hold: for any $\alpha \in \mathbb{N}_0^d$, $\varrho \geq 0$,

$$\|\partial_x^\alpha S_k u\|_{L^\infty} \lesssim 2^{k(|\alpha|-\varrho)} \|u\|_{W^{\varrho,\infty}} , \tag{2.2.27}$$

and, for any $\varrho > 0$,

$$\|u - \chi_\theta(D)u\|_{L^\infty} \lesssim \theta^{-\varrho} \|u\|_{W^{\varrho,\infty}} . \tag{2.2.28}$$

In this way it results as in $\mathbb{R}^d$ that the Hölder norms $\| \, \|_{W^{\varrho,\infty}}$ satisfy interpolation estimates. In particular we shall use that, given $\varrho, \varrho_1, \varrho_2 \geq 0$,

$$\begin{aligned} &\|uv\|_{W^{\varrho,\infty}} \lesssim \|u\|_{W^{\varrho,\infty}} \|v\|_{L^\infty} + \|u\|_{L^\infty} \|v\|_{W^{\varrho,\infty}} \\ &\|u\|_{W^{\varrho,\infty}} \lesssim \|u\|_{W^{\varrho_1,\infty}}^{\theta} \|u\|_{W^{\varrho_2,\infty}}^{1-\theta} , \quad \varrho = \theta\varrho_1 + (1-\theta)\varrho_2 , \, \theta \in (0,1) . \end{aligned} \tag{2.2.29}$$

Hölder estimates of regularized symbols. In order to prove estimates of the regularized symbol a_χ defined in (2.2.10) in Hölder spaces (Lemma 2.2.8) we represent it as a convolution integral on $\mathbb{R}^d$, by Lemma 2.2.5,

$$a_\chi(x,\xi) = \int_{\mathbb{R}^d} a(x-y,\xi) \, \psi_{\epsilon\langle\xi\rangle}(y) \, \mathrm{d}y \tag{2.2.30}$$

where $\psi_\theta(z) = \theta^d \psi(\theta z)$ and ψ is the anti-Fourier transform of χ.

In the proof of Lemma 2.2.8 we shall use the following estimate.

Lemma 2.2.7. *For any $\beta \in \mathbb{N}_0^d$, $u \in L^\infty(\mathbb{T}^d)$, we have*

$$\|\partial_\xi^\beta \chi_{\epsilon\langle\xi\rangle}(D)u\|_{L^\infty} \lesssim \langle\xi\rangle^{-|\beta|} \|u\|_{L^\infty} . \tag{2.2.31}$$

Proof. By (2.2.30) we have, for all $\beta \in \mathbb{N}_0^d$,

$$\partial_\xi^\beta \chi_{\epsilon\langle\xi\rangle}(D)u = \int_{\mathbb{R}^d} u(x-y) \, \partial_\xi^\beta \psi_{\epsilon\langle\xi\rangle}(y) \, \mathrm{d}y . \tag{2.2.32}$$

By the definition $\psi_{\epsilon\langle\xi\rangle}(y) = (\epsilon\langle\xi\rangle)^d \psi(\epsilon\langle\xi\rangle y)$ and Faà di Bruno formula, we have that

$$\int_{\mathbb{R}^d} \big|\partial_\xi^\beta \psi_{\epsilon\langle\xi\rangle}(y)\big| \, \mathrm{d}y \lesssim \langle\xi\rangle^{-|\beta|}, \quad \forall \xi \in \mathbb{R}^d . \tag{2.2.33}$$

Then (2.2.31) follows by (2.2.32) and (2.2.33). ∎

The next lemma provides estimates of the regularized symbol a_χ in terms of the symbol a.

Lemma 2.2.8. (Estimates on regularized symbols) *Let $m \in \mathbb{R}$, $N \in \mathbb{N}_0$.*

1. *If $a \in \Gamma^m_{L^\infty}$, $m \in \mathbb{R}$, then a_χ defined in (2.2.10) belongs to $\Sigma^m_{L^\infty}$ and*

$$|a_\chi|_{m,L^\infty,N} \lesssim |a|_{m,L^\infty,N} \,. \tag{2.2.34}$$

2. *If $a \in \Gamma^m_{H^{s_0-\varrho}}$, $\varrho \geq 0$, $s_0 > \frac{d}{2}$, then a_χ belongs to $\Gamma^{m+\varrho}_{L^\infty}$ and*

$$|a_\chi|_{m+\varrho,L^\infty,N} \lesssim |a|_{m,H^{s_0-\varrho},N} \,. \tag{2.2.35}$$

3. *If $a \in \Gamma^m_{W^{\varrho,\infty}}$, $\varrho > 0$, then, for any $\beta \in \mathbb{N}^d_0$, $\partial^\beta_\xi a_\chi - (\partial^\beta_\xi a)_\chi \in \Sigma^{m-|\beta|-\varrho}_{L^\infty}$ and*

$$\left|\partial^\beta_\xi a_\chi - (\partial^\beta_\xi a)_\chi\right|_{m-|\beta|-\varrho,L^\infty,N} \lesssim |a|_{m,W^{\varrho,\infty},N+|\beta|} \,. \tag{2.2.36}$$

4. *If $a \in \Gamma^m_{W^{\varrho,\infty}}$, $\varrho \geq 0$, then, for any $\alpha \in \mathbb{N}^d_0$ with $|\alpha| \geq \varrho$, $\partial^\alpha_x a_\chi = (\partial^\alpha_x a)_\chi \in \Sigma^{m+|\alpha|-\varrho}_{L^\infty}$ and*

$$|\partial^\alpha_x a_\chi|_{m+|\alpha|-\varrho,L^\infty,N} \lesssim |a|_{m,W^{\varrho,\infty},N} \,. \tag{2.2.37}$$

5. *If $a \in \Gamma^m_{W^{\varrho,\infty}}$, $\varrho > 0$, then, $a - a_\chi \in \Gamma^{m-\varrho}_{L^\infty}$ and*

$$|a - a_\chi|_{m-\varrho,L^\infty,N} \lesssim |a|_{m,W^{\varrho,\infty},N} \,. \tag{2.2.38}$$

Proof. PROOF OF (2.2.34). Differentiating (2.2.10) for any $\beta \in \mathbb{N}^d_0$, we have

$$\partial^\beta_\xi a_\chi(x,\xi) = \sum_{\beta_1+\beta_2=\beta} C_{\beta_1,\beta_2} \partial^{\beta_1}_\xi \chi_{\epsilon\langle\xi\rangle}(D)\, \partial^{\beta_2}_\xi a(\cdot,\xi)\,.$$

Then (2.2.3) and (2.2.31) directly imply (2.2.34).

PROOF OF (2.2.35) By the Cauchy-Schwartz inequality

$$\begin{aligned} |a_\chi(x,\xi)| &= \Big|\sum_{n\in\mathbb{Z}^d} \chi_\epsilon\left(\frac{n}{\langle\xi\rangle}\right) \widehat{a}(n,\xi)\, e^{\mathrm{i}n\cdot x}\Big| \leq \sum_{n\in\mathbb{Z}^d} \chi_\epsilon\Big(\frac{n}{\langle\xi\rangle}\Big) \frac{\langle n\rangle^\varrho}{\langle n\rangle^{s_0}} \langle n\rangle^{s_0-\varrho} |\widehat{a}(n,\xi)| \\ &\lesssim \Big(\sum_{n\in\mathbb{Z}^d} \chi^2_\epsilon\Big(\frac{n}{\langle\xi\rangle}\Big) \frac{\langle n\rangle^{2\varrho}}{\langle n\rangle^{2s_0}}\Big)^{1/2} \|a(\cdot,\xi)\|_{H^{s_0-\varrho}} \lesssim \langle\xi\rangle^{m+\varrho} |a|_{m,H^{s_0-\varrho},0}\,. \end{aligned}$$

The case $N \geq 1$ follows in the same way.

PROOF OF (2.2.36). First, for any $\xi \in \mathbb{R}^d$, we define $k \in \mathbb{N}$ such that $2^{k-1} \leq 2\epsilon\langle\xi\rangle \leq 2^k$. Then, by the properties of the cut-off function χ in (2.2.9) and the projector S_k in (2.2.26) we have

$$\partial^\beta_\xi \chi_\epsilon\left(\frac{\eta}{\langle\xi\rangle}\right) = \left(\partial^\beta_\xi \chi_\epsilon\Big(\frac{\eta}{\langle\xi\rangle}\Big)\right) S_k\,, \quad \forall\eta \in \mathbb{R}^d\,, \quad \forall\beta \in \mathbb{N}^d_0\,. \tag{2.2.39}$$

Differentiating (2.2.10) and using (2.2.39) we get

$$\partial^\beta_\xi a_\chi - (\partial^\beta_\xi a)_\chi = \sum_{\beta_1+\beta_2=\beta,\beta_1\neq 0} C_{\beta_1\beta_2}\, \partial^{\beta_1}_\xi \chi_{\epsilon\langle\xi\rangle}(D)\, S_k\, \partial^{\beta_2}_\xi a(\cdot,\xi)\,,$$

and, using (2.2.31) and (2.2.27)

$$\begin{aligned}\big\| \big(\partial_\xi^\beta a_\chi - (\partial_\xi^\beta a)_\chi\big)(\cdot,\xi)\big\|_{L^\infty} &\lesssim \sum_{\beta_1+\beta_2=\beta, \beta_1\neq 0} \langle\xi\rangle^{-|\beta_1|} 2^{-k\varrho} \big\|\partial_\xi^{\beta_2} a(\cdot,\xi)\big\|_{W^{\varrho,\infty}} \\ &\lesssim \langle\xi\rangle^{m-|\beta|} 2^{-k\varrho} |a|_{m,W^{\varrho,\infty},|\beta|} \lesssim \langle\xi\rangle^{m-|\beta|-\varrho} |a|_{m,W^{\varrho,\infty},|\beta|}\end{aligned}$$

because $\langle\xi\rangle \lesssim 2^k$. This proves (2.2.36) for $N=0$. For $N\geq 1$ the estimate is similar.

PROOF OF (2.2.37). For any $\xi\in\mathbb{R}^d$, we define $k\in\mathbb{N}$ such that $2^{k-1}\leq 2\epsilon\langle\xi\rangle\leq 2^k$. By (2.2.10) and (2.2.39) with $\beta=0$, we write $a_\chi(\cdot,\xi)=\chi_{\epsilon\langle\xi\rangle}(D)a(\cdot,\xi)=\chi_{\epsilon\langle\xi\rangle}(D)S_k a(\cdot,\xi)$, and then

$$\begin{aligned}\|\partial_x^\alpha a_\chi(\cdot,\xi)\|_{L^\infty} &= \|\chi_{\epsilon\langle\xi\rangle}(D)\partial_x^\alpha S_k a(\cdot,\xi)\|_{L^\infty} \overset{(2.2.31)}{\lesssim} \|\partial_x^\alpha S_k a(\cdot,\xi)\|_{L^\infty} \\ &\overset{(2.2.27)}{\lesssim} 2^{k(|\alpha|-\varrho)}\|a(\cdot,\xi)\|_{W^{\varrho,\infty}} \overset{\langle\xi\rangle\sim 2^k}{\lesssim} \langle\xi\rangle^{|\alpha|-\varrho}\|a(\cdot,\xi)\|_{W^{\varrho,\infty}} \lesssim \langle\xi\rangle^{m+|\alpha|-\varrho}|a|_{m,W^{\varrho,\infty},0}\end{aligned}$$

by (2.2.3). This proves (2.2.37) with $N=0$. For $N\geq 1$ the estimate is similar.

PROOF OF (2.2.38). For any $\beta\in\mathbb{N}_0^d$ we write $\partial_\xi^\beta(a-a_\chi)=\big[\partial_\xi^\beta a-(\partial_\xi^\beta a)_\chi\big]+\big[(\partial_\xi^\beta a)_\chi-\partial_\xi^\beta a_\chi\big]$. The first term is bounded, using (2.2.28) with $\theta=\epsilon\langle\xi\rangle$, as

$$\big\|\big(\partial_\xi^\beta a-(\partial_\xi^\beta a)_\chi\big)(\cdot,\xi)\big\|_{L^\infty} \lesssim \langle\xi\rangle^{-\varrho}\|\partial_\xi^\beta a(\cdot,\xi)\|_{W^{\varrho,\infty}} \lesssim |a|_{m,W^{\varrho,\infty},|\beta|}\langle\xi\rangle^{m-\varrho-|\beta|}$$

The second term satisfies the same bound by (2.2.36). This proves (2.2.38). □

Change of quantization. In order to prove the boundedness Theorem 2.2.10 and the composition Theorem 2.2.11, it is convenient to pass from the Weyl quantization of a symbol $a(x,\xi)$, defined in (2.2.11), to the standard quantization which is defined, given a symbol $b(x,\xi)$, as

$$\mathrm{Op}(b)[u] := \sum_{j\in\mathbb{Z}^d}\Big(\sum_{k\in\mathbb{Z}^d}\widehat{b}(j-k,k)\,u_k\Big)e^{\mathrm{i}j\cdot x} = \sum_{k\in\mathbb{Z}^d} b(x,k)\,u_k e^{\mathrm{i}k\cdot x}\,. \tag{2.2.40}$$

We have the change of quantization formula

$$\mathrm{Op}^W(a)=\mathrm{Op}(b) \qquad \Leftrightarrow \qquad \widehat{b}(n,\xi) := \widehat{a}\big(n,\xi+\frac{n}{2}\big)\,. \tag{2.2.41}$$

In the next lemma we estimate the norms of b in terms of those of a. We remind that $\Sigma^m_{\mathscr{W}}$ denotes the set of *spectrally localized* symbols, i.e. satisfying (2.2.2).

Lemma 2.2.9. (Change of quantization) *Let $a\in\Sigma^m_{L^\infty}$, $m\in\mathbb{R}$. If $\delta>0$ in (2.2.2) is small enough, then (cfr. (2.2.41))*

$$b(x,\xi) := \sum_{n\in\mathbb{Z}^d}\widehat{a}\big(n,\xi+\frac{n}{2}\big)\,e^{\mathrm{i}n\cdot x} \tag{2.2.42}$$

is a symbol in $\Sigma^m_{L^\infty}$ satisfying

$$|b|_{m,L^\infty,N} \lesssim |a|_{m,L^\infty,N+d+1}\,, \quad \forall N\in\mathbb{N}_0\,. \tag{2.2.43}$$

Proof. Since a satisfies (2.2.2) with δ small enough, it follows that b satisfies (2.2.2). In order to prove (2.2.43) we differentiate (2.2.42) obtaining that, for any $\beta\in\mathbb{N}_0^d$,

$$\partial_\xi^\beta b(x,\xi) = \sum_{n\in\mathbb{Z}^d}\widehat{\partial_\xi^\beta a}\big(n,\xi+\frac{n}{2}\big)e^{\mathrm{i}n\cdot x} = \sum_{n\in\mathbb{Z}^d}\widehat{\partial_\xi^\beta a}\big(n,\xi+\frac{n}{2}\big)\,\chi_\epsilon\Big(\frac{n}{\langle\xi\rangle}\Big)\,e^{\mathrm{i}n\cdot x}$$

for some $\epsilon = \epsilon(\delta') > 0$, where in the last equality we used that the sum is actually restricted over the indexes for which $|n| \le \delta'\langle\xi\rangle$, $\delta' \in (0,1)$. Then we represent $\partial_\xi^\beta b$ as the integral

$$\partial_\xi^\beta b(x,\xi) = \int_{\mathbb{T}^d} K(x,y)\,\mathrm{d}y\,, \quad K(x,y) := \frac{1}{(2\pi)^d}\sum_{n\in\mathbb{Z}^d} (\partial_\xi^\beta a)\big(x-y,\xi+\frac{n}{2}\big)\,\chi_\epsilon\left(\frac{n}{\langle\xi\rangle}\right) e^{\mathrm{i}n\cdot y}\,. \tag{2.2.44}$$

We are going to estimate the L^1-norm of $K(x,\cdot)$ using Lemma 2.2.4. First note that, since $a \in \Sigma^m_{L^\infty}$, we have $\langle \xi + \frac{n}{2}\rangle \sim \langle\xi\rangle$ on the support of $\widehat{\partial_\xi^\beta a}(n, \xi + (n/2))$, and then we bound (2.2.44) as

$$|K(x,y)| \lesssim \sum_{|n|\le\delta'\langle\xi\rangle} |a|_{m,L^\infty,|\beta|}\langle\xi\rangle^{m-|\beta|} \lesssim |a|_{m,L^\infty,|\beta|}\,\langle\xi\rangle^{d+m-|\beta|}\,, \tag{2.2.45}$$

uniformly in x. Moreover, using Abel resummation formula (2.2.16) and the Leibniz rule (2.2.15) for finite differences, we get, for any $h = 1,\dots,d$,

$$K(x,y) = \frac{1}{(e^{\mathrm{i}y_h}-1)^{d+1}} \sum_{k_1+k_2=d+1} C_{k_1,k_2} \sum_{|n|\le\delta'\langle\xi\rangle} \partial_h^{k_1}(\partial_\xi^\beta a)\big(x-y,\xi+\frac{n}{2}\big)\,\partial_h^{k_2}\chi_\epsilon\left(\frac{n}{\langle\xi\rangle}\right) e^{\mathrm{i}n\cdot y}\,.$$

Then, using (2.2.3) and that $\big|\partial_h^k\chi_\epsilon\big(\frac{n}{\langle\xi\rangle}\big)\big| \lesssim \langle\xi\rangle^{-k}$, $\forall h = 1,\dots,d$, we estimate

$$|K(x,y)| \lesssim \frac{\langle\xi\rangle^{m-(d+1)-|\beta|}}{|2\sin(y_h/2)|^{d+1}}\,|a|_{m,L^\infty,|\beta|+d+1} \sum_{|n|\le\delta'\langle\xi\rangle} 1 \lesssim \frac{\langle\xi\rangle^{d+m-|\beta|}\,|a|_{m,L^\infty,|\beta|+d+1}}{|\langle\xi\rangle\,2\sin(y_h/2)|^{d+1}} \tag{2.2.46}$$

uniformly in x. In view of (2.2.45)-(2.2.46) we apply Lemma 2.2.4 with $A = \langle\xi\rangle$ and $B = \langle\xi\rangle^{m-|\beta|}\,|a|_{m,L^\infty,|\beta|+d+1}$ obtaining

$$\big|\partial_\xi^\beta b(x,\xi)\big| \le \int_{\mathbb{T}^d} |K(x,y)|\mathrm{d}y \lesssim \langle\xi\rangle^{m-|\beta|}\,|a|_{m,L^\infty,|\beta|+d+1}\,, \quad \forall (x,\xi) \in \mathbb{T}^d\times\mathbb{R}^d,$$

that proves (2.2.43). ☐

Continuity. We now prove boundedness estimates in Sobolev spaces of operators with spectrally localized symbols, requiring derivatives in ξ of the symbol and no derivatives in x.

Theorem 2.2.10. (Continuity) *Let $a \in \Sigma^m_{L^\infty}$ with $m \in \mathbb{R}$. Then $\mathrm{Op}(a)$ defined in (2.2.40) extends to a bounded operator from $H^s \to H^{s-m}$, for any $s \in \mathbb{R}$, satisfying*

$$\|\mathrm{Op}(a)\,u\|_{s-m} \lesssim |a|_{m,L^\infty,d+1}\,\|u\|_s. \tag{2.2.47}$$

Moreover, if a fulfills (2.2.2) with $\delta > 0$ small enough, then the operator $\mathrm{Op}^W(a)$ defined in (2.2.11) satisfies

$$\|\mathrm{Op}^W(a)\,u\|_{s-m} \lesssim |a|_{m,L^\infty,2(d+1)}\,\|u\|_s\,. \tag{2.2.48}$$

Proof. We first recall the Littlewood-Paley characterization of the Sobolev norm

$$\|u\|_s^2 \sim \sum_{k\ge0} 2^{2ks}\|\Delta_k u\|_0^2 \tag{2.2.49}$$

where Δ_k are defined in (2.2.25). The norm $\| \; \|_0 = \| \; \|_{L^2}$. We first prove (2.2.47).

Step 1: according to (2.2.24), we perform the Littlewood-Paley decomposition of $\mathrm{Op}(a)$,

$$\mathrm{Op}(a)\,v = \sum_{k\geq 0} \mathrm{Op}(a_k)\,v\,, \tag{2.2.50}$$

where

$$a_0(x,\xi) := a(x,\xi)\chi(\xi)\,, \quad a_k(x,\xi) := a(x,\xi)\varphi(2^{-k}\xi)\,, \quad k \geq 1\,. \tag{2.2.51}$$

In order to prove (2.2.47), it is sufficient to prove that

$$\|\mathrm{Op}(a_k)\,v\|_0 \lesssim |a|_{m,L^\infty,d+1}\, 2^{km}\, \|v\|_0, \quad \forall k \in \mathbb{N}_0\,, \quad \forall v \in L^2\,. \tag{2.2.52}$$

Indeed, decomposing v in Paley-Littlewood packets as in (2.2.25),

$$v = \sum_{j\geq 0} \Delta_j v\,, \quad \Delta_0 = \chi(D)\,,\ \Delta_j = \varphi(2^{-j}D)\,, \tag{2.2.53}$$

which are *almost orthogonal* in L^2 (namely $\Delta_k\Delta_j = 0$ for any $|j-k| \geq 3$), using the fact that $\mathrm{Op}(a_k)\,v = \mathrm{Op}(a)\,\Delta_k v$, and since the action of $\mathrm{Op}(a_k)$ does not spread much the Fourier support of functions being a spectrally localized, according to (2.2.13), we have

$$\begin{aligned}
\|\mathrm{Op}(a)\,v\|_{s-m}^2 &\overset{(2.2.50)}{=} \Big\| \sum_{k\geq 0} \mathrm{Op}(a_k)\,v \Big\|_{s-m}^2 \overset{(2.2.53),(2.2.51)}{=} \Big\| \sum_{|j-k|<3} \mathrm{Op}(a_k)\,\Delta_j v \Big\|_{s-m}^2 \\
&\sim \sum_{|j-k|<3} 2^{2k(s-m)}\, \|\mathrm{Op}(a_k)\,\Delta_j v\|_0^2 \overset{(2.2.52)}{\lesssim} |a|_{m,L^\infty,d+1}^2 \sum_{|j-k|<3} 2^{2ks}\, \|\Delta_j v\|_0^2 \\
&\lesssim |a|_{m,L^\infty,d+1}^2 \sum_{k\geq 0} 2^{2ks} \|\Delta_k v\|_0^2 \overset{(2.2.49)}{\sim} |a|_{m,L^\infty,d+1}^2 \|v\|_s^2\,.
\end{aligned}$$

Step 2: By (2.2.51) and (2.2.40) we write $\mathrm{Op}(a_k)$ as the integral operator

$$(\mathrm{Op}(a_k)\,v)(x) = \int_{\mathbb{T}^d} K_k(x, x-y)\, v(y)\, \mathrm{d}y \tag{2.2.54}$$

with kernel

$$K_k(x,z) := \frac{1}{(2\pi)^d} \sum_{\ell\in\mathbb{Z}^d} e^{\mathrm{i}\ell\cdot z}\, a(x,\ell)\, \varphi(2^{-k}\ell)\,. \tag{2.2.55}$$

We shall deduce (2.2.52) by applying the Schur lemma: if

$$\sup_{x\in\mathbb{T}^d} \int_{\mathbb{T}^d} |K(x,x-y)| \mathrm{d}y =: C_1 < +\infty\,, \quad \sup_{y\in\mathbb{T}^d} \int_{\mathbb{T}^d} |K(x,x-y)| \mathrm{d}x =: C_2 < +\infty \tag{2.2.56}$$

then Schur lemma guarantees that the integral operator (2.2.54) is bounded on $L^2(\mathbb{T}^d)$ and

$$\|\mathrm{Op}(a_k)\,v\|_0 \leq (C_1C_2)^{1/2} \|v\|_0\,. \tag{2.2.57}$$

Let us prove (2.2.56) and estimate the constants C_1, C_2. By (2.2.55) we have that

$$|K_k(x,z)| \lesssim \sum_{\ell\in\mathbb{Z}^d} |a(x,\ell)| \varphi(2^{-k}\ell) \overset{(2.2.3)}{\lesssim} |a|_{m,L^\infty,0} \sum_{\ell\in\mathbb{Z}^d} \langle\ell\rangle^m \varphi(2^{-k}\ell) \lesssim 2^{k(d+m)} |a|_{m,L^\infty,0}\,. \tag{2.2.58}$$

Then, applying $(d+1)$-times Abel resummation formula (2.2.16) to (2.2.55), we obtain, for any $h = 1,\ldots,d$,

$$K_k(x,z) = \frac{1}{(2\pi)^d}\frac{1}{(e^{\mathrm{i}z_h}-1)^{d+1}}\sum_{\ell\in\mathbb{Z}^d} e^{\mathrm{i}\ell\cdot z}\,\partial_h^{d+1}(a(x,\ell)\varphi(2^{-k}\ell))$$

and we deduce, using (2.2.3), (2.2.15), $|K_k(x,z)| \lesssim |2\sin(z_h/2)|^{-d-1}|a|_{m,L^\infty,d+1}2^{k(m-1)}$ for any $h = 1,\ldots,d$, thus

$$|K_k(x,z)| \lesssim 2^{k(d+m)}\,|a|_{m,L^\infty,d+1}\min_{h=1,\ldots,d}\frac{1}{(2^k 2\,|\sin(z_h/2)|)^{d+1}}\,. \tag{2.2.59}$$

By (2.2.58), (2.2.59) we apply Lemma 2.2.4 with $A = 2^k$ and $B = 2^{km}|a|_{m,L^\infty,d+1}$, deducing that

$$\int_{\mathbb{T}^d}|K_k\big(x,x-y\big)|\mathrm{d}y = \int_{\mathbb{T}^d}|K_k\big(x,z\big)|\mathrm{d}z \lesssim 2^{km}\,|a|_{m,L^\infty,d+1} \tag{2.2.60}$$

uniformly for $x \in \mathbb{T}^d$. Similarly

$$\int_{\mathbb{T}^d}|K_k\big(x,x-y\big)|\mathrm{d}x \lesssim 2^{km}\,|a|_{m,L^\infty,d+1} \tag{2.2.61}$$

uniformly for $y \in \mathbb{T}^d$. Finally (2.2.60), (2.2.61), (2.2.57) prove (2.2.52) completing the proof of (2.2.47).

PROOF OF (2.2.48). By Lemma 2.2.9 we have $\mathrm{Op}^W(a) = \mathrm{Op}(b)$ for a spectrally localized symbol $b \in \Sigma^m_{L^\infty}$ which fulfills estimate (2.2.43). Then (2.2.48) follows by (2.2.47). □

Composition of para-differential operators. We finally prove a composition result for para-differential operators. The difference with respect to Theorem 6.1.1 and 6.1.4 in [99] is to have periodic symbols and the use of the Weyl quantization.

We shall use that, in view of the interpolation inequality (2.2.29), if $a \in \Gamma^m_{W^{\varrho,\infty}}$ and $b \in \Gamma^{m'}_{W^{\varrho,\infty}}$ then $ab \in \Gamma^{m+m'}_{W^{\varrho,\infty}}$ and, for any $N \in \mathbb{N}_0$, any $0 \le \varrho_1 \le \alpha \le \beta \le \varrho_2$ such that $\varrho_1 + \varrho_2 = \alpha + \beta$

$$\begin{aligned}&|ab|_{m+m',W^{\varrho,\infty},N} \lesssim |a|_{m,W^{\varrho,\infty},N}\,|b|_{m',L^\infty,N} + |a|_{m,L^\infty,N}\,|b|_{m',W^{\varrho,\infty},N}\,,\\ &|a|_{m,W^{\alpha,\infty},N}\,|b|_{m',W^{\beta,\infty},N} \lesssim |a|_{m,W^{\varrho_1,\infty},N}\,|b|_{m',W^{\varrho_2,\infty},N} + |a|_{m,W^{\varrho_2,\infty},N}\,|b|_{m',W^{\varrho_1,\infty},N}\,.\end{aligned} \tag{2.2.62}$$

Theorem 2.2.11. (Composition) *Let $a \in \Gamma^m_{W^{\varrho,\infty}}$, $b \in \Gamma^{m'}_{W^{\varrho,\infty}}$ with $m, m' \in \mathbb{R}$ and $\varrho \in (0,2]$. Then*

$$\mathrm{Op}^{\mathrm{BW}}(a)\mathrm{Op}^{\mathrm{BW}}(b) = \mathrm{Op}^{\mathrm{BW}}(a\#_\varrho b) + R^{-\varrho}(a,b) \tag{2.2.63}$$

where the linear operator $R^{-\varrho}(a,b)\colon \dot{H}^s \to \dot{H}^{s-(m+m')+\varrho}$, $\forall s \in \mathbb{R}$, satisfies

$$\|R^{-\varrho}(a,b)u\|_{s-(m+m')+\varrho} \lesssim \Big(|a|_{m,W^{\varrho,\infty},N}\,|b|_{m',L^\infty,N} + |a|_{m,L^\infty,N}\,|b|_{m',W^{\varrho,\infty},N}\Big)\,\|u\|_s \tag{2.2.64}$$

with $N \ge 3d+4$.

Proof. We give the proof in the case $\varrho \in (1,2]$. We first compute $\mathrm{Op}^{\mathrm{BW}}(a)\mathrm{Op}^{\mathrm{BW}}(b)$. Recalling the definition (2.2.12) we obtain

$$\mathrm{Op}^{\mathrm{BW}}(a)\mathrm{Op}^{\mathrm{BW}}(b)u = \mathrm{Op}^W(a_\chi)\,\mathrm{Op}^W(b_\chi) = \sum_{j,k,\ell}\widehat{a}_\chi\Big(j-k,\frac{j+k}{2}\Big)\,\widehat{b}_\chi\Big(k-\ell,\frac{k+\ell}{2}\Big)\,u_\ell\,e^{\mathrm{i}j\cdot x}\,.$$

We now perform a Taylor expansion of $\widehat{a}_\chi\big(j-k,\frac{j+k}{2}\big)$ in the second variable, around the point $\frac{j+\ell}{2}$. Writing $j+k=j+\ell+(k-\ell)$, we obtain

$$\widehat{a}_\chi\Big(j-k,\frac{j+k}{2}\Big)=\widehat{a}_\chi\Big(j-k,\frac{j+\ell}{2}\Big)+\Big(\frac{k-\ell}{2}\Big)\cdot\partial_\xi\widehat{a}_\chi\Big(j-k,\frac{j+\ell}{2}\Big)\\ +\sum_{\alpha\in\mathbb{N}_0^d,|\alpha|=2}\Big(\frac{k-\ell}{2}\Big)^\alpha\int_0^1(1-t)\,\partial_\xi^\alpha\widehat{a}_\chi\Big(j-k,\frac{j+\ell+t(k-\ell)}{2}\Big)\mathrm{d}t\,.$$

We expand analogously $\widehat{b}_\chi\big(k-\ell,\frac{k+\ell}{2}\big)$ around the point $\frac{j+\ell}{2}$. Writing $k+\ell=j+\ell-(j-k)$, we obtain

$$\widehat{b}_\chi\Big(k-\ell,\frac{k+\ell}{2}\Big)=\widehat{b}_\chi\Big(k-\ell,\frac{j+\ell}{2}\Big)-\Big(\frac{j-k}{2}\Big)\cdot\partial_\xi\widehat{b}_\chi\Big(k-\ell,\frac{j+\ell}{2}\Big)\\ +\sum_{\beta\in\mathbb{N}_0^d,|\beta|=2}\Big(\frac{k-j}{2}\Big)^\beta\int_0^1(1-t)\,\partial_\xi^\beta\widehat{b}_\chi\Big(k-\ell,\frac{j+\ell+t(k-j)}{2}\Big)\mathrm{d}t\,.$$

Moreover, recalling (2.2.87) and (2.2.11), we write $\mathrm{Op}^{\mathrm{BW}}(a\#_\varrho b)u=\mathrm{Op}^W\big((ab+\frac{1}{2\mathrm{i}}\{a,b\})_\chi\big)u$ and, by the previous expansions,

$$\Big(\mathrm{Op}^{\mathrm{BW}}(a)\mathrm{Op}^{\mathrm{BW}}(b)-\mathrm{Op}^{BW}\Big(ab+\frac{1}{2\mathrm{i}}\{a,b\}\Big)\Big)u=\sum_{i=1}^4R_i(a,b)u$$

where

$$R_1(a,b)u:=\mathrm{Op}^{\mathrm{W}}\Big(a_\chi b_\chi-(ab)_\chi+\frac{1}{2\mathrm{i}}\big(\{a_\chi,b_\chi\}-(\{a,b\})_\chi\big)\Big)u \tag{2.2.65}$$

$$R_2(a,b)u:=\sum_{j,k,\ell}\widehat{b}_\chi\Big(k-\ell,\frac{k+\ell}{2}\Big)\sum_{|\alpha|=2}\Big(\frac{k-\ell}{2}\Big)^\alpha\int_0^1(1-t)\,\partial_\xi^\alpha\widehat{a}_\chi\Big(j-k,\frac{j+\ell+t(k-\ell)}{2}\Big)\mathrm{d}tu_\ell\,e^{\mathrm{i}j\cdot x} \tag{2.2.66}$$

$$R_3(a,b)u:=\sum_{j,k,\ell}-\Big(\frac{k-\ell}{2}\Big)\cdot\partial_\xi\widehat{a}_\chi\Big(j-k,\frac{j+\ell}{2}\Big)\Big(\frac{j-k}{2}\Big)\cdot\int_0^1\partial_\xi\widehat{b}_\chi\Big(k-\ell,\frac{j+\ell+t(k-j)}{2}\Big)\mathrm{d}tu_\ell\,e^{\mathrm{i}j\cdot x} \tag{2.2.67}$$

$$R_4(a,b)u:=\sum_{j,k,\ell}\widehat{a}_\chi\Big(j-k,\frac{j+\ell}{2}\Big)\sum_{|\beta|=2}\Big(\frac{k-j}{2}\Big)^\beta\int_0^1(1-t)\,\partial_\xi^\beta\widehat{b}_\chi\Big(k-\ell,\frac{j+\ell+t(k-j)}{2}\Big)\mathrm{d}tu_\ell\,e^{\mathrm{i}j\cdot x}. \tag{2.2.68}$$

We show now that the operators $R_i(a,b)$, $i=1,\dots,4$ fulfill estimate (2.2.64).

Estimate of $R_1(a,b)$. By exchanging the role of a and b it is enough to prove that the symbols $\partial_\xi^\alpha a_\chi\partial_x^\alpha b_\chi-(\partial_\xi^\alpha a\,\partial_x^\alpha b)_\chi$, $|\alpha|\le 1$, belong to $\Sigma_{L^\infty}^{m+m'-\varrho}$ and then apply Theorem 2.2.10. The spectral localization property follows because of the cut-off χ_ϵ and ϵ small. As ∂_x^α commutes with the Fourier multiplier $\chi_{\epsilon\langle\xi\rangle}(D)$ we have that $\partial_x^\alpha b_\chi=(\partial_x^\alpha b)_\chi$ and we write $\partial_\xi^\alpha a_\chi\,\partial_x^\alpha b_\chi-(\partial_\xi^\alpha a\,\partial_x^\alpha b)_\chi$ as

$$(\partial_\xi^\alpha a)_\chi\,[(\partial_x^\alpha b)_\chi-\partial_x^\alpha b]+\big[(\partial_\xi^\alpha a)_\chi-\partial_\xi^\alpha a\big]\,\partial_x^\alpha b+\big[\partial_\xi^\alpha a\,\partial_x^\alpha b-(\partial_\xi^\alpha a\,\partial_x^\alpha b)_\chi\big] \tag{2.2.69}$$

$$+\big[\partial_\xi^\alpha a_\chi-(\partial_\xi^\alpha a)_\chi\big]\,(\partial_x^\alpha b)_\chi\,. \tag{2.2.70}$$

Consider first the term in (2.2.70). By Lemma 2.2.8, $\partial_\xi^\alpha a_\chi - (\partial_\xi^\alpha a)_\chi \in \Gamma_{L^\infty}^{m-\varrho-|\alpha|}$ and $(\partial_x^\alpha b)_\chi \in \Gamma_{L^\infty}^{m'+|\alpha|}$ and by remark (v) after Definition 2.2.1, for any $n \in \mathbb{N}_0$,

$$\big|\big[\partial_\xi^\alpha a_\chi - (\partial_\xi^\alpha a)_\chi\big]\,(\partial_x^\alpha b)_\chi\big|_{m+m'-\varrho,L^\infty,n} \leq \big|\partial_\xi^\alpha a_\chi - (\partial_\xi^\alpha a)_\chi\big|_{m-|\alpha|-\varrho,L^\infty,n} |(\partial_x^\alpha b)_\chi|_{m'+|\alpha|,L^\infty,n} \overset{(2.2.36),(2.2.37)}{\lesssim} |a|_{m,W^{\varrho,\infty},n+|\alpha|}\,|b|_{m',L^\infty,n}\,.$$

Next consider the terms in (2.2.69). By remarks (iii), (iv) after Definition 2.2.1, we have $\partial_\xi^\alpha a \in \Gamma_{W^{\varrho,\infty}}^{m-|\alpha|} \subset \Gamma_{W^{\varrho-|\alpha|,\infty}}^{m-|\alpha|}$, $\partial_x^\alpha b \in \Gamma_{W^{\varrho-|\alpha|,\infty}}^{m'}$, so we can apply Lemma 2.2.8, property (2.2.62) and (2.2.6) to obtain

$$\begin{aligned} |(2.2.69)|_{m+m'-\varrho,L^\infty,n} &\lesssim |a|_{m,W^{\varrho-|\alpha|,\infty},n+|\alpha|}\,|b|_{m',W^{|\alpha|,\infty},N} + |a|_{m,L^\infty,n+|\alpha|}\,|b|_{m',W^{\varrho,\infty},n} \\ &\lesssim |a|_{m,W^{\varrho,\infty},n+1}\,|b|_{m',L^\infty,n+1} + |a|_{m,L^\infty,n+1}\,|b|_{m',W^{\varrho,\infty},n+1} \end{aligned} \tag{2.2.71}$$

where to pass from the first to the second line we used the second interpolation inequality in (2.2.62). Altogether we have proved that the symbol in (4.7.24) belongs to $\Sigma_{L^\infty}^{m+m'-\varrho}$ and its semi-norms are bounded by (2.2.71). Then Theorem 2.2.10 proves that $R_1(a,b)$ fulfills estimate (2.2.64).

<u>Estimate of $R_2(a,b)$</u>. First we rewrite (2.2.66) as

$$R_2(a,b)u = \frac{1}{4}\sum_{j,\ell}\Big(\int_0^1 (1-t)\sum_{|\alpha|=2}\widehat{f_t^\alpha}(j-\ell,\ell)\,\mathrm{d}t\Big)u_\ell\, e^{\mathrm{i}j\cdot x}$$

where

$$\begin{aligned} \widehat{f_t^\alpha}(n,\ell) &:= \sum_{k\in\mathbb{Z}^d}\widehat{D_x^\alpha b_\chi}\Big(k-\ell,\frac{k+\ell}{2}\Big)\partial_\xi^\alpha\widehat{a}_\chi\Big(n+\ell-k,\,\ell+\frac{n+t(k-\ell)}{2}\Big) \\ &\overset{j=k-\ell}{=} \sum_{j\in\mathbb{Z}^d}\widehat{D_x^\alpha b_\chi}\Big(j,\ell+\frac{j}{2}\Big)\partial_\xi^\alpha\widehat{a}_\chi\Big(n-j,\,\ell+\frac{n+tj}{2}\Big) \end{aligned}$$

and $D_{x_n} := \partial_{x_n}/\mathrm{i}$ and $D_x^\alpha := D_{x_1}^{\alpha_1}\cdots D_{x_d}^{\alpha_d}$. Then, recalling (2.2.40),

$$R_2(a,b)u = \frac{1}{4}\int_0^1 (1-t)\sum_{|\alpha|=2}\mathrm{Op}(f_t^\alpha)\,u\,\mathrm{d}t$$

where

$$f_t^\alpha(x,\xi) := \sum_{n,j}\widehat{D_x^\alpha b_\chi}\Big(j,\xi+\frac{j}{2}\Big)\partial_\xi^\alpha\widehat{a}_\chi\Big(n-j,\,\xi+\frac{n+tj}{2}\Big)e^{\mathrm{i}n\cdot x}. \tag{2.2.72}$$

We claim that $f_t^\alpha(x,\xi)$ is spectrally localized, namely

$$\exists\delta\in(0,1):\quad |n|\leq\delta\langle\xi\rangle,\quad \forall(n,\xi)\in\operatorname{supp}\widehat{f_t^\alpha}\,. \tag{2.2.73}$$

In fact on the support of $\widehat{b}_\chi\big(j,\xi+\frac{j}{2}\big)$ we have, for some $\delta'\in(0,1)$,

$$|j|\leq\delta'\langle\xi\rangle\,, \tag{2.2.74}$$

whereas, on the support of $\partial_\xi^\alpha\widehat{a}_\chi\big(n-j,\xi+\frac{n+tj}{2}\big)$, $t\in[0,1]$,

$$|n-j|\leq\delta\langle\xi\rangle+\delta\langle n\rangle+\delta\langle j\rangle \overset{(2.2.74)}{\leq} (\delta+\delta\delta')\langle\xi\rangle+\delta\langle n\rangle\,. \tag{2.2.75}$$

The estimates (2.2.74)-(2.2.75) then give $|n| \leq |j| + |n-j| \leq \delta'\langle\xi\rangle + (\delta + \delta\delta')\langle\xi\rangle + \delta\langle n\rangle$, which implies (2.2.73).

In order to apply Theorem 2.2.10 it remains to prove that, for any $N \geq 3d+4$,

$$|f_t^\alpha(x,\xi)|_{m+m'-\varrho,L^\infty,d+1} \lesssim |b|_{m',W^{\varrho,\infty},N}\; |a|_{m,L^\infty,N}\,, \tag{2.2.76}$$

which implies, for any $s \in \mathbb{R}$, $u \in \dot{H}^s$, $\|R_2(a,b)u\|_{s-m-m'+\varrho} \lesssim |b|_{m',W^{\varrho,\infty},N}\; |a|_{m,L^\infty,N}\|u\|_s$. Thus $R_2(a,b)$ satisfies the estimate (2.2.64).

In order to prove (2.2.76) note that, differentiating (2.2.72), for any $\beta \in \mathbb{N}_0^d$,

$$\begin{aligned}\partial_\xi^\beta f_t^\alpha(x,\xi) &= \sum_{\beta_1+\beta_2=\beta} C_{\beta_1,\beta_2} \sum_{n,j} \widehat{\partial_\xi^{\beta_1} D_x^\alpha b_\chi}\Big(j,\xi+\frac{j}{2}\Big)\partial_\xi^{\alpha+\beta_2}\widehat{a}_\chi\Big(n-j,\,\xi+\frac{n+tj}{2}\Big)e^{\mathrm{i}n\cdot x}\\ &= \sum_{\beta_1+\beta_2=\beta} C_{\beta_1,\beta_2}\int_{\mathbb{T}^{2d}} K_t^{\beta_1,\beta_2}(x,y,z)\,\mathrm{d}y\,\mathrm{d}z\end{aligned} \tag{2.2.77}$$

where C_{β_1,β_2} are binomial coefficients and

$$K_t^{\beta_1,\beta_2}(x,y,z) := \frac{1}{(2\pi)^{2d}}\sum_{n,j}(\partial_\xi^{\beta_1}D_x^\alpha b_\chi)\Big(x-z-y,\xi+\frac{j}{2}\Big)\partial_\xi^{\alpha+\beta_2}a_\chi\Big(x-z,\,\xi+\frac{n+tj}{2}\Big)e^{\mathrm{i}(n\cdot z+j\cdot y)}\,. \tag{2.2.78}$$

By (2.2.73) and (2.2.74) the sum over n in (2.2.72) is restricted to indexes satisfying

$$|n| \ll \langle\xi\rangle\,,\ |j| \ll \langle\xi\rangle\,, \qquad \text{and therefore} \qquad \langle\xi+\frac{j}{2}\rangle \sim \langle\xi+\frac{n+tj}{2}\rangle \sim \langle\xi\rangle\,.$$

We deduce that the sum in (2.2.78) is bounded by

$$\begin{aligned}\big|K_t^{\beta_1,\beta_2}(x,y,z)\big| &\lesssim \langle\xi\rangle^{2d+m+m'-|\beta|-\varrho}\,|\partial_\xi^{\beta_1}D_x^\alpha b_\chi|_{m'-|\beta_1|+|\alpha|-\varrho,L^\infty,0}\,|\partial_\xi^{\alpha+\beta_2}a_\chi|_{m-|\alpha|-|\beta_2|,L^\infty,0}\\ &\overset{(2.2.6),(2.2.37),(2.2.34)}{\lesssim} \langle\xi\rangle^{2d+m+m'-|\beta|-\varrho}\,|b|_{m',W^{\varrho,\infty},|\beta|}\,|a|_{m,L^\infty,2+|\beta|}\,,\end{aligned} \tag{2.2.79}$$

recalling that $|\alpha| = 2$. We also estimate $K_t^{\beta_1,\beta_2}(x,y,z)$ applying Abel resummation formula (2.2.16) in the sum (2.2.78), in the index n and in the index j separately, obtaining, using (2.2.37), (2.2.34), (2.2.15) and (2.2.6),

$$\begin{aligned}\big|K_t^{\beta_1,\beta_2}(x,y,z)\big| &\lesssim \langle\xi\rangle^{2d+m+m'-|\beta|-\varrho}\,|b|_{m',W^{\varrho,\infty},2d+1+|\beta|}|a|_{m,L^\infty,2d+3+|\beta|}\\ &\quad\times\min_{1\leq h\leq d}\Big(\Big|\langle\xi\rangle 2\sin\frac{y_h}{2}\Big|^{-(2d+1)},\ \Big|\langle\xi\rangle 2\sin\frac{z_h}{2}\Big|^{-(2d+1)}\Big).\end{aligned} \tag{2.2.80}$$

In view of (2.2.79)-(2.2.80) and $|\beta| \leq d+1$, we apply Lemma 2.2.4 with $d \rightsquigarrow 2d$, choosing $A = \langle\xi\rangle$, $B = \langle\xi\rangle^{m+m'-|\beta|-\varrho}|b|_{m',W^{\varrho,\infty},2d+1+|\beta|}\,|a|_{m,L^\infty,2d+3+|\beta|}$ and we obtain

$$\|\partial_\xi^\beta f_t^\alpha(\cdot,\xi)\|_{L^\infty} \lesssim \int_{\mathbb{T}^{2d}}|K_t^{\beta_1,\beta_2}(x,y,z)|\,\mathrm{d}y\,\mathrm{d}z \lesssim \langle\xi\rangle^{m+m'-\varrho-|\beta|}\,|b|_{m',W^{\varrho,\infty},3d+2}\,|a|_{m,L^\infty,3d+4}$$

proving (2.2.76).

The proof that $R_3(a,b)$ and $R_4(a,b)$ satisfy the estimate (2.2.64) follows similarly. $\square$

2.2.3 Para-differential calculus in Sobolev spaces

The Sobolev norms $\| \; \|_s$ satisfy interpolation inequalities (see e.g. section 3.5 in [26]):
(i) for all $s \geq s_0 > \frac{d}{2}$, $u, v \in H^s$,

$$\|uv\|_s \lesssim \|u\|_{s_0}\|v\|_s + \|u\|_s\|v\|_{s_0}\,. \tag{2.2.81}$$

(ii) For all $0 \leq s \leq s_0$, $v \in H^s$, $u \in H^{s_0}$,

$$\|uv\|_s \lesssim \|u\|_{s_0}\|v\|_s\,. \tag{2.2.82}$$

(iii) For all $s_1 < s_2$, $\theta \in [0,1]$ and $u \in H^{s_2}$,

$$\|u\|_{\theta s_1+(1-\theta)s_2} \leq \|u\|_{s_1}^{\theta}\|u\|_{s_2}^{1-\theta}\,. \tag{2.2.83}$$

(iv) For all $a \leq \alpha \leq \beta \leq b$, $u, v \in H^b$,

$$\|u\|_\alpha\|v\|_\beta \leq \|u\|_a\|v\|_b + \|u\|_b\|v\|_a\,. \tag{2.2.84}$$

Para-differential operator act on Sobolev spaces, namely the following result holds true .

Theorem 2.2.12. (Continuity of Bony-Weyl operators) *Let $a \in \Gamma^m_{s_0}$, resp. $a \in \Gamma^m_{L^\infty}$, with $m \in \mathbb{R}$. Then $\mathrm{Op}^{\mathrm{BW}}(a)$ extends to a bounded operator $\dot{H}^s \to \dot{H}^{s-m}$ for any $s \in \mathbb{R}$ satisfying the estimate, for any $u \in \dot{H}^s$,*

$$\|\mathrm{Op}^{\mathrm{BW}}(a)u\|_{s-m} \lesssim |a|_{m,s_0,2(d+1)}\,\|u\|_s \tag{2.2.85}$$

Moreover, for any $\varrho \geq 0$, $s \in \mathbb{R}$, $u \in \dot{H}^s(\mathbb{T}^d)$,

$$\|\mathrm{Op}^{\mathrm{BW}}(a)u\|_{s-m-\varrho} \lesssim |a|_{m,s_0-\varrho,2(d+1)}\,\|u\|_s\,. \tag{2.2.86}$$

Proof. Since $\mathrm{Op}^{\mathrm{BW}}(a) = \mathrm{Op}^W(a_\chi)$, the estimate (2.2.85) follows by (2.2.48), (2.2.34) and $|a|_{m,L^\infty,N} \lesssim |a|_{m,s_0,N}$. Note that the condition on the Fourier support of a_χ in Theorem 2.2.10 is automatically satisfied provided ϵ in (2.2.9) is sufficiently small. To prove (2.2.86) we use also (2.2.35). □

The second result of symbolic calculus that we shall use regards composition for Bony-Weyl para-differential operators at the second order with mild smoothness assumptions for the symbols in the space variable x. Given symbols $a \in \Gamma^m_{s_0+\varrho}$, $b \in \Gamma^{m'}_{s_0+\varrho}$ with $m, m' \in \mathbb{R}$ and $\varrho \in (0,2]$ we define

$$a\#_\varrho b := \begin{cases} ab\,, & \varrho \in (0,1] \\ ab + \frac{1}{2\mathrm{i}}\{a,b\}\,, & \varrho \in (1,2]\,, \end{cases} \quad \text{where} \quad \{a,b\} := \nabla_\xi a \cdot \nabla_x b - \nabla_x a \cdot \nabla_\xi b\,, \tag{2.2.87}$$

is the Poisson bracket between $a(x,\xi)$ and $b(x,\xi)$. By (2.2.6) and (2.2.8) we have that ab is a symbol in $\Gamma^{m+m'}_{s_0+\varrho}$ and $\{a,b\}$ is in $\Gamma^{m+m'-1}_{s_0+\varrho-1}$. The next result follows directly by Theorem 2.2.11 and (2.2.5).

Theorem 2.2.13. (Composition) *Let $a \in \Gamma^m_{s_0+\varrho}$, $b \in \Gamma^{m'}_{s_0+\varrho}$ with $m, m' \in \mathbb{R}$ and $\varrho \in (0,2]$. Then*

$$\mathrm{Op}^{\mathrm{BW}}(a)\mathrm{Op}^{\mathrm{BW}}(b) = \mathrm{Op}^{\mathrm{BW}}(a\#_\varrho b) + R^{-\varrho}(a,b) \tag{2.2.88}$$

where the linear operator $R^{-\varrho}(a,b)\colon \dot{H}^s \to \dot{H}^{s-(m+m')+\varrho}$, $\forall s \in \mathbb{R}$, satisfies, for any $u \in \dot{H}^s$,

$$\|R^{-\varrho}(a,b)u\|_{s-(m+m')+\varrho} \lesssim \Big(|a|_{m,s_0+\varrho,N}\,|b|_{m',s_0,N} + |a|_{m,s_0,N}\,|b|_{m',s_0+\varrho,N}\Big)\|u\|_s \tag{2.2.89}$$

where $N \geq 3d+4$.

A useful corollary of Theorems 2.2.13 and 2.2.12 (using also (2.2.6)-(2.2.8)) is the following:

Corollary 2.2.14. *Let* $a \in \Gamma^m_{s_0+2}$, $b \in \Gamma^{m'}_{s_0+2}$, $c \in \Gamma^{m''}_{s_0+2}$ *with* $m, m', m'' \in \mathbb{R}$. *Then*

$$\mathrm{Op}^{\mathrm{BW}}(a) \circ \mathrm{Op}^{\mathrm{BW}}(b) \circ \mathrm{Op}^{\mathrm{BW}}(c) = \mathrm{Op}^{\mathrm{BW}}(abc) + R_1(a,b,c) + R_0(a,b,c), \tag{2.2.90}$$

where

$$R_1(a,b,c) := \frac{1}{2\mathrm{i}}\mathrm{Op}^{BW}\big(\{a,c\}b + \{b,c\}a + \{a,b\}c\big) \tag{2.2.91}$$

satisfies $R_1(a,b,c) = -R_1(c,b,a)$ *and* $R_0(a,b,c)$ *is a bounded operator* $\dot{H}^s \to \dot{H}^{s-(m+m'+m'')+2}$, $\forall s \in \mathbb{R}$, *satisfying, for any* $u \in \dot{H}^s$,

$$\|R_0(a,b,c)\|_{s-(m+m'+m'')+2} \lesssim |a|_{m,s_0+2,N}\, |b|_{m',s_0+2,N}\, |c|_{m'',s_0+2,N}\, \|u\|_s \tag{2.2.92}$$

where $N \geq 3d+5$.

We now provide the Bony-para-product decomposition for the product of Sobolev functions in the Bony-Weyl quantization. Recall that $\Pi_0^\perp$ denotes the projector on the subspace H^s_0.

Lemma 2.2.15. (Bony para-product decomposition) *Let* $u \in H^s$, $v \in H^r$ *with* $s + r \geq 0$. *Then*

$$uv = \mathrm{Op}^{\mathrm{BW}}(u)v + \mathrm{Op}^{\mathrm{BW}}(v)u + R(u,v) \tag{2.2.93}$$

where the bi-linear operator $R\colon H^s \times H^r \to H^{s+r-s_0}$ *is symmetric and satisfies the estimate*

$$\|R(u,v)\|_{s+r-s_0} \lesssim \|u\|_s \|v\|_r \,. \tag{2.2.94}$$

Moreover $R(u,v) = R(\Pi_0^\perp u, \Pi_0^\perp v) - u_0 v_0$ *and then*

$$\|\Pi_0^\perp R(u,v)\|_{s+r-s_0} \lesssim \|\Pi_0^\perp u\|_s \|\Pi_0^\perp v\|_r \,. \tag{2.2.95}$$

Proof. Introduce the function $\theta_\epsilon(j,k)$ by

$$1 = \chi_\epsilon\Big(\frac{j-k}{\langle j+k\rangle}\Big) + \chi_\epsilon\Big(\frac{k}{\langle 2j-k\rangle}\Big) + \theta_\epsilon(j,k)\,. \tag{2.2.96}$$

Note that $|\theta_\epsilon(j,k)| \leq 1$. Let $\Sigma := \{(j,k) \in \mathbb{Z}^d \times \mathbb{Z}^d \,:\, \theta_\epsilon(j,k) \neq 0\}$ denote the support of θ_ϵ. We claim that

$$(j,k) \in \Sigma \qquad \Longrightarrow \qquad |j| \leq C_\epsilon \min(|j-k|,\ |k|)\,. \tag{2.2.97}$$

Indeed, recalling the definition of the cut-off function χ in (2.2.9), we first note that[1]

$$\Sigma = \{(0,0)\} \cup \Big\{|j-k| \geq \epsilon\langle j+k\rangle\,,\ |k| \geq \epsilon\langle 2j-k\rangle\Big\}.$$

Thus, for any $(j,k) \in \Sigma$,

$$|j| \leq \frac{1}{2}|j-k| + \frac{1}{2}|j+k| \leq \left(\frac{1}{2} + \frac{1}{2\epsilon}\right)|j-k|\,, \quad |j| \leq \frac{1}{2}|2j-k| + \frac{1}{2}|k| \leq \left(\frac{1}{2}+\frac{1}{2\epsilon}\right)|k|$$

[1] For δ sufficiently small, if $|j-k| \leq \delta\langle j+k\rangle$ and $|k| \leq \delta\langle 2j-k\rangle$ then $(j,k) = (0,0)$.

proving (2.2.97). Using (2.2.96) we decompose

$$uv = \sum_{j,k} \widehat{u}_{j-k}\,\chi_\epsilon\Big(\frac{j-k}{\langle j+k\rangle}\Big)\widehat{v}_k\,e^{\mathrm{i}j\cdot x} + \sum_{j,k}\widehat{v}_k\,\chi_\epsilon\Big(\frac{k}{\langle 2j-k\rangle}\Big)\,\widehat{u}_{j-k}\,e^{\mathrm{i}j\cdot x} + \sum_{j,k}\theta_\epsilon(j,k)\widehat{u}_{j-k}\,\widehat{v}_k\,e^{\mathrm{i}j\cdot x}$$
$$= \mathrm{Op}^{BW}(u)v + \mathrm{Op}^{BW}(v)u + R(u,v)\,.$$

By (2.2.97), $s+r\geq 0$, and the Cauchy-Schwartz inequality, we get

$$\|R(u,v)\|^2_{s+r-s_0} \leq \sum_j \langle j\rangle^{2(s+r-s_0)}\Big|\sum_k \theta_\epsilon(j,k)\widehat{u}_{j-k}\,\widehat{v}_k\Big|^2$$
$$\lesssim \sum_j \langle j\rangle^{-2s_0}\Big|\sum_k \langle j-k\rangle^s\,|\widehat{u}_{j-k}|\,\langle k\rangle^r\,|\widehat{v}_k|\Big|^2 \lesssim \|u\|_s^2\,\|v\|_r^2$$

proving (2.2.94). Finally, since on the support of θ_ϵ we have or $(j,k)=(0,0)$ or $j-k\neq 0$ and $k\neq 0$, we deduce that

$$R(u,v) = \theta_\epsilon(0,0)\widehat{u}_0\,\widehat{v}_0 + \sum_{j-k\neq 0, k\neq 0}\theta_\epsilon(j,k)\widehat{u}_{j-k}\,\widehat{v}_k\,e^{\mathrm{i}j\cdot x} = -\widehat{u}_0\,\widehat{v}_0 + R(\Pi_0^\perp u,\Pi_0^\perp v)$$

and we deduce (2.2.95). □

Composition estimates. We will use the following Moser estimates for composition of functions in Sobolev spaces.

Theorem 2.2.16. *Let $I\subseteq\mathbb{R}$ be an open interval and $F\in C^\infty(I;\mathbb{C})$ a smooth function. Let $J\subset I$ be a compact interval. For any function $u,v\in H^s(\mathbb{T}^d,\mathbb{R})$, $s>\frac{d}{2}$, with values in J, we have*

$$\begin{aligned}&\|F(u)\|_s \leq C(s,F,J)\,(1+\|u\|_s)\,,\\ &\|F(u)-F(v)\|_s \leq C(s,F,J)\,(\|u-v\|_s + (\|u\|_s+\|v\|_s)\|u-v\|_{L^\infty})\\ &\|F(u)\|_s\leq C(s,F,J)\|u\|_s \quad \text{if} \quad F(0)=0\,.\end{aligned} \tag{2.2.98}$$

Proof. Take an extension $\tilde{F}\in C^\infty(\mathbb{R};\mathbb{C})$ such that $\tilde{F}_{|I}=F$. Then $F(u)=\tilde{F}(u)$ for any $u\in H^s(\mathbb{T}^d;\mathbb{R})$ with values in J, and apply the usual Moser estimate, see e.g. [6], replacing the Littlewood-Paley decomposition on $\mathbb{R}^d$ with the one on $\mathbb{T}^d$ in (2.2.25). □

2.3 Para-linearization of (EK)-system and complex form

In this section we para-linearize the Euler-Korteweg system (2.1.6) and write it in terms of the complex variable

$$u := \frac{1}{\sqrt{2}}\left(\frac{\mathtt{m}}{K(\mathtt{m})}\right)^{-1/4}\rho + \frac{\mathrm{i}}{\sqrt{2}}\left(\frac{\mathtt{m}}{K(\mathtt{m})}\right)^{1/4}\phi,\quad \rho\in\dot{H}^s\,,\ \phi\in\dot{H}^s\,. \tag{2.3.1}$$

The variable $u\in\dot{H}^s$. We denote this change of coordinates in $\dot{H}^s\times\dot{H}^s$ by

$$\begin{aligned}&\begin{pmatrix}u\\ \overline{u}\end{pmatrix} = \boldsymbol{C}^{-1}\begin{pmatrix}\rho\\ \phi\end{pmatrix},\\ &\boldsymbol{C} := \frac{1}{\sqrt{2}}\begin{pmatrix}\left(\frac{\mathtt{m}}{K(\mathtt{m})}\right)^{\frac14} & \left(\frac{\mathtt{m}}{K(\mathtt{m})}\right)^{\frac14}\\ -\mathrm{i}\left(\frac{\mathtt{m}}{K(\mathtt{m})}\right)^{-\frac14} & \mathrm{i}\left(\frac{\mathtt{m}}{K(\mathtt{m})}\right)^{-\frac14}\end{pmatrix},\quad \boldsymbol{C}^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix}\left(\frac{\mathtt{m}}{K(\mathtt{m})}\right)^{-\frac14} & \mathrm{i}\left(\frac{\mathtt{m}}{K(\mathtt{m})}\right)^{\frac14}\\ \left(\frac{\mathtt{m}}{K(\mathtt{m})}\right)^{-\frac14} & -\mathrm{i}\left(\frac{\mathtt{m}}{K(\mathtt{m})}\right)^{\frac14}\end{pmatrix}.\end{aligned} \tag{2.3.2}$$

We also define the matrices

$$J := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad \mathbb{J} := \begin{bmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{bmatrix}, \qquad \mathrm{Id} := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{2.3.3}$$

Proposition 2.3.1. (Para-linearized Euler-Korteweg equations in complex coordinates) *The (EK)-system (2.1.6) can be written in terms of the complex variable* $U := \begin{pmatrix} u \\ \overline{u} \end{pmatrix}$ *with* u *defined in (2.3.1), in the para-linearized form*

$$\partial_t U = \mathbb{J}\Big[\mathrm{Op}^{\mathrm{BW}}\left(A_2(U;x,\xi) + A_1(U;x,\xi)\right)\Big]U + R(U) \tag{2.3.4}$$

where, for any function $U \in \dot{\mathbf{H}}^{s_0+2}$ *such that*

$$\rho(U) := \frac{1}{\sqrt{2}}\left(\frac{\mathtt{m}}{K(\mathtt{m})}\right)^{1/4}\Pi_0^{\perp}(u+\overline{u}) \in \mathcal{Q} \ \ (\text{see } (2.1.12)), \tag{2.3.5}$$

(i) $A_2(U;x,\xi) \in \Gamma^2_{s_0+2} \otimes \mathcal{M}_2(\mathbb{C})$ *is the matrix of symbols*

$$A_2(U;x,\xi) := \sqrt{\mathtt{m}K(\mathtt{m})}|\xi|^2 \begin{bmatrix} 1 + \mathtt{a}_+(U;x) & \mathtt{a}_-(U;x) \\ \mathtt{a}_-(U;x) & 1 + \mathtt{a}_+(U;x) \end{bmatrix} \tag{2.3.6}$$

where $\mathtt{a}_\pm(U;x) \in \Gamma^0_{s_0+2}$ *are the* ξ*-independent functions*

$$\mathtt{a}_\pm(U;x) := \frac{1}{2}\left(\frac{K(\rho+\mathtt{m}) - K(\mathtt{m})}{K(\mathtt{m})} \pm \frac{\rho}{\mathtt{m}}\right). \tag{2.3.7}$$

(ii) $A_1(U;x,\xi) \in \Gamma^1_{s_0+1} \otimes \mathcal{M}_2(\mathbb{C})$ *is the diagonal matrix of symbols*

$$A_1(U;x,\xi) := \begin{bmatrix} \mathtt{b}(U;x)\cdot\xi & 0 \\ 0 & -\mathtt{b}(U;x)\cdot\xi \end{bmatrix}, \qquad \mathtt{b}(U;x) := \nabla\phi \in \Gamma^0_{s_0+1} \otimes \mathbb{R}^d. \tag{2.3.8}$$

Moreover for any $\sigma \geq 0$ *there exists a non decreasing function* $\mathtt{C}(\,) : \mathbb{R}_+ \to \mathbb{R}_+$ *(depending on* K*) such that, for any* $U, V \in \dot{\mathbf{H}}^{s_0}$ *with* $\rho(U), \rho(V) \in \mathcal{Q}$*,* $W \in \dot{\mathbf{H}}^{\sigma+2}$ *and* $j = 1, 2$*, we have*

$$\|\mathrm{Op}^{\mathrm{BW}}(A_j(U))W\|_\sigma \leq \mathtt{C}(\|U\|_{s_0})\|W\|_{\sigma+2} \tag{2.3.9}$$

$$\|\mathrm{Op}^{\mathrm{BW}}(A_j(U) - A_j(V))W\|_\sigma \leq \mathtt{C}(\|U\|_{s_0}, \|V\|_{s_0})\|W\|_{\sigma+2}\|U-V\|_{s_0} \tag{2.3.10}$$

where in (2.3.10) we denoted by $\mathtt{C}(\cdot,\cdot) := \mathtt{C}(\max\{\cdot,\cdot\})$.

(iii) The vector field $R(U)$ *satisfies the following "semi-linear" estimates: for any* $\sigma \geq s_0 > d/2$ *there exists a non decreasing function* $\mathtt{C}(\,) : \mathbb{R}_+ \to \mathbb{R}_+$ *(depending also on* g, K*) such that, for any* $U, V \in \dot{\mathbf{H}}^{\sigma+2}$ *such that* $\rho(U), \rho(V) \in \mathcal{Q}$*, we have*

$$\|R(U)\|_\sigma \leq \mathtt{C}(\|U\|_{s_0+2})\|U\|_\sigma, \qquad \|R(U)\|_\sigma \leq \mathtt{C}(\|U\|_{s_0})\|U\|_{\sigma+2}, \tag{2.3.11}$$

$$\|R(U) - R(V)\|_\sigma \leq \mathtt{C}(\|U\|_{s_0+2}, \|V\|_{s_0+2})\|U-V\|_\sigma + \mathtt{C}(\|U\|_\sigma, \|V\|_\sigma)\|U-V\|_{s_0+2} \tag{2.3.12}$$

$$\|R(U) - R(V)\|_{s_0} \leq \mathtt{C}(\|U\|_{s_0+2}, \|V\|_{s_0+2})\|U-V\|_{s_0}, \tag{2.3.13}$$

where in (2.3.12) and (2.3.13) we denoted again by $\mathtt{C}(\cdot,\cdot) := \mathtt{C}(\max\{\cdot,\cdot\})$.

Proof. We first para-linearize the original equations (2.1.6), then we switch to complex coordinates.

Step 1: para-linearization of (2.1.6). We apply several times the para-product Lemma 4.2.12 and the composition Theorem 2.2.13. In the following we denote by R^p the remainder that comes from Lemma 4.2.12, and by $R^{-\varrho}$, $\varrho = 1, 2$, the remainder that comes from Theorem 2.2.13. We shall adopt the following convention: given $\mathbb{R}^d$-valued symbols $a = (a_j)_{j=1,\dots,d}$, $b = (b_j)_{j=1,\dots,d}$ in some class $\Gamma^m_s \otimes \mathbb{R}^d$, we denote $R^p(a, b) := \sum_{j=1}^d R^p(a_j, b_j)$,

$$R^{-\varrho}(a, b) := \sum_{j=1}^{d} R^{-\varrho}(a_j, b_j) \quad \text{and} \quad \mathrm{Op}^{\mathrm{BW}}(a) \cdot \mathrm{Op}^{\mathrm{BW}}(b) := \sum_{j=1}^{d} \mathrm{Op}^{\mathrm{BW}}(a_j)\mathrm{Op}^{\mathrm{BW}}(b_j)\,.$$

We para-linearize the terms in the first line of (2.1.6). We have $\Delta\phi = -\mathrm{Op}^{\mathrm{BW}}\big(|\xi|^2\big)\phi$ and $\operatorname{div}(\rho\nabla\phi) = \nabla\rho \cdot \nabla\phi + \rho\Delta\phi$ can be written as

$$\begin{aligned} \rho\Delta\phi &= -\mathrm{Op}^{\mathrm{BW}}\big(\rho|\xi|^2 + \nabla\rho \cdot \mathrm{i}\xi\big)\phi \\ &\quad + \mathrm{Op}^{\mathrm{BW}}(\Delta\phi)\rho + R^p(\rho, \Delta\phi) + R^{-2}(\rho, |\xi|^2)\phi\,, \end{aligned} \tag{2.3.14}$$

$$\begin{aligned} \nabla\rho \cdot \nabla\phi &= \mathrm{Op}^{\mathrm{BW}}(\nabla\rho \cdot \mathrm{i}\xi)\phi + \mathrm{Op}^{\mathrm{BW}}(\nabla\phi \cdot \mathrm{i}\xi)\rho \\ &\quad + R^p(\nabla\rho, \nabla\phi) + R^{-1}(\nabla\rho, \mathrm{i}\xi)\phi + R^{-1}(\nabla\phi, \mathrm{i}\xi)\rho\,. \end{aligned} \tag{2.3.15}$$

Then we para-linearize the terms in the second line of (2.1.6). We have

$$\begin{aligned} \frac{1}{2}|\nabla\phi|^2 &= \mathrm{Op}^{\mathrm{BW}}(\nabla\phi \cdot \mathrm{i}\xi)\phi \\ &\quad + \frac{1}{2}R^p(\nabla\phi, \nabla\phi) + R^{-1}(\nabla\phi, \mathrm{i}\xi)\phi\,. \end{aligned} \tag{2.3.16}$$

Using (2.1.10) we regard the semi-linear term

$$g(\mathtt{m} + \rho) = g(\mathtt{m} + \rho) - g(\mathtt{m}) =: R(\rho) \tag{2.3.17}$$

directly as a remainder. Moreover, writing $\Delta\rho = -\mathrm{Op}^{\mathrm{BW}}\big(|\xi|^2\big)\rho$, we get

$$\begin{aligned} K(\mathtt{m} + \rho)\Delta\rho &= \mathrm{Op}^{\mathrm{BW}}(K(\mathtt{m} + \rho))\Delta\rho + \mathrm{Op}^{\mathrm{BW}}(\Delta\rho)K(\mathtt{m} + \rho) + R^p(\Delta\rho, K(\mathtt{m} + \rho)) \\ &= -\mathrm{Op}^{\mathrm{BW}}\big(K(\mathtt{m} + \rho)|\xi|^2 + K'(\mathtt{m} + \rho)\nabla\rho \cdot \mathrm{i}\xi\big)\rho \\ &\quad + \mathrm{Op}^{\mathrm{BW}}(\Delta\rho)K(\mathtt{m} + \rho) + R^p(\Delta\rho, K(\mathtt{m} + \rho)) - R^{-2}(K(\mathtt{m} + \rho), |\xi|^2)\rho\,. \end{aligned} \tag{2.3.18}$$

Finally, using for $\frac{1}{2}|\nabla\rho|^2$ the expansion (2.3.16) for ρ instead of ϕ, we obtain

$$\begin{aligned} \frac{1}{2}K'(\mathtt{m} + \rho)|\nabla\rho|^2 &= \frac{1}{2}\mathrm{Op}^{\mathrm{BW}}\big(K'(\mathtt{m} + \rho)\big)|\nabla\rho|^2 + \frac{1}{2}\mathrm{Op}^{\mathrm{BW}}\big(|\nabla\rho|^2\big)K'(\mathtt{m} + \rho) \\ &\quad + \frac{1}{2}R^p(|\nabla\rho|^2, K'(\mathtt{m} + \rho)) = \mathrm{Op}^{\mathrm{BW}}\big(K'(\mathtt{m} + \rho)\nabla\rho \cdot \mathrm{i}\xi\big)\rho + \mathtt{R}(\rho) \end{aligned}$$

where

$$\mathtt{R}(\rho) := \frac{1}{2}\mathrm{Op}^{\mathrm{BW}}\big(|\nabla\rho|^2\big)K'(\mathtt{m} + \rho) + \frac{1}{2}R^p(|\nabla\rho|^2, K'(\mathtt{m} + \rho)) \tag{2.3.19}$$

$$+ \frac{1}{2}\mathrm{Op}^{\mathrm{BW}}\big(K'(\mathtt{m} + \rho)\big)R^p(\nabla\rho, \nabla\rho) \tag{2.3.20}$$

$$+ \mathrm{Op}^{\mathrm{BW}}\big(K'(\mathtt{m} + \rho)\big)R^{-1}(\nabla\rho, \mathrm{i}\xi)\rho + R^{-1}(K'(\mathtt{m} + \rho), \mathrm{i}\nabla\rho \cdot \xi)\rho\,. \tag{2.3.21}$$

Collecting all the above expansions and recalling the definition of the symplectic matrix J in (2.3.3), the system (2.1.6) can be written in the para-linearized form

$$\partial_t \begin{pmatrix} \rho \\ \phi \end{pmatrix} = J\mathrm{Op}^{BW}\left(\begin{bmatrix} K(\mathtt{m}+\rho)|\xi|^2 & \nabla\phi\cdot \mathrm{i}\xi \\ -\nabla\phi\cdot\mathrm{i}\xi & (\mathtt{m}+\rho)|\xi|^2 \end{bmatrix}\right)\begin{pmatrix} \rho \\ \phi \end{pmatrix} + R(\rho,\phi) \tag{2.3.22}$$

where we collected in $R(\rho,\phi)$ all the terms in lines (2.3.14)–(2.3.21).

Step 2: complex coordinates. We now write system (2.3.22) in the complex coordinates $U = \boldsymbol{C}^{-1}\begin{pmatrix} \rho \\ \phi \end{pmatrix}$. Note that $\boldsymbol{C}^{-1}$ conjugates the Poisson tensor J to $\mathbb{J}$ defined in (2.3.3), i.e. $\boldsymbol{C}^{-1}J = \mathbb{J}\boldsymbol{C}^*$ and therefore system (2.3.22) is conjugated to

$$\partial_t U = \mathbb{J}\boldsymbol{C}^*\mathrm{Op}^{BW}\left(\begin{bmatrix} K(\mathtt{m}+\rho)|\xi|^2 & \nabla\phi\cdot\mathrm{i}\xi \\ -\nabla\phi\cdot\mathrm{i}\xi & \rho|\xi|^2 \end{bmatrix}\right)\boldsymbol{C}U + \boldsymbol{C}^{-1}R(\boldsymbol{C}U)\,. \tag{2.3.23}$$

Using (2.3.2), system (2.3.23) reads as system (2.3.4)-(4.5.39) with $R(U) := \boldsymbol{C}^{-1}R(\boldsymbol{C}U)$.

We note also that estimates (2.3.9) and (2.3.10) for $j = 2$ follow by (2.2.85) and (2.2.98), whereas in case $j = 1$ follow by (2.2.86) applied with $m = 1$, $\varrho = 1$.

Step 3: Estimate of the remainder $R(U)$. We now prove (2.3.11)-(2.3.13). Since $\|\rho\|_\sigma, \|\phi\|_\sigma \sim \|U\|_\sigma$ for any $\sigma \in \mathbb{R}$ by (2.3.2), the estimates (2.3.11)-(2.3.13) directly follow from those of $R(\rho,\phi)$ in (2.3.22). We now estimate each term in (2.3.14)–(2.3.21). In the sequel $\sigma \geq s_0 > d/2$.

ESTIMATE OF THE TERMS IN LINE (2.3.14). Applying first (2.2.85) with $m = 0$, and then (2.2.86) with $\varrho = 2$, we have

$$\|\mathrm{Op}^{BW}(\Delta\phi)\rho\|_\sigma \lesssim \|\phi\|_{s_0+2}\|\rho\|_\sigma\,, \quad \|\mathrm{Op}^{BW}(\Delta\phi)\rho\|_\sigma \lesssim \|\phi\|_{s_0}\|\rho\|_{\sigma+2}\,. \tag{2.3.24}$$

By (2.2.94), the smoothing remainder in line (2.3.14) satisfies the estimates

$$\|R^p(\rho,\Delta\phi)\|_\sigma \lesssim \|\phi\|_{s_0+2}\|\rho\|_\sigma\,, \quad \|R^p(\rho,\Delta\phi)\|_\sigma \lesssim \|\phi\|_{s_0}\|\rho\|_{\sigma+2}\,, \tag{2.3.25}$$

and, by (2.2.89) with $\varrho = 2$, and the interpolation estimate (2.2.84),

$$\|R^{-2}(\rho,|\xi|^2)\phi\|_\sigma \lesssim \|\rho\|_{s_0+2}\|\phi\|_\sigma \lesssim \|\phi\|_{s_0}\|\rho\|_{\sigma+2} + \|\rho\|_{s_0}\|\phi\|_{\sigma+2}\,. \tag{2.3.26}$$

By (2.3.24)-(2.3.26) and $\|\rho\|_\sigma, \|\phi\|_\sigma \sim \|U\|_\sigma$ we deduce that the terms in line (2.3.14), written in function of U, satisfy (2.3.11). Next we write

$$\mathrm{Op}^{BW}(\Delta\phi_1)\rho_1 - \mathrm{Op}^{BW}(\Delta\phi_2)\rho_2 = \mathrm{Op}^{BW}(\Delta\phi_1)[\rho_1-\rho_2] + \mathrm{Op}^{BW}(\Delta\phi_1-\Delta\phi_2)\rho_2$$

and, applying (2.2.85) with $m = 0$, and (2.2.86) with $\varrho = 2$ to $\mathrm{Op}^{BW}(\Delta\phi_1 - \Delta\phi_2)\rho_2$, we get

$$\begin{aligned} &\|\mathrm{Op}^{BW}(\Delta\phi_1)\rho_1 - \mathrm{Op}^{BW}(\Delta\phi_2)\rho_2\|_\sigma \lesssim \|\phi_1\|_{s_0+2}\|\rho_1-\rho_2\|_\sigma + \|\phi_1-\phi_2\|_{s_0+2}\|\rho_2\|_\sigma \\ &\|\mathrm{Op}^{BW}(\Delta\phi_1)\rho_1 - \mathrm{Op}^{BW}(\Delta\phi_2)\rho_2\|_\sigma \lesssim \|\phi_1\|_{s_0+2}\|\rho_1-\rho_2\|_\sigma + \|\phi_1-\phi_2\|_{s_0}\|\rho_2\|_{\sigma+2}\,. \end{aligned} \tag{2.3.27}$$

Concerning the remainder $R^p(\rho,\Delta\phi)$, we write $R^p(\rho_1,\Delta\phi_1) - R^p(\rho_2,\Delta\phi_2) = R^p(\rho_1-\rho_2,\Delta\phi_1) + R^p(\rho_2,\Delta\phi_2-\Delta\phi_1)$ and, applying (2.2.94), we get

$$\begin{aligned} &\|R^p(\rho_1,\Delta\phi_1) - R^p(\rho_2,\Delta\phi_2)\|_\sigma \lesssim \|\phi_1\|_{s_0+2}\|\rho_1-\rho_2\|_\sigma + \|\rho_2\|_\sigma\|\phi_1-\phi_2\|_{s_0+2} \\ &\|R^p(\rho_1,\Delta\phi_1) - R^p(\rho_2,\Delta\phi_2)\|_\sigma \lesssim \|\phi_1\|_{s_0+2}\|\rho_1-\rho_2\|_\sigma + \|\rho_2\|_{\sigma+2}\|\phi_1-\phi_2\|_{s_0}\,. \end{aligned} \tag{2.3.28}$$

Finally we write $R^{-2}(\rho_1,|\xi|^2)\phi_1 - R^{-2}(\rho_2,|\xi|^2)\phi_2 = R^{-2}(\rho_1-\rho_2,|\xi|^2)\phi_1 + R^{-2}(\rho_2,|\xi|^2)[\phi_1-\phi_2]$. Using (2.2.89) we get

$$\|R^{-2}(\rho_1,|\xi|^2)\phi_1 - R^{-2}(\rho_2,|\xi|^2)\phi_2\|_\sigma \lesssim \|\phi_1\|_\sigma\|\rho_1-\rho_2\|_{s_0+2} + \|\phi_1-\phi_2\|_\sigma\|\rho_2\|_{s_0+2}\,. \tag{2.3.29}$$

We also claim that

$$\|R^{-2}(\rho_1,|\xi|^2)\phi_1 - R^{-2}(\rho_2,|\xi|^2)\phi_2\|_\sigma \lesssim \|\rho_1-\rho_2\|_{s_0}\|\phi_1\|_{\sigma+2} + \|\phi_1-\phi_2\|_\sigma\|\rho_2\|_{s_0+2}\,. \tag{2.3.30}$$

Indeed, we bound

$$\|R^{-2}(\rho_1,|\xi|^2)\phi_1 - R^{-2}(\rho_2,|\xi|^2)\phi_2\|_\sigma \lesssim \|R^{-2}(\rho_1-\rho_2,|\xi|^2)\phi_1\|_\sigma + \|\phi_1-\phi_2\|_\sigma\|\rho_2\|_{s_0+2}$$

and, to control $R^{-2}(\rho_1-\rho_2,|\xi|^2)\phi_1$, we use that, by definition, it equals

$$\mathrm{Op}^{BW}(\rho_1-\rho_2)\mathrm{Op}^{BW}\big(|\xi|^2\big)\phi_1 - \mathrm{Op}^{BW}\big((\rho_1-\rho_2)|\xi|^2\big)\phi_1 - \mathrm{Op}^{BW}\left(\nabla(\rho_1-\rho_2)\cdot \mathrm{i}\xi\right)\phi_1$$

and we estimate the first two terms using (2.2.86) with $\varrho=0$ and the last term with $\varrho=1$, by $\|R^{-2}(\rho_1-\rho_2,|\xi|^2)\phi_1\|_\sigma \lesssim \|\rho_1-\rho_2\|_{s_0}\|\phi_1\|_{\sigma+2}$, proving (2.3.30). By (2.3.27)-(2.3.30) and $\|\rho\|_\sigma, \|\phi\|_\sigma \sim \|U\|_\sigma$ we deduce that the terms in line (2.3.14), written in function of U, satisfy (2.3.12)-(2.3.13).

The estimates (2.3.11)-(2.3.13) for the terms in lines (2.3.15), (2.3.16), (2.3.18) and (2.3.17), follow by similar arguments, using also (2.2.98).

ESTIMATES OF $\mathtt{R}(\rho)$ DEFINED IN (2.3.19)-(2.3.21).

Writing $\mathrm{Op}^{BW}\big(|\nabla\rho|^2\big)K'(\mathtt{m}+\rho) = \mathrm{Op}^{BW}\big(|\nabla\rho|^2\big)(K'(\mathtt{m}+\rho)-K'(\mathtt{m}))$ (in the homogeneous spaces $\dot H^s$), we have, by (2.2.85), the fact that $\rho\in\mathcal{Q}$, Theorem 2.2.16, (2.2.94), (2.2.82), (2.2.89) with $\varrho=1$,

$$\|\mathtt{R}(\rho)\|_\sigma \le \mathtt{C}\big(\|\rho\|_{s_0+2}\big)\|\rho\|_\sigma\,.$$

Thus $\mathtt{R}(\rho)$, written as a function of U, satisfies (2.3.11). The estimates (2.3.12)-(2.3.13) follow by

$$\|\mathtt{R}(\rho_1)-\mathtt{R}(\rho_2)\|_\sigma \le \mathtt{C}\big(\|\rho_1\|_{s_0+2},\|\rho_2\|_{s_0+2}\big)\|\rho_1-\rho_2\|_\sigma + \mathtt{C}\big(\|\rho_1\|_\sigma,\|\rho_2\|_\sigma\big)\|\rho_1-\rho_2\|_{s_0+2} \tag{2.3.31}$$

$$\|\mathtt{R}(\rho_1)-\mathtt{R}(\rho_2)\|_{s_0} \le \mathtt{C}\big(\|\rho_1\|_{s_0+2},\|\rho_2\|_{s_0+2}\big)\|\rho_1-\rho_2\|_{s_0}\,. \tag{2.3.32}$$

PROOF OF (2.3.31). Defining $w:=\nabla(\rho_1+\rho_2)$, $v:=\nabla(\rho_1-\rho_2)$, then we have, by (2.2.81),

$$\||\nabla\rho_1|^2-|\nabla\rho_2|^2\|_{s_0} = \|w\cdot v\|_{s_0} \lesssim \big(\|\rho_1\|_{s_0+1}+\|\rho_2\|_{s_0+1}\big)\|\rho_1-\rho_2\|_{s_0+1} \tag{2.3.33}$$

$$\||\nabla\rho_1|^2-|\nabla\rho_2|^2\|_{s_0-1} = \|w\cdot v\|_{s_0-1} \overset{(2.2.82)}{\lesssim} \big(\|\rho_1\|_{s_0+1}+\|\rho_2\|_{s_0+1}\big)\|\rho_1-\rho_2\|_{s_0}\,. \tag{2.3.34}$$

Let us prove (2.3.31) for the first term in (2.3.19). Remind that ρ_1,ρ_2 are in $\mathcal{Q}$. We have

$$\begin{aligned}
&\|\mathrm{Op}^{BW}\big(|\nabla\rho_1|^2\big)K'(\mathtt{m}+\rho_1) - \mathrm{Op}^{BW}\big(|\nabla\rho_2|^2\big)K'(\mathtt{m}+\rho_2)\|_\sigma \\
&\le \|\mathrm{Op}^{BW}(w\cdot v)\big(K'(\mathtt{m}+\rho_1)\|_\sigma + \|\mathrm{Op}^{BW}\big(|\nabla\rho_2|^2\big)\big[K'(\mathtt{m}+\rho_1)-K'(\mathtt{m}+\rho_2)\big]\|_\sigma \\
&\overset{(2.2.85)}{\lesssim} \|w\cdot v\|_{s_0}\|K'(\mathtt{m}+\rho_1)-K'(\mathtt{m})\|_\sigma + \|\rho_2\|^2_{s_0+1}\|K'(\mathtt{m}+\rho_1)-K'(\mathtt{m}+\rho_2)\|_\sigma \\
&\overset{(2.2.84),(2.2.98),(2.3.33)}{\lesssim} \|\rho_1\|_\sigma(\|\rho_1\|_{s_0+1}+\|\rho_2\|_{s_0+1})\|\rho_1-\rho_2\|_{s_0+1} + \mathtt{C}\big(\|\rho_1\|_{s_0+1},\|\rho_2\|_{s_0+1}\big)\|\rho_1-\rho_2\|_\sigma \\
&\quad + \|\rho_2\|_{s_0}\|\rho_2\|_{s_0+2}(\|\rho_1\|_\sigma+\|\rho_2\|_\sigma)\|\rho_1-\rho_2\|_{s_0} \\
&\overset{(2.2.84)}{\le} \mathtt{C}\big(\|\rho_1\|_\sigma,\|\rho_2\|_\sigma\big)\|\rho_1-\rho_2\|_{s_0+2} + \mathtt{C}\big(\|\rho_1\|_{s_0+2},\|\rho_2\|_{s_0+2}\big)\|\rho_1-\rho_2\|_\sigma\,.
\end{aligned} \tag{2.3.35}$$

In the same way the second term in (2.3.19) is bounded by (2.3.35). Regarding the term in (2.3.20), using that $R^p(\cdot,\cdot)$ is bi-linear and symmetric, we have

$$\begin{aligned}
&\|\mathrm{Op}^{\mathrm{BW}}\big(K'(\mathtt{m}+\rho_1)\big)R^p(\nabla\rho_1,\nabla\rho_1)-\mathrm{Op}^{\mathrm{BW}}\big(K'(\mathtt{m}+\rho_2)\big)R^p(\nabla\rho_2,\nabla\rho_2)\|_\sigma\\
&\le\|\mathrm{Op}^{\mathrm{BW}}\big(K'(\mathtt{m}+\rho_1)-K'(\mathtt{m}+\rho_2)\big)R^p(\nabla\rho_1,\nabla\rho_1)\|_\sigma+\|\mathrm{Op}^{\mathrm{BW}}\big(K'(\mathtt{m}+\rho_2)\big)R^p(w,v)\|_\sigma\\
&\overset{(2.2.85),(2.2.94)}{\lesssim}\|\rho_1\|_\sigma\|\rho_1\|_{s_0+2}\|K'(\mathtt{m}+\rho_1)-K'(\mathtt{m}+\rho_2)\|_{s_0}+\|w\|_{s_0+1}\|v\|_{\sigma-1}\|K'(\mathtt{m}+\rho_2)\|_{s_0}\\
&\overset{(2.2.98),(2.3.33)}{\le}\mathtt{C}\big(\|\rho_1\|_\sigma,\|\rho_2\|_\sigma\big)\|\rho_1-\rho_2\|_{s_0}+\mathtt{C}\big(\|\rho_1\|_{s_0+2},\|\rho_2\|_{s_0+2}\big)\|\rho_1-\rho_2\|_\sigma\,.
\end{aligned}\tag{2.3.36}$$

Also the terms in (2.3.21) are bounded by (2.3.35), proving that $\mathtt{R}(\rho)$ satisfies (2.3.31).

PROOF OF (2.3.32). Regarding the first term (2.3.19), we have

$$\begin{aligned}
&\|\mathrm{Op}^{\mathrm{BW}}\big(|\nabla\rho_1|^2\big)K'(\mathtt{m}+\rho_1)-\mathrm{Op}^{\mathrm{BW}}\big(|\nabla\rho_2|^2\big)K'(\mathtt{m}+\rho_2)\|_{s_0}\\
&\le\|\mathrm{Op}^{\mathrm{BW}}(w\cdot v)K'(\mathtt{m}+\rho_1)\|_{s_0}+\|\mathrm{Op}^{\mathrm{BW}}\big(|\nabla\rho_2|^2\big)\big[K'(\mathtt{m}+\rho_1)-K'(\mathtt{m}+\rho_2)\big]\|_{s_0}\\
&\overset{(2.2.85),(2.2.86)}{\lesssim}\|w\cdot v\|_{s_0-1}\|K'(\mathtt{m}+\rho_1)\|_{s_0+1}+\|\rho_2\|_{s_0+1}^2\|K'(\mathtt{m}+\rho_1)-K'(\mathtt{m}+\rho_2)\|_{s_0}\\
&\overset{(2.2.98),(2.3.34)}{\le}\mathtt{C}\big(\|\rho_1\|_{s_0+1},\|\rho_2\|_{s_0+1}\big)\|\rho_1-\rho_2\|_{s_0}\,.
\end{aligned}\tag{2.3.37}$$

Similarly we deduce that the second term in (2.3.19) is bounded as in (2.3.37). Regarding the term in (2.3.20), note that the bound (2.3.32) follows from (2.3.36) applied for $\sigma=s_0$. The estimate for last two terms in (2.3.21) follows in the same way so we analyze the last one. First we have

$$\begin{aligned}
&\|R^{-1}(K'(\mathtt{m}+\rho_1),\mathrm{i}\nabla\rho_1\cdot\xi)\rho_1-R^{-1}(K'(\mathtt{m}+\rho_2),\mathrm{i}\nabla\rho_2\cdot\xi)\rho_2\|_{s_0}\\
&\quad\le\|\big[R^{-1}(K'(\mathtt{m}+\rho_1),\nabla\rho_1\cdot\mathrm{i}\xi)-R^{-1}(K'(\mathtt{m}+\rho_2),\mathrm{i}\nabla\rho_2\cdot\xi)\big]\rho_1\|_{s_0}\\
&\quad+\|R^{-1}(K'(\mathtt{m}+\rho_2),\nabla\rho_2\cdot\mathrm{i}\xi)(\rho_1-\rho_2)\|_{s_0}\\
&\overset{(2.2.89),(2.2.98)}{\le}\|\big[R^{-1}(K'(\mathtt{m}+\rho_1),\nabla\rho_1\cdot\mathrm{i}\xi)-R^{-1}(K'(\mathtt{m}+\rho_2),\mathrm{i}\nabla\rho_2\cdot\xi)\big]\rho_1\|_{s_0}+\mathtt{C}\big(\|\rho_2\|_{s_0+2}\big)\|\rho_1-\rho_2\|_{s_0}.
\end{aligned}$$

On the other hand, by definition, we have

$$\begin{aligned}
&\big[R^{-1}(K'(\mathtt{m}+\rho_1),\nabla\rho_1\cdot\mathrm{i}\xi)-R^{-1}(K'(\mathtt{m}+\rho_2),\mathrm{i}\nabla\rho_2\cdot\xi)\big]\rho_1\\
&\quad=\big[\mathrm{Op}^{\mathrm{BW}}\big(K'(\mathtt{m}+\rho_1)\big)\mathrm{Op}^{\mathrm{BW}}(\nabla\rho_1\cdot\mathrm{i}\xi)-\mathrm{Op}^{\mathrm{BW}}\big(K'(\mathtt{m}+\rho_2)\big)\mathrm{Op}^{\mathrm{BW}}(\nabla\rho_2\cdot\mathrm{i}\xi)\big]\rho_1\\
&\qquad+\mathrm{Op}^{\mathrm{BW}}\big(K'(\mathtt{m}+\rho_1)\nabla\rho_1\cdot\mathrm{i}\xi-K'(\mathtt{m}+\rho_2)\nabla\rho_2\cdot\mathrm{i}\xi\big)\rho_1\\
&\quad=\mathrm{Op}^{\mathrm{BW}}\big(K'(\mathtt{m}+\rho_1)-K'(\mathtt{m}+\rho_2)\big)\mathrm{Op}^{\mathrm{BW}}(\nabla\rho_1\cdot\mathrm{i}\xi)\rho_1\\
&\qquad+\mathrm{Op}^{\mathrm{BW}}\big(K'(\mathtt{m}+\rho_2)\big)\mathrm{Op}^{\mathrm{BW}}(\nabla(\rho_1-\rho_2)\cdot\mathrm{i}\xi)\rho_1\\
&\qquad+\mathrm{Op}^{\mathrm{BW}}(\nabla(K(\mathtt{m}+\rho_1)-K(\mathtt{m}+\rho_2))\cdot\mathrm{i}\xi)\rho_1\,.
\end{aligned}\tag{2.3.38}$$

Then, applying first (2.2.85) to the first term and then (2.2.86) with $\varrho=1$, $m=1$ and (2.2.98) to each term, we deduce that the $\|\ \|_{s_0}$-norm of (2.3.38) is bounded by $\mathtt{C}\big(\|\rho_1\|_{s_0+2},\|\rho_2\|_{s_0+2}\big)\|\rho_1-\rho_2\|_{s_0}$. Thus (2.3.32) is proved. □

2.4 Local existence

In this section we prove the existence of a local in time solution of system (2.3.4). For any $s\in\mathbb{R}$ and $T>0$, we denote $L^\infty_T\dot{\mathbf{H}}^s:=L^\infty([0,T],\dot{\mathbf{H}}^s)$. For $\delta>0$ we also introduce

$$\mathcal{Q}_\delta:=\big\{\rho\in H_0^{s_0}\ :\ \mathtt{m}_1+\delta\le\mathtt{m}+\rho(x)\le\mathtt{m}_2-\delta\big\}\subset\mathcal{Q}\tag{2.4.1}$$

where $\mathcal{Q}$ is defined in (2.1.12).

Proposition 2.4.1. (Local well-posedness in $\mathbb{T}^d$) *For any $s > \frac{d}{2}+2$, any initial datum $U_0 \in \dot{\mathbf{H}}^s$ with $\rho(U_0) \in \mathcal{Q}_\delta$ for some $\delta > 0$, there exist $T := T(\|U\|_{s_0+2}, \delta) > 0$ and a unique solution $U \in C^0\big([0,T], \dot{\mathbf{H}}^s\big) \cap C^1\big([0,T], \dot{\mathbf{H}}^{s-2}\big)$ of (2.3.4) satisfying $\rho(U) \in \mathcal{Q}$, for any $t \in [0,T]$. Moreover the solution depends continuously with respect to the initial datum in $\dot{\mathbf{H}}^s$.*

Proposition 2.4.1 proves Theorem 2.1.2 and thus Theorem 2.1.1.

The first step is to prove the local well-posedness result of a linear in-homogeneous problem.

Proposition 2.4.2. (Linear local well-posedness) *Let $\Theta \geq r > 0$ and U be a function in $C^0([0,T], \dot{\mathbf{H}}^{s_0+2}) \cap C^1([0,T], \dot{\mathbf{H}}^{s_0})$ satisfying*

$$\|U\|_{L^\infty_T \dot{\mathbf{H}}^{s_0+2}} + \|\partial_t U\|_{L^\infty_T \dot{\mathbf{H}}^{s_0}} \leq \Theta, \quad \|U\|_{L^\infty_T \dot{\mathbf{H}}^{s_0}} \leq r, \quad \rho(U(t)) \in \mathcal{Q}, \; \forall t \in [0,T]. \tag{2.4.2}$$

Let $\sigma \geq 0$ and $t \mapsto R(t)$ be a function in $C^0([0,T], \dot{\mathbf{H}}^\sigma)$. Then there exists a unique solution $V \in C^0([0,T], \dot{\mathbf{H}}^\sigma) \cap C^1([0,T], \dot{\mathbf{H}}^{\sigma-2})$ of the linear in-homogeneous system

$$\partial_t V = \mathbb{J}\,\mathrm{Op}^{BW}\left(A_2(U(t);x,\xi) + A_1(U(t);x,\xi)\right)V + R(t), \quad V(0,x) = V_0(x) \in \dot{\mathbf{H}}^\sigma, \tag{2.4.3}$$

satisfying, for some $C_\Theta := C_{\Theta,\sigma} > 0$ and $C_r := C_{r,\sigma} > 0$, the estimate

$$\|V\|_{L^\infty_T \dot{\mathbf{H}}^\sigma} \leq C_r e^{C_\Theta T} \|V_0\|_\sigma + C_\Theta e^{C_\Theta T} T \|R\|_{L^\infty_T \dot{\mathbf{H}}^\sigma}. \tag{2.4.4}$$

The following two sections are devoted to the proof of Proposition 2.4.2. The key step is the construction of a modified energy which is controlled by the $\dot{\mathbf{H}}^\sigma$-norm, and whose time variation is bounded by the $\dot{\mathbf{H}}^\sigma$ norm of the solution, as done e.g. in [2]. In order to construct such modified energy, the first step is to diagonalize the matrix $\mathbb{J}A_2$ in (2.4.3).

2.4.1 Diagonalization at highest order

We diagonalize the matrix of symbols $\mathbb{J}A_2(U;x,\xi)$. The eigenvalues of the matrix

$$\mathbb{J}\begin{bmatrix} 1 + \mathtt{a}_+(U;x) & \mathtt{a}_-(U;x) \\ \mathtt{a}_-(U;x) & 1 + \mathtt{a}_+(U;x) \end{bmatrix} \tag{2.4.5}$$

with $\mathtt{a}_\pm(U;x)$ defined in (2.3.7) are given by $\pm \mathrm{i}\lambda(U;x)$ with

$$\lambda(U;x) := \sqrt{(1+\mathtt{a}_+(U;x))^2 - \mathtt{a}_-(U;x)^2} = \sqrt{\frac{(\mathtt{m} + \rho(U))\,K(\mathtt{m}+\rho(U))}{\mathtt{m}\,K(\mathtt{m})}}. \tag{2.4.6}$$

These eigenvalues are purely imaginary because $\rho(U) \in \mathcal{Q}$ (see (2.1.12)) and (2.1.9), which guarantees that $\lambda(U;x)$ is real valued and fulfills

$$0 < \lambda_{\min} := \sqrt{\frac{\mathtt{m}_1 c_K}{\mathtt{m}K(\mathtt{m})}} \leq \lambda(U;x) \leq \sqrt{\frac{\mathtt{m}_2 C_K}{\mathtt{m}K(\mathtt{m})}} =: \lambda_{\max}. \tag{2.4.7}$$

A matrix which diagonalizes (2.4.5) is

$$F := \begin{pmatrix} f(U;x) & g(U;x) \\ g(U;x) & f(U;x) \end{pmatrix}, \quad f := \frac{1 + \mathtt{a}_+ + \lambda}{\sqrt{(1+\mathtt{a}_+ + \lambda)^2 - \mathtt{a}_-^2}}, \quad g := \frac{-\mathtt{a}_-}{\sqrt{(1+\mathtt{a}_+ + \lambda)^2 - \mathtt{a}_-^2}}. \tag{2.4.8}$$

Note that $F(U;x)$ is well defined because

$$\begin{aligned}(1+\mathtt{a}_+ +\lambda)^2-\mathtt{a}_-^2 &= \Big(\frac{K(\mathtt{m}+\rho(U))}{K(\mathtt{m})}+\lambda\Big)\Big(\frac{\mathtt{m}+\rho(U)}{\mathtt{m}}+\lambda\Big)\\ &> \frac{(\mathtt{m}+\rho(U))K(\mathtt{m}+\rho(U))}{\mathtt{m}K(\mathtt{m})}\ge\frac{\mathtt{m}_1 c_K}{\mathtt{m}K(\mathtt{m})}\end{aligned} \tag{2.4.9}$$

by (2.1.12) and (2.1.9). The matrix $F(U;x)$ has $\det F(U;x)=f^2-g^2=1$ and its inverse is

$$F(U;x)^{-1}:=\begin{pmatrix} f(U;x) & -g(U;x)\\ -g(U;x) & f(U;x)\end{pmatrix}. \tag{2.4.10}$$

We have that

$$F(U;x)^{-1}\,\mathbb{J}\begin{bmatrix}1+\mathtt{a}_+(U;x) & \mathtt{a}_-(U;x)\\ \mathtt{a}_-(U;x) & 1+\mathtt{a}_+(U;x)\end{bmatrix}F(U;x)=\mathbb{J}\lambda(U;x). \tag{2.4.11}$$

By (2.2.98) and (2.4.9) we deduce the following estimates: for any $N\in\mathbb{N}_0$, $s\ge 0$ and $\sigma>\frac{d}{2}$,

$$\begin{aligned}&\|\mathtt{a}_\pm(U)\|_\sigma,\ \|f(U)\|_\sigma,\ \|g(U)\|_\sigma\le \mathtt{C}\big(\|U\|_\sigma\big),\\ &|\lambda(U;x)|\xi|^{2s}|_{2s,\sigma,N}\le\mathtt{C}_N\big(\|U\|_\sigma\big),\quad |\mathtt{b}(U)\cdot\xi|_{1,\sigma,N}\le\mathtt{C}_N\big(\|U\|_{\sigma+1}\big).\end{aligned} \tag{2.4.12}$$

For any $\varepsilon>0$, consider the regularized matrix symbol

$$A^\varepsilon(U;x,\xi):=\big(A_2(U;x,\xi)+A_1(U;x,\xi)\big)\,\chi(\varepsilon\lambda(U;x)|\xi|^2), \tag{2.4.13}$$

where χ is the cut-off function in (2.2.9) and $\lambda(U;x)$ is the function defined in (2.4.6). In what follows we will denote by $\chi_\varepsilon:=\chi(\varepsilon\lambda(U;x)|\xi|^2)$. Note that, by (2.2.98), (2.4.7) and by the fact that the function $y\mapsto\langle\xi\rangle^{|\alpha|}\partial_\xi^\alpha\chi(\varepsilon y|\xi|^2)$ is bounded together with its derivatives uniformly in $\varepsilon\in(0,1)$, $\xi\in\mathbb{R}^d$ and $y\in[\lambda_{\min},\lambda_{\max}]$, the symbol χ_ε satisfies, for any $N\in\mathbb{N}_0$, $\sigma>d/2$

$$|\chi_\varepsilon|_{0,\sigma,N}\le\mathtt{C}\left(\|U\|_\sigma\right),\quad\text{uniformly in }\varepsilon. \tag{2.4.14}$$

The diagonalization (2.4.11) has the following operatorial consequence.

Lemma 2.4.3. *We have*

$$\mathrm{Op}^{\mathrm{BW}}\left(F^{-1}\right)\mathbb{J}\mathrm{Op}^{\mathrm{BW}}\left(A^\varepsilon\right)\mathrm{Op}^{\mathrm{BW}}\left(F\right)=\mathbb{J}\mathrm{Op}^{\mathrm{BW}}\Big(\big(\sqrt{\mathtt{m}K(\mathtt{m})}\lambda|\xi|^2+\mathtt{b}\cdot\xi\big)\chi_\varepsilon\Big)+\mathcal{F}(U) \tag{2.4.15}$$

where $\mathcal{F}(U):=\mathcal{F}_\varepsilon(U)\colon\dot{\mathbf{H}}^\sigma\to\dot{\mathbf{H}}^\sigma$, $\forall\sigma\ge 0$, *satisfies, uniformly in* ε,

$$\|\mathcal{F}(U)W\|_\sigma\le\mathtt{C}(\|U\|_{s_0+2})\|W\|_\sigma,\quad\forall W\in\dot{\mathbf{H}}^\sigma. \tag{2.4.16}$$

Proof. We have that

$$\mathrm{Op}^{\mathrm{BW}}\left(F^{-1}\right)\mathbb{J}\mathrm{Op}^{\mathrm{BW}}\left(A_2\chi_\varepsilon\right)\mathrm{Op}^{\mathrm{BW}}\left(F\right)=\mathbb{J}\sqrt{\mathtt{m}K(\mathtt{m})}\begin{bmatrix}D_2 & B_2\\ B_2 & D_2\end{bmatrix},$$

where

$$\begin{aligned}D_2=&\ \mathrm{Op}^{\mathrm{BW}}(f)\mathrm{Op}^{\mathrm{BW}}\big(|\xi|^2(1+\mathtt{a}_+)\chi_\varepsilon\big)\mathrm{Op}^{\mathrm{BW}}(f)+\mathrm{Op}^{\mathrm{BW}}(g)\mathrm{Op}^{\mathrm{BW}}\big(|\xi|^2(1+\mathtt{a}_+)\chi_\varepsilon\big)\mathrm{Op}^{\mathrm{BW}}(g)\\ &+\mathrm{Op}^{\mathrm{BW}}(f)\mathrm{Op}^{\mathrm{BW}}\big(|\xi|^2\mathtt{a}_-\chi_\varepsilon\big)\mathrm{Op}^{\mathrm{BW}}(g)+\mathrm{Op}^{\mathrm{BW}}(g)\mathrm{Op}^{\mathrm{BW}}\big(|\xi|^2\mathtt{a}_-\chi_\varepsilon\big)\mathrm{Op}^{\mathrm{BW}}(f)\\ B_2=&\ \mathrm{Op}^{\mathrm{BW}}(f)\mathrm{Op}^{\mathrm{BW}}\big(|\xi|^2(1+\mathtt{a}_+)\chi_\varepsilon\big)\mathrm{Op}^{\mathrm{BW}}(g)+\mathrm{Op}^{\mathrm{BW}}(g)\mathrm{Op}^{\mathrm{BW}}\big(|\xi|^2(1+\mathtt{a}_+)\chi_\varepsilon\big)\mathrm{Op}^{\mathrm{BW}}(f)\\ &+\mathrm{Op}^{\mathrm{BW}}(f)\mathrm{Op}^{\mathrm{BW}}\big(|\xi|^2\mathtt{a}_-\chi_\varepsilon\big)\mathrm{Op}^{\mathrm{BW}}(f)+\mathrm{Op}^{\mathrm{BW}}(g)\mathrm{Op}^{\mathrm{BW}}\big(|\xi|^2\mathtt{a}_-\chi_\varepsilon\big)\mathrm{Op}^{\mathrm{BW}}(g).\end{aligned}$$

By Corollary 2.2.14 we obtain

$$
\begin{aligned}
D_2 &= \mathrm{Op}^{\mathrm{BW}}\left(\left[(f^2+g^2)(1+\mathtt{a}_+)+2fg\mathtt{a}_-\right]|\xi|^2\chi_\varepsilon\right)+\mathcal{F}_1(U)=\mathrm{Op}^{\mathrm{BW}}\left(\lambda(U)|\xi|^2\chi_\varepsilon\right)+\mathcal{F}_1(U)\,,\\
B_2 &= \mathrm{Op}^{\mathrm{BW}}\left(\left[(f^2+g^2)\mathtt{a}_-+2fg(1+\mathtt{a}_+)\right]|\xi|^2\chi_\varepsilon\right)+\mathcal{F}_2(U)=\mathcal{F}_2(U)\,,
\end{aligned}
$$

where $\mathcal{F}_1,\mathcal{F}_2$ satisfy (2.4.16) by (2.2.92), (2.4.12), and (2.4.14) and since, by the definition of f and g in (2.4.8) and λ in (2.4.6), we have $(f^2+g^2)(1+\mathtt{a}_+)+2fg\mathtt{a}_-=\lambda$ and $(f^2+g^2)\mathtt{a}_-+2fg(1+\mathtt{a}_+)=0$. Moreover

$$
\mathrm{Op}^{\mathrm{BW}}\left(F^{-1}\right)\mathbb{J}\mathrm{Op}^{\mathrm{BW}}(A_1\chi_\varepsilon)\mathrm{Op}^{\mathrm{BW}}(F)=\mathbb{J}\begin{bmatrix} D_1 & B_1\\ -B_1 & -D_1\end{bmatrix},
$$

where

$$
\begin{aligned}
D_1 &= \mathrm{Op}^{\mathrm{BW}}(f)\mathrm{Op}^{\mathrm{BW}}(\mathtt{b}\cdot\xi\chi_\varepsilon)\mathrm{Op}^{\mathrm{BW}}(f)-\mathrm{Op}^{\mathrm{BW}}(g)\mathrm{Op}^{\mathrm{BW}}(\mathtt{b}\cdot\xi\chi_\varepsilon)\mathrm{Op}^{\mathrm{BW}}(g)\\
B_1 &= \mathrm{Op}^{\mathrm{BW}}(f)\mathrm{Op}^{\mathrm{BW}}(\mathtt{b}\cdot\xi\chi_\varepsilon)\mathrm{Op}^{\mathrm{BW}}(g)-\mathrm{Op}^{\mathrm{BW}}(g)\mathrm{Op}^{\mathrm{BW}}(\mathtt{b}\cdot\xi\chi_\varepsilon)\mathrm{Op}^{\mathrm{BW}}(f)\,.
\end{aligned}
$$

Applying Theorem 2.2.13, (2.4.12), (2.4.14), using that $f^2-g^2=1$ we obtain $D_1=\mathrm{Op}^{\mathrm{BW}}(\mathtt{b}\cdot\xi\,\chi_\varepsilon)+\mathcal{F}_1(U)$ and $B_1=\mathcal{F}_2(U)$ with $\mathcal{F}_1,\mathcal{F}_2$ satisfying (2.4.16). □

2.4.2 Energy estimate for smoothed system

We first solve (2.4.3) in the case $R(t)=0$ and $V_0\in\dot{C}^\infty:=\cap_{\sigma\in\mathbb{R}}\dot{\mathbf{H}}^\sigma$. Consider the regularized Cauchy problem

$$
\partial_t V^\varepsilon=\mathbb{J}\mathrm{Op}^{\mathrm{BW}}(A^\varepsilon(U(t);x,\xi))V^\varepsilon\,,\quad V^\varepsilon(0)=V_0\in\dot{C}^\infty\,, \tag{2.4.17}
$$

where $A^\varepsilon(U;x,\xi)$ is defined in (2.4.13). As the operator $\mathrm{Op}^{\mathrm{BW}}(A^\varepsilon(U;x,\xi))$ is bounded for any $\varepsilon>0$, and $U(t)$ satisfies (2.4.2), the differential equation (2.4.17) has a unique solution $V^\varepsilon(t)$ which belongs to $C^2([0,T],\dot{\mathbf{H}}^\sigma)$ for any $\sigma\geq 0$. The important fact is that it admits the following ε-independent energy estimate.

Proposition 2.4.4. (Energy estimate) *Let U satisfy (2.4.2). For any $\sigma\geq 0$, there exist constants $C_r,C_\Theta>0$ (depending also on σ), such that for any $\varepsilon>0$, the unique solution $V^\varepsilon(t)$ of (2.4.17) fulfills*

$$
\|V^\varepsilon(t)\|_\sigma^2\leq C_r\|V_0\|_\sigma^2+C_\Theta\int_0^t\|V^\varepsilon(\tau)\|_\sigma^2\,\mathrm{d}\tau\,,\quad\forall t\in[0,T]\,. \tag{2.4.18}
$$

As a consequence, there are constants C_r,C_Θ independent of ε, such that

$$
\|V^\varepsilon(t)\|_\sigma\leq C_r e^{C_\Theta t}\|V_0\|_\sigma\,,\quad\forall t\in[0,T]\,. \tag{2.4.19}
$$

In order to prove Proposition 2.4.4, we define, for any $\sigma\geq 0$, the *modified energy*

$$
\|V\|_{\sigma,U}^2:=\langle\mathrm{Op}^{\mathrm{BW}}\left(\lambda^\sigma(U;x)|\xi|^{2\sigma}\right)\mathrm{Op}^{\mathrm{BW}}\left(F^{-1}(U;x)\right)V,\mathrm{Op}^{\mathrm{BW}}\left(F^{-1}(U;x)\right)V\rangle\,, \tag{2.4.20}
$$

where we introduce the *real* scalar product

$$
\langle V,W\rangle:=2\mathrm{Re}\int_{\mathbb{T}^d}v(x)\overline{w}(x)\,\mathrm{d}x\,,\qquad V=\begin{bmatrix}v\\ \overline{v}\end{bmatrix},\quad W=\begin{bmatrix}w\\ \overline{w}\end{bmatrix}.
$$

Lemma 2.4.5. *Fix $\sigma\geq 0$, $r>0$. There exists a constant $C_r>0$ (depending also on σ) such that for any $U\in\dot{\mathbf{H}}^{s_0}$ with $\|U\|_{s_0}\leq r$ and $\rho(U)\in\mathcal{Q}$ we have*

$$
C_r^{-1}\|V\|_\sigma^2-\|V\|_{-2}^2\leq\|V\|_{\sigma,U}^2\leq C_r\|V\|_\sigma^2\,,\quad\forall V\in\dot{\mathbf{H}}^\sigma\,. \tag{2.4.21}
$$

Proof. We first prove the upper bound in (2.4.21). We note that, by (2.4.12), $\lambda^\sigma(U;x)|\xi|^{2\sigma} \in \Gamma^{2\sigma}_{s_0}$ and $F^{-1}(U;x) \in \Gamma^0_{s_0} \otimes \mathcal{M}_2(\mathbb{C})$ and, by Theorem 2.2.12 and (2.4.12) we have

$$\|V\|^2_{\sigma,U} \le \|\mathrm{Op}^{\mathrm{BW}}\left(\lambda^\sigma(U;x)|\xi|^{2\sigma}\right)\mathrm{Op}^{\mathrm{BW}}\left(F^{-1}(U;x)\right)V\|_{-\sigma}\|\mathrm{Op}^{\mathrm{BW}}\left(F^{-1}(U;x)\right)V\|_\sigma \le C_r\|V\|^2_\sigma .$$

In order to prove the lower bound, we fix $\delta \in (0,1)$ such that $s_0 - \delta > \frac{d}{2}$ and, due to (2.4.7), we have $\lambda^{-\frac{\sigma}{2}} \in \Gamma^0_{s_0-\delta}$. So, applying Theorem 2.2.13 and (2.4.12) with $s_0-\delta$ instead of s_0 and with $\varrho = \delta$, we have

$$\mathrm{Op}^{\mathrm{BW}}\left(\lambda^{-\frac{\sigma}{2}}\right)\mathrm{Op}^{\mathrm{BW}}(F)\mathrm{Op}^{\mathrm{BW}}\left(\lambda^{\frac{\sigma}{2}}\right)\mathrm{Op}^{\mathrm{BW}}\left(F^{-1}\right) = \mathrm{Id} + \mathcal{F}^{-\delta}(U), \tag{2.4.22}$$

where for any $\sigma' \in \mathbb{R}$ there exists a constant $C_{r,\sigma'} > 0$ such that

$$\|\mathcal{F}^{-\delta}(U)f\|_{\sigma'} \le C_{r,\sigma'}\|f\|_{\sigma'-\delta}, \quad \forall f \in \dot{\mathbf{H}}^{\sigma'-\delta}. \tag{2.4.23}$$

Again, applying Theorem 2.2.13 with $s_0-\delta$ instead of s_0 and with $\varrho = \delta$, we have also

$$\mathrm{Op}^{\mathrm{BW}}\left(\lambda^{\frac{\sigma}{2}}\right)\mathrm{Op}^{\mathrm{BW}}\left(|\xi|^{2\sigma}\right)\mathrm{Op}^{\mathrm{BW}}\left(\lambda^{\frac{\sigma}{2}}\right) = \mathrm{Op}^{\mathrm{BW}}\left(\lambda^\sigma|\xi|^{2\sigma}\right) + \mathcal{F}^{2\sigma-\delta}(U), \tag{2.4.24}$$

where for any $\sigma' \in \mathbb{R}$ there exists a constant $C_{r,\sigma'} > 0$ such that

$$\|\mathcal{F}^{2\sigma-\delta}(U)f\|_{\sigma'-2\sigma+\delta} \le C_{r,\sigma'}\|f\|_{\sigma'}, \quad \forall f \in \dot{\mathbf{H}}^{\sigma'}. \tag{2.4.25}$$

By (2.4.22)–(2.4.25), Theorem 2.2.12 and (2.4.12) and using also that $\mathrm{Op}^{\mathrm{BW}}\left(\lambda^{\frac{\sigma}{2}}\right)$ is symmetric with respect to $\langle\cdot,\cdot\rangle$, we have

$$\begin{aligned}
\|V\|^2_\sigma &\le 2\|\mathrm{Op}^{\mathrm{BW}}\left(\lambda^{-\frac{\sigma}{2}}\right)\mathrm{Op}^{\mathrm{BW}}(F)\mathrm{Op}^{\mathrm{BW}}\left(\lambda^{\frac{\sigma}{2}}\right)\mathrm{Op}^{\mathrm{BW}}\left(F^{-1}\right)V\|^2_\sigma + 2\|\mathcal{F}^{-\delta}(U)V\|^2_\sigma \\
&\le C_r\left(\|\mathrm{Op}^{\mathrm{BW}}\left(\lambda^{\frac{\sigma}{2}}\right)\mathrm{Op}^{\mathrm{BW}}\left(F^{-1}\right)V\|^2_\sigma + \|V\|^2_{\sigma-\delta}\right) \\
&= C_r\left(\langle\mathrm{Op}^{\mathrm{BW}}\left(\lambda^{\frac{\sigma}{2}}\right)\mathrm{Op}^{\mathrm{BW}}\left(|\xi|^{2\sigma}\right)\mathrm{Op}^{\mathrm{BW}}\left(\lambda^{\frac{\sigma}{2}}\right)\mathrm{Op}^{\mathrm{BW}}\left(F^{-1}\right)V, \mathrm{Op}^{\mathrm{BW}}\left(F^{-1}\right)V\rangle + \|V\|^2_{\sigma-\delta}\right) \\
&= C_r\left(\|V\|^2_{\sigma,U} + \langle\mathcal{F}^{2\sigma-\delta}(U)\mathrm{Op}^{\mathrm{BW}}\left(F^{-1}\right)V, \mathrm{Op}^{\mathrm{BW}}\left(F^{-1}\right)V\rangle + \|V\|^2_{\sigma-\delta}\right) \\
&\le C_r\left(\|V\|^2_{\sigma,U} + \|V\|^2_{\sigma-\frac{\delta}{2}}\right).
\end{aligned}$$

Now we use (2.2.83) and the asymmetric Young inequality to get, for any $\epsilon > 0$,

$$\|V\|^2_{\sigma-\frac{\delta}{2}} \le \|V\|_{-2}^{\frac{\delta}{\sigma+2}}\,\|V\|_\sigma^{\frac{2(\sigma+2)-\delta}{\sigma+2}} \le \epsilon^{-\frac{2(\sigma+2)}{\delta}}\|V\|^2_{-2} + \epsilon^{\frac{2(\sigma+2)}{2(\sigma+2)-\delta}}\|V\|^2_\sigma;$$

we choose ϵ so small so that $\epsilon^{\frac{2(\sigma+2)}{2(\sigma+2)-\delta}}C_r = \frac{1}{2}$ and we get $\|V\|^2_\sigma \le 2C_r\left(\|V\|^2_{\sigma,U} + \|V\|^2_{-2}\right)$. This proves the lower bound in (2.4.21). □

Proof of Proposition 2.4.4. The time derivative of the modified energy (2.4.20) along a solution $V^\varepsilon(t)$ of (2.4.17) is

$$\begin{aligned}
\frac{d}{dt}\|V^\varepsilon\|^2_{\sigma,U(t)} &= \langle\mathrm{Op}^{\mathrm{BW}}\left(\partial_t(\lambda^\sigma)|\xi|^{2\sigma}\right)\mathrm{Op}^{\mathrm{BW}}\left(F^{-1}\right)V^\varepsilon, \mathrm{Op}^{\mathrm{BW}}\left(F^{-1}\right)V^\varepsilon\rangle &(2.4.26)\\
&\quad + 2\langle\mathrm{Op}^{\mathrm{BW}}\left(\lambda^\sigma|\xi|^{2\sigma}\right)\mathrm{Op}^{\mathrm{BW}}\left(\partial_t F^{-1}\right)V^\varepsilon, \mathrm{Op}^{\mathrm{BW}}\left(F^{-1}\right)V^\varepsilon\rangle &(2.4.27)\\
&\quad + 2\langle\mathrm{Op}^{\mathrm{BW}}\left(\lambda^\sigma|\xi|^{2\sigma}\right)\mathrm{Op}^{\mathrm{BW}}\left(F^{-1}\right)\partial_t V^\varepsilon, \mathrm{Op}^{\mathrm{BW}}\left(F^{-1}\right)V^\varepsilon\rangle. &(2.4.28)
\end{aligned}$$

By Theorem 2.2.12 and using that $\forall\sigma \geq 0$, $N \in \mathbb{N}_0$,

$$\left|\partial_t \lambda^\sigma(U)|\xi|^{2\sigma}\right|_{2\sigma,s_0,N}, \quad \left|\partial_t F^{-1}(U)\right|_{0,s_0,N} \leq \mathtt{C}_N(\|U\|_{s_0}, \|\partial_t U\|_{s_0})$$

and the assumption (2.4.2), there exists a constant $C_\Theta > 0$ (depending also on σ) such that

$$(2.4.26) + (2.4.27) \leq C_\Theta \|V^\varepsilon\|_\sigma^2 . \tag{2.4.29}$$

We now estimate (2.4.28). By Theorem 2.2.13 with $\varrho = 2$ and (2.4.2) we have

$$\mathrm{Op}^{\mathrm{BW}}(F)\mathrm{Op}^{\mathrm{BW}}\left(F^{-1}\right) = \mathrm{Id} + \mathcal{F}_+^{-2}(U), \quad \mathrm{Op}^{\mathrm{BW}}\left(F^{-1}\right)\mathrm{Op}^{\mathrm{BW}}(F) = \mathrm{Id} + \mathcal{F}_-^{-2}(U), \tag{2.4.30}$$

where $\mathcal{F}_\pm^{-2}(U)$ are bounded operators from $\dot{\mathbf{H}}^{\sigma'}$ to $\dot{\mathbf{H}}^{\sigma'+2}$, $\forall \sigma' \in \mathbb{R}$, satisfying

$$\|\mathcal{F}_\pm^{-2}(U)W\|_{\sigma'+2} \leq C_{\Theta,\sigma'}\|W\|_{\sigma'}, \quad \forall W \in \dot{\mathbf{H}}^{\sigma'} . \tag{2.4.31}$$

Thus, denoting $\widetilde{V}^\varepsilon := \mathrm{Op}^{\mathrm{BW}}\left(F^{-1}\right)V^\varepsilon$, by (2.4.30), we have

$$\mathrm{Op}^{\mathrm{BW}}(F)\widetilde{V}^\varepsilon = V^\varepsilon + \mathcal{F}_+^{-2}(U)V^\varepsilon . \tag{2.4.32}$$

Recalling (2.4.17) we have

$$\begin{aligned}
(2.4.28) &= 2\langle \mathrm{Op}^{\mathrm{BW}}\left(\lambda^\sigma|\xi|^{2\sigma}\right)\mathrm{Op}^{\mathrm{BW}}\left(F^{-1}\right)\mathbb{J}\mathrm{Op}^{\mathrm{BW}}(A^\varepsilon)V^\varepsilon, \widetilde{V}^\varepsilon\rangle \\
&\overset{(2.4.32)}{=} 2\langle \mathrm{Op}^{\mathrm{BW}}\left(\lambda^\sigma|\xi|^{2\sigma}\right)\mathrm{Op}^{\mathrm{BW}}\left(F^{-1}\right)\mathbb{J}\mathrm{Op}^{\mathrm{BW}}(A^\varepsilon)\mathrm{Op}^{\mathrm{BW}}(F)\widetilde{V}^\varepsilon, \widetilde{V}^\varepsilon\rangle \\
&\quad - 2\langle \mathrm{Op}^{\mathrm{BW}}\left(\lambda^\sigma|\xi|^{2\sigma}\right)\mathrm{Op}^{\mathrm{BW}}\left(F^{-1}\right)\mathbb{J}\mathrm{Op}^{\mathrm{BW}}(A^\varepsilon)\mathcal{F}_+^{-2}V^\varepsilon, \widetilde{V}^\varepsilon\rangle
\end{aligned}$$

and by Lemma 2.4.3 we get

$$(2.4.28) \overset{(2.4.15)}{=} \langle \mathbb{J}\left[\mathrm{Op}^{\mathrm{BW}}\left(\lambda^\sigma|\xi|^{2\sigma}\right), \mathrm{Op}^{BW}\left(\sqrt{\mathtt{m}K(\mathtt{m})}\lambda|\xi|^2\chi_\varepsilon\right)\right]\widetilde{V}^\varepsilon, \widetilde{V}^\varepsilon\rangle \tag{2.4.33}$$

$$+ \langle \mathbb{J}\left[\mathrm{Op}^{\mathrm{BW}}\left(\lambda^\sigma|\xi|^{2\sigma}\right), \mathrm{Op}^{\mathrm{BW}}(\mathtt{b}\cdot\xi\,\chi_\varepsilon)\right]\widetilde{V}^\varepsilon, \widetilde{V}^\varepsilon\rangle \tag{2.4.34}$$

$$+ 2\langle \mathrm{Op}^{\mathrm{BW}}\left(\lambda^\sigma|\xi|^{2\sigma}\right)\mathcal{F}\widetilde{V}^\varepsilon, \widetilde{V}^\varepsilon\rangle \tag{2.4.35}$$

$$- 2\langle \mathrm{Op}^{\mathrm{BW}}\left(\lambda^\sigma|\xi|^{2\sigma}\right)\mathrm{Op}^{\mathrm{BW}}\left(F^{-1}\right)\mathbb{J}\mathrm{Op}^{\mathrm{BW}}(A^\varepsilon)\mathcal{F}_+^{-2}V^\varepsilon, \widetilde{V}^\varepsilon\rangle \tag{2.4.36}$$

where in line (2.4.35) the operator $\mathcal{F}(U)$ is the bounded remainder of Lemma 2.4.3. We estimate each contribution. First we consider line (2.4.33). Using Theorem 2.2.13 with $\varrho = 2$, the principal symbol of the commutator is

$$\mathrm{i}^{-1}\left\{\lambda^\sigma|\xi|^{2\sigma}, \sqrt{\mathtt{m}K(\mathtt{m})}\lambda|\xi|^2\chi\left(\varepsilon\lambda|\xi|^2\right)\right\} = 0,$$

and, using (2.4.14), (2.4.12) and assumption (2.4.2), we get

$$|(2.4.33)| \leq C'_\Theta\|\widetilde{V}^\varepsilon\|_\sigma^2 \leq C_\Theta\|V^\varepsilon\|_\sigma^2 . \tag{2.4.37}$$

Similarly, using Theorem 2.2.13 with $\varrho = 1$, Theorem 2.2.12, (2.4.12) and estimates (2.4.31) and (2.4.16), we obtain

$$|(2.4.34)| + |(2.4.35)| + |(2.4.36)| \leq C_\Theta\|V^\varepsilon\|_\sigma^2 . \tag{2.4.38}$$

In conclusion, by (2.4.29), (2.4.37), (2.4.38), we deduce the bound $\frac{d}{dt}\|V^\varepsilon(t)\|_{\sigma,U(t)}^2 \leq C_\Theta\|V^\varepsilon(t)\|_\sigma^2$, that gives, for any $t \in [0, T]$

$$\begin{aligned}
\|V^\varepsilon(t)\|_{\sigma,U(t)}^2 &\leq \|V^\varepsilon(0)\|_{\sigma,U(0)}^2 + C_\Theta\int_0^t \|V^\varepsilon(\tau)\|_\sigma^2\,\mathrm{d}\tau \\
&\overset{(2.4.21)}{\leq} C_r\|V^\varepsilon(0)\|_\sigma^2 + C_\Theta\int_0^t \|V^\varepsilon(\tau)\|_\sigma^2\,\mathrm{d}\tau .
\end{aligned} \tag{2.4.39}$$

Since $V^\varepsilon(t)$ solves (2.4.17), by Theorem 2.2.12, (2.4.12), (2.4.14) there exists a constant $C_\Theta > 0$ (independent on ε) such that $\|\partial_t V^\varepsilon(t)\|_{-2}^2 \le C_\Theta \|V^\varepsilon(t)\|_0^2 \le C_\Theta \|V^\varepsilon(t)\|_\sigma^2$ and therefore

$$\|V^\varepsilon(t)\|_{-2}^2 \le \|V^\varepsilon(0)\|_{-2}^2 + C_\Theta \int_0^t \|V^\varepsilon(\tau)\|_\sigma^2 \, \mathrm{d}\tau \,, \quad \forall t \in [0,T] \,. \tag{2.4.40}$$

We finally deduce (2.4.18) by (2.4.39), the lower bound in (2.4.21) and (2.4.40). The estimate (2.4.19) follows by Gronwall inequality. □

Proof of Proposition 2.4.2. By Proposition 2.4.4, Ascoli-Arzelá theorem ensures that, for any $\sigma \ge 0$, V^ε converges up to subsequence to a limit V in $C^1([0,T], \dot{\mathbf{H}}^\sigma)$, as $\varepsilon \to 0$ that solves (2.4.3) with $R(t) = 0$, initial datum $V_0 \in \dot{C}^\infty$, and satisfies $\|V(t)\|_\sigma \le C_r e^{C_\Theta t} \|V_0\|_\sigma$, for any $\sigma \ge 0$. The case $V_0 \in \dot{\mathbf{H}}^\sigma$ follows by a classical approximation argument with smooth initial data. This shows that the propagator of $\mathbb{J}\mathrm{Op}^{BW}(A_2(U(t);x,\xi) + A_1(U(t);x,\xi))$ is, for any $\sigma \ge 0$, a well defined bounded linear operator

$$\Phi(t) : \dot{\mathbf{H}}^\sigma \mapsto \dot{\mathbf{H}}^\sigma \,, \; V_0 \mapsto \Phi(t)V_0 := V(t) \,, \; \forall t \in [0,T] \,, \; \text{satisfying} \; \|\Phi(t)V_0\|_\sigma \le C_r e^{C_\Theta t} \|V_0\|_\sigma \,.$$

In the in-homogeneous case $R \neq 0$, the solutions of (2.4.3) is given by the Duhamel formula $V(t) = \Phi(t)V_0 + \Phi(t)\int_0^t \Phi^{-1}(\tau) R(\tau) \, \mathrm{d}\tau$, and the estimate (2.4.4) follows.

2.4.3 Iterative scheme

In order to prove that the nonlinear system (2.3.4) has a local in time solution we consider the sequence of linear Cauchy problems

$$\mathcal{P}_1 := \begin{cases} \partial_t U_1 = -\mathbb{J}\sqrt{\mathtt{m}K(\mathtt{m})}\,\Delta U_1 \\ U_1(0) = U_0 \,, \end{cases} \qquad \mathcal{P}_n := \begin{cases} \partial_t U_n = \mathbb{J}\,\mathrm{Op}^{BW}(A(U_{n-1};x,\xi))U_n + R(U_{n-1}) \\ U_n(0) = U_0 \,, \end{cases}$$

for $n \ge 2$, where $A := A_2 + A_1$, cfr. (2.3.6), (4.5.39). The strategy is to prove that the sequence of solutions U_n of the approximated problems $\mathcal{P}_n$ converges to a solution U of system (2.3.4).

Lemma 2.4.6. *Let $U_0 \in \dot{\mathbf{H}}^s$, $s > 2 + \frac{d}{2}$, such that $\rho(U_0) \in \mathcal{Q}_\delta$ for some $\delta > 0$ (recall (2.3.5) and (2.4.1)) and define $r := 2\|U_0\|_{s_0}$. Then there exists a time $T := T(\|U_0\|_{s_0+2}, \delta) > 0$ such that, for any $n \in \mathbb{N}$:*

$(S0)_n$: *The problem $\mathcal{P}_n$ admits a unique solution $U_n \in C^0([0,T], \dot{\mathbf{H}}^s) \cap C^1([0,T], \dot{\mathbf{H}}^{s-2})$.*

$(S1)_n$: *For any $t \in [0,T]$, $\rho(U_n(t))$ belongs to $\mathcal{Q}_{\frac{\delta}{2}}$.*

$(S2)_n$: *There exists a constant $C_r \ge 1$ (depending also on s) such that, defining $\Theta := 4C_r\|U_0\|_{s_0+2}$ and $M := 4C_r\|U_0\|_s$, for any $1 \le m \le n$ one has*

$$\|U_m\|_{L_T^\infty \dot{\mathbf{H}}^{s_0}} \le r \,; \tag{2.4.41}$$

$$\|U_m\|_{L_T^\infty \dot{\mathbf{H}}^{s_0+2}} \le \Theta \,, \quad \|\partial_t U_m\|_{L_T^\infty \dot{\mathbf{H}}^{s_0}} \le C_r \Theta \,; \tag{2.4.42}$$

$$\|U_m\|_{L_T^\infty \dot{\mathbf{H}}^{s}} \le M \,, \quad \|\partial_t U_m\|_{L_T^\infty \dot{\mathbf{H}}^{s-2}} \le C_r M \,. \tag{2.4.43}$$

$(S3)_n$: *For $1 \le m \le n$ one has*

$$\|U_1\|_{L_T^\infty \dot{\mathbf{H}}^{s_0}} = r/2 \,, \qquad \|U_m - U_{m-1}\|_{L_T^\infty \dot{\mathbf{H}}^{s_0}} \le 2^{-m} r \,, \; m \ge 2 \,.$$

Proof. We prove the statement by induction on $n \in \mathbb{N}$. Given $r > 0$, we define

$$C_r := \max\{1, C_{r,s_0}, C_{r,s_0+2}, C_{r,s}, 2\mathtt{C}(r)\},$$

where $C_{r,\sigma}$ is the constant in Proposition 2.4.2 (where we stress that it depends also on σ) and $\mathtt{C}(\cdot)$ is the function in (2.3.9) and (2.3.11). In the following we shall denote by C_Θ all the constants depending on Θ, which can vary from line to line.

Proof of $(S0)_1$**:** The problem $\mathcal{P}_1$ admits a unique global solution which preserves the Sobolev norms.

Proof of $(S1)_1$**:** We have $\rho(U_0) \in \mathcal{Q}_\delta$. In addition

$$\|\rho(U_1(t) - U_0)\|_{L^\infty(\mathbb{T}^d)} \lesssim \|U_1(t) - U_0\|_{s_0} \lesssim T\|U_0\|_{s_0+2} \leq \delta/2$$

for $T := T(\|U_0\|_{s_0+2}, \delta) > 0$ sufficiently small, which implies $\rho(U_1(t)) \in \mathcal{Q}_{\frac{\delta}{2}}$, for any $t \in [0, T]$.

Proof of $(S2)_1$ **and** $(S3)_1$**:** The flow of $\mathcal{P}_1$ is an isometry and $M \geq \|U_0\|_s$, $\Theta \geq \|U_0\|_{s_0+2}$.

Suppose that $(S0)_{n-1}$–$(S3)_{n-1}$ hold true. We prove $(S0)_n$–$(S3)_n$.

Proof of $(S0)_n$**:** We apply Proposition 2.4.2 with $\sigma = s$, $U \rightsquigarrow U_{n-1}$ and $R(t) := R(U_{n-1}(t))$. By $(S1)_{n-1}$ and $(S2)_{n-1}$, the function U_{n-1} satisfies assumption (2.4.2) with $\Theta \rightsquigarrow (1 + C_r)\Theta$. In addition $R(U_{n-1}(t))$ belongs to $C^0([0,T], \dot{\mathbf{H}}^s)$ thanks to (2.3.12) and $U_{n-1} \in C^0([0,T]; \dot{\mathbf{H}}^s)$. Thus Proposition 2.4.2 with $\sigma = s$ implies $(S0)_n$. In particular U_n satisfies the estimate (2.4.4).

Proof of $(S2)_n$**:** We first prove (2.4.42). The estimate (2.4.4) with $\sigma = s_0 + 2$, the bound (2.3.11) of $R(U_{n-1}(t))$ and (2.4.42) at the step $n - 1$, imply

$$\|U_n\|_{L^\infty_T \dot{\mathbf{H}}^{s_0+2}} \leq C_r e^{C_\Theta T}\|U_0\|_{s_0+2} + TC_\Theta e^{C_\Theta T}\Theta. \tag{2.4.44}$$

As $\Theta = 4C_r\|U_0\|_{s_0+2}$, we take $T > 0$ small such that

$$C_\Theta T \leq 1, \quad TC_\Theta e^{C_\Theta T} \leq 1/4, \tag{2.4.45}$$

which, by (2.4.44), gives $\|U_n\|_{L^\infty_T \dot{\mathbf{H}}^{s_0+2}} \leq \Theta$. This proves the first estimate of (2.4.42). Regarding the control of $\partial_t U_n$, we use the equation $\mathcal{P}_n$, the second estimate in (2.3.11) and (2.3.9) with $\sigma = s_0$ to obtain

$$\|\partial_t U_n(t)\|_{s_0} \leq \mathtt{C}(\|U_{n-1}(t)\|_{s_0})\|U_n(t)\|_{s_0+2} + \mathtt{C}(\|U_{n-1}(t)\|_{s_0})\|U_{n-1}(t)\|_{s_0+2} \leq C_r\Theta \tag{2.4.46}$$

which proves the second estimate of (2.4.42).

Next we prove (2.4.43). Applying estimate (2.4.4) with $\sigma = s$, we have

$$\|U_n\|_{L^\infty_T \dot{\mathbf{H}}^s} \leq C_r e^{C_\Theta T}\|U_0\|_s + TC_\Theta e^{C_\Theta T} M \leq M$$

for $M = 4C_r\|U_0\|_s$ and since $T > 0$ is chosen as in (2.4.45). The estimate for $\|\partial_t U_n\|_{s-2}$ is similar to (2.4.46), and we omit it. Estimate (2.4.41) is a consequence of $(S3)_n$, which we prove below.

Proof of $(S1)_n$**:** We use estimate (2.4.46) to get

$$\|\rho(U_n(t) - U_0)\|_{L^\infty(\mathbb{T}^d)} \leq C\|U_n(t) - U_0\|_{s_0} \leq C\int_0^T \|\partial_t U_n(t)\|_{s_0}\,\mathrm{d}t \leq C\,C_r T\Theta \leq \delta/2$$

provided that $T < \delta/(2CC_r\Theta)$. This shows that $\rho(U_n(t)) \in \mathcal{Q}_{\frac{\delta}{2}}$.

Proof of $(S3)_n$**:** Define $V_n := U_n - U_{n-1}$ if $n \geq 2$ and $V_1 := U_1$. Note that V_n, $n \geq 2$, solves

$$\partial_t V_n = \mathbb{J}\mathrm{Op}^{\mathrm{BW}}(A(U_{n-1}))V_n + f_n, \quad V_n(0) = 0, \tag{2.4.47}$$

where $A := A_2 + A_1$ and

$$f_n := \mathbb{J}\mathrm{Op}^{\mathrm{BW}}\left(A(U_{n-1}) - A(U_{n-2})\right)U_{n-1} + R(U_{n-1}) - R(U_{n-2})\,, \text{ for } n > 2\,,$$
$$f_2 := \mathbb{J}\mathrm{Op}^{BW}\left(A(U_1) - \sqrt{\mathtt{m}K(\mathtt{m})}|\xi|^2\right)U_1 + R(U_1)\,.$$

Applying estimates (2.3.13), (2.3.10), (2.3.11) and (2.4.42) we obtain, for $n \geq 2$,

$$\|f_n\|_{s_0} \leq C_\Theta \|V_{n-1}\|_{s_0}\,, \quad \forall t \in [0,T]\,. \tag{2.4.48}$$

We apply Proposition 2.4.2 to (2.4.47) with $\sigma = s_0$. Thus by (2.4.4) and (2.4.48) we get

$$\|V_n\|_{L^\infty_T \dot{\mathbf{H}}^{s_0}} \leq C_\Theta e^{C_\Theta T} T \|f_n\|_{L^\infty_T \dot{\mathbf{H}}^{s_0}} \leq C_\Theta e^{C_\Theta T} T \|V_{n-1}\|_{L^\infty_T \dot{\mathbf{H}}^{s_0}} \leq \frac{1}{2}\|V_{n-1}\|_{L^\infty_T \dot{\mathbf{H}}^{s_0}}$$

provided $C_\Theta e^{C_\Theta T} T \leq \frac{1}{2}$. The proof of Lemma 2.4.6 is complete. □

Corollary 2.4.7. *With the same assumptions of Lemma 2.4.6, for any $s_0 + 2 \leq s' < s$:*

(i) $(U_n)_{n\geq 1}$ *is a Cauchy sequence in* $C^0([0,T], \dot{\mathbf{H}}^{s'}) \cap C^1([0,T], \dot{\mathbf{H}}^{s'-2})$ *with* $T = T(\|U_0\|_{s_0+2}, \delta)$ *given by Lemma 2.4.6. It converges to the unique solution* $U(t)$ *of (2.3.4) with initial datum* U_0*,* $U(t)$ *is in* $C^0([0,T], \dot{\mathbf{H}}^{s'}) \cap C^1([0,T], \dot{\mathbf{H}}^{s'-2})$*. Moreover* $\rho(U(t)) \in \mathcal{Q}$*,* $\forall t \in [0,T]$*.*

(ii) For any $t \in [0,T]$*,* $U(t) \in \dot{\mathbf{H}}^s$ *and* $\|U(t)\|_s \leq 4C_r\|U_0\|_s$ *where* C_r *is the constant of* $(S2)_n$*.*

Proof. (i) If $s' = s_0$ it is the content of $(S3)_n$. For $s' \in (s_0, s)$, we use interpolation estimate (2.2.83), (2.4.43) and $(S3)_n$ to get, for $n \geq 2$,

$$\|U_n - U_{n-1}\|_{L^\infty_T \dot{\mathbf{H}}^{s'}} \leq \|U_n - U_{n-1}\|^\theta_{L^\infty_T \dot{\mathbf{H}}^{s_0}} \|U_n - U_{n-1}\|^{1-\theta}_{L^\infty_T \dot{\mathbf{H}}^{s}} \leq 2^{-n\theta} C_M\,,$$

where $\theta \in (0,1)$ is chosen so that $s' = \theta s_0 + (1-\theta)s$. Thus $(U_n)_{n\geq 1}$ is a Cauchy sequence in $C^0([0,T], \dot{\mathbf{H}}^{s'})$; we denote by $U(t) \in C^0([0,T], \dot{\mathbf{H}}^{s'})$ its limit. Similarly using that $\partial_t U_n$ solves $\mathcal{P}_n$, one proves that $\partial_t U_n$ is a Cauchy sequence in $C^0([0,T], \dot{\mathbf{H}}^{s'-2})$ that converges to $\partial_t U$ in $C^0([0,T], \dot{\mathbf{H}}^{s'-2})$. In order to prove that $U(t)$ solves (2.3.4), it is enough to show that

$$\mathcal{R}(U, U_{n-1}, U_n) := \mathbb{J}\mathrm{Op}^{\mathrm{BW}}\left(A(U_{n-1})\right)U_n - \mathbb{J}\mathrm{Op}^{\mathrm{BW}}\left(A(U)\right)U + R(U_{n-1}) - R(U)$$

converges to 0 in $L^\infty_T \dot{\mathbf{H}}^{s'-2}$. This holds true because by estimates (2.2.85), (2.3.10), (2.3.12), $(S2)_n$, and the fact that $U(t) \in C^0([0,T], \dot{\mathbf{H}}^{s'})$, we have

$$\|\mathcal{R}(U, U_{n-1}, U_n)\|_{L^\infty_T \dot{\mathbf{H}}^{s'-2}} \leq C_M \left(\|U - U_{n-1}\|_{L^\infty_T \dot{\mathbf{H}}^{s'}} + \|U - U_{n-1}\|_{L^\infty_T \dot{\mathbf{H}}^{s_0+2}} + \|U - U_n\|_{L^\infty_T \dot{\mathbf{H}}^{s'}} \right)$$

which converges to 0 as $n \to \infty$.

Let us now prove the uniqueness. Suppose that $V_1, V_2 \in C^0([0,T], \dot{\mathbf{H}}^{s'}) \cap C^1([0,T], \dot{\mathbf{H}}^{s'-2})$ are solutions of (2.3.4) with initial datum U_0. Then $W := V_1 - V_2$ solves

$$\partial_t W = \mathbb{J}\mathrm{Op}^{\mathrm{BW}}\left(A(V_1)\right)W + \mathbf{R}(t)\,, \quad W(0) = 0\,,$$

where $\mathbf{R}(t) := \mathbb{J}\mathrm{Op}^{\mathrm{BW}}\left(A(V_1) - A(V_2)\right)V_2 + R(V_1) - R(V_2)$. Applying Proposition 2.4.2 with $\sigma = s_0$ and Θ, r defined by

$$\Theta := \max_{j=1,2}\left(\|V_j\|_{L^\infty_T \dot{\mathbf{H}}^{s_0+2}} + \|\partial_t V_j\|_{L^\infty_T \dot{\mathbf{H}}^{s_0}}\right)\,, \quad r := \max_{j=1,2}\|V_j\|_{L^\infty_T \dot{\mathbf{H}}^{s_0}}\,,$$

together with estimates (2.3.13) and (2.3.10) we have, for any $t \in [0,T]$,

$$\|W\|_{L^\infty_t \dot{\mathbf{H}}^{s_0}} \leq C_\Theta e^{C_\Theta t} t \|\mathbf{R}\|_{L^\infty_t \dot{\mathbf{H}}^{s_0}} \leq C_\Theta e^{C_\Theta t} t \|W\|_{L^\infty_t \dot{\mathbf{H}}^{s_0}} .$$

Therefore, provided t is so small that $C_\Theta e^{C_\Theta t} t < 1$, we get $V_1(\tau) = V_2(\tau)$ $\forall \tau \in [0,t]$. As (2.3.4) is autonomous, actually one has $V_1(t) = V_2(t)$ for all $t \in [0,T]$. This proves the uniqueness.
Finally, as $\rho(U_n(t)) \in \mathcal{Q}_{\frac{\delta}{2}}$ and $U_n(t) \to U(t)$ in $\dot{\mathbf{H}}^{s_0}$, then $\rho(U(t)) \in \mathcal{Q}_{\frac{\delta}{2}} \subset \mathcal{Q}$.

(ii) Since $\|U_n(t)\|_s \leq 4C_r \|U_0\|_s$ and $U_n(t) \to U(t)$ in $\dot{\mathbf{H}}^{s'}$ then $\|U(t)\|_s \leq 4C_r \|U_0\|_s$. □

Let $\Pi_N U := \Big(\sum_{1 \leq |j| \leq N} u_j e^{\mathrm{i} j \cdot x}, \sum_{1 \leq |j| \leq N} \overline{u_j} e^{-\mathrm{i} j \cdot x} \Big)$. We need below the following technical lemma.

Lemma 2.4.8. *Let $U_0 \in \dot{\mathbf{H}}^s$, $s > 2 + \frac{d}{2}$, with $\rho(U_0) \in \mathcal{Q}_\delta$ for some $\delta > 0$. Then there exists a time $\tilde{T} := \tilde{T}(\|U_0\|_{s_0+2}, \delta) > 0$ and $N_0 > 0$ such that for any $N > N_0$:*

(i) system (2.3.4) with initial datum $\Pi_N U_0$ has a unique solution $U_N \in C^0([0,\tilde{T}], \dot{\mathbf{H}}^{s+2})$.

(ii) Let U be the unique solution of (2.3.4) with initial datum U_0 defined in the time interval $[0,T]$ (which exists by Corollary 2.4.7). Then there is $\breve{T} < \min\{T, \tilde{T}\}$, depending on $\|U_0\|_s$, independent of N, such that

$$\|U - U_N\|_{L^\infty_{\breve{T}} \dot{\mathbf{H}}^s} \leq \mathtt{C}(\|U_0\|_s) \left(\|U_0 - \Pi_N U_0\|_s + N^{s_0+2-s} \right). \tag{2.4.49}$$

In particular $U_N \to U$ in $C^0([0,\breve{T}], \dot{\mathbf{H}}^s)$ when $N \to \infty$.

Proof. Clearly $\Pi_N U_0 \in \dot{C}^\infty$. Moreover, as $\|\rho(U_0 - \Pi_N U_0)\|_{L^\infty(\mathbb{T}^d)} \to 0$ when $N \to \infty$, one has $\rho(\Pi_N U_0) \in \mathcal{Q}_{\frac{\delta}{2}}$ provided $N \geq N_0$ is sufficiently large. So we can apply Corollary 2.4.7 and obtain a time $\tilde{T} > 0$, independent on N, and a unique solution $U_N \in C^0([0,\tilde{T}], \dot{\mathbf{H}}^{s+2})$ of (2.3.4) with initial datum $\Pi_N U_0$. Moreover, by item (ii) of that corollary, setting $r = 2\|\Pi_N U_0\|_{s_0}$,

$$\|U_N\|_{L^\infty_{\tilde{T}} \dot{\mathbf{H}}^s} \leq 4C_r \|\Pi_N U_0\|_s \leq \mathtt{C}(\|U_0\|_{s_0}) \|U_0\|_s , \tag{2.4.50}$$

$$\|U_N\|_{L^\infty_{\tilde{T}} \dot{\mathbf{H}}^{s+2}} \leq 4C_r \|\Pi_N U_0\|_{s+2} \leq \mathtt{C}(\|U_0\|_{s_0}) N^2 \|U_0\|_s . \tag{2.4.51}$$

This proves item (i). In the following let $\breve{T} \leq \min\{\tilde{T}, T\}$.
Let us prove (ii). Let $\Theta := \|U\|_{L^\infty_{\tilde{T}} \dot{\mathbf{H}}^{s_0+2}} + \|\partial_t U\|_{L^\infty_{\tilde{T}} \dot{\mathbf{H}}^{s_0}}$ and $r := \|U\|_{L^\infty_{\tilde{T}} \dot{\mathbf{H}}^{s_0}}$. The function $W_N(t) := U(t) - U_N(t)$ satisfies $\|W_N(t)\|_s \leq \|U(t)\|_s + \|U_N(t)\|_s \leq \mathtt{C}(\|U_0\|_s)$, $\forall t \in [0,\breve{T}]$, by Corollary 2.4.7-$(ii)$. Moreover, W_N solves

$$\partial_t W_N = \mathbb{J} \mathrm{Op}^{\mathrm{BW}}(A(U)) W_N + \mathbf{R}(t) , \quad W_N(0) = U_0 - \Pi_N U_0$$

where $\mathbf{R}(t) := \mathbb{J} \mathrm{Op}^{\mathrm{BW}}(A(U) - A(U_N)) U_N + R(U) - R(U_N)$. Applying Proposition 2.4.2 with $\sigma = s_0$ and estimates (2.3.10), (2.3.13), (2.4.50) one obtains

$$\|W_N\|_{L^\infty_{\breve{T}} \dot{\mathbf{H}}^{s_0}} \leq C_r e^{C_\Theta \breve{T}} \|U_0 - \Pi_N U_0\|_{s_0} + \breve{T} C_\Theta e^{C_\Theta \breve{T}} \mathtt{C}(\|U_0\|_{s_0+2}) \|W_N\|_{L^\infty_{\breve{T}} \dot{\mathbf{H}}^{s_0}} ,$$

which, provided $\breve{T}$ is so small that $\breve{T} C_\Theta e^{C_\Theta \breve{T}} \mathtt{C}(\|U_0\|_{s_0+2}) \leq \frac{1}{2}$ (eventually shrinking it), gives

$$\|W_N\|_{L^\infty_{\breve{T}} \dot{\mathbf{H}}^{s_0}} \leq C_r \|U_0 - \Pi_N U_0\|_{s_0} \leq C_r N^{s_0 - s} \|U_0\|_s . \tag{2.4.52}$$

Similarly one estimates $\|W_N(t)\|_{\dot{\mathbf{H}}^s}$, getting

$$\begin{aligned}
&\|W_N\|_{L^\infty_{\breve{T}}\dot{\mathbf{H}}^s}\\
&\leq C_r e^{C_\Theta \breve{T}}\|U_0-\Pi_N U_0\|_s + C_\Theta e^{C_\Theta \breve{T}}\breve{T}\mathtt{C}(\|U_0\|_s)\Big(\|W_N\|_{L^\infty_{\breve{T}}\dot{\mathbf{H}}^{s_0}}\|U_N\|_{L^\infty_{\breve{T}}\dot{\mathbf{H}}^{s+2}} + \|W_N\|_{L^\infty_{\breve{T}}\dot{\mathbf{H}}^s}\Big)\\
&\overset{(2.4.51),(2.4.52)}{\leq} C_r e^{C_\Theta \breve{T}}\|U_0-\Pi_N U_0\|_s + C_\Theta e^{C_\Theta \breve{T}}\breve{T}\mathtt{C}(\|U_0\|_s)\big(N^{s_0-s+2} + \|W_N\|_{L^\infty_{\breve{T}}\dot{\mathbf{H}}^s}\big)
\end{aligned}$$

from which (2.4.49) follows provided $\breve{T}$ (depending on $\|U_0\|_s$) is sufficiently small. □

Proof of Proposition 2.4.1: Given an initial datum $U_0 \in \dot{\mathbf{H}}^s$ with $\rho(U_0) \in \mathcal{Q}$, choose $\delta > 0$ so small that $\rho(U_0) \in \mathcal{Q}_\delta$. Then Corollary 2.4.7 gives us a time $T = T(\|U_0\|_{s_0+2}, \delta) > 0$ and a unique solution $U \in C^0([0,T], \dot{\mathbf{H}}^{s'}) \cap C^1([0,T], \dot{\mathbf{H}}^{s'-2})$, $\forall s_0+2 \leq s' < s$, of (2.3.4) with initial datum U_0. Now take an open neighborhood $\mathcal{U} \subset \dot{\mathbf{H}}^s$ of U_0 such that $\forall V \in \mathcal{U}$ one has $\rho(V) \in \mathcal{Q}_{\frac{\delta}{2}}$ and $\|V\|_s \leq 2\|U_0\|_s$. Then there exists $\tilde{T} \in (0,T]$ such that the flow map of (2.3.4),

$$\Omega^t : \mathcal{U} \to \dot{\mathbf{H}}^s \cap \big\{U \in \dot{\mathbf{H}}^s : \rho(U) \in \mathcal{Q}_{\frac{\delta}{4}}\big\}, \quad U_0 \mapsto \Omega^t(U_0) := U(t),$$

is well defined for any $t \in [0,\tilde{T}]$, it satisfies the group property

$$\Omega^{t+\tau} = \Omega^t \circ \Omega^\tau, \quad \forall t, \tau, t+\tau \in [0,\tilde{T}], \tag{2.4.53}$$

and $\|\Omega^t(U_0)\|_s \leq \mathtt{C}(\|U_0\|_s)$ for all $U_0 \in \mathcal{U}$, $t \in [0,\tilde{T}]$. For simplicity of notation in the sequel we denote by T a time, independent of N, smaller than $\tilde{T}$.

Continuity of $t \mapsto U(t)$: We show that $U \in C^0([0,T], \dot{\mathbf{H}}^s)$. By (2.4.53), it is enough to prove that $t \mapsto U(t)$ is continuous in a neighborhood of $t=0$. This follows by Lemma 2.4.8, as U is the uniform limit of continuous functions.

Continuity of the flow map: We shall follow the method by [39, 22]. Let $U_0^n \to U_0 \in \dot{\mathbf{H}}^s$ and pick $\delta > 0$ such that $\rho(U_0^n)$, $\rho(U_0)$, $\rho(\Pi_N U_0^n)$, $\rho(\Pi_N U_0) \in \mathcal{Q}_\delta$, for any $n \geq n_0$, $N \geq N_0$ sufficiently large. Denote by $U^n, U \in C^0([0,T], \dot{\mathbf{H}}^s)$ the solutions of (2.3.4) with initial data U_0^n, respectively U_0, and $U_N(t) := \Omega^t(\Pi_N U_0)$, $U_N^n(t) := \Omega^t(\Pi_N U_0^n)$. Note that these solutions are well defined in $\dot{\mathbf{H}}^s$ up to a common time $T' \in (0,T]$, depending on $\|U_0\|_s$, thanks to Lemma 2.4.8. By triangular inequality we have, by (2.4.49), for any $n \geq n_0$, $N \geq N_0$,

$$\begin{aligned}
&\|U^n - U\|_{L^\infty_T\dot{\mathbf{H}}^s} \leq \|U^n - U_N^n\|_{L^\infty_T\dot{\mathbf{H}}^s} + \|U_N^n - U_N\|_{L^\infty_T\dot{\mathbf{H}}^s} + \|U_N - U\|_{L^\infty_T\dot{\mathbf{H}}^s}\\
&\leq \mathtt{C}(\|U_0\|_s)\big(\|(\mathrm{Id}-\Pi_N)U_0^n\|_s + \|(\mathrm{Id}-\Pi_N)U_0\|_s + N^{s_0+2-s}\big) + \|U_N^n - U_N\|_{L^\infty_T\dot{\mathbf{H}}^s}\\
&\leq \mathtt{C}(\|U_0\|_s)\big(\|(\mathrm{Id}-\Pi_N)U_0\|_s + N^{s_0+2-s}\big) + \mathtt{C}(\|U_0\|_s)\|U_0^n - U_0\|_s + \|U_N^n - U_N\|_{L^\infty_T\dot{\mathbf{H}}^s}.
\end{aligned} \tag{2.4.54}$$

For any $\varepsilon > 0$, since $s > s_0+2$, there exists $N_\varepsilon \in \mathbb{N}$ (independent of n) such that

$$\mathtt{C}(\|U_0\|_s)\big(\|(\mathrm{Id}-\Pi_{N_\varepsilon})U_0\|_s + N_\varepsilon^{s_0+2-s}\big) \leq \varepsilon/2. \tag{2.4.55}$$

Consider now the term $\|U_N^n - U_N\|_{L^\infty_T\dot{\mathbf{H}}^s}$. As $\Pi_N U_0, \Pi_N U_0^n \in \dot{C}^\infty$, the solutions $U_N(t)$, $U_N^n(t)$ belong actually to $\dot{\mathbf{H}}^{s+2}$. By interpolation and by item (ii) of Corollary 2.4.7 applied with $s \rightsquigarrow s+2$ one has, for $s+2 = \theta s_0 + (1-\theta)(s+2)$,

$$\begin{aligned}
\|U_{N_\varepsilon}^n - U_{N_\varepsilon}\|_{L^\infty_T\dot{\mathbf{H}}^s} &\leq \mathtt{C}\big(\|\Pi_{N_\varepsilon}U_0\|_{s+2}, \|\Pi_{N_\varepsilon}U_0^n\|_{s+2}\big)\|U_{N_\varepsilon}^n - U_{N_\varepsilon}\|^\theta_{L^\infty_T\dot{\mathbf{H}}^{s_0}}\\
&\leq \mathtt{C}\big(N_\varepsilon^2\|U_0\|_s\big)\|U_{N_\varepsilon}^n - U_{N_\varepsilon}\|^\theta_{L^\infty_T\dot{\mathbf{H}}^{s_0}}.
\end{aligned} \tag{2.4.56}$$

Arguing in the same way of the proof of (2.4.52) one obtains

$$\|U^n_{N_\varepsilon} - U_{N_\varepsilon}\|_{L^\infty_T \dot{\mathbf{H}}^{s_0}} \le \mathtt{C}\big(\|U_0\|_{s_0+2}\big)\|\Pi_{N_\varepsilon}(U^n_0 - U_0)\|_{s_0} . \tag{2.4.57}$$

By (2.4.54)-(2.4.57), we have $\limsup_{n\to\infty}\|U^n - U\|_{L^\infty_T \dot{\mathbf{H}}^s} \le \varepsilon, \forall \varepsilon \in (0,1)$. □

Chapter 3

Long-time stability of QHD

In this chapter we prove the long-time result outlined in Section 1.1.2. In Section 3.1 we provide an introduction to the problem, the statement of the main result already described in Section 1.1.2 as well as a more detailed literature.

3.1 Introduction to Chapter 3

We consider the quantum hydrodynamic system on an irrational torus of dimension 2 or 3

$$\begin{cases} \partial_t \rho = -\mathtt{m}\Delta\phi - \operatorname{div}(\rho\nabla\phi) \\ \partial_t \phi = -\frac{1}{2}|\nabla\phi|^2 - g(\mathtt{m}+\rho) + \dfrac{\kappa}{\mathtt{m}+\rho}\Delta\rho - \dfrac{\kappa}{2(\mathtt{m}+\rho)^2}|\nabla\rho|^2 , \end{cases} \tag{QHD}$$

where $\mathtt{m} > 0$, $\kappa > 0$, the function g belongs to $C^\infty(\mathbb{R}_+;\mathbb{R})$ and $g(\mathtt{m}) = 0$. The function $\rho(t,x)$ is such that $\rho(t,x) + \mathtt{m} > 0$ and it has zero average in x. The space variable x belongs to the irrational torus

$$\mathbb{T}^d_\nu := (\mathbb{R}/2\pi\nu_1\mathbb{Z}) \times \cdots \times (\mathbb{R}/2\pi\nu_d\mathbb{Z}) , \qquad d = 2,3 , \tag{3.1.1}$$

with $\nu = (\nu_1,\dots,\nu_d) \in [1,2]^d$. We assume the *strong* ellipticity condition

$$g'(\mathtt{m}) > 0. \tag{3.1.2}$$

We shall consider an initial condition (ρ_0,ϕ_0) having small size $\varepsilon \ll 1$ in the standard Sobolev space $H^s(\mathbb{T}^d_\nu)$ with $s \gg 1$. Since the equation has a quadratic nonlinear term, the local existence theory (which may be obtained in the spirit of [32, 72]) implies that the solution of (QHD) remains of size ε for times of magnitude $O(\varepsilon^{-1})$. The aim of the present chapter is to prove that, for *generic irrational tori*, the solution remains of size ε for longer times.

For $\phi \in H^s(\mathbb{T}^d_\nu)$ we define

$$\Pi_0\phi := \frac{1}{(2\pi)^d\nu_1\cdots\nu_d}\int_{\mathbb{T}^d_\nu}\phi(x)dx , \qquad \Pi_0^\perp := \mathrm{id} - \Pi_0 . \tag{3.1.3}$$

Our main result is the following.

Theorem 3.1.1. *Let $d = 2$ or $d = 3$. There exists $s_0 \equiv s_0(d) \in \mathbb{R}$ such that for almost all $\nu \in [1,2]^d$, for any $s \geq s_0$, $\mathtt{m} > 0$, $\kappa > 0$ there exist $C > 0$, $\varepsilon_0 > 0$ such that for any $0 < \varepsilon \leq \varepsilon_0$ we have the following. For any initial data $(\rho_0,\phi_0) \in H^s_0(\mathbb{T}^d_\nu) \times H^s(\mathbb{T}^d_\nu)$ such that*

$$\|\rho_0\|_{H^s(\mathbb{T}^d_\nu)} + \|\Pi_0^\perp\phi_0\|_{H^s(\mathbb{T}^d_\nu)} \leq \varepsilon , \tag{3.1.4}$$

there exists a unique solution of (QHD) *with* $(\rho(0), \phi(0)) = (\rho_0, \phi_0)$ *such that*

$$\begin{aligned} &(\rho(t), \phi(t)) \in C^0\big([0, T_\varepsilon); H^s(\mathbb{T}^d_\nu) \times H^s(\mathbb{T}^d_\nu)\big) \bigcap C^1\big([0, T_\varepsilon); H^{s-2}(\mathbb{T}^d_\nu) \times H^{s-2}(\mathbb{T}^d_\nu)\big)\,, \\ &\sup_{t\in[0,T_\varepsilon)} \Big(\|\rho(t,\cdot)\|_{H^s(\mathbb{T}^d_\nu)} + \|\Pi_0^\perp \phi(t,\cdot)\|_{H^s(\mathbb{T}^d_\nu)} \Big) \le C\varepsilon\,, \qquad T_\varepsilon \ge \varepsilon^{-1-\frac{1}{d-1}} \log^{-d-2}\big(1 + \varepsilon^{\frac{1}{1-d}}\big)\,. \end{aligned} \tag{3.1.5}$$

Derivation from Euler-Korteweg system. The (QHD) is derived from the compressible Euler-Korteweg system[1]

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \vec{u}) = 0 \\ \partial_t \vec{u} + \vec{u} \cdot \nabla \vec{u} + \nabla g(\rho) = \nabla\big(K(\rho)\Delta\rho + \frac{1}{2}K'(\rho)|\nabla\rho|^2\big)\,, \end{cases} \tag{EK}$$

where the function $\rho(t, x) > 0$ is the density of the fluid and $\vec{u}(t, x) \in \mathbb{R}^d$ is the time dependent velocity field; we assume that $K(\rho), g(\rho) \in C^\infty(\mathbb{R}_+; \mathbb{R})$ and that $K(\rho) > 0$. In particular, in (QHD), we assumed

$$K(\rho) = \frac{\kappa}{\rho}\,, \qquad \kappa \in \mathbb{R}_+\,. \tag{3.1.6}$$

We look for solutions $\vec{u}$ which stay irrotational for all times, i.e.,

$$\vec{u} = \vec{c}(t) + \nabla\phi\,, \quad \vec{c}(t) \in \mathbb{R}^d\,, \quad \vec{c}(t) = \frac{1}{(2\pi)^d \nu_1 \cdots \nu_d} \int_{\mathbb{T}^d_\nu} \vec{u}\, dx\,, \tag{3.1.7}$$

where $\phi : \mathbb{T}^d_\nu \to \mathbb{R}$ is a scalar potential. By the second equation in (EK) and using that $\mathrm{rot}\vec{u} = 0$ we deduce

$$\partial_t \vec{c}(t) = -\frac{1}{(2\pi)^d \nu_1 \cdots \nu_d} \int_{\mathbb{T}^d_\nu} \vec{u} \cdot \nabla \vec{u}\, dx = 0 \qquad \Rightarrow \qquad \vec{c}(t) = \vec{c}(0)\,.$$

The system (EK) is Galilean invariant, i.e., if $(\rho(t, x), \vec{u}(t, x))$ solves (EK) then also

$$\rho_{\vec{c}}(t, x) := \rho(t, x + \vec{c}t)\,, \qquad \vec{u}_{\vec{c}}(t, x) := \vec{u}(t, x + \vec{c}t) - \vec{c},$$

solves (EK). Then we can always assume that $\vec{u} = \nabla\phi$ for some scalar potential $\phi : \mathbb{T}^d_\nu \to \mathbb{R}$. The system (EK) reads

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \nabla\phi) = 0 \\ \partial_t \phi + \frac{1}{2}|\nabla\phi|^2 + g(\rho) = K(\rho)\Delta\rho + \frac{1}{2}K'(\rho)|\nabla\rho|^2\,. \end{cases} \tag{3.1.8}$$

Notice that the average

$$\frac{1}{(2\pi)^d \nu_1 \cdots \nu_d} \int_{\mathbb{T}^d_\nu} \rho(x)\, dx = \mathtt{m} \in \mathbb{R}\,, \tag{3.1.9}$$

is a constant of motion of (3.1.8). Notice also that the vector field of (3.1.8) depends only on $\Pi_0^\perp \phi$ (see (3.1.3)). In view of (3.1.9) we rewrite $\rho \rightsquigarrow \mathtt{m} + \rho$ where ρ is a function with zero average. Then, the system (3.1.8) (recall also (3.1.6)) becomes (QHD).

Phase space and notation. In this chapter we work with functions belonging to the Sobolev space

$$H^s(\mathbb{T}^d_\nu) := \Big\{ u(x) = \frac{1}{(2\pi)^{d/2}} \sum_{j\in\mathbb{Z}^d_\nu} u_j e^{\mathrm{i} j\cdot x} \ : \ \|u(\cdot)\|^2_{H^s(\mathbb{T}^d_\nu)} := \sum_{j\in\mathbb{Z}^d_\nu} \langle j \rangle^{2s} |u_j|^2 < +\infty \Big\}\,, \tag{3.1.10}$$

[1] Some authors prefer to write the second equation in terms of the current density $J := \rho\vec{u}$, see for instance [9].

where $\langle j\rangle := \sqrt{1+|j|^2}$ for $j \in \mathbb{Z}^d_\nu$ with $\mathbb{Z}^d_\nu := (\mathbb{Z}/\nu_1)\times\cdots\times(\mathbb{Z}/\nu_d)$. The natural phase space for (QHD) is $H^s_0(\mathbb{T}^d_\nu)\times \dot{H}^s(\mathbb{T}^d_\nu)$ where $\dot{H}^s(\mathbb{T}^d_\nu) := H^s(\mathbb{T}^d_\nu)/_\sim$ is the homogeneous Sobolev space obtained by the equivalence relation $\psi_1(x)\sim\psi_2(x)$ if and only if $\psi_1(x)-\psi_2(x)=c$ is a constant; $H^s_0(\mathbb{T}^d_\nu)$ is the subspace of $H^s(\mathbb{T}^d_\nu)$ of functions with zero average. Despite this fact we prefer to work with a couple of variable $(\rho,\phi)\in H^s_0(\mathbb{T}^d_\nu)\times H^s(\mathbb{T}^d_\nu)$ but at the end we control only the norm $\|\Pi_0^\perp\phi\|_{H^s(\mathbb{T}^d_\nu)}$ which in fact is the relevant quantity for (QHD). To lighten the notation we shall write $\|\cdot\|_{H^s_\nu}$ to denote $\|\cdot\|_{H^s(\mathbb{T}^d_\nu)}$.

In the following we will use the notation $A\lesssim B$ to denote $A\le CB$ where C is a positive constant depending on parameters fixed once for all, for instance d and s. We will emphasize by writing $\lesssim_q$ when the constant C depends on some other parameter q.

Ideas of the proof. The general (EK) is a system of quasi-linear equations. The case (QHD), i.e., the system (EK) with the particular choice (3.1.6), reduces, for small solutions, to a semi-linear equation, more precisely to a nonlinear Schrödinger equation. This is a consequence of the fact that the Madelung transform (introduced for the first time in the seminal work by Madelung [96]) is well defined for small solutions. In other words one can introduce the new variable $\psi := \sqrt{\mathtt{m}+\rho}\,e^{\mathrm{i}\phi/\hbar}$ (see Section 3.2 for details), where $\hbar = 2\sqrt{\kappa}$, obtaining the equation

$$\partial_t\psi = \mathrm{i}\Big(\frac{\hbar}{2}\Delta\psi - \frac{1}{\hbar}g(|\psi|^2)\psi\Big).$$

Since $g(\mathtt{m})=0$, such an equation has an equilibrium point at $\psi=\sqrt{\mathtt{m}}$. The study of the stability of small solutions for (QHD) is equivalent to the study of the stability of the variable $z=\psi-\sqrt{\mathtt{m}}$. The equation for the variable z reads

$$\partial_t z = -\mathrm{i}\big(\frac{\hbar|D|^2_\nu}{2}+\frac{\mathtt{m}g'(\mathtt{m})}{\hbar}\big)z - \mathrm{i}\frac{\mathtt{m}g'(\mathtt{m})}{\hbar}\bar{z} + f(z),$$

where f is a smooth function having a zero of order 2 at $z=0$, i.e., $|f(z)|\lesssim |z|^2$, and $|D|^2_\nu$ is the Fourier multiplier with symbol

$$|\xi|^2_\nu := \sum_{i=1}^d a_i|\xi_i|^2\,,\quad a_i := \nu_i^2\,,\quad \forall\,\xi\in\mathbb{Z}^d\,. \tag{3.1.11}$$

The aim is to use a Birkhoff normal form/modified energy technique in order to reduce the size of the non-linearity $f(z)$. To do that, it is convenient to perform some preliminary reductions. First of all we want to eliminate the addendum $-\mathrm{i}\frac{\mathtt{m}g'(\mathtt{m})}{\hbar}\bar{z}$. In other words we want to diagonalize the matrix

$$\mathcal{L} = \begin{pmatrix} \frac{\hbar}{2}|D|^2_\nu + \frac{1}{\hbar}\mathtt{m}g'(\mathtt{m}) & \frac{1}{\hbar}\mathtt{m}g'(\mathtt{m}) \\ \frac{1}{\hbar}\mathtt{m}g'(\mathtt{m}) & \frac{\hbar}{2}|D|^2_\nu + \frac{1}{\hbar}\mathtt{m}g'(\mathtt{m})\end{pmatrix}. \tag{3.1.12}$$

To achieve the diagonalization of this matrix it is necessary to rewrite the equation in a system of coordinates which does not involve the zero mode. We perform this reduction in Section 3.2.2: we use the gauge invariance of the equation as well as the L^2 norm preservation to eliminate the dynamics of the zero mode. This idea has been introduced for the first time in [65]. After the diagonalization of the matrix in (3.1.12) we end up with a diagonal, quadratic, semi-linear equation with dispersion law

$$\omega(j) := \sqrt{\frac{\hbar^2}{4}|j|^4_\nu + \mathtt{m}g'(\mathtt{m})|j|^2_\nu}\,, \tag{3.1.13}$$

where j is a vector in $\mathbb{Z}^d\setminus\{0\}$. At this point we are ready to define a suitable modified energy. Our primary aim is to control the derivative of the H^s-norm of the solution

$$\frac{d}{dt}\|\tilde{z}(t)\|^2_{H^s}\,, \tag{3.1.14}$$

where $\tilde{z}$ is the variable of the diagonalized system, for the longest time possible. Using the equation, such a quantity may be rewritten as the sum of tri-linear expressions in $\tilde{z}$. We perturb the Sobolev energy by expressions homogeneous of degree at least 3 such that their time derivatives cancel out the main contribution (i.e., the one coming from cubic terms) in (3.1.14), up to remainders of higher order. In trying to do this small dividers appear, i.e. denominators of the form

$$\pm\omega(j_1) \pm \omega(j_2) \pm \omega(j_3) .$$

It is fundamental that the perturbations we define is bounded by some power of $\|\tilde{z}\|_{H^s}$, with the same s in (3.1.14), otherwise we obtain an estimate with loss of derivatives. Therefore we need to impose some lower bounds on the small dividers. Here it enters in the game the irrationality of the torus ν. We prove indeed that for almost any $\nu \in [1,2]^d$, there exists $\gamma > 0$ such that

$$| \pm \omega(j_1) \pm \omega(j_2) \pm \omega(j_3)| \geq \frac{\gamma}{\mu_1^{d-1} \log^{d+1}\left(1+\mu_1^2\right)\mu_3^{M(d)}} ,$$

if $\pm j_1 \pm j_2 \pm j_3 = 0$, we denoted by $M(d)$ a positive constant depending on the dimension d and μ_i the i-st largest integer among $|j_1|, |j_2|$ and $|j_3|$. It is nowadays well known, see for instance Section 1.1.5, that the power of μ_3 is not dangerous if we work in H^s with s big enough. Unfortunately we have also a power of the highest frequency μ_1 which represents, in principle, a loss of derivatives. However, this loss of derivatives may be transformed in a loss of length of the lifespan through partition of frequencies, as done for instance in [56, 87, 68, 23].

Some comments. As already mentioned, an estimate on small divisors involving only powers of μ_3 is not dangerous. We may obtain such an estimate when the equation is considered on the squared torus $\mathbb{T}^d$, using as a parameter the mass $\mathtt{m}$. In this case, indeed, one can obtain better estimates by following the proof in [65]. This is a consequence of the fact that the set of differences of eigenvalues is discrete. This is not the case of irrational tori with fixed mass, where the set of eigenvalues is not discrete. Having estimates involving only μ_3 one could actually prove an almost-global stability. More precisely one can prove, for instance, that there exists a zero Lebesgue measure set $\mathcal{N} \subset [1,+\infty)$, such that if $\mathtt{m}$ is in $[1,+\infty) \setminus \mathcal{N}$, then for any $N \geq 1$ if the initial condition is sufficiently regular (w.r.t. N) and of size ε sufficiently small (w.r.t. N) then the solution stays of size ε for a time of order ε^{-N}. The proof follows the lines of classical papers such as [15, 19, 17] by using the Hamiltonian structure of the equation. More precisely, the system (QHD) can be written in the form

$$\partial_t \begin{pmatrix} \rho \\ \phi \end{pmatrix} = X_H(\rho,\phi) = \begin{pmatrix} \partial_\phi H(\rho,\phi) \\ -\partial_\rho H(\rho,\phi) \end{pmatrix} , \tag{3.1.15}$$

where ∂ denotes the L^2-gradient and $H(\rho,\phi)$ is the Hamiltonian function

$$H(\rho,\phi) = \frac{1}{2}\int_{\mathbb{T}^d_\nu} (\mathtt{m}+\rho)|\nabla\phi|^2 dx + \int_{\mathbb{T}^d_\nu} \Big(\frac{1}{2}\frac{\kappa}{\mathtt{m}+\rho}|\nabla\rho|^2 + G(\mathtt{m}+\rho)\Big) dx \tag{3.1.16}$$

where $G'(\rho) = g(\rho)$.

We do not know if the solution of (QHD) are globally defined. There are positive answers in the case that the equation is posed on the Euclidean space $\mathbb{R}^d$ with $d \geq 3$, see for instance [13] for strong global solutions arising from small initial data (the local well posedness was previously analyzed by Benzoni Gavage, Danchin and Descombes [22]). Exploiting the Madelung transformation Antonelli-Marcati [10] proved the existence of global in time weak solutions of finite energy. Here the dispersive character of the equation is taken into account. An overview of recent results, a discussion of the Madelung transform including vacuum regions can be found in Antonelli-Hientzsch-Marcati-Zheng [12] see also [9] and reference therein. It is worth mentioning also the scattering result for the Gross–Pitaevskii equation [78]. Since we are considering the equation on a compact manifold, the dispersive estimates are not available.

3.2 From (QHD) to nonlinear Schrödinger

3.2.1 Madelung transform

For $\lambda \in \mathbb{R}_+$, we define the change of variable (*Madelung transform*)

$$\psi := \mathcal{M}_\psi(\rho,\phi) := \sqrt{\mathtt{m}+\rho}\, e^{\mathrm{i}\lambda\phi}\,, \quad \overline{\psi} := \mathcal{M}_{\overline{\psi}}(\rho,\phi) := \sqrt{\mathtt{m}+\rho}\, e^{-\mathrm{i}\lambda\phi}\,. \qquad (\mathcal{M})$$

Notice that the inverse map has the form

$$\mathtt{m}+\rho = \mathcal{M}_\rho^{-1}(\psi,\overline{\psi}) := |\psi|^2\,, \qquad \phi = \mathcal{M}_\phi^{-1}(\psi,\overline{\psi}) := \frac{1}{\lambda}\arctan\left(\frac{-\mathrm{i}(\psi-\overline{\psi})}{\psi+\overline{\psi}}\right). \qquad (3.2.1)$$

In the following lemma we provide a well-posedness result for the Madelung transform.

Lemma 3.2.1. *Define*

$$\frac{1}{4\lambda^2} = \kappa\,, \qquad \hbar := \frac{1}{\lambda} = 2\sqrt{\kappa}\,. \qquad (3.2.2)$$

The following holds.
(i) *Let* $s > \frac{d}{2}$ *and*

$$\delta := \frac{1}{\mathtt{m}}\|\rho\|_{H^s_\nu} + \frac{1}{\sqrt{\kappa}}\|\Pi_0^\perp\phi\|_{H^s_\nu} \quad \sigma := \Pi_0\phi\,.$$

There is $C = C(s) > 1$ *such that, if* $C(s)\delta \le 1$, *then the function* ψ *in* ($\mathcal{M}$) *satisfies*

$$\|\psi - \sqrt{\mathtt{m}}e^{\mathrm{i}\lambda\sigma}\|_{H^s_\nu} \le 2\sqrt{\mathtt{m}}\delta\,. \qquad (3.2.3)$$

(ii) *Define*

$$\delta' := \inf_{\sigma\in\mathbb{T}}\|\psi - \sqrt{\mathtt{m}}e^{\mathrm{i}\sigma}\|_{H^s_\nu}\,.$$

There is $C' = C'(s) > 1$ *such that, if* $C'(s)\delta'(\sqrt{\mathtt{m}})^{-1} \le 1$, *then the functions* ρ,

$$\frac{1}{\mathtt{m}}\|\rho\|_{H^s_\nu} + \frac{1}{\sqrt{\kappa}}\|\Pi_0^\perp\phi\|_{H^s_\nu} \le 8\frac{1}{\sqrt{\mathtt{m}}}\delta'\,. \qquad (3.2.4)$$

Proof. The bound (3.2.3) follows by ($\mathcal{M}$) and classical estimates on composition operators on Sobolev spaces (see for instance [102]). Let us check the (3.2.4). By the first of (3.2.1), for any $\sigma \in \mathbb{T}$, we have

$$\|\rho\|_{H^s_\nu} \le \|\sqrt{\mathtt{m}}(\psi e^{-\mathrm{i}\sigma} - \sqrt{\mathtt{m}})\|_{H^s_\nu} + \|\sqrt{\mathtt{m}}(\overline{\psi}e^{\mathrm{i}\sigma} - \sqrt{\mathtt{m}})\|_{H^s_\nu} + \|(\psi e^{-\mathrm{i}\sigma} - \sqrt{\mathtt{m}})(\overline{\psi}e^{\mathrm{i}\sigma} - \sqrt{\mathtt{m}})\|_{H^s_\nu} \qquad (3.2.5)$$

$$\le \mathtt{m}\Big(\frac{2}{\sqrt{\mathtt{m}}}\|\psi - \sqrt{\mathtt{m}}e^{\mathrm{i}\sigma}\|_{H^s_\nu} + \big(\frac{1}{\sqrt{\mathtt{m}}}\|\psi - \sqrt{\mathtt{m}}e^{\mathrm{i}\sigma}\|_{H^s_\nu}\big)^2\Big)\,. \qquad (3.2.6)$$

Therefore, by the arbitrariness of σ and using that $(\sqrt{\mathtt{m}})^{-1}\delta' \ll 1$, one deduces

$$\frac{1}{\mathtt{m}}\|\rho\|_{H^s_\nu} \le 3\frac{1}{\sqrt{\mathtt{m}}}\delta'\,.$$

Moreover we note that

$$\|(\psi-\overline{\psi})(\psi+\overline{\psi})^{-1}\|_{H^s_\nu} \le 2\frac{1}{\sqrt{\mathtt{m}}}\|\psi - \sqrt{\mathtt{m}}\|_{H^s_\nu}\,.$$

Then by the second in (3.2.1), (3.2.2), composition estimates on Sobolev spaces and the smallness condition $(\sqrt{\mathtt{m}})^{-1}\delta' \ll 1$, one deduces, for any $\sigma \in \mathbb{T}$ such that $(\sqrt{\mathtt{m}})^{-1}\|\psi - \sqrt{\mathtt{m}}e^{\mathrm{i}\sigma}\|_{H^s_\nu} \ll 1$, that

$$\begin{aligned}\frac{1}{\sqrt{\kappa}}\|\Pi_0^\perp\phi\|_{H^s_\nu} + 2\|\Pi_0^\perp \arctan\big(\frac{-\mathrm{i}(\psi-\overline{\psi})}{\psi+\overline{\psi}}\big)\|_{H^s_\nu} \\ = 2\|\Pi_0^\perp \arctan\big(\frac{-\mathrm{i}(\psi e^{-\mathrm{i}\sigma}-\overline{\psi}e^{\mathrm{i}\sigma})}{\psi e^{-\mathrm{i}\sigma}+\overline{\psi}e^{\mathrm{i}\sigma}}\big)\|_{H^s_\nu} \\ \le \frac{5}{\sqrt{\mathtt{m}}}\|\psi - \sqrt{\mathtt{m}}e^{\mathrm{i}\sigma}\|_{H^s_\nu}\,.\end{aligned}$$

Therefore the (3.2.4) follows. □

We now rewrite equation (QHD) in the variable $(\psi, \overline{\psi})$.

Lemma 3.2.2. *Let $(\rho, \phi) \in H^s_0(\mathbb{T}^d_\nu) \times H^s(\mathbb{T}^d_\nu)$ be a solution of (QHD) defined over a time interval $[0, T]$, $T > 0$, such that*

$$\sup_{t\in[0,T)}\Big(\frac{1}{\mathtt{m}}\|\rho(t,\cdot)\|_{H^s_\nu} + \frac{1}{\sqrt{\kappa}}\|\Pi_0^\perp\phi(t,\cdot)\|_{H^s_\nu}\Big) \le \varepsilon \tag{3.2.7}$$

for some $\varepsilon > 0$ small enough. Then the function ψ defined in ($\mathcal{M}$) solves

$$\begin{cases}\partial_t\psi = -\mathrm{i}\big(-\frac{\hbar}{2}\Delta\psi + \frac{1}{\hbar}g(|\psi|^2)\psi\big)\\ \psi(0) = \sqrt{\mathtt{m}+\rho(0)}e^{\mathrm{i}\phi(0)}\,.\end{cases} \tag{3.2.8}$$

Remark 3.2.3. We remark that the assumption of Lemma 3.2.2 can be verified in the same spirit of the local well-posedness results in [72] and [32].

Proof of Lemma 3.2.2. The smallness condition (3.2.7) implies that the function ψ is well-defined and satisfies a bound like (3.2.3). We first note

$$\nabla\psi = (\frac{1}{2\sqrt{\mathtt{m}+\rho}}\nabla\rho + \mathrm{i}\lambda\sqrt{\mathtt{m}+\rho}\nabla\phi)e^{\mathrm{i}\lambda\phi}\,, \tag{3.2.9}$$

$$\frac{1}{2\lambda^2}|\nabla\psi|^2 = \frac{1}{2}\frac{1}{4\lambda^2}\frac{1}{\mathtt{m}+\rho}|\nabla\rho|^2 + \frac{1}{2}(\mathtt{m}+\rho)|\nabla\phi|^2\,. \tag{3.2.10}$$

Moreover, using (QHD), (3.2.2), ($\mathcal{M}$) and that

$$\mathrm{div}(\rho\nabla\phi) = \nabla\rho\cdot\nabla\phi + \rho\Delta\phi\,,$$

we get

$$\begin{aligned}\partial_t\psi = \mathrm{i}e^{\mathrm{i}\lambda\phi}\Big(\frac{\Delta\rho}{4\lambda\sqrt{\mathtt{m}+\rho}} - \frac{|\nabla\rho|^2}{8\lambda(\mathtt{m}+\rho)^{\frac{3}{2}}} + \frac{\mathrm{i}\sqrt{\mathtt{m}+\rho}\Delta\phi}{2} - \frac{\sqrt{\mathtt{m}+\rho}\lambda|\nabla\phi|^2}{2} + \frac{\mathrm{i}\nabla\rho\cdot\nabla\phi}{2\sqrt{\mathtt{m}+\rho}}\Big) \\ - \mathrm{i}\lambda g(|\psi|^2)\psi\,.\end{aligned} \tag{3.2.11}$$

On the other hand, by (3.2.9), we have

$$\mathrm{i}\Delta\psi = \mathrm{i}e^{\lambda\phi}\Big(\frac{\Delta\rho}{2\sqrt{\mathtt{m}+\rho}} - \frac{|\nabla\rho|^2}{4(\mathtt{m}+\rho)^{\frac{3}{2}}} + \mathrm{i}\lambda\sqrt{\mathtt{m}+\rho}\Delta\phi - \lambda^2\sqrt{\mathtt{m}+\rho}|\nabla\phi|^2 + \frac{\mathrm{i}\lambda\nabla\rho\cdot\nabla\phi}{\sqrt{\mathtt{m}+\rho}}\Big)\,. \tag{3.2.12}$$

Therefore, by (3.2.11), (3.2.12), we deduce

$$\partial_t\psi = \frac{\mathrm{i}}{2\lambda}\Delta\psi - \mathrm{i}\lambda g(|\psi|^2)\psi\,, \quad \text{where} \quad \frac{1}{\lambda} = \hbar\,, \tag{3.2.13}$$

which is the (3.2.8). □

Notice that the (3.2.8) is an Hamiltonian equation of the form

$$\partial_t \psi = -\mathrm{i}\partial_{\overline{\psi}}\mathcal{H}(\psi,\overline{\psi}), \qquad \mathcal{H}(\psi,\overline{\psi}) = \int_{\mathbb{T}^d_\nu} \big(\frac{\hbar}{2}|\nabla\psi|^2 + \frac{1}{\hbar}G(|\psi|^2)\big)dx, \tag{3.2.14}$$

where $\partial_{\overline{\psi}} = (\partial_{\mathrm{Re}\,\psi} + \mathrm{i}\partial_{\mathrm{Im}\,\psi})/2$ and the function G is a primitive of g. The Poisson bracket is

$$\{\mathcal{H},\mathcal{G}\} := -\mathrm{i}\int_{\mathbb{T}^d_\nu} \partial_\psi\mathcal{H}\partial_{\overline{\psi}}\mathcal{G} - \partial_{\overline{\psi}}\mathcal{H}\partial_\psi\mathcal{G}dx. \tag{3.2.15}$$

3.2.2 Elimination of the zero mode

In the following it would be convenient to rescale the space variables $x \in \mathbb{T}^d_\nu \rightsquigarrow \nu\cdot x$ with $x \in \mathbb{T}^d$ and work with functions belonging to the Sobolev space $H^s(\mathbb{T}^d) := H^s(\mathbb{T}^d_1)$, i.e., the Sobolev space in (3.1.10) with $\nu = (1,\dots,1)$. By using the notation $\psi = (2\pi)^{-\frac{d}{2}}\sum_{j\in\mathbb{Z}^d}\psi_j e^{\mathrm{i}j\cdot x}$, we introduce the set of variables

$$\begin{cases}\psi_0 = \alpha e^{-\mathrm{i}\theta} & \alpha\in[0,+\infty),\,\theta\in\mathbb{T}\\ \psi_j = z_j e^{-\mathrm{i}\theta} & j\neq 0,\end{cases} \tag{3.2.16}$$

which are the polar coordinates for $j = 0$ and a phase translation for $j \neq 0$. Rewriting (3.2.14) in Fourier coordinates one has

$$\mathrm{i}\partial_t\psi_j = \partial_{\overline{\psi_j}}\mathcal{H}(\psi,\overline{\psi}), \quad j\in\mathbb{Z}^d, \tag{3.2.17}$$

where $\mathcal{H}$ is defined in (3.2.14). We define also the zero mean variable

$$z := (2\pi)^{-\frac{d}{2}}\sum_{j\in\mathbb{Z}^d\setminus\{0\}} z_j e^{\mathrm{i}j\cdot x}. \tag{3.2.18}$$

By (3.2.16) and (3.2.18) one has

$$\psi = (\alpha+z)e^{\mathrm{i}\theta}, \tag{3.2.19}$$

and it is easy to prove that the quantity

$$\mathtt{m} := \sum_{j\in\mathbb{Z}^d}|\psi_j|^2 = \alpha^2 + \sum_{j\neq 0}|z_j|^2$$

is a constant of motion for (3.2.8). Using (3.2.16), one can completely recover the real variable α in terms of $\{z_j\}_{j\in\mathbb{Z}^d\setminus\{0\}}$ as

$$\alpha = \sqrt{\mathtt{m} - \sum_{j\neq 0}|z_j|^2}. \tag{3.2.20}$$

Note also that the (ρ,ϕ) variables in (3.2.1) do not depend on the angular variable θ defined above. This implies that system (QHD) is completely described by the complex variable z. On the other hand, using

$$\partial_{\overline{\psi_j}}\mathcal{H}(\psi e^{\mathrm{i}\theta},\overline{\psi e^{\mathrm{i}\theta}}) = \partial_{\overline{\psi_j}}\mathcal{H}(\psi,\overline{\psi})e^{\mathrm{i}\theta},$$

one obtains

$$\begin{cases}\mathrm{i}\partial_t\alpha + \partial_t\theta\alpha = \Pi_0\left(g(|\alpha+z|^2)(\alpha+z)\right)\\ \mathrm{i}\partial_t z_j + \partial_t\theta z_j = \frac{\partial\mathcal{H}}{\partial\overline{\psi}_j}(\alpha+z,\alpha+\overline{z}).\end{cases} \tag{3.2.21}$$

Taking the real part of the first equation in (3.2.21) we obtain

$$\partial_t \theta = \frac{1}{\alpha}\Pi_0\left(\frac{1}{\hbar}g(|\alpha+z|^2)\mathrm{Re}\,(\alpha+z)\right) = \frac{1}{2\alpha}\partial_{\overline{\alpha}}\mathcal{H}(\alpha,z,\overline{z}), \tag{3.2.22}$$

where (recall (3.1.11))

$$\tilde{\mathcal{H}}(\alpha,z,\overline{z}) := \frac{\hbar}{2}\int_{\mathbb{T}^d}|D|_\nu^2 z\cdot\overline{z}\mathrm{d}x + \frac{1}{\hbar}\int_{\mathbb{T}^d}G(|\alpha+z|^2)\,\mathrm{d}x. \tag{3.2.23}$$

By (3.2.22), (3.2.21) and using that

$$\partial_{\overline{\psi_j}}\mathcal{H}(\alpha+z,\alpha+\overline{z}) = \partial_{\overline{z_j}}\tilde{\mathcal{H}}(\alpha,z,\overline{z}),$$

one obtains

$$\mathrm{i}\partial_t z_j = \partial_{\overline{z_j}}\tilde{\mathcal{H}}(\alpha,z,\overline{z}) - \frac{z_j}{2\alpha}\partial_\alpha\tilde{\mathcal{H}}(\alpha,z,\overline{z}) = \partial_{\overline{z_j}}\mathcal{K}_{\mathtt{m}}(z,\overline{z}), \quad j\neq 0, \tag{3.2.24}$$

where

$$\mathcal{K}_{\mathtt{m}}(z,\overline{z}) := \tilde{\mathcal{H}}(\alpha,z,\overline{z})_{|\alpha=\sqrt{\mathtt{m}-\sum_{j\neq 0}|z_j|^2}}.$$

We resume the above discussion in the following lemma.

Lemma 3.2.4. *The following holds.*

(i) *Let* $s>\frac{d}{2}$ *and*

$$\delta := \frac{1}{\mathtt{m}}\|\rho\|_{H^s} + \frac{1}{\sqrt{\kappa}}\|\Pi_0^\perp\phi\|_{H^s}, \quad \theta := \Pi_0\phi.$$

There is $C=C(s)>1$ *such that, if* $C(s)\delta\leq 1$*, then the function* z *in* (3.2.18) *satisfies*

$$\|z\|_{H^s} \leq 2\sqrt{\mathtt{m}}\delta. \tag{3.2.25}$$

(ii) *Define*

$$\delta' := \|z\|_{H^s}.$$

There is $C'=C'(s)>1$ *such that, if* $C'(s)\delta'(\sqrt{\mathtt{m}})^{-1}\leq 1$*, then the functions* ρ*,*

$$\frac{1}{\mathtt{m}}\|\rho\|_{H^s} + \frac{1}{\sqrt{\kappa}}\|\Pi_0^\perp\phi\|_{H^s} \leq 16\frac{1}{\sqrt{\mathtt{m}}}\delta'. \tag{3.2.26}$$

(iii) *Let* $(\rho,\phi)\in H_0^s(\mathbb{T}_\nu^d)\times H^s(\mathbb{T}_\nu^d)$ *be a solution of* (QHD) *defined over a time interval* $[0,T]$*,* $T>0$*, such that*

$$\sup_{t\in[0,T)}\left(\frac{1}{\mathtt{m}}\|\rho(t,\cdot)\|_{H^s} + \frac{1}{\sqrt{\kappa}}\|\Pi_0^\perp\phi(t,\cdot)\|_{H^s}\right)\leq\varepsilon$$

for some $\varepsilon>0$ *small enough. Then the function* $z\in H_0^s(\mathbb{T}_\nu^d)$ *defined in* (3.2.18) *solves* (3.2.24).

Proof. We note that

$$\|z\|_{H^s} = \|\Pi_0^\perp\psi\|_{H^s} \leq \|\psi-\sqrt{m}e^{\mathrm{i}\theta}\|_{H_s} \overset{(3.2.3)}{\leq} 2\sqrt{\mathtt{m}}\delta, \tag{3.2.27}$$

which proves (3.2.25). In order to prove (3.2.26) we note that

$$\begin{aligned}\inf_{\sigma\in\mathbb{T}}\|\psi-\sqrt{\mathtt{m}}e^{\mathrm{i}\sigma}\|_{H^s} &\leq \|\psi-\sqrt{\mathtt{m}}e^{\mathrm{i}\theta}\|_{H^s} = \|\alpha-\sqrt{\mathtt{m}}+z\|_{H^s}\\ &\leq \left|\sqrt{\mathtt{m}-\|z\|_{L^2}^2}-\sqrt{\mathtt{m}}\right| + \|z\|_{H^s} \leq 2\delta',\end{aligned}$$

so that the (3.2.26) follows by (3.2.4). The point (iii) follows by (3.2.21) and (3.2.22). □

Remark 3.2.5. Using (3.2.1) and (3.2.19) one can study the system (QHD) near the equilibrium point $(\rho,\phi)=(0,0)$ by studying the complex Hamiltonian system

$$\mathrm{i}\partial_t z = \partial_{\overline{z}}\mathcal{K}_{\mathtt{m}}(z,\overline{z}) \tag{3.2.28}$$

near the equilibrium $z=0$. Note also that the natural phase-space for (3.2.28) is the complex Sobolev space $H^s_0(\mathbb{T}^d;\mathbb{C})$, $s\in\mathbb{R}$, of complex Sobolev functions with zero mean.

3.2.3 Taylor expansion of the Hamiltonian

In order to study the stability of $z=0$ for (3.2.28) it is useful to expand $\mathcal{K}_{\mathtt{m}}$ at $z=0$. We have

$$\begin{aligned}\mathcal{K}_{\mathtt{m}}(z,\overline{z}) &= \frac{\hbar}{2}\int_{\mathbb{T}^d}|D|^2_\nu z\cdot\overline{z}\,\mathrm{d}x+\frac{1}{\hbar}\int_{\mathbb{T}^d}G\Big(\Big|\sqrt{\mathtt{m}-\sum_{j\neq 0}|z_j|^2}+z\Big|^2\Big)\,\mathrm{d}x\\ &=(2\pi)^d\frac{G(\mathtt{m})}{\hbar}+\mathcal{K}^{(2)}_{\mathtt{m}}(z,\overline{z})+\sum_{r=3}^{N-1}\mathcal{K}^{(r)}_{\mathtt{m}}(z,\overline{z})+R^{(N)}(z,\overline{z}),\end{aligned} \tag{3.2.29}$$

where, in view of the identity $G'(\mathtt{m})=g(\mathtt{m})=0$, the quadratic Hamiltonian has the form

$$\mathcal{K}^{(2)}_{\mathtt{m}}(z,\overline{z})=\frac{1}{2}\int_{\mathbb{T}^d}\frac{\hbar}{2}|D|^2_\nu z\cdot\overline{z}\,\mathrm{d}x+\frac{g'(\mathtt{m})\mathtt{m}}{\hbar}\int_{\mathbb{T}^d}\frac{1}{2}(z+\overline{z})^2\,\mathrm{d}x, \tag{3.2.30}$$

for any $r=3,\cdots,N-1$, $\mathcal{K}^{(r)}_{\mathrm{m}}(z,\overline{z})$ is an homogeneous multi-linear Hamiltonian function of degree r of the form

$$\mathcal{K}^{(r)}_{\mathtt{m}}(z,\overline{z})=\sum_{\substack{\sigma\in\{-1,1\}^r,\ j\in(\mathbb{Z}^d\setminus\{0\})^r\\ \sum_{i=1}^r\sigma_i j_i=0}}(\mathcal{K}^{(r)}_{\mathtt{m}})_{\sigma,j}z^{\sigma_1}_{j_1}\cdots z^{\sigma_r}_{j_r},\qquad |(\mathcal{K}^{(r)}_{\mathtt{m}})_{\sigma,j}|\lesssim_r 1,$$

and

$$\|X_{R^{(N)}}(z)\|_{H^s}\lesssim_s\|z\|^{r-1}_{H^s},\qquad\forall z\in B_1(H^s_0(\mathbb{T}^d;\mathbb{C})). \tag{3.2.31}$$

The vector field of the Hamiltonian in (3.2.29) has the form (recall (3.1.15))

$$\partial_t\begin{bmatrix}z\\ \overline{z}\end{bmatrix}=\begin{bmatrix}-\mathrm{i}\partial_{\overline{z}}\mathcal{K}_{\mathtt{m}}\\ \mathrm{i}\partial_z\mathcal{K}_{\mathtt{m}}\end{bmatrix}=-\mathrm{i}\begin{pmatrix}\frac{\hbar|D|^2_\nu}{2}+\frac{\mathtt{m}g'(\mathtt{m})}{\hbar} & \frac{\mathtt{m}g'(\mathtt{m})}{\hbar}\\ -\frac{\mathtt{m}g'(\mathtt{m})}{\hbar} & -\frac{\hbar|D|^2_\nu}{2}-\frac{\mathtt{m}g'(\mathtt{m})}{\hbar}\end{pmatrix}\begin{bmatrix}z\\ \overline{z}\end{bmatrix}+\sum_{r=3}^{N-1}\begin{bmatrix}-\mathrm{i}\partial_{\overline{z}}\mathcal{K}^{(r)}_{\mathtt{m}}\\ \mathrm{i}\partial_z\mathcal{K}^{(r)}_{\mathtt{m}}\end{bmatrix}+\begin{bmatrix}-\mathrm{i}\partial_{\overline{z}}R^{(N)}\\ \mathrm{i}\partial_z R^{(N)}\end{bmatrix}. \tag{3.2.32}$$

Let us now introduce the 2×2 matrix of operators

$$\mathcal{C}:=\frac{1}{\sqrt{2\omega(D)A(D,\mathtt{m})}}\begin{pmatrix}A(D,\mathtt{m}) & -\frac{1}{2}\mathtt{m}g'(\mathtt{m})\\ -\frac{1}{2}\mathtt{m}g'(\mathtt{m}) & A(D,\mathtt{m})\end{pmatrix},$$

with

$$A(D,\mathtt{m}):=\omega(D)+\tfrac{\hbar}{2}|D|^2_\nu+\tfrac{1}{2}\mathtt{m}g'(\mathtt{m}),$$

and where $\omega(D)$ is the Fourier multiplier with symbol

$$\omega(j):=\sqrt{\frac{\hbar^2}{4}|j|^4_\nu+\mathtt{m}g'(\mathtt{m})|j|^2_\nu}. \tag{3.2.33}$$

Notice that, by using (C.0.13), the matrix $\mathcal{C}$ is bounded, invertible and symplectic, with estimates

$$\|\mathcal{C}^{\pm 1}\|_{\mathcal{L}(H^s_0\times H^s_0,\,H^s_0\times H^s_0)}\le 1+\sqrt{k}\beta,\quad \beta:=\frac{\mathtt{m}g'(\mathtt{m})}{k}. \tag{3.2.34}$$

Consider the change of variables

$$\begin{bmatrix} w \\ \overline{w} \end{bmatrix} := \mathcal{C}^{-1} \begin{bmatrix} z \\ \overline{z} \end{bmatrix}. \tag{3.2.35}$$

then the Hamiltonian (3.2.29) reads

$$\begin{aligned} &\widetilde{\mathcal{K}}_{\mathtt{m}}(w,\overline{w}) := \widetilde{\mathcal{K}}_{\mathtt{m}}^{(2)}(w,\overline{w}) + \widetilde{\mathcal{K}}_{\mathtt{m}}^{(3)}(w,\overline{w}) + \widetilde{\mathcal{K}}_{\mathtt{m}}^{(\geq 4)}(w,\overline{w}), \\ &\widetilde{\mathcal{K}}_{\mathtt{m}}^{(2)}(w,\overline{w}) := \mathcal{K}_{\mathtt{m}}^{(2)}\Big(\mathcal{C}\begin{bmatrix} w \\ \overline{w} \end{bmatrix}\Big) := \frac{1}{2}\int_{\mathbb{T}^d} \omega(D) z \cdot \overline{z} \mathrm{d}x, \\ &\widetilde{\mathcal{K}}_{\mathtt{m}}^{(3)}(w,\overline{w}) := \mathcal{K}_{\mathtt{m}}^{(3)}\Big(\mathcal{C}\begin{bmatrix} w \\ \overline{w} \end{bmatrix}\Big), \qquad \widetilde{\mathcal{K}}_{\mathtt{m}}^{(\geq 4)}(w,\overline{w}) := \sum_{r=4}^{N-1} \mathcal{K}_{\mathtt{m}}^{(r)}\Big(\mathcal{C}\begin{bmatrix} w \\ \overline{w} \end{bmatrix}\Big) + R^{(N)}\Big(\mathcal{C}\begin{bmatrix} w \\ \overline{w} \end{bmatrix}\Big). \end{aligned} \tag{3.2.36}$$

Therefore system (3.2.32) becomes

$$\partial_t w = -\mathrm{i}\omega(D) w - \mathrm{i}\partial_{\overline{w}} \widetilde{\mathcal{K}}_{\mathtt{m}}^{(3)}(w,\overline{w}) - \mathrm{i}\partial_{\overline{w}} \widetilde{\mathcal{K}}_{\mathtt{m}}^{(\geq 4)}(w,\overline{w}). \tag{3.2.37}$$

3.3 Small divisors

As explained in the introduction we shall study the long time behavior of solutions of (3.2.37) by means of Birkhoff normal form approach. Therefore we have to provide suitable non resonance conditions among linear frequencies of oscillations $\omega(j)$ in (3.2.33). This is actually the aim of this section.

Let $\mathtt{a} = (\mathtt{a}_1, \ldots, \mathtt{a}_d) = (\nu_1^2, \ldots, \nu_d^2) \in (1,4)^d$, $d = 2,3$. If $j \in \mathbb{Z}^d \setminus \{0\}$ we define

$$|j|_{\mathtt{a}}^2 = \sum_{k=1}^{d} \mathtt{a}_k j_k^2. \tag{3.3.1}$$

We consider the dispersion relation

$$\omega(j) := \sqrt{k|j|_{\mathtt{a}}^4 + \mathtt{m}g'(\mathtt{m})|j|_{\mathtt{a}}^2}, \tag{3.3.2}$$

we note that $\omega(j) = \sqrt{k}\big(|j|_{\mathtt{a}}^2 + \frac{\beta}{2} - \frac{\beta^2}{8}\frac{1}{|j|_{\mathtt{a}}^2} + O(\frac{\beta^3}{|j|_{\mathtt{a}}^4})\big)$ for any j big enough with respect to $\beta := \frac{\mathtt{m}g'(\mathtt{m})}{k}$.

Throughout this section we assume, without loss of generality, $|j_1|_{\mathtt{a}} \geq |j_2|_{\mathtt{a}} \geq |j_3|_{\mathtt{a}} > 0$, for any j_i in $\mathbb{Z}^d$, moreover, in order to lighten the notation, we adopt the convention $\omega_i := \omega(j_i)$ for any $i = 1,2,3$. The main result is the following.

Proposition 3.3.1 (Measure estimates)**.** *There exists a full Lebesgue measure set $\mathfrak{A} \subset (1,4)^d$ such that for any $\mathtt{a} \in \mathfrak{A}$ there exists $\gamma > 0$ such that the following holds true. If $\sigma_1 j_1 + \sigma_2 j_2 + \sigma_3 j_3 = 0$, $\sigma_i \in \{\pm 1\}$ we have the estimate*

$$k^{-\frac{1}{2}}\big|\sigma_3\omega_3 + \sigma_2\omega_2 + \sigma_1\omega_1\big| \gtrsim_d \begin{cases} \frac{\gamma}{|j_1|^{d-1}\log^{d+1}(1+|j_1|^2)|j_3|^{M(d)}}, & \text{if } \sigma_1\sigma_2 = -1 \\ 1, & \text{if } \sigma_1\sigma_2 = 1 \end{cases}. \tag{3.3.3}$$

for any $|j_1|_{\mathtt{a}} \geq |j_2|_{\mathtt{a}} \geq |j_3|_{\mathtt{a}}$, $j_i \in \mathbb{Z}^d$ and where $M(d)$ is a constant depending only on d.

The proof of this proposition is divided in several steps and it is postponed to the end of the section. The main ingredient is the following standard proposition which follows the lines of [23]. Here we give weak lower bounds of the small divisors, these estimates will be improved later.

Proposition 3.3.2. *Consider I and J two bounded intervals of $\mathbb{R}^+ \setminus \{0\}$; $r \geq 2$ and $j_1, \dots, j_r \in \mathbb{Z}^d$ such that $j_i \neq \pm j_k$ if $i \neq k$, $n_1, \dots n_r \in \mathbb{Z} \setminus \{0\}$ and $h : J^{d-1} \to \mathbb{R}$ measurable. Then for any $\gamma > 0$ we have*

$$\mu\Big\{(\mathtt{p}, \mathtt{b}) \in I \times J^{d-1} : \big|h(\mathtt{b}) + \sum_{k=1}^{r} n_k \sqrt{|j_k|^4_{(1,\mathtt{b})} + \mathtt{p}|j_k|^2_{(1,\mathtt{b})}}\big| \leq \gamma\Big\} \lesssim_{I,J,d,r,n} \gamma^{\frac{1}{2r}} (\langle j_1 \rangle \cdots \langle j_r \rangle)^{\frac{2}{r}},$$

with $(1, \mathtt{b}) = (1, \mathtt{b}_1, \dots, \mathtt{b}_{d-1}) \in \mathbb{R}^d$ and where μ is the Lebesgue measure.

Remark 3.3.3. We shall apply this general proposition only in the case $r = 3$, however we preferred to write it in general for possible future applications.

Proof of Prop. 3.3.2. For simplicity in the proof we assume $|j_1|_{(1,\mathtt{b})} > \ldots > |j_r|_{(1,\mathtt{b})}$. Since by assumption we have $j_i \neq j_k$ for any $i \neq k$ then one could easily prove that for any $\eta > 0$ (later it will be chosen in function of γ) we have

$$\mu(P^{i,k}_\eta) < \eta \mu(J^{d-2}), \quad P^{i,k}_\eta := \{\mathtt{b} \in J^{d-1} : \big||j_i|^2_{(1,\mathtt{b})} - |j_k|^2_{(1,\mathtt{b})}\big| < \eta\}.$$

We define $P_\eta = \cup_{i \neq k} P^{i,k}_\eta$, and

$$B_\gamma := \Big\{(\mathtt{p}, \mathtt{b}) \in I \times J^{d-1} : \big|h(\mathtt{b}) + \sum_{k=1}^{r} n_k \sqrt{|j_k|^4_{(1,\mathtt{b})} + \mathtt{p}|j_k|^2_{(1,\mathtt{b})}}\big| \leq \gamma\Big\},$$

then we have

$$\begin{aligned}
\mu(B_\gamma) &\leq \mu(B_\gamma \cap P_\eta) + \mu(B_\gamma \cap (P_\eta)^c) \\
&\leq \mu(I)\mu(P_\eta) + \mu(J^{d-1}) \sup_{\substack{i \neq k \\ \mathtt{b} \notin P_\eta}} \mu\Big(\{\mathtt{p} \in I : \big|h(\mathtt{b}) + \sum_{k=1}^{r} n_k \sqrt{|j_k|^4_{(1,\mathtt{b})} + \mathtt{p}|j_k|^2_{(1,\mathtt{b})}}\big| \leq \gamma\}\Big) \\
&\lesssim_r \mu(I)\mu(J^{d-2})\eta + \mu(J^{d-1}) \sup_{\substack{i \neq k \\ \mathtt{b} \notin P_\eta}} \mu\Big(\{\mathtt{p} \in I : \big|h(\mathtt{b}) + \sum_{k=1}^{r} n_k \sqrt{|j_k|^4_{(1,\mathtt{b})} + \mathtt{p}|j_k|^2_{(1,\mathtt{b})}}\big| \leq \gamma\}\Big).
\end{aligned}$$

We have to estimate from above the measure of the last set. We define the function

$$g(\mathtt{p}) := h(\mathtt{b}) + \sum_{k=1}^{r} n_k \sqrt{|j_k|^4_{(1,\mathtt{b})} + \mathtt{p}|j_k|^2_{(1,\mathtt{b})}}.$$

For any $\ell \geq 1$ we have

$$\frac{d^\ell}{d\mathtt{p}^\ell} g(\mathtt{p}) = c_\ell \sum_{k=1}^{r} n_k |j_k|_{(1,\mathtt{b})} (\mathtt{p} + |j_k|^2_{(1,\mathtt{b})})^{\frac{1}{2} - \ell}, \quad c_\ell := \prod_{i=1}^{\ell} (\tfrac{1}{2} - i).$$

Therefore we can write the system of equations

$$\begin{pmatrix} c_1^{-1} \partial^1_{\mathtt{p}} g(\mathtt{p}) \\ \vdots \\ c_r^{-1} \partial^r_{\mathtt{p}} g(\mathtt{p}) \end{pmatrix} = \begin{pmatrix} (\mathtt{p} + |j_1|^2_{(1,\mathtt{b})})^0 & \ldots & (\mathtt{p} + |j_r|^2_{(1,\mathtt{b})})^0 \\ \vdots & \ddots & \vdots \\ (\mathtt{p} + |j_1|^2_{(1,\mathtt{b})})^{1-r} & \ldots & (\mathtt{p} + |j_r|^2_{(1,\mathtt{b})})^{1-r} \end{pmatrix} \begin{pmatrix} n_1 |j_1|_{(1,\mathtt{b})} (\mathtt{p} + |j_1|^2_{(1,\mathtt{b})})^{-1/2} \\ \vdots \\ n_r |j_r|_{(1,\mathtt{b})} (\mathtt{p} + |j_r|^2_{(1,\mathtt{b})})^{-1/2} \end{pmatrix}.$$

We denote by V the Vandermonde matrix above. We have that V is invertible since

$$|\det(V)| = \prod_{1\le i<k\le r} \Big|\frac{1}{\mathtt{p}+|j_i|^2_{(1,\mathtt{b})}} - \frac{1}{\mathtt{p}+|j_k|^2_{(1,\mathtt{b})}}\Big| \ge \prod_{1\le i<k\le r} \frac{\big||j_i|^2_{(1,\mathtt{b})} - |j_k|^2_{(1,\mathtt{b})}\big|}{(\mathtt{p}+|j_i|^2_{(1,\mathtt{b})})(\mathtt{p}+|j_k|^2_{(1,\mathtt{b})})}$$
$$\gtrsim \prod_{1\le k\le r} \frac{\eta}{(\mathtt{p}+|j_k|^2_{(1,\mathtt{b})})^2} \gtrsim \eta^r \frac{1}{\langle j_1\rangle^4 \cdots \langle j_r\rangle^4},$$

where in the penultimate passage we have used that $\mathtt{b}\notin P_\eta$ and $|j_i|_{(1,\mathtt{b})} \le |j_k|_{(1,\mathtt{b})}$ if $i>k$. Therefore we have

$$\max_{\ell=1}^{r} |c_\ell \partial^\ell_{\mathtt{p}} g(\mathtt{p})| \gtrsim_r |\det(V)| \max_{\ell=1}^{r} \Big| n_\ell |j_\ell|_{(1,\mathtt{b})} (\mathtt{p}+|j_\ell|)^{-\frac12}_{(1,\mathtt{b})}\Big|$$
$$\gtrsim_{r,n} \eta^r \frac{|j_1|^{1/2}_{(1,\mathtt{b})}}{\langle j_1\rangle^4 \cdots \langle j_r\rangle^4} \gtrsim_{r,n} \frac{\eta^r}{\langle j_1\rangle^4 \cdots \langle j_r\rangle^4}.$$

At this point we are ready to use Lemma 7 in appendix A of the paper [108], we obtain

$$\mu\Big(\{\mathtt{p}\in I : \big|h(\mathtt{b}) + \sum_{k=1}^{r} \sqrt{|j_k|^4_{(1,\mathtt{b})} + \mathtt{p}|j_k|^2_{(1,\mathtt{b})}}\big| \le \gamma\}\Big) \le \Big(\frac{\gamma\langle j_1\rangle^4 \cdots \langle j_r\rangle^4}{\eta^r}\Big)^{\frac1r}.$$

Summarizing we obtained

$$\mu(B_\gamma) \lesssim_{I,J,d,r,n} \eta + \eta^{-1}\gamma^{\frac1r}(\langle j_1\rangle^4 \cdots \langle j_r\rangle^4)^{\frac1r},$$

we may optimize by choosing $\eta = \gamma^{\frac{1}{2r}}(\langle j_1\rangle^2 \cdots \langle j_r\rangle^2)^{\frac1r}$ and we obtain the thesis. □

As a consequence of the preceding proposition we have the following.

Corollary 3.3.4. *Let $r\ge 1$, consider $j_1,\dots,j_r\in\mathbb{Z}^d$ such that $j_k\ne j_i$ if $i\ne k$ and $n_1,\dots,n_k\in\mathbb{Z}\setminus\{0\}$. For any $\gamma>0$ we have*

$$\mu\Big(\Big\{a\in(1,4) : \sum_{i=1}^{r} n_i\sqrt{k|j_i|^4_{\mathtt{a}} + \mathtt{m}g'(\mathtt{m})|j_i|^2_{\mathtt{a}}} \le \gamma\Big\}\Big) \lesssim_{d,r,n} \Big(\frac{\gamma}{\sqrt{k}}\Big)^{\frac{1}{2r}} (\langle j_1\rangle \dots \langle j_r\rangle)^{\frac2r}.$$

Proof. We write

$$\sum_{i=1}^{r} n_i\sqrt{k|j_i|^4_{\mathtt{a}} + \mathtt{m}g'(\mathtt{m})|j_i|^2_{\mathtt{a}}}\Big) = \sqrt{k}\mathtt{a}_1 \sum_{i=1}^{r} n_i \sqrt{|j_i|^4_{(1,\mathtt{b})} + \frac{\beta}{\mathtt{a}_1}|j_i|^2_{(1,\mathtt{b})}},$$

where we have set

$$\beta := \frac{\mathtt{m}g'(\mathtt{m})}{k}, \quad \mathtt{b} := (\frac{\mathtt{a}_2}{\mathtt{a}_1},\dots,\frac{\mathtt{a}_d}{\mathtt{a}_1}). \tag{3.3.4}$$

The map $(\mathtt{a}_1,\dots,\mathtt{a}_d)\mapsto(\frac{1}{\mathtt{a}_1},\mathtt{b})$ is invertible onto its image, which is contained in $(\frac14,1)\times(\frac14,4)^{d-1}$. The determinant of its inverse is bounded by a constant depending only on d. Therefore the result follows by applying Prop. 3.3.2 and the change of coordinates $(\mathtt{a}_1,\dots,\mathtt{a}_d)\mapsto(\frac{1}{\mathtt{a}_1},\mathtt{b})$. □

Owing to the corollary above we may reduce in the following to the study of the small dividers when we have 2 frequencies much larger then the other.

Lemma 3.3.5. *Consider $\tilde\Lambda := \sqrt{k}|j_1|_{\mathtt a}^2 - \sqrt{k}|j_2|_{\mathtt a}^2 - \omega_3$ and β defined in (3.3.4). If there exists $i \in \{1, \dots, d\}$ such that*

$$|j_{3,i}|\sqrt{1+\frac{\beta}{|j_3|_{\mathtt a}^2}} \le \frac{1}{2}|j_{1,i}+j_{2,i}|\,, \tag{3.3.5}$$

then for any $\tilde\gamma > 0$ we have

$$\mu\Big(\big\{\mathtt a \in (1,4)^d : |\tilde\Lambda| \le \tilde\gamma\big\}\Big) \le \frac{2\tilde\gamma}{\sqrt{k}|j_{1,i}+j_{2,i}|}\,.$$

Proof. We give a lower bound for the derivative of the function $\tilde\Lambda$ with respect to the parameter a_i.

$$|\partial_{a_i}\tilde\Lambda| \ge \sqrt{k}\Big[|j_{3,i}(j_{1,i}+j_{2,i})| - j_{3,i}^2\sqrt{1+\frac{\beta}{|j_3|_a^2}}\Big] \ge \sqrt{k}\frac{1}{2}|j_{3,i}||j_{1,i}+j_{2,i}| \ge \sqrt{k}\frac{1}{2}|j_{1,i}+j_{2,i}|.$$

Therefore $a_i \mapsto \tilde\Lambda$ is a diffeomorphism and applying this change of variable we get the thesis. □

Proposition 3.3.6. *There exists a set of full Lebesgue measure $\mathfrak A_3 \subset (1,4)^d$ such that for any $\mathtt a$ in $\mathfrak A_3$ there exists $\gamma > 0$ such that*

$$\big|\sigma\omega_3 + \omega_2 - \omega_1\big| \ge \frac{\sqrt{k}\gamma}{|j_1|^{d-1}\log^{d+1}(1+|j_1|^2)|j_3|^{d+1}}\,,$$

for any $\sigma \in \pm 1$, for any j_1, j_2, j_3 in $\mathbb Z^d$ satisfying $|j_1|_{\mathtt a} > |j_2|_{\mathtt a} \ge |j_3|_{\mathtt a}$, the momentum condition $\sigma j_3 + j_2 - j_1 = 0$ and

$$\mathfrak J(j_1,\beta) = \min\left\{\frac{\sqrt{||j_1|^2 - 4d^2\beta|}}{2d}, \min\Big\{\Big(\frac{\gamma}{4\beta^2}\Big)^{\frac{1}{d+2}}, \Big(\frac{\gamma}{2\beta^3}\Big)^{\frac{1}{d+1}}\Big\}\Big(\frac{|j_1|^{4-d}}{\log(1+|j_1|)^{d+1}}\Big)^{\frac{1}{d+2}}\right\} > |j_3|\,, \tag{3.3.6}$$

where β is defined in (3.3.4).

Proof. We suppose $\sigma = 1$, we set $\Lambda := \omega_1 - \omega_2 - \omega_3$ and

$$L(\gamma) := \frac{\sqrt{k}\gamma}{\big(|j_3|^{d+1}|j_1|^{d-1}\log^{d+1}(1+|j_1|)\big)}.$$

From the first condition in (3.3.6) we deduce that $\beta/|j_1|^2 < 1$, therefore, by Taylor expanding the (3.3.2), we obtain

$$\Lambda = \sqrt{k}\Big(|j_1|_{\mathtt a}^2 - |j_2|_{\mathtt a}^2 + \frac{\beta^2}{8}\frac{|j_1|_{\mathtt a}^2 - |j_2|_{\mathtt a}^2}{|j_2|_{\mathtt a}^2|j_1|_{\mathtt a}^2} + R\Big) - \omega_3\,, \tag{3.3.7}$$

where $|R| \le \frac{1}{8}\frac{\beta^3}{|j_2|_{\mathtt a}^4}$. We define $\tilde\Lambda := \sqrt{k}|j_1|_{\mathtt a}^2 - \sqrt{k}|j_2|_{\mathtt a}^2 - \omega_3$ and the following *good sets*

$$\mathcal G_\gamma := \{\mathtt a \in (1,4)^d : |\Lambda| > L(\gamma), \forall j_1, j_3 \in \mathbb Z^d\}, \quad \tilde{\mathcal G}_\gamma := \{\mathtt a \in (1,4)^d : \big|\tilde\Lambda\big| > 3L(\gamma), \forall j_1, j_3 \in \mathbb Z^d\}.$$

We claim that, thanks to (3.3.6), we have the inclusion $\tilde{\mathcal G}_\gamma \subset \mathcal G_\gamma$. First of all we have

$$\big|\Lambda\big| \ge \big|\omega_3 + \sqrt{k}|j_2|_{\mathtt a}^2 - \sqrt{k}|j_1|_{\mathtt a}^2\big| - \sqrt{k}\frac{\beta^2}{8}\frac{|j_1|_{\mathtt a}^2 - |j_2|_{\mathtt a}^2}{|j_1|_{\mathtt a}^2|j_2|_{\mathtt a}^2} - \sqrt{k}|R|\,. \tag{3.3.8}$$

From the momentum condition $j_1 - j_2 = j_3$ and the ordering $|j_1|_{\mathtt{a}} > |j_2|_{\mathtt{a}} \geq |j_3|_{\mathtt{a}}$ we have that $|j_1|_{\mathtt{a}} \leq 2|j_2|_{\mathtt{a}}$, which implies

$$\frac{|j_1|_{\mathtt{a}}^2 - |j_2|_{\mathtt{a}}^2}{|j_1|_{\mathtt{a}}^2 |j_2|_{\mathtt{a}}^2} = \frac{\sum_{k=1}^d a_k j_{3,k}(j_{1,k} + j_{2,k})}{|j_1|_{\mathtt{a}}^2 |j_2|_{\mathtt{a}}^2} \leq 2\frac{|j_3|_{\mathtt{a}}|j_1|_{\mathtt{a}}}{|j_1|_{\mathtt{a}}^2 |j_2|_{\mathtt{a}}^2} \leq 2\frac{|j_3|_{\mathtt{a}}}{|j_1|_{\mathtt{a}}|j_2|_{\mathtt{a}}^2} \leq 32\frac{|j_3|}{|j_1|^3}\,, \tag{3.3.9}$$

where we used $|\cdot| < |\cdot|_{\mathtt{a}} < 4|\cdot|$. Therefore from (3.3.6), more precisely from

$$\left(\frac{\gamma}{4\beta^2}\right)^{\frac{1}{d+2}} \left(\frac{|j_1|^{4-d}}{\log(1+|j_1|)^{d+1}}\right)^{\frac{1}{d+2}} > |j_3|,$$

we deduce that

$$\sqrt{k}\frac{\beta^2}{8}\frac{|j_1|_{\mathtt{a}}^2 - |j_2|_{\mathtt{a}}^2}{|j_1|_{\mathtt{a}}^2 |j_2|_{\mathtt{a}}^2} < L.$$

Analogously one proves that $\sqrt{k}|R| < L$. We have eventually proved that $\tilde{\mathcal{G}}_\gamma \subset \mathcal{G}_\gamma$ using (3.3.8).

We define the *bad sets* $\tilde{\mathcal{B}}_\gamma := \big((1,4)^d \setminus \tilde{\mathcal{G}}_\gamma\big) \supset \mathcal{B}_\gamma := \big((1,4)^d \setminus \mathcal{G}_\gamma\big)$ and we prove that the Lebesgue measure of $\cap_\gamma \tilde{\mathcal{B}}_\gamma$ equals to zero, this implies the thesis.

We want to apply Lemma 3.3.5 with $\tilde{\gamma} \rightsquigarrow L$. We know that there exists $i \in \{1,\dots,d\}$ such that $d|j_{1,i}| \geq |j_1|$. We claim that, thanks to (3.3.6), we satisfy condition (3.3.5) for the same index i. Let us suppose by contradiction that

$$|j_{3,i}|\sqrt{1 + \tfrac{\beta}{|j_3|_{\mathtt{a}}^2}} > \tfrac{1}{2}|j_{1,i} + j_{2,i}| = \tfrac{1}{2}|2j_{1,i} - j_{3,1}| \geq |j_{1,i}| - \tfrac{1}{2}|j_{3,i}| > \tfrac{|j_1|}{d} - \tfrac{1}{2}|j_{3,i}|\,,$$

from which we obtain $|j_1| \leq 2d|j_3|\sqrt{1 + \beta/|j_3|_{\mathtt{a}}^2}$. Taking the squares we get

$$|j_1|^2 \leq 4d^2|j_3|^2 + 4d^2\beta\frac{|j_3|^2}{|j_3|_{\mathtt{a}}^2}\,,$$

which, recalling that $|\cdot| < |\cdot|_{\mathtt{a}} < 4|\cdot|$, contradicts (3.3.6).

Therefore, by using Lemma 3.3.5, we have

$$\begin{aligned}\mu(\tilde{\mathcal{B}_\gamma}) &= \mu\big(\big\{\mathtt{a} \in (1,4)^d | \exists j_1, j_3 \in \mathbb{Z}^d : |\tilde{\Lambda}| \leq \sqrt{k}\gamma|j_3|^{-d-1}|j_1|^{1-d}\log(|j_1|)^{-d-1}\big\}\big) \\ &\leq \sum_{j_3\in\mathbb{Z}^d}\frac{1}{|j_3|^{d+1}}\sum_{j_1\in\mathbb{Z}^d}\frac{\gamma}{|j_1|^{d-1}|j_{1,i}|\log(|j_1|)^{d+1}} \lesssim_d \gamma\,.\end{aligned}$$

This implies that $meas(\cap_\gamma \mathcal{B}_\gamma) = 0$, hence we can set $\mathfrak{A}_3 = \cup_\gamma \mathcal{G}_\gamma$. □

We are now in position to prove Prop. 3.3.1.

Proof of Prop. 3.3.1. The case $\sigma_1\sigma_2 = 1$ is trivial, we give the proof if $\sigma_1\sigma_2 = -1$. From Prop. 3.3.6 we know that there exists a full Lebesgue measure set $\mathfrak{A}_3$ and $\gamma > 0$ such that the statement is proven if $|j_3| \leq \mathfrak{J}(j_1,\gamma)$. Let us now assume $|j_3| > \mathfrak{J}(j_1,\gamma)$. Let us define

$$\mathcal{B}_\gamma := \bigcup_{j_1,j_3\in\mathbb{Z}^d}\left\{\mathtt{a} \in (1,4)^d : |\sigma_3\omega_3 + \omega_2 - \omega_1| \leq \sqrt{k}\frac{\tilde{\gamma}}{|j_3|^{M(d)}}\right\},$$

where $\tilde{\gamma}$ will be chosen in function of γ and $M(d)$ big enough w.r.t. d.

Let us set $p := (\frac{M(d)}{6} - d - 1)\frac{1}{d+2}$ suppose for the moment $\big(\gamma/4\beta^2\big)^{\frac{1}{d+2}} \le \big(\gamma/2\beta^3\big)^{\frac{1}{d+1}}$. From $|j_3| > \mathfrak{J}(j_1,\beta)$ (see (3.3.6)) and Corollary 3.3.4 with $r = 3$, we have

$$\mu(\mathcal{B}_\gamma) \lesssim_d \sum_{j_1,j_3\in\mathbb{Z}^d} \frac{\tilde\gamma^{\frac{1}{6}}}{|j_3|^{M(d)/6}}\langle j_1\rangle^2 \lesssim_d \tilde\gamma^{1/6}\gamma^{-p}(4\beta^2)^p \sum_{j_1\in\mathbb{Z}^d} \frac{\log^{p(d+1)}(1+|j_1|)}{|j_1|^{(4-d)p-2}} \sum_{j_3} |j_3|^{-d-1}.$$

If the exponent $M(d)$ (and hence p) is chosen large enough we get the summability in the r.h.s. of the inequality above. We now choose $\tilde\gamma^{1/6}\gamma^{-p} = \gamma^m$, we eventually obtain $\mu(\mathcal{B}_\gamma) \lesssim \gamma^m$. If $\big(\gamma/4\beta^2\big)^{\frac{1}{d+2}} > \big(\gamma/2\beta^3\big)^{\frac{1}{d+1}}$ one can reason similarly. The wanted set of full Lebesgue measure is therefore obtained by choosing $\mathfrak{A} := \mathfrak{A}_3 \cap (\cup_{\gamma>0}\mathcal{B}_\gamma^c)$. $\square$

3.4 Energy estimates

In this section we construct a modified energy for the Hamiltonian $\widetilde{\mathcal{K}}_{\mathtt{m}}$ in (3.2.36). We first need some convenient notation.

Definition 3.4.1. *If $j \in (\mathbb{Z}^d)^r$ for some $r \ge k$ then $\mu_k(j)$ denotes the k^{st} largest number among $|j_1|,\dots,|j_r|$ (multiplicities being taken into account).*

Definition 3.4.2 (Formal Hamiltonians)**.** *We denote by $\mathcal{L}_3$ the set of Hamiltonian having homogeneity 3 and such that they may be written in the form*

$$G_3(w) = \sum_{\substack{\sigma_i\in\{-1,1\},\ j_i\in\mathbb{Z}^d\setminus\{0\} \\ \sigma_1 j_1+\sigma_2 j_2+\sigma_3 j_3=0}} (G_3)_{\sigma,j} w_{j_1}^{\sigma_1} w_{j_2}^{\sigma_2} w_{j_3}^{\sigma_3}\,, \quad (G_3)_{\sigma,j}\in\mathbb{C}\,, \qquad \begin{matrix}\sigma := (\sigma_1,\sigma_2,\sigma_3)\\ j := (j_1,j_2,j_3)\end{matrix} \tag{3.4.1}$$

with symmetric coefficients $(G_3)_{\sigma,j}$ (i.e. for any $\rho\in\mathfrak{S}_3$ one has $(G_3)_{\sigma,j} = (G_3)_{\sigma\circ\rho,j\circ\rho}$) and where we denoted

$$w_j^\sigma := w_j\,, \quad \text{if } \ \sigma = +\,, \qquad w_j^\sigma := \overline{w_j}\,, \quad \text{if } \ \sigma = -\,.$$

The Hamiltonian in (3.2.36) has the form (see (3.2.33))

$$\widetilde{\mathcal{K}}_{\mathtt{m}} := \widetilde{\mathcal{K}}_{\mathtt{m}}^{(2)} + \widetilde{\mathcal{K}}_{\mathtt{m}}^{(3)} + \widetilde{\mathcal{K}}_{\mathtt{m}}^{(\ge 4)}\,, \qquad \widetilde{\mathcal{K}}_{\mathtt{m}}^{(2)} = \sum_{j\in\mathbb{Z}^d\setminus\{0\}} \omega(j) w_j \overline{w}_j\,, \tag{3.4.2}$$

where $\widetilde{\mathcal{K}}_{\mathtt{m}}^{(3)}$ is a tri-linear Hamiltonian in $\mathcal{L}_3$ with coefficients satisfying

$$|(\widetilde{\mathcal{K}}_{\mathtt{m}}^{(3)})_{\sigma,j}| \lesssim 1\,, \qquad \forall\,\sigma\in\{-1,+1\}^3\,,\ \ j\in(\mathbb{Z}^d)^3\setminus\{0\}\,, \tag{3.4.3}$$

and where $\widetilde{\mathcal{K}}_{\mathtt{m}}^{(\ge 4)}$ satisfies for any $s > d/2$

$$\|X_{\widetilde{\mathcal{K}}_{\mathtt{m}}^{(\ge 4)}}(w)\|_{H^s} \lesssim_s \|w\|_{H^s}^3\,, \qquad \text{if } \ \|w\|_{H^s} < 1\,. \tag{3.4.4}$$

The main result of this section is the following.

Proposition 3.4.3. *Let $\mathfrak{A}$ and M given by Proposition 3.3.1. Consider $\mathtt{a}\in\mathfrak{A}$. For any $N > 1$ and any $s \ge \tilde s_0$, for some $\tilde s_0 = \tilde s_0(M) > 0$, there exist $\varepsilon_0 \lesssim_{s,\delta} \log^{-d-1}(1+N)$ and a tri-linear function E_3 in the class $\mathcal{L}_3$ such that the following holds:*

• *the coefficients* $(E_3)_{\sigma,j}$ *satisfies*

$$|(E_3)_{\sigma,j}| \lesssim_s N^{d-2} \log^{d+1}(1+N) \mu_3(j)^{M+1} \mu_1(j)^{2s}, \tag{3.4.5}$$

for $\sigma \in \{-1,1\}^3$, $j \in (\mathbb{Z}^d)^3 \setminus \{0\}$*;*
• *for any* w *in the ball of radius* ε_0 *of* $H_0^s(\mathbb{T}^d;\mathbb{C})$ *one has*

$$|\{N_s + E_3, \widetilde{\mathcal{K}}_{\mathtt{m}}\}| \lesssim_s N^{d-2} \log^{d+1}(1+N) \|w\|_{H^s}^4 + N^{-1} \|w\|_{H^s}^3 . \tag{3.4.6}$$

where N_s *is defined as*

$$N_s(w) := \|w\|_{H^s}^2 = \sum_{j \in \mathbb{Z}^d} \langle j \rangle^{2s} |w_j|^2, \tag{3.4.7}$$

and $\widetilde{\mathcal{K}}_{\mathtt{m}}$ *in* (3.4.2).

In subsection 3.4.1 we study some properties of the Hamiltonians in $\mathcal{L}_3$ of Def. 3.4.2. Then in subsection 3.4.2 we give the proof of Proposition 3.4.3. Finally, in subsection 3.4.3, we conclude the proof of the main theorem.

3.4.1 Tri-linear Hamiltonians

We now recall some properties of tri-linear Hamiltonians introduced in Definition 3.4.2. We first give some further definitions.

Definition 3.4.4. *Let* $N \in \mathbb{R}$ *with* $N \geq 1$.
(i) *If* $G_3 \in \mathcal{L}_3$ *then* $G_3^{>N}$ *denotes the element of* $\mathcal{L}_3$ *defined by*

$$(G_3^{>N})_{\sigma,j} := \begin{cases} (G_3)_{\sigma,j}\,, & \text{if } \ \mu_2(j) > N\,, \\ 0\,, & \text{else}\,. \end{cases} \tag{3.4.8}$$

We set $G_3^{\leq N} := G_3 - G_3^{>N}$.
(ii) *We define* $G_3^{(+1)} \in \mathcal{L}_3$ *by*

$$\begin{aligned} &(G_3^{(+1)})_{\sigma,j} := (G_3)_{\sigma,j}\,, \ \text{ when } \ \exists i,p = 1,2,3, \text{ s.t.} \\ &\mu_1(j) = |j_i|\,, \ \ \mu_2(j) = |j_p| \ \text{ and } \ \sigma_i\sigma_p = +1\,. \end{aligned}$$

We define $G_3^{(-1)} := G_3 - G_3^{(+1)}$.

Consider the quadratic Hamiltonian $\widetilde{\mathcal{K}}_{\mathtt{m}}^{(2)}$ in (3.4.2). Given a tri-linear Hamiltonian G_3 in $\mathcal{L}_3$ we define the adjoint action

$$\mathrm{ad}_{\widetilde{\mathcal{K}}_{\mathtt{m}}^{(2)}} G_3 := \{\widetilde{\mathcal{K}}_{\mathtt{m}}^{(2)}, G_3\}$$

(see (3.2.15)) as the Hamiltonian in $\mathcal{L}_3$ with coefficients

$$\bullet\ (\textbf{adjoint action}) \qquad (\mathrm{ad}_{\widetilde{\mathcal{K}}_{\mathtt{m}}^{(2)}} G_3)_{\sigma,j} := \Big(\mathrm{i} \sum_{i=1}^{3} \sigma_i \omega(j_i)\Big) (G_3)_{\sigma,j}\,. \tag{3.4.9}$$

The following lemma is the counterpart of Lemma 3.5 in [23]. We omit its proof.

Lemma 3.4.5. *Let $N \geq 1$, $q_i \in \mathbb{R}$, consider $G_3^i(u)$ in $\mathcal{L}_3$. Assume that the coefficients $(G_3^i)_{\sigma,j}$ satisfy (recall Def. 3.4.1)*

$$|(G_3^i)_{\sigma,j}| \leq C_i \mu_3(j)^{\beta_i} \mu_1(j)^{-q_i}\,, \qquad \forall \sigma \in \{-1,+1\}^3\,,\ j \in \mathbb{Z}^d \setminus \{0\}\,,$$

for some $\beta_i > 0$ and $C_i > 0$, $i = 1,2$.
(i) (Estimates on Sobolev spaces). Set $\delta = \delta_i$, $q = q_i$, $\beta = \beta_i$, $C = C_i$ and $G_3^i = G_3$ for $i = 1,2$. There is $s_0 = s_0(\beta,d)$ such that for $s \geq s_0$, G_3 defines naturally a smooth function from $H_0^s(\mathbb{T}^d;\mathbb{C})$ to $\mathbb{R}$. In particular one has the following estimates:

$$\begin{aligned} |G_3(w)| &\lesssim_s C\|w\|_{H^s}^3\,, \\ \|X_{G_3}(w)\|_{H^{s+q}} &\lesssim_s C\|w\|_{H^s}^2\,, \\ \|X_{G_3^{>N}}(w)\|_{H^s} &\lesssim_s CN^{-q}\|w\|_{H^s}^2\,, \end{aligned}$$

for any $w \in H_0^s(\mathbb{T}^d;\mathbb{C})$.
(ii) (Poisson bracket). The Poisson bracket between G_3^1 and G_3^2 satisfies the estimate

$$|\{G_3^1, G_3^2\}| \lesssim_s C_1 C_2 \|w\|_{H^s}^4\,.$$

Let $F : H_0^s(\mathbb{T}^d;\mathbb{C}) \to \mathbb{R}$ a C^1 Hamiltonian function such that

$$\|X_F(w)\|_{H^s} \lesssim_s C_3 \|w\|_{H^s}^3\,,$$

for some $C_3 > 0$. Then one has

$$|\{G_3^1, F\}| \lesssim_s C_1 C_3 \|w\|_{H^s}^5\,.$$

We have the following result.

Lemma 3.4.6 (Energy estimate)**.** *Consider the Hamiltonians N_s in (3.4.7), $G_3 \in \mathcal{L}_3$ and write $G_3 = G_3^{(+1)} + G_3^{(-1)}$ (recall Definition 3.4.2). Assume also that the coefficients of G_3 satisfy*

$$|(G_3^{(\eta)})_{\sigma,j}| \leq C\mu_3(j)^{\beta}\mu_1(j)^{-q}\,, \qquad \forall \sigma \in \{-1,+1\}^3\,,\ j \in \mathbb{Z}^d \setminus \{0\}\,, \eta \in \{-1,+1\},$$

for some $\beta > 0$, $C > 0$ and $q \geq 0$. We have that the Hamiltonian $Q_3^{(\eta)} := \{N_s, G_3^{(\eta)}\}$, $\eta \in \{-1,1\}$, belongs to the class $\mathcal{L}_3$ and has coefficients satisfying

$$|(Q_3^{(\eta)})_{\sigma,j}| \lesssim_s C\mu_3(j)^{\beta+1}\mu_1(j)^{2s}\mu_1(j)^{-q-\alpha}\,, \qquad \alpha := \begin{cases} 1\,, & \text{if } \eta = -1 \\ 0\,, & \text{if } \eta = +1\,. \end{cases}$$

Proof. One can reason as in the proof of Lemma 4.2 in [23]. □

Remark 3.4.7. As a consequence of Lemma 3.4.6 we have the following. The action of the operator $\{N_s, \cdot\}$ on multi-linear Hamiltonian functions as in (3.4.1) where the two highest indexes have opposite sign (i.e., $G_3^{(-1)}$), provides a decay property of the coefficients w.r.t. the highest index. This implies (by Lemma 3.4.5-(ii)) a smoothing property of the Hamiltonian $\{N_s, G_3^{(-1)}\}$.

3.4.2 Proof of Proposition 3.4.3

Recalling Definitions 3.4.2, 3.4.4 and considering the Hamiltonian $\widetilde{\mathcal{K}}_{\mathtt{m}}^{(3)}$ in (3.4.2), (3.2.36), we write $\widetilde{\mathcal{K}}_{\mathtt{m}}^{(3)} = \widetilde{\mathcal{K}}_{\mathtt{m}}^{(3,+1)} + \widetilde{\mathcal{K}}_{\mathtt{m}}^{(3,-1)}$. We define (see (3.4.9))

$$E_3^{(+1)} := (\mathrm{ad}_{\widetilde{\mathcal{K}}_{\mathtt{m}}^{(2)}})^{-1}\{N_s, \widetilde{\mathcal{K}}_{\mathtt{m}}^{(3,+1)}\}, \qquad E_3^{(-1)} := (\mathrm{ad}_{\widetilde{\mathcal{K}}_{\mathtt{m}}^{(2)}})^{-1}\{N_s, (\widetilde{\mathcal{K}}_{\mathtt{m}}^{(3,-1)})^{(\le N)}\}, \tag{3.4.10}$$

and we set $E_3 := E_3^{(+1)} + E_3^{(-1)}$. It is easy to note that $E_3 \in \mathcal{L}_3$. Moreover, using that $|(\widetilde{\mathcal{K}}_{\mathtt{m}}^{(3)})_{\sigma,j}| \lesssim 1$ (see (3.4.3)), Lemma 3.4.6 and Proposition 3.3.1, one can check that the coefficients $(E_3)_{\sigma,j}$ satisfy the (3.4.5). Using (3.4.10) we notice that

$$\{N_s, \widetilde{\mathcal{K}}_{\mathtt{m}}^{(3)}\} + \{E_3, \widetilde{\mathcal{K}}_{\mathtt{m}}^{(2)}\} = \{N_s, (\widetilde{\mathcal{K}}_{\mathtt{m}}^{(3,-1)})^{(>N)}\}. \tag{3.4.11}$$

Combining Lemmata 3.4.5 and 3.4.6 we deduce

$$|\{N_s, (\widetilde{\mathcal{K}}_{\mathtt{m}}^{(3,-1)})^{(>N)}\}(w)| \lesssim_s N^{-1}\|w\|_{H^s}^3, \tag{3.4.12}$$

for s large enough with respect to M. We now prove the estimate (3.4.6). We have

$$\begin{aligned} \{N_s + E_3, \widetilde{\mathcal{K}}_{\mathtt{m}}\} &\overset{(3.4.2)}{=} \{N_s + E_3, \widetilde{\mathcal{K}}_{\mathtt{m}}^{(2)} + \widetilde{\mathcal{K}}_{\mathtt{m}}^{(3)} + \widetilde{\mathcal{K}}_{\mathtt{m}}^{(\ge 4)}\} && (3.4.13)\\ &= \{N_s, \widetilde{\mathcal{K}}_{\mathtt{m}}^{(2)}\} && (3.4.14)\\ &\quad + \{N_s, \widetilde{\mathcal{K}}_{\mathtt{m}}^{(3)}\} + \{E_3, \widetilde{\mathcal{K}}_{\mathtt{m}}^{(2)}\} && (3.4.15)\\ &\quad + \{E_3, \widetilde{\mathcal{K}}_{\mathtt{m}}^{(3)} + \widetilde{\mathcal{K}}_{\mathtt{m}}^{(\ge 4)}\} + \{N_s, \widetilde{\mathcal{K}}_{\mathtt{m}}^{(\ge 4)}\}. && (3.4.16) \end{aligned}$$

We study each summand separately. First of all note that (recall (3.4.7), (3.4.2)) the term (3.4.14) vanishes. By (3.4.4), (3.4.5) and Lemma 3.4.5-(ii) we obtain

$$|(3.4.16)| \lesssim_s N^{d-2}\log^{d+1}(1+N)\|w\|_{H^s}^4.$$

Moreover, by (3.4.11), (3.4.12), we deduce

$$|(3.4.15)| \lesssim_s N^{-1}\|w\|_{H^s}^3.$$

The discussion above implies the bound (3.4.6).

3.4.3 Proof of the main result

Consider the Hamiltonian $\widetilde{\mathcal{K}}_{\mathtt{m}}(w, \overline{w})$ in (3.4.2) and the associated Cauchy problem

$$\begin{cases} \mathrm{i}\partial_t w = \partial_{\overline{w}}\widetilde{\mathcal{K}}_{\mathtt{m}}(w, \overline{w}) \\ w(0) = w_0 \in H_0^s(\mathbb{T}^d; \mathbb{C}), \end{cases} \tag{3.4.17}$$

for some $s > 0$ large. We shall prove the following.

Lemma 3.4.8 (Main bootstrap). *Let $s_0 = s_0(d)$ given by Proposition 3.4.3. For any $s \ge s_0$, there exists $\varepsilon_0 = \varepsilon_0(s)$ such that the following holds. Let $w(t,x)$ be a solution of (3.4.17) with $t \in [0, T)$, $T > 0$ and initial condition $w(0,x) = w_0(x) \in H_0^s(\mathbb{T}^d; \mathbb{C})$. For any $\varepsilon \in (0, \varepsilon_0)$ if*

$$\|w_0\|_{H^s} \le \varepsilon, \quad \sup_{t\in[0,T)} \|w(t)\|_{H^s} \le 2\varepsilon, \quad T \le \varepsilon^{-1-\frac{1}{d-1}}\log^{-d-2}\left(1 + \varepsilon^{\frac{1}{1-d}}\right), \tag{3.4.18}$$

then we have the improved bound $\sup_{t\in[0,T)} \|w(t)\|_{H^s} \le 3/2\varepsilon$.

First of all we show that the energy $N_s + E_3$ constructed by Proposition 3.4.3 provides an equivalent Sobolev norm.

Lemma 3.4.9 (Equivalence of the energy norm)**.** *Let $N \geq 1$. Let $w(t,x)$ as in (3.4.18) with $s \gg 1$ large enough. Then, for any $0 < c_0 < 1$, there exists $C = C(s,d,c_0) > 0$ such that, if we have the smallness condition*

$$\varepsilon C N^{d-2} \log^{(d+1)}(1+N) \leq 1\,, \tag{3.4.19}$$

the following holds true. Define

$$\mathcal{E}_s(w) := (N_s + E_3)(w) \tag{3.4.20}$$

with N_s is in (3.4.7), E_3 is given by Proposition 3.4.3. We have

$$1/(1+c_0)\mathcal{E}_s(w) \leq \|w\|_{H^s}^2 \leq (1+c_0)\mathcal{E}_s(w)\,, \quad \forall t \in [0,T]\,. \tag{3.4.21}$$

Proof. Fix $c_0 > 0$. By (3.4.5) and Lemma 3.4.5, we deduce

$$|E_3(w)| \leq \tilde{C}\|w\|_{H^s}^3 N^{d-2}\log^{(d+1)}(1+N)\,, \tag{3.4.22}$$

for some $\tilde{C} > 0$ depending on s. Then, recalling (3.4.20), we get

$$|\mathcal{E}_s(w)| \leq \|w\|_{H^s}^2(1+\tilde{C}\|w\|_{H^s}N^{d-2}\log^{(d+1)}(1+N)) \overset{(3.4.19)}{\leq} \|w\|_{H^s}^2(1+c_0)\,,$$

where we have chosen C in (3.4.19) large enough. This implies the first inequality in (3.4.21). On the other hand, using (3.4.22) and (3.4.18), we have

$$\|w\|_{H^s}^2 \leq \mathcal{E}_s(w) + \tilde{C}N^{d-2}\log^{(d+1)}(1+N)\varepsilon\|w\|_{H^s}^2\,.$$

Then, taking C in (3.4.19) large enough, we obtain the second inequality in (3.4.21). ☐

Proof of Lemma 3.4.8. We study how the equivalent energy norm $\mathcal{E}_s(w)$ defined in (3.4.20) evolves along the flow of (3.4.17). Notice that

$$\partial_t \mathcal{E}_s(w) = -\{\mathcal{E}_s, \mathcal{H}\}(w)\,.$$

Therefore, for any $t \in [0,T]$, we have that

$$\left|\int_0^T \partial_t \mathcal{E}_s(w)\,\mathrm{d}t\right| \overset{(3.4.6),(3.4.18)}{\lesssim_s} TN^{d-2}\log^{(d+1)}(1+N)\varepsilon^4 + N^{-1}\varepsilon^3\,.$$

Let $0 < \alpha$ and set $N := \varepsilon^{-\alpha}$. Hence we have

$$\left|\int_0^T \partial_t \mathcal{E}_s(w)\,\mathrm{d}t\right| \lesssim_s \varepsilon^2 T\big(\varepsilon^{2-\alpha(d-2)}\log^{(d+1)}(1+\varepsilon^{-\alpha}) + \varepsilon^{1+\alpha}\big)\,. \tag{3.4.23}$$

We choose $\alpha > 0$ such that

$$2-\alpha(d-2) = 1+\alpha\,, \quad \text{i.e.,} \quad \alpha := \frac{1}{d-1}\,. \tag{3.4.24}$$

Therefore estimate (3.4.23) becomes

$$\left|\int_0^T \partial_t \mathcal{E}_s(w)\,\mathrm{d}t\right| \lesssim_s \varepsilon^2 T \varepsilon^{\frac{d}{d-1}}\log^{d+1}(1+\varepsilon^{-\alpha})\,.$$

Since ε can be chosen small with respect to s, with this choices we get

$$\left|\int_0^T \partial_t \mathcal{E}_s(w)\,\mathrm{d}t\right| \leq \varepsilon^2/4$$

as long as

$$T \leq \varepsilon^{-d/(d-1)} \log^{-d-2}\left(1+\varepsilon^{\frac{1}{1-d}}\right). \tag{3.4.25}$$

Then, using the equivalence of norms (3.4.21) and choosing $c_0 > 0$ small enough, we have

$$\begin{aligned}\|w(t)\|_{H^s}^2 &\leq (1+c_0)\mathcal{E}_0(w(t)) \leq (1+c_0)\left[\mathcal{E}_s(w(0)) + \left|\int_0^T \partial_t \mathcal{E}_s(w)\,\mathrm{d}t\right|\right] \\ &\leq (1+c_0)^2\varepsilon^2 + (1+c_0)\varepsilon^2/4 \leq \varepsilon^2 3/2\,,\end{aligned}$$

for times T as in (3.4.25). This implies the thesis. □

Proof of Theorem 3.1.1. In the same spirit of [72], [32] we have that for any initial condition (ρ_0, ϕ_0) as in (3.1.4) there exists a solution of (QHD) satisfying

$$\sup_{t\in[0,T)} \left(\frac{1}{\mathtt{m}}\|\rho(t,\cdot)\|_{H^s} + \frac{1}{\sqrt{k}}\|\Pi_0^{\perp}\phi(t,\cdot)\|_{H^s}\right) \leq 2\varepsilon$$

for some $T > 0$ possibly small. The result follows by Lemma 3.4.8. By Lemma 3.2.4 and estimates (3.2.34) we deduce that the function w solving the equation (3.2.37) is defined over the time interval $[0, T]$ and satisfies

$$\sup_{t\in[0,T]} \|w(t)\|_{H^s} \leq 4\sqrt{\mathtt{m}}(1+\sqrt{k}\beta)\varepsilon\,.$$

As long as $\nu \in [1,2]^d$ (defined as at the beginning of section 3.3) belongs to the full Lebesgue measure set given by Proposition 3.3.1, we can apply Proposition 3.4.3 if ε is small enough. Then by Lemma 3.4.8 and by a standard bootstrap argument we deduce that the solution $w(t)$ is defined for $t \in [0, T_\varepsilon]$, T_ε as in (3.1.5), and

$$\sup_{t\in[0,T_\varepsilon]} \|w(t)\|_{H^s} \leq 8\sqrt{\mathtt{m}}(1+\sqrt{k}\beta)\varepsilon\,.$$

Using again Lemma 3.2.4 and (3.2.34) one can deduce the bound (3.1.5). Hence the thesis follows. □

Chapter 4

Hamiltonian Birkhoff normal form for water waves

4.1 Introduction to Chapter 4

We consider the Euler equations of hydrodynamics for a 2-dimensional perfect and incompressible fluid with constant *vorticity* γ, under the action of gravity and capillary forces at the free surface. The fluid fills the time dependent region

$$\mathcal{D}_\eta := \big\{(x,y) \in \mathbb{T} \times \mathbb{R} \, : \, -\mathtt{h} < y < \eta(t,x)\big\} , \quad \mathbb{T} := \mathbb{T}_x := \mathbb{R}/(2\pi\mathbb{Z}) , \tag{4.1.1}$$

with depth $\mathtt{h} > 0$, possibly infinite, and space periodic boundary conditions. The unknowns are the free surface $y = \eta(t,x)$ of $\mathcal{D}_\eta$ and the divergence free velocity field $\binom{u(t,x,y)}{v(t,x,y)}$. In case of a fluid with constant vorticity $v_x - u_y = \gamma$ (a property which is preserved along the time evolution), the velocity field is the sum of the Couette flow $\binom{-\gamma y}{0}$, which carries all the vorticity γ, and an irrotational field, expressed as the gradient of a harmonic function Φ, called the generalized velocity potential.

We study the water waves problem in the *Hamiltonian* Zakharov-Craig-Sulem [124, 50] formulation, extended by Constantin, Ivanov, Prodanov [45] and Wahlén [115] for constant vorticity fluids. Denoting by $\psi(t,x)$ the evaluation of the generalized velocity potential at the free interface $\psi(t,x) := \Phi(t,x,\eta(t,x))$, one recovers Φ as the unique harmonic function $\Delta\Phi = 0$ in $\mathcal{D}_\eta$ with Dirichlet boundary condition $\Phi = \psi$ at $y = \eta(t,x)$ and Neumann boundary condition $\Phi_y(t,x,y) \to 0$ as $y \to -\mathtt{h}$. Imposing that the fluid particles at the free surface remain on it along the evolution (kinematic boundary condition) and that the pressure of the fluid plus the capillary forces at the free surface is equal to the constant atmospheric pressure (dynamic boundary condition), the time evolution of the fluid is determined by the non-local quasi-linear equations

$$\begin{cases} \partial_t \eta = G(\eta)\psi + \gamma\eta\eta_x \\ \partial_t \psi = -g\eta - \dfrac{1}{2}\psi_x^2 + \dfrac{1}{2}\dfrac{(\eta_x\psi_x + G(\eta)\psi)^2}{1+\eta_x^2} + \kappa\partial_x\Big[\dfrac{\eta_x}{(1+\eta_x^2)^{\frac{1}{2}}}\Big] + \gamma\eta\psi_x + \gamma\partial_x^{-1}G(\eta)\psi \end{cases} \tag{4.1.2}$$

where $g > 0$ is the gravity constant, $\kappa > 0$ is the surface tension coefficient, $\partial_x\big[\frac{\eta_x}{(1+\eta_x^2)^{1/2}}\big]$ is the curvature of the surface and $G(\eta)$ is the Dirichlet-Neumann operator

$$G(\eta)\psi := (-\Phi_x\eta_x + \Phi_y)|_{y=\eta(x)} . \tag{4.1.3}$$

We will derive the equation of motion (4.1.2) for the water waves problem in Appendix C.

The quantity $\int_{\mathbb{T}} \eta(x)\,dx$ is a prime integral of (4.1.2) (indeed $\int_{\mathbb{T}} G(\eta)\psi dx = 0$) and then, with no loss of generality, we restrict to interfaces with zero average $\int_{\mathbb{T}} \eta(x)\,dx = 0$. The component η of the solution of (4.1.2) will lie in a Sobolev space $H_0^{s+\frac{1}{4}}(\mathbb{T})$ of periodic functions with zero mean. Moreover the vector field on the right hand side of (4.1.2) depends only on η and $\psi - \frac{1}{2\pi}\int_{\mathbb{T}} \psi\,dx$ (indeed $G(\eta)[1] = 0$) and therefore ψ will evolve in a homogeneous Sobolev space $\dot{H}^{s-\frac{1}{4}}(\mathbb{T})$ of periodic functions modulo constants.

By [124, 50, 45, 115] the equations (4.1.2) are the Hamiltonian system

$$\partial_t \begin{pmatrix} \eta \\ \psi \end{pmatrix} = J_\gamma \begin{pmatrix} \nabla_\eta H_\gamma(\eta,\psi) \\ \nabla_\psi H_\gamma(\eta,\psi) \end{pmatrix} \qquad \text{where} \qquad J_\gamma := \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & \gamma\partial_x^{-1} \end{pmatrix} \tag{4.1.4}$$

and

$$H_\gamma(\eta,\psi) := \frac{1}{2}\int_{\mathbb{T}} \big(\psi G(\eta)\psi + g\eta^2\big)\,dx + \kappa\int_{\mathbb{T}} \sqrt{1+\eta_x^2}\,dx + \frac{\gamma}{2}\int_{\mathbb{T}} \big(-\psi_x\eta^2 + \frac{\gamma}{3}\eta^3\big)\,dx\,. \tag{4.1.5}$$

The L^2-gradients $(\nabla_\eta H_\gamma, \nabla_\psi H_\gamma)$ in (4.1.4) belong to (a dense subspace of) $\dot{L}^2(\mathbb{T}) \times L_0^2(\mathbb{T})$.

Since the bottom of $\mathcal{D}_\eta$ in (4.1.1) is flat, the Hamiltonian vector field X_γ, defined by the right hand side of (4.1.2), is translation invariant, namely

$$X_\gamma \circ \tau_\varsigma = \tau_\varsigma \circ X_\gamma\,, \quad \forall \varsigma \in \mathbb{R}\,, \qquad \text{where} \qquad \tau_\varsigma \colon f(x) \mapsto f(x+\varsigma) \tag{4.1.6}$$

is the translation operator. Equivalently the Hamiltonian H_γ in (4.1.5) satisfies $H_\gamma \circ \tau_\varsigma = H_\gamma$ for any $\varsigma \in \mathbb{R}$. The associated conservation law induced by Noether theorem is the momentum $\int_{\mathbb{T}} \psi(x)\eta_x(x)\,dx$.

The main result of the present chapter (Theorem 4.1.1) is that, for almost all surface tension coefficients κ, for any integer N, the solutions of the water waves equations (4.1.2) with initial data (smooth enough) of size ε small enough, are defined over a time interval of length at least $c\varepsilon^{-N-1}$. This is the most general almost global existence in time result for the solutions of the water waves equations with periodic boundary conditions known so far. We present below the mathematical literature concerning the local and global well posedness theory of water waves, focusing on the maximal time life span of the solutions.

In order to state precisely the main theorem we define, for any $s \in \mathbb{R}$, the Sobolev spaces

$$H_0^s(\mathbb{T},\mathbb{C}) = \Big\{ u(x) \in H^s(\mathbb{T},\mathbb{C}) \,:\, \int_{\mathbb{T}} u(x)\,dx = 0 \Big\}, \quad \dot{H}^s(\mathbb{T},\mathbb{C}) = H^s(\mathbb{T},\mathbb{C})/\mathbb{C}\,,$$

equipped with the same norm

$$\|u\|_{H_0^s} = \|u\|_{\dot{H}^s} = \Big(\sum_{n\in\mathbb{N}} \|\Pi_n u\|_{L^2}^2 n^{2s}\Big)^{\frac{1}{2}} = \Big(\sum_{j\in\mathbb{Z}\setminus\{0\}} |u_j|^2 |j|^{2s}\Big)^{\frac{1}{2}}$$

where Π_n denote the orthogonal projectors from $L^2(\mathbb{T},\mathbb{C})$ on the sub-spaces spanned by $\{e^{-inx}, e^{inx}\}$ and u_j are the Fourier coefficients of $u(x)$. The quotient map induces an isometry between H_0^s and $\dot{H}^s$ and we shall often identify H_0^s with $\dot{H}^s$. Our main result is the following.

Theorem 4.1.1. **(Almost global in time gravity-capillary water waves with constant vorticity)** *For any value of the gravity $g > 0$, depth $\mathtt{h} \in (0,+\infty]$ and vorticity $\gamma \in \mathbb{R}$, there is a zero measure set $\mathcal{K} \subset (0,+\infty)$ such that, for any surface tension coefficient $\kappa \in (0,+\infty) \setminus \mathcal{K}$, for any N in $\mathbb{N}_0$, there is $s_0 > 0$ and, for any $s \geq s_0$, there are $\varepsilon_0 > 0, c > 0, C > 0$ such that, for any $0 < \varepsilon < \varepsilon_0$, any initial datum*

$$(\eta_0,\psi_0) \in H_0^{s+\frac{1}{4}}(\mathbb{T},\mathbb{R}) \times \dot{H}^{s-\frac{1}{4}}(\mathbb{T},\mathbb{R}) \quad \textit{with} \quad \|\eta_0\|_{H_0^{s+\frac{1}{4}}} + \|\psi_0\|_{\dot{H}^{s-\frac{1}{4}}} < \varepsilon\,,$$

system (4.1.2) *has a unique classical solution* (η, ψ) *in*

$$C^0\Big([-T_\varepsilon, T_\varepsilon], H_0^{s+\frac{1}{4}}(\mathbb{T},\mathbb{R}) \times \dot{H}^{s-\frac{1}{4}}(\mathbb{T},\mathbb{R})\Big) \quad \textit{with} \quad T_\varepsilon \geq c\varepsilon^{-N-1}, \tag{4.1.7}$$

satisfying the initial condition $\eta|_{t=0} = \eta_0, \psi|_{t=0} = \psi_0$. *Moreover*

$$\sup_{t\in[-T_\varepsilon, T_\varepsilon]} \big(\|\eta\|_{H_0^{s+\frac{1}{4}}} + \|\psi\|_{\dot{H}^{s-\frac{1}{4}}}\big) \leq C\varepsilon. \tag{4.1.8}$$

Here are comments on the result.

1. COMPARISON WITH [27]. We first discuss the relation between Theorem 4.1.1 and the result in Berti-Delort [27]. Theorem 4.1.1 extends the one in [27] in two ways: (i) the equations (4.1.2) may have a *non zero vorticity*, whereas the water waves in [27] are irrotational, i.e. $\gamma = 0$. (ii) Also in the irrotational case Theorem 4.1.1 is new since the almost global existence result in [27] holds for initial data (η_0, ψ_0) even in x, whereas Theorem 4.1.1 applies to *any* (η_0, ψ_0). We remark that, in the irrotational case, the subspace of functions even in x -the so called standing waves- is invariant under evolution, whereas for $\gamma \neq 0$ it is not invariant under the flow of (4.1.2) and the approach of [27] can not be applied.

2. PERIODIC SETTING VS $\mathbb{R}^d$. Global (and almost global) in time results [119, 76, 120, 77, 85, 4, 82, 86, 62] have been proved for irrotational water waves equations on $\mathbb{R}^d$ for sufficiently small, localized and smooth initial data, exploiting the dispersive effects of the linear flow. So far no global existence is known for the solutions of (4.1.2) in $\mathbb{R}^2$, not even for irrotational fluids ([62] applies in $\mathbb{R}^3$). The periodic setting is deeply different, as the linear waves oscillate without decaying in time, and the long time dynamics of the equations strongly depends on the presence of N-*wave resonant interactions* and the Hamiltonian and reversible nature of the equations.

3. DISPERSION RELATION AND NON-RESONANT PARAMETERS. The water waves equations (4.1.2) may be regarded as a *quasi-linear* complex PDE of the form

$$\partial_t u = -\mathrm{i}\Omega(D)u + \mathcal{N}(u, \overline{u}), \quad u(x) = \tfrac{1}{\sqrt{2\pi}} \sum\nolimits_{j\in\mathbb{Z}\setminus\{0\}} u_j \, e^{\mathrm{i}jx},$$

where $\mathcal{N}$ is a quadratic non-linearity (depending on derivatives of u) and $\Omega_j(\kappa)$ is the *dispersion relation*

$$\Omega_j(\kappa) := \omega_j(\kappa) + \frac{\gamma}{2}\frac{\mathtt{G}(j)}{j}, \qquad \omega_j(\kappa) := \sqrt{\mathtt{G}(j)\Big(g + \kappa j^2 + \frac{\gamma^2}{4}\frac{\mathtt{G}(j)}{j^2}\Big)}, \tag{4.1.9}$$

where $\mathtt{G}(\xi) = |\xi|\tanh(\mathtt{h}|\xi|)$ ($= |\xi|$ in infinite depth) is the symbol of the Dirichlet-Neumann operator $G(0)$. The linear frequencies $\Omega_j(\kappa)$ actually depend on $(\kappa, g, \mathtt{h}, \gamma)$. The restriction on the parameters required in Theorem 4.1.1 arises to ensure the absence of N-wave resonant interactions

$$\Omega_{j_1}(\kappa) \pm \ldots \pm \Omega_{j_N}(\kappa) \neq 0 \tag{4.1.10}$$

(with quantitative lower bounds as (4.1.13) below) among integer indices $j_1, \ldots, j_N$ which are not super-action preserving, cfr. Definition 4.7.4. In Theorem 4.1.1 we fix arbitrary $(g, \mathtt{h}, \gamma)$ and require $\kappa \notin \mathcal{K}$, but other choices are possible.

4. ENERGY ESTIMATES. The life span estimate (4.1.7) and the bound (4.1.8) for the solutions of (4.1.2) follow by an energy estimate for $\|(\eta,\psi)\|_{X^s} := \|\eta\|_{H_0^{s+\frac{1}{4}}} + \|\psi\|_{\dot{H}^{s-\frac{1}{4}}}$ of the form

$$\|(\eta,\psi)(t)\|_{X^s}^2 \lesssim_{s,N} \|(\eta,\psi)(0)\|_{X^s}^2 + \int_0^t \|(\eta,\psi)(\tau)\|_{X^s}^{N+3}\,\mathrm{d}\tau. \tag{4.1.11}$$

The fact that the right hand side in (4.1.11) contains the same norm $\| \|_{X^s}$ of the left hand side is non trivial because the equations (4.1.2) are quasi-linear. The presence of the exponent N is not trivial at all because the non-linearity in (4.1.2) vanishes only quadratically for $(\eta, \psi) = (0, 0)$. Actually it will be a major consequence of our Hamiltonian Birkhoff normal form reduction, as we explain below.

5. Long time existence of water waves. We now describe the long time existence results proved in literature for *space* periodic water waves, with or without capillarity and vorticity.

- (i) $T_\varepsilon \geq c\varepsilon^{-1}$. The local well posedness theory for free boundary Euler equations has been developed along several years in different scenarios in [104, 122, 49, 117, 118, 92, 2, 3, 35, 100, 109, 95, 46, 111, 112, 84, 7]. As a whole they prove the existence, for sufficiently nice initial data, of classical smooth solutions on a small time interval. When specialized to initial data of size ε in some Sobolev space, imply a time of existence larger than $c\varepsilon^{-1}$ (the non-linearity in (4.1.2) vanishes quadratically at zero). We remark that other large initial data can lead to breakdown in finite time, see for example the papers [43, 47] on "splash" singularities.

- (ii) $T_\varepsilon \geq c\varepsilon^{-2}$. Wu [119], Ionescu-Pusateri [85], Alazard-Delort [4] for pure gravity waves, and Ifrim-Tataru [83], Ionescu-Pusateri [86] for $\kappa > 0$, $g = 0$ and $\mathtt{h} = +\infty$, proved that small data of size ε (periodic or on the line) give rise to irrotational solutions defined on a time interval at least $c\varepsilon^{-2}$. We quote [84] for $\kappa = 0$, $g > 0$, infinite depth and constant vorticity, [80] for irrotational fluids, and [79] in finite depth. All the previous results hold in absence of three wave interactions. Exploiting the Hamiltonian nature of the water waves equations, Berti-Feola-Franzoi [28] proved, for any value of gravity, capillarity and depth, an energy estimate as (4.1.11) with $N = 1$, and so a $c\varepsilon^{-2}$ lower bound for the time of existence. The interesting fact is that in these cases three wave interactions may occur, giving rise to the well known Wilton ripples in fluid mechanics literature. We finally mention the $\varepsilon^{-\frac{5}{3}+}$ long time existence result [87] for periodic 2D gravity-capillary water waves (see [68, 74] for NLS).

- (iii) $T_\varepsilon \geq c\varepsilon^{-3}$. A time of existence larger than $c\varepsilon^{-3}$ has been recently proved for the pure gravity water waves equations in deep water in Berti-Feola-Pusateri [29]. In this case four wave interactions may occur, but the Hamiltonian Birkhoff normal form turns out to be completely integrable by the formal computation in Zakharov-Dyachenko [125],implying an energy estimate as (4.1.11) with $N = 2$. This result has been recently extended by S. Wu [121] for a larger class of initial data, developing a novel approach in configuration space, and, even more recently, by Deng-Ionescu-Pusateri [63] for waves with large period.

- (iv) $T_\varepsilon \geq c_N \varepsilon^{-N}$ *for any* N. Berti-Delort [27] proved, for almost all the values of the surface tension $\kappa \in (0, +\infty)$, an almost global existence result as in Theorem 4.1.1 for the solutions of (4.1.2) in the case of *zero* vorticity $\gamma = 0$ *and* for initial data (η_0, ψ_0) even in x. The restriction on the capillary parameter arises to imply the absence of N-wave interactions, for any N. As already said, Theorem 4.1.1 extends this result for *any* γ and for *any* periodic initial data, see comment 1.

The results [27, 29, 28] are based on para-differential calculus. We remark that all the transformations performed to get energy estimates, as the celebrated Alinhac good unknown [5, 2, 3, 4], are *not* symplectic. In [29, 28] an a-posteriori identification argument allows to prove that the corresponding quadratic and cubic Poincaré-Birkhoff normal forms are nevertheless Hamiltonian. This argument does not work for any N. We note that also the local well posedness approach of S.Wu [117, 118] introduces coordinates which break the Hamiltonian nature of the equations.

The lack of preservation of the Hamiltonian structure is a substantial difficulty in order to deduce long time existence results. A major novelty of this our work is to provide an effective tool to recover, in the framework of para-differential calculus, the nonlinear Hamiltonian structure, at any degree of homogeneity N. The present approach is in principle applicable to a wide range of quasi-linear PDEs.

6. OPEN PROBLEM: we do not know if the almost global solutions of the Cauchy problem proved in Theorem 4.1.1 are global in time or not, being (4.1.2) a quasi-linear system of equations with periodic boundary conditions (no dispersive effects of the flow nor conservation laws at the regularity level of the local well posedness theory can be exploited). Nevertheless several families of time periodic/quasi-periodic solutions of (4.1.2) have been constructed in the last years in [116, 1, 30, 34] (other KAM results for pure gravity water waves are proved in [106, 88, 14, 31, 69]). We point out that it could also happen that small, smooth and localized initial data lead to solutions which blow-up in finite time (as it happen for quasi-linear wave equations [89] and for compressible Euler equations [110]). The following natural question therefore remains still open: what happens to the solutions of (4.1.2) which do not start on a KAM invariant torus for times longer than the ones provided by Theorem 4.1.1?

We now illustrate some of the main ideas of our approach.

1. PARA-DIFFERENTIAL HAMILTONIAN BIRKHOFF NORMAL FORM. For PDEs on a compact manifold (where dispersion is not available) a natural tool to extend the life span of solutions is normal form ideas. This approach has been developed for Hamiltonian semi-linear PDEs starting with the seminal works by Bambusi [15], Bambusi-Grebért [19], Bambusi-Delort-Grebért-Szeftel [17], and for quasi-linear ones by Delort [58, 59]. These methods do not work for the quasi-linear equations (4.1.2), as we explain below. The long time existence result of Theorem 4.1.1 relies on a novel *para-differential Hamiltonian Birkhoff normal form* reduction for quasi-linear PDEs in presence of resonant wave interactions.

The situation is substantially more difficult than in [27] which exploits only the reversible structure of the water waves, and it is preserved by usual para-differential calculus. On the subspace of functions even in x, it implies that its normal form possesses the actions $|u_n|^2$ as prime integrals (on the subspace of even functions the linear frequencies $\omega_j(\kappa)$ in the dispersion relation are simple). On the other hand, without this restriction, the $\omega_j(\kappa)$ in (4.1.9) are double and the approach in [27] fails. We remark that, in view of the 8-wave resonant interactions (4.1.15) described below, also for $\gamma \neq 0$ the approach in [27] fails. In order to prove Theorem 4.1.1, it is necessary to change strategy and *preserve the Hamiltonian nature of the normal form* to show that the *super-actions*

$$\|\Pi_n u\|_{L^2}^2 = |u_{-n}|^2 + |u_n|^2, \quad \forall n \in \mathbb{N}, \tag{4.1.12}$$

are prime integrals. This is a major difficulty since usual para-differential calculus transformations performed to get energy estimates do not preserve the Hamiltonian structure.

2. SYMPLECTIC DARBOUX CORRECTOR. In order to preserve the Hamiltonian structure along the normal form reduction -it is sufficient up to homogeneity N- we construct *symplectic correctors* of usual para-differential transformations. We remind that the first step to apply para-differential calculus to PDEs relies on a suitable para-linearization of the equations. For Hamiltonian PDEs, the para-differential part inherits a linear Hamiltonian structure that is preserved by performing “linearly symplectic” transformations. The aim of the abstract Theorem 4.4.1 is to correct para-differential (more generally spectrally localized) linearly symplectic maps (up to homogeneity N) to *nonlinear* symplectic ones, up to an arbitrary degree of homogeneity. Theorem 4.4.1 is proved via Darboux-type arguments. The Darboux corrector turns out to be a smoothing perturbation of the identity. As a consequence it only slightly modifies the para-differential structure of the PDE.

Symplectic corrections via Darboux-type arguments have been used in different contexts by Kuksin-Perelman [91], Bambusi [16], Cuccagna [54, 55], Bambusi-Maspero [20, 21]. The present case is much more delicate since the symplectic form to be corrected might be an unbounded perturbation of the standard one (in all the above works it is a smoothing perturbation). This requires a novel analysis that we describe below. The present approach is quite efficient in PDE applications, since it systematically allows to symplectically correct usual para-differential transformations which lead to energy estimates.

Para-differential calculus has been also developed by Delort [58, 59] for Hamiltonian quasi-linear Klein-Gordon equations on spheres, with a different approach. Also in these works the Hamiltonian structure is preserved only up to homogeneity N.

3. NON-RESONANCE CONDITIONS. A key ingredient to achieve the Hamiltonian Birkhoff normal form reduction which possesses the super-actions (4.1.12) as prime integrals, are the non-resonance conditions (B.0.1) for the linear frequencies $\Omega_j(\kappa)$ in (4.1.9) proved in Theorem B.0.1, which exclude, for almost all surface tension coefficients, N-*wave interactions*,

$$|\Omega_{j_1}(\kappa) \pm \ldots \pm \Omega_{j_N}(\kappa)| \gtrsim \max(|j_1|, \ldots, |j_N|)^{-\tau}, \tag{4.1.13}$$

for all integer indices $j_1, \ldots, j_N$ which are *not super-action preserving*. Their proof is based on the Delort-Szeftel Theorem 5.1 in [60] about measure estimates for sub-levels of sub-analytic functions.

4. SAP-HAMILTONIANS. Thanks to the non-resonance conditions (B.0.1) we eliminate the Hamiltonian monomials which do not Poisson commute with the super-actions (4.1.12), cfr. Lemma 4.7.7. The remaining monomials, which we call *super-action-preserving* (SAP) (Definition 4.7.8), have either the *integrable* form $|z_{j_1}|^2 \ldots |z_{j_m}|^2$ or the form

$$z_{j_1}\overline{z_{-j_1}} \ldots z_{j_m}\overline{z_{-j_m}} \times \text{integrable monomial} \tag{4.1.14}$$

(with not necessarily distinct indexes $j_1, \ldots, j_m$). The not integrable monomials (4.1.14) allow an exchange of energy between the Fourier modes $\{z_{j_a}, z_{-j_a}\}$, $a = 1, \ldots, m$, but, thanks to the Hamiltonian structure, each super-action $|z_{j_a}|^2 + |z_{-j_a}|^2$ remains constant in time.

We may *not* expect to get an integrable Hamiltonian Birkhoff normal form for the water waves equations (4.1.2) starting from the degree of homogeneity 8. Actually, using the conservation of momentum, the fourth order Hamiltonian Birkhoff normal form is integrable, see Remark 4.7.16. The same holds if $\gamma \neq 0$ also at degree 6. But there are 8-wave resonant interactions corresponding to SAP *not* integrable monomials

$$z_{n_1}\overline{z_{-n_1}} z_{n_2}\overline{z_{-n_2}} z_{-n_3}\overline{z_{n_3}} z_{-n_4}\overline{z_{n_4}}$$

(which are momentum preserving if $n_1 + n_2 = n_3 + n_4$) for *any* $\kappa > 0$ and *any* $\gamma \in \mathbb{R}$. Indeed, for any positive integer n_1, n_2, n_3, n_4 we have, if $\mathtt{h} = +\infty$,

$$\begin{aligned} &\Omega_{n_1}(\kappa) - \Omega_{-n_1}(\kappa) + \Omega_{n_2}(\kappa) - \Omega_{-n_2}(\kappa) + \Omega_{-n_3}(\kappa) - \Omega_{n_3}(\kappa) + \Omega_{-n_4}(\kappa) - \Omega_{n_4}(\kappa) \\ &\stackrel{(4.1.9)}{=} \gamma(\mathrm{sign}(n_1) + \mathrm{sign}(n_2) - \mathrm{sign}(n_3) - \mathrm{sign}(n_4)) \equiv 0\,. \end{aligned} \tag{4.1.15}$$

The analytical difficulties of the *loss of derivatives* caused by the quasi-linearity of the equations and the small divisors in (4.1.10) along the Birkhoff normal form reduction is overcome by preserving the para-differential structure of the equations. The final outcome is that the water waves system in Hamiltonian Birkhoff normal form satisfies an *energy estimate* of the form

$$\|z(t)\|^2_{\dot H^s} \leq \|z(0)\|^2_{\dot H^s} + C(s)\int_0^t \|z(\tau)\|^{N+3}_{\dot H^s}\,\mathrm{d}\tau\,. \tag{4.1.16}$$

5. COMPARISON WITH THE APPROACH IN [58, 59] AND [27]. The Hamiltonian approach to para-differential calculus in [58, 59] is developed for quasi-linear Klein-Gordon equations and can not be applied to prove Theorem 4.1.1. Indeed, since the Klein-Gordon dispersion relation is asymptotically linear, it is not required a reduction to x-independent para-differential operators up to smoothing remainders: since the commutator between first order para-differential operators is still a first order para-differential operator, it

is possible to implement a Hamiltonian Birkhoff normal form reduction in degrees of homogeneity, in the same spirit of semi-linear PDEs. This approach can not be applied for (4.1.2) since the dispersion relation (4.1.9) is super-linear. It is for this reason that we first reduce in Proposition 4.7.2 the paralinearized water waves equations to x-independent symbols up to smoothing remainders. This was done in [27] for $\gamma = 0$ (in a different way) but breaking the Hamiltonian structure (see [71] for NLS). Incidentally we mention that the para-differential normal form in [27] is not a Birkhoff normal form: for standing waves it is not needed to reduce the x-independent symbols to deduce that the actions $|u_n|^2$ are prime integrals.

Summarizing, the proof of Theorem 4.1.1 demands

- a reduction of the water waves equations (4.1.2) to para-differential x-independent symbols up to smoothing remainders, done in [27] for $\gamma = 0$ (in a different way) losing the Hamiltonian structure, and, additionally, reduce the x-independent symbols to super-action preserving Birkhoff normal form;
- preserve the Hamiltonian structure of the Birkhoff normal form, goal achieved in [58, 59] but only for Klein-Gordon equations.

The resolution of these requirements is a main achievement of our work.

Before presenting further ideas of the proof of Theorem 4.1.1 we state the following byproduct of the Darboux-type Theorem 4.4.1 concerning a symplectic version of the Alinhac good unknown. Such result may be of separate interest and use for water waves results in other contexts.

Symplectic good unknown up to homogeneity N. The celebrated Alazard, Burq, Zuily approach [2, 3, 5] to local well posedness extends Lannes [92] introducing the nonlinear, *not* symplectic, Alinach good unknown

$$\omega := \psi - \mathrm{Op}^{BW}\left(B(\eta,\psi)\right)\eta \qquad \text{where} \qquad B(\eta,\psi) := (\Phi_y)(x,y)_{|y=\eta(x)}$$

and Φ is the generalized harmonic velocity potential in (4.1.3) (the notation $\mathrm{Op}^{BW}(\cdot)$ refers to a para-differential operator in the Weyl quantization, according to Definition 4.2.4). The nonlinear map

$$\mathcal{G}_A \begin{pmatrix} \eta \\ \psi \end{pmatrix} = \mathrm{Op}^{BW}\begin{pmatrix} 1 & 0 \\ -B(\eta,\psi) & 1 \end{pmatrix}\begin{pmatrix} \eta \\ \psi \end{pmatrix}, \tag{4.1.17}$$

although not symplectic, is linearly symplectic, namely

$$\mathrm{Op}^{BW}\begin{pmatrix} 1 & 0 \\ -B(\eta,\psi) & 1 \end{pmatrix}^{\top} E_0\, \mathrm{Op}^{BW}\begin{pmatrix} 1 & 0 \\ -B(\eta,\psi) & 1 \end{pmatrix} = E_0 \qquad \text{where} \qquad E_0 := \begin{pmatrix} 0 & -\mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix}. \tag{4.1.18}$$

A direct corollary of Theorem 4.4.1 is the following result, proved at the end of Section 4.4. We refer to Definition 4.2.7 for the precise definition of smoothing operators.

Theorem 4.1.2. (Symplectic good unknown up to homogeneity N) *Let $N \in \mathbb{N}$. There exists a pluri-homogeneous matrix of real smoothing operators $R_{\le N}(\cdot)$ in $\Sigma_1^N \widetilde{\mathcal{R}}_q^{-\varrho} \otimes \mathcal{M}_2(\mathbb{C})$ for any $\varrho \ge 0$ such that*

$$\big(\mathrm{Id} + R_{\le N}(\cdot)\big) \circ \mathcal{G}_A(\eta,\psi) \tag{4.1.19}$$

is symplectic up to homogeneity N, according to (4.3.40).

Let us make some comments about Theorem 4.1.2 and the more general Theorem 4.4.1.

1. The pluri-homogeneous smoothing correcting operators $R_{\le N}(\cdot)$ in (4.1.19) are constructed in Proposition 4.4.7 by a Darboux deformation argument à la Moser. More precisely the $R_{\le N}(\cdot)$ are defined as approximate inverses of approximate flows, up to homogeneity N, generated by smoothing vector fields, which are

algorithmically determined by the Darboux mechanism and depend only on the pluri-homogeneous components of $\mathcal{G}$ up to degree N (more in general of $\mathbf{B}_{\leq N}$ in (4.4.1)).

2. The Alinhac good unknown map (4.1.17) is bounded, but Theorem 4.4.1 also holds for a (spectrally localized) map $\mathbf{B}_{\leq N}(U)$ in (4.4.1) which is *unbounded*. This is the case for example when $\mathbf{B}_{\leq N}(U)$ is the Taylor expansion of a linear flow generated by an unbounded operator, as we discuss later in (4.1.23).

3. We do not expect to find in Theorem 4.4.1 a corrector which produces a completely symplectic transformation of the phase space, but only up to an arbitrary degree of homogeneity N. In the Darboux approach of Section 4.4 this is because the equation (4.4.28) for the smoothing vector field Y^τ, whose flow defines the symplectic corrector, can be solved only in homogeneity having the form $E_c Y^\tau = R(V, Y^\tau, \mathrm{d}_V Y^\tau)$ and so losing derivatives, see Remark 4.4.6. We remark that also the transformations in [58, 59] are symplectic at degree of homogeneity $\leq N$. A similar problem appears in [73].

4. Darboux perturbative methods for Hamiltonian PDEs have been developed in different contexts in [16, 20, 21, 54, 55, 91]. In all these cases, the perturbed symplectic form is a smoothing perturbation of the standard one and thus Darboux correctors are symplectic maps. On the other hand, in this work the perturbed symplectic tensor is a (possibly) *unbounded perturbation* of the standard one,

$$E_{\leq N}(V) = E_c + \text{(possibly) unbounded operator}\,. \tag{4.1.20}$$

A key tool to overcome this difficulty is the structural Lemma 4.4.5.

5. A symplectic map up to homogeneity N, transforms a Hamiltonian system up to homogeneity N into another Hamiltonian system up to homogeneity N, see Lemma 4.3.15.

Further ideas of proof and plan of the chapter

The chapter is divided in

1. Part I) containing the abstract functional setting and the Darboux result;
2. Part II) with the proof of the almost global in time Theorem 4.1.1.

We first illustrate the way we proceed to preserve the Hamiltonian structure, up to homogeneity N, in a generic transformation step along the proof of Theorem 4.1.1.

Symplectic conjugation step up to homogeneity N. Consider a real-to-real system in para-differential form

$$\partial_t U = X(U) = \mathrm{Op}^{BW}(A(U;t,x,\xi))[U] + R(U;t)[U]\,, \quad U = \begin{bmatrix} u \\ \overline{u} \end{bmatrix}\,, \tag{4.1.21}$$

where $A(U;t,x,\xi)$ is a matrix of symbols and $R(U;t)$ are ϱ-smoothing operators, which admit a homogeneous expansion up to homogeneity N, whereas the terms with homogeneity $> N$ are dealt, as in [27], as time dependent symbols and remainders, see Section 4.2.1. This is quite convenient from a technical point of view because it does not demand much information about the higher degree terms. Moreover this enables to directly use the paralinearization of the Dirichlet-Neumann operator proved in [27]. System (4.1.21) is Hamiltonian up to homogeneity N, namely the homogeneous components of the vector field $X(U)$ of degree $\leq N+1$ have the Hamiltonian form

$$J_c \nabla H(U) \qquad \text{where} \qquad J_c := \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix} \tag{4.1.22}$$

is the Poisson tensor and $H(U)$ is a real valued pluri-homogeneous Hamiltonian of degree $\leq N+2$. Moreover the para-differential operator $\mathrm{Op}^{\mathrm{BW}}(A(U))$ in (4.1.21) is a linear Hamiltonian operator, up to homogeneity N, namely of the form $\mathrm{Op}^{\mathrm{BW}}(A(U)) = J_c \mathrm{Op}^{\mathrm{BW}}(B(U))$ where $B(U)$ is a symmetric operator up to homogeneity N, see Definition 4.3.6.

In order to prove energy estimates for (4.1.21) we transform it under several changes of variables. Actually we do not really perform changes of variables of the phase space, but we proceed in the time dependent setting due to the high homogeneity terms. Let us discuss a typical transformation step. Let $\mathcal{G}(U;t) := \mathcal{G}^\tau(U;t)|_{\tau=1}$ be the time 1-flow

$$\partial_\tau \mathcal{G}^\tau(U;t) = J_c \mathrm{Op}^{\mathrm{BW}}\big(B(U;\tau,t,x,\xi)\big)\mathcal{G}^\tau(U;t), \quad \mathcal{G}^0(U;t) = \mathrm{Id}, \tag{4.1.23}$$

generated by a linearly Hamiltonian operator $J_c \mathrm{Op}^{\mathrm{BW}}\big(B(U;\tau,t,x,\xi)\big)$ up to homogeneity N. The transformation $\mathcal{G}(U;t)$ is invertible and bounded on $\dot{H}^s(\mathbb{T}) \times \dot{H}^s(\mathbb{T})$ for any $s \in \mathbb{R}$ and it admits a pluri-homogeneous expansion $\mathcal{G}_{\leq N}(U)$, which is an unbounded operator if the generator $J_c \mathrm{Op}^{\mathrm{BW}}\big(B\big)$ is unbounded, see Section 4.3.3. If U solves (4.1.21) then the variable

$$W := \mathcal{G}(U;t)U \tag{4.1.24}$$

solves a new system in para-differential form

$$\partial_t W = X_+(W) = \mathrm{Op}^{\mathrm{BW}}(A_+(W;t,x,\xi))[W] + R_+(W;t)[W] \tag{4.1.25}$$

(actually the symbols and remainders of homogeneity $> N$ in (4.1.25) are still expressed in terms of U, but for simplicity we skip to discuss this issue here). In Section 4.6 we perform several transformations of this kind, choosing suitable generators $J_c \mathrm{Op}^{\mathrm{BW}}\big(B\big)$ (either bounded or unbounded) in order to obtain a diagonal matrix A_+ with x-independent symbols.

We remark that, with this procedure, since the time one flow map $\mathcal{G}(U;t)$ of the linear Hamiltonian system (4.1.23) is only *linearly* symplectic up to homogeneity N, namely

$$\mathcal{G}(U;t)^\top E_c \mathcal{G}(U;t) = E_c + E_{>N}(U;t), \quad E_{>N}(U;t) = O(\|U\|^{N+1}),$$

where $E_c := J_c^{-1}$ is the standard symplectic tensor, the new system (4.1.25) *is not* Hamiltonian anymore, not even its pluri-homogeneous components of degree $\leq N+1$. The new system (4.1.25) is only linearly symplectic, up to homogeneity N, see Lemma 4.3.9. In order to obtain a new Hamiltonian system up to homogeneity N, we use the Darboux results of Section 4.4 to construct perturbatively a "symplectic corrector" of the transformation (4.1.24).

Let us say some words about the construction of the symplectic corrector. We remark that the perturbed symplectic tensor $E_{\leq N}(V)$ induced by the non-symplectic transformation $\mathcal{G}_{\leq N}(U)$ is *not* a smoothing perturbation of the standard Poisson tensor E_c, cfr. (4.1.20). However, Lemmata 4.4.4 and 4.4.5 prove that, for any pluri-homogeneous vector field $X(V)$, we have

$$E_{\leq N}(V)[X(V)] = E_c X(V) + \nabla \mathcal{W}(V) + \text{ smoothing vector fields} + \text{high homogeneity terms}$$

where $\mathcal{W}(V)$ is a scalar function. This algebraic structural property enables to prove the Darboux Proposition 4.4.7, thus Theorem 4.4.1, via a deformation argument à la Moser. We also remark that the operators $R_{\leq N}(\cdot)$ of Theorem 4.4.1 are smoothing for arbitrary $\varrho \geq 0$, since they have 2 equivalent frequencies, namely $\max_2(n_1, \ldots, n_{p+1}) \sim \max(n_1, \ldots, n_{p+1})$ in (4.2.38), arising by applications of Lemma 4.2.21. This property compensates the presence of unbounded operators in $\mathcal{G}_{\leq N}(U)$.

In conclusion, Theorem 4.4.1 provides a nonlinear map $W + R_{\le N}(W)W$, where $R_{\le N}(W)$ are pluri-homogeneous ϱ-smoothing operators for arbitrary $\varrho > 0$, such that the pluri-homogeneous map

$$\mathcal{D}_N(U) := (\mathrm{Id} + R_{\le N}(\cdot)) \circ \mathcal{G}_{\le N}(U)U$$

is symplectic up to homogeneity N, i.e.

$$\left[\mathrm{d}_U \mathcal{D}_N(U)\right]^\top E_c \left[\mathrm{d}_U \mathcal{D}_N(U)\right] = E_c + E_{>N}(U) \tag{4.1.26}$$

where $E_{>N}(U)$ is an operator of homogeneity degree $\ge N+1$. As a consequence, since (4.1.21) is Hamiltonian up to homogeneity N, the variable

$$Z(t) := \mathcal{D}(U(t);t) := W(t) + R_{\le N}(W(t)) = (\mathrm{Id} + R_{\le N}(\cdot)) \circ \mathcal{G}(U(t);t)U(t)$$

satisfies a system which is *Hamiltonian up to homogeneity* N as well, and which has, since $R_{\le N}(\cdot)$ are smoothing operators, the same para-differential form as in (4.1.25),

$$\partial_t Z = X_{++}(Z) = \mathrm{Op}^{BW}\left(A_{++}(Z;t,x,\xi)\right)[Z] + R_{++}(Z;t)[Z]. \tag{4.1.27}$$

This is the content of Theorem 4.7.1. Note that the matrix of symbols $A_{++}(Z;t,x,\xi)$ in (4.1.27) is obtained by substituting in $A_+(W;t,x,\xi)$ the relation $W = Z - R_{\le N}(Z) + \ldots$ obtained inverting $Z = W + R_{\le N}(W)$ approximately up to homogeneity N. This procedure is rigorously justified in Lemmata A.0.4 and A.0.5.

Scheme of proof of Theorem 4.1.1. In part II we apply the abstract formalism developed in part I to prove Theorem 4.1.1. We proceed as follows.

Section 4.5: paralinearization of the water waves equations. In Section 4.5 we first paralinearize the water waves equations (4.1.2), we introduce the Wahlén variables (η, ζ) in (4.5.2) and the complex variable U in (4.5.6) which diagonalizes the linearized equations at zero. The resulting paralinearized equations (4.5.37) are a Hamiltonian system of the form

$$\partial_t U = J_c \nabla H_\gamma(U) \tag{4.1.28}$$

where H_γ is the Hamiltonian in (4.1.5) written in the variable U. Our goal is to perform several changes of variable to prove energy estimates for (4.5.37), i.e. (4.1.28), valid up to times of order ε^{-N-1}. We split the proof in two major steps.

Section 4.6: Hamiltonian para-differential normal form. In Section 4.6.1 we introduce the good unknown of Alinhac $\mathcal{G}(U)$ (written in complex coordinates), obtaining a system which has energy estimates for times of order ε^{-1}. The Alinhac good unknown is *not* symplectic and therefore the transformed system (4.6.4) is not Hamiltonian anymore. Next we transform (4.6.4) into a diagonal matrix of x-independent symbols up to smoothing remainders, in order to compensate along the Birkhoff normal form reduction process the loss of derivatives due to the small divisors and the quasi-linear nature of the water waves equations, see Proposition 4.6.1. The resulting system

$$\partial_t W = \mathrm{Op}^{BW}_{\mathtt{vec}}\left(\mathtt{im}_{\frac{3}{2}}(U;t,\xi)\right)W + R(U;t)W \tag{4.1.29}$$

is *no longer Hamiltonian*. In (4.1.29) the imaginary part of the symbol $\mathtt{m}_{\frac{3}{2}}$ has order zero and homogeneity larger than N, whereas $R(U;t)$ is a smoothing remainder vanishing linearly in U.

Section 4.7: Hamiltonian Birkhoff normal form. In order to recover the Hamiltonian structure we apply the symplectic corrector given by Theorem 4.4.1: using Theorem 4.7.1 and Lemmata A.0.4 and A.0.5, we obtain in Proposition 4.7.2 system (4.7.4) which is Hamiltonian up to homogeneity N. We perform the Hamiltonian Birkhoff normal form reduction for any value of the surface tension κ outside the set $\mathcal{K}$ defined in Theorem B.0.1. Iteratively we first reduce the p–homogeneous x-independent para-differential symbol to its super-action-preserving component, via the linear flow generated by an unbounded Fourier multiplier, see (4.7.42). Since such transformation is only linearly symplectic, we apply again Theorem 4.7.1 to recover a Hamiltonian system up to homogeneity N, see system (4.7.56). Finally we reduce the $(p+1)$-homogeneous component of the Hamiltonian smoothing vector field to its super-action preserving part, see (4.7.70). The key property is that a super-action preserving Hamiltonian Poisson commutes with the super-actions defined in (4.1.12). After N iterations, the final outcome is the Hamiltonian Birkhoff normal form system (4.7.21), which has the form

$$\partial_t Z = J_c \nabla H^{(\mathtt{SAP})}(Z) + \mathrm{Op}^{BW}_{\mathtt{vec}}\Big(-\mathrm{i}(\mathtt{m}_{\frac{3}{2}})_{>N}(U;t,\xi)\Big)Z + R_{>N}(U;t)U \tag{4.1.30}$$

where $H^{(\mathtt{SAP})}(Z)$ is a super-action preserving Hamiltonian (Definition 4.7.8) and the higher order homogeneity para-differential and smoothing terms admit energy estimates in Sobolev spaces (the imaginary part of the symbol $(\mathtt{m}_{\frac{3}{2}})_{>N}$ has order zero).

Section 4.8: energy estimates. The Hamiltonian Birkhoff normal form equation $\partial_t Z = J_c\nabla H^{(\mathtt{SAP})}(Z)$ obtained neglecting the terms of homogeneity larger than N in (4.1.30) possesses the super–actions $|z_{-n}|^2 + |z_n|^2$, for any $n \in \mathbb{N}$, as prime integrals. Thus it preserves the Sobolev norms and the solutions of (4.1.30) with initial data of size ε have energy estimates up to times of order ε^{-N-1}. In conclusion, since the Sobolev norms of U in (4.1.28) and Z in (4.1.30) are equivalent, we deduce energy estimates for (4.1.28),

$$\|U(t)\|^2_{\dot H^s} \lesssim_{s,K} \|U(0)\|^2_{\dot H^s} + \int_0^t \|U(\tau)\|^{N+3}_{\dot H^s}\,\mathrm{d}\tau$$

valid up to times of order ε^{-N-1}. A standard bootstrap argument concludes the proof of Theorem 4.1.1.

Notation: The notation $A \lesssim B$ means that there exists a constant $C \geq 0$ such that $A \leq CB$. We denote $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Part I

Abstract setting and Darboux symplectic corrector

4.2 Functional Setting

This section contains the abstract functional setting necessary for this chapter.

In Section 4.2.1 we present definitions and results about para-differential calculus following Berti-Delort [27], but defining the different notion of m-operators. Using the same classes of symbols and smoothing operators of [27] has the advantage to directly rely on the result in [27] concerning the paralinearization of the Dirichlet-Neumann operator with multilinear expansions. In Section 4.2.2 we introduce the notion of spectrally localized maps which includes, as a particular case, para-differential operators of any order. Then we prove several properties of spectrally localized maps among which that the transpose of the differential of a homogeneous spectrally localized map is smoothing, see Lemma 4.2.21. This result generalizes a lemma which has been proved in Feola-Iandoli [73] for para-differential operators, and it is relevant for producing the Hamiltonian corrections to the homogeneous components of the vector field in Section 4.4, by means of a Darboux approximate procedure. In Section 4.2.3 we construct approximate inverses of non–linear maps and approximate flows up to an arbitrary degree of homogeneity. In Section 4.2.4 we introduce the formalism of pluri-homogeneous k-forms, Lie derivatives and Cartan's magic formula. Let us first fix some notation used along the chapter.

Function spaces. Along the chapter we deal with real parameters

$$s \geq s_0 \gg K \gg \varrho \gg N \tag{4.2.1}$$

where $N \in \mathbb{N}_0$ is the constant in Theorem 4.1.1.

Given an interval $I \subset \mathbb{R}$ symmetric with respect to $t = 0$ and $s \in \mathbb{R}$, we define the space

$$C_*^K(I, \dot{H}^s(\mathbb{T}, \mathbb{C}^2)) := \bigcap_{k=0}^{K} C^k\big(I, \dot{H}^{s-\frac{3}{2}k}(\mathbb{T}, \mathbb{C}^2)\big)$$

endowed with the norm

$$\sup_{t\in I} \|U(t,\cdot)\|_{K,s} \qquad \text{where} \qquad \|U(t,\cdot)\|_{K,s} := \sum_{k=0}^{K} \|\partial_t^k U(t,\cdot)\|_{\dot{H}^{s-\frac{3}{2}k}}, \tag{4.2.2}$$

and we also consider its subspace

$$C_{*\mathbb{R}}^K(I, \dot{H}^s(\mathbb{T}, \mathbb{C}^2)) := \left\{ U \in C_*^K(I, \dot{H}^s(\mathbb{T}, \mathbb{C}^2)) \;:\; U = \begin{pmatrix} u \\ \overline{u} \end{pmatrix} \right\}.$$

Given $r > 0$ we set $B_s^K(I; r)$ the ball of radius r in $C_*^K(I, \dot{H}^s(\mathbb{T}, \mathbb{C}^2))$ and by $B_{s,\mathbb{R}}^K(I; r)$ the ball of radius r in $C_{*\mathbb{R}}^K(I, \dot{H}^s(\mathbb{T}, \mathbb{C}^2))$.

The parameter s in (4.2.2) denotes the spatial Sobolev regularity of the solution $U(t,\cdot)$ and K its regularity in the time variable. The gravity-capillary water waves vector field loses $3/2$-derivatives, and therefore, differentiating the solution $U(t)$ for k-times in the time variable, there is a loss of $\frac{3}{2}k$-spatial derivatives. The parameter ϱ in (4.2.1) denotes the order where we decide to stop our regularization of the system and depends on the number N of steps of Birkhoff normal form that we will perform and the smallness of the small divisors due to the resonances.

We denote $\dot{L}^2(\mathbb{T}, \mathbb{C}) := \dot{H}^0(\mathbb{T}, \mathbb{C})$ and $\dot{L}_r^2 := \dot{L}^2(\mathbb{T}, \mathbb{R}) = \dot{H}^0(\mathbb{T}, \mathbb{R})$ the subspace of $\dot{L}^2(\mathbb{T}, \mathbb{C})$ made by real valued functions. Given $u, v \in \dot{L}^2(\mathbb{T}, \mathbb{C})$ we define

$$\langle u, v \rangle_{\dot{L}_r^2} := \int_{\mathbb{T}} \Pi_0^{\perp} u(x)\, \Pi_0^{\perp} v(x)\, \mathrm{d}x, \quad \text{respectively} \quad \langle u, v \rangle_{\dot{L}^2} := \int_{\mathbb{T}} \Pi_0^{\perp} u(x)\, \overline{\Pi_0^{\perp} v(x)}\, \mathrm{d}x, \tag{4.2.3}$$

where $\Pi_0^{\perp} u := u - \frac{1}{2\pi}\int u(x)\,\mathrm{d}x$ is the projector onto the zero mean functions.

We also consider the non-degenerate bilinear form on $\dot L^2(\mathbb{T};\mathbb{C}^2)$

$$\Big\langle \begin{pmatrix} v_1^+ \\ v_1^- \end{pmatrix}, \begin{pmatrix} v_2^+ \\ v_2^- \end{pmatrix} \Big\rangle_r := \langle v_1^+, v_2^+ \rangle_{\dot L^2_r} + \langle v_1^-, v_2^- \rangle_{\dot L^2_r}\,. \tag{4.2.4}$$

Fourier expansions. Given a 2π-periodic function $u(x)$ in the homogeneous space $\dot L^2(\mathbb{T},\mathbb{C})$, we identify $u(x)$ with its zero average representative and we expand it in Fourier series as

$$u(x) = \sum_{j\in\mathbb{Z}\setminus\{0\}} \widehat{u}(j)\frac{e^{\mathrm{i}jx}}{\sqrt{2\pi}}, \quad \widehat{u}(j) := \frac{1}{\sqrt{2\pi}}\int_{\mathbb{T}} u(x)e^{-\mathrm{i}jx}\,\mathrm{d}x\,. \tag{4.2.5}$$

We shall expand a function $\begin{pmatrix} u^+ \\ u^- \end{pmatrix}$ as

$$\begin{pmatrix} u^+ \\ u^- \end{pmatrix} = \sum_{\sigma\in\pm}\sum_{j\in\mathbb{Z}\setminus\{0\}} \mathtt{q}^\sigma u_j^\sigma \frac{e^{\mathrm{i}\sigma jx}}{\sqrt{2\pi}}, \quad u_j^\sigma := \widehat{u^\sigma}(\sigma j) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{T}} u^\sigma(x)e^{-\mathrm{i}\sigma jx}\,\mathrm{d}x \tag{4.2.6}$$

where

$$\mathtt{q}^+ := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathtt{q}^- := \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{4.2.7}$$

For $n \in \mathbb{N}$ we denote by Π_n the orthogonal projector from $L^2(\mathbb{T},\mathbb{C})$ to the linear subspace spanned by $\{e^{\mathrm{i}nx}, e^{-\mathrm{i}nx}\}$,

$$(\Pi_n u)(x) := \widehat{u}(n)\frac{e^{\mathrm{i}nx}}{\sqrt{2\pi}} + \widehat{u}(-n)\frac{e^{-\mathrm{i}nx}}{\sqrt{2\pi}}\,, \tag{4.2.8}$$

and we denote by Π_n also the corresponding projector in $L^2(\mathbb{T},\mathbb{C}^2)$.

If $\mathcal{U} = (U_1,\ldots,U_p)$ is a p-tuple of functions and $\vec n = (n_1,\ldots,n_p) \in \mathbb{N}^p$, we set

$$\Pi_{\vec n}\mathcal{U} := \big(\Pi_{n_1}U_1,\ldots,\Pi_{n_p}U_p\big)\,, \qquad \tau_\varsigma\mathcal{U} := \big(\tau_\varsigma U_1,\ldots,\tau_\varsigma U_p\big)\,.$$

For $\vec\jmath_p = (j_1,\ldots,j_p) \in (\mathbb{Z}\setminus\{0\})^p$ and $\vec\sigma_p = (\sigma_1,\ldots,\sigma_p) \in \{\pm\}^p$ we denote $|\vec\jmath_p| := \max(|j_1|,\ldots,|j_p|)$ and

$$u_{\vec\jmath_p}^{\vec\sigma_p} := u_{j_1}^{\sigma_1}\ldots u_{j_p}^{\sigma_p}\,, \qquad \vec\sigma_p\cdot\vec\jmath_p := \sigma_1 j_1 + \ldots + \sigma_p j_p\,. \tag{4.2.9}$$

Note that, under the translation operator τ_ς defined in (4.1.6), the Fourier coefficients of $\tau_\varsigma u$ transform as

$$(\tau_\varsigma u)_j^\sigma = e^{\mathrm{i}\sigma j\varsigma}u_j^\sigma\,.$$

We finally denote

$$\mathfrak{T}_p := \Big\{(\vec\jmath_p,\vec\sigma_p) \in (\mathbb{Z}\setminus\{0\})^p \times \{\pm\}^p : \ \vec\sigma_p\cdot\vec\jmath_p = 0\Big\}\,. \tag{4.2.10}$$

Real-to-real operators and vector fields. Given a linear operator $R(U)[\cdot]$ acting on $\dot L^2(\mathbb{T};\mathbb{C})$ we associate the linear operator defined by the relation

$$\overline{R}(U)[v] := \overline{R(U)[\overline{v}]}\,, \quad \forall v : \mathbb{T}\to\mathbb{C}\,. \tag{4.2.11}$$

An operator $R(U)$ is *real* if $R(U) = \overline{R}(U)$. We say that a matrix of operators acting on $\dot L^2(\mathbb{T};\mathbb{C}^2)$ is *real-to-real*, if it has the form

$$R(U) = \begin{pmatrix} R_1(U) & R_2(U) \\ \overline{R_2}(U) & \overline{R_1}(U) \end{pmatrix}\,, \tag{4.2.12}$$

for any U in

$$\dot{L}^2_{\mathbb{R}}(\mathbb{T},\mathbb{C}^2) := \left\{ V \in \dot{L}^2(\mathbb{T},\mathbb{C}^2) \,:\, V = \begin{pmatrix} v \\ \overline{v} \end{pmatrix} \right\}. \tag{4.2.13}$$

We define similarly $\dot{H}^s_{\mathbb{R}}(\mathbb{T},\mathbb{C}^2)$. A real-to-real matrix of operators $R(U)$ acts in the subspace $\dot{L}^2_{\mathbb{R}}(\mathbb{T},\mathbb{C}^2)$.

If $R_1(U)$ and $R_2(U)$ are real-to-real operators then also $R_1(U) \circ R_2(U)$ is real-to-real.

Similarly we will say that a vector field

$$X(U) := \begin{pmatrix} X(U)^+ \\ X(U)^- \end{pmatrix} \quad \text{is real-to-real if} \quad \overline{X(U)^+} = X(U)^-, \quad \forall U \in \dot{L}^2_{\mathbb{R}}(\mathbb{T},\mathbb{C}^2). \tag{4.2.14}$$

4.2.1 Para-differential calculus

We first introduce the para-differential operators (Definition 4.2.4) following [27]. Then we define the new class of m-Operators (Definition 4.2.5) that, for $m \le 0$, are the smoothing ones (Definition 4.2.7), and we prove properties of m-operators under transposition and composition.

Classes of symbols. We give the definition of the classes of symbols that we use. Roughly speaking the class $\widetilde{\Gamma}^m_p$ contains symbols of order m and homogeneity p in U, whereas the class $\Gamma^m_{K,K',p}$ contains non-homogeneous symbols of order m that vanishes at degree at least p in U and that are $(K-K')$-times differentiable in t; we can think the parameter K' like the number of time derivatives of U that are contained in the symbols. In the following we denote $\dot{H}^\infty(\mathbb{T};\mathbb{C}^2) := \bigcap_{s\in\mathbb{R}} \dot{H}^s(\mathbb{T};\mathbb{C}^2)$.

Definition 4.2.1. *Let $m \in \mathbb{R}$, $p, N \in \mathbb{N}_0$, $K, K' \in \mathbb{N}_0$ with $K' \le K$, and $r > 0$.*

*(i) p-**homogeneous symbols.** We denote by $\widetilde{\Gamma}^m_p$ the space of symmetric p-linear maps from $(\dot{H}^\infty(\mathbb{T};\mathbb{C}^2))^p$ to the space of C^∞ functions from $\mathbb{T}\times\mathbb{R}$ to $\mathbb{C}$, $(x,\xi) \mapsto a(\mathcal{U};x,\xi)$, satisfying the following: there exist $\mu > 0$ and, for any $\alpha, \beta \in \mathbb{N}_0$, there is a constant $C > 0$ such that*

$$|\partial_x^\alpha \partial_\xi^\beta a(\Pi_{\vec{n}}\mathcal{U};x,\xi)| \le C|\vec{n}|^{\mu+\alpha}\langle\xi\rangle^{m-\beta} \prod_{j=1}^{p} \|\Pi_{n_j} U\|_{L^2} \tag{4.2.15}$$

for any $\mathcal{U} = (U_1,\ldots,U_p) \in (\dot{H}^\infty(\mathbb{T};\mathbb{C}^2))^p$ and $\vec{n} = (n_1,\ldots,n_p) \in \mathbb{N}^p$. Moreover we assume that, if for some $(n_0,\ldots,n_p) \in \mathbb{N}_0 \times \mathbb{N}^p$, $\Pi_{n_0} a\left(\Pi_{n_1}U_1,\ldots\Pi_{n_p}U_p;\cdot\right) \neq 0$, then there is a choice of signs $\sigma_0,\ldots,\sigma_p \in \{-1,1\}$ such that $\sum_{j=0}^p \sigma_j n_j = 0$. In addition we require the translation invariance property

$$a(\tau_\varsigma \mathcal{U};x,\xi) = a(\mathcal{U};x+\varsigma,\xi), \quad \forall \varsigma \in \mathbb{R}, \tag{4.2.16}$$

where τ_ς is the translation operator in (4.1.6).

For $p = 0$ we denote by $\widetilde{\Gamma}^m_0$ the space of constant coefficients symbols $\xi \mapsto a(\xi)$ which satisfy (4.2.15) with $\alpha = 0$ and the right hand side replaced by $C\langle\xi\rangle^{m-\beta}$.

We denote by $\Sigma_p^N \widetilde{\Gamma}^m_q$ the class of pluri-homogeneous symbols $\sum_{q=p}^N a_q$ with $a_q \in \widetilde{\Gamma}^m_q$. For $p \ge N+1$ we mean that the sum is empty.

*(ii) **Non-homogeneous symbols.** We denote by $\Gamma^m_{K,K',p}[r]$ the space of functions $(U;t,x,\xi) \mapsto a(U;t,x,\xi)$, defined for $U \in B^{K'}_{s_0}(I;r)$ for some s_0 large enough, with complex values, such that for any $0 \le k \le K-K'$, any $\sigma \ge s_0$, there are $C > 0$, $0 < r(\sigma) < r$ and for any $U \in B^K_{s_0}(I;r(\sigma)) \cap C^{k+K'}_*(I,\dot{H}^\sigma(\mathbb{T};\mathbb{C}^2))$ and any $\alpha, \beta \in \mathbb{N}_0$, with $\alpha \le \sigma - s_0$ one has the estimate*

$$|\partial_t^k \partial_x^\alpha \partial_\xi^\beta a(U;t,x,\xi)| \le C\langle\xi\rangle^{m-\beta} \|U\|^{p-1}_{k+K',s_0} \|U\|_{k+K',\sigma}. \tag{4.2.17}$$

If $p=0$ the right hand side has to be replaced by $C\langle\xi\rangle^{m-\beta}$.

(iii) **Symbols.** *We denote by $\Sigma\Gamma^m_{K,K',p}[r,N]$ the space of functions $(U;t,x,\xi)\mapsto a(U;t,x,\xi)$, with complex values such that there are homogeneous symbols $a_q\in\widetilde{\Gamma}^m_q$, $q=p,\dots,N$ and a non-homogeneous symbol $a_{>N}\in\Gamma^m_{K,K',N+1}$ such that*

$$a(U;t,x,\xi)=\sum_{q=p}^{N}a_q(U,\dots,U;x,\xi)+a_{>N}(U;t,x,\xi)\,. \tag{4.2.18}$$

We denote by $\Sigma\Gamma^m_{K,K',p}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ the space of 2×2 matrices with entries in $\Sigma\Gamma^m_{K,K',p}[r,N]$.

We say that a symbol $a(U;t,x,\xi)$ is real *if it is real valued for any $U\in B^{K'}_{s_0,\mathbb{R}}(I;r)$.*

- If $a(\mathcal{U};\cdot)$ is a homogeneous symbol in $\widetilde{\Gamma}^m_p$ then $a(U,\dots,U;\cdot)$ belongs to $\Gamma^m_{K,0,p}[r]$, for any $r>0$.
- If a is a symbol in $\Sigma\Gamma^m_{K,K',p}[r,N]$ then $\partial_x a\in\Sigma\Gamma^m_{K,K',p}[r,N]$ and $\partial_\xi a\in\Sigma\Gamma^{m-1}_{K,K',p}[r,N]$. If in addition b is a symbol in $\Sigma\Gamma^{m'}_{K,K',p'}[r,N]$ then $ab\in\Sigma\Gamma^{m+m'}_{K,K',p+p'}[r,N]$.
- **Notation for p-homogeneous symbols:** If $a_p(U_1,\dots,U_p;x,\xi)$ is a p-homogeneous symbol, with a slightly abuse of notation, we also denote by $a_p(U;x,\xi):=a_p(U,\dots,U;x,\xi)$ the corresponding polynomial and say that $a_p(U;x,\xi)$ is in $\widetilde{\Gamma}^m_p$.

Remark 4.2.2. (Fourier representation of symbols) The translation invariance property (4.2.16) means that the dependence with respect to the variable x of a symbol $a(\mathcal{U};x,\xi)$ enters only through the functions $\mathcal{U}(x)$, implying that a symbol $a_q(U;x,\xi)$ in $\widetilde{\Gamma}^m_q$, $m\in\mathbb{R}$, has the form (recall notation (4.2.9))

$$a_q(U;x,\xi)=\sum_{\vec{\jmath}\in(\mathbb{Z}\setminus\{0\})^q,\vec{\sigma}\in\{\pm1\}^q}(a_q)^{\vec{\sigma}}_{\vec{\jmath}}(\xi)u^{\vec{\sigma}}_{\vec{\jmath}}e^{\mathrm{i}\vec{\sigma}\cdot\vec{\jmath}x} \tag{4.2.19}$$

where $(a_q)^{\vec{\sigma}}_{\vec{\jmath}}(\xi)\in\mathbb{C}$ are Fourier multipliers of order m satisfying: there exists $\mu>0$, and for any $\beta\in\mathbb{N}_0$, there is $C_\beta>0$ such that

$$|\partial^\beta_\xi(a_q)^{\vec{\sigma}}_{\vec{\jmath}}(\xi)|\le C_\beta|\vec{\jmath}|^\mu\langle\xi\rangle^{m-\beta},\quad\forall(\vec{\jmath},\vec{\sigma})\in(\mathbb{Z}\setminus\{0\})^q\times\{\pm\}^q\,. \tag{4.2.20}$$

A symbol $a_q(U;x,\xi)$ as in (4.2.19) is real if

$$\overline{(a_q)^{\vec{\sigma}}_{\vec{\jmath}}(\xi)}=(a_q)^{-\vec{\sigma}}_{\vec{\jmath}}(\xi)\,. \tag{4.2.21}$$

By (4.2.19) a symbol a_1 in $\widetilde{\Gamma}^m_1$ can be written as $a_1(U;x,\xi)=\sum_{j\in\mathbb{Z}\setminus\{0\},\sigma=\pm}(a_1)^\sigma_j(\xi)u^\sigma_je^{\mathrm{i}\sigma jx}$, and therefore, if a_1 is independent of x, it is actually $a_1\equiv0$.

We also define classes of functions in analogy with our classes of symbols.

Definition 4.2.3. (Functions) *Let $p,N\in\mathbb{N}_0$, $K,K'\in\mathbb{N}_0$ with $K'\le K$, $r>0$. We denote by $\widetilde{\mathcal{F}}_p$, resp. $\mathcal{F}_{K,K',p}[r]$, $\Sigma\mathcal{F}_{K,K',p}[r,N]$, the subspace of $\widetilde{\Gamma}^0_p$, resp. $\Gamma^0_{K,K',p}[r]$, resp. $\Sigma\Gamma^0_{K,K',p}[r,N]$, made of those symbols which are independent of ξ. We write $\widetilde{\mathcal{F}}^{\mathbb{R}}_p$, resp. $\mathcal{F}^{\mathbb{R}}_{K,K',p}[r]$, $\Sigma\mathcal{F}^{\mathbb{R}}_{K,K',p}[r,N]$, to denote functions in $\widetilde{\mathcal{F}}_p$, resp. $\mathcal{F}_{K,K',p}[r]$, $\Sigma\mathcal{F}_{K,K',p}[r,N]$, which are real valued for any $U\in B^{K'}_{s_0,\mathbb{R}}(I;r)$.*

Para-differential quantization. Given $p \in \mathbb{N}_0$ we consider functions $\chi_p \in C^\infty(\mathbb{R}^p \times \mathbb{R};\mathbb{R})$ and $\chi \in C^\infty(\mathbb{R} \times \mathbb{R};\mathbb{R})$, even with respect to each of their arguments, satisfying, for $0 < \delta \ll 1$,

$$\begin{aligned} &\operatorname{supp}\chi_p \subset \{(\xi',\xi) \in \mathbb{R}^p \times \mathbb{R}; |\xi'| \le \delta\langle\xi\rangle\}, \qquad \chi_p(\xi',\xi) \equiv 1 \text{ for } |\xi'| \le \delta\langle\xi\rangle/2, \\ &\operatorname{supp}\chi \subset \{(\xi',\xi) \in \mathbb{R} \times \mathbb{R}; |\xi'| \le \delta\langle\xi\rangle\}, \qquad \chi(\xi',\xi) \equiv 1 \text{ for } |\xi'| \le \delta\langle\xi\rangle/2. \end{aligned}$$

For $p = 0$ we set $\chi_0 \equiv 1$. We assume moreover that

$$|\partial_\xi^\alpha \partial_{\xi'}^\beta \chi_p(\xi',\xi)| \le C_{\alpha,\beta}\langle\xi\rangle^{-\alpha-|\beta|}, \ \forall \alpha \in \mathbb{N}_0, \beta \in \mathbb{N}_0^p, \ \ |\partial_\xi^\alpha \partial_{\xi'}^\beta \chi(\xi',\xi)| \le C_{\alpha,\beta}\langle\xi\rangle^{-\alpha-\beta}, \ \forall \alpha, \beta \in \mathbb{N}_0.$$

If $a(x,\xi)$ is a smooth symbol we define its Weyl quantization as the operator acting on a 2π-periodic function $u(x)$ (written as in (4.2.5)) as

$$\operatorname{Op}^W(a)u = \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \Big(\sum_{j\in\mathbb{Z}} \widehat{a}\big(k-j, \frac{k+j}{2}\big)\widehat{u}(j)\Big) \frac{e^{\mathrm{i}kx}}{\sqrt{2\pi}}$$

where $\widehat{a}(k,\xi)$ is the $k^{th}-$Fourier coefficient of the $2\pi-$periodic function $x \mapsto a(x,\xi)$.

Definition 4.2.4. (Bony-Weyl quantization) *If a is a symbol in $\widetilde{\Gamma}^m_p$, respectively in $\Gamma^m_{K,K',p}[r]$, we set*

$$\begin{aligned} a_{\chi_p}(\mathcal{U};x,\xi) &:= \sum_{\vec{n}\in\mathbb{N}^p} \chi_p(\vec{n},\xi)\, a(\Pi_{\vec{n}}\mathcal{U};x,\xi), \\ a_\chi(U;t,x,\xi) &:= \frac{1}{2\pi}\int_{\mathbb{R}} \chi(\xi',\xi)\widehat{a}(U;t,\xi',\xi)e^{\mathrm{i}\xi' x}\mathrm{d}\xi', \end{aligned} \tag{4.2.22}$$

where in the last equality $\widehat{a}$ stands for the Fourier transform with respect to the x variable, and we define the Bony-Weyl *quantization of a as*

$$\operatorname{Op}^{BW}(a(\mathcal{U};\cdot)) = \operatorname{Op}^W(a_{\chi_p}(\mathcal{U};\cdot)), \qquad \operatorname{Op}^{BW}(a(U;t,\cdot)) = \operatorname{Op}^W(a_\chi(U;t,\cdot)). \tag{4.2.23}$$

If a is a symbol in $\Sigma\Gamma^m_{K,K',p}[r,N]$, we define its Bony-Weyl *quantization*

$$\operatorname{Op}^{BW}(a(U;t,\cdot)) = \sum_{q=p}^{N} \operatorname{Op}^{BW}(a_q(U,\ldots,U;\cdot)) + \operatorname{Op}^{BW}(a_{>N}(U;t,\cdot)).$$

We will use also the notation

$$\operatorname{Op}^{BW}_{\mathtt{vec}}(a(U;t,x,\xi)) := \operatorname{Op}^{BW}\left(\begin{bmatrix} a(U;t,x,\xi) & 0 \\ 0 & \overline{a^\vee(U;t,x,\xi)} \end{bmatrix}\right), \quad a^\vee(x,\xi) := a(x,-\xi). \tag{4.2.24}$$

• The operator $\operatorname{Op}^{BW}(a)$ acts on homogeneous spaces of functions, see Proposition 3.8 of [27].
• If a is a homogeneous symbol, the two definitions of quantization in (4.2.23) differ by a smoothing operator according to Definition 4.2.7 below. With the first regularization in (4.2.22) we guarantee the important property that $\operatorname{Op}^{BW}(a)$ is a spectrally localized map according to Definition 4.2.16 below.
• The action of $\operatorname{Op}^{BW}(a)$ on homogeneous spaces only depends on the values of the symbol $a = a(U;t,x,\xi)$ (or $a(\mathcal{U};t,x,\xi)$) for $|\xi| \ge 1$. Therefore, we may identify two symbols $a(U;t,x,\xi)$ and $b(U;t,x,\xi)$ if they agree for $|\xi| \ge 1/2$. In particular, whenever we encounter a symbol that is not smooth at $\xi = 0$, such as, for example, $a = g(x)|\xi|^m$ for $m \in \mathbb{R}\setminus\{0\}$, or $\operatorname{sign}(\xi)$, we will consider its smoothed out version $\chi(\xi)a$, where $\chi \in C^\infty(\mathbb{R};\mathbb{R})$ is an even and positive cut-off function satisfying

$$\chi(\xi) = 0 \ \text{ if } \ |\xi| \le \tfrac{1}{8}, \quad \chi(\xi) = 1 \ \text{ if } \ |\xi| > \tfrac{1}{4}, \quad \partial_\xi \chi(\xi) > 0 \quad \forall \xi \in \big(\tfrac{1}{8}, \tfrac{1}{4}\big).$$

• Definition 4.2.4 is independent of the cut-off functions χ_p, χ, up to smoothing operators that we define below (see Definition 4.2.7), see the remark at pag. 50 of [27].
• If for some $(n_0, \dots, n_{p+1}) \in \mathbb{N}^{p+2}$, $\Pi_{n_0} \mathrm{Op}^{\mathrm{BW}}(a(\Pi_{\vec{n}}\mathcal{U}; \cdot))\Pi_{n_{p+1}} U_{p+1} \neq 0$, then there exist signs $\epsilon_j \in \{\pm\}$, $j = 0, \dots, p+1$, such that $\sum_0^{p+1} \epsilon_j n_j = 0$ and the indices satisfy (see Proposition 3.8 in [27])

$$n_0 \sim n_{p+1}, \quad n_j \leq C\delta n_{p+1}, \quad n_j \leq C\delta n_0, \quad j = 1, \dots, p. \tag{4.2.25}$$

• Given a para-differential operator $A = \mathrm{Op}^{\mathrm{BW}}(a(x,\xi))$ it results

$$\overline{A} = \mathrm{Op}^{\mathrm{BW}}\left(\overline{a(x,-\xi)}\right), \quad A^\top = \mathrm{Op}^{\mathrm{BW}}(a(x,-\xi)), \quad A^* = \mathrm{Op}^{\mathrm{BW}}\left(\overline{a(x,\xi)}\right), \tag{4.2.26}$$

where $A^\top$ and A^* denote respectively the transposed and adjoint operator with respect to the complex, respectively real, scalar product of $\dot{L}^2$ in (4.2.3). It results $A^* = \overline{A}^\top$.
• A para-differential operator $A = \mathrm{Op}^{\mathrm{BW}}(a(x,\xi))$ is *real* (i.e. $A = \overline{A}$) if

$$\overline{a(x,\xi)} = a^\vee(x,\xi) \quad \text{where} \quad a^\vee(x,\xi) := a(x,-\xi). \tag{4.2.27}$$

• A matrix of para-differential operators $\mathrm{Op}^{\mathrm{BW}}(A(U;t,x,\xi))$ is real-to-real, i.e. (4.2.12) holds, if and only if the matrix of symbols $A(U;t,x,\xi)$ has the form

$$A(U;x,\xi) = \begin{pmatrix} a(U;t,x,\xi) & b(U;t,x,\xi) \\ \overline{b^\vee(U;t,x,\xi)} & \overline{a^\vee(U;t,x,\xi)} \end{pmatrix}. \tag{4.2.28}$$

Classes of m-Operators and smoothing Operators. Given integers $(n_1, \dots, n_{p+1}) \in \mathbb{N}^{p+1}$, we denote by $\max_2(n_1, \dots, n_{p+1})$ the second largest among $n_1, \dots, n_{p+1}$. We shall often use that $\max_2$ is monotone in each component, i.e. if $n'_j \geq n_j$ for some j, then

$$\max{}_2(n_1, \dots, n_j, \dots, n_p) \leq \max{}_2(n_1, \dots, n'_j, \dots, n_p). \tag{4.2.29}$$

In addition $\max_2$ is non decreasing by adding elements, namely

$$\max{}_2(n_1, \dots, n_p) \leq \max{}_2(n_1, \dots, n_p, n_{p+1}). \tag{4.2.30}$$

We now define the m-operators. The class $\widetilde{\mathcal{M}}_p^m$ denotes multilinear operators that lose m derivatives and are p-homogeneous in U, while the class $\mathcal{M}^m_{K,K',p}$ contains non-homogeneous operators which lose m derivatives, vanish at degree at least p in U, satisfy tame estimates and are $(K-K')$-times differentiable in t. The constant μ in (4.2.31) takes into account possible loss of derivatives in the "low" frequencies.

Definition 4.2.5. (Classes of m-operators) *Let $m \in \mathbb{R}$, $p, N \in \mathbb{N}_0$ $K, K' \in \mathbb{N}_0$ with $K' \leq K$, and $r > 0$.*

(i) **p-homogeneous m-operators.** *We denote by $\widetilde{\mathcal{M}}_p^m$ the space of $(p+1)$-linear operators M from $(\dot{H}^\infty(\mathbb{T};\mathbb{C}^2))^p \times \dot{H}^\infty(\mathbb{T};\mathbb{C})$ to $\dot{H}^\infty(\mathbb{T};\mathbb{C})$ which are symmetric in $(U_1, \dots, U_p)$, of the form*

$$(U_1, \dots, U_{p+1}) \to M(U_1, \dots, U_p)U_{p+1}$$

that satisfy the following. There are $\mu \geq 0$, $C > 0$ such that

$$\|\Pi_{n_0} M(\Pi_{\vec{n}}\mathcal{U})\Pi_{n_{p+1}} U_{p+1}\|_{L^2} \leq C \max{}_2(n_1, \dots, n_{p+1})^\mu \max(n_1, \dots, n_{p+1})^m \prod_{j=1}^{p+1} \|\Pi_{n_j} U_j\|_{L^2} \tag{4.2.31}$$

for any $\mathcal{U} = (U_1, \ldots, U_p) \in (\dot{H}^\infty(\mathbb{T};\mathbb{C}^2))^p$*, any* $U_{p+1} \in \dot{H}^\infty(\mathbb{T};\mathbb{C})$*,* $\vec{n} = (n_1, \ldots, n_p)$ *in* $\mathbb{N}^p$*, any* $n_0, n_{p+1} \in \mathbb{N}$*. Moreover, if*

$$\Pi_{n_0} M(\Pi_{n_1} U_1, \ldots, \Pi_{n_p} U_p) \Pi_{n_{p+1}} U_{p+1} \neq 0 \,, \tag{4.2.32}$$

then there is a choice of signs $\sigma_0, \ldots, \sigma_{p+1} \in \{\pm 1\}$ *such that* $\sum_{j=0}^{p+1} \sigma_j n_j = 0$*. In addition we require the translation invariance property*

$$M(\tau_\varsigma \mathcal{U})[\tau_\varsigma U_{p+1}] = \tau_\varsigma\big(M(\mathcal{U}) U_{p+1}\big) \,, \quad \forall \varsigma \in \mathbb{R} \,. \tag{4.2.33}$$

We denote $\widetilde{\mathcal{M}}_p := \cup_{m \geq 0} \widetilde{\mathcal{M}}_p^m$ *and* $\Sigma_p^N \widetilde{\mathcal{M}}_q^m$ *the class of pluri-homogeneous operators* $\sum_{q=p}^N M_q$ *with* M_q *in* $\widetilde{\mathcal{M}}_q^m$*. For* $p \geq N+1$ *we mean that the sum is empty. We set* $\Sigma_p \widetilde{\mathcal{M}}_q^m := \bigcup_{N \in \mathbb{N}} \Sigma_p^N \widetilde{\mathcal{M}}_q^m$*.*

(ii) **Non-homogeneous** m**-operators.** *We denote by* $\mathcal{M}_{K,K',p}^m[r]$ *the space of operators* $(U,t,V) \mapsto M(U;t)V$ *defined on* $B_{s_0}^{K'}(I;r) \times I \times C_*^0(I, \dot{H}^{s_0}(\mathbb{T},\mathbb{C}))$ *for some* $s_0 > 0$*, which are linear in the variable* V *and such that the following holds true. For any* $s \geq s_0$ *there are* $C > 0$ *and* $r(s) \in]0, r[$ *such that for any* $U \in B_{s_0}^K(I; r(s)) \cap C_*^K(I, \dot{H}^s(\mathbb{T},\mathbb{C}^2))$*, any* $V \in C_*^{K-K'}(I, \dot{H}^s(\mathbb{T},\mathbb{C}))$*, any* $0 \leq k \leq K - K'$*,* $t \in I$*, we have that*

$$\|\partial_t^k\left(M(U;t)V\right)\|_{\dot{H}^{s-\frac{3}{2}k-m}} \leq C \sum_{k'+k''=k} \|V\|_{k'',s} \|U\|_{k'+K',s_0}^p + \|V\|_{k'',s_0} \|U\|_{k'+K',s_0}^{p-1} \|U\|_{k'+K',s} \,. \tag{4.2.34}$$

In case $p = 0$ *we require the estimate* $\|\partial_t^k\left(M(U;t)V\right)\|_{\dot{H}^{s-\frac{3}{2}k-m}} \leq C\|V\|_{k,s}$*.*

(iii) m**-Operators.** *We denote by* $\Sigma\mathcal{M}_{K,K',p}^m[r,N]$*, the space of operators* $(U,t,V) \to M(U;t)V$ *such that there are homogeneous* m*-operators* M_q *in* $\widetilde{\mathcal{M}}_q^m$*,* $q = p, \ldots, N$ *and a non–homogeneous* m*-operator* $M_{>N}$ *in* $\mathcal{M}_{K,K',N+1}^m[r]$ *such that*

$$M(U;t)V = \sum_{q=p}^N M_q(U, \ldots, U)V + M_{>N}(U;t)V \,. \tag{4.2.35}$$

We denote

$$\widetilde{\mathcal{M}}_p := \bigcup_{m \geq 0} \widetilde{\mathcal{M}}_p^m \,, \ \ \mathcal{M}_{K,K',p}[r] := \bigcup_{m \geq 0} \mathcal{M}_{K,K',p}^m[r] \,, \ \ \Sigma\mathcal{M}_{K,K',p}[r,N] := \bigcup_{m \geq 0} \Sigma\mathcal{M}_{K,K',p}^m[r,N] \,,$$

and $\Sigma\mathcal{M}_{K,K',p}^m[r,N] \otimes \mathcal{M}_2(\mathbb{C})$ *the space of* 2×2 *matrices whose entries are operators in* $\Sigma\mathcal{M}_{K,K',p}^m[r,N]$*.*

- If $M(U, \ldots, U)$ is a p–homogeneous m-operator in $\widetilde{\mathcal{M}}_p^m$ then the differential of the non–linear map $M(U, \ldots, U)U$, $\mathrm{d}_U\big(M(U, \ldots, U)U\big)V = pM(V, U, \ldots, U)U + M(U, \ldots, U)V$ is a p–homogeneous m-operator in $\widetilde{\mathcal{M}}_p^m$. This follows because the right hand side of (4.2.31) is symmetric in $(n_1, \ldots, n_{p+1})$.
- If $m_1 \leq m_2$ then $\Sigma\mathcal{M}_{K,K',p}^{m_1}[r,N] \subseteq \Sigma\mathcal{M}_{K,K',p}^{m_2}[r,N]$.
- **Notation for** p**-homogeneous** m**-operators:** if $M(U_1, \ldots, U_p)$ is a p-homogeneous m-operator, we shall often denote by $M(U) := M(U, \ldots, U)$ the corresponding polynomial and say that $M(U)$ is in $\widetilde{\mathcal{M}}_p^m$. Viceversa, a polynomial can be represented by a $(p+1)$-linear form $M(U_1, \ldots, U_p)U_{p+1}$ not necessarily symmetric in the internal variables. If it fulfills the symmetric estimate (4.2.31), the polynomial is generated by the m-operator in $\widetilde{\mathcal{M}}_p^m$ obtained by symmetrization of the internal variables. We will do this consistently without mentioning it further.
- **Notation for projection on homogeneous components:** given an operator $M(U;t)$ in $\Sigma\mathcal{M}_{K,K',p}^m[r,N]$ of the form (4.2.35) we denote by

$$\mathcal{P}_{\leq N}[M(U;t)] := \sum_{q=p}^N M_q(U) \,, \quad \text{resp.} \quad \mathcal{P}_q[M(U;t)] := M_q(U) \,, \tag{4.2.36}$$

the projections on the pluri-homogeneous, resp. homogeneous, operators in $\Sigma_p^N \widetilde{\mathcal{M}}_q^m$, resp. in $\widetilde{\mathcal{M}}_q^m$. Given an integer $p \le p' \le N$ we also denote

$$\mathcal{P}_{\ge p'}[M(U;t)] := \sum_{q=p'}^{N} M_q(U), \quad \mathcal{P}_{\le p'}[M(U;t)] := \sum_{q=p}^{p'} M_q(U)\,.$$

The same notation will be also used to denote pluri-homogeneous/homogeneous components of symbols.

Remark 4.2.6. Definition 4.2.5 of homogeneous m-operators is different than the one in Definition 3.9 in [27], due to the different bound (4.2.31). However for $m \ge 0$ the class of homogeneous m-operators contains the class of homogeneous maps of order m in Definition 3.9 of [27], and in view of (4.2.31) is contained in the class of maps of order $m+\mu$ of [27]. On the other hand the class of non-homogeneous m-operators coincides with the class of non-homogeneous maps in Definition 3.9 of [27].

If $m \le 0$ the operators in $\Sigma\mathcal{M}^m_{K,K',p}[r,N]$ are referred to as smoothing operators.

Definition 4.2.7. (Smoothing operators) *Let $\varrho \ge 0$. A $(-\varrho)$-operator $R(U)$ belonging to $\Sigma\mathcal{M}^{-\varrho}_{K,K',p}[r,N]$ is called a smoothing operator. Along the chapter will use also the notation*

$$\begin{aligned} &\widetilde{\mathcal{R}}_p^{-\varrho} := \widetilde{\mathcal{M}}_p^{-\varrho}, \quad \Sigma_p^N \widetilde{\mathcal{R}}_q^{-\varrho} := \Sigma_p^N \widetilde{\mathcal{M}}_q^{-\varrho}, \\ &\mathcal{R}^{-\varrho}_{K,K',p}[r] := \mathcal{M}^{-\varrho}_{K,K',p}[r], \quad \Sigma\mathcal{R}^{-\varrho}_{K,K',p}[r,N] := \Sigma\mathcal{M}^{-\varrho}_{K,K',p}[r,N]\,. \end{aligned} \tag{4.2.37}$$

• Given $\varrho \ge 0$, an operator $R(U)$ belongs to $\widetilde{\mathcal{R}}_p^{-\varrho}$ if and only if there is $\mu = \mu(\varrho) \ge 0$ and $C > 0$ such that

$$\|\Pi_{n_0} R(\Pi_{\vec{n}}\mathcal{U})\Pi_{n_{p+1}} U_{p+1}\|_{L^2} \le C \frac{\max_2(n_1,\dots,n_{p+1})^{\mu}}{\max(n_1,\dots,n_{p+1})^{\varrho}} \prod_{j=1}^{p+1} \|\Pi_{n_j} U_j\|_{L^2}\,. \tag{4.2.38}$$

We remark that Definition 4.2.7 of smoothing operators coincides with Definition 3.7 in [27].
• In view of (4.2.31) and (4.2.38) a homogeneous m-operator in $\widetilde{\mathcal{M}}_p^m$ with the property that, on its support, $\max_2(n_1,\dots,n_{p+1}) \sim \max(n_1,\dots,n_{p+1})$ is actually a *smoothing* operator in $\widetilde{\mathcal{R}}_p^{-\varrho}$ for *any* $\varrho \ge 0$.
• The Definition 4.2.7 of smoothing operators is modeled to gather remainders which satisfy either the property $\max_2(n_1,\dots,n_{p+1}) \sim \max(n_1,\dots,n_{p+1})$ or arise as remainders of compositions of para-differential operators, see Proposition 4.2.14 below, and thus have a fixed order ϱ of regularization.

Lemma 4.2.8. *If $M(U_1,\dots,U_p)$ is a p–homogeneous m-operator in $\widetilde{\mathcal{M}}_p^m$ then for any $K \in \mathbb{N}_0$ and $0 \le k \le K$ there exists $s_0 > 0$ such that for any $s \ge s_0$, for any $U \in C^K_*(I, \dot{H}^s(\mathbb{T},\mathbb{C}^2))$, any $v \in C^K_*(I, \dot{H}^s(\mathbb{T},\mathbb{C}))$, one has*

$$\begin{aligned} \|\partial_t^k(M(U_1,\dots,U_p)v)\|_{\dot{H}^{s-m-\frac{3}{2}k}} \lesssim_K \sum_{k_1+\dots+k_{p+1}=k} \Big(&\|v\|_{k_{p+1},s} \prod_{a=1}^{p} \|U_a\|_{k_a,s_0} \\ &+ \|v\|_{k_{p+1},s_0} \sum_{\overline{a}=1}^{p} \|U_{\overline{a}}\|_{k_{\overline{a}},s} \prod_{\substack{a=1\\ a\ne\overline{a}}}^{p} \|U_a\|_{k_a,s_0} \Big)\,. \end{aligned} \tag{4.2.39}$$

In particular $M(U)$ is a non-homogeneous m-operator in $\mathcal{M}^m_{K,0,p}[r]$ for any $r > 0$ and $K \in \mathbb{N}_0$.

Proof. For any $0 \le k \le K$ we estimate

$$\|\partial_t^k (M(U_1,\dots,U_p)v)\|_{\dot H^{s-m-\frac32 k}} \lesssim_K \sum_{k_1+\dots+k_{p+1}=k} \|M(\partial_t^{k_1}U_1,\dots,\partial_t^{k_p}U_p)\partial_t^{k_{p+1}}v\|_{\dot H^{s-m-\frac32 k}}.$$

We now estimate each term in the above sum. We denote $\vec n_{p+1} := (n_1,\dots,n_{p+1}) \in \mathbb{N}^{p+1}$, $\vec\sigma_{p+1} := (\sigma_1,\dots,\sigma_{p+1}) \in \{\pm\}^{p+1}$ and $\mathcal{I}(n_0,\vec\sigma_{p+1}) := \{\vec n_{p+1}\in\mathbb{N}^{p+1} \colon n_0 = \sigma_1 n_1 + \dots + \sigma_{p+1}n_{p+1}\}$. We get

$$\begin{aligned}
&\|M(\partial_t^{k_1}U_1,\dots,\partial_t^{k_p}U_p)\partial_t^{k_{p+1}}v\|_{\dot H^{s-m-\frac32 k}} = \Big\| n_0^{s-m-\frac32 k}\|\Pi_{n_0}(M(\partial_t^{k_1}U_1,\dots,\partial_t^{k_p}U_p)\partial_t^{k_{p+1}}v)\|_{L^2}\Big\|_{\ell^2_{n_0}}\\
&\overset{(4.2.32)}{\le} \Big\| n_0^{s-m-\frac32 k} \sum_{\substack{\vec\sigma_{p+1}\in\{\pm\}^{p+1}\\ \vec n_{p+1}\in\mathcal{I}(n_0,\vec\sigma_{p+1})}} \|\Pi_{n_0}(M(\Pi_{n_1}\partial_t^{k_1}U_1,\dots,\Pi_{n_p}\partial_t^{k_p}U_p)\Pi_{n_{p+1}}\partial_t^{k_{p+1}}v)\|_{L^2}\Big\|_{\ell^2_{n_0}}\\
&\overset{(4.2.31)}{\lesssim} \sum_{\vec\sigma_{p+1}} \Big\| \sum_{\vec n_{p+1}\in\mathcal{I}(n_0,\vec\sigma_{p+1})} \max{}_2(n_1,\dots,n_{p+1})^{\mu} \max(n_1,\dots,n_{p+1})^{s-\frac32 k} \prod_{a=1}^p \|\Pi_{n_a}\partial_t^{k_a}U_a\|_{L^2}\|\Pi_{n_{p+1}}\partial_t^{k_{p+1}}v\|_{L^2}\Big\|_{\ell^2_{n_0}}
\end{aligned}$$

where in the last inequality we also used that $n_0 \lesssim \max\{n_1,\dots,n_{p+1}\}$ and $s-m-\frac32 k \ge 0$ to bound $n_0^{s-m-\frac32 k}$. For any choice of $\vec\sigma_{p+1}\in\{\pm\}^{p+1}$, we split the internal sum in $p+1$ components

$$\sum_{\overline a=1}^{p+1} \Sigma^{(\overline a)}_{\vec\sigma_{p+1}}, \qquad \Sigma^{(\overline a)}_{\vec\sigma_{p+1}} := \sum_{\substack{\vec n_{p+1}\in\mathcal{I}(n_0,\vec\sigma_{p+1})\\ \max(n_1,\dots,n_{p+1})=n_{\overline a}}}.$$

We first deal with the term $\Sigma^{(p+1)}_{\vec\sigma_{p+1}}$. In this case we bound

$$\begin{aligned}
\|\Pi_{n_{p+1}}\partial_t^{k_{p+1}}v\|_{L^2} &\le \frac{\widetilde c_{n_{p+1}}}{n_{p+1}^{s-\frac32 k_{p+1}}}\|\partial_t^{k_{p+1}}v\|_{\dot H^{s-\frac32 k_{p+1}}} \le \frac{\widetilde c_{n_{p+1}}}{n_{p+1}^{s-\frac32 k_{p+1}}}\|v\|_{k_{p+1},s}\\
\|\Pi_{n_a}\partial_t^{k_a}U_a\|_{L^2} &\le \frac{c^{(a)}_{n_a}}{n_a^{\mu+1-k_a}}\|\partial_t^{k_a}U_a\|_{\dot H^{\mu+1-k_a}} \le \frac{c^{(a)}_{n_a}}{n_a^{\mu+1-k_a}}\|U_a\|_{k_a,\mu+1}, \quad a=1,\dots,p
\end{aligned}$$

for some sequences $(\widetilde c_n)_{n\in\mathbb{N}}$, $(c^{(a)}_n)_{n\in\mathbb{N}}$ in $\ell^2(\mathbb{N})$. With these bounds, and using $\max(n_1,\dots,n_{p+1})^{s-\frac32 k} = n_{p+1}^{s-\frac32 k} \le n_{p+1}^{s-\frac32 k_{p+1}} n_1^{-\frac32 k_1}\dots n_p^{-\frac32 k_p}$, we get

$$\Big\|\Sigma^{(p+1)}_{\vec\sigma_{p+1}}\Big\|_{\ell^2_{n_0}} \lesssim \Big\| \sum_{\substack{(\vec n_{p+1})\in\mathbb{N}^{p+1}\\ n_0=\sigma_1 n_1+\dots+\sigma_{p+1}n_{p+1}}} \widetilde c_{n_{p+1}} \frac{c^{(1)}_{n_1}}{n_1}\times\dots\times\frac{c^{(p)}_{n_p}}{n_p}\Big\|_{\ell^2_{n_0}} \|v\|_{k_{p+1},s}\prod_{a=1}^p \|U_a\|_{k_a,\mu+1}.$$

Applying Young inequality for convolution of sequences and using that $\big(c^{(a)}_n n^{-1}\big)_{n\in\mathbb{N}} \in \ell^1(\mathbb{N})$, we finally arrive at

$$\Big\|\Sigma^{(p+1)}_{\vec\sigma_{p+1}}\Big\|_{\ell^2_{n_0}} \lesssim \|v\|_{k_{p+1},s}\prod_{a=1}^p \|U_a\|_{k_a,s_0}, \qquad k_1+\dots+k_{p+1}=k,$$

which is the first term of inequality (4.2.39) with $s_0 = \max(m+\frac32 K, \mu+1)$. Proceeding similarly we obtain, for any $\overline a = 1,\dots,p$,

$$\|\Sigma^{(\overline a)}_{\vec\sigma_{p+1}}\|_{\ell^2_{n_0}} \lesssim \|v\|_{k_{p+1},\mu+1}\|U_{\overline a}\|_{k_{\overline a},s}\prod_{a\ne\overline a}\|U_a\|_{k_a,\mu+1}$$

which are terms in the sum in the second line of (4.2.39). If $U_a = U$ for any a, we deduce by (4.2.39) and the estimate $\|U\|_{k_a,\sigma} \le \|U\|_{k-k_{p+1},\sigma}$ that $M(U)$ fulfills (4.2.34) with $K' = 0$, $k'' := k_{p+1}$, $k' = k - k_{p+1}$. Hence $M(U)$ belongs to $\mathcal{M}^m_{K,0,p}[r]$ for any $r > 0$ and $K \in \mathbb{N}_0$. □

• A pluri-homogeneous nonlinear map $Z + R_{\le N}(Z)Z$ where $R_{\le N}(Z)$ is in $\Sigma_1^N \widetilde{\mathcal{R}}^{-\varrho}_q \otimes \mathcal{M}_2(\mathbb{C})$ satisfies the following bound: for any $K \in \mathbb{N}_0$ there is $s_0 > 0$ such that for any $s \ge s_0$, $0 < r < r_0(s,K)$ small enough and any $Z \in B^K_{s_0}(I;r) \cap C^K_{*\mathbb{R}}(I;\dot{H}^s(\mathbb{T},\mathbb{C}^2))$ one has

$$2^{-1}\|Z\|_{k,s} \le \|Z + R_{\le N}(Z)Z\|_{k,s} \le 2\|Z\|_{k,s}\,, \quad \forall 0 \le k \le K\,. \tag{4.2.40}$$

Fourier representation of m-operators. We may also represent a matrix of operators

$$M(U) = \begin{pmatrix} M_+^+(U) & M_+^-(U) \\ M_-^+(U) & M_-^-(U) \end{pmatrix} \in \widetilde{\mathcal{M}}_p \otimes \mathcal{M}_2(\mathbb{C}) \tag{4.2.41}$$

through their Fourier matrix elements, see (4.2.6), writing

$$M(U)V = \begin{pmatrix} (M(U)V)^+ \\ (M(U)V)^- \end{pmatrix}, \quad (M(U)V)^\sigma = \sum_{\substack{(\vec{\jmath}_p,j,k)\in(\mathbb{Z}\setminus\{0\})^{p+2} \\ (\vec{\sigma}_p,\sigma')\in\{\pm\}^{p+1} \\ \sigma k = \vec{\sigma}_p\cdot\vec{\jmath}_p + \sigma' j}} M^{\vec{\sigma}_p,\sigma',\sigma}_{\vec{\jmath}_p,j,k}\, u^{\vec{\sigma}_p}_{\vec{\jmath}_p} v^{\sigma'}_j \frac{e^{\mathrm{i}\sigma k x}}{\sqrt{2\pi}}\,, \tag{4.2.42}$$

where [1]

$$\begin{aligned} M^{\vec{\sigma}_p,\sigma',\sigma}_{\vec{\jmath}_p,j,k} &:= \int_{\mathbb{T}} M\Big(\mathsf{q}^{\sigma_1}\frac{e^{\mathrm{i}\sigma_1 j_1 x}}{\sqrt{2\pi}},\dots,\mathsf{q}^{\sigma_p}\frac{e^{\mathrm{i}\sigma_p j_p x}}{\sqrt{2\pi}}\Big)\Big[\frac{\mathsf{q}^{\sigma'}e^{\mathrm{i}\sigma' j x}}{\sqrt{2\pi}}\Big]\cdot \mathsf{q}^{\sigma}\frac{e^{-\mathrm{i}\sigma k x}}{\sqrt{2\pi}}\,\mathrm{d}x \\ &= \int_{\mathbb{T}} M^{\sigma'}_{\sigma}\Big(\mathsf{q}^{\sigma_1}\frac{e^{\mathrm{i}\sigma_1 j_1 x}}{\sqrt{2\pi}},\dots,\mathsf{q}^{\sigma_p}\frac{e^{\mathrm{i}\sigma_p j_p x}}{\sqrt{2\pi}}\Big)\Big[\frac{e^{\mathrm{i}\sigma' j x}}{\sqrt{2\pi}}\Big]\frac{e^{-\mathrm{i}\sigma k x}}{\sqrt{2\pi}}\,\mathrm{d}x \in \mathbb{C}\,, \end{aligned} \tag{4.2.43}$$

and $\mathsf{q}^{\pm}$ are defined in (4.2.7). In (4.2.42) we have exploited the translation invariance property (4.2.33) which implies that if $M^{\vec{\sigma}_p,\sigma',\sigma}_{\vec{\jmath}_p,j,k} \ne 0$ then

$$\sigma k = \vec{\sigma}_p\cdot\vec{\jmath}_p + \sigma' j\,. \tag{4.2.44}$$

Note also that since M is symmetric in the internal entries, the coefficients $M^{\vec{\sigma}_p,\sigma',\sigma}_{\vec{\jmath}_p,j,k}$ in (4.2.43) satisfy the following symmetric property: for any permutation π of $\{1,\dots,p\}$, it results

$$M^{\sigma_{\pi(1)},\dots,\sigma_{\pi(p)},\sigma',\sigma}_{j_{\pi(1)},\dots,j_{\pi(p)},j,k} = M^{\sigma_1,\dots,\sigma_p,\sigma',\sigma}_{j_1,\dots,j_p,j,k}\,. \tag{4.2.45}$$

The operator $M(U)$ is real-to-real, according to definition (4.2.12), if and only if its coefficients fulfill

$$\overline{M^{\vec{\sigma}_p,\sigma',\sigma}_{\vec{\jmath}_p,j,k}} = M^{-\vec{\sigma}_p,-\sigma',-\sigma}_{\vec{\jmath}_p,j,k}\,. \tag{4.2.46}$$

The matrix entries of the transpose operator $M(U)^\top$ with respect to the non-degenerate bilinear form (4.2.4) are

$$\begin{aligned} (M^\top)^{\vec{\sigma}_p,\sigma',\sigma}_{\vec{\jmath}_p,j,k} &= \int_{\mathbb{T}} M^\top\Big(\mathsf{q}^{\sigma_1}\frac{e^{\mathrm{i}\sigma_1 j_1 x}}{\sqrt{2\pi}},\dots,\mathsf{q}^{\sigma_p}\frac{e^{\mathrm{i}\sigma_p j_p x}}{\sqrt{2\pi}}\Big)\Big[\frac{\mathsf{q}^{\sigma'}e^{\mathrm{i}\sigma' j x}}{\sqrt{2\pi}}\Big]\cdot \frac{\mathsf{q}^{\sigma}e^{-\mathrm{i}\sigma k x}}{\sqrt{2\pi}}\,\mathrm{d}x \\ &= \int_{\mathbb{T}} M\Big(\mathsf{q}^{\sigma_1}\frac{e^{\mathrm{i}\sigma_1 j_1 x}}{\sqrt{2\pi}},\dots,\mathsf{q}^{\sigma_p}\frac{e^{\mathrm{i}\sigma_p j_p x}}{\sqrt{2\pi}}\Big)\Big[\frac{\mathsf{q}^{\sigma}e^{-\mathrm{i}\sigma k x}}{\sqrt{2\pi}}\Big]\cdot \frac{\mathsf{q}^{\sigma'}e^{\mathrm{i}\sigma' j x}}{\sqrt{2\pi}}\,\mathrm{d}x = M^{\vec{\sigma}_p,\sigma,\sigma'}_{\vec{\jmath}_p,-k,-j}\,. \end{aligned} \tag{4.2.47}$$

[1] Given $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{C}^2$ we set $u\cdot v := u_1v_1 + u_2v_2$.

One can directly verify that $(M^\top)^{\vec\sigma_p,\sigma',\sigma}_{\vec\jmath_p,j,k}$ fulfill (4.2.44), (4.2.45). If $M(U)$ is real-to-real (i.e. (4.2.46) holds) then $M(U)^\top$ is real-to-real as well.

Lemma 4.2.9. (Characterization of m-operators in Fourier basis) *Let $m \in \mathbb{R}$. A real-to-real linear operator $M(U)$ as in (4.2.42)-(4.2.43) is a matrix of p-homogeneous m-operators in $\widetilde{\mathcal{M}}^m_p \otimes \mathcal{M}_2(\mathbb{C})$ if and only if its coefficients $M^{\vec\sigma_p,\sigma',\sigma}_{\vec\jmath_p,j,k}$ satisfy (4.2.44), (4.2.45), (4.2.46) and there exist $\mu > 0$ and $C > 0$ such that for any $(\vec\jmath_p, j) \in (\mathbb{Z}\setminus\{0\})^{p+1}$, $(\vec\sigma_p,\sigma) \in \{\pm\}^{p+1}$,*

$$|M^{\vec\sigma_p,\sigma',\sigma}_{\vec\jmath_p,j,k}| \le C\max_2\{|j_1|,\dots,|j_p|,|j|\}^\mu \max\{|j_1|,\dots,|j_p|,|j|\}^m . \tag{4.2.48}$$

Proof. Let $M(U)$ be a matrix of m-operators in $\widetilde{\mathcal{M}}^m_p \otimes \mathcal{M}_2(\mathbb{C})$. Then by (4.2.43), applying Cauchy-Schwartz inequality and recalling (4.2.8) we get

$$\begin{aligned}
|M^{\vec\sigma_p,\sigma',\sigma}_{\vec\jmath_p,j,k}| &\le \Big\| M^{\sigma'}_\sigma\Big(\mathtt{q}^{\sigma_1}\Pi_{|j_1|}\frac{e^{\mathrm{i}\sigma_1 j_1 x}}{\sqrt{2\pi}},\dots,\mathtt{q}^{\sigma_p}\Pi_{|j_p|}\frac{e^{\mathrm{i}\sigma_p j_p x}}{\sqrt{2\pi}}\Big)\Big[\Pi_{|j|}\frac{e^{\mathrm{i}\sigma' j x}}{\sqrt{2\pi}}\Big]\Big\|_{L^2} \\
&\overset{(4.2.32)}{\le} \sum_{\substack{\epsilon_1,\dots,\epsilon_p,\epsilon\in\{\pm\}\\ n_0=\epsilon_1|j_1|+\dots+\epsilon_p|j_p|+\epsilon|j|}} \Big\|\Pi_{n_0} M^{\sigma'}_\sigma\Big(\mathtt{q}^{\sigma_1}\Pi_{|j_1|}\frac{e^{\mathrm{i}\sigma_1 j_1 x}}{\sqrt{2\pi}},\dots,\mathtt{q}^{\sigma_p}\Pi_{|j_p|}\frac{e^{\mathrm{i}\sigma_p j_p x}}{\sqrt{2\pi}}\Big)\Big[\Pi_{|j|}\frac{e^{\mathrm{i}\sigma' j x}}{\sqrt{2\pi}}\Big]\Big\|_{L^2} \\
&\overset{(4.2.31)}{\le} C2^{p+1}\max_2\{|j_1|,\dots,|j_p|,|j|\}^\mu \max\{|j_1|,\dots,|j_p|,|j|\}^m
\end{aligned}$$

proving (4.2.48). Viceversa suppose that $M(U)$ is an operator as in (4.2.41)-(4.2.43) with coefficients satisfying (4.2.48). Then, for any $\sigma,\sigma' \in \{\pm\}$,

$$\begin{aligned}
&\|\Pi_{n_0} M^{\sigma'}_\sigma(\Pi_{n_1}U_1,\dots,\Pi_{n_p}U_p)\Pi_{n_{p+1}}v^{\sigma'}\|_{L^2} = \Big\|\sum_{\substack{j_1=\pm n_1,\dots,j_p=\pm n_p\\ j=\pm n_{p+1},k=\pm n_0}} M^{\vec\sigma_p,\sigma',\sigma}_{\vec\jmath_p,j,k}(u_1)^{\sigma_1}_{j_1}\cdots(u_p)^{\sigma_p}_{j_p} v^{\sigma'}_j \frac{e^{\mathrm{i}\sigma k x}}{\sqrt{2\pi}}\Big\|_{L^2} \\
&\overset{(4.2.48)}{\le} C \sum_{\substack{j_1=\pm n_1,\dots,j_p=\pm n_p\\ j=\pm n_{p+1},k=\pm n_0}} \max_2\{|j_1|,\dots,|j_p|,|j|\}^\mu \max\{|j_1|,\dots,|j_p|,|j|\}^m |(u_1)^{\sigma_1}_{j_1}|\cdots|(u_p)^{\sigma_p}_{j_p}||v^{\sigma'}_j| \\
&\le C2^{p+2}\max_2\{n_1,\dots,n_p,n_{p+1}\}^\mu \max\{n_1,\dots,n_p,n_{p+1}\}^m \prod_{\ell=1}^p \|\Pi_{n_\ell}U_\ell\|_{L^2}\|\Pi_{n_{p+1}}v\|_{L^2}
\end{aligned}$$

proving (4.2.31). □

The transpose of a matrix of m-operators is a m'-operator.

Lemma 4.2.10. (Transpose of m-Operators) *Let $p \in \mathbb{N}_0$, $m \in \mathbb{R}$. If $M(U)$ is a matrix of p–homogeneous m-operators in $\widetilde{\mathcal{M}}^m_p \otimes \mathcal{M}_2(\mathbb{C})$ then $M(U)^\top$ (where the transpose is computed with respect to the non-degenerate bilinear form (4.2.4)) is a matrix of p-homogeneous operators in $\widetilde{\mathcal{M}}^{m'}_p \otimes \mathcal{M}_2(\mathbb{C})$ for some $m' \ge \max(m,0)$.*

If in addition there exists $C > 1$ such that

$$M^{\vec\sigma_p,\sigma,\sigma'}_{\vec\jmath_p,k,j} \ne 0 \quad\Rightarrow\quad C^{-1}|k| \le |j| \le C|k| \tag{4.2.49}$$

then $M(U)^\top \in \widetilde{\mathcal{M}}^m_p \otimes \mathcal{M}_2(\mathbb{C})$.

Proof. By (4.2.47), (4.2.48) (applied to $M^{\vec{\sigma}_p,\sigma,\sigma'}_{\vec{\jmath}_p,-k,-j}$) and since, by (4.2.44), $|k| \le (p+1)\max\{|j|,|j_1|,\dots,|j_p|\}$, we deduce that

$$|(M^\top)^{\vec{\sigma}_p,\sigma',\sigma}_{\vec{\jmath}_p,j,k}| \lesssim \max{}_2\{|j_1|,\dots,|j_p|,|k|\}^\mu \max\{|j_1|,\dots,|j_p|,|k|\}^m \\ \lesssim \max\{|j_1|,\dots,|j_p|,|j|\}^{\max(m,0)+\mu} \tag{4.2.50}$$

proving (4.2.48) for $M(U)^\top$ with $m \rightsquigarrow m' := \max(m,0)+\mu$.

If in addition (4.2.49) holds true then $\max\{|j_1|,\dots,|j_p|,|k|\}^m \sim \max\{|j_1|,\dots,|j_p|,|j|\}^m$ for any $m \in \mathbb{R}$ and similarly $\max_2\{|j_1|,\dots,|j_p|,|k|\}^\mu \sim \max_2\{|j_1|,\dots,|j_p|,|j|\}^\mu$ for any $\mu \ge 0$. We deduce by (4.2.50) that $M(U)^\top$ is in $\widetilde{\mathcal{M}}^m_p \otimes \mathcal{M}_2(\mathbb{C})$. □

Remark 4.2.11. (Transpose of Smoothing operators) If $R(U)$ is a matrix of smoothing operators in $\widetilde{\mathcal{R}}^{-\varrho}_p \otimes \mathcal{M}_2(\mathbb{C})$ and the spectral condition (4.2.49) holds true, then $R(U)^\top$ is a smoothing operator in the same class. Without the spectral condition (4.2.49) this might fail: for example consider $R(U)$ such that its transpose is

$$R(U_1,\dots,U_p)^\top V := \mathrm{Op}^{BW}(A(V,U_1,\dots,U_{p-1}))U_p\,,\ \ A(\cdot) \in \widetilde{\Gamma}^\mu_p \otimes \mathcal{M}_2(\mathbb{C})\,. \tag{4.2.51}$$

As a consequence of Lemma 4.2.21 below, we have that $R(U) = [R(U)^\top]^\top$ is in $\widetilde{\mathcal{R}}^{-\varrho}_p \otimes \mathcal{M}_2(\mathbb{C})$ for any $\varrho \ge 0$, but $R(U)^\top$ in (4.2.51) is a μ-operator.

We conclude this subsection with the paralinearization of the product (see e.g. Lemma 7.2 in [27]).

Lemma 4.2.12. (Bony paraproduct decomposition) *Let u_1, u_2 be functions in $H^\sigma(\mathbb{T};\mathbb{C})$ with $\sigma > \frac{1}{2}$. Then*

$$u_1u_2 = \mathrm{Op}^{BW}(u_1)u_2 + \mathrm{Op}^{BW}(u_2)u_1 + R_1(u_1)u_2 + R_2(u_2)u_1 \tag{4.2.52}$$

where for $j = 1,2$, R_j is a homogeneous smoothing operator in $\widetilde{\mathcal{R}}^{-\varrho}_1$ for any $\varrho \ge 0$.

Composition theorems. Let $\sigma(D_x,D_\xi,D_y,D_\eta) := D_\xi D_y - D_x D_\eta$ where $D_x := \frac{1}{\mathrm{i}}\partial_x$ and D_ξ, D_y, D_η are similarly defined. The following is Definition 3.11 in [27].

Definition 4.2.13. (Asymptotic expansion of composition symbol) *Let p, p' in $\mathbb{N}_0$, $K, K' \in \mathbb{N}_0$ with $K' \le K$, $\varrho \ge 0$, $m, m' \in \mathbb{R}$, $r > 0$. Consider symbols $a \in \Sigma\Gamma^m_{K,K',p}[r,N]$ and $b \in \Sigma\Gamma^{m'}_{K,K',p'}[r,N]$. For U in $B^K_\sigma(I;r)$ we define, for $\varrho < \sigma - s_0$, the symbol*

$$(a\#_\varrho b)(U;t,x,\xi) := \sum_{k=0}^{\varrho} \frac{1}{k!}\left(\frac{\mathrm{i}}{2}\sigma(D_x,D_\xi,D_y,D_\eta)\right)^k \Big[a(U;t,x,\xi)b(U;t,y,\eta)\Big]_{|x=y,\xi=\eta} \tag{4.2.53}$$

modulo symbols in $\Sigma\Gamma^{m+m'-\varrho}_{K,K',p+p'}[r,N]$.

- The symbol $a\#_\varrho b$ belongs to $\Sigma\Gamma^{m+m'}_{K,K',p+p'}[r,N]$.
- We have that $a\#_\varrho b = ab + \frac{1}{2\mathrm{i}}\{a,b\}$ up to a symbol in $\Sigma\Gamma^{m+m'-2}_{K,K',p+p'}[r,N]$, where

$$\{a,b\} := \partial_\xi a \partial_x b - \partial_x a \partial_\xi b$$

denotes the Poisson bracket.
- If c is a symbol in $\Sigma\Gamma^{m''}_{K,K',p''}[r,N]$ then $a\#_\varrho b\#_\varrho c + c\#_\varrho b\#_\varrho a - 2abc$ is a symbol in $\Sigma\Gamma^{m+m'+m''-2}_{K,K',p+p'+p''}[r,N]$.
- $\overline{a^\vee}\#_\varrho\overline{b^\vee} = \overline{a\#_\varrho b}^\vee$ where $a^\vee$ is defined in (4.2.27).

The following result is proved in Proposition 3.12 in [27].

Proposition 4.2.14. (Composition of Bony-Weyl operators) *Let $p, q, N, K, K' \in \mathbb{N}_0$ with $K' \le K$, $\varrho \ge 0$, $m, m' \in \mathbb{R}$, $r > 0$. Consider symbols $a \in \Sigma\Gamma^m_{K,K',p}[r,N]$ and $b \in \Sigma\Gamma^{m'}_{K,K',q}[r,N]$. Then*

$$\mathrm{Op}^{\mathrm{BW}}\left(a(U;t,x,\xi)\right) \circ \mathrm{Op}^{\mathrm{BW}}\left(b(U;t,x,\xi)\right) - \mathrm{Op}^{\mathrm{BW}}\left((a\#_\varrho b)(U;t,x,\xi)\right) \tag{4.2.54}$$

is a smoothing operator in $\Sigma\mathcal{R}^{-\varrho+m+m'}_{K,K',p+q}[r,N]$.

We now prove other composition results concerning m-operators.

Proposition 4.2.15. (Compositions of m-operators) *Let $p, p', N, K, K' \in \mathbb{N}_0$ with $K' \le K$ and $r > 0$. Let $m, m' \in \mathbb{R}$. Then*

- (i) *If $M(U;t)$ is in $\Sigma\mathcal{M}^m_{K,K',p}[r,N]$ and $M'(U;t)$ is in $\Sigma\mathcal{M}^{m'}_{K,K',p'}[r,N]$ then the composition $M(U;t) \circ M'(U;t)$ is in $\Sigma\mathcal{M}^{m+\max(m',0)}_{K,K',p+p'}[r,N]$.*
- (ii) *If $M(U)$ is a homogeneous m-operator in $\widetilde{\mathcal{M}}^m_p$ and $M^{(\ell)}(U;t)$, $\ell = 1, \dots, p+1$, are matrices of m_ℓ-operators in $\Sigma\mathcal{M}^{m_\ell}_{K,K',q_\ell}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$ with $m_\ell \in \mathbb{R}$, $q_\ell \in \mathbb{N}_0$, then*

  $$M(M^{(1)}(U;t)U, \dots, M^{(p)}(U;t)U)M^{(p+1)}(U;t)$$

  belongs to $\Sigma\mathcal{M}^{m+\overline{m}}_{K,K',p+\overline{q}}[r,N]$ with $\overline{m} := \sum_{\ell=1}^{p+1} \max(m_\ell, 0)$ and $\overline{q} := \sum_{\ell=1}^{p+1} q_\ell$.
- (iii) *If $M(U;t)$ is in $\mathcal{M}^m_{K,0,p}[\check{r}]$ for any $\check{r} \in \mathbb{R}^+$ and $\mathbf{M}_0(U;t)$ belongs to $\mathcal{M}^0_{K,K',0}[r] \otimes \mathcal{M}_2(\mathbb{C})$, then $M(\mathbf{M}_0(U;t)U;t)$ is in $\mathcal{M}^m_{K,K',p}[r]$.*
- (iv) *Let c be a homogeneous symbol in $\widetilde{\Gamma}^m_p$ and $M^{(\ell)}(U;t)$, $\ell = 1, \dots, p$, be operators in $\Sigma\mathcal{M}_{K,K',q_\ell}[r,N]$ with $q_\ell \in \mathbb{N}_0$. Then*

  $$U \to b(U;t,x,\xi) := c(M^{(1)}(U;t)U, \dots, M^{(p)}(U;t)U;t,x,\xi)$$

  is a symbol in $\Sigma\Gamma^m_{K,K',p+\overline{q}}[r,N]$ with $\overline{q} := q_1 + \dots + q_p$ and

  $$\mathrm{Op}^{\mathrm{BW}}(c(W_1, \dots, W_p;t,x,\xi))_{|W_\ell = M^{(\ell)}(U;t)U} = \mathrm{Op}^{\mathrm{BW}}(b(U;t,x,\xi)) + R(U;t)$$

  where $R(U;t)$ is a smoothing operator in $\Sigma\mathcal{R}^{-\varrho}_{K,K',p+\overline{q}}[r,N]$ for any $\varrho \ge 0$.

Proof. PROOF OF (ii): It is sufficient to prove the thesis for $m_\ell \ge 0$ otherwise we regard $M^{(\ell)}(U;t)$ as a 0-operator in $\Sigma\mathcal{M}^0_{K,K',p}[r,N]$. We decompose, for any $\ell = 1, \dots, p+1$, $M^{(\ell)} = \sum_{a=q_\ell}^N M^{(\ell)}_a + M^{(\ell)}_{>N}$ as in (4.2.35). Given integers $a_\ell \in [q_\ell, N]$, $\ell = 1, \dots p+1$, we use the notation $\overline{a}_\ell := a_1 + \dots + a_\ell + \ell$ and $\overline{a}_0 := 0$. Note that $\overline{a}_{p+1} - 1 = a_1 + \ldots + a_{p+1} + p \ge \overline{q} + p$. We also denote the vector with a_ℓ elements

$$\mathcal{U}_\ell := (U_{\overline{a}_{\ell-1}+1}, \dots, U_{\overline{a}_\ell - 1}), \quad \ell = 1, \dots, p+1.$$

By multi-linearity we have to show on the one hand that, if $\overline{a}_{p+1} - 1 \le N$ then

$$M\Big(M^{(1)}_{a_1}(\mathcal{U}_1)U_{a_1+1}, \dots, M^{(p)}_{a_p}(\mathcal{U}_p)U_{\overline{a}_p}\Big)M^{(p+1)}_{a_{p+1}}(\mathcal{U}_{p+1}) \tag{4.2.55}$$

is a homogeneous operator in $\widetilde{\mathcal{M}}^{m+\overline{m}}_{\overline{a}_{p+1}-1}$. On the other hand, if $\overline{a}_{p+1} - 1 \ge N+1$ then

$$M\Big(M^{(1)}_{a_1}(U, \dots, U)U, \dots, M^{(p)}_{a_p}(U, \dots, U)U\Big)M^{(p+1)}_{a_{p+1}}(U, \dots, U) \tag{4.2.56}$$

is a non–homogeneous operator in $\mathcal{M}^{m+\overline{m}}_{K,K',N+1}[r]$, having included for notational convenience the inhomogeneous term as $M^{(\ell)}_{N+1}(U,\dots,U) := M^{(\ell)}_{>N}(U;t)$ which belongs to $\mathcal{M}^{m_\ell}_{K,K',N+1}[r]$.

We first study (4.2.55). First of all, using the notation $\vec{n}_\ell := (n_{\overline{a}_{\ell-1}+1},\dots,n_{\overline{a}_\ell-1})$ one has

$$\begin{aligned}
&\|\Pi_{n_0} M\Big(M^{(1)}_{a_1}(\Pi_{\vec{n}_1}\mathcal{U}_1)\Pi_{n_{\overline{a}_1}}U_{\overline{a}_1},\dots,M^{(p)}_{a_p}(\Pi_{\vec{n}_p}\mathcal{U}_p)\Pi_{n_{\overline{a}_p}}U_{\overline{a}_p}\Big)M^{(p+1)}_{a_{p+1}}(\Pi_{\vec{n}_{p+1}}\mathcal{U}_{p+1})\Pi_{n_{\overline{a}_{p+1}}}U_{\overline{a}_{p+1}}\|_{L^2}\\
&\le \sum_{n'_1,\dots,n'_{p+1}} \|\Pi_{n_0} M\Big(\Pi_{n'_1}M^{(1)}_{a_1}(\Pi_{\vec{n}_1}\mathcal{U}_1)\Pi_{n_{\overline{a}_1}}U_{\overline{a}_1},\dots,\Pi_{n'_p}M^{(p)}_{a_p}(\Pi_{\vec{n}_p}\mathcal{U}_p)\Pi_{n_{\overline{a}_p}}U_{\overline{a}_p}\Big)\\
&\qquad\qquad \circ\,\Pi_{n'_{p+1}}M^{(p+1)}_{a_{p+1}}(\Pi_{\vec{n}_{p+1}}\mathcal{U}_{p+1})\Pi_{n_{\overline{a}_{p+1}}}U_{\overline{a}_{p+1}}\|_{L^2}\,. \qquad (4.2.57)
\end{aligned}$$

Thanks to the conditions (4.2.32) for M and $M^{(\ell)}$ the indices in the above sum satisfy, for some choice of signs σ_b,ϵ_j, $b=1,\dots,\overline{a}_{p+1}$, $j=1,\dots,p+1$, the restrictions

$$n_0=\sum_{j=1}^{p+1}\epsilon_j n'_j\,,\quad n'_\ell=\sum_{b=\overline{a}_{\ell-1}+1}^{\overline{a}_\ell}\sigma_b n_b\,,\quad \ell=1,\dots,p+1\,. \qquad (4.2.58)$$

As a consequence (4.2.55) satisfies the corresponding condition (4.2.32) and

$$n'_\ell \lesssim \max\{\vec{n}_\ell, n_{\overline{a}_\ell}\}\,,\qquad \forall \ell=1,\dots,p+1\,, \qquad (4.2.59)$$

then we have

$$\begin{aligned}
&\max\{n'_1,\dots,n'_{p+1}\}\lesssim \max\{n_1,\dots,n_{\overline{a}_{p+1}}\} \qquad (4.2.60)\\
&\max{}_2\{n'_1,\dots,n'_{p+1}\}\overset{(4.2.59)}{\lesssim}\max{}_2\big\{\max\{\vec{n}_1,n_{\overline{a}_1}\},\dots,\max\{\vec{n}_{p+1},n_{\overline{a}_{p+1}}\}\big\}\lesssim \max{}_2\{n_1,\dots,n_{\overline{a}_{p+1}}\}
\end{aligned}$$

where in the last inequality we used that $\{n_1,\dots,n_{\overline{a}_{p+1}}\}$ is the disjoint union of the sets $\{\vec{n}_\ell,n_{\overline{a}_\ell}\}_{\ell=1,\dots,p+1}$. Using (4.2.58), (4.2.31) for M and $M^{(\ell)}$ we get, with $\vec{n}':=(n'_1,\dots,n'_{p+1})$

$$\begin{aligned}
(4.2.57)&\lesssim \max{}_2\{\vec{n}'\}^{\mu}\max\{\vec{n}'\}^{m}\prod_{\ell=1}^{p+1}\max{}_2\{\vec{n}_\ell,n_{\overline{a}_\ell}\}^{\mu_\ell}\max\{\vec{n}_\ell,n_{\overline{a}_\ell}\}^{m_\ell}\prod_{b=1}^{\overline{a}_{p+1}}\|\Pi_{n_b}U_b\|_{L^2}\\
&\overset{(4.2.60),(4.2.30),m_\ell\ge 0}{\lesssim}\max{}_2\{n_1,\dots,n_{\overline{a}_{p+1}}\}^{\overline{\mu}}\max\{n_1,\dots,n_{\overline{a}_{p+1}}\}^{\overline{m}}\max\{\vec{n}'\}^{m}\prod_{b=1}^{\overline{a}_{p+1}}\|\Pi_{n_b}U_b\|_{L^2}, \qquad (4.2.61)
\end{aligned}$$

where $\overline{\mu}:=\mu+\mu_1+\dots+\mu_{p+1}$ and recall $\overline{m}=m_1+\dots+m_{p+1}$. We claim that

$$\max\{\vec{n}'\}^m\lesssim \max{}_2\{n_1,\dots,n_{\overline{a}_{p+1}}\}^{\mu''}\max\{n_1,\dots,n_{\overline{a}_{p+1}}\}^m \qquad (4.2.62)$$

for some $\mu''\ge 0$. Then (4.2.31), (4.2.61) and (4.2.62) imply that the operator in (4.2.55) belongs to $\widetilde{\mathcal{M}}^{m+\overline{m}}_{\overline{a}_{p+1}-1}$.

We now prove (4.2.62). If $m\ge 0$ it follows by (4.2.60) with $\mu''=0$. So from now on we consider $m<0$. We fix $\overline{\ell}$ such that

$$\max\{n_1,\dots,n_{\overline{a}_{p+1}}\}=\max\{\vec{n}_{\overline{\ell}},\, n_{\overline{a}_{\overline{\ell}}}\} \qquad (4.2.63)$$

and we distinguish two cases:

Case 1: $n'_{\overline{\ell}}\ge \frac{1}{2}\max\{\vec{n}_{\overline{\ell}},\, n_{\overline{a}_{\overline{\ell}}}\}$. In this case

$$\max\{\vec{n}'\}^m\overset{m<0}{\le}(n'_{\overline{\ell}})^m\overset{\text{Case 1}}{\le}2^{|m|}\max\{\vec{n}_{\overline{\ell}},n_{\overline{a}_{\overline{\ell}}}\}^m\overset{(4.2.63)}{=}2^{|m|}\max\{n_1,\dots,n_{\overline{a}_{p+1}}\}^m$$

which proves (4.2.62) with $\mu''=0$.
Case 2**:** $n'_{\overline{\ell}} < \frac{1}{2}\max\{\vec{n}_{\overline{\ell}},\, n_{\overline{a}_{\overline{\ell}}}\}$. In view of the momentum condition (4.2.58),

$$\begin{aligned}\max\{\vec{n}_{\overline{\ell}},\, n_{\overline{a}_{\overline{\ell}}}\} &\leq a_{\overline{\ell}}\max{}_2\{\vec{n}_{\overline{\ell}},\, n_{\overline{a}_{\overline{\ell}}}\} + n'_{\overline{\ell}}\\ &\overset{\text{Case 2}}{\leq} a_{\overline{\ell}}\max{}_2\{\vec{n}_{\overline{\ell}},\, n_{\overline{a}_{\overline{\ell}}}\} + \frac{1}{2}\max\{\vec{n}_{\overline{\ell}},\, n_{\overline{a}_{\overline{\ell}}}\}\end{aligned}$$

and consequently

$$\max\{\vec{n}_{\overline{\ell}},\, n_{\overline{a}_{\overline{\ell}}}\} \leq 2a_{\overline{\ell}}\max{}_2\{\vec{n}_{\overline{\ell}},\, n_{\overline{a}_{\overline{\ell}}}\}\,. \tag{4.2.64}$$

Then, since $m<0$,

$$\begin{aligned}\max\{\vec{n}'\}^m \leq 1 &\overset{(4.2.64)}{\leq} (2a_{\overline{\ell}})^{|m|}\max{}_2\{\vec{n}_{\overline{\ell}},\, n_{\overline{a}_{\overline{\ell}}}\}^{|m|}\max\{\vec{n}_{\overline{\ell}},\, n_{\overline{a}_{\overline{\ell}}}\}^m\\ &\overset{(4.2.63),(4.2.30)}{\leq} (2a_{\overline{\ell}})^{|m|}\max{}_2\{n_1,\ldots,n_{\overline{a}_{p+1}}\}^{|m|}\max\{n_1,\ldots,n_{\overline{a}_{p+1}}\}^m\end{aligned}$$

which proves (4.2.62) with $\mu''=|m|$. This concludes the proof of (4.2.62) and then that the operator in (4.2.55) is in $\widetilde{\mathcal{M}}^{m+\overline{m}}_{\overline{a}_{p+1}-1}$.

Now we prove that the operator in (4.2.56) is in $\mathcal{M}^{m+\overline{m}}_{K,K',N+1}[r]$. We have to verify (4.2.34) with $m\rightsquigarrow m+\overline{m}$ and $p\rightsquigarrow N+1$. For simplicity we denote $M^{(\ell)}_{a_\ell}(U)=M^{(\ell)}_{a_\ell}(U,\ldots,U)$. First we apply (4.2.39) to $M(U)\in\widetilde{\mathcal{M}}^m_p$ (with $U_\ell\rightsquigarrow M^{(\ell)}_{a_\ell}(U)U$, $v\rightsquigarrow M^{(p+1)}_{a_{p+1}}(U)v$ and $s\rightsquigarrow s-\overline{m}$) getting, for any $k=0,\ldots,K-K'$,

$$\|\partial_t^k\big((4.2.56)v\big)\|_{\dot H^{s-m-\overline{m}-\frac{3}{2}k}} \lesssim_K \sum_{k_1+\cdots+k_{p+1}=k}\Big(\|M^{(p+1)}_{a_{p+1}}(U)v\|_{k_{p+1},s-\overline{m}}\prod_{\ell=1}^{p}\|M^{(\ell)}_{a_\ell}(U)U\|_{k_\ell,s_0} \tag{4.2.65}$$

$$+\|M^{(p+1)}_{a_{p+1}}(U)v\|_{k_{p+1},s_0}\sum_{\overline{\ell}=1}^{p}\|M^{(\overline{\ell})}_{a_{\overline{\ell}}}(U)U\|_{k_{\overline{\ell}},s-\overline{m}}\prod_{\substack{\ell=1\\ \ell\neq\overline{\ell}}}^{p}\|M^{(\ell)}_{a_\ell}(U)U\|_{k_\ell,s_0}\Big)\,. \tag{4.2.66}$$

Then we estimate line (4.2.65) where, by Lemma 4.2.8, each $M^{(\ell)}_{a_\ell}(U)$ is in $\mathcal{M}^{m_\ell}_{K,0,a_\ell}[r]$ and $M^{(\ell)}_{N+1}(U)$ belongs to $\mathcal{M}^{m_\ell}_{K,K',N+1}[r]$. For any $\ell=1,\ldots,p$, using (4.2.34) (with $m\rightsquigarrow m_\ell$ and $p\rightsquigarrow a_\ell$) we bound (since $k_\ell\leq k-k_{p+1}$ and $0\leq k\leq K-K'$)

$$\begin{aligned}&\|M^{(\ell)}_{a_\ell}(U)U\|_{k_\ell,s_0}\leq C\|U\|^{a_\ell+1}_{k_\ell+K',s_0+\overline{m}}\leq C\|U\|^{a_\ell+1}_{k-k_{p+1}+K',s_0+\overline{m}}\,, \qquad (4.2.67)\\ &\|M^{(p+1)}_{a_{p+1}}(U)v\|_{k_{p+1},s-\overline{m}}\leq C\sum_{k'+k''\leq k_{p+1}}\|v\|_{k'',s}\|U\|^{a_{p+1}}_{k'+K',s_0}+\|v\|_{k'',s_0}\|U\|^{a_{p+1}-1}_{k'+K',s_0}\|U\|_{k'+K',s}\\ &\qquad\leq C\sum_{k''\leq k_{p+1}}\|v\|_{k'',s}\|U\|^{a_{p+1}}_{k-k''+K',s_0}+\|v\|_{k'',s_0}\|U\|^{a_{p+1}-1}_{k-k''+K',s_0}\|U\|_{k-k''+K',s}\end{aligned}$$

since $k'\leq k-k''$ (being $k_{p+1}\leq k$). By (4.2.67) and since $k-k_{p+1}+K'\leq k-k''+K'$, we get

$$\begin{aligned}(4.2.65) &\lesssim_{K,s}\sum_{k''\leq k}\|v\|_{k'',s}\|U\|^{a_1+\cdots+a_{p+1}+p}_{k-k''+K',s_0+\overline{m}}+\|v\|_{k'',s_0}\|U\|^{a_1+\cdots+a_{p+1}+p-1}_{k-k''+K',s_0+\overline{m}}\|U\|_{k-k''+K',s}\\ &\lesssim_{K,s}\sum_{k''\leq k}\|v\|_{k'',s}\|U\|^{N+1}_{k-k''+K',s_0+\overline{m}}+\|v\|_{k'',s_0}\|U\|^{N}_{k-k''+K',s_0+\overline{m}}\|U\|_{k-k''+K',s} \qquad (4.2.68)\end{aligned}$$

because $a_1+\cdots+a_{p+1}+p=\overline{a}_{p+1}-1\geq N+1$ (for $\|U\|_{K,s_0+\overline{m}}<1$). Regarding (4.2.66), we first bound by (4.2.34) (and (4.2.67) with $s=s_0+\overline{m}$)

$$\|M^{(\overline{\ell})}_{a_{\overline{\ell}}}(U)U\|_{k_{\overline{\ell}},s-\overline{m}}\leq C\|U\|^{a_{\overline{\ell}}}_{k_{\overline{\ell}}+K',s_0}\|U\|_{k_{\overline{\ell}}+K',s}\leq C\|U\|^{a_{\overline{\ell}}}_{k-k_{p+1}+K',s_0}\|U\|_{k-k_{p+1}+K',s}$$
$$\|M^{(p+1)}_{a_{p+1}}(U)v\|_{k_{p+1},s_0}\leq C\sum_{k''\leq k_{p+1}}\|v\|_{k'',s_0+\overline{m}}\|U\|^{a_{p+1}}_{k-k''+K',s_0+\overline{m}}$$

and, proceeding similarly to the previous computation, we deduce

$$(4.2.66)\lesssim_{K,s}\sum_{k''\leq k}\|v\|_{k'',s_0+\overline{m}}\|U\|^{N}_{k-k''+K',s_0+\overline{m}}\|U\|_{k-k''+K',s}\,. \tag{4.2.69}$$

Hence (4.2.68), (4.2.69) imply that for any $k=0,\dots,K-K'$

$$\|\partial_t^k\big((4.2.56)v\big)\|_{\dot H^{s-m-\overline{m}-\frac32 k}}\lesssim_{K,s}\sum_{k'+k''=k}\|v\|_{k'',s}\|U\|^{N+1}_{k'+K',s_0+\overline{m}}+\|v\|_{k'',s_0+\overline{m}}\|U\|^{N}_{k'+K',s_0+\overline{m}}\|U\|_{k'+K',s} \tag{4.2.70}$$

proving that the operator (4.2.56) satisfies (4.2.34) with $m\rightsquigarrow m+\overline{m}$, $p\rightsquigarrow N+1$ and $s_0\rightsquigarrow s_0+\overline{m}$.
PROOF OF (i): Decomposing $M=\sum_{q=p}^{N}M_q+M_{>N}$ and $M'=\sum_{q=p'}^{N}M'_q+M'_{>N}$ as in (4.2.35), by item (ii) we deduce that $M_{q_1}(U)M'_{q_2}(U)$ is in $\widetilde{\mathcal{M}}^{m+\max\{m',0\}}_{q_1+q_2}$ if $q_1+q_2\leq N$, and $M_{q_1}(U)M'_{q_2}(U)$, $q_1+q_2\geq N+1$ and $M_{q_1}(U)M'_{>N}(U;t)$, $q_1=p,\dots,N$, are in $\mathcal{M}^{m+\max\{m',0\}}_{K,K',N+1}[r]$. Furthermore, proceeding as in the proof of (4.2.70), one shows that $M_{>N}(U;t)M'_{q_2}(U)$, $q_2=p',\dots,N$, and $M_{>N}(U;t)M'_{>N}(U;t)$ are operators in $\mathcal{M}^{m+\max\{m',0\}}_{K,K',N+1}[r]$.
PROOF OF (iii): As $\mathbf{M}_0(U;t)\in\mathcal{M}^0_{K,K',0}[r]\otimes\mathcal{M}_2(\mathbb{C})$, for any $U\in B^{K'}_{s_0}(I;r)$, the function $\mathbf{M}_0(U;t)U\in B^0_{s_0}(I;Cr)$, for some $C\geq 1$. Hence the composition $M(\mathbf{M}_0(U;t)U)$ is a well defined operator. Moreover, for any $U\in B^K_{s_0}(I;r(\sigma))$, we have the quantitative estimate $\|\mathbf{M}_0(U;t)U\|_{k,\sigma}\lesssim\|U\|_{k,\sigma}$, for $k=0,\dots,K-K'$ and $\sigma\geq s_0$. To bound the time derivatives of $M(\mathbf{M}_0(U;t)U)$ we use (4.2.34) and the previous estimate to get, for any $k=0,\dots,K-K'$

$$\begin{aligned}&\|\partial_t^k\left(M(\mathbf{M}_0(U;t)U)V\right)\|_{s-\frac32 k-m}\\&\lesssim\sum_{k'+k''=k}\|\mathbf{M}_0(U;t)U\|^p_{k',s_0}\|V\|_{k'',s}+\|\mathbf{M}_0(U;t)U\|_{k',s}\|\mathbf{M}_0(U;t)U\|^{p-1}_{k',s_0}\|V\|_{k'',s_0}\\&\lesssim\sum_{k'+k''=k}\|V\|_{k'',s}\|U\|^p_{k'+K',s_0}+\|V\|_{k'',s_0}\|U\|^{p-1}_{k'+K',s_0}\|U\|_{k'+K',s}\end{aligned}$$

proving (4.2.34).
PROOF OF (iv): It follows, in view of Remark 4.2.6, by Proposition 3.17-(i) and Proposition 3.18 in [27]. □

We shall use the following facts which follow by Proposition 4.2.15.

- If $a(U;t,x,\xi)$ is a symbol in $\Sigma\Gamma^m_{K,K',p}[r,N]$ and U is a solution of $\partial_t U=M(U;t)U$ for some $M(U;t)$ in $\Sigma\mathcal{M}_{K,0,0}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$, then $\partial_t a(U;t,x,\xi)$ is a symbol in $\Sigma\Gamma^m_{K,K'+1,p}[r,N]$.
- If $R(U;t)$ is a smoothing operator in $\Sigma\mathcal{R}^{-\varrho}_{K,K',p}[r,N]$ and U is a solution of $\partial_t U=M(U;t)U$ for some $M(U;t)$ in $\Sigma\mathcal{M}_{K,0,0}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$, then $\partial_t R(U;t)$ is a smoothing operator $\Sigma\mathcal{R}^{-\varrho}_{K,K'+1,p}[r,N]$.
- If $M(U;t)$ is in $\Sigma\mathcal{M}^m_{K,K',p}[r,N]$ and $R(U;t)$ is in $\Sigma\mathcal{R}^{-\varrho}_{K,K',p'}[r,N]$ then the composition $M(U;t)\circ R(U;t)$ is in $\Sigma\mathcal{M}^m_{K,K',p+p'}[r,N]$, and so it is not a smoothing map.

4.2.2 Spectrally localized maps

We introduce the notion of a "spectrally localized" map. The class $\widetilde{\mathcal{S}}_p^m$ denotes p-linear m-operators with the spectral support similar to a para–differential operator (compare (4.2.25) and (4.2.71)), i.e. the " internal frequencies" are controlled by the "external" ones which are equivalent. On the other hand the class $\mathcal{S}_{K,K',p}^m$ contains non-homogeneous m-operators which vanish at degree at least p in U, are $(K-K')$-times differentiable in t, and satisfy estimates similar to para–differential operators, see (4.2.72). These maps include para–differential operators, smoothing remainders which come from compositions of para–differential operators (see (4.2.54)) and also linear flows generated by para–differential operators. The class of spectrally localized maps do not enjoy a symbolic calculus and it is reminiscent of the maps introduced in Definition 1.2.1 in [59].

The class of spectrally localized maps is closed under transposition (Lemma 4.2.18) and under "external" and "internal" compositions, see Proposition 4.2.19. A key property is that the transpose of the internal differential of a spectrally localized map is a smoothing operator, see Lemma 4.2.21.

Definition 4.2.16. (Spectrally localized maps) *Let $m \in \mathbb{R}$, $p, N \in \mathbb{N}_0$, $K, K' \in \mathbb{N}_0$ with $K' \leq K$ and $r > 0$.*

(i) **Spectrally localized p-homogeneous maps.** *We denote by $\widetilde{\mathcal{S}}_p^m$ the subspace of m-operators $S(\mathcal{U})$ in $\widetilde{\mathcal{M}}_p^m$ satisfying the following spectral condition: there exist $\delta > 0$, $C > 1$ such that for any $(U_1, \ldots, U_p) \in (\dot{H}^\infty(\mathbb{T};\mathbb{C}^2))^p$, for any $U_{p+1} \in \dot{H}^\infty(\mathbb{T};\mathbb{C})$ and for any $n_0, \ldots, n_{p+1} \in \mathbb{N}$ such that*

$$\Pi_{n_0} S(\Pi_{n_1} U_1, \ldots, \Pi_{n_p} U_p) \Pi_{n_{p+1}} U_{p+1} \neq 0 \,,$$

it results

$$\max\{n_1, \ldots, n_p\} \leq \delta n_{p+1} \,, \qquad C^{-1} n_0 \leq n_{p+1} \leq C n_0 \,. \tag{4.2.71}$$

We denote $\widetilde{\mathcal{S}}_p := \bigcup_m \widetilde{\mathcal{S}}_p^m$ and by $\Sigma_p^N \widetilde{\mathcal{S}}_q^m$ the class of pluri-homogeneous spectrally localized maps of the form $\sum_{q=p}^N S_q$ with $S_q \in \widetilde{\mathcal{S}}_q^m$ and $\Sigma_p \widetilde{\mathcal{S}}_q^m := \bigcup_{N \in \mathbb{N}} \Sigma_p^N \widetilde{\mathcal{S}}_q^m$. For $p \geq N+1$ we mean that the sum is empty.

(ii) **Non-homogeneous spectrally localized maps.** *We denote $\mathcal{S}_{K,K',p}^m[r]$ the space of maps $(U,t,V) \mapsto S(U;t)V$ defined on $B_{s_0}^{K'}(I;r) \times I \times C_*^0(I, \dot{H}^{s_0}(\mathbb{T},\mathbb{C}))$ for some $s_0 > 0$, which are linear in the variable V and such that the following holds true. For any $s \in \mathbb{R}$ there are $C > 0$ and $r(s) \in]0, r[$ such that for any $U \in B_{s_0}^K(I; r(s)) \cap C_*^K(I, \dot{H}^s(\mathbb{T},\mathbb{C}^2))$, any $V \in C_*^{K-K'}(I, \dot{H}^s(\mathbb{T},\mathbb{C}))$, any $0 \leq k \leq K-K'$, $t \in I$, we have that*

$$\|\partial_t^k \left(S(U;t)V\right)(t,\cdot)\|_{\dot{H}^{s-\frac{3}{2}k-m}} \leq C \sum_{k'+k''=k} \|U\|_{k'+K',s_0}^p \|V\|_{k'',s} \,. \tag{4.2.72}$$

In case $p = 0$ we require the estimate $\|\partial_t^k \left(S(U;t)V\right)\|_{\dot{H}^{s-\frac{3}{2}k-m}} \leq C\|V\|_{k,s}$.

We denote $\mathcal{S}_{K,K',N}[r] = \bigcup_m \mathcal{S}_{K,K',N}^m[r]$.

(iii) **Spectrally localized Maps.** *We denote by $\Sigma\mathcal{S}_{K,K',p}^m[r,N]$, the space of maps $(U,t,V) \to S(U;t)V$ of the form*

$$S(U;t)V = \sum_{q=p}^N S_q(U, \ldots, U)V + S_{>N}(U;t)V \tag{4.2.73}$$

where S_q are spectrally localized homogeneous maps in $\widetilde{\mathcal{S}}_q^m$, $q = p, \ldots, N$ and $S_{>N}$ is a non–homogeneous spectrally localized map in $\mathcal{S}_{K,K',N+1}^m[r]$. We denote by $\Sigma\mathcal{S}_{K,K',p}^m[r,N] \otimes \mathcal{M}_2(\mathbb{C})$ the space of 2×2 matrices whose entries are spectrally localized maps in $\Sigma\mathcal{S}_{K,K',p}^m[r,N]$. We will use also the notation $\Sigma\mathcal{S}_{K,K',p}[r,N] := \bigcup_{m \geq 0} \Sigma\mathcal{S}_{K,K',p}^m[r,N]$.

• Note that (4.2.71) implies that $\max\{n_1,\dots,n_{p+1}\}\sim n_{p+1}$ and $\max_2\{n_1,\dots,n_{p+1}\}\sim\max\{n_1,\dots,n_p\}$ and therefore, by (4.2.31), $S\in\widetilde{\mathcal{S}}^m_p$ if and only if (4.2.71) holds and there are $\mu\geq 0$, $C>0$ such that

$$\|\Pi_{n_0}S(\Pi_{n_1}U_1,\dots,\Pi_{n_p}U_p)\Pi_{n_{p+1}}U_{p+1}\|_{L^2}\leq C\max(n_1,\dots,n_p)^\mu\, n_{p+1}^m\prod_{j=1}^{p+1}\|\Pi_{n_j}U_j\|_{L^2} \tag{4.2.74}$$

for any $n_0,\dots,n_{p+1}\in\mathbb{N}$.

• In view of Lemma 4.2.9 a matrix of spectrally localized maps $S(U)$ is characterized in terms of its Fourier coefficients as follows: $S(U)$ is in $\widetilde{\mathcal{S}}^m_p\otimes\mathcal{M}_2(\mathbb{C})$ if and only if $S^{\vec\sigma_p,\sigma',\sigma}_{\vec\jmath_p,j,k}$ (defined as in (4.2.43)) satisfy, for some $\mu\geq 0$, $C>0$,

$$|S^{\vec\sigma_p,\sigma',\sigma}_{\vec\jmath_p,j,k}|\leq C\max\{|j_1|,\dots,|j_p|\}^\mu|j|^m\,,\quad\forall(\vec\jmath_p,j,k)\in(\mathbb{Z}\setminus\{0\})^{p+2},\ (\vec\sigma_p,\sigma',\sigma)\in\{\pm\}^{p+2}\,, \tag{4.2.75}$$

and if $S^{\vec\sigma_p,\sigma',\sigma}_{\vec\jmath_p,j,k}\neq 0$ then (4.2.44), (4.2.45) hold and, for some $\delta>0$,

$$\max\{|j_1|,\dots,|j_p|\}\leq\delta|j|\,,\quad C^{-1}|k|\leq|j|\leq C\,|k|\,. \tag{4.2.76}$$

• If S is a p–homogeneous spectrally localized map in $\widetilde{\mathcal{S}}^m_p$ then $S(U)[V]$ defines a non–homogeneous spectrally localized map in $\mathcal{S}^m_{K,0,p}[r]$ for any $r>0$. The proof is similar to the one of Lemma 4.2.8 noting that the estimate (4.2.72) holds for any $s\in\mathbb{R}$ because of the equivalence $n_0\sim n_{p+1}\sim\max\{n_1,\dots,n_{p+1}\}$ in (4.2.71).

• **(Para-differential operators as spectrally localized maps)** If $a(U;t,x,\xi)$ is a symbol in $\Sigma\Gamma^m_{K,K',p}[r,N]$ then the para-differential operator $\mathrm{Op}^{BW}(a(U;t,x,\xi))$ is a spectrally localized map in $\Sigma\mathcal{S}^m_{K,K',p}[r,N]$. This is a consequence of Proposition 3.8 in [27]. We remark that for a homogeneous symbol this is a consequence of the choice of the first quantization in (4.2.23).

• If $S(U,\dots,U)$ is a p–homogeneous spectrally localized map in $\widetilde{\mathcal{S}}_p$, then the differential of the non–linear map $S(U,\dots,U)U$, $\mathrm{d}_U\big(S(U,\dots,U)U\big)V=pS(V,U,\dots,U)U+S(U,\dots,U)V$ is a p–homogeneous operator in $\widetilde{\mathcal{M}}_p$, *not* necessarily spectrally localized. Indeed the operator $S(V,U,\dots,U)U$ is *not*, in general, spectrally localized.

• If $m_1\leq m_2$ then $\Sigma\mathcal{S}^{m_1}_{K,K',p}[r,N]\subseteq\Sigma\mathcal{S}^{m_2}_{K,K',p}[r,N]$. If $K_1'\leq K_2'$ then $\Sigma\mathcal{S}^m_{K,K_1',p}[r,N]\subseteq\Sigma\mathcal{S}^m_{K,K_2',p}[r,N]$.

Remark 4.2.17. The constant $\delta>0$ in the spectral condition (4.2.71) is not assumed to be small (unlike the one in (4.2.25) for para-differential operators).

The class of matrices of spectrally localized homogeneous maps is closed under transposition.

Lemma 4.2.18. *Let $p\in\mathbb{N}_0$, $m\in\mathbb{R}$. If $S(U)$ is a matrix of p–homogeneous spectrally localized maps in $\widetilde{\mathcal{S}}^m_p\otimes\mathcal{M}_2(\mathbb{C})$ then $S(U)^\top$ is in $\widetilde{\mathcal{S}}^m_p\otimes\mathcal{M}_2(\mathbb{C})$, where the transpose is computed with respect to the non-degenerate real bilinear form* (4.2.4).

Proof. It results

$$|(S^\top)^{\vec\sigma_p,\sigma',\sigma}_{\vec\jmath_p,j,k}|\overset{(4.2.47)}{=}|S^{\vec\sigma_p,\sigma,\sigma'}_{\vec\jmath_p,-k,-j}|\overset{(4.2.75)}{\leq}C\max\{|j_1|,\dots,|j_p|\}^\mu|k|^m\overset{(4.2.76)}{\leq}C'\max\{|j_1|,\dots,|j_p|\}^\mu|j|^m$$

which means that $(S^\top)^{\vec\sigma_p,\sigma',\sigma}_{\vec\jmath_p,j,k}$ satisfies (4.2.75). Since $|j|\sim|k|$ then $\max\{|j_1|,\dots,|j_p|\}\leq\delta C|k|$ hence (4.2.76) holds for $(S^\top)^{\vec\sigma_p,\sigma',\sigma}_{\vec\jmath_p,j,k}$. □

We now prove some further composition results for spectrally localized maps.

Proposition 4.2.19. (Compositions of spectrally localized maps) *Let $p, p', p'', N, K, K' \in \mathbb{N}_0$ with $K' \leq K$ and $r > 0$. Let $m, m' \in \mathbb{R}$ and $S(U;t)$ be a spectrally localized map in $\Sigma\mathcal{S}^m_{K,K',p}[r,N]$. Then*

(i) if $R(U;t)$ is a smoothing operator in $\Sigma\mathcal{R}^{-\varrho}_{K,K',p''}[r,N]$ for some $\varrho \geq 0$, then $S(U;t) \circ R(U;t)$ and $R(U;t) \circ S(U;t)$ are smoothing operators in $\Sigma\mathcal{R}^{-\varrho+\max(m,0)}_{K,K',p+p''}[r,N]$.

(ii) If $S'(U;t)$ is in $\Sigma\mathcal{S}^{m'}_{K,K',p'}[r,N]$ then the composition $S(U;t) \circ S'(U;t)$ is in $\Sigma\mathcal{S}^{m+m'}_{K,K',p+p'}[r,N]$.

(iii) If S is in $\widetilde{\mathcal{S}}^m_p$ and $S^{(a)}(U;t)$ are spectrally localized maps in $\Sigma\mathcal{S}^{\ell_a}_{K,K',q_a}[r,N]$ for some $\ell_a \in \mathbb{R}$ and $q_a \in \mathbb{N}$, $a = 1, \ldots, p$, then also the internal composition

$$S(S^{(1)}(U;t)U, \ldots, S^{(p)}(U;t)U) \tag{4.2.77}$$

is a spectrally localized map in $\Sigma\mathcal{S}^m_{K,K',p+\overline{q}}[r,N]$ with $\overline{q} := \sum_{a=1}^p q_a$.

(iv) If $S(U;t)$ is in $\mathcal{S}^m_{K,0,p}[\check{r}]$ for any $\check{r} \in \mathbb{R}^+$ and $\mathbf{M}_0(U;t)$ belongs to $\mathcal{M}^0_{K,K',0}[r] \otimes \mathcal{M}_2(\mathbb{C})$, then $S(\mathbf{M}_0(U;t)U;t)$ is in $\mathcal{S}^m_{K,K',p}[r]$.

(v) If $S(U)$ is a matrix of p-homogeneous spectrally localized maps in $\widetilde{\mathcal{S}}^m_p \otimes \mathcal{M}_2(\mathbb{C})$ and $S_a(U;t)$ are in $\mathcal{S}^{m_a}_{K,K',q_a}[r] \otimes \mathcal{M}_2(\mathbb{C})$, $a = 1, 2$, $m_a \in \mathbb{R}$, then

$$\mathrm{d}_U \left. (S(U)U)\right|_{S_1(U;t)U} [S_2(U;t)U]$$

is in $\mathcal{S}^{m+\max(m_1,m_2)}_{K,K',p+pq_1+q_2}[r] \otimes \mathcal{M}_2(\mathbb{C})$.

Proof. PROOF OF (i). The operator $R(U;t) \circ S(U;t)$ is in $\Sigma\mathcal{R}^{-\varrho+\max(m,0)}_{K,K',p+p''}[r,N]$ by Proposition 4.2.15-(i) since $S(U;t)$ is in $\Sigma\mathcal{M}^m_{K,K',p}[r,N]$. Then we prove that $S(U;t) \circ R(U;t)$ is in $\Sigma\mathcal{R}^{-\varrho+\max(m,0)}_{K,K',p+p''}[r,N]$. It is sufficient to consider the case $m \geq 0$ since, if $m < 0$, we regard $S(U;t)$ as a spectrally localized map in $\Sigma\mathcal{S}^0_{K,K',p}[r,N]$. Decomposing $S = \sum_{q=p}^N S_q + S_{>N}$ as in (4.2.73) and $R = \sum_{q=p''}^N R_q + R_{>N}$ as in (4.2.35), we have to show, on the one hand that

$$S_{q_1}(U_1, \ldots, U_{q_1}) R_{q_2}(U_{q_1+1}, \ldots, U_{q_1+q_2}) \tag{4.2.78}$$

is a homogeneous smoothing operator in $\widetilde{\mathcal{R}}^{-\varrho+m}_{q_1+q_2}$ if $q_1 + q_2 \leq N$ and, on the other hand, that

$$\begin{aligned} &S_{q_1}(U, \ldots, U) R_{q_2}(U, \ldots, U),\ q_1 + q_2 \geq N+1, \\ &S_{>N}(U;t) R_{q_2}(U, \ldots, U),\ q_2 = p'', \ldots, N, \\ &S_{q_1}(U, \ldots, U) R_{>N}(U;t),\ q_1 = p, \ldots, N, \\ &S_{>N}(U;t) R_{>N}(U;t) \end{aligned} \tag{4.2.79}$$

are non–homogeneous smoothing operators in $\mathcal{R}^{-\varrho+m}_{K,K',N+1}[r]$. We first study (4.2.78). First of all one has

$$\begin{aligned} &\|\Pi_{n_0} S_{q_1}(\Pi_{n_1}U_1, \ldots, \Pi_{n_{q_1}}U_{q_1}) R_{q_2}(\Pi_{n_{q_1+1}}U_{q_1+1}, \ldots, \Pi_{n_{q_1+q_2}}U_{q_1+q_2}) \Pi_{n_{q_1+q_2+1}}U_{q_1+q_2+1}\|_{L^2} \\ &\leq \sum_{n'} \|\Pi_{n_0} S_{q_1}(\Pi_{n_1}U_1, \ldots, \Pi_{n_{q_1}}U_{q_1}) \Pi_{n'} R_{q_2}(\Pi_{n_{q_1+1}}U_{q_1+1}, \ldots, \Pi_{n_{q_1+q_2}}U_{q_1+q_2}) \Pi_{n_{q_1+q_2+1}}U_{q_1+q_2+1}\|_{L^2}. \end{aligned} \tag{4.2.80}$$

Thanks to the conditions (4.2.32), (4.2.71) for S and (4.2.32) for R the indices in the above sum satisfy, for some choice of signs σ_a, $a = 1, \dots, q_1 + q_2 + 1$, the restrictions

$$n' = \sum_{a=0}^{q_1} \sigma_a n_a, \quad n' = \sum_{a=q_1+1}^{q_1+q_2+1} \sigma_a n_a, \quad \max\{n_1, \dots, n_{q_1}\} \lesssim n', \quad n_0 \sim n'. \tag{4.2.81}$$

As a consequence (4.2.78) satisfies the corresponding condition (4.2.32) and

$$n' \lesssim \max\{n_{q_1+1}, \dots, n_{q_1+q_2+1}\}, \quad \max\{n_1, \dots, n_{q_1+q_2+1}\} \sim \max\{n_{q_1+1}, \dots, n_{q_1+q_2+1}\}. \tag{4.2.82}$$

We also claim that

$$\max\{n_1, \dots, n_{q_1}\} \lesssim \max{}_2\{n_1, \dots, n_{q_1+q_2+1}\}. \tag{4.2.83}$$

Indeed, by (4.2.81) and (4.2.82), we have

$$\max\{n_1, \dots, n_{q_1}\} \lesssim \max\{n_{q_1+1}, \dots, n_{q_1+q_2+1}\}, \tag{4.2.84}$$

and we distinguish two cases. If $\max\{n_1, \dots, n_{q_1+q_2+1}\} = \max\{n_{q_1+1}, \dots, n_{q_1+q_2+1}\}$ then we directly obtain $\max\{n_1, \dots, n_{q_1}\} \leq \max_2\{n_1, \dots, n_{q_1+q_2+1}\}$ which gives (4.2.83). If $\max\{n_1, \dots, n_{q_1+q_2+1}\} = \max\{n_1, \dots, n_{q_1}\}$ then $\max\{n_{q_1+1}, \dots, n_{q_1+q_2+1}\} \leq \max_2\{n_1, \dots, n_{q_1+q_2+1}\}$ which together with (4.2.84) proves (4.2.83). Using (4.2.80), (4.2.81), (4.2.74), (4.2.31) (with $m = -\varrho$ and $\mu \rightsquigarrow \mu'$) we get

$$\begin{aligned}
&\|\Pi_{n_0} S_{q_1}(\Pi_{n_1} U_1, \dots, \Pi_{n_{q_1}} U_{q_1}) R_{q_2}(\Pi_{n_{q_1+1}} U_{q_1+1}, \dots, \Pi_{n_{q_1+q_2}} U_{q_1+q_2}) \Pi_{n_{q_1+q_2+1}} U_{q_1+q_2+1}\|_{L^2} \\
&\leq C \sum_{n'} \max\{n_1, \dots, n_{q_1}\}^{\mu} (n')^m \frac{\max_2\{n_{q_1+1}, \dots, n_{q_1+q_2+1}\}^{\mu'}}{\max\{n_{q_1+1}, \dots, n_{q_1+q_2+1}\}^{\varrho}} \prod_{a=1}^{q_1+q_2+1} \|\Pi_{n_a} U_a\|_{L^2} \\
&\overset{(4.2.83),(4.2.82),m\geq 0}{\leq} C \frac{\max_2\{n_1, \dots, n_{q_1+q_2+1}\}^{\mu'+\mu}}{\max\{n_1, \dots, n_{q_1+q_2+1}\}^{\varrho-m}} \prod_{a=1}^{q_1+q_2+1} \|\Pi_{n_a} U_a\|_{L^2}.
\end{aligned}$$

This proves that (4.2.78) is in $\widetilde{\mathcal{R}}^{-\varrho+m}_{q_1+q_2}$.

In order to prove that the operators in (4.2.79) satisfy (4.2.34) (with $m \rightsquigarrow -\varrho + m$ and $p \rightsquigarrow N$) we recall that $S_{q_1}(U, \dots, U)$ defines a spectrally localized map in $\mathcal{S}^m_{K,0,q_1}[r]$ (as remarked below Definition 4.2.16) and $R_{q_2}(U, \dots, U)$ defines a smoothing operator in $\mathcal{R}^{-\varrho}_{K,0,q_2}[r]$ thanks to Lemma 4.2.8. Then the thesis follows by estimates (4.2.72) and (4.2.34). For instance consider the last term in (4.2.79). For any $k = 0, \dots, K - K'$

$$\begin{aligned}
&\|\partial_t^k (S_{>N}(U;t) R_{>N}(U;t) V)\|_{\dot{H}^{s-\frac{3}{2}k-m+\varrho}} \leq C \sum_{k'+k''=k} \|U\|^{N+1}_{k'+K',s_0} \|R_{>N}(U;t)V\|_{k'',s+\varrho} \\
&\leq C \sum_{k'+k''=k} \sum_{\substack{0 \leq j \leq k'' \\ j'+j''=j}} \|U\|^{N+1}_{k'+K',s_0} \Big(\|V\|_{j'',s} \|U\|^{N+1}_{j'+K',s_0} + \|V\|_{j'',s_0} \|U\|^{N}_{j'+K',s_0} \|U\|_{j'+K',s} \Big) \\
&\leq C \sum_{j''=0}^{k} \|V\|_{j'',s} \|U\|^{2N+2}_{k-j''+K',s_0} + \|V\|_{j'',s_0} \|U\|^{2N+1}_{k-j''+K',s_0} \|U\|_{k-j''+K',s}
\end{aligned}$$

using that $k' \leq k - j''$ and $j' \leq k - j''$. This proves that (4.2.78) is in $\mathcal{R}^{-\varrho+m}_{K,K',N+1}[r]$ (as $\|U\|_{K,s_0} < 1$).

PROOF OF (ii). Decomposing $S = \sum_{q=p}^{N} S_q + S_{>N}$ and $S' = \sum_{q=p'}^{N} S'_q + S'_{>N}$ as in (4.2.73), we have to show, on the one hand that

$$S_{q_1}(U_1, \dots, U_{q_1}) S'_{q_2}(U_{q_1+1}, \dots, U_{q_1+q_2}) \tag{4.2.85}$$

is a homogeneous spectrally localized map in $\widetilde{\mathcal{S}}^{m+m'}_{q_1+q_2}$ if $q_1+q_2\le N$ and, on the other hand, that

$$\begin{aligned}
&S_{q_1}(U,\dots,U)S'_{q_2}(U,\dots,U),\ q_1+q_2\ge N+1,\\
&S_{>N}(U;t)S'_{q_2}(U,\dots,U),\ q_2=p',\dots,N,\\
&S_{q_1}(U,\dots,U)S'_{>N}(U;t),\ q_1=p,\dots,N,\\
&S_{>N}(U;t)S'_{>N}(U;t)
\end{aligned}\tag{4.2.86}$$

are non–homogeneous spectrally localized map in $\mathcal{S}^{m+m'}_{K,K',N+1}[r]$. We first prove that (4.2.85) is in $\widetilde{\mathcal{S}}^{m+m'}_{q_1+q_2}$. First of all one has

$$\begin{aligned}
&\|\Pi_{n_0}S_{q_1}(\Pi_{n_1}U_1,\dots,\Pi_{n_{q_1}}U_{q_1})S'_{q_2}(\Pi_{n_{q_1+1}}U_{q_1+1},\dots,\Pi_{n_{q_1+q_2}}U_{q_1+q_2})\Pi_{n_{q_1+q_2+1}}U_{q_1+q_2+1}\|_{L^2}\\
&\le\sum_{n'}\|\Pi_{n_0}S_{q_1}(\Pi_{n_1}U_1,\dots,\Pi_{n_{q_1}}U_{q_1})\Pi_{n'}S'_{q_2}(\Pi_{n_{q_1+1}}U_{q_1+1},\dots,\Pi_{n_{q_1+q_2}}U_{q_1+q_2})\Pi_{n_{q_1+q_2+1}}U_{q_1+q_2+1}\|_{L^2}.
\end{aligned}\tag{4.2.87}$$

Thanks to the conditions (4.2.32), (4.2.71) for S and S' the indices in the above sum satisfy, for some choice of signs σ_a, $a=1,\dots,q_1+q_2+1$, the restriction

$$\begin{aligned}
&n'=\sum_{a=0}^{q_1}\sigma_a n_a,\quad n'=\sum_{a=q_1+1}^{q_1+q_2+1}\sigma_a n_a,\\
&\begin{cases}\max\{n_1,\dots,n_{q_1}\}\le\delta n'\\ \frac{n_0}{C}\le n'\le Cn_0\,,\end{cases}\qquad
\begin{cases}\max\{n_{q_1+1},\dots,n_{q_1+q_2}\}\le\delta' n_{q_1+q_2+1}\\ \frac{n'}{C'}\le n_{q_1+q_2+1}\le C'n'\,,\end{cases}
\end{aligned}\tag{4.2.88}$$

for some $C,C'>1$ and $\delta,\delta'>0$. Therefore

$$\begin{cases}\max\{n_1,\dots,n_{q_1+q_2}\}\le\max\{\delta',\delta C'\}n_{q_1+q_2+1}\\ \frac{n_0}{CC'}\le n_{q_1+q_2+1}\le CC'n_0\,.\end{cases}$$

This proves that $S_{q_1}\circ S'_{q_2}$ fulfills the localization property of Definition 4.2.16 (i) (see (4.2.71)). In addition (4.2.85) satisfies the corresponding condition (4.2.32). Using (4.2.87), (4.2.74) we get

$$\begin{aligned}
&\|\Pi_{n_0}S_{q_1}(\Pi_{n_1}U_1,\dots,\Pi_{n_{q_1}}U_{q_1})S'_{q_2}(\Pi_{n_{q_1+1}}U_{q_1+1},\dots,\Pi_{n_{q_1+q_2}}U_{q_1+q_2})\Pi_{n_{q_1+q_2+1}}U_{q_1+q_2+1}\|_{L^2}\\
&\le C\max\{n_1,\dots,n_{q_1}\}^{\mu}(n')^m\max\{n_{q_1+1},\dots,n_{q_1+q_2}\}^{\mu'}n^{m'}_{q_1+q_2+1}\prod_{a=1}^{q_1+q_2+1}\|\Pi_{n_a}U_a\|_{L^2}\\
&\overset{(4.2.88)}{\le}C\max\{n_1,\dots,n_{q_1+q_2}\}^{\mu'+\mu}(n_{q_1+q_2+1})^{m+m'}\prod_{a=1}^{q_1+q_2+1}\|\Pi_{n_a}U_a\|_{L^2}
\end{aligned}$$

which proves that $S_{q_1}\circ S'_{q_2}$ satisfies (4.2.74). In order to prove that the terms in (4.2.79) satisfy (4.2.72) we first note that, thanks to (i) of Lemma 4.2.18, we have that $S_{q_1}(U,\dots,U)\in\mathcal{S}^m_{K,0,q_1}[r]$ and $S'_{q_2}(U,\dots,U)\in\mathcal{S}^{m'}_{K,0,q_2}[r]$ and then the thesis follows using (4.2.72). For instance consider the first term in (4.2.86). Using twice (4.2.72) (with $K'=0$) we get, for any $k=0,\dots,K$,

$$\begin{aligned}
\|\partial_t^k\big(S_{q_1}(U;t)S'_{q_2}(U;t)V\big)\|_{\dot H^{s-\frac32k-m_1-m_2}}&\le C\sum_{k'+k''=k}\|U\|^{q_1}_{k',s_0}\|S'_{q_2}(U;t)V\|_{k'',s-m_2}\\
&\le C\sum_{k'+k''=k}\sum_{\substack{0\le j\le k''\\ j'+j''=j}}\|U\|^{q_1}_{k',s_0}\|U\|^{q_2}_{j',s_0}\|V\|_{j'',s}\\
&\le C\sum_{j''=0}^{k}\|U\|^{q_1+q_2}_{k-j'',s_0}\|V\|_{j'',s}
\end{aligned}$$

where in the last step we used that $j' \leq k - j''$ and $k' \leq k - j''$. The last line is (4.2.72) with p replaced by $q_1 + q_2 \geq N + 1$.

PROOF OF (iii): We now consider the internal composition (4.2.77) in the homogeneous case. For simplicity of notation we consider the case $q_2 = \cdots = q_p = 0$, $S^{(2)} = \cdots = S^{(p)} = \mathrm{Id}$ and $q_1 = \overline{q} =: q$, the general case follows in the same way. So we need to show that $S(S^{(1)}(U)U, U, \ldots, U)$ is a spectrally localized map in $\widetilde{\mathcal{S}}^m_{p+q}$. We first estimate

$$\begin{aligned}
&\|\Pi_{n_0} S\big(S^{(1)}(\Pi_{n_1}U_1, \ldots, \Pi_{n_q}U_q)\Pi_{n_{q+1}}U_{q+1}, \Pi_{n_{q+2}}U_{q+2}, \ldots, \Pi_{n_{q+p}}U_{q+p}\big)\Pi_{n_{p+q+1}}U_{p+q+1}\|_{L^2} \\
&\leq \sum_{n'} \|\Pi_{n_0} S\big(\Pi_{n'} S^{(1)}(\Pi_{n_1}U_1, \ldots, \Pi_{n_q}U_q)\Pi_{n_{q+1}}U_{q+1}, \Pi_{n_{q+2}}U_{q+2}, \ldots, \Pi_{n_{q+p}}U_{q+p}\big)\Pi_{n_{p+q+1}}U_{p+q+1}\|_{L^2} .
\end{aligned} \tag{4.2.89}$$

Thanks to the conditions (4.2.32) for S and $S^{(1)}$ the indices in the above sum satisfy, for some choice of signs σ_a, $a = 1, \ldots, p+q+1$, the restrictions $n' = \sum_{a=1}^{q+1} \sigma_a n_a$ and $n' = \sigma_0 n_0 + \sum_{a=q+2}^{p+q+1} \sigma_a n_a$, proving that $S(S^{(1)}(U)U, U, \ldots, U)$ fulfills (4.2.32). Moreover the condition (4.2.71) for S and $S^{(1)}$ imply the existence of $\delta, \delta_1 > 0$, $C, C_1 > 1$ such that

$$\begin{cases} \max\{n', n_{q+2}, \ldots, n_{q+p}\} \leq \delta n_{q+p+1} \\ \frac{n_0}{C} \leq n_{p+q+1} \leq C n_0 , \end{cases} \qquad \begin{cases} \max\{n_1, \ldots, n_q\} \leq \delta_1 n_{q+1} \\ \frac{n'}{C_1} \leq n_{q+1} \leq C_1 n' , \end{cases} \tag{4.2.90}$$

and therefore $\max\{n_1, \ldots, n_{p+q}\} \leq \max(\delta\delta_1 C_1, \delta C_1) n_{q+p+1}$ and $\frac{n_0}{C} \leq n_{p+q+1} \leq C n_0$, proving that $S(S^{(1)}(U)U, U, \ldots, U)$ fulfills (4.2.71). We now prove it fulfills also (4.2.74). We get

$$\begin{aligned}
&\|\Pi_{n_0} S(S^{(1)}(\Pi_{n_1}U_1, \ldots, \Pi_{n_q}U_q)\Pi_{n_{q+1}}U_{q+1}, \Pi_{n_{q+2}}U_{q+2}, \ldots, \Pi_{n_{q+p}}U_{q+p})\Pi_{n_{p+q+1}}U_{p+q+1}\|_{L^2} \\
&\overset{(4.2.89),(4.2.74)}{\lesssim} \sum_{n'} \max(n', n_{q+2}, \ldots, n_{q+p})^{\mu} n_{p+q+1}^m \max(n_1, \ldots, n_q)^{\mu'} n_{q+1}^{\ell_1} \prod_{a=1}^{p+q+1} \|\Pi_{n_a}U_a\|_{L^2} \\
&\overset{(4.2.90), n_{q+1} \sim n'}{\lesssim} \sum_{n'} \max(n_{q+1}, n_{q+2}, \ldots, n_{q+p})^{\mu + \max(0,\ell_1)} n_{p+q+1}^m \max(n_1, \ldots, n_q)^{\mu'} \prod_{a=1}^{p+q+1} \|\Pi_{n_a}U_a\|_{L^2} \\
&\lesssim \max(n_1, \ldots, n_{p+q})^{\mu+\mu'+\max(0,\ell_1)} n_{p+q+1}^m \prod_{a=1}^{p+q+1} \|\Pi_{n_a}U_a\|_{L^2} ,
\end{aligned}$$

proving that $S(S^{(1)}(U)U, U, \ldots, U)$ is a spectrally localized map in $\widetilde{\mathcal{S}}^m_{p+q}$. Finally $S(S^{(1)}(U)U, U, \ldots, U)$ satisfies also (4.2.33), concluding the proof that it is a spectrally localized map in $\widetilde{\mathcal{S}}^m_{p+q}$.

PROOF OF (iv): By the estimate below (4.2.34) for $\mathbf{M}_0(U;t)$, for any $U \in B^K_{s_0}(I;r)$ and any $k = 0, \ldots, K - K'$, $\|\mathbf{M}_0(U;t)U\|_{k,s_0} \lesssim \|U\|_{k,s_0}$. Then estimate (4.2.72) for $S(\mathbf{M}_0(U;t)U;t)$ for any $0 \leq k \leq K - K'$ follows from the ones for $S(U;t)$ arguing as in (iii) of Proposition 4.2.15.

PROOF OF (v): It follows computing explicitly the differential $\mathrm{d}_U(S(U)U)[V]$, evaluating it at $U \rightsquigarrow S_1(U;t)U$ and $V \rightsquigarrow S_2(U;t)U$ and using item (ii) and (iii) of the proposition. $\square$

The following lemma proves that the internal composition of a spectrally localized map with a map, is a spectrally localized map plus a smoothing operator whose transpose is another smoothing operator.

Lemma 4.2.20. *Let $p \in \mathbb{N}$, $q \in \mathbb{N}_0$, $m \in \mathbb{R}$ and $m' \geq 0$. Let $S(U)$ be a matrix of spectrally localized homogeneous maps in $\widetilde{\mathcal{S}}^m_p \otimes \mathcal{M}_2(\mathbb{C})$ and $M(U)$ be a matrix of homogeneous m'-operators in $\widetilde{\mathcal{M}}^{m'}_q \otimes \mathcal{M}_2(\mathbb{C})$. Then*

$$S(M(U)U, U, \ldots, U) = S'(U) + R(U) \tag{4.2.91}$$

where

- *$S'(U)$ is a matrix of spectrally localized homogeneous maps in $\widetilde{\mathcal{S}}^m_{p+q} \otimes \mathcal{M}_2(\mathbb{C})$;*
- *$R(U)$ is a matrix of homogeneous smoothing operators in $\widetilde{\mathcal{R}}^{-\varrho}_{p+q} \otimes \mathcal{M}_2(\mathbb{C})$ for any $\varrho \geq 0$, and $R(U)^\top$ is a matrix of homogeneous smoothing operators in $\widetilde{\mathcal{R}}^{-\varrho}_{p+q} \otimes \mathcal{M}_2(\mathbb{C})$ for any $\varrho \geq 0$ as well.*

Proof. By multilinearity we expand

$$\begin{aligned} & S\big(M(U_1, \dots, U_q)U_{q+1}, U_{q+2}, \dots, U_{p+q}\big) \\ & = \sum_{(n', n_0, \dots, n_{p+q+1}) \in \mathcal{N}} \Pi_{n_0} S\big(\Pi_{n'} M(\Pi_{n_1} U_1, \dots, \Pi_{n_q} U_q)\Pi_{n_{q+1}} U_{q+1}, \Pi_{n_{q+2}} U_{q+2}, \dots, \Pi_{n_{p+q}} U_{p+q}\big)\Pi_{n_{p+q+1}} \end{aligned} \tag{4.2.92}$$

where $\mathcal{N}$ is a subset of $\mathbb{N}^{p+q+3}$ made by indexes fulfilling, by restrictions (4.2.71) and (4.2.32), the following conditions: there exist signs $\{\sigma_j, \epsilon_j, \epsilon'\} \subset \{\pm\}$ and constants $\delta > 0$, $C > 1$ such that

$$\begin{cases} \max(n', n_{q+2}, \dots, n_{p+q}) \leq \delta n_{p+q+1} & (a) \\ \quad C^{-1} n_0 \leq n_{p+q+1} \leq C n_0 & (b) \\ n' = \sum_{j=1}^{q+1} \sigma_j n_j, \quad n_0 = \epsilon' n' + \sum_{j=q+2}^{p+q+1} \epsilon_j n_j & (c)\,. \end{cases} \tag{4.2.93}$$

We fix $\delta' > \delta$ and we denote by $\mathcal{N}'$ the subset of $\mathcal{N}$ made by indices which satisfy the additional restriction

$$\max(n_1, \dots, n_{q+1}) \leq \delta' n_{p+q+1}\,. \tag{4.2.94}$$

Then we define

$$\begin{aligned} & S'(U_1, \dots, U_{p+q}) \\ & := \sum_{(n', n_0, \dots, n_{p+q+1}) \in \mathcal{N}'} \Pi_{n_0} S\big(\Pi_{n'} M(\Pi_{n_1} U_1, \dots, \Pi_{n_q} U_q)\Pi_{n_{q+1}} U_{q+1}, \Pi_{n_{q+2}} U_{q+2}, \dots, \Pi_{n_{p+q}} U_{p+q}\big)\Pi_{n_{p+q+1}}\,. \end{aligned} \tag{4.2.95}$$

By (4.2.93) and (4.2.94) one has that $S'(U)$ fulfills the spectral condition $n_0 \sim n_{p+q+1}$, $\max(n_1, \dots, n_{p+q}) \leq \delta' n_{p+q+1}$ which is the condition (4.2.71). Moreover using (4.2.74) and (4.2.31) we bound

$$\begin{aligned} & \Big\| \Pi_{n_0} S\big(\Pi_{n'} M(\Pi_{n_1} U_1, \dots, \Pi_{n_q} U_q)\Pi_{n_{q+1}} U_{q+1}, \Pi_{n_{q+2}} U_{q+2}, \dots, \Pi_{n_{p+q}} U_{p+q}\big)\Pi_{n_{p+q+1}} U_{p+q+q} \Big\|_{L^2} \\ & \qquad \lesssim \max(n_1, \dots, n_{p+q})^{\mu + m'} n_{p+q+1}^m \prod_{a=1}^{p+q+1} \|\Pi_{n_a} U_a\|_{L^2} \end{aligned} \tag{4.2.96}$$

for some $\mu > 0$. Finally by (4.2.93)(c) one has also that (4.2.32) holds. One checks that also (4.2.33) holds true. We have proved that $S'(U)$ is a matrix of spectrally localized maps in $\widetilde{\mathcal{S}}^m_{p+q} \otimes \mathcal{M}_2(\mathbb{C})$.

Then, recalling (4.2.92) and (4.2.95), we define

$$\begin{aligned} & R(U_1, \dots, U_{p+q}) \\ & := \sum_{(n', n_0, \dots, n_{p+q+1}) \in \mathcal{N} \setminus \mathcal{N}'} \Pi_{n_0} S\big(\Pi_{n'} M(\Pi_{n_1} U_1, \dots, \Pi_{n_q} U_q)\Pi_{n_{q+1}} U_{q+1}, \Pi_{n_{q+2}} U_{q+2}, \dots, \Pi_{n_{p+q}} U_{p+q}\big)\Pi_{n_{p+q+1}}\,. \end{aligned} \tag{4.2.97}$$

We claim that there is $C' > 0$ such that if $(n', n_0, \dots, n_{p+q+1}) \in \mathcal{N} \setminus \mathcal{N}'$ then

$$\begin{cases} \max(n_{q+2}, \dots, n_{p+q}) \leq \delta' n_{p+q+1} & (a) \\ C^{-1} n_0 \leq n_{p+q+1} \leq C n_0 & (b) \\ n_{p+q+1} \leq C' \max_2(n_1, \dots, n_{q+1}) & (c) \\ \max(n_1, \dots, n_{q+1}) \leq C' \max_2(n_1, \dots, n_{q+1}) & (d)\,. \end{cases} \tag{4.2.98}$$

Before proving (4.2.98) we note that it implies

$$\max(n_1,\dots,n_{p+q+1}) \overset{(a)}{\le} (1+\delta')\max(n_1,\dots,n_{q+1},n_{p+q+1})$$
$$\overset{(c)+(d)}{\le} (1+\delta')C'\max{}_2(n_1,\dots,n_{q+1}) \le (1+\delta')C'\max{}_2(n_1,\dots,n_{p+q+1})$$

proving that $\max(n_1,\dots,n_{p+q+1}) \sim \max_2(n_1,\dots,n_{p+q+1})$. Then by (4.2.97) and (4.2.96) we obtain

$$\|\Pi_{n_0}R(\Pi_{n_1}U_1,\dots,\Pi_{n_{p+q}}U_{p+q})\Pi_{n_{p+q+1}}U_{p+q+1}\|_{L^2} \lesssim \max(n_1,\dots,n_{p+q+1})^{\widetilde{\mu}} \prod_{a=1}^{p+q+1} \|\Pi_{n_a}U_a\|_{L^2}$$
$$\lesssim \frac{\max_2(n_1,\dots,n_{p+q+1})^{\widetilde{\mu}+\varrho}}{\max(n_1,\dots,n_{p+q+1})^{\varrho}} \prod_{a=1}^{p+q+1} \|\Pi_{n_a}U_a\|_{L^2} \qquad \text{with} \qquad \widetilde{\mu} = \mu + m' + \max(m,0)\,,$$

showing that $R(U)$ is a $(p+q)$–homogeneous smoothing operator in $\widetilde{\mathcal{R}}^{-\varrho}_{p+q} \otimes \mathcal{M}_2(\mathbb{C})$ for any $\varrho \ge 0$.

We now prove (4.2.98). Note that (a) and (b) of (4.2.98) follow by (a) and (b) of (4.2.93) and $\delta' > \delta$. Then note that if $(n', n_0, \dots, n_{p+q+1}) \in \mathcal{N} \setminus \mathcal{N}'$ then

$$\max(n_1,\dots,n_{q+1}) > \delta' n_{p+q+1}\,. \tag{4.2.99}$$

Then, by (c) of (4.2.93), one has

$$\max(n_1,\dots,n_{q+1}) = \epsilon' n' + \sum_{n_j \le \max_2(n_1,\dots,n_{q+1})} \epsilon_j n_j \le n' + q\max{}_2(n_1,\dots,n_{q+1}) \tag{4.2.100}$$

so that $\max_2(n_1,\dots,n_{q+1}) \ge \frac{1}{q}\big(\max(n_1,\dots,n_{q+1}) - n'\big)$. We deduce using (4.2.99) and (4.2.93)(a) that

$$\max{}_2(n_1,\dots,n_{q+1}) \ge \frac{\delta'-\delta}{q}\, n_{p+q+1} \tag{4.2.101}$$

thus proving (4.2.98) (c). Finally using (4.2.100), (4.2.93) (a) and (4.2.101) we get

$$\max(n_1,\dots,n_{q+1}) \le q\big(\frac{\delta}{\delta'-\delta} + 1\big)\max{}_2(n_1,\dots,n_{q+1})$$

which proves (d) of (4.2.98).

We finally prove that $R(U)^\top$ is a smoothing operator. First note that if $\Pi_{n_0}R(\Pi_{\vec{n}}\mathcal{U})^\top\Pi_{n_{p+q+1}} \neq 0$ then $\Pi_{n_{p+q+1}}R(\Pi_{\vec{n}}\mathcal{U})\Pi_{n_0} \neq 0$, which implies that $C^{-1}n_{p+q+1} \le n_0 \le Cn_{p+q+1}$ by (4.2.98) (b). Then $R(U)^\top$ is a smoothing operator in $\widetilde{\mathcal{R}}^{-\varrho}_{p+q} \otimes \mathcal{M}_2(\mathbb{C})$ for any $\varrho \ge 0$ by Lemma 4.2.10 and Remark 4.2.11. □

We finally prove the following lemma which generalizes a result in [73] for para-differential operators: the transpose of the internal differential of a spectrally localized map is a smoothing operator (with two equivalent frequencies).

Lemma 4.2.21. *Let $p \in \mathbb{N}$ and $m \in \mathbb{R}$. Given a matrix of spectrally localized p–homogeneous maps $S(U) \in \widetilde{\mathcal{S}}^m_p \otimes \mathcal{M}_2(\mathbb{C})$, consider*

$$V \mapsto L(U)V := \frac{1}{p}\mathrm{d}_U S(U)[V]U = S(V,U,\dots,U)U\,. \tag{4.2.102}$$

Then the transposed map $L(U)^\top$ is a matrix of smoothing operators in $\widetilde{\mathcal{R}}^{-\varrho}_p \otimes \mathcal{M}_2(\mathbb{C})$ for any $\varrho \ge 0$.

Proof. We first note that the matrix entries $L(U)^\top$ defined as in (4.2.43) are

$$
\begin{aligned}
[L^\top]_{\vec{\jmath}_p,j,k}^{\vec{\sigma}_p,\sigma',\sigma} &\overset{(4.2.47)}{=} L_{\vec{\jmath}_p,-k,-j}^{\vec{\sigma}_p,\sigma,\sigma'} \\
&\overset{(4.2.43)}{=} \int_{\mathbb{T}} L\Big(\mathtt{q}^{\sigma_1}\frac{e^{\mathrm{i}\sigma_1 j_1 x}}{\sqrt{2\pi}},\dots,\mathtt{q}^{\sigma_p}\frac{e^{\mathrm{i}\sigma_p j_p x}}{\sqrt{2\pi}}\Big)\Big[\frac{\mathtt{q}^{\sigma}e^{-\mathrm{i}\sigma k x}}{\sqrt{2\pi}}\Big]\cdot \mathtt{q}^{\sigma'}\frac{e^{\mathrm{i}\sigma' j x}}{\sqrt{2\pi}}\,\mathrm{d}x \\
&\overset{(4.2.102)}{=} \int_{\mathbb{T}} S\Big(\mathtt{q}^{\sigma}\frac{e^{-\mathrm{i}\sigma k x}}{\sqrt{2\pi}},\mathtt{q}^{\sigma_1}\frac{e^{\mathrm{i}\sigma_1 j_1 x}}{\sqrt{2\pi}}\dots,\mathtt{q}^{\sigma_{p-1}}\frac{e^{\mathrm{i}\sigma_{p-1} j_{p-1} x}}{\sqrt{2\pi}}\Big)\Big[\frac{\mathtt{q}^{\sigma_p}e^{\mathrm{i}\sigma_p j_p x}}{\sqrt{2\pi}}\Big]\cdot \mathtt{q}^{\sigma'}\frac{e^{\mathrm{i}\sigma' j x}}{\sqrt{2\pi}}\,\mathrm{d}x \\
&\overset{(4.2.43)}{=} S_{-k,j_1,\dots,j_{p-1},j_p,-j}^{\sigma,\sigma_1,\dots,\sigma_{p-1},\sigma_p,\sigma'}
\end{aligned} \tag{4.2.103}
$$

which, by (4.2.76), are different from zero only if

$$
\max\{|k|,|j_1|,\dots,|j_{p-1}|\}\le \delta|j_p|\,,\quad C^{-1}|j|\le|j_p|\le C|j|\,. \tag{4.2.104}
$$

The restriction (4.2.104) implies that

$$
\max(|j_1|,\dots,|j_p|,|j|)\sim \max{}_2(|j_1|,\dots,|j_p|,|j|) \tag{4.2.105}
$$

because

$$
\begin{aligned}
\max(|j_1|,\dots,|j_p|,|j|) &\le (1+\delta)\max(|j_p|,|j|) \\
&\le C(1+\delta)\max{}_2(|j_p|,|j|)\le C(1+\delta)\max{}_2(|j_1|,\dots,|j_p|,|j|)\,.
\end{aligned}
$$

Finally by (4.2.103), we estimate

$$
\begin{aligned}
\big|[L^\top]_{\vec{\jmath}_p,j,k}^{\vec{\sigma}_p,\sigma',\sigma}\big| = \big|S_{-k,j_1,\dots,j_{p-1},j_p,-j}^{\sigma,\sigma_1,\dots,\sigma_{p-1},\sigma_p,\sigma'}\big| &\overset{(4.2.75)}{\le} C\max\{|k|,|j_1|,\dots,|j_{p-1}|\}^{\mu}|j_p|^m \\
&\overset{(4.2.104)}{\le} C'\max\{|j_1|,\dots,|j_p|,|j|\}^{\mu+\max(m,0)} \\
&\overset{(4.2.105)}{\le} C''\frac{\max_2\{|j_1|,\dots,|j_p|,|j|\}^{\mu+\max(m,0)+\varrho}}{\max\{|j_1|,\dots,|j_p|,|j|\}^{\varrho}}
\end{aligned}
$$

implying, in view of Lemma 4.2.9 (with $m\rightsquigarrow-\varrho$ and $\mu\rightsquigarrow\mu+\max(m,0)+\varrho$), that $L(U)^\top$ is a matrix of smoothing operators in $\widetilde{\mathcal{R}}^{-\varrho}_p\otimes\mathcal{M}_2(\mathbb{C})$ for any $\varrho\ge 0$. □

We conclude this section with a lemma which shall be used in Section 4.3.4.

Lemma 4.2.22. *Let $p\in\mathbb{N}$ and $\varrho\ge 0$. Let $S(U)$ be a matrix of p-homogeneous spectrally localized maps in $\widetilde{\mathcal{S}}_p\otimes\mathcal{M}_2(\mathbb{C})$ of the form*

$$
S(U)=L(U)+R(U)\,, \tag{4.2.106}
$$

where $L(U)^\top$ and $R(U)$ are matrices of p-homogeneous smoothing operators in $\widetilde{\mathcal{R}}^{-\varrho}_p\otimes\mathcal{M}_2(\mathbb{C})$. Then $S(U)$ is a matrix of p–homogeneous smoothing operators in $\widetilde{\mathcal{R}}^{-\varrho}_p\otimes\mathcal{M}_2(\mathbb{C})$.

Proof. In view of (4.2.47) and (4.2.48) (for $L(U)^\top\in\widetilde{\mathcal{R}}^{-\varrho}_p\otimes\mathcal{M}_2(\mathbb{C})$),

$$
|L_{\vec{\jmath}_p,j,k}^{\vec{\sigma}_p,\sigma',\sigma}| = |(L^\top)_{\vec{\jmath}_p,-k,-j}^{\vec{\sigma}_p,\sigma,\sigma'}|\le C\max{}_2\{|j_1|,\dots,|j_p|,|k|\}^{\mu}\max\{|j_1|,\dots,|j_p|,|k|\}^{-\varrho}\,. \tag{4.2.107}
$$

Since $S(U)$ is spectrally localized, its non zero Fourier coefficients $S^{\vec{\sigma}_p,\sigma',\sigma}_{\vec{j}_p,j,k}$ (defined as in (4.2.43)) satisfy $|j| \sim |k|$ (see (4.2.76)) and, in view of (4.2.106),

$$|S^{\vec{\sigma}_p,\sigma',\sigma}_{\vec{j}_p,j,k}| \leq |L^{\vec{\sigma}_p,\sigma',\sigma}_{\vec{j}_p,j,k}| + |R^{\vec{\sigma}_p,\sigma',\sigma}_{\vec{j}_p,j,k}| \overset{(4.2.107),(4.2.48)}{\lesssim} \frac{\max_2\{|j_1|,\dots,|j_p|,|k|\}^\mu}{\max\{|j_1|,\dots,|j_p|,|k|\}^\varrho} + \frac{\max_2\{|j_1|,\dots,|j_p|,|j|\}^\mu}{\max\{|j_1|,\dots,|j_p|,|j|\}^\varrho}$$
$$\overset{|j|\sim|k|}{\lesssim} \frac{\max_2\{|j_1|,\dots,|j_p|,|j|\}^\mu}{\max\{|j_1|,\dots,|j_p|,|j|\}^\varrho}$$

proving that $S(U)$ is a smoothing operator in $\widetilde{\mathcal{R}}^{-\varrho}_p \otimes \mathcal{M}_2(\mathbb{C})$. □

4.2.3 Approximate inverse of non–linear maps and flows

In this section we construct an approximate version of two fundamental non–linear operations that we will need: the inverse of a non–linear map and the flow generated by a non–linear vector field. We first provide the definition of an approximate inverse of a map, up to homogeneity N.

Definition 4.2.23. (Approximate inverse up to homogeneity N) *Let $p, N \in \mathbb{N}$ with $p \leq N$. Consider*

$$\Psi_{\leq N}(U) = U + M_{\leq N}(U)U \quad \textit{where} \quad M_{\leq N}(U) \in \Sigma_p^N \widetilde{\mathcal{M}}_q \otimes \mathcal{M}_2(\mathbb{C}) \tag{4.2.108}$$

is a matrix of pluri–homogeneous operators. We say that

$$\Phi_{\leq N}(V) = V + \breve{M}_{\leq N}(V)V \quad \textit{where} \quad \breve{M}_{\leq N}(V) \in \Sigma_p^N \widetilde{\mathcal{M}}_q \otimes \mathcal{M}_2(\mathbb{C}), \tag{4.2.109}$$

is an approximate inverse of $\Psi_{\leq N}(U)$ up to homogeneity N if

$$\begin{aligned}\big(\mathrm{Id} + M_{\leq N}(\Phi_{\leq N}(V))\big)\big(\mathrm{Id} + \breve{M}_{\leq N}(V)\big) &= \mathrm{Id} + M'_{>N}(V)\\ \big(\mathrm{Id} + \breve{M}_{\leq N}(\Psi_{\leq N}(U))\big)\big(\mathrm{Id} + M_{\leq N}(U)\big) &= \mathrm{Id} + M''_{>N}(U)\end{aligned} \tag{4.2.110}$$

where $M'_{>N}(V)$ and $M''_{>N}(U)$ are pluri–homogeneous matrices of operators in $\Sigma_{N+1}\widetilde{\mathcal{M}}_q \otimes \mathcal{M}_2(\mathbb{C})$.

Note that, if $\Phi_{\leq N}(V)$ is an approximate inverse up to homogeneity N of $\Psi_{\leq N}(U)$ then

$$\Psi_{\leq N} \circ \Phi_{\leq N}(V) = V + M'_{>N}(V)V\,, \quad \Phi_{\leq N} \circ \Psi_{\leq N}(U) = U + M''_{>N}(U)U\,, \tag{4.2.111}$$

and, by differentiation and taking the transpose

$$\begin{aligned}&\mathrm{d}_U\Psi_{\leq N}(\Phi_{\leq N}(V))\mathrm{d}_V\Phi_{\leq N}(V) = \mathrm{Id} + M^1_{>N}(V)\\ &\mathrm{d}_V\Phi_{\leq N}(\Psi_{\leq N}(U))\mathrm{d}_U\Psi_{\leq N}(U) = \mathrm{Id} + M^2_{>N}(U)\\ &\mathrm{d}_V\Phi_{\leq N}(V)^\top \mathrm{d}_U\Psi_{\leq N}(\Phi_{\leq N}(V))^\top = \mathrm{Id} + M^3_{>N}(V)\\ &\mathrm{d}_U\Psi_{\leq N}(\Phi_{\leq N}(V))^\top \mathrm{d}_V\Phi_{\leq N}(V)^\top = \mathrm{Id} + M^4_{>N}(V)\end{aligned} \tag{4.2.112}$$

where $M^a_{>N}(V)$, $a = 1,\dots,4$ are other pluri-homogeneous operators in $\Sigma_{N+1}\widetilde{\mathcal{M}}_q \otimes \mathcal{M}_2(\mathbb{C})$ (the differential of a homogeneous m-operator is a homogeneous m-operator by the first remark after Definition 4.2.5, so is its transpose by Lemma 4.2.10).

The following lemma ensures the existence of an approximate inverse.

Lemma 4.2.24. (Approximate inverse) *Let $p, N \in \mathbb{N}$ with $p \le N$. Consider $\Psi_{\le N}(U) = U + M_{\le N}(U)U$ as in (4.2.108). Then there exists an approximate inverse of $\Psi_{\le N}(U)$ up to homogeneity N (according to Definition 4.2.23) of the form (4.2.109) with*

$$\mathcal{P}_p[\breve{M}_{\le N}(V)] = -\mathcal{P}_p[M_{\le N}(V)]\,. \tag{4.2.113}$$

Moreover, if $M_{\le N}(U)$ in (4.2.108) is a matrix of pluri–homogeneous

(i) spectrally localized maps in $\Sigma_p^N \widetilde{\mathcal{S}}_q^m \otimes \mathcal{M}_2(\mathbb{C})$, $m \ge 0$, then $\breve{M}_{\le N}$ in (4.2.109) is a matrix of pluri-homogeneous spectrally localized maps in $\Sigma_p^N \widetilde{\mathcal{S}}_q^{m(N-p+1)} \otimes \mathcal{M}_2(\mathbb{C})$ and $M'_{>N}$, $M''_{>N}$ in (4.2.110) belong to $\Sigma_{N+1} \widetilde{\mathcal{S}}_q^{m(N-p+2)} \otimes \mathcal{M}_2(\mathbb{C})$;

(ii) smoothing operators in $\Sigma_p^N \widetilde{\mathcal{R}}_q^{-\varrho} \otimes \mathcal{M}_2(\mathbb{C})$ for some $\varrho \ge 0$, then $\breve{M}_{\le N}$ in (4.2.109) is a matrix of pluri-homogeneous smoothing operators in $\Sigma_p^N \widetilde{\mathcal{R}}_q^{-\varrho} \otimes \mathcal{M}_2(\mathbb{C})$ and $M'_{>N}$, $M''_{>N}$ in (4.2.110) belong to $\Sigma_{N+1} \widetilde{\mathcal{R}}_q^{-\varrho} \otimes \mathcal{M}_2(\mathbb{C})$.

Proof. We expand in homogeneous components

$$M(U) := \mathrm{Id} + M_{\le N}(U) = \mathrm{Id} + \sum_{q=p}^{N} M_q(U) \quad \text{with} \quad M_q(U) \in \widetilde{\mathcal{M}}_q \otimes \mathcal{M}_2(\mathbb{C})\,. \tag{4.2.114}$$

In order to solve the first equation in (4.2.110) we look for a pluri–homogeneous operator

$$\breve{M}(V) = \mathrm{Id} + \breve{M}_{\le N}(V) = \breve{M}_0(V) + \sum_{a=p}^{N} \breve{M}_a(V) \quad \text{with} \quad \breve{M}_0(V) := \mathrm{Id}\,,\ \breve{M}_a(V) \in \widetilde{\mathcal{M}}_q \otimes \mathcal{M}_2(\mathbb{C})\,, \tag{4.2.115}$$

such that

$$M\big(\breve{M}(V)V\big)\breve{M}(V) = \mathrm{Id} + M_{>N}(V) \quad \text{with} \quad M_{>N}(V) \in \Sigma_{N+1}\widetilde{\mathcal{M}}_q \otimes \mathcal{M}_2(\mathbb{C})\,. \tag{4.2.116}$$

By (4.2.114), (4.2.115) we get

$$\begin{aligned} &M\big(\breve{M}(V)V\big)\breve{M}(V) \\ &= \mathrm{Id} + \sum_{\ell=p}^{N} \breve{M}_\ell(V) + \sum_{q=p}^{N} \sum_{0 \le a_1, \dots a_{q+1} \le N} M_q(\breve{M}_{a_1}(V)V, \dots, \breve{M}_{a_q}(V)V)\breve{M}_{a_{q+1}}(V) \\ &= \mathrm{Id} + \sum_{\ell=p}^{N} \Big[\breve{M}_\ell(V) + \sum_{\substack{p \le q \le \ell \\ q + a_1 + \dots + a_{q+1} = \ell}} M_q(\breve{M}_{a_1}(V)V, \dots, \breve{M}_{a_q}(V)V)\breve{M}_{a_{q+1}}(V)\Big] + M_{>N}(V) \end{aligned}$$

and therefore equation (4.2.116) is recursively solved by defining, for any $\ell = p, \dots, N$,

$$\breve{M}_\ell(V, \dots, V) := -\sum_{q + a_1 + \dots + a_{q+1} = \ell} M_q(\breve{M}_{a_1}(V)V, \dots, \breve{M}_{a_q}(V)V)\breve{M}_{a_{q+1}}(V)\,. \tag{4.2.117}$$

Note that each $\breve{M}_\ell(V, \dots, V)$ is a matrix of homogeneous operators by (ii) of Proposition 4.2.15. We proved the first identity in (4.2.110).

We now prove the second identity in (4.2.110). Using the same recursive procedure we find a matrix of pluri–homogeneous operators of the form

$$M''(U) = \mathrm{Id} + \sum_{n=p}^{N} M''_n(U) \quad \text{with} \quad M''_n(U) \in \widetilde{\mathcal{M}}_n \otimes \mathcal{M}_2(\mathbb{C}) \tag{4.2.118}$$

such that $M''(U)$ is an approximate right inverse of $\breve{M}(V)$, i.e.

$$\breve{M}\big(M''(U)U\big)M''(U) = \mathrm{Id} + M_{>N}(U) \quad \text{with} \quad M_{>N}(U) \in \Sigma_{N+1}\widetilde{\mathcal{M}}_q \otimes \mathcal{M}_2(\mathbb{C})\,. \tag{4.2.119}$$

Applying (4.2.116) with $V = M''(U)U$ and right–composing it with $M''(U)$ defined in (4.2.118), we obtain by (4.2.119) and Proposition 4.2.15 that

$$M\big(U + M_{>N}(U)U\big) = M''(U) + M'_{>N}(U) \tag{4.2.120}$$

where $M_{>N}(U)$ and $M'_{>N}(U)$ are operators in $\Sigma_{N+1}\widetilde{\mathcal{M}}_q \otimes \mathcal{M}_2(\mathbb{C})$. Then, expanding the left hand side of (4.2.120) by multilinearity, we get

$$M\big(U\big) - M''(U) = \sum_{q=p}^{N}\sum_{\ell=1}^{q}\binom{q}{\ell} M_q(\underbrace{M_{>N}(U)U,\ldots,M_{>N}(U)U}_{\ell-times},U,\ldots,U) + M'_{>N}(U) = \tilde{M}_{>N}(U)$$

where $\tilde{M}_{>N}(U)$ is in $\Sigma_{N+1}\widetilde{\mathcal{M}}_q \otimes \mathcal{M}_2(\mathbb{C})$ thanks to (ii) of Proposition 4.2.15. This implies, since both $M(U)$ and $M''(U)$ are pluri–homogeneous operators up to homogeneity N (cfr. (4.2.114), (4.2.118)), that $M\big(U\big) = M''(U)$ and, by (4.2.119), we conclude that

$$\breve{M}\big(M(U)U\big)M(U) = \mathrm{Id} + M_{>N}(U)\,. \tag{4.2.121}$$

This proves, recalling the notation (4.2.114), (4.2.115), (4.2.108), the second identity in (4.2.110). Moreover for $\ell = p$ the sum in (4.2.117) reduces to the unique element with $q = p$, $a_1 = \cdots = a_{q+1} = 0$ and $\breve{M}_p(V) = -M_p(V)$, proving (4.2.113).

(i) If $M_{\leq N}(U)$ is a spectrally localized map in $\Sigma_p^N \widetilde{\mathcal{S}}_q^m \otimes \mathcal{M}_2(\mathbb{C})$ we claim that $\breve{M}_\ell(V)$ is a spectrally localized map in $\widetilde{\mathcal{S}}_\ell^{m+m(\ell-p)} \otimes \mathcal{M}_2(\mathbb{C})$ for $\ell = p,\ldots,N$. For $\ell = p$ by (4.2.113) we have $\breve{M}_p(V) = -M_p(V)$ which is in $\widetilde{\mathcal{S}}_\ell^m \otimes \mathcal{M}_2(\mathbb{C})$. Then supposing inductively that $\breve{M}_a(V)$ is in $\widetilde{\mathcal{S}}_a^{m+m(a-p)} \otimes \mathcal{M}_2(\mathbb{C})$ for $a = p,\ldots \ell-1$, we deduce by (ii) and (iii) of Proposition 4.2.19, that each term in the sum in (4.2.117) is a spectrally localized map in $\widetilde{\mathcal{S}}_\ell^{m+(m+m(a_{q+1}-p))} \otimes \mathcal{M}_2(\mathbb{C})$ which is included in $\widetilde{\mathcal{S}}_\ell^{m+m(\ell-p)} \otimes \mathcal{M}_2(\mathbb{C})$ using that $a_{q+1} \leq \ell - 1$.

(ii) In the same way, if $M_{\leq N}(U)$ is a smoothing operator in $\Sigma_p^N \widetilde{\mathcal{R}}_q^{-\varrho} \otimes \mathcal{M}_2(\mathbb{C})$, thanks to (ii) of Proposition 4.2.15 one proves recursively that $\breve{M}_\ell(V)$ are smoothing operators in $\widetilde{\mathcal{R}}_\ell^{-\varrho} \otimes \mathcal{M}_2(\mathbb{C})$. □

Vector fields: We introduce the following definition of vector fields.

Definition 4.2.25 (Homogeneous vector fields). *Let $m \in \mathbb{R}$ and $p, N \in \mathbb{N}_0$. We denote by $\widetilde{\mathfrak{X}}_{p+1}^m$ the space of $(p+1)$-homogeneous vector fields of the form $X(U) = M(U)U$ where $M(U)$ is a matrix of p-homogeneous m-operators in $\widetilde{\mathcal{M}}_p^m \otimes \mathcal{M}_2(\mathbb{C})$. We denote $\widetilde{\mathfrak{X}}_{p+1} := \cup_{m\geq 0}\widetilde{\mathfrak{X}}_{p+1}^m$ and $\Sigma_{p+1}^{N+1}\widetilde{\mathfrak{X}}_q^m$ the class of pluri-homogeneous vector fields. We also set $\Sigma_{p+1}\widetilde{\mathfrak{X}}_q := \bigcup_{N\in\mathbb{N}} \Sigma_{p+1}^{N+1}\widetilde{\mathfrak{X}}_q$. The vector fields in $\widetilde{\mathfrak{X}}_{p+1}^{-\varrho}$, $\varrho \geq 0$, are called smoothing.*

Note that $X(U) = M(U)U$ is real-to-real in the sense of (4.2.14) if and only if the operator $M(U)$ is real-to-real in the sense of (4.2.12).

• **Fourier representation of $(p+1)$-homogeneous vector fields:** A $(p+1)$-homogeneous vector field can be expressed in Fourier as: for any $\sigma = \pm$,

$$X(U)^\sigma = \sum_{k\in\mathbb{Z}\setminus\{0\}} X(U)_k^\sigma \frac{e^{\mathrm{i}\sigma kx}}{\sqrt{2\pi}}, \qquad X(U)_k^\sigma = \sum_{(\vec{\jmath}_{p+1},k,\vec{\sigma}_{p+1},-\sigma)\in\mathfrak{T}_{p+2}} X_{\vec{\jmath}_{p+1},k}^{\vec{\sigma}_{p+1},\sigma} u_{\vec{\jmath}_{p+1}}^{\vec{\sigma}_{p+1}}, \tag{4.2.122}$$

the last sum being in $(\vec{\jmath}_{p+1}, \vec{\sigma}_{p+1})$, with coefficients $X_{\vec{\jmath}_{p+1},k}^{\vec{\sigma}_{p+1},\sigma} \in \mathbb{C}$ given by

$$X_{j_1,\dots,j_p,j_{p+1},k}^{\sigma_1,\dots,\sigma_p,\sigma_{p+1},\sigma} = \frac{1}{p+1}\Big(M_{j_1,\dots,j_p,j_{p+1},k}^{\sigma_1,\dots,\sigma_p,\sigma_{p+1},\sigma} + M_{j_{p+1},\dots,j_p,j_1,k}^{\sigma_{p+1},\dots,\sigma_p,\sigma_1,\sigma} + \cdots + M_{j_1,\dots,j_{p+1},j_p,k}^{\sigma_1,\dots,\sigma_{p+1},\sigma_p,\sigma}\Big), \tag{4.2.123}$$

namely they are obtained symmetrizing with respect to the first $p+1$ indices the coefficients $M_{\vec{\jmath}_p,j,k}^{\vec{\sigma}_p,\sigma',\sigma}$ of $M(U)$.

In particular they satisfy the symmetry condition: for any permutation π of $\{1,\dots,p+1\}$,

$$X_{j_{\pi(1)},\dots,j_{\pi(p+1)},k}^{\sigma_{\pi(1)},\dots,\sigma_{\pi(p+1)},\sigma} = X_{j_1,\dots,j_{p+1},k}^{\sigma_1,\dots,\sigma_{p+1},\sigma}. \tag{4.2.124}$$

In addition, if $X(U)$ is real-to-real, see (4.2.14), then one has

$$\overline{X(U)_k^+} = X(U)_k^- \quad \text{i.e.} \quad \overline{X_{\vec{\jmath}_{p+1},k}^{\vec{\sigma}_{p+1},+}} = X_{\vec{\jmath}_{p+1},k}^{-\vec{\sigma}_{p+1},-}. \tag{4.2.125}$$

By Lemma 4.2.9 we obtain the following characterization of vector fields.

Lemma 4.2.26. (Characterization of vector fields in Fourier basis) *Let $m \in \mathbb{R}$. A real-to-real vector field $X(U)$ belongs to $\widetilde{\mathfrak{X}}_{p+1}^m$ if and only if its coefficients $X_{\vec{\jmath}_{p+1},k}^{\vec{\sigma}_{p+1},\sigma}$ (defined as in (4.2.122)) fulfill the symmetric and real-to-real conditions (4.2.124), (4.2.125) and: there exist $\mu \geq 0$ and $C > 0$ such that*

$$|X_{\vec{\jmath}_{p+1},k}^{\vec{\sigma}_{p+1},\sigma}| \leq C \max{}_2\{|j_1|,\dots,|j_{p+1}|\}^\mu \max\{|j_1|,\dots,|j_{p+1}|\}^m \tag{4.2.126}$$

for any $(\vec{\jmath}_{p+1}, k, \vec{\sigma}_{p+1}, -\sigma) \in \mathfrak{T}_{p+2}$ (cfr. (4.2.10)).

We now define the approximate flow of a smoothing τ–dependent vector field.

Definition 4.2.27. (Approximate flow of a smoothing vector field up to homogeneity N) *Let $p, N \in \mathbb{N}$ with $p \leq N$. An* approximate flow up to homogeneity N *of a τ–dependent pluri-homogeneous smoothing vector field $X^\tau(Z)$ in $\Sigma_{p+1}\widetilde{\mathfrak{X}}_q^{-\varrho}$, defined for $\tau \in [0,1]$ and some $\varrho \geq 0$, is a non–linear map,*

$$\mathcal{F}_{\leq N}^\tau(Z) = Z + F_{\leq N}^\tau(Z)Z, \qquad \tau \in [0,1], \tag{4.2.127}$$

where $F_{\leq N}^\tau(Z)$ is a matrix of pluri–homogeneous, τ–dependent, smoothing operators in $\Sigma_p^N \widetilde{\mathcal{R}}_q^{-\varrho} \otimes \mathcal{M}_2(\mathbb{C})$, with estimates uniform in $\tau \in [0,1]$, solving

$$\partial_\tau \mathcal{F}_{\leq N}^\tau(Z) = X^\tau(\mathcal{F}_{\leq N}^\tau(Z)) + R_{>N}^\tau(Z)Z, \quad \mathcal{F}_{\leq N}^0(Z) = Z, \tag{4.2.128}$$

where $R_{>N}^\tau(Z)$ is a matrix of τ–dependent, smoothing operators in $\Sigma_{N+1}\widetilde{\mathcal{R}}_q^{-\varrho} \otimes \mathcal{M}_2(\mathbb{C})$, with estimates uniform in $\tau \in [0,1]$.

The following lemma ensures the existence of an approximate flow.

Lemma 4.2.28. **(Approximate flow)** *Let $p, N \in \mathbb{N}$ with $p \leq N$. Consider a pluri–homogeneous τ–dependent smoothing vector field $X^\tau(Z)$ in $\Sigma_{p+1}\widetilde{\mathfrak{X}}_q^{-\varrho}$, defined for $\tau \in [0,1]$ and some $\varrho \geq 0$. Then*

- *there exists an approximate flow $\mathcal{F}^\tau_{\leq N}$ according to Definition 4.2.27;*
- *denoting by $G_p^\tau(Z)Z$ with $G_p^\tau(Z) \in \widetilde{\mathcal{R}}_p^{-\varrho} \otimes \mathcal{M}_2(\mathbb{C})$ the (p+1)-homogeneous component of $X^\tau(Z)$, then the p-homogeneous component of the smoothing operator $F^\tau_{\leq N}(Z)$ in (4.2.127) is*

$$\mathcal{P}_p(F^\tau_{\leq N}(Z)) = \int_0^\tau G_p^{\tau'}(Z)\mathrm{d}\tau'. \tag{4.2.129}$$

Proof. We write $X^\tau(Z) := G^\tau(Z)Z$ with $G^\tau(Z) = \sum_{q=p}^M G_q^\tau(Z)$ and $G_q^\tau(Z) \in \widetilde{\mathcal{R}}_q^{-\varrho} \otimes \mathcal{M}_2(\mathbb{C})$, and we look for an approximate flow solution of (4.2.128) of the form

$$\mathcal{F}^\tau_{\leq N}(Z) = Z + \sum_{\ell=p}^N F_\ell^\tau(Z,\ldots,Z)Z \qquad \text{with} \qquad F_\ell^\tau(Z) \in \widetilde{\mathcal{R}}_\ell^{-\varrho} \otimes \mathcal{M}_2(\mathbb{C}).$$

Since $G^\tau(Z) = \sum_{q=p}^M G_q^\tau(Z)$ then, using the notation $F_0^\tau(Z) := \mathrm{Id}$, we expand by multilinearity

$$X^\tau(\mathcal{F}^\tau_{\leq N}(Z)) = \sum_{q=p}^M G_q^\tau(\mathcal{F}^\tau_{\leq N}(Z),\ldots,\mathcal{F}^\tau_{\leq N}(Z))\mathcal{F}^\tau_{\leq N}(Z) = \sum_{\mathtt{a}=p}^{N+(N+1)M} \breve{X}_{\mathtt{a}}^\tau(Z)Z$$

where

$$\breve{X}_{\mathtt{a}}^\tau(Z) := \sum_{\substack{q=p,\ldots,M \\ \ell_1,\ldots,\ell_{q+1}\in\{0,p,\ldots,N\} \\ \ell_1+\cdots+\ell_{q+1}+q=\mathtt{a}}} G_q^\tau\big(F_{\ell_1}^\tau(Z)Z,\ldots,F_{\ell_q}^\tau(Z)Z\big)F_{\ell_{q+1}}^\tau(Z). \tag{4.2.130}$$

Then we solve (4.2.128) defining recursively for $\mathtt{a} = p,\ldots,N$,

$$F_{\mathtt{a}}^\tau(Z) := \int_0^\tau \breve{X}_{\mathtt{a}}^{\tau'}(Z)\,\mathrm{d}\tau', \qquad R^\tau_{>N}(Z) := \sum_{\mathtt{a}=N+1}^{N+(N+1)M} X_{\mathtt{a}}^\tau(Z).$$

Using recursively formula (4.2.130) and Proposition 4.2.15 one verifies that each $\breve{X}_{\mathtt{a}}^\tau(Z)$ is a $\mathtt{a}$-homogeneous smoothing operator in $\widetilde{\mathcal{R}}_{\mathtt{a}}^{-\varrho} \otimes \mathcal{M}_2(\mathbb{C})$, so is $F_{\mathtt{a}}^\tau(Z)$, and $R^\tau_{>N}(Z)$ is a pluri-homogeneous smoothing operator in $\Sigma_{N+1}\widetilde{\mathcal{R}}_{\mathtt{a}}^{-\varrho} \otimes \mathcal{M}_2(\mathbb{C})$. Note that for $\mathtt{a} = p$ the sum in (4.2.130) reduces to the indices $q = p$, $\ell_1 = \cdots = \ell_{q+1} = 0$. As a consequence $F_p^\tau(Z) = \int_0^\tau G_p^{\tau'}(Z)\,\mathrm{d}\tau'$ proving (4.2.129). ☐

4.2.4 Pluri-homogeneous differential geometry

In this section we introduce pluri-homogeneous k-forms. We revisit the classical identities of differential geometry ($\mathrm{d}^2 = 0$, Cartan's magic formula) for $k = 0, 1, 2$ which are the only cases needed our purpose.

Definition 4.2.29. **(r-homogeneous k-form)** *Let $p \in \mathbb{N}_0$, $k = 0, 1, 2$ and set $r := p + 2 - k$. A r-homogeneous k-form is a $(r + k)$-linear map from $(\dot{H}^\infty(\mathbb{T};\mathbb{C}^2))^r \times (\dot{H}^\infty(\mathbb{T};\mathbb{C}^2))^k$ to $\mathbb{C}$ of the form $\Lambda(U_1,\ldots,U_r)[V_1,\ldots,V_k]$, symmetric in the variables $\mathcal{U} := (U_1,\ldots,U_r)$ and antisymmetric in the entries $\mathcal{V} := (V_1,\ldots,V_k)$, satisfying the following: there are constants $C > 0$ and $m \geq 0$ such that*

$$\begin{aligned} &|\Lambda(\Pi_{n_1}U_1,\ldots,\Pi_{n_r}U_r)[\Pi_{n_{r+1}}V_1,\ldots,\Pi_{n_{r+k}}V_k]| \\ &\qquad \leq C\max\{n_1,\ldots,n_{r+k}\}^m \prod_{j=1}^r \|\Pi_{n_j}U_j\|_{L^2} \prod_{\ell=1}^k \|\Pi_{n_{r+\ell}}V_\ell\|_{L^2} \end{aligned} \tag{4.2.131}$$

for any $\mathcal{U} \in (\dot{H}^{\infty}(\mathbb{T};\mathbb{C}^2))^r$*, any* $\mathcal{V} \in (\dot{H}^{\infty}(\mathbb{T};\mathbb{C}^2))^k$*, and any* $(n_1,\ldots,n_{r+k})$ *in* $\mathbb{N}^{r+k}$*. Moreover, if*

$$\Lambda(\Pi_{n_1}U_1,\ldots,\Pi_{n_r}U_r)[\Pi_{n_{r+1}}V_1,\ldots,\Pi_{n_{r+k}}V_k] \neq 0\,,$$

then there is a choice of signs $\sigma_1,\ldots,\sigma_{r+k} \in \{\pm 1\}$ *such that* $\sum_{j=1}^{r+k}\sigma_j n_j = 0$*. In addition we require the translation invariant property:*

$$\Lambda(\tau_\varsigma\mathcal{U})[\tau_\varsigma\mathcal{V}] = \Lambda(\mathcal{U})[\mathcal{V}]\,, \quad \forall\varsigma \in \mathbb{R}\,. \tag{4.2.132}$$

We also require that $\Lambda(\mathcal{U})[\mathcal{V}]$ *is real valued for any* $\mathcal{U} \in (L^2_{\mathbb{R}}(\mathbb{T},\mathbb{C}^2))^r$ *and* $\mathcal{V} \in (L^2_{\mathbb{R}}(\mathbb{T},\mathbb{C}^2))^k$*, cfr.* (4.2.13)*. We denote by* $\widetilde{\Lambda}^k_r$ *the space of* r*-homogeneous* k*-forms and by* $\Sigma^N_r\widetilde{\Lambda}^k_q$ *the space of pluri–homogeneous* k*-forms. We set* $\Sigma_r\widetilde{\Lambda}^k_q := \cup_{N\geq r}\Sigma^N_r\widetilde{\Lambda}^k_q$*.*

• A r-homogeneous 0-form is also called a homogeneous *Hamiltonian*.

• **Fourier representation of** $(p+2)$**-homogeneous** 0**-forms:** Let $p \in \mathbb{N}_0$. A $(p+2)$-homogeneous 0-form can be expressed in Fourier as (recall (4.2.10))

$$H(U) = \frac{1}{p+2}\sum_{(\vec{\jmath}_{p+2},\vec{\sigma}_{p+2})\in\mathfrak{T}_{p+2}} H^{\vec{\sigma}_{p+2}}_{\vec{\jmath}_{p+2}} u^{\vec{\sigma}_{p+2}}_{\vec{\jmath}_{p+2}}\,. \tag{4.2.133}$$

The reality condition $H(U) \in \mathbb{R}$ for any $U \in L^2_{\mathbb{R}}(\mathbb{T},\mathbb{C}^2)$ amounts to

$$\overline{H^{\vec{\sigma}_{p+2}}_{\vec{\jmath}_{p+2}}} = H^{-\vec{\sigma}_{p+2}}_{\vec{\jmath}_{p+2}}\,. \tag{4.2.134}$$

Moreover the scalar coefficients $H^{\vec{\sigma}_{p+2}}_{\vec{\jmath}_{p+2}} \equiv H^{\sigma_1,\ldots,\sigma_{p+2}}_{j_1,\ldots,j_{p+2}} \in \mathbb{C}$ satisfy the symmetric condition: for any permutation π of $\{1,\ldots,p+2\}$

$$H^{\sigma_{\pi(1)},\ldots,\sigma_{\pi(p+2)}}_{j_{\pi(1)},\ldots,j_{\pi(p+2)}} = H^{\sigma_1,\ldots,\sigma_{p+2}}_{j_1,\ldots,j_{p+2}} \tag{4.2.135}$$

and, for some $m \geq 0$, the bound

$$|H^{\vec{\sigma}_{p+2}}_{\vec{\jmath}_{p+2}}| \lesssim \max(|j_1|,\ldots,|j_{p+2}|)^m\,. \tag{4.2.136}$$

• **Fourier representation of** $(p+1)$**-homogeneous** 1**-forms:** A $(p+1)$-homogeneous 1-form can be expressed in Fourier as

$$\theta(U)[V] = \sum_{(\vec{\jmath}_{p+1},k,\vec{\sigma}_{p+1},\sigma)\in\mathfrak{T}_{p+2}} \Theta^{\vec{\sigma}_{p+1},\sigma}_{\vec{\jmath}_{p+1},k} u^{\vec{\sigma}_{p+1}}_{\vec{\jmath}_{p+1}} v^{\sigma}_k\,. \tag{4.2.137}$$

The reality condition $\theta(U)[V] \in \mathbb{R}$ for any $U,V \in L^2_{\mathbb{R}}(\mathbb{T},\mathbb{C}^2)$ amounts to

$$\overline{\Theta^{\vec{\sigma}_{p+1},\sigma}_{\vec{\jmath}_{p+1},k}} = \Theta^{-\vec{\sigma}_{p+1},-\sigma}_{\vec{\jmath}_{p+1},k}\,. \tag{4.2.138}$$

Moreover the coefficients satisfy, for some $m \geq 0$,

$$|\Theta^{\vec{\sigma}_{p+1},\sigma}_{\vec{\jmath}_{p+1},k}| \lesssim \max(|j_1|,\ldots,|j_{p+1}|)^m\,. \tag{4.2.139}$$

• **Fourier representation of** p**-homogeneous** 2**-forms:** A p-homogeneous 2-form can be expressed in Fourier as

$$\Lambda(U)[V_1,V_2] = \sum_{(\vec{\jmath}_p,j,k,\vec{\sigma}_p,\sigma',\sigma)\in\mathfrak{T}_{p+2}} \Lambda^{\vec{\sigma}_p,\sigma',\sigma}_{\vec{\jmath}_p,j,k} u^{\vec{\sigma}_p}_{\vec{\jmath}_p}(v_1)^{\sigma'}_j(v_2)^{\sigma}_k\,. \tag{4.2.140}$$

The antisymmetry condition $\Lambda(U)[V_1,V_2] = -\Lambda(U)[V_2,V_1]$ amounts to

$$\Lambda^{\vec{\sigma}_p,\sigma',\sigma}_{\vec{\jmath}_p,j,k} = -\Lambda^{\vec{\sigma}_p,\sigma,\sigma'}_{\vec{\jmath}_p,k,j} . \tag{4.2.141}$$

The reality condition $\Lambda(U)[V_1,V_2] \in \mathbb{R}$ for any $U,V_1,V_2 \in L^2_{\mathbb{R}}(\mathbb{T},\mathbb{C}^2)$ amounts to

$$\overline{\Lambda^{\vec{\sigma}_p,\sigma',\sigma}_{\vec{\jmath}_p,j,k}} = \Lambda^{-\vec{\sigma}_p,-\sigma',-\sigma}_{\vec{\jmath}_p,j,k} . \tag{4.2.142}$$

Moreover the coefficients satisfy, for some $m \geq 0$,

$$|\Lambda^{\vec{\sigma}_p,\sigma',\sigma}_{\vec{\jmath}_p,j,k}| \lesssim \max(|\vec{\jmath}_p|,|j|)^m . \tag{4.2.143}$$

The following lemma characterizes 0, 1 and 2 forms.

Lemma 4.2.30. (Operatorial characterization of Hamiltonians and 1- 2 forms) *Let $p \in \mathbb{N}_0$. Then*

(i) *A 0-form $H(U)$ belongs to $\widetilde{\Lambda}^0_{p+2}$ if and only if there exists a matrix of p-homogeneous real-to-real operators $M(U)$ in $\widetilde{\mathcal{M}}_p \otimes \mathcal{M}_2(\mathbb{C})$, such that,*

$$H(U) = \langle M(U)U,U\rangle_r, \quad \forall U \in \dot{H}^\infty(\mathbb{T};\mathbb{C}^2) . \tag{4.2.144}$$

(ii) *A 1-form $\theta(U)$ belongs to $\widetilde{\Lambda}^1_{p+1}$ if and only if there exists a matrix of pluri–homogeneous real-to-real operators $M(U)$ in $\widetilde{\mathcal{M}}_p \otimes \mathcal{M}_2(\mathbb{C})$, such that,*

$$\theta(U)[V] := \langle M(U)U,V\rangle_r , \quad \forall V \in \dot{H}^\infty(\mathbb{T};\mathbb{C}^2) . \tag{4.2.145}$$

(iii) *A 2-form $\Lambda(U)$ belongs to $\widetilde{\Lambda}^2_p$ if and only if there exists a matrix of pluri–homogeneous real-to-real operators $M(U)$ in $\widetilde{\mathcal{M}}_p \otimes \mathcal{M}_2(\mathbb{C})$, satisfying $M(U)^\top = -M(U)$, such that*

$$\Lambda(U)[V_1,V_2] := \langle M(U)V_1,V_2\rangle_r , \quad \forall (V_1,V_2) \in (\dot{H}^\infty(\mathbb{T};\mathbb{C}^2))^2 . \tag{4.2.146}$$

Proof. PROOF OF (i): Identity (4.2.144) follows with an operator $M(U)$ which has Fourier entries

$$M^{\vec{\sigma}_p,\sigma',\sigma}_{\vec{\jmath}_p,j,k} = \frac{1}{p+2} H^{\vec{\sigma}_p,\sigma',\sigma}_{\vec{\jmath}_p,j,-k} \tag{4.2.147}$$

where the Fourier coefficients of H are defined in (4.2.133). By (4.2.136) and Lemma 4.2.9 the operator $M(U)$ defined by (4.2.147) is a matrix of m-operators in $\widetilde{\mathcal{M}}^m_p \otimes \mathcal{M}_2(\mathbb{C})$. Note that, in view of (4.2.147), the entries of the operator $M(U)$ satisfy the corresponding momentum condition $\vec{\sigma}_p \cdot \vec{\jmath}_p + \sigma' j = \sigma k$ thanks to the restriction in (4.2.133). The reality condition (4.2.134) is equivalent to (4.2.46).
PROOF OF (ii): Identity (4.2.145) follows with an operator $M(U)$ which has Fourier entries

$$M^{\vec{\sigma}_p,\sigma',\sigma}_{\vec{\jmath}_p,j,k} = \Theta^{\vec{\sigma}_p,\sigma',\sigma}_{\vec{\jmath}_p,j,-k} \tag{4.2.148}$$

where the Fourier coefficients of Θ are defined in (4.2.137). By (4.2.139) and Lemma 4.2.9 the operator $M(U)$ defined by (4.2.148) is a matrix of m-operators in $\widetilde{\mathcal{M}}^m_p \otimes \mathcal{M}_2(\mathbb{C})$. Note that, in view of (4.2.149), the entries of the operator $M(U)$ satisfy the corresponding momentum condition $\vec{\sigma}_p \cdot \vec{\jmath}_p + \sigma' j = \sigma k$ thanks to the restriction in (4.2.134). The reality condition (4.2.138) comes from (4.2.46).

PROOF OF (iii): Identity (4.2.146) follows with an operator $M(U)$ which has Fourier entries (cfr. (4.2.43))

$$M^{\vec{\sigma}_p,\sigma',\sigma}_{\vec{\jmath}_p,j,k} = \Lambda^{\vec{\sigma}_p,\sigma',\sigma}_{\vec{\jmath}_p,j,-k} \tag{4.2.149}$$

where the Fourier coefficients of Λ are defined in (4.2.140). By (4.2.143) and Lemma 4.2.9 the operator $M(U)$ defined by (4.2.149) is a matrix of m-operators in $\widetilde{\mathcal{M}}^m_p \otimes \mathcal{M}_2(\mathbb{C})$. Note that, in view of (4.2.149), the entries of the operator $M(U)$ satisfy the corresponding momentum condition $\vec{\sigma}_p \cdot \vec{\jmath}_p + \sigma' j = \sigma k$ thanks to the restriction in (4.2.140). The antisymmetry of $\Lambda(U)$ amounts to $M(U)^\top = -M(U)$ and the reality condition (4.2.142) comes from (4.2.46). □

We now extend to pluri-homogeneous k-forms the typical "operations" of differential geometry.

Definition 4.2.31. (Exterior derivative) *We define the exterior derivative of a r-homogeneous k-form $\Lambda(U)$ in $\widetilde{\Lambda}^k_r$ as*

$$\mathrm{d}\Lambda(U)[V_1,\dots,V_{k+1}] := \sum_{j=1}^{k+1}(-1)^{j-1}\mathrm{d}_U\big(\Lambda(U)[V_1,\dots,\widehat{V}_j,\dots,V_{k+1}]\big)[V_j] \tag{4.2.150}$$

where the notation $[V_1,\dots,\widehat{V}_j,\dots,V_{k+1}]$ denotes the k-tuple obtained excluding the j-th component.

• If $H(U)$ is a $p+2$-homogeneous 0-form in $\widetilde{\Lambda}^0_{p+2}$ then its exterior differential coincides with the usual differential of functions, namely $\mathrm{d}H(U)[V] = \mathrm{d}_U H(U)[V]$. Moreover $\mathrm{d}H(U)$ is a 1-form in $\widetilde{\Lambda}^1_{p+1}$ and we define the gradient $\nabla H(U) := \nabla_U H(U)$ as the vector field in $\widetilde{\mathfrak{X}}_{p+1}$ such that, cfr. (4.2.145),

$$\mathrm{d}H(U)[V] := \langle \nabla H(U), V\rangle_r\,, \quad \forall V \in \dot{H}^\infty(\mathbb{T};\mathbb{C}^2)\,. \tag{4.2.151}$$

• If $\theta(U)$ is a $(p+1)$-homogeneous 1-form in $\widetilde{\Lambda}^1_{p+1}$ written as in (4.2.145) then its exterior differential is

$$\mathrm{d}\theta(U)[V_1,V_2] = \Big\langle \big(\mathrm{d}_U X(U) - \mathrm{d}_U X(U)^\top\big)V_1, V_2\Big\rangle_r\,, \quad X(U) := M(U)U \tag{4.2.152}$$

where $\mathrm{d}_U X(U)$ and $\mathrm{d}_U X(U)^\top$ are, by the first remark below Definition 4.2.5 and Lemma 4.2.10, matrices of operators in $\widetilde{\mathcal{M}}_p \otimes \mathcal{M}_2(\mathbb{C})$. Moreover $\mathrm{d}\theta(U)$ belongs to $\widetilde{\Lambda}^2_p$.

Definition 4.2.32. *Let $p, p' \in \mathbb{N}_0$ and set $r := p+2-k$. Given a r-homogeneous k-form $\Lambda(U)$ in $\widetilde{\Lambda}^k_r$ and a matrix of homogeneous operators $M(U)$ in $\widetilde{\mathcal{M}}_{p'} \otimes \mathcal{M}_2(\mathbb{C})$ we define the*

• **Pull back** *of $\Lambda(U)$ via the map $\varphi(U) := M(U)U$ as*

$$(\varphi^*\Lambda)(U)[V_1,\dots,V_k] := \Lambda(\varphi(U))[\mathrm{d}_U\varphi(U)V_1,\dots,\mathrm{d}_U\varphi(U)V_k]\,. \tag{4.2.153}$$

• **Lie derivative** *of $\Lambda(U)$ along the vector field $X(U) := M(U)U$ as*

$$(\mathcal{L}_X\Lambda)(U)[V_1,\dots,V_k] := \mathrm{d}_U\Lambda(U)[X(U)][V_1,\dots,V_k] + \sum_{j=1}^{k}\Lambda(U)[V_1,\dots,\mathrm{d}_U X(U)[V_j],\dots,V_k]. \tag{4.2.154}$$

• **Contraction** *of $\Lambda(U)$ with the vector field $X(U) = M(U)U$ as*

$$(i_X\Lambda)(U)[V_1,\dots,V_{k-1}] := \Lambda(U)[X(U),V_1,\dots,V_{k-1}]\,. \tag{4.2.155}$$

Let $\Lambda(U)$ be a r-homogeneous k-form in $\widetilde{\Lambda}^k_r$, $k = 0,1,2$. Thanks to the first bullet below Definition 4.2.5, Lemma 4.2.10 and (i) and (ii) of Proposition 4.2.15 (see also (4.2.160)-(4.2.161)), one has the following:
• if $\varphi(U) = M(U)U$ is a map where $M(U)$ is a pluri-homogeneous operator in $\Sigma_0^N\widetilde{\mathcal{M}}_q \otimes \mathcal{M}_2(\mathbb{C})$ then $(\varphi^*\Lambda)(U)$ defined in (4.2.153) belongs to $\Sigma_r\widetilde{\Lambda}^k_q$;
• if $X(U)$ is a homogeneous vector field in $\Sigma_{p'+1}\widetilde{\mathfrak{X}}_q$ for some $p' \in \mathbb{N}_0$, then $(\mathcal{L}_X\Lambda)(U)$ defined in (4.2.154) belongs to $\Sigma_{r+p'}\widetilde{\Lambda}^k_q$;
• if $k = 1,2$ then $(i_X\Lambda)(U)$ defined in (4.2.155) belongs to $\Sigma_{r+p'+1}\widetilde{\Lambda}^{k-1}_q$;
• the basic identities of differential geometry are directly verified for pluri-homogeneous k-forms: Let $p \in \mathbb{N}_0$, $k = 0,1,2$. Then for any Λ in $\widetilde{\Lambda}^k_{p+2}$ it results

$$\mathrm{d}^2\Lambda = 0\,. \tag{4.2.156}$$

Given $\varphi(U) := M(U)U$ with $M(U)$ in $\Sigma_0\widetilde{\mathcal{M}}_q \otimes \mathcal{M}_2(\mathbb{C})$, it results

$$\mathrm{d}(\varphi^*\Lambda) = (\varphi^*\mathrm{d}\Lambda)\,. \tag{4.2.157}$$

Given also $\phi(U) := M'(U)U$ with $M'(U) \in \Sigma_0\widetilde{\mathcal{M}}_q \otimes \mathcal{M}(\mathbb{C})$, it results

$$\phi^*\varphi^*\Lambda = (\varphi\circ\phi)^*\Lambda\,. \tag{4.2.158}$$

Given $X(U)$ in $\Sigma_1\widetilde{\mathfrak{X}}_q$ then

$$\mathcal{L}_X\Lambda = \mathrm{d}\circ i_X\Lambda + i_X\circ\mathrm{d}\Lambda\,; \tag{4.2.159}$$

• if $\varphi(U) = M'(U)U$ is a map where $M'(U)$ is a pluri-homogeneous operator in $\Sigma_0\widetilde{\mathcal{M}}_q \otimes \mathcal{M}_2(\mathbb{C})$ and $\theta(U) \in \widetilde{\Lambda}^1_{p+1}$ and $\Lambda(U) \in \widetilde{\Lambda}^2_p$ are represented as in (4.2.145), (4.2.146) then

$$(\varphi^*\theta)(U)[V] = \langle \mathrm{d}_U\varphi(U)^\top\, M(\varphi(U))\,\varphi(U), V\rangle\,; \tag{4.2.160}$$
$$(\varphi^*\Lambda)(U)[V_1,V_2] = \langle \mathrm{d}_U\varphi(U)^\top\, M(\varphi(U))\,\mathrm{d}_U\varphi(U)V_1, V_2\rangle\,. \tag{4.2.161}$$

In Section 4.4 we shall use the following result about Lie derivatives and approximate flows.

Lemma 4.2.33. *Let $p,N \in \mathbb{N}$ with $p \le N$. Let θ^τ be a τ-dependent family of 1-forms in $\Sigma_1\widetilde{\Lambda}^1_q$ defined for $\tau \in [0,1]$. Let $\mathcal{F}^\tau_{\le N}$ be the approximate flow generated by a pluri–homogeneous, τ–dependent smoothing vector field $Y^\tau(U)$ in $\Sigma_{p+1}\widetilde{\mathfrak{X}}^{-\varrho}_q$, defined for $\tau \in [0,1]$ and some $\varrho \ge 0$ (cfr. Lemma 4.2.28). Then*

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(\mathcal{F}^\tau_{\le N})^*\theta^\tau = (\mathcal{F}^\tau_{\le N})^*\big[\mathcal{L}_{Y^\tau}\theta^\tau + \partial_\tau\theta^\tau\big] + \theta^\tau_{>N+1} \tag{4.2.162}$$

where $\theta^\tau_{>N+1}$ is a pluri–homogeneous 1–form in $\Sigma_{N+2}\widetilde{\Lambda}^1_q$, with estimates uniform in $\tau \in [0,1]$.

Proof. Recalling the definition of pullback (4.2.153) and using that $\mathcal{F}^\tau_{\le N}$ fulfills the approximate equation (4.2.128) (with Y^τ replacing X^τ) we get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\tau}(\mathcal{F}^\tau_{\le N})^*\theta^\tau(U)[\widehat{U}] = \frac{\mathrm{d}}{\mathrm{d}\tau}\big(\theta^\tau(\mathcal{F}^\tau_{\le N}(U))[\mathrm{d}_U\mathcal{F}^\tau_{\le N}(U)\widehat{U}]\big)\\
&= \partial_\tau\theta^\tau(\mathcal{F}^\tau_{\le N}(U))[\mathrm{d}_U\mathcal{F}^\tau_{\le N}(U)\widehat{U}] + \mathrm{d}_V\theta^\tau(\mathcal{F}^\tau_{\le N}(U))[Y^\tau(\mathcal{F}^\tau_{\le N}(U)) + R^\tau_{>N}(U)U][\mathrm{d}_U\mathcal{F}^\tau_{\le N}(U)\widehat{U}]\\
&\quad + \theta^\tau(\mathcal{F}^\tau_{\le N}(U))[\mathrm{d}_V Y^\tau(\mathcal{F}^\tau_{\le N}(U))\mathrm{d}_U\mathcal{F}^\tau_{\le N}(U)\widehat{U} + \mathrm{d}_U(R^\tau_{>N}(U)U)\widehat{U}]\\
&\overset{(4.2.154),(4.2.153)}{=} (\mathcal{F}^\tau_{\le N})^*\big[\mathcal{L}_{Y^\tau}\theta^\tau + \partial_\tau\theta^\tau\big](U)[\widehat{U}] + \theta^\tau_{>N+1}(U)[\widehat{U}]
\end{aligned}$$

where

$$\theta^{\tau}_{>N+1}(U)[\widehat{U}] := \mathrm{d}_V\theta^{\tau}(\mathcal{F}^{\tau}_{\le N}(U))[R^{\tau}_{>N}(U)U][\mathrm{d}_U\mathcal{F}^{\tau}_{\le N}(U)\widehat{U}] + \theta^{\tau}(\mathcal{F}^{\tau}_{\le N}(U))[\mathrm{d}_U(R^{\tau}_{>N}(U)U)\widehat{U}]\,. \tag{4.2.163}$$

We now verify that $\theta^{\tau}_{>N+1}$ is a 1-form in $\Sigma_{N+2}\widetilde{\Lambda}^1_q$. Representing, by Lemma 4.2.30,

$$\theta^{\tau}(U)[\widehat{U}] = \langle M^{\tau}(U)U, \widehat{U}\rangle_r \quad \text{with} \quad M^{\tau}(U) \in \Sigma_0\widetilde{\mathcal{M}}_q \otimes \mathcal{M}_2(\mathbb{C})\,,$$

and since $\mathcal{F}^{\tau}_{\le N}(U) = (\mathrm{Id}+R^{\tau}_{\le N}(U))U$ with $R^{\tau}_{\le N}(U)$ in $\Sigma^N_p\widetilde{\mathcal{R}}^{-\varrho}_q \otimes \mathcal{M}_2(\mathbb{C})$, the 1-form $\theta^{\tau}_{>N+1}$ in (4.2.163) reads

$$\theta^{\tau}_{>N+1}(U)[\widehat{U}] = \langle M'(U)U, \widehat{U}\rangle_r \quad \text{where} \quad M'(U) := \left[\mathrm{d}_U\mathcal{F}^{\tau}_{\le N}(U)\right]^{\top} \mathrm{d}_V(M^{\tau}(V)V)|_{V=\mathcal{F}^{\tau}_{\le N}(U)}\, R^{\tau}_{>N}(U) + [\mathrm{d}_U(R^{\tau}_{>N}(U)U)]^{\top} M^{\tau}(\mathcal{F}^{\tau}_{\le N}(U))(\mathrm{Id} + R^{\tau}_{\le N}(U))$$

is a τ-dependent matrix of operators in $\Sigma_{N+1}\widetilde{\mathcal{M}}_q \otimes \mathcal{M}_2(\mathbb{C})$, with estimates uniform in $\tau \in [0,1]$, because $R^{\tau}_{>N}(U)$ are smoothing operators in $\Sigma_{N+1}\widetilde{\mathcal{R}}^{-\varrho}_q \otimes \mathcal{M}_2(\mathbb{C})$, and using (i), (ii) of Proposition 4.2.15 and Lemma 4.2.10. Thus $\theta^{\tau}_{>N+1}$ is a 1-form in $\Sigma_{N+2}\widetilde{\Lambda}^1_q$, with estimates uniform in $\tau \in [0,1]$. □

4.3 Hamiltonian formalism

Along the chapter we consider real Hamiltonian systems and their symplectic structures in real, complex and Fourier coordinates, that we describe in Section 4.3.1. In Section 4.3.2 we introduce the notion of vector fields which are Hamiltonian up to homogeneity N and we prove that the classical Hamiltonian theory is preserved "up to homogeneity N". In Section 4.3.3 we present results about linear symplectic flows. In Section 4.3.4 we discuss Hamiltonian systems with a para-differential structure.

4.3.1 Hamiltonian and symplectic structures

Real Hamiltonian systems. We equip the real phase space $\dot{L}^2_r \times \dot{L}^2_r$ with the scalar product in (4.2.4) and the symplectic form

$$\Omega_0\left(\begin{pmatrix}\eta_1\\ \zeta_1\end{pmatrix}, \begin{pmatrix}\eta_2\\ \zeta_2\end{pmatrix}\right) := \left\langle E_0\begin{pmatrix}\eta_1\\ \zeta_1\end{pmatrix}, \begin{pmatrix}\eta_2\\ \zeta_2\end{pmatrix}\right\rangle_r = -\langle\zeta_1,\eta_2\rangle_{\dot{L}_r} + \langle\eta_1,\zeta_2\rangle_{\dot{L}_r} \tag{4.3.1}$$

where E_0 is the symplectic operator acting on $\dot{L}^2_r \times \dot{L}^2_r$ defined by

$$E_0 := \begin{pmatrix}0 & -\mathrm{Id}\\ \mathrm{Id} & 0\end{pmatrix}. \tag{4.3.2}$$

The Hamiltonian vector field X_H associated to a (densely defined) Hamiltonian function $H : \dot{L}^2_r \times \dot{L}^2_r \to \mathbb{R}$ is characterized as the unique vector field satisfying

$$\Omega_0\left(X_H(\eta,\zeta), \begin{pmatrix}\breve{\eta}\\ \breve{\zeta}\end{pmatrix}\right) = \mathrm{d}H(\eta,\zeta)\left[\begin{pmatrix}\breve{\eta}\\ \breve{\zeta}\end{pmatrix}\right], \quad \forall\begin{pmatrix}\breve{\eta}\\ \breve{\zeta}\end{pmatrix} \in \dot{L}^2_r \times \dot{L}^2_r\,. \tag{4.3.3}$$

As

$$\mathrm{d}H(\eta,\zeta)\left[\begin{pmatrix}\breve{\eta}\\ \breve{\zeta}\end{pmatrix}\right] = \mathrm{d}_\eta H(\eta,\zeta)[\breve{\eta}] + \mathrm{d}_\zeta H(\eta,\zeta)[\breve{\zeta}] = \left\langle \begin{pmatrix}\nabla_\eta H\\ \nabla_\zeta H\end{pmatrix}, \begin{pmatrix}\breve{\eta}\\ \breve{\zeta}\end{pmatrix}\right\rangle_r \tag{4.3.4}$$

where $(\nabla_\eta H, \nabla_\zeta H) \in \dot{L}^2_r \times \dot{L}^2_r$ denote the $\dot{L}^2_r$- gradients, the Hamiltonian vector field is given by

$$X_H = J \begin{pmatrix} \nabla_\eta H \\ \nabla_\zeta H \end{pmatrix} = \begin{pmatrix} \nabla_\zeta H \\ -\nabla_\eta H \end{pmatrix} \quad \text{where} \quad J := E_0^{-1} = \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix}. \tag{4.3.5}$$

We also denote by

$$\theta_0(\eta, \zeta)\Big[\begin{pmatrix} \breve{\eta} \\ \breve{\zeta} \end{pmatrix} \Big] := \frac{1}{2} \Big\langle E_0 \begin{pmatrix} \eta \\ \zeta \end{pmatrix}, \begin{pmatrix} \breve{\eta} \\ \breve{\zeta} \end{pmatrix} \Big\rangle_r \tag{4.3.6}$$

the Liouville 1–form. Note that $\mathrm{d}\theta_0 = \Omega_0$, where the exterior differential is recalled in Section 4.2.4.

Real linear Hamiltonian systems. We now consider the most general quadratic real Hamiltonian

$$H(\eta, \zeta) = \frac{1}{2} \Big\langle \mathbf{A} \begin{pmatrix} \eta \\ \zeta \end{pmatrix}, \begin{pmatrix} \eta \\ \zeta \end{pmatrix} \Big\rangle_r, \quad \mathbf{A} := \begin{pmatrix} A & B \\ B^\top & D \end{pmatrix}, \quad \mathbf{A}^\top = \mathbf{A}, \tag{4.3.7}$$

where A, B, D are linear real operators acting on $\dot{L}^2_r$ and the operators A, D are symmetric, i.e. $A^\top = A$, $D^\top = D$, where $A^\top$ denotes the transpose operator with respect to the real scalar product $\langle \, , \, \rangle_{\dot{L}^2_r}$. $\mathbf{A}^\top$ is the transpose with respect to the scalar product $\langle \cdot, \cdot \rangle_r$ in (4.2.4).

Definition 4.3.1. (Linear Hamiltonian operator) *A linear operator $\mathcal{A}$ acting on (a dense subspace) of $\dot{L}^2_r \times \dot{L}^2_r$ is Hamiltonian if it has the form*

$$\mathcal{A} = J\mathbf{A} = J \begin{pmatrix} A & B \\ B^\top & D \end{pmatrix}, \qquad \mathbf{A} = \mathbf{A}^\top, \tag{4.3.8}$$

with A, B, D real operators satisfying $A = A^\top$ and $D = D^\top$; equivalently if $E_0 \mathcal{A} = \mathbf{A}$ is symmetric with respect to the real scalar product $\langle \cdot, \cdot \rangle_r$ defined in (4.2.4).

We now provide the characterization of a real linear Hamiltonian para-differential operator. In view of (4.3.8) and (4.2.26) a matrix of para-differential operators is Hamiltonian if it has the form

$$J \mathrm{Op}^{BW} \left(\begin{bmatrix} a(x,\xi) & b(x,\xi) \\ b(x,-\xi) & d(x,\xi) \end{bmatrix} \right) = \mathrm{Op}^{BW} \left(\begin{bmatrix} b(x,-\xi) & d(x,\xi) \\ -a(x,\xi) & -b(x,\xi) \end{bmatrix} \right) \tag{4.3.9}$$

with

$$a(x,\xi) \in \mathbb{R}, \; a(x,\xi) = a(x,-\xi), \quad d(x,\xi) \in \mathbb{R}, \; d(x,\xi) = d(x,-\xi), \quad b(x,-\xi) = \overline{b(x,\xi)}. \tag{4.3.10}$$

Real Hamiltonian systems in complex coordinates. We now describe the above real Hamiltonian systems in the complex coordinates defined by the change of variables

$$\begin{pmatrix} \eta \\ \zeta \end{pmatrix} = \mathcal{C} \begin{pmatrix} u \\ \overline{u} \end{pmatrix}, \quad \mathcal{C} := \frac{1}{\sqrt{2}} \begin{pmatrix} \mathrm{Id} & \mathrm{Id} \\ -\mathrm{i} & \mathrm{i} \end{pmatrix}, \quad \mathcal{C}^{-1} := \frac{1}{\sqrt{2}} \begin{pmatrix} \mathrm{Id} & \mathrm{i} \\ \mathrm{Id} & -\mathrm{i} \end{pmatrix}, \quad \mathcal{C}^\top := \frac{1}{\sqrt{2}} \begin{pmatrix} \mathrm{Id} & -\mathrm{i} \\ \mathrm{Id} & \mathrm{i} \end{pmatrix}. \tag{4.3.11}$$

Note that $\mathcal{C}$ is a map between the real subspace of vector functions $\dot{L}^2_{\mathbb{R}}(\mathbb{T}, \mathbb{C}^2)$ into $\dot{L}^2_r \times \dot{L}^2_r$. In the sequel to save space we denote $\begin{pmatrix} u \\ \overline{u} \end{pmatrix}$ also as $(u, \overline{u})$.

The pull-back $\Omega_c := \mathcal{C}^* \Omega_0$ of the symplectic form Ω_0 in (4.3.1) is

$$\Omega_c \left(\begin{pmatrix} u_1 \\ \overline{u}_1 \end{pmatrix}, \begin{pmatrix} u_2 \\ \overline{u}_2 \end{pmatrix} \right) = \Big\langle \mathcal{C}^\top E_0 \mathcal{C} \begin{pmatrix} u_1 \\ \overline{u}_1 \end{pmatrix}, \begin{pmatrix} u_2 \\ \overline{u}_2 \end{pmatrix} \Big\rangle_r = \Big\langle E_c \begin{pmatrix} u_1 \\ \overline{u}_1 \end{pmatrix}, \begin{pmatrix} u_2 \\ \overline{u}_2 \end{pmatrix} \Big\rangle_r \tag{4.3.12}$$

where E_c is the symplectic operator acting on $\dot{L}^2_{\mathbb{R}}(\mathbb{T},\mathbb{C}^2)$

$$E_c := \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} = \mathrm{i}E_0 \,. \tag{4.3.13}$$

Remark that $E_c^\top = -E_c$ and $E_c^2 = \mathrm{Id}$. Similarly the Liouville 1-form θ_0 in (4.3.6) is transformed into the symplectic form $\theta_c := \mathcal{C}^*\theta_0$ given by

$$\theta_c(U)[V] = \frac{1}{2}\langle E_c U, V\rangle_r \,, \quad \forall\, U = \begin{pmatrix} u \\ \overline{u} \end{pmatrix}, \, V = \begin{pmatrix} v \\ \overline{v} \end{pmatrix} \tag{4.3.14}$$

and it results

$$\mathrm{d}\theta_c = \Omega_c \,. \tag{4.3.15}$$

Next we show how the differential and gradient of a Hamiltonian transform under the complex change of coordinates. The pull-back under $\mathcal{C}$ of the 1-form (cfr. (4.3.4)) $\mathrm{d}H(\eta,\zeta)[\cdot] = \Big\langle \begin{pmatrix} \nabla_\eta H \\ \nabla_\zeta H \end{pmatrix}, \cdot \Big\rangle_r$ is

$$(\mathcal{C}^*\mathrm{d}H)(u,\overline{u})[(v,\overline{v})] = \Big\langle \begin{pmatrix} \nabla_u H \\ \nabla_{\overline{u}} H \end{pmatrix}\Big|_{\mathcal{C}(u,\overline{u})}, \begin{pmatrix} v \\ \overline{v} \end{pmatrix} \Big\rangle_r \tag{4.3.16}$$

where

$$\nabla_u H := \frac{1}{\sqrt{2}}(\nabla_\eta H - \mathrm{i}\nabla_\zeta H)\,|_{\mathcal{C}(u,\overline{u})} \,, \quad \nabla_{\overline{u}} H := \frac{1}{\sqrt{2}}(\nabla_\eta H + \mathrm{i}\nabla_\zeta H)\,|_{\mathcal{C}(u,\overline{u})} \,. \tag{4.3.17}$$

Furthermore, by (4.3.11),

$$(\mathcal{C}^*\mathrm{d}H)(u,\overline{u})[(v,\overline{v})] = \mathrm{d}_u H|_{\mathcal{C}(u,\overline{u})}[v] + \mathrm{d}_{\overline{u}} H|_{\mathcal{C}(u,\overline{u})}[\overline{v}]$$

having defined

$$\mathrm{d}_u H := \frac{1}{\sqrt{2}}(\mathrm{d}_\eta H - \mathrm{i}\mathrm{d}_\zeta H)\,|_{\mathcal{C}(u,\overline{u})} \,, \quad \mathrm{d}_{\overline{u}} H := \frac{1}{\sqrt{2}}(\mathrm{d}_\eta H + \mathrm{i}\mathrm{d}_\zeta H)\,|_{\mathcal{C}(u,\overline{u})} \,.$$

In the sequel we also use the compact notation, given $U = (u,\overline{u})$,

$$\mathrm{d}_U H(U)[\widehat{U}] := \mathrm{d}_u H(U)[\widehat{u}] + \mathrm{d}_{\overline{u}} H(U)[\overline{\widehat{u}}] \,, \quad \forall \widehat{U} = \begin{pmatrix} \widehat{u} \\ \overline{\widehat{u}} \end{pmatrix} .$$

Real Hamiltonian vector fields in complex coordinates. Given a real valued Hamiltonian $H(\eta,\zeta)$, consider the Hamiltonian in complex coordinates $H_c := H \circ \mathcal{C}$ which is a function of $(u,\overline{u})$. Recalling the characterization (4.3.3) of the Hamiltonian vector field and (4.3.16), the associated Hamiltonian vector field is

$$X_{H_c}(U) := \mathcal{C}^{-1}(X_H)_{|\mathcal{C}(U)} = J_c \begin{pmatrix} \nabla_u H_c \\ \nabla_{\overline{u}} H_c \end{pmatrix} = J_c \nabla H_c(U) = \begin{pmatrix} -\mathrm{i}\nabla_{\overline{u}} H_c \\ \mathrm{i}\nabla_u H_c \end{pmatrix} \tag{4.3.18}$$

where $J_c := E_c^{-1} = E_c$ is the Poisson tensor in (4.1.22). One has also the characterization

$$\Omega_c(X_{H_c},\cdot) = \mathrm{d}_U H_c(U)[\cdot] \,. \tag{4.3.19}$$

In case H is the quadratic form (4.3.7), the transformed Hamiltonian $H_c = H \circ \mathcal{C}$ is given by

$$H_c(u,\overline{u}) = \frac{1}{2}\Big\langle \mathbf{R}\begin{pmatrix} u \\ \overline{u} \end{pmatrix}, \begin{pmatrix} u \\ \overline{u} \end{pmatrix} \Big\rangle_r, \quad \mathbf{R} := \mathcal{C}^\top \mathbf{A} \mathcal{C} := \begin{pmatrix} R_1 & R_2 \\ \overline{R_2} & \overline{R_1} \end{pmatrix} \tag{4.3.20}$$

where $R_1 := (A-D) - \mathrm{i}(B+B^\top)$, $R_2 := (A+D) + \mathrm{i}(B-B^\top)$. The operator $\mathbf{R}$ is real-to-real according to (4.2.12). In addition, since $\mathbf{A}$ is symmetric, cfr. (4.3.7), the operator $\mathbf{R}$ is symmetric with respect to the real non-degenerate bilinear form $\langle\cdot,\cdot\rangle_r$, namely

$$\mathbf{R} = \mathbf{R}^\top\,, \quad \text{i.e. } R_1 = R_1^\top\,, \ R_2^* = R_2\,. \tag{4.3.21}$$

Definition 4.3.2. (Linear Hamiltonian operator in complex coordinates) *A real-to-real linear operator $J_c\mathbf{M}$ is* linearly Hamiltonian *if $\mathbf{M} = \mathbf{M}^\top$ is symmetric with respect to the non-degenerate bilinear form $\langle\cdot,\cdot\rangle_r$, cfr.* (4.3.21).

In view of (4.2.26) and (4.3.21) a matrix of para-differential real-to-real complex operators is linearly Hamiltonian if

$$J_c \mathrm{Op}^{BW}\begin{pmatrix} b_1(U;t,x,\xi) & b_2(U;t,x,\xi) \\ \overline{b_2(U;t,x,-\xi)} & \overline{b_1(U;t,x,-\xi)} \end{pmatrix}, \quad \begin{cases} b_1(U;t,x,-\xi) = b_1(U;t,x,\xi)\,, \\ b_2(U;t,x,\xi) \in \mathbb{R}\,, \end{cases} \tag{4.3.22}$$

namely b_1 is even in ξ and b_2 is real valued.

Definition 4.3.3. (Linearly symplectic map) *A real-to-real linear transformation $\mathcal{A}$ is linearly symplectic if $\mathcal{A}^*\Omega_c = \Omega_c$ where Ω_c is defined in* (4.3.12)*, namely $\mathcal{A}^\top E_c \mathcal{A} = E_c$, where E_c is the symplectic operator defined in* (4.3.13).

Hamiltonian systems in Fourier basis. Given a Hamiltonian $H(U)$ expanded as in (4.2.133) we characterize its Hamiltonian vector field. We decompose each Fourier coefficients as $u_j = \frac{x_j + \mathrm{i}y_j}{\sqrt{2}}$, $\overline{u_j} = \frac{x_j - \mathrm{i}y_j}{\sqrt{2}}$, where $x_j := \sqrt{2}\mathrm{Re}(u_j)$ and $y_j := \sqrt{2}\mathrm{Im}(u_j)$ and we define

$$\partial_{u_j} := \frac{1}{\sqrt{2}}\big(\partial_{x_j} - \mathrm{i}\partial_{y_j}\big)\,, \quad \partial_{\overline{u_j}} := \frac{1}{\sqrt{2}}\big(\partial_{x_j} + \mathrm{i}\partial_{y_j}\big)\,, \tag{4.3.23}$$

so that $\partial_{u_j^\sigma} u_j^\sigma = 1$, for any $\sigma = \pm$, and $\partial_{u_j^\sigma} u_j^{\sigma'} = 0$, for any $\sigma\sigma' = -1$. For a real valued Hamiltonian H it results

$$\overline{\partial_{u_j} H} = \partial_{\overline{u_j}} H\,. \tag{4.3.24}$$

We now write a Hamiltonian vector field (4.3.18) in the coordinates $(u_j)_{j\in\mathbb{Z}\setminus\{0\}}$. For notational simplicity we also denote $u \equiv (u_j)_{j\in\mathbb{Z}\setminus\{0\}}$. We first note that, by (4.3.12) and (4.2.5), the symplectic form (4.3.12) reads, for any $U = (u,\overline{u})$, $V = (v,\overline{v})$,

$$\Omega_c\big(U,V\big) = \sum_{j\in\mathbb{Z}\setminus\{0\}} -\mathrm{i}\overline{u_j}v_j + \mathrm{i}u_j\overline{v_j} = -\mathrm{i}\sum_{\substack{j\in\mathbb{Z}\setminus\{0\} \\ \sigma\in\{\pm\}}} \sigma u_j^{-\sigma} v_j^\sigma\,. \tag{4.3.25}$$

Lemma 4.3.4. (Fourier expansion of a Hamiltonian vector field) *The Fourier components of the Hamiltonian vector field associated to a real Hamiltonian $H(U)$ are, for any $\sigma = \pm$, $k \in \mathbb{Z}\setminus\{0\}$,*

$$\big(J_c\nabla H(U)\big)_k^\sigma = -\mathrm{i}\sigma\partial_{u_k^{-\sigma}} H(U)\,. \tag{4.3.26}$$

In particular, if the Hamiltonian H is expanded as in (4.2.133)*, then*

$$\big(J_c\nabla H(U)\big)_k^\sigma = -\mathrm{i}\sigma \sum_{(\vec{\jmath}_{p+1},k,\vec{\sigma}_{p+1},-\sigma)\in\mathfrak{T}_{p+2}} H_{\vec{\jmath}_{p+1},k}^{\vec{\sigma}_{p+1},-\sigma} u_{\vec{\jmath}_{p+1}}^{\vec{\sigma}_{p+1}}\,. \tag{4.3.27}$$

Proof. The expression (4.3.26) is a consequence of (4.3.25) and the definition (4.3.19) of a Hamiltonian vector field, using (4.3.24). Finally (4.3.27) is a consequence of (4.3.26), (4.2.134), (4.2.135). □

By (4.3.27) and (4.2.122) we deduce the following characterization of Hamiltonian vector fields:

Lemma 4.3.5. (Characterization of $(p+1)$–homogeneous Hamiltonian vector fields) *A $(p+1)$–homogeneous real-to-real vector field $X(U) \in \widetilde{\mathfrak{X}}_{p+1}$ of the form (4.2.122) is Hamiltonian if and only if the coefficients*

$$H^{\vec{\sigma}_{p+1},\sigma}_{\vec{\jmath}_{p+1},k} := -\mathrm{i}\sigma X^{\vec{\sigma}_{p+1},-\sigma}_{\vec{\jmath}_{p+1},k}, \qquad \forall(\vec{\jmath}_{p+1}, k, \vec{\sigma}_{p+1}, \sigma) \in \mathfrak{T}_{p+2}, \tag{4.3.28}$$

satisfy (4.2.135), (4.2.134) and (4.2.136). In such a case $X(U)$ is the Hamiltonian vector field generated by

$$H(U) = \frac{1}{p+2} \sum_{(\vec{\jmath}_{p+2},\vec{\sigma}_{p+2})\in\mathfrak{T}_{p+2}} H^{\vec{\sigma}_{p+2}}_{\vec{\jmath}_{p+2}} u^{\vec{\sigma}_{p+2}}_{\vec{\jmath}_{p+2}}, \quad H^{\vec{\sigma}_{p+2}}_{\vec{\jmath}_{p+2}} := -\mathrm{i}\sigma_{p+2} X^{\vec{\sigma}_{p+1},-\sigma_{p+2}}_{\vec{\jmath}_{p+1},j_{p+2}}.$$

The Poisson bracket between two real functions F, G is, by (4.3.19), (4.3.25), (4.3.26), (4.3.24),

$$\{F, G\} := \mathrm{d}F(X_G) = \Omega_c(X_F, X_G) = \sum_{j\in\mathbb{Z}\setminus\{0\}} \mathrm{i}\Big(\partial_{\overline{u_j}} F \partial_{u_j} G - \partial_{u_j} F \partial_{\overline{u_j}} G\Big). \tag{4.3.29}$$

Note that the right hand side of (4.3.29) is well–defined also for complex valued functions F and G and, with a small abuse of notation, we shall still refer to it as the Poisson bracket between F and G.

4.3.2 Hamiltonian systems up to homogeneity N

Along the chapter we encounter vector fields which are Hamiltonian up to homogeneity N. We distinguish between linear and nonlinear ones.

Linear Hamiltonian operators. In the sequel let $p, N, K, K' \in \mathbb{N}_0$ and $K' \le K$, $r > 0$.

Definition 4.3.6. (Linearly Hamiltonian operator up to homogeneity N) *A real-to-real matrix of spectrally localized maps $J_c\mathbf{B}(U;t)$ in $\Sigma\mathcal{S}_{K,K',p}[r, N] \otimes \mathcal{M}_2(\mathbb{C})$ is* linearly Hamiltonian up to homogeneity N *if the pluri-homogeneous component $\mathcal{P}_{\le N}(\mathbf{B}(U;t))$ (defined in (4.2.36)) is symmetric, namely*

$$\mathcal{P}_{\le N}(\mathbf{B}(U;t)) = \mathcal{P}_{\le N}(\mathbf{B}(U;t)^\top). \tag{4.3.30}$$

In particular, a matrix of para-differential real-to-real complex operators is linearly Hamiltonian up to homogeneity N if it has the form (cfr. (4.3.22))

$$J_c \mathrm{Op}^{BW} \begin{pmatrix} b_1(U;t,x,\xi) & b_2(U;t,x,\xi) \\ \overline{b_2(U;t,x,-\xi)} & \overline{b_1(U;t,x,-\xi)} \end{pmatrix}, \begin{cases} b_1(U;t,x,-\xi) - b_1(U;t,x,\xi) \in \Gamma^m_{K,K',N+1}[r] \\ \mathrm{Im}\, b_2(U;t,x,\xi) \in \Gamma^{m'}_{K,K',N+1}[r] \end{cases} \tag{4.3.31}$$

for some m, m' in $\mathbb{R}$.

Definition 4.3.7. (Linearly symplectic map up to homogeneity N) *A real-to-real matrix of spectrally localized maps $\mathbf{S}(U;t)$ in $\Sigma\mathcal{S}_{K,K',0}[r, N] \otimes \mathcal{M}_2(\mathbb{C})$ is* linearly symplectic up to homogeneity N *if*

$$\mathbf{S}(U;t)^\top E_c \mathbf{S}(U;t) = E_c + S_{>N}(U;t) \tag{4.3.32}$$

where E_c is the symplectic operator defined in (4.3.13) and $S_{>N}(U;t)$ is a matrix of spectrally localized maps in $\mathcal{S}_{K,K',N+1}[r] \otimes \mathcal{M}_2(\mathbb{C})$.

The approximate inverse up to homogeneity N of a linearly symplectic map up to homogeneity N is still linearly symplectic up to homogeneity N.

Lemma 4.3.8. *Let $p, N \in \mathbb{N}$ with $p \leq N$. Let $\Phi_{\leq N}(U) := \mathbf{B}_{\leq N}(U)U$ be such that $\mathbf{B}_{\leq N}(U) - \mathrm{Id} \in \Sigma_p^N \widetilde{\mathcal{S}}_q \otimes \mathcal{M}_2(\mathbb{C})$ and $\mathbf{B}_{\leq N}(U)$ is linearly symplectic up to homogeneity N (Definition 4.3.7). Then its approximate inverse $\Psi_{\leq N}(V)$, constructed in Lemma 4.2.24, has the form $\Psi_{\leq N}(V) = \mathbf{A}_{\leq N}(V)V$ where $\mathbf{A}_{\leq N}(V) - \mathrm{Id}$ is in $\Sigma_p^N \widetilde{\mathcal{S}}_q \otimes \mathcal{M}_2(\mathbb{C})$ and $\mathbf{A}_{\leq N}(V)$ is linearly symplectic up to homogeneity N, more precisely*

$$\mathbf{A}_{\leq N}(V)^\top E_c \mathbf{A}_{\leq N}(V) = E_c + S_{>N}(V) \tag{4.3.33}$$

where $S_{>N}(V)$ is a matrix of pluri–homogeneous spectrally localized maps in $\Sigma_{N+1} \widetilde{\mathcal{S}}_q \otimes \mathcal{M}_2(\mathbb{C})$.

Proof. As $\mathbf{B}_{\leq N}(U)$ is symplectic up to homogeneity N, one has

$$\mathbf{B}_{\leq N}(U)^\top E_c \mathbf{B}_{\leq N}(U) = E_c + S_{>N}(U) \tag{4.3.34}$$

where $S_{>N}(U)$ is a pluri-homogeneous operator in $\Sigma_{N+1} \widetilde{\mathcal{S}}_q \otimes \mathcal{M}_2(\mathbb{C})$, being the left hand side above a pluri-homogeneous operator). Then we evaluate (4.3.34) at $U = \Psi_{\leq N}(V)$, apply $\mathbf{A}_{\leq N}(V)$ to the right and $\mathbf{A}_{\leq N}(V)^\top$ to the left and use (4.2.110) and the composition properties in Proposition 4.2.19. The operator $S_{>N}(V)$ is pluri-homogeneous as the left-hand side of (4.3.33). □

The class of linearly Hamiltonian operators up to homogeneity N is closed under conjugation under a linearly symplectic up to homogeneity N map.

Lemma 4.3.9. *Let $J_c \mathbf{B}(U;t)$ be a linearly Hamiltonian operator up to homogeneity N (Definition 4.3.6) and $\mathcal{G}(U;t)$ be an invertible map, linearly symplectic to homogeneity N (Definition 4.3.7). Then the operators $\mathcal{G}(U;t) J_c \mathbf{B}(U;t) \mathcal{G}(U;t)^{-1}$ and $(\partial_t \mathcal{G}(U;t)) \mathcal{G}^{-1}(U;t)$ are linearly Hamiltonian up to homogeneity N.*

Proof. Set $\mathbf{B} := \mathbf{B}(U;t)$ and $\mathcal{G} := \mathcal{G}(U;t)$ for brevity. As $\mathcal{G}$ is invertible, we deduce from (4.3.32) that $\mathcal{P}_{\leq N}(\mathcal{G} J_c) = \mathcal{P}_{\leq N}\left(J_c [\mathcal{G}^{-1}]^\top\right)$. Then

$$\mathcal{P}_{\leq N}\left(\mathcal{G} J_c \mathbf{B} \mathcal{G}^{-1}\right) = \mathcal{P}_{\leq N}\left(\mathcal{G} J_c \mathcal{P}_{\leq N}[\mathbf{B}] \mathcal{G}^{-1}\right) \overset{(4.3.32)}{=} \mathcal{P}_{\leq N}\left(J_c [\mathcal{G}^{-1}]^\top \mathcal{P}_{\leq N}[\mathbf{B}] \mathcal{G}^{-1}\right) = J_c \mathbf{M}$$

where $\mathbf{M} := \mathcal{P}_{\leq N}\left([\mathcal{G}^{-1}]^\top \mathcal{P}_{\leq N}[\mathbf{B}] \mathcal{G}^{-1}\right)$ is symmetric since $\mathcal{P}_{\leq N}[\mathbf{B}^\top] = \mathcal{P}_{\leq N}[\mathbf{B}]$. This proves that $\mathcal{G} J_c \mathbf{B} \mathcal{G}^{-1}$ is linearly Hamiltonian up to homogeneity N.

Next, differentiating (4.3.32) (with $\mathcal{G}(U;t)$ replacing $\mathbf{S}(U;t)$), we get

$$\mathcal{P}_{\leq N}\left[E_c (\partial_t \mathcal{G}) \mathcal{G}^{-1}\right] = -\mathcal{P}_{\leq N}\left[(\mathcal{G}^{-1})^\top (\partial_t \mathcal{G})^\top E_c\right] = \mathcal{P}_{\leq N}\left[\left(E_c (\partial_t \mathcal{G}) \mathcal{G}^{-1}\right)^\top\right]$$

showing that $(\partial_t \mathcal{G}) \mathcal{G}^{-1}$ is linearly Hamiltonian up to homogeneity N. □

Nonlinear Hamiltonian systems up to homogeneity N. Let $K, K' \in \mathbb{N}_0$ with $K' \leq K$, $r > 0$ and $U \in B^K_{s_0}(I;r)$. Let

$$Z := \mathbf{M}_0(U;t)U \quad \text{with} \quad \mathbf{M}_0(U;t) \in \mathcal{M}^0_{K,K',0}[r] \otimes \mathcal{M}_2(\mathbb{C}). \tag{4.3.35}$$

Definition 4.3.10. (Hamiltonian system up to homogeneity N) *Let $N, K, K' \in \mathbb{N}_0$ with $K \geq K' + 1$ and assume (4.3.35). A U–dependent system*

$$\partial_t Z = J_c \nabla H(Z) + M_{>N}(U;t)[U] \tag{4.3.36}$$

is Hamiltonian up to homogeneity N *if*

- *$H(Z)$ is a pluri-homogeneous Hamiltonian in $\Sigma_2^{N+2} \widetilde{\Lambda}^0_q$;*
- *$M_{>N}(U;t)$ is a matrix of non-homogeneous operators in $\mathcal{M}_{K,K'+1,N+1}[r] \otimes \mathcal{M}_2(\mathbb{C})$.*

In view of the first bullet after Definition 4.2.31 the Hamiltonian vector field $J_c \nabla H(Z)$ is in $\Sigma_1^{N+1}\widetilde{\mathfrak{X}}_q$.

We shall perform nonlinear changes of variables which are symplectic up to homogeneity N according to the following definition.

Definition 4.3.11. **(Symplectic map up to homogeneity** N**)** *Let $p, N \in \mathbb{N}$ with $p \le N$. We say that*

$$\mathcal{D}(Z;t) = M(Z;t)Z \qquad \textit{with} \qquad M(Z;t) - \mathrm{Id} \in \Sigma\mathcal{M}_{K,K',p}[r,N] \otimes \mathcal{M}_2(\mathbb{C}), \tag{4.3.37}$$

is symplectic up to homogeneity N, *if its pluri-homogeneous component $\mathcal{D}_{\le N}(Z) := \big(\mathcal{P}_{\le N} M(Z;t)\big)Z$ satisfies*

$$(\mathcal{D}_{\le N})^* \Omega_c = \Omega_c + \Omega_{>N} \tag{4.3.38}$$

where $\Omega_{>N}$ is a pluri-homogeneous 2-form in $\Sigma_{N+1}\widetilde{\Lambda}^2_q$.

Equivalently, by (4.3.12) and the operatorial representation (4.2.146) of 2-forms, the nonlinear map $\mathcal{D}(Z;t)$ is symplectic up to homogeneity N, if

$$[\mathrm{d}_Z \mathcal{D}_{\le N}(Z)]^\top E_c \, \mathrm{d}_Z \mathcal{D}_{\le N}(Z) = E_c + E_{>N}(Z) \quad \text{with} \quad E_{>N}(Z) \in \Sigma_{N+1}\widetilde{\mathcal{M}}_q \otimes \mathcal{M}_2(\mathbb{C}). \tag{4.3.39}$$

Remark 4.3.12. In the real setting we say that a map $\mathcal{D}(\eta,\zeta)$ is symplectic up to homogeneity if its pluri-homogeneous component $\mathcal{D}_{\le N}(\eta,\zeta)$ satisfies

$$\left[\mathrm{d}_{(\eta,\zeta)} \mathcal{D}_{\le N}(\eta,\zeta)\right]^\top E_0 \left[\mathrm{d}_{(\eta,\zeta)} \mathcal{D}_{\le N}(\eta,\zeta)\right] = E_0 + E_{>N}(\eta,\zeta) \tag{4.3.40}$$

where E_0 is the real symplectic tensor defined in (4.3.2) and $E_{>N}$ is matrix of real operators in $\Sigma_{N+1}\widetilde{\mathcal{M}}_q \otimes \mathcal{M}_2(\mathbb{C})$.

We now show that the usual properties of symplectic maps still hold, up to homogeneity N. For example the approximate inverse of a symplectic up to homogeneity N map is symplectic up to homogeneity N as well.

Lemma 4.3.13. *Let $p, N \in \mathbb{N}$ with $p \le N$. Let $\mathcal{D}_{\le N}(Z) = Z + M_{\le N}(Z)Z$ as in (4.2.108) be symplectic up to homogeneity N. Then its approximate inverse $\mathcal{E}_{\le N}(V) = V + \breve{M}_{\le N}(V)V$ up to homogeneity N as in (4.2.109) (provided by Lemma 4.2.24) is symplectic up to homogeneity N as well. Moreover*

$$\big[\mathrm{d}_Z \mathcal{D}_{\le N}(Z)\big] J_c \big[\mathrm{d}_Z \mathcal{D}_{\le N}(Z)\big]^\top = J_c + J_{>N}(Z), \quad J_{>N}(Z) \in \Sigma_{N+1}\widetilde{\mathcal{M}}_q \otimes \mathcal{M}_2(\mathbb{C}). \tag{4.3.41}$$

Proof. As $\mathcal{D}_{\le N}(Z)$ is symplectic up to homogeneity N, we get that, using also the first bullet after Definition 4.2.32,

$$(\mathcal{E}_{\le N})^*(\mathcal{D}_{\le N})^*\Omega_c = (\mathcal{E}_{\le N})^*\left[\Omega_c + \Omega_{>N}\right] = (\mathcal{E}_{\le N})^*\Omega_c + \widetilde{\Omega}_{>N} \tag{4.3.42}$$

for some pluri-homogeneous 2-forms $\Omega_{>N}, \widetilde{\Omega}_{>N}$ in $\Sigma_{N+1}\widetilde{\Lambda}^2_q$. Now recall that, being $\mathcal{E}_{\le N}(V)$ the approximate inverse of $\mathcal{D}_{\le N}(Z)$ up to homogeneity N, by (4.2.111) one has $\mathcal{D}_{\le N} \circ \mathcal{E}_{\le N} = \mathrm{Id} + F_{>N}$ for some $F_{>N}(V) = M_{>N}(V)V$ with $M_{>N}(V)$ in $\Sigma_{N+1}\widetilde{\mathcal{M}}_q \otimes \mathcal{M}_2(\mathbb{C})$. Thus we can also write

$$(\mathcal{E}_{\le N})^*(\mathcal{D}_{\le N})^*\Omega_c \overset{(4.2.158)}{=} (\mathcal{D}_{\le N} \circ \mathcal{E}_{\le N})^*\,\Omega_c = (\mathrm{Id} + F_{>N})^*\,\Omega_c \overset{(4.2.161)}{=} \Omega_c + \Omega'_{>N}(V) \tag{4.3.43}$$

for some pluri-homogeneous 2-form $\Omega'_{>N}$ in $\Sigma_{N+1}\widetilde{\Lambda}^2_q$ (by the first bullet below Definition 4.2.5, Lemma 4.2.10 and Proposition 4.2.15). Then (4.3.42)-(4.3.43) prove that $\mathcal{E}_{\le N}$ is symplectic up to homogeneity N.

Next we prove (4.3.41). We start from (4.3.39) for $\mathcal{E}_{\leq N}$ evaluated at $\mathcal{D}_{\leq N}(Z)$, i.e.

$$\big[\mathrm{d}_V \mathcal{E}_{\leq N}(\mathcal{D}_{\leq N}(Z))\big]^\top E_c \mathrm{d}_V \mathcal{E}_{\leq N}(\mathcal{D}_{\leq N}(Z)) = E_c + E_{>N}(Z) \tag{4.3.44}$$

with $E_{>N}(Z)$ in $\Sigma_{N+1}\widetilde{\mathcal{M}}_q \otimes \mathcal{M}_2(\mathbb{C})$, by Proposition 4.2.15 and Lemma 4.2.10. Then apply J_c to the left of (4.3.44) and $\big[\mathrm{d}_Z \mathcal{D}_{\leq N}(Z)\big] J_c \big[\mathrm{d}_Z \mathcal{D}_{\leq N}(Z)\big]^\top$ to the right of it and use the first and last of (4.2.112), and Proposition 4.2.15 to deduce (4.3.41). □

The approximate flow of a Hamiltonian smoothing vector field is symplectic up to homogeneity N.

Lemma 4.3.14. *Let $p, N \in \mathbb{N}$ with $p \leq N$. Let $Y(U)$ be a homogeneous Hamiltonian smoothing vector field in $\widetilde{\mathfrak{X}}^{-\varrho}_{p+1}$ for some $\varrho \geq 0$. Then its approximate flow $\mathcal{F}^\tau_{\leq N}$ (provided by Lemma 4.2.28) is symplectic up to homogeneity N (Definition 4.3.11).*

Proof. Recalling that $\Omega_c = \mathrm{d}\theta_c$ we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tau}(\mathcal{F}^\tau_{\leq N})^* \Omega_c &\overset{(4.2.157)}{=} \mathrm{d}\frac{\mathrm{d}}{\mathrm{d}\tau}(\mathcal{F}^\tau_{\leq N})^* \theta_c \\
&\overset{(4.2.162)}{=} \mathrm{d}(\mathcal{F}^\tau_{\leq N})^* \mathcal{L}_Y \theta_c + \mathrm{d}\theta^\tau_{>N+1} \\
&\overset{(4.2.159),(4.2.157),(4.2.156)}{=} (\mathcal{F}^\tau_{\leq N})^* \mathrm{d}(i_Y \Omega_c) + \mathrm{d}\theta^\tau_{>N+1} \\
&\overset{(4.3.19)}{=} (\mathcal{F}^\tau_{\leq N})^* \mathrm{d}^2 H_{p+2} + \mathrm{d}\theta^\tau_{>N+1} \overset{(4.2.156)}{=} \mathrm{d}\theta^\tau_{>N+1}
\end{aligned} \tag{4.3.45}$$

where H_{p+2} is the Hamiltonian of $Y(U)$ and $\theta^\tau_{>N+1}$ is a pluri-homogeneous 1-form in $\Sigma_{N+2}\widetilde{\Lambda}^1_q$. Integrating (4.3.45) from 0 to τ, and using that $\mathcal{F}^0_{\leq N} = \mathrm{Id}$, we get

$$(\mathcal{F}^\tau_{\leq N})^* \Omega_c = \Omega_c + \Omega^\tau_{>N}, \quad \Omega^\tau_{>N} := \int_0^\tau \mathrm{d}\theta^t_{>N+1}\, \mathrm{d}t$$

where $\Omega_{>N}$ is in $\Sigma_{N+1}\widetilde{\Lambda}^2_q$. This proves that $\mathcal{F}^\tau_{\leq N}$ is symplectic up to homogeneity N. □

A symplectic map up to homogeneity N transforms a Hamiltonian system up to homogeneity N into another Hamiltonian system up to homogeneity N.

Lemma 4.3.15. *Let $p, N \in \mathbb{N}$ with $p \leq N$, $K, K' \in \mathbb{N}_0$ with $K \geq K'+1$. Let $Z := \mathbf{M}_0(U;t)U$ as in (4.3.35). Assume $\mathcal{D}(Z;t) = M(Z;t)Z$ is a symplectic map up to homogeneity N (Definition 4.3.11) such that*

$$M(Z;t) - \mathrm{Id} \in \begin{cases} \Sigma\mathcal{M}_{K,K',p}[r,N] \otimes \mathcal{M}_2(\mathbb{C}) & \textit{if} \quad \mathbf{M}_0(U;t) = \mathrm{Id}, \\ \Sigma\mathcal{M}_{K,0,p}[\breve{r},N] \otimes \mathcal{M}_2(\mathbb{C}), \forall \breve{r} > 0 & \textit{otherwise}. \end{cases} \tag{4.3.46}$$

If $Z(t)$ solves a U-dependent Hamiltonian system up to homogeneity N (Definition 4.3.10), then the variable $W := \mathcal{D}(Z;t)$ solves another U-dependent Hamiltonian system up to homogeneity N (generated by the transformed Hamiltonian).

Proof. Decompose $\mathcal{D}(Z;t) = \mathcal{D}_{\leq N}(Z) + M^{\mathcal{D}}_{>N}(Z;t)Z$ where $\mathcal{D}_{\leq N}(Z) := \mathcal{P}_{\leq N}[M(Z;t)]Z$ is its pluri-homogeneous component and

$$M^{\mathcal{D}}_{>N}(Z;t) \in \begin{cases} \mathcal{M}_{K,K',N+1}[r] \otimes \mathcal{M}_2(\mathbb{C}) & \text{if} \quad \mathbf{M}_0(U;t) = \mathrm{Id}, \\ \mathcal{M}_{K,0,N+1}[\breve{r}] \otimes \mathcal{M}_2(\mathbb{C}), \forall \breve{r} > 0 & \text{otherwise}. \end{cases}$$

By Definition 4.3.11 the map $\mathcal{D}_{\le N}(Z)$ satisfies (4.3.38). If $Z(t)$ solves (4.3.36) then $W = \mathcal{D}(Z;t)$ solves

$$\begin{aligned}\partial_t W &= \big(\mathrm{d}_Z\mathcal{D}_{\le N}(Z) + M^{\mathcal{D}}_{>N}(Z;t)\big)\left[J_c\nabla_Z H(Z) + M_{>N}(U;t)U\right] + (\partial_t M^{\mathcal{D}}_{>N}(Z;t))Z \\ &= \mathrm{d}_Z\mathcal{D}_{\le N}(Z)J_c\nabla_Z H(Z) + M'_{>N}(U;t)U \end{aligned} \tag{4.3.47}$$

where, by the first bullet below Definition 4.2.5 and Proposition 4.2.15,

$$M'_{>N}(U;t) \in \mathcal{M}_{K,K'+1,N+1}[r] \otimes \mathcal{M}_2(\mathbb{C})\,. \tag{4.3.48}$$

Denote by $\breve{\mathcal{D}}_{\le N}(W)$ the approximate inverse up to homogeneity N of $\mathcal{D}_{\le N}(Z)$ (see Lemma 4.2.24). Then

$$\breve{\mathcal{D}}_{\le N}(W) = \breve{\mathcal{D}}_{\le N}\big(\mathcal{D}_{\le N}(Z) + M^{\mathcal{D}}_{>N}(Z;t)Z\big) = Z + \breve{M}'_{>N}(Z;t)Z$$

where, by (4.2.111) and Proposition 4.2.15,

$$\breve{M}'_{>N}(Z;t) \in \begin{cases}\mathcal{M}_{K,K',N+1}[r]\otimes\mathcal{M}_2(\mathbb{C}) & \text{if} \quad \mathbf{M}_0(U;t) = \mathrm{Id}\,, \\ \mathcal{M}_{K,0,N+1}[\breve{r}]\otimes\mathcal{M}_2(\mathbb{C})\,, \forall \breve{r}>0 & \text{otherwise}\,.\end{cases}$$

Finally we substitute $Z = \mathbf{M}_0(U;t)U$, cfr. (4.3.35), in the non–homogeneous term $\breve{M}'_{>N}(Z;t)Z$ and using (iii) and (i) Proposition 4.2.15 we get

$$Z = \breve{\mathcal{D}}_{\le N}(W) + \breve{M}_{>N}(U;t)U \tag{4.3.49}$$

with $\breve{M}_{>N}(U;t) \in \mathcal{M}_{K,K',N+1}[r]\otimes\mathcal{M}_2(\mathbb{C})$. We substitute (4.3.49) in the term $\nabla_Z H(Z)$ in (4.3.47) to obtain

$$\begin{aligned}\partial_t W &= \mathrm{d}_Z\mathcal{D}_{\le N}(Z)J_c\nabla_Z H(\breve{\mathcal{D}}_{\le N}(W)) + M''_{>N}(U;t)U \\ &\overset{(4.2.112),(4.3.35)}{=} \mathrm{d}_Z\mathcal{D}_{\le N}(Z)J_c[\mathrm{d}_Z\mathcal{D}_{\le N}(Z)]^\top[\mathrm{d}_W\breve{\mathcal{D}}_{\le N}(\mathcal{D}_{\le N}(Z))]^\top\nabla_Z H(\breve{\mathcal{D}}_{\le N}(W)) + M'''_{>N}(U;t)U \\ &\overset{(4.3.41),(4.3.35)}{=} J_c[\mathrm{d}_W\breve{\mathcal{D}}_{\le N}(W)]^\top\nabla_Z H(\breve{\mathcal{D}}_{\le N}(W)) + M''''_{>N}(U;t)U \\ &= J_c\nabla_W(H\circ\breve{\mathcal{D}}_{\le N})(W) + M''''_{>N}(U;t)U\end{aligned} \tag{4.3.50}$$

where $M''_{>N}(U;t)$, $M'''_{>N}(U;t)$, $M''''_{>N}(U;t)$ are matrices of operators as in (4.3.48). Note that in the very last passage we also substituted $\mathcal{D}_{\le N}(Z) = W + M_{>N}(U;t)U$ where $M_{>N}(U;t)$ is a matrix of operators as in (4.3.48). This proves that system (4.3.50) is Hamiltonian up to homogeneity N. ☐

4.3.3 Linear symplectic flows

We consider the flow of a linearly Hamiltonian up to homogeneity N para-differential operator.

Lemma 4.3.16. (Linear symplectic flow) *Let $p \in \mathbb{N}$, $N, K, K' \in \mathbb{N}_0$ with $K' \le K$, $m \le 1$, $r > 0$. Let $J_c\mathrm{Op}^{\mathrm{BW}}(B)$ be a linearly Hamiltonian operator up to homogeneity N (Definition 4.3.6) where $B(\tau, U; t, x, \xi)$ is a matrix of symbols*

$$B(\tau,U;t,x,\xi) := \begin{pmatrix} b_1(\tau,U;t,x,\xi) & b_2(\tau,U;t,x,\xi) \\ \overline{b_2(\tau,U;t,x,-\xi)} & \overline{b_1(\tau,U;t,x,-\xi)}\end{pmatrix}, \quad \begin{cases} b_1 \in \Sigma\Gamma^0_{K,K',p}[r,N] \\ b_2 \in \Sigma\Gamma^m_{K,K',p}[r,N]\,,\end{cases} \tag{4.3.51}$$

with $b_1^\vee - b_1$ in $\Gamma^0_{K,K',N+1}[r]$ and the imaginary part $\operatorname{Im} b_2$ in $\Gamma^0_{K,K',N+1}[r]$ (cfr. (4.3.31)) uniformly in $|\tau| \le 1$. Then there exists $s_0 > 0$ such that, for any $U \in B^K_{s_0,\mathbb{R}}(I;r)$, the system

$$\begin{cases}\partial_\tau\mathcal{G}^\tau_B(U;t) = J_c\mathrm{Op}^{\mathrm{BW}}\big(B(\tau,U;t,x,\xi)\big)\mathcal{G}^\tau_B(U;t) \\ \mathcal{G}^0_B(U;t) = \mathrm{Id}\,,\end{cases} \tag{4.3.52}$$

has a unique solution $\mathcal{G}^\tau_B(U)$ defined for all $|\tau| \le 1$, satisfying the following properties:

(i) **Boundedness:** *For any $s \in \mathbb{R}$ the linear map $\mathcal{G}_B^\tau(U;t)$ is invertible and there is $r(s) \in]0,r[$ such that for any $U \in B^K_{s_0,\mathbb{R}}(I;r(s))$ for any $0 \le k \le K-K'$, $V \in C_*^{K-K'}(I;\dot{H}^s(\mathbb{T},\mathbb{C}^2))$,*

$$\|\partial_t^k(\mathcal{G}_B^\tau(U;t)V)\|_{\dot{H}^{s-\frac{3}{2}k}} + \|\partial_t^k(\mathcal{G}_B^\tau(U;t)^{-1}V)\|_{\dot{H}^{s-\frac{3}{2}k}} \le \big(1 + C_{s,k}\|U\|_{k+K',s_0}\big)\|V\|_{k,s} \quad (4.3.53)$$

uniformly in $|\tau| \le 1$.

In particular $\mathcal{G}_B^\tau(U;t)$ and $\mathcal{G}_B^\tau(U;t)^{-1}$ are non–homogeneous spectrally localized maps in $\mathcal{S}^0_{K,K',0}[r] \otimes \mathcal{M}_2(\mathbb{C})$ according to Definition 4.2.16.

(ii) **Linear symplecticity:** *The map $\mathcal{G}_B^\tau(U;t)$ is linearly symplectic up to homogeneity N (Definition 4.3.7). If $J_c \mathrm{Op}^{\mathrm{BW}}(B)$ is linearly Hamiltonian (Definition 4.3.2), then $\mathcal{G}_B^\tau(U;t)$ is linearly symplectic (Definition 4.3.3).*

(iii) **Homogeneous expansion:** *$\mathcal{G}_B^\tau(U;t)$ and its inverse are spectrally localized maps and $\mathcal{G}_B^\tau(U;t)^{\pm} - \mathrm{Id}$ belong to $\Sigma\mathcal{S}^{(N+1)m_0}_{K,K',p}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$ with $m_0 := \max(m,0)$, uniformly in $|\tau| \le 1$.*

Proof. Since the symbols b_1 and $\operatorname{Im} b_2$ have order 0 and $\operatorname{Re} b_2$ has order $m \le 1$, the existence of the flow $\mathcal{G}_B^\tau(U;t)$ and the estimates (4.3.53) (actually with loss of k derivatives instead of $\frac{3}{2}k$) are classical and follow as in Lemma 3.22 of [27]. In view of (4.2.72), the bounds (4.3.53) imply that $\mathcal{G}_B^\tau(U;t)$ is in $\mathcal{S}^0_{K,K',0}[r] \otimes \mathcal{M}_2(\mathbb{C})$. The inverse $\mathcal{G}_B^\tau(U;t)^{-1}$ satisfies the same estimates regarding it as the time τ-flow $\mathcal{G}_{B^-}^{\tau'}(U;t)|_{\tau'=\tau}$ of the system

$$\partial_{\tau'}\mathcal{G}_{B^-}^{\tau'}(U;t) = J_c\mathrm{Op}^{\mathrm{BW}}(B^-(\tau,\tau',U;t,x,\xi))\mathcal{G}_{B^-}^{\tau'}(U;t)\,, \quad \mathcal{G}_{B^-}^{0}(U;t) = \mathrm{Id}\,, \quad (4.3.54)$$

where $B^-(\tau,U;t,x,\xi) := -B(\tau-\tau',U;t,x,\xi)$.

Let us prove item (ii). Set $\mathbf{B} := \mathrm{Op}^{\mathrm{BW}}(B(\tau,U;t,x,\xi))$ and $\mathcal{G}^\tau := \mathcal{G}_B^\tau(U;t)$ for brevity. By (4.3.52) we get, for any τ,

$$\begin{aligned}\partial_\tau(\mathcal{G}^\tau)^\top E_c\mathcal{G}^\tau &= -(\mathcal{G}^\tau)^\top\mathbf{B}^\top J_cE_c\mathcal{G}^\tau + (\mathcal{G}^\tau)^\top E_cJ_c\mathbf{B}\mathcal{G}^\tau \\ &\overset{J_cE_c=\mathrm{Id},(4.3.13)}{=} (\mathcal{G}^\tau)^\top(\mathbf{B}-\mathbf{B}^\top)\mathcal{G}^\tau \overset{(4.3.30)}{=} (\mathcal{G}^\tau)^\top(\mathbf{B}_{>N}-\mathbf{B}^\top_{>N})\mathcal{G}^\tau\,.\end{aligned}$$

Therefore

$$(\mathcal{G}^\tau)^\top E_c\mathcal{G}^\tau = E_c + S_{>N} \quad \text{where} \quad S_{>N} := \int_0^\tau (\mathcal{G}^{\tau'})^\top(\mathbf{B}_{>N}-\mathbf{B}^\top_{>N})\mathcal{G}^{\tau'}\,\mathrm{d}\tau'$$

is a matrix of spectrally localized maps in $\mathcal{S}_{K,K',N+1}[r] \otimes \mathcal{M}_2(\mathbb{C})$ because $\mathcal{G}_B^\tau(U;t)$ and $\mathcal{G}_B^\tau(U;t)^{-1}$ are in $\mathcal{S}^0_{K,K',0}[r] \otimes \mathcal{M}_2(\mathbb{C})$, the para-differential operator $\mathbf{B}$ belongs to $\Sigma\mathcal{S}^{m_0}_{K,K',p}[r] \otimes \mathcal{M}_2(\mathbb{C})$ (see the fourth bullet after Definition 4.2.16), and (ii) of Proposition 4.2.19. This proves that the $\mathcal{G}_B^\tau(U;t)$ is linearly symplectic up to homogeneity N according to Definition 4.3.7. The same proof shows that, if $J_c\mathrm{Op}^{\mathrm{BW}}(B)$ is linearly Hamiltonian, then $\mathcal{G}_B^\tau(U;t)$ is linearly symplectic.

Let us prove item (iii). By (4.3.52), iterating N–times the fundamental theorem of calculus we get the expansion

$$\mathcal{G}_B^\tau(U;t) = \mathrm{Id} + \sum_{j=1}^N S_j^\tau(U) + S^\tau_{>(pN)}(U;t) \quad (4.3.55)$$

where

$$S_j^\tau(U) := \int_0^\tau\int_0^{\tau_1}\cdots\int_0^{\tau_{j-1}} J_c\mathrm{Op}^{\mathrm{BW}}(B(\tau_1,U;t,x,\xi))\cdots J_c\mathrm{Op}^{\mathrm{BW}}(B(\tau_j,U;t,x,\xi))\,\mathrm{d}\tau_1\cdots\mathrm{d}\tau_j$$

and, writing for brevity $\mathrm{Op}^{\mathrm{BW}}(B(\tau_j,U)) := \mathrm{Op}^{\mathrm{BW}}(B(\tau_j,U;t,x,\xi))$,

$$S^{\tau}_{>(pN)}(U;t) = \int_0^{\tau}\int_0^{\tau_1}\cdots\int_0^{\tau_N} J_c\mathrm{Op}^{\mathrm{BW}}(B(\tau_1,U))\cdots J_c\mathrm{Op}^{\mathrm{BW}}(B(\tau_{N+1},U))\mathcal{G}_B^{\tau_{N+1}}(U;t)\,\mathrm{d}\tau_1\cdots\mathrm{d}\tau_{N+1}.$$

Since each $\mathrm{Op}^{\mathrm{BW}}(B(\tau_j,U))$ belongs to $\Sigma\mathcal{S}^{m_0}_{K,K',p}[r]\otimes\mathcal{M}_2(\mathbb{C})$ and $\mathcal{G}_B^{\tau_{N+1}}(U)$ is in $\mathcal{S}^0_{K,K',0}\otimes\mathcal{M}_2(\mathbb{C})$ we deduce, by (ii) of Proposition 4.2.19, that $\mathcal{G}^{\tau}_B(U;t)-\mathrm{Id}$ in (4.3.55) is a matrix of spectrally localized maps in $\Sigma\mathcal{S}^{(N+1)m_0}_{K,K',p}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$, uniformly in $|\tau|\leq 1$. The analogous statement for $\mathcal{G}^{\tau}_B(U;t)^{-1}-\mathrm{Id}$ follows by (4.3.54). □

The flow generated by a Fourier multiplier satisfies similar properties.

Lemma 4.3.17. (Flow of a Fourier multiplier) *Let $p\in\mathbb{N}$ and $g_p(Z;\xi)$ be a p–homogeneous, x-independent, real symbol in $\widetilde{\Gamma}^{\frac32}_p$. Then the flow $\mathcal{G}^{\tau}_{g_p}(Z)$ defined by*

$$\partial_\tau\mathcal{G}^{\tau}_{g_p}(Z) = \mathrm{Op}^{BW}_{\mathtt{vec}}(\mathrm{i}g_p(Z;\xi))\mathcal{G}^{\tau}_{g_p}(Z)\,,\quad \mathcal{G}^{0}_{g_p}(Z)=\mathrm{Id}\,, \tag{4.3.56}$$

is well defined for any $|\tau|\leq 1$ and satisfies the following properties:

(i) **Boundedness:** *For any $K\in\mathbb{N}$ and $r>0$ the flow $\mathcal{G}^{\tau}_{g_p}(Z)$ is a real-to-real diagonal matrix of spectrally localized maps in $\mathcal{S}^0_{K,0,0}[r]\otimes\mathcal{M}_2(\mathbb{C})$. Moreover there is $s_0>0$ such that for any $s\in\mathbb{R}$, there is $r(s)\in(0,r)$ such that for any functions $Z\in B^K_{s_0,\mathbb{R}}(I;r(s))$ and $W\in C^K_*(I;\dot H^s(\mathbb{T},\mathbb{C}^2))$, it results, for any $0\leq k\leq K$,*

$$\|\partial_t^k(\mathcal{G}^{\tau}_{g_p}(Z)W)\|_{\dot H^{s-\frac32 k}} + \|\partial_t^k(\mathcal{G}^{\tau}_{g_p}(Z)^{-1}W)\|_{\dot H^{s-\frac32 k}} \leq \big(1+C_{s,k}\|Z\|_{k,s_0}\big)\|W\|_{k,s} \tag{4.3.57}$$

uniformly in $|\tau|\leq 1$.

(ii) **Linear symplecticity:** *The flow map $\mathcal{G}^{\tau}_{g_p}(Z)$ is linearly symplectic (Definition 4.3.3).*

(iii) **Homogeneous expansion:** *The flow map $\mathcal{G}^{\tau}_{g_p}(Z)$ and its inverse $\mathcal{G}^{-\tau}_{g_p}(Z)$ are matrices of spectrally localized maps such that $\mathcal{G}^{\pm\tau}_{g_p}(Z)-\mathrm{Id}$ belong to $\Sigma\mathcal{S}^{\frac32(N+1)}_{K,0,p}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$, uniformly in $|\tau|\leq 1$.*

Proof. Since $g_p(Z;\xi)$ is real and independent of x, then the flow $\mathcal{G}^{\tau}_{g_p}(Z)$ is well defined in $\dot H^s$ and it is unitary, namely $\|\mathcal{G}^{\tau}_{g_p}(Z)W\|_{\dot H^s}=\|W\|_{\dot H^s}$. Moreover, since g_p is a Fourier multiplier of order $\frac32$, we have

$$\begin{aligned}\|\partial_t(\mathcal{G}^{\tau}_{g_p}(Z)W)\|_{\dot H^{s-\frac32}} &= \|\mathcal{G}^{\tau}_{g_p}(Z)\partial_t W\|_{\dot H^{s-\frac32}} + \|\mathrm{Op}^{BW}_{\mathtt{vec}}(\mathrm{i}\partial_t g_p(Z;\xi))\mathcal{G}^{\tau}_{g_p}(Z)W\|_{\dot H^{s-\frac32}}\\ &\leq \|W\|_{1,s}+C\|Z\|^p_{1,s_0}\|W\|_{0,s}\,.\end{aligned}$$

The estimates for the k–th derivative follow similarly using also that $\mathcal{G}^{\tau}_{g_p}(Z)^{-1}=\mathcal{G}^{-\tau}_{g_p}(Z)$.

To prove (ii) we use that, in view of (4.2.24), (4.3.22) and since $g_p(Z;\xi)$ is real valued, the operator $\mathrm{Op}^{BW}_{\mathtt{vec}}(\mathrm{i}g_p(Z;\xi))$ is linearly Hamiltonian, according to Definition 4.3.2. Then, as for item (ii) of Lemma 4.3.16, the flow $\mathcal{G}^{\tau}_{g_p}(Z)$ is linearly symplectic. Finally also item (iii) follows as for item (iii) of Lemma 4.3.16, since $\mathrm{Op}^{BW}_{\mathtt{vec}}(\mathrm{i}g_p(Z;\xi))^k$ is in $\widetilde{\mathcal{S}}^{\frac32 k}_{kp}\otimes\mathcal{M}_2(\mathbb{C})$ and $\mathcal{G}^{\tau}_{g_p}(Z)$ is in $\mathcal{S}^0_{K,0,0}[r]\otimes\mathcal{M}_2(\mathbb{C})$, uniformly in $|\tau|\leq 1$. □

4.3.4 Para-differential Hamiltonian structure

In order to compute the Hamiltonian vector field associated to a para-differential Hamiltonian we provide the following result.

Lemma 4.3.18. *Let $p \in \mathbb{N}$, $m \in \mathbb{R}$. Let $S(U)$ be a real-to-real symmetric matrix of p-homogeneous spectrally localized maps in $\widetilde{\mathcal{S}}_p^m \otimes \mathcal{M}_2(\mathbb{C})$ and define the Hamiltonian function*

$$H(U) := \frac{1}{2}\big\langle S(U)U, U\big\rangle_r \,. \tag{4.3.58}$$

Then its gradient

$$\nabla H(U) = S(U)U + R(U)U \tag{4.3.59}$$

where $R(U)$ is a real-to-real matrix of homogeneous smoothing operators in $\widetilde{\mathcal{R}}_p^{-\varrho} \otimes \mathcal{M}_2(\mathbb{C})$ for any $\varrho \geq 0$.

Proof. By the definition (4.2.151), the gradient $\nabla H(U)$ is the vector field

$$\nabla H(U) = S(U)U + L(U)^\top U \quad \text{where} \quad L(U)W := \frac{1}{2}\mathrm{d}_U S(U)[W]U \,. \tag{4.3.60}$$

As $S(U)$ is a spectrally localized map in $\widetilde{\mathcal{S}}_p^m \otimes \mathcal{M}_2(\mathbb{C})$, by Lemma 4.2.21 the transposed of its internal differential, namely $L(U)^\top$, is a smoothing operator in $\widetilde{\mathcal{R}}_p^{-\varrho} \otimes \mathcal{M}_2(\mathbb{C})$ for any $\varrho \geq 0$. Then (4.3.59) follows from (4.3.60). □

As a corollary we obtain the Hamiltonian vector field associated to a para-differential Hamiltonian.

Lemma 4.3.19. *Let $p \in \mathbb{N}$, $m \in \mathbb{R}$ and $a(U;x,\xi)$ a real valued homogeneous symbol in $\widetilde{\Gamma}_p^m$. Then the Hamiltonian vector field generated by the Hamiltonian*

$$H(U) := \mathrm{Re}\,\langle A(U)u, \overline{u}\rangle_{\dot{L}^2_r} = \frac{1}{2}\Big\langle \begin{pmatrix} 0 & \overline{A(U)} \\ A(U) & 0 \end{pmatrix} U, U\Big\rangle_r, \quad A(U) := \mathrm{Op}^{\mathrm{BW}}(a(U;x,\xi)),$$

is

$$J_c \nabla H(U) = \mathrm{Op}^{BW}_{\mathtt{vec}}(-\mathrm{i}a(U;x,\xi))U + R(U)U$$

where $R(U)$ is a real-to-real matrix of homogeneous smoothing operators in $\widetilde{\mathcal{R}}_p^{-\varrho} \otimes \mathcal{M}_2(\mathbb{C})$ for any $\varrho \geq 0$.

We now prove that if a homogeneous Hamiltonian vector field $X(U) = J_c\nabla H(U)$ can be written in para-differential form

$$X(U) = J_c \mathrm{Op}^{\mathrm{BW}}(A(U))U + R(U)U$$

where $A(U)$ is a matrix of symbols and $R(U)$ is a smoothing operator, then $\mathrm{Op}^{\mathrm{BW}}(A(U)) = \mathrm{Op}^{\mathrm{BW}}(A(U))^\top$ up to a smoothing operator. As a consequence we may always assume, up to modifying the smoothing operator, that the para-differential operator $\mathrm{Op}^{\mathrm{BW}}(A(U))$ is symmetric, namely that $J_c\mathrm{Op}^{\mathrm{BW}}(A(U))$ is linearly Hamiltonian.

Lemma 4.3.20. *Let $p \in \mathbb{N}$, $m \in \mathbb{R}$ and $\varrho \geq 0$. Let*

$$X(U) = J_c \mathrm{Op}^{\mathrm{BW}}(A(U;x,\xi))U + R(U)U = J_c \nabla H(U) \tag{4.3.61}$$

be a $(p+1)$-homogeneous Hamiltonian vector field, where (cfr. (4.2.28))

$$A(U;x,\xi) = \begin{pmatrix} a(U;x,\xi) & b(U;x,\xi) \\ \overline{b(U;x,-\xi)} & \overline{a(U;x,-\xi)} \end{pmatrix} \tag{4.3.62}$$

is matrix of symbols in $\widetilde{\Gamma}^m_p \otimes \mathcal{M}_2(\mathbb{C})$ *and* $R(U)$ *is a real-to-real matrix of smoothing operators in* $\widetilde{\mathcal{R}}^{-\varrho}_p \otimes \mathcal{M}_2(\mathbb{C})$. *Then we may write*

$$X(U) = J_c \mathrm{Op}^{\mathrm{BW}}(A_1(U;x,\xi))U + R_1(U)U \tag{4.3.63}$$

where the matrix of para-differential operators $\mathrm{Op}^{\mathrm{BW}}(A_1(U;x,\xi))$ *is symmetric, with matrix of symbols*

$$A_1(U;x,\xi) = \frac{1}{2}\begin{pmatrix} a + a^{\vee} & b + \overline{b} \\ \overline{b}^{\vee} + b^{\vee} & \overline{a + a^{\vee}} \end{pmatrix} \tag{4.3.64}$$

and $R_1(U)$ *is another real-to-real matrix of smoothing operators in* $\widetilde{\mathcal{R}}^{-\varrho}_p \otimes \mathcal{M}_2(\mathbb{C})$.

Proof. The linear vector field $\mathrm{d}_U X(U)$, obtained linearizing a Hamiltonian vector field $X(U) = J_c \nabla H(U)$, is Hamiltonian, namely

$$\mathrm{d}_U X(U) = J_c S(U) \quad \text{where} \quad S(U) = S(U)^{\top} \quad \text{is symmetric}.$$

On the other hand, by linearizing (4.3.61),

$$S(U) = \mathrm{Op}^{\mathrm{BW}}(A(U)) + \mathrm{Op}^{\mathrm{BW}}(\mathrm{d}_U A(U)[\cdot])U + J_c \breve{R}(U)$$

where $\breve{R}(U) := R(U) + \mathrm{d}_U R(U)[\cdot]U$ is a matrix of smoothing operators in $\widetilde{\mathcal{R}}^{-\varrho}_p \otimes \mathcal{M}_2(\mathbb{C})$ (see the remark after Definition 4.2.5). Then, since $S(U)$ is symmetric, it results, writing for brevity $A := A(U)$,

$$\mathrm{Op}^{\mathrm{BW}}(A) - \mathrm{Op}^{\mathrm{BW}}(A)^{\top} = \big((J_c\breve{R}(U))^{\top} - \mathrm{Op}^{\mathrm{BW}}(\mathrm{d}_U A(U)[\cdot])U\big) + \big(\big[\mathrm{Op}^{\mathrm{BW}}(\mathrm{d}_U A(U)[\cdot])U\big]^{\top} - J_c\breve{R}(U)\big).$$

We now apply Lemma 4.2.22 to the spectrally localized map $\mathrm{Op}^{\mathrm{BW}}(A(U)) - \mathrm{Op}^{\mathrm{BW}}(A(U))^{\top}$ which has the form (4.2.106) with

$$L := \big((J_c\breve{R}(U))^{\top} - \mathrm{Op}^{\mathrm{BW}}(\mathrm{d}_U A(U)[\cdot])U\big), \quad R := \big(\big[\mathrm{Op}^{\mathrm{BW}}(\mathrm{d}_U A(U)[\cdot])U\big]^{\top} - J_c\breve{R}(U)\big). \tag{4.3.65}$$

By Lemma 4.2.21, the operator $\big[\mathrm{Op}^{\mathrm{BW}}(\mathrm{d}_U A(U)[\cdot])U\big]^{\top}$ is in $\widetilde{\mathcal{R}}^{-\varrho}_p \otimes \mathcal{M}_2(\mathbb{C})$, and therefore both $L^{\top}$ and R in (4.3.65) are p-homogeneous smoothing operators in $\widetilde{\mathcal{R}}^{-\varrho}_p \otimes \mathcal{M}_2(\mathbb{C})$. The assumptions of Lemma 4.2.22 are satisfied, implying that

$$\mathrm{Op}^{\mathrm{BW}}(A(U)) - \mathrm{Op}^{\mathrm{BW}}(A(U))^{\top} =: R'(U) \in \widetilde{\mathcal{R}}^{-\varrho}_p \otimes \mathcal{M}_2(\mathbb{C}).$$

In conclusion we deduce (4.3.63) with

$$\mathrm{Op}^{\mathrm{BW}}(A_1(U)) := \frac{1}{2}\Big(\mathrm{Op}^{\mathrm{BW}}(A(U)) + \mathrm{Op}^{\mathrm{BW}}(A(U))^{\top}\Big), \quad R_1(U) := \frac{1}{2}J_c R'(U) + R(U),$$

and (4.3.64) follows recalling (4.2.26). □

Another consequence of Lemma 4.2.22 is the following.

Lemma 4.3.21. *Let* $p \in \mathbb{N}$, $m \in \mathbb{R}$ *and* $\varrho \geq 0$. *Let* $S(U)$ *be a matrix of spectrally localized homogeneous maps in* $\widetilde{\mathcal{S}}_p \otimes \mathcal{M}_2(\mathbb{C})$ *which is linearly Hamiltonian (Definition 4.3.2) of the form*

$$S(U) = J_c \mathrm{Op}^{\mathrm{BW}}(A(U;x,\xi)) + R(U), \tag{4.3.66}$$

where $A(U;x,\xi)$ *is a real-to-real matrix of symbols in* $\widetilde{\Gamma}^m_p \otimes \mathcal{M}_2(\mathbb{C})$ *as in* (4.3.62)*, and* $R(U)$ *is a real-to-real matrix of smoothing operators in* $\widetilde{\mathcal{R}}^{-\varrho}_p \otimes \mathcal{M}_2(\mathbb{C})$. *Then we may write*

$$S(U) = J_c \mathrm{Op}^{\mathrm{BW}}(A_1(U;x,\xi)) + R_1(U)$$

where the matrix of symbols $A_1(U;x,\xi)$ *in* $\widetilde{\Gamma}^m_p \otimes \mathcal{M}_2(\mathbb{C})$ *has the form* (4.3.64) *and* $R_1(U)$ *is another matrix of real-to-real smoothing operators in* $\widetilde{\mathcal{R}}^{-\varrho}_p \otimes \mathcal{M}_2(\mathbb{C})$. *In particular the homogeneous operator* $J_c \mathrm{Op}^{\mathrm{BW}}(A_1(U))$ *is linearly Hamiltonian.*

Proof. It is enough to prove that the operator $\mathrm{Op}^{\mathrm{BW}}(A(U))$ is equal to $\mathrm{Op}^{\mathrm{BW}}(A(U))^{\top}$ up to a matrix of smoothing operators. To prove this claim, recall that $S(U)$ linearly Hamiltonian means that $E_c S(U)$ is symmetric, so that by (4.3.66) one gets

$$\mathrm{Op}^{\mathrm{BW}}(A(U)) - \mathrm{Op}^{\mathrm{BW}}(A(U))^{\top} = -R(U)^{\top} E_c - E_c R(U).$$

Now, since $S(U)$ and $\mathrm{Op}^{\mathrm{BW}}(A(U))$ are spectrally localized maps, so is $R(U)$ in (4.3.66). By Remark 4.2.11 the transpose $R(U)^{\top}$ is also a smoothing operator in $\widetilde{\mathcal{R}}_p^{-\varrho} \otimes \mathcal{M}_2(\mathbb{C})$, proving the claim. □

4.4 Construction of a Darboux symplectic corrector

If $\mathbf{B}(U;t)$ is a spectrally localized map which is linearly symplectic up to homogeneity N, then the associated nonlinear map $\Phi_{\leq N}(U) := \mathbf{B}_{\leq N}(U)U$, where $\mathbf{B}_{\leq N}(U) = \mathcal{P}_{\leq N}(\mathbf{B}(U;t))$, is *not* symplectic up to homogeneity N. In this section we provide a systematic procedure to construct a nearby nonlinear map which is symplectic up to homogeneity N according to Definition 4.3.11.

Theorem 4.4.1. (Symplectic correction up to homogeneity N**)** *Let $p, N \in \mathbb{N}$ with $p \leq N$. Consider a nonlinear map*

$$\Phi_{\leq N}(U) := \mathbf{B}_{\leq N}(U)U, \tag{4.4.1}$$

where

(i) $\mathbf{B}_{\leq N}(U) - \mathrm{Id}$ *is a matrix of pluri–homogeneous spectrally localized maps in* $\Sigma_p^N \widetilde{\mathcal{S}}_q \otimes \mathcal{M}_2(\mathbb{C})$*;*

(ii) $\mathbf{B}_{\leq N}(U)$ *is linearly symplectic up to homogeneity N (Definition 4.3.7).*

Then there exists a real-to-real map

$$\mathcal{C}_{\leq N}(W) = W + R_{\leq N}(W)W \quad \textit{with} \quad R_{\leq N}(W) \in \Sigma_p^N \widetilde{\mathcal{R}}_q^{-\varrho} \otimes \mathcal{M}_2(\mathbb{C}), \ \textit{for any } \varrho \geq 0, \tag{4.4.2}$$

such that the Darboux correction

$$\mathcal{D}_N(U) := \big(\mathcal{C}_{\leq N} \circ \Phi_{\leq N}\big)(U) = \big(\mathrm{Id} + R_{\leq N}(\Phi_{\leq N}(U))\big)\Phi_{\leq N}(U) \tag{4.4.3}$$

is symplectic up to homogeneity N, according to Definition 4.3.11.

Remark 4.4.2. The first assumption implies that the operator in (4.4.7) is smoothing for any $\varrho > 0$. This fact and the second assumption allow to deduce that the vector field representing the perturbed symplectic 1-form $\theta_{\leq N}$ in (4.4.10) is a smoothing perturbation of $E_c V$, see (4.4.11). These properties are crucial to guarantee that the vector field $Y^{\tau}(V)$ solving the Darboux equation (4.4.26) is smoothing (see Lemma 4.4.9), which in turn implies that the Darboux corrector $\mathcal{C}_{\leq N}(W)$ in (4.4.2) is a smoothing perturbation of the identity.

The rest of this section is devoted to the proof of Theorem 4.4.1.

In order to correct the nonlinear map $\Phi_{\leq N}$ defined in (4.4.1) we develop a perturbative Darboux procedure to construct a nearby symplectic map up to homogeneity N. The map $\Phi_{\leq N}$ induces the nonstandard symplectic 2-form

$$\Omega_{\leq N} := \Psi_{\leq N}^{*} \, \Omega_c \tag{4.4.4}$$

where $\Psi_{\leq N}$ is the approximate inverse of $\Phi_{\leq N}$ defined by Lemma 4.2.24 and Ω_c is the standard symplectic form in (4.3.12). The next lemma describes properties of the approximate inverse $\Psi_{\leq N}$.

Lemma 4.4.3. (Approximate inverse) *The approximate inverse up to homogeneity N of the map $\Phi_{\leq N}(U) = \mathbf{B}_{\leq N}(U)U$ defined in (4.4.1) has the form*

$$\Psi_{\leq N}(V) = \mathbf{A}_{\leq N}(V)V \tag{4.4.5}$$

where

(i) $\mathbf{A}_{\leq N}(V) - \mathrm{Id}$ *is a matrix of pluri-homogeneous spectrally localized maps in* $\Sigma_p^N \widetilde{\mathcal{S}}_q \otimes \mathcal{M}_2(\mathbb{C})$*;*

(ii) $\mathbf{A}_{\leq N}(V)$ *is linearly symplectic up to homogeneity N (Definition 4.3.7), more precisely (4.3.33) holds.*

In addition

$$\mathrm{d}_V \Psi_{\leq N}(V) = \mathbf{A}_{\leq N}(V) + \mathbf{G}_{\leq N}(V), \qquad \mathbf{G}_{\leq N}(V)\widehat{V} := \mathrm{d}_V \mathbf{A}_{\leq N}(V)[\widehat{V}]V, \tag{4.4.6}$$

and

(iii) $\mathbf{G}_{\leq N}(V)$ *is a matrix of pluri–homogeneous operators in* $\Sigma_p^N \widetilde{\mathcal{M}}_q \otimes \mathcal{M}_2(\mathbb{C})$*;*

(iv) the transposed operator

$$\mathbf{G}^\top_{\leq N}(V) := [\mathbf{G}_{\leq N}(V)]^\top \in \Sigma_p^N \widetilde{\mathcal{R}}_q^{-\varrho} \otimes \mathcal{M}_2(\mathbb{C}) \tag{4.4.7}$$

is a matrix of ϱ-smoothing operators for arbitrary $\varrho \geq 0$.

Proof. Items (i) and (ii) are proved in Lemma 4.3.8, and (iii) follows by the fifth bullet after Definition 4.2.16. Finally (iv) follows applying Lemma 4.2.21 to each spectrally localized map $\mathcal{P}_q(\mathbf{A}_{\leq N}(V))$ for $q = p, \ldots, N$ (with $U \rightsquigarrow V$ and $V \rightsquigarrow \widehat{V}$). □

We now compute $\Omega_{\leq N}$.

Lemma 4.4.4. (Non-standard symplectic form $\Omega_{\leq N}$) *The symplectic 2-form $\Omega_{\leq N} = \Psi^*_{\leq N}\, \Omega_c$ in (4.4.4) is represented as $\Omega_{\leq N}(V) = \langle E_{\leq N}(V)\cdot, \cdot\rangle_r$ with symplectic tensor*

$$E_{\leq N}(V) = E_c + \mathbf{A}^\top_{\leq N}(V)E_c\mathbf{G}_{\leq N}(V) + \mathbf{G}^\top_{\leq N}(V)E_c\mathbf{A}_{\leq N}(V) + \mathbf{G}^\top_{\leq N}(V)E_c\mathbf{G}_{\leq N}(V) + \mathbf{S}_{>N}(V) \tag{4.4.8}$$

where

(i) $\mathbf{G}^\top_{\leq N}(V)E_c\mathbf{A}_{\leq N}(V)$ *and* $\mathbf{G}^\top_{\leq N}(V)E_c\mathbf{G}_{\leq N}(V)$ *are matrices of pluri–homogeneous smoothing operators in* $\Sigma_p^{2N} \widetilde{\mathcal{R}}_q^{-\varrho} \otimes \mathcal{M}_2(\mathbb{C})$ *for any $\varrho \geq 0$;*

(ii) $\mathbf{S}_{>N}(V) := \mathbf{A}^\top_{\leq N}(V)E_c\mathbf{A}_{\leq N}(V) - E_c$ *is a matrix of pluri–homogeneous spectrally localized maps in* $\Sigma_{N+1} \widetilde{\mathcal{S}}_q \otimes \mathcal{M}_2(\mathbb{C})$.

Moreover

$$\Omega_{\leq N} = \mathrm{d}\theta_{\leq N}, \tag{4.4.9}$$

where the 1-form

$$\theta_{\leq N} := \Psi^*_{\leq N}\, \theta_c, \qquad \theta_c(V) := \frac{1}{2}\big\langle E_c V, \cdot \big\rangle_r, \tag{4.4.10}$$

has the form

$$\theta_{\leq N}(V) = \frac{1}{2}\Big\langle [Z_{\leq N}(V) + \mathbf{S}_{>N}(V)]V, \cdot \Big\rangle_r \quad \textit{with} \quad Z_{\leq N}(V) := E_c + \mathbf{G}^\top_{\leq N}(V)\, E_c\, \mathbf{A}_{\leq N}(V). \tag{4.4.11}$$

Proof. By (4.2.161) we have that

$$\Omega_{\leq N}(V)[X,Y] = \left\langle \mathrm{d}\Psi_{\leq N}(V)^\top E_c \mathrm{d}\Psi_{\leq N}(V)X, Y\right\rangle_r$$

which, using (4.4.6) and the fact that $\mathbf{A}_{\leq N}(V)$ is linearly symplectic up to homogeneity N (cfr. (4.3.33)), provides formula (4.4.8). Then items (i)-(ii) follow by (4.4.6), (4.4.7) and Proposition 4.2.15. The identity (4.4.9) follows by (4.3.15) and (4.2.157). Finally (4.4.11) follows similarly computing $\Psi^*_{\leq N}\theta_c$ by (4.2.160),

$$\theta_{\leq N}(V)[X] = \frac{1}{2}\left\langle \mathrm{d}_V \Psi_{\leq N}(V)^\top E_c \Psi_{\leq N}(V), X\right\rangle,$$

and using (4.4.5), (4.4.6) and $\mathbf{A}^\top_{\leq N}(V)E_c\mathbf{A}_{\leq N}(V) = E_c + \mathbf{S}_{>N}(V)$. □

The key step is to implement a Darboux–type procedure to transform the symplectic form $\Omega_{\leq N}$ back to the standard symplectic form Ω_c up to arbitrary high degree of homogeneity. It turns out that the required transformation is a smoothing perturbation of the identity as claimed in Theorem 4.4.1, see Proposition 4.4.7. This is not at all obvious, since in the expression (4.4.8) of $E_{\leq N}(V)$ the second operator $\mathbf{A}^\top_{\leq N}(V)E_c\mathbf{G}_{\leq N}(V)$ is *not* smoothing. However it has a nice structure that we now describe.

Lemma 4.4.5. *Let $X(V)$ be a pluri-homogeneous vector field in $\Sigma_{p'+1}\widetilde{\mathfrak{X}}_q$ for some $p' \in \mathbb{N}_0$. Then*

$$\mathbf{A}^\top_{\leq N}(V)E_c\mathbf{G}_{\leq N}(V)[X(V)] = \nabla\mathcal{W}(V) + R(V)V + M_{>N}(V)V \tag{4.4.12}$$

where

- *$\mathcal{W}(V)$ is 0-form in $\Sigma_{p+p'+2}\widetilde{\Lambda}^0_q$;*
- *$R(V)$ is matrix of pluri–homogeneous smoothing operators in $\Sigma_{p+p'}\widetilde{\mathcal{R}}^{-\varrho}_q \otimes \mathcal{M}_2(\mathbb{C})$ for any $\varrho \geq 0$;*
- *$M_{>N}(V)$ is a matrix of pluri–homogeneous operators in $\Sigma_{N+1}\widetilde{\mathcal{M}}_q \otimes \mathcal{M}_2(\mathbb{C})$.*

Proof. For simplicity of notation we set $\mathbf{A}(V) := \mathbf{A}_{\leq N}(V)$ and $\mathbf{G}(V) := \mathbf{G}_{\leq N}(V)$.
STEP 1: *For any vector W, the linear operator*

$$\mathbf{K}_W(V) := \mathbf{A}(V)^\top E_c\left[\mathrm{d}_V\mathbf{A}(V)[W]\right] \tag{4.4.13}$$

is symmetric up to homogeneity N, precisely

$$\mathbf{K}_W(V) - \mathbf{K}_W(V)^\top = \mathrm{d}_V\mathbf{S}_{>N}(V)[W] \tag{4.4.14}$$

where $\mathbf{S}_{>N}(V)$ is the spectrally localized map in $\Sigma_{N+1}\widetilde{\mathcal{S}}_q \otimes \mathcal{M}_2(\mathbb{C})$ of Lemma 4.4.4.
Indeed, differentiating the relation $\mathbf{A}(V)^\top E_c\mathbf{A}(V) = E_c + \mathbf{S}_{>N}(V)$ (see Lemma 4.4.4 (ii)), in direction W, we get

$$\begin{aligned}\mathrm{d}_V\mathbf{S}_{>N}(V)[W] &= \mathbf{A}(V)^\top E_c\left[\mathrm{d}_V\mathbf{A}(V)[W]\right] + \left[\mathrm{d}_V\mathbf{A}(V)[W]\right]^\top E_c\mathbf{A}(V)\\ &= \mathbf{A}(V)^\top E_c\left[\mathrm{d}_V\mathbf{A}(V)[W]\right] - \Big(\mathbf{A}(V)^\top E_c\left[\mathrm{d}_V\mathbf{A}(V)[W]\right]\Big)^\top\end{aligned}$$

proving, in view of (4.4.13), (4.4.14).
STEP 2: *The linear operator*

$$\mathbf{K}(V) := \mathbf{K}_{X(V)}(V) = \mathbf{A}(V)^\top E_c\big[\mathrm{d}_V\mathbf{A}(V)[X(V)]\big] \tag{4.4.15}$$

can be decomposed as

$$\mathbf{K}(V) = S(V) + \breve{R}(V) + M_{>N}(V) \tag{4.4.16}$$

where

- $S(V)$ *is a symmetric matrix of spectrally localized pluri–homogeneous maps in* $\Sigma_{p+p'}\widetilde{\mathcal{S}}_q \otimes \mathcal{M}_2(\mathbb{C})$*;*
- $\breve{R}(V)$ *is a symmetric matrix of pluri–homogeneous smoothing operators in* $\Sigma_{p+p'}\widetilde{\mathcal{R}}_q^{-\varrho} \otimes \mathcal{M}_2(\mathbb{C})$ *for any* $\varrho \geq 0$ *as well as its transpose;*
- $M_{>N}(V) := \frac{1}{2}\mathrm{d}_V \mathbf{S}_{>N}(V)[X(V)]$ *is a matrix of pluri–homogeneous operators in* $\Sigma_{N+1}\widetilde{\mathcal{M}}_q \otimes \mathcal{M}_2(\mathbb{C})$*.*

We apply Lemma 4.2.20 to each component $\mathbf{A}_q(V) := \mathcal{P}_q[\mathbf{A}(V)]$, $q = p, \ldots, N$, each of which is a map in $\widetilde{\mathcal{S}}_q \otimes \mathcal{M}_2(\mathbb{C})$, see Lemma 4.4.3 (i). Lemma 4.2.20 (with $M(U)U \rightsquigarrow X(V)$) gives the decomposition

$$\mathrm{d}_V \mathbf{A}(V)[X(V)] = \sum_{q=p}^{N} q\mathbf{A}_q\big(X(V), V, \ldots, V\big) = S'(V) + R'(V), \tag{4.4.17}$$

where $S'(V)$ is a matrix of spectrally localized pluri–homogeneous maps in $\Sigma_{p+p'}\widetilde{\mathcal{S}}_q \otimes \mathcal{M}_2(\mathbb{C})$ and $R'(V)$ is a matrix of pluri–homogeneous smoothing operators in $\Sigma_{p+p'}\widetilde{\mathcal{R}}_q^{-\varrho} \otimes \mathcal{M}_2(\mathbb{C})$ for any $\varrho \geq 0$, as well as its transpose. Then we obtain by (4.4.14) (with $W = X(V)$),

$$\begin{aligned}\mathbf{K}(V) &= \frac{1}{2}(\mathbf{K}(V) + \mathbf{K}(V)^\top) + \frac{1}{2}\mathrm{d}_V \mathbf{S}_{>N}(V)[X(V)] \\ &\overset{(4.4.15),(4.4.17)}{=} \underbrace{\frac{1}{2}\big[\mathbf{A}^\top E_c S' - S'^\top E_c \mathbf{A}\big](V)}_{:=S(V),\ S(V)=S(V)^\top} + \underbrace{\frac{1}{2}\big[\mathbf{A}^\top E_c R' - R'^\top E_c \mathbf{A}\big](V)}_{:=\breve{R}(V),\ \breve{R}(V)=\breve{R}(V)^\top} + M_{>N}(V),\end{aligned}$$

where $M_{>N}(V) = \frac{1}{2}\mathrm{d}_V \mathbf{S}_{>N}(V)[X(V)]$ is in $\Sigma_{N+1}\widetilde{\mathcal{M}}_q \otimes \mathcal{M}_2(\mathbb{C})$ by Proposition 4.2.15. Since the maps $S'(V)$ and $\mathbf{A}(V)$ are spectrally localized then $S(V)$ belongs to $\Sigma_{p+p'}\widetilde{\mathcal{S}}_q \otimes \mathcal{M}_2(\mathbb{C})$ by Proposition 4.2.19 (ii) and Lemma 4.2.18. The operator $\breve{R}(V)$ is in $\Sigma_{p+p'}\widetilde{\mathcal{R}}_q^{-\varrho} \otimes \mathcal{M}_2(\mathbb{C})$ for any $\varrho \geq 0$ by Proposition 4.2.19 (i), Lemma 4.2.18 and the fact that $R'(V)$ belongs $\Sigma_{p+p'}\widetilde{\mathcal{R}}_q^{-\varrho} \otimes \mathcal{M}_2(\mathbb{C})$ for any $\varrho \geq 0$ as well as its transpose.
CONCLUSION: The identity (4.4.12) follows by Step 2 defining

$$\mathcal{W}(V) := \frac{1}{2}\big\langle S(V)V, V\big\rangle_r, \quad S(V) \text{ in } (4.4.16) .$$

Indeed by Lemma 4.3.18, $\nabla \mathcal{W}(V) = S(V)V + R''(V)V$ for some smoothing operator $R''(V)$ belonging to $\widetilde{\mathcal{R}}_{p+p'}^{-\varrho} \otimes \mathcal{M}_2(\mathbb{C})$ for any $\varrho > 0$. Then we get

$$\begin{aligned}\mathbf{A}_{\leq N}^\top(V) E_c \mathbf{G}_{\leq N}(V)[X(V)] &\overset{(4.4.12),(4.4.6)}{=} \mathbf{K}(V)V \overset{(4.4.16)}{=} S(V)V + \breve{R}(V)V + M_{>N}(V)V \\ &= \nabla \mathcal{W}(V) - R''(V)V + \breve{R}(V)V + M_{>N}(V)V ,\end{aligned}$$

proving (4.4.12) with $R(V) := \breve{R}(V) - R''(V)$ which belongs to $\Sigma_{p+p'}\widetilde{\mathcal{R}}_q^{-\varrho} \otimes \mathcal{M}_2(\mathbb{C})$ for any $\varrho \geq 0$. □

Remark 4.4.6. The vector field $R(V)V$ in (4.4.12) depends on $X(V)$ and its differential $\mathrm{d}_V X(V)$. This is because $S(V)$ depends linearly on $X(V)$ (actually it is the term $S'(V)$ that depends linearly on $X(V)$, see (4.4.17)). Hence the smoothing vector field $R''(V)V$ coming from the gradient $\nabla \mathcal{W}(V)$, given explicitly by $R''(V)V := L(V)^\top V$ where $L(V)W = \frac{1}{2}\mathrm{d}_V S(V)[W]V$ (see (4.3.60)), depends on the differential of $X(V)$.

Now we present the main Darboux procedure.

Proposition 4.4.7. **(Darboux procedure)** *There exists a τ–dependent pluri-homogeneous smoothing vector field $Y^\tau(V)$ in $\Sigma_{p+1}\widetilde{\mathfrak{X}}^{-\varrho}_{p+1}$, for any $\varrho \geq 0$, defined for $\tau \in [0,1]$, such that its approximate time 1–flow*

$$\mathcal{F}_{\leq N}(V) = V + R_{\leq N}(V)V \quad \textit{with} \quad R_{\leq N}(V) \in \Sigma_p^N \widetilde{\mathcal{R}}^{-\varrho}_q \otimes \mathcal{M}_2(\mathbb{C}), \ \forall \varrho \geq 0, \tag{4.4.18}$$

(given by Lemma 4.2.28) satisfies

$$\mathcal{F}^*_{\leq N}\Omega_{\leq N} = \Omega_c + \Omega_{>N} \tag{4.4.19}$$

where $\Omega_{>N}$ is a pluri-homogeneous 2-form in $\Sigma_{N+1}\widetilde{\Lambda}^2_q$.

Proof. We follow the famous deformation argument by Moser. We define the homothety between the symplectic 2-forms Ω_c and $\Omega_{\leq N}$ defined in (4.4.4) by setting

$$\Omega^\tau := \Omega_c + \tau(\Omega_{\leq N} - \Omega_c), \quad \forall \tau \in [0,1]. \tag{4.4.20}$$

Equivalently $\Omega^\tau = \langle E^\tau(V)\cdot, \cdot\rangle_r$ with associated symplectic tensor

$$E^\tau(V) = E_c + \tau(E_{\leq N}(V) - E_c) \tag{4.4.21}$$

$$\overset{(4.4.8)}{=} E_c + \tau \mathbf{R}_{\leq N}(V) + \tau \mathbf{A}^\top_{\leq N}(V)\, E_c\, \mathbf{G}_{\leq N}(V) + \tau \mathbf{S}_{>N}(V) \tag{4.4.22}$$

where

- $\mathbf{R}_{\leq N}(V)$ is the matrix of pluri–homogeneous smoothing operators

$$\mathbf{R}_{\leq N}(V) := \mathbf{G}^\top_{\leq N}(V)\, E_c \mathbf{A}_{\leq N}(V) + \mathbf{G}^\top_{\leq N}(V)\, E_c\, \mathbf{G}_{\leq N}(V)$$

belonging to $\Sigma_p \widetilde{\mathcal{R}}^{-\varrho}_q \otimes \mathcal{M}_2(\mathbb{C})$ for any $\varrho \geq 0$;
- $\mathbf{S}_{>N}(V)$ is the map in $\Sigma_{N+1}\widetilde{\mathcal{S}}_q \otimes \mathcal{M}_2(\mathbb{C})$ of Lemma 4.4.4.

In addition, by (4.4.9) and (4.3.15), we have

$$\Omega^\tau = \mathrm{d}\theta^\tau, \quad \theta^\tau := \theta_c + \tau(\theta_{\leq N} - \theta_c) \tag{4.4.23}$$

where $\theta_{\leq N}$ is given in (4.4.11).

We look for a τ–dependent pluri-homogeneous smoothing vector field $Y^\tau(V)$ in $\Sigma_{p+1}\widetilde{\mathfrak{X}}^{-\varrho}_q$, for any $\varrho \geq 0$, such that its approximate flow $\mathcal{F}^\tau_{\leq N}$ up to homogeneity N (defined by Lemma 4.2.28), satisfies

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(\mathcal{F}^\tau_{\leq N})^*\Omega^\tau = \mathrm{d}\theta^\tau_{>N+1}, \quad \forall \tau \in [0,1], \tag{4.4.24}$$

for a certain 1-form $\theta^\tau_{>N+1}$ in $\Sigma_{N+2}\widetilde{\Lambda}^1_q$. Then, integrating (4.4.24) and recalling (4.4.20), we deduce

$$(\mathcal{F}^1_{\leq N})^*\Omega_{\leq N} = \Omega_c + \int_0^1 \mathrm{d}\theta^\tau_{>N+1}\,\mathrm{d}\tau,$$

which proves (4.4.19) with $\mathcal{F}_{\leq N} := \mathcal{F}^1_{\leq N}$ and $\Omega_{>N} := \int_0^1 \mathrm{d}\theta^\tau_{>N+1}\,\mathrm{d}\tau$.

We now construct the vector field $Y^\tau(V)$. Using the definition of Lie derivative and the Cartan magic formula, we derive the chain of identities

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tau}(\mathcal{F}^\tau_{\leq N})^*\Omega^\tau &\overset{(4.4.23),(4.2.157)}{=} \mathrm{d}\Big(\frac{\mathrm{d}}{\mathrm{d}\tau}(\mathcal{F}^\tau_{\leq N})^*\theta^\tau\Big) \\
&\overset{(4.2.162)}{=} \mathrm{d}\Big((\mathcal{F}^\tau_{\leq N})^*(\mathcal{L}_{Y^\tau}\theta^\tau + \frac{\mathrm{d}}{\mathrm{d}\tau}\theta^\tau)\Big) + \mathrm{d}\tilde{\theta}^\tau_{>N+1} \\
&\overset{(4.2.159),(4.2.157)}{=} (\mathcal{F}^\tau_{\leq N})^*\mathrm{d}\big((i_{Y^\tau}\circ \mathrm{d} + \mathrm{d}\circ i_{Y^\tau})\theta^\tau + \theta_{\leq N} - \theta_c\big) + \mathrm{d}\tilde{\theta}^\tau_{>N+1} \\
&\overset{(4.4.23),(4.2.156)}{=} (\mathcal{F}^\tau_{\leq N})^*\mathrm{d}\big(i_{Y^\tau}\Omega^\tau + \theta_{\leq N} - \theta_c\big) + \mathrm{d}\tilde{\theta}^\tau_{>N+1}
\end{aligned} \tag{4.4.25}$$

where $\tilde{\theta}^{\tau}_{>N+1}$ is a 1-form in $\Sigma_{N+2}\widetilde{\Lambda}^1_q$ by Lemma 4.2.33. We look for a vector field $Y^{\tau}(V)$ and a 0-form $\mathcal{W}^{\tau}(V)$ such that

$$i_{Y^{\tau}}\Omega^{\tau} + \theta_{\leq N} - \theta_c = \breve{\theta}^{\tau}_{>N+1} + \mathrm{d}\mathcal{W}^{\tau}, \tag{4.4.26}$$

for some pluri–homogeneous 1–form $\breve{\theta}^{\tau}_{>N+1}$ in $\Sigma_{N+2}\widetilde{\Lambda}^1_q$. If (4.4.26) holds, then, in view of (4.4.25), equation (4.4.24) is satisfied with

$$\theta^{\tau}_{>N+1} := (\mathcal{F}^{\tau}_{\leq N})^{*}\breve{\theta}^{\tau}_{>N+1} + \tilde{\theta}^{\tau}_{>N+1} \in \Sigma_{N+2}\widetilde{\Lambda}^1_q.$$

We turn to solve equation (4.4.26). Using (4.4.21), (4.4.11), recalling that $\theta_c(V) = \frac{1}{2}\langle E_c V, \cdot\rangle_r$, and writing $\breve{\theta}^{\tau}_{>N+1}(V) = \frac{1}{2}\langle \breve{Z}^{\tau}_{>N}(V)V, \cdot\rangle_r$, we first rewrite (4.4.26) as the equation

$$E^{\tau}(V)Y^{\tau}(V) + \frac{1}{2}\Big(Z_{\leq N}(V)V - E_c V + \mathbf{S}_{>N}(V)V\Big) = \frac{1}{2}\breve{Z}^{\tau}_{>N}(V)V + \nabla\mathcal{W}^{\tau}(V)\,. \tag{4.4.27}$$

Remark 4.4.8. This equation is linear in $Y^{\tau}(V)$. In the works [91, 16, 54, 55, 20, 21] the operator $E^{\tau}(V)$ is a smoothing perturbation of E_c, so is its inverse and the vector field $Y^{\tau}(V)$ is immediately a smoothing vector field. In our case, $E^{\tau}(V)$ is a (possibly) unbounded perturbation of E_c, and its (approximate) inverse is only an m-operator. Hence, the composition of the (approximate) inverse of $E^{\tau}(V)$ with the smoothing operator $Z_{\leq N}(V) - E_c$ (see (4.4.11)) is only an m-operator, not a smoothing one (see the bullet at pag. 124). Therefore we cannot directly conclude that $Y^{\tau}(V)$ is a smoothing vector field. We proceed differently and solve the equation (4.4.27) in homogeneity, exploiting the freedom given by the function $\nabla\mathcal{W}^{\tau}(V)$ to remove the non-smoothing components of the equation, thanks to structural Lemma 4.4.5.

By (4.4.22) and (4.4.11), equation (4.4.27) becomes

$$\begin{aligned} E_c Y^{\tau}(V) = &- \tfrac{1}{2}\mathbf{G}^{\top}_{\leq N}(V)\, E_c\, \mathbf{A}_{\leq N}(V)V - \tau\mathbf{R}_{\leq N}(V)Y^{\tau}(V) \\ &- \tau\mathbf{A}^{\top}_{\leq N}(V)E_c\mathbf{G}_{\leq N}(V)Y^{\tau}(V) + \nabla\mathcal{W}^{\tau}(V) \\ &+ \tfrac{1}{2}\breve{Z}^{\tau}_{>N}(V)V - \tfrac{1}{2}\mathbf{S}_{>N}(V)V - \tau\mathbf{S}_{>N}(V)Y^{\tau}(V)\,. \end{aligned} \tag{4.4.28}$$

We now solve (4.4.28) for a smoothing vector field $Y^{\tau}(V)$, a suitable function $\mathcal{W}^{\tau}(V)$ and a high homogeneity pluri–homogeneous map $\breve{Z}^{\tau}_{>N}(V)$ by an iterative procedure in increasing order of homogeneity. Note that $\mathbf{G}^{\top}_{\leq N}(V)\, E_c\, \mathbf{A}_{\leq N}(V)$ and $\mathbf{R}_{\leq N}(V)$ are smoothing operators unlike $\mathbf{A}^{\top}_{\leq N}(V)E_0\mathbf{G}_{\leq N}(V)Y^{\tau}(V)$ that will be canceled using $\nabla\mathcal{W}^{\tau}(V)$, thanks to the structure property explicated in Lemma 4.4.5.

Lemma 4.4.9. *Fix $\overline{N} \in \mathbb{N}$ such that $(\overline{N}+2)p \geq N+1$. There exist*

- *a pluri-homogeneous smoothing vector field $Y^{\tau}(V) = \sum_{a=0}^{\overline{N}} Y^{\tau}_{(a)}(V)$, defined for any $\tau \in [0,1]$, with $Y^{\tau}_{(a)}(V)$ in $\Sigma_{(a+1)p+1}\widetilde{\mathfrak{X}}^{-\varrho}_q$ for any $\varrho \geq 0$, uniformly in $\tau \in [0,1]$;*
- *a pluri-homogeneous Hamiltonian $\mathcal{W}^{\tau}(V) = \sum_{a=0}^{\overline{N}} \mathcal{W}^{\tau}_{(a)}(V)$, defined for any $\tau \in [0,1]$, with $\mathcal{W}^{\tau}_{(a)}(V)$ in $\Sigma_{(a+1)p+2}\widetilde{\Lambda}^0_q$, uniformly in $\tau \in [0,1]$;*
- *a pluri-homogeneous matrix of operators $\breve{Z}^{\tau}_{>N}(V)$, defined for any $\tau \in [0,1]$, in $\Sigma_{N+1}\widetilde{\mathcal{M}}_q \otimes \mathcal{M}_2(\mathbb{C})$, uniformly in $\tau \in [0,1]$;*

which solve equation (4.4.28).

Proof. We define

$$\begin{cases} Y_{(0)}(V) := -\frac{1}{2}E_c^{-1}\mathbf{G}^{\top}_{\leq N}(V)\, E_c\, \mathbf{A}_{\leq N}(V)V \\ \mathcal{W}^{\tau}_{(0)}(V) := 0 \end{cases} \tag{4.4.29}$$

Note that $Y_{(0)}(V)$ is smoothing vector field in $\Sigma_{p+1}\widetilde{\mathfrak{X}}_q^{-\varrho}$ for any $\varrho \geq 0$, since $\mathbf{G}_{\leq N}^{\top}(V)\ E_c\ \mathbf{A}_{\leq N}(V)$ are smoothing operators in $\Sigma_p\widetilde{\mathcal{R}}_q^{-\varrho} \otimes \mathcal{M}_2(\mathbb{C})$ for any $\varrho \geq 0$, by Lemma 4.4.4-(i).

For $a \geq 0$, we prove the following recursive statements: *there exist a*
(S1)$_a$ *pluri-homogeneous smoothing vector field* $Y_{(a)}^{\tau}(V)$ *belonging to* $\Sigma_{(a+1)p+1}\widetilde{\mathfrak{X}}_q^{-\varrho}$ *for any* $\varrho \geq 0$*;*
(S2)$_a$ *pluri-homogeneous Hamiltonian* $\mathcal{W}_{(a)}^{\tau}(V)$ *in* $\Sigma_{(a+1)p+2}\widetilde{\Lambda}_q^0$*;*
(S3)$_a$ *matrix of pluri-homogeneous operators* $Z_{>N,(a)}^{\tau}(V)$ *in* $\Sigma_{N+1}\widetilde{\mathcal{M}}_q \otimes \mathcal{M}_2(\mathbb{C})$*;*
uniformly in $\tau \in [0,1]$*, with* $(Y_{(0)}^{\tau}(V), \mathcal{W}_{(0)}^{\tau}(V), Z_{>N,(0)}^{\tau}(V))$ *defined in* (4.4.29)*, satisfying, for any* $a \geq 1$,

$$\begin{aligned} E_c Y_{(a)}^{\tau}(V) = &-\tau \mathbf{R}_{\leq N}(V) Y_{(a-1)}^{\tau}(V) \\ &- \tau \mathbf{A}_{\leq N}^{\top}(V) E_c \mathbf{G}_{\leq N}(V)[Y_{(a-1)}^{\tau}(V)] + \nabla \mathcal{W}_{(a)}^{\tau}(V) \\ &+ Z_{>N,(a)}^{\tau}(V)V - \tau \mathbf{S}_{>N}(V) Y_{(a-1)}^{\tau}(V). \end{aligned} \tag{4.4.30}$$

Given $(Y_{(a-1)}^{\tau}(V), \mathcal{W}_{(a-1)}^{\tau}(V), Z_{>N,(a-1)}^{\tau}(V))$ we now prove **(S1)**$_a$-**(S3)**$_a$. Note that the first term in (4.4.30) is a smoothing vector field of homogeneity $(a+1)p+1$ while the first term in the second line of (4.4.30) has homogeneity $(a+1)p+1$ but it is not a smoothing vector field. However by Lemma 4.4.5 we have the decomposition

$$\mathbf{A}_{\leq N}^{\top}(V) E_c \mathbf{G}_{\leq N}(V)[Y_{(a-1)}^{\tau}(V)] = \nabla \breve{\mathcal{W}}_{(a-1)}^{\tau}(V) + \breve{R}_{(a-1)}^{\tau}(V)V + \breve{M}_{>N,(a-1)}^{\tau}(V)V\,,$$

where $\breve{\mathcal{W}}_{(a-1)}^{\tau}(V)$ is a Hamiltonian in $\Sigma_{(a+1)p+2}\widetilde{\Lambda}_q^0$, $\breve{R}_{(a-1)}^{\tau}(V)$ is a pluri–homogeneous smoothing operator in $\Sigma_{(a+1)p}\widetilde{\mathcal{R}}_q^{-\varrho} \otimes \mathcal{M}_2(\mathbb{C})$ and $\breve{M}_{>N,(a-1)}^{\tau}(V)$ is a pluri–homogeneous operator in $\Sigma_{N+1}\widetilde{\mathcal{M}}_q \otimes \mathcal{M}_2(\mathbb{C})$. Then equation (4.4.30) becomes

$$\begin{aligned} E_c Y_{(a)}^{\tau}(V) = &-\tau \mathbf{R}_{\leq N}(V) Y_{(a-1)}^{\tau}(V) - \tau \breve{R}_{(a-1)}^{\tau}(V)V \\ &+ \nabla\big(\mathcal{W}_{(a)}^{\tau}(V) - \tau \breve{\mathcal{W}}_{(a-1)}^{\tau}(V)\big) \\ &+ Z_{>N,(a)}^{\tau}(V)V - \tau \breve{M}_{>N,(a-1)}^{\tau}(V)V - \tau \mathbf{S}_{>N}(V) Y_{(a-1)}^{\tau}(V) \end{aligned}$$

which is solved by

$$\begin{cases} Y_{(a)}^{\tau}(V) := -E_c^{-1}\big[\tau \mathbf{R}_{\leq N}(V) Y_{(a-1)}^{\tau}(V) + \tau \breve{R}_{(a-1)}^{\tau}(V)V\big] \\ \mathcal{W}_{(a)}^{\tau}(V) := \tau \breve{\mathcal{W}}_{(a-1)}^{\tau}(V) \\ Z_{>N,(a)}^{\tau}(V)V := \tau \breve{M}_{>N,(a-1)}^{\tau}(V)V + \tau \mathbf{S}_{>N}(V) Y_{(a-1)}^{\tau}(V) \end{cases}$$

proving **(S1)**$_a$-**(S3)**$_a$.

Summing (4.4.29) and (4.4.30) for any $a = 1, \ldots, \overline{N}$ we find that $Y^{\tau}(V) = \sum_{a=0}^{\overline{N}} Y_{(a)}^{\tau}(V)$ and $\mathcal{W}^{\tau}(V) = \sum_{a=0}^{\overline{N}} \mathcal{W}_{(a)}^{\tau}(V)$ solve (4.4.28) with

$$\frac{1}{2}\breve{Z}_{>N}^{\tau}(V)V := \frac{1}{2}\mathbf{S}_{>N}(V)V + \sum_{a=1}^{\overline{N}} Z_{>N,(a)}^{\tau}(V)V + \tau\big[\mathbf{R}_{\leq N}(V) + \mathbf{A}_{\leq N}^{\top}(V) E_c \mathbf{G}_{\leq N}(V) + \mathbf{S}_{>N}(V)\big] Y_{(\overline{N})}^{\tau}(V)$$

which is an operator in $\Sigma_{N+1}\widetilde{\mathcal{M}}_q \otimes \mathcal{M}_2(\mathbb{C})$, since $(\overline{N}+2)p \geq N+1$. Lemma 4.4.9 is proved. □

The approximate flow up to homogeneity N of the smoothing vector field Y^{τ} defined by Lemma 4.4.9 solves (4.4.24). This concludes the proof of Proposition 4.4.7. □

Proof of Theorem 4.4.1. The map $\mathcal{E}_N := \Psi_{\leq N} \circ \mathcal{F}_{\leq N}$, where $\mathcal{F}_{\leq N}$ is defined in Proposition 4.4.7, fulfills

$$\mathcal{E}_N^* \Omega_c = \mathcal{F}_{\leq N}^* \Psi_{\leq N}^* \Omega_c \overset{(4.4.4)}{=} \mathcal{F}_{\leq N}^* \Omega_{\leq N} \overset{(4.4.19)}{=} \Omega_c + \Omega_{>N}$$

and so $\mathcal{E}_N$ is symplectic up to homogeneity N. We define the map $\mathcal{C}_{\leq N}$ in (4.4.2) as the approximate inverse (given by Lemma 4.2.24) of the nonlinear map $\mathcal{F}_{\leq N}$ in (4.4.18), hence it has the claimed form. Since $\Psi_{\leq N}$ is an approximate inverse of $\Phi_{\leq N}$, the map $\mathcal{D}_N := \mathcal{C}_{\leq N} \circ \Phi_{\leq N}$ is an approximate inverse of $\mathcal{E}_N$, and so it is symplectic up to homogeneity N by Lemma 4.3.13. □

Proof of Theorem 4.1.2. We write the good-unknown of Alinhac (4.1.17) in complex variables $(u, \overline{u})$ induced by the transformation $\mathcal{C}$ defined in (4.3.11), obtaining the real-to-real spectrally localized, linearly symplectic map (according to Definition 4.3.3)

$$\mathcal{G}_c(U) := \mathcal{C}\mathrm{Op}^{BW}\left(\begin{bmatrix} 1 & 0 \\ -B(\mathcal{C}U) & 1 \end{bmatrix}\right)\mathcal{C}^{-1}, \quad (\eta, \psi) = \mathcal{C}U\,,$$

where $B(\eta, \psi)$ is the real function defined in (4.5.14) which, as stated in Lemma 4.5.1, belongs to $\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[r, N]$. Then Theorem 4.1.2 follows by applying Theorem 4.4.1 to the pluri-homogeneous spectrally localized map $\mathbf{B}_{\leq N}(U) = \mathcal{P}_{\leq N}(\mathcal{G}_c(U)) = \mathcal{C}\mathrm{Op}^{BW}\left(\begin{bmatrix} 1 & 0 \\ \mathcal{P}_{\leq N}[-B(\mathcal{C}U)] & 1 \end{bmatrix}\right)\mathcal{C}^{-1}$. □

Part II

Almost global existence of water waves

We now begin the proof of the almost global existence Theorem 4.1.1 for solutions of the gravity-capillary water waves equations (4.1.2) with constant vorticity.

After further describing the Hamiltonian structure of the water waves equations (4.1.2) and diagonalizing the linearized system at the equilibrium, we paralinearize the water waves equations (4.1.2) with constant vorticity, written in the Zakharov-Craig-Sulem (η,ψ) variables, which are the Hamiltonian system (4.1.4) with the non–standard Poisson tensor J_γ. Then we express such paralinearized system in the Wahlén coordinates (η,ζ) in (4.5.2), which coincides with the Hamiltonian system in (4.5.3) in standard Darboux form. Finally we write such paralinearized system in the complex variable U defined in (4.5.6), i.e. (4.5.36). The final system (4.5.37) is *Hamiltonian in the complex sense*, i.e. has the form (4.3.18).

4.5 Paralinearization of the water waves equations

From now on we consider (4.1.2) as a system on (a dense subspace of) the *homogeneous* space $\dot L^2_r \times \dot L^2_r$, namely, denoting $X_\gamma(\eta,\psi)$ the right hand side in (4.1.2), we consider

$$\partial_t(\eta,\psi) = X_\gamma(\Pi_0^\perp \eta,\psi) \tag{4.5.1}$$

where $\Pi_0^\perp$ is the L^2-projector onto the space of functions with zero average. For simplicity of notation we shall not distinguish between (4.5.1) and (4.1.2), which are equivalent via the isometric isomorphism $\Pi_0^\perp$ between $\dot L^2(\mathbb{T};\mathbb{R})$ and $L^2_0(\mathbb{T};\mathbb{R})$. System (4.5.1) is the Hamiltonian system as in (4.1.4) defined on (a dense subspace of) $\dot L^2_r \times \dot L^2_r$ generated by the Hamiltonian $H_\gamma(\Pi_0^\perp\eta,\psi)$, with H_γ in (4.1.5), computing the $\dot L^2_r$-gradients $(\nabla_\eta H_\gamma, \nabla_\psi H_\gamma)$ with respect to the scalar product $\langle\cdot,\cdot\rangle_{\dot L^2_r}$ in (4.2.3) and regarding the Poisson tensor J_γ in (4.1.4) as a linear operator acting in $\dot L^2_r \times \dot L^2_r$. We shall not insist more on this detail.

Wahlén variables. The variables (η,ψ) are not Darboux coordinates, since the Poisson tensor J_γ in (4.1.4) is not the canonical one when $\gamma \neq 0$. Wahlén noted in [115] that, introducing the variable $\zeta := \psi - \frac{\gamma}{2}\partial_x^{-1}\eta$, the coordinates (η,ζ) are canonical coordinates. Precisely, under the linear change of variables

$$\begin{pmatrix}\eta\\ \psi\end{pmatrix} = \mathcal{W}\begin{pmatrix}\eta\\ \zeta\end{pmatrix}, \quad \mathcal{W} := \begin{pmatrix}\mathrm{Id} & 0\\ \frac{\gamma}{2}\partial_x^{-1} & \mathrm{Id}\end{pmatrix}, \quad \mathcal{W}^{-1} = \begin{pmatrix}\mathrm{Id} & 0\\ -\frac{\gamma}{2}\partial_x^{-1} & \mathrm{Id}\end{pmatrix}, \tag{4.5.2}$$

the Poisson tensor J_γ becomes the standard one,

$$\mathcal{W}^{-1} J_\gamma (\mathcal{W}^{-1})^\top = J, \quad J := \begin{pmatrix}0 & \mathrm{Id}\\ -\mathrm{Id} & 0\end{pmatrix},$$

and the Hamiltonian system (4.1.4) assumes the standard Darboux form

$$\partial_t\begin{pmatrix}\eta\\ \zeta\end{pmatrix} = J\begin{pmatrix}\nabla_\eta \mathcal{H}_\gamma(\eta,\zeta)\\ \nabla_\zeta \mathcal{H}_\gamma(\eta,\zeta)\end{pmatrix}, \quad \mathcal{H}_\gamma(\eta,\zeta) := H_\gamma\big(\eta, \zeta + \frac{\gamma}{2}\partial_x^{-1}\eta\big). \tag{4.5.3}$$

Note that the new Hamiltonian $\mathcal{H}_\gamma$ is still translation invariant so is its Hamiltonian vector field.

Linearized equation at the equilibrium. The linearized equations (4.5.3) at the equilibrium $(\eta,\zeta) = (0,0)$ are obtained by conjugating the linearized equations (4.1.2) at $(\eta,\psi) = (0,0)$, namely

$$\partial_t\begin{pmatrix}\eta\\ \zeta\end{pmatrix} = \mathcal{W}^{-1}\begin{pmatrix}0 & G(0)\\ -(g+\kappa D^2) & \gamma G(0)\partial_x^{-1}\end{pmatrix}\mathcal{W} = \begin{pmatrix}\frac{\gamma}{2}G(0)\partial_x^{-1} & G(0)\\ -(g+\kappa D^2 + \frac{\gamma^2}{4}G(0)D^{-2}) & \frac{\gamma}{2}\partial_x^{-1}G(0)\end{pmatrix}\begin{pmatrix}\eta\\ \zeta\end{pmatrix} \tag{4.5.4}$$

where $D := \frac{1}{\mathrm{i}}\partial_x$ and the Dirichlet-Neumann operator $G(0)$ at the flat surface $\eta = 0$ is the Fourier multiplier with symbol

$$\mathtt{G}(\xi) := \begin{cases} \xi \tanh(\mathtt{h}\xi) & 0 < \mathtt{h} < +\infty \\ |\xi| & \mathtt{h} = +\infty \,. \end{cases} \tag{4.5.5}$$

We diagonalize system (4.5.4) introducing the complex variables

$$\begin{gathered} \begin{pmatrix} u \\ \overline{u} \end{pmatrix} := \mathcal{M}^{-1} \begin{pmatrix} \eta \\ \zeta \end{pmatrix}, \\ \mathcal{M} := \frac{1}{\sqrt{2}} \begin{pmatrix} M(D) & M(D) \\ -\mathrm{i}M^{-1}(D) & \mathrm{i}M^{-1}(D) \end{pmatrix}, \quad \mathcal{M}^{-1} := \frac{1}{\sqrt{2}} \begin{pmatrix} M(D)^{-1} & \mathrm{i}M(D) \\ M^{-1}(D) & -\mathrm{i}M(D) \end{pmatrix}, \end{gathered} \tag{4.5.6}$$

where $M(D)$ is the Fourier multiplier

$$M(D) := \left(\frac{G(0)}{g + \kappa D^2 + \frac{\gamma^2}{4} G(0) D^{-2}} \right)^{\frac{1}{4}} . \tag{4.5.7}$$

A direct computation (cfr. Section 2.2. in [30]), using the identities

$$M(D)\big(g + \kappa D^2 + \frac{\gamma^2}{4} G(0) D^{-2}\big) M(D) = \omega(D) = M^{-1}(D) G(0) M^{-1}(D) \tag{4.5.8}$$

where $\omega(D)$ is the Fourier multiplier with symbol

$$\omega(\xi) := \sqrt{\mathtt{G}(\xi)\Big(g + \kappa\xi^2 + \frac{\gamma^2}{4}\frac{\mathtt{G}(\xi)}{\xi^2}\Big)} \tag{4.5.9}$$

(with $\mathtt{G}(\xi)$ defined in (4.5.5)) shows that the variables $(u, \overline{u})$ in (4.5.6) solve the diagonal linear system

$$\partial_t \begin{pmatrix} u \\ \overline{u} \end{pmatrix} = -\mathrm{i}\boldsymbol{\Omega}(D) \begin{pmatrix} u \\ \overline{u} \end{pmatrix}, \quad \boldsymbol{\Omega}(D) := \begin{pmatrix} \Omega(D) & 0 \\ 0 & -\overline{\Omega}(D) \end{pmatrix} \tag{4.5.10}$$

where

$$\Omega(D) := \omega(D) + \mathrm{i}\frac{\gamma}{2} G(0) \partial_x^{-1}, \quad \overline{\Omega}(D) := \omega(D) - \mathrm{i}\frac{\gamma}{2} G(0) \partial_x^{-1} . \tag{4.5.11}$$

The real-to-real system (4.5.10) amounts to the scalar equation

$$\partial_t u = -\mathrm{i}\Omega(D) u \,, \quad u(x) = \frac{1}{\sqrt{2\pi}} \sum_{j \in \mathbb{Z}\setminus\{0\}} u_j \, e^{\mathrm{i}jx} \,,$$

which, written in Fourier basis, decouples in infinitely many harmonic oscillators

$$\partial_t u_j = -\mathrm{i}\Omega_j(\kappa) u_j \,, \quad j \in \mathbb{Z} \setminus \{0\} \,,$$

where

$$\Omega_j(\kappa) := \omega_j(\kappa) + \frac{\gamma}{2}\frac{\mathtt{G}(j)}{j}, \qquad \omega_j(\kappa) := \sqrt{\mathtt{G}(j)\Big(g + \kappa j^2 + \frac{\gamma^2}{4}\frac{\mathtt{G}(j)}{j^2}\Big)} . \tag{4.5.12}$$

Note that the map $j \to \Omega_j(\kappa)$ is not even because of the vorticity term $\frac{\gamma}{2}\frac{\mathtt{G}(j)}{j}$ which is odd.

A fundamental property that we prove in Appendix B is that the linear frequencies $\{\Omega_j(\kappa)\}_{j \in \mathbb{Z}\setminus\{0\}}$ satisfy the non-resonance conditions of Theorem B.0.1. Thus one can think to implement a Birkhoff normal form procedure. Since the water waves equations (4.1.2) are a quasilinear system we first paralinearize them.

Paralinearization of the water waves We denote the horizontal and vertical components of the velocity field at the free interface by

$$V = V(\eta,\psi) := (\partial_x \Phi)(x,\eta(x)) = \psi_x - \eta_x B\,, \tag{4.5.13}$$

$$B = B(\eta,\psi) := (\partial_y \Phi)(x,\eta(x)) = \frac{G(\eta)\psi + \eta_x\psi_x}{1+\eta_x^2}\,. \tag{4.5.14}$$

Lemma 4.5.1. (Water-waves equations in Zakharov-Craig-Sulem variables) *Let $N \in \mathbb{N}_0$ and $\varrho \geq 0$. For any $K \in \mathbb{N}_0$ there exist $s_0, r > 0$ such that, if $(\eta,\psi) \in B^K_{s_0}(I;r)$ solves (4.1.2), then*

$$\begin{aligned}\partial_t \eta &= G(0)\psi + \mathrm{Op}^{\mathrm{BW}}\left(-B(\eta,\psi;x)|\xi| - \mathrm{i}V_\gamma(\eta,\psi;x)\xi + a_0(\eta,\psi;x,\xi)\right)\eta + \mathrm{Op}^{\mathrm{BW}}\left(b_{-1}(\eta;x,\xi)\right)\psi \\ &\quad + R_1(\eta)\psi + R_1'(\eta,\psi)\eta\,, \end{aligned}\tag{4.5.15}$$

$$\begin{aligned}\partial_t \psi &= -(g+\kappa D^2)\eta + \gamma G(0)\partial_x^{-1}\psi + \mathrm{Op}^{\mathrm{BW}}\left(-\kappa \mathtt{f}(\eta;x)\xi^2 - B^2(\eta,\psi;x)|\xi| + c_0(\eta,\psi;x,\xi)\right)\eta \\ &\quad + \mathrm{Op}^{\mathrm{BW}}\left(B(\eta,\psi;x)|\xi| - \mathrm{i}V_\gamma(\eta,\psi;x)\xi + d_0(\eta,\psi;x,\xi)\right)\psi + R_2(\eta,\psi)\psi + R_2'(\eta,\psi)\eta \end{aligned}\tag{4.5.16}$$

where

- *$V_\gamma(\eta,\psi;x) := V(\eta,\psi;x) - \gamma\eta(x)$ is a function in $\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[r,N]$ as well as the functions V, B defined in (4.5.13)-(4.5.14);*
- *a_0, c_0, d_0 are symbols in $\Sigma\Gamma^0_{K,0,1}[r,N]$ and $b_{-1}(\eta;x,\xi)$ is a symbol in $\Sigma\Gamma^{-1}_{K,0,1}[r,N]$ satisfying (4.2.27);*
- *the function $\mathtt{f}(\eta;x) := (1+\eta_x^2(x))^{-\frac{3}{2}} - 1$ belongs to $\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,2}[r,N]$;*
- *R_1, R_1', R_2, R_2' are real smoothing operators in $\Sigma\mathcal{R}^{-\varrho}_{K,0,1}[r,N]$.*

Moreover (4.5.15)–(4.5.16) are the Hamiltonian *system (4.1.4).*

Proof. By Proposition 7.4 of [27], the function B defined in (4.5.14) belongs to $\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[r,N]$, as well as the function V in (4.5.13) and $V_\gamma = V - \gamma\eta$.

PARALINEARIZATION OF THE FIRST EQUATION IN (4.1.2). We use the paralinearization of the Dirichlet-Neumann operator $G(\eta)\psi$ proved in [27]. By Propositions 7.5 and 8.3 in [27] where $\omega := \psi - \mathrm{Op}^{\mathrm{BW}}(B)\eta$ is the "good unknown" of Alinhac, using Propositions 4.2.14 and 4.2.19-(i), the second bullet below (4.2.53), and noting that $\xi\tanh(\mathtt{h}\xi) - |\xi| \in \widetilde{\Gamma}^{-\varrho}_0$, for any $\varrho > 0$, we get

$$\begin{aligned} G(\eta)\psi &= G(0)\big(\psi - \mathrm{Op}^{\mathrm{BW}}(B)\eta\big) + \mathrm{Op}^{\mathrm{BW}}\big(-\mathrm{i}V\xi + \breve{a}_0\big)\eta + \mathrm{Op}^{\mathrm{BW}}(b_{-1})\big(\psi - \mathrm{Op}^{\mathrm{BW}}(B)\eta\big) \\ &\quad + R'(\eta,\psi)\eta + R(\eta)\psi \\ &= G(0)\psi + \mathrm{Op}^{\mathrm{BW}}\big(-B|\xi| - \mathrm{i}V\xi + a_0'\big)\eta + \mathrm{Op}^{\mathrm{BW}}(b_{-1})\psi + R(\eta)\psi + R'(\eta,\psi)\eta \end{aligned}\tag{4.5.17}$$

where $\breve{a}_0, a_0'$ are symbols in $\Sigma\Gamma^0_{K,0,1}[r,N]$, b_{-1} is a symbol in $\Gamma^{-1}_{K,0,1}[r,N]$ depending only on η, and $R(\eta)$, $R'(\eta,\psi)$ are smoothing operators in $\Sigma\mathcal{R}^{-\varrho}_{K,0,1}[r,N]$.

We now paralinearize the term with the vorticity. Using Lemma 4.2.12, Proposition 4.2.14, the identity $\eta_x = \mathrm{Op}^{\mathrm{BW}}(\mathrm{i}\xi)\eta$ and (i) of Proposition 4.2.15 we get

$$\eta\eta_x = \mathrm{Op}^{\mathrm{BW}}(\eta)\mathrm{Op}^{\mathrm{BW}}(\mathrm{i}\xi)\eta + \mathrm{Op}^{\mathrm{BW}}(\eta_x)\eta + R(\eta)\eta = \mathrm{Op}^{\mathrm{BW}}\big(\mathrm{i}\eta\xi + \tfrac{1}{2}\eta_x\big)\eta + R(\eta)\eta \tag{4.5.18}$$

where $R(\eta)$ is a homogeneous smoothing operator in $\widetilde{\mathcal{R}}^{-\varrho}_1$. Then (4.5.17) and (4.5.18) imply (4.5.15) with symbol $a_0 := a_0' + \frac{\gamma}{2}\eta_x$ in $\Sigma\Gamma^0_{K,0,1}[r,N]$. Furthermore, since (4.5.15) is a real equation we may assume that a_0 and b_{-1} satisfy (4.2.27) eventually replacing them with $\frac{1}{2}(a_0 + \overline{a}_0^\vee)$ and $\frac{1}{2}(b_{-1} + \overline{b}_{-1}^\vee)$ and replacing the smoothing remainders with $\frac{1}{2}(R_1+\overline{R}_1)$ and $\frac{1}{2}(R'_1+\overline{R}'_1)$.

PARALINEARIZATION OF THE SECOND EQUATION IN (4.1.2). By Lemma 4.2.12 and Proposition 4.2.14 we get

$$-\frac{1}{2}\psi_x^2 = -\mathrm{Op}^{BW}\big(\mathrm{i}\psi_x\xi - \tfrac{1}{2}\psi_{xx}\big)\psi + R(\psi)\psi \tag{4.5.19}$$

where $R(\psi)$ is a smoothing operator in $\widetilde{\mathcal{R}}_1^{-\varrho}$. Next, recalling (4.5.14) and using Lemma 4.2.12, we get

$$\begin{aligned}\frac{1}{2}\frac{(G(\eta)\psi+\eta_x\psi_x)^2}{(1+\eta_x^2)} &= \frac{1}{2}(1+\eta_x^2)B^2 \\ &= \frac{1}{2}\mathrm{Op}^{BW}\big((1+\eta_x^2)B\big)B + \frac{1}{2}\mathrm{Op}^{BW}(B)\big[(1+\eta_x^2)B\big] + R(\eta,\psi)\psi + R'(\eta,\psi)\eta\end{aligned} \tag{4.5.20}$$

where $R(\eta,\psi)$, $R'(\eta,\psi)$ are smoothing operators in $\Sigma\mathcal{R}^{-\varrho}_{K,0,1}[r,N]$. Consider the second term in the right hand side of (4.5.20). Applying Lemma 4.2.12, Propositions 4.2.14 and 4.2.15 (and since $\mathrm{Op}^{BW}(B)[1]$ is a constant which we neglect because we consider (4.1.2) posed in homogeneous spaces) we get

$$\tfrac{1}{2}\mathrm{Op}^{BW}(B)\big[(1+\eta_x^2)B\big] = \tfrac{1}{2}\mathrm{Op}^{BW}\big((1+\eta_x^2)B\big)B + \mathrm{i}\mathrm{Op}^{BW}\big(B^2\eta_x\xi+\breve{c}_0\big)\eta + R(\eta,\psi)\psi + R'(\eta,\psi)\eta \tag{4.5.21}$$

where $\breve{c}_0$ is a symbol in $\Sigma\Gamma^0_{K,0,1}[r,N]$ and $R(\eta,\psi)$, $R'(\eta,\psi)$ are smoothing operators in $\Sigma\mathcal{R}^{-\varrho}_{K,0,1}[r,N]$. Then by (4.5.20)-(4.5.21) and (4.5.14) we deduce that

$$\frac{1}{2}(1+\eta_x^2)B^2 = \mathrm{Op}^{BW}(B)[G(\eta)\psi+\eta_x\psi_x] - \mathrm{i}\mathrm{Op}^{BW}\big(B^2\eta_x\xi+\breve{c}_0\big)\eta + R(\eta,\psi)\psi + R'(\eta,\psi)\eta\,. \tag{4.5.22}$$

In order to expand this term we first write

$$\eta_x\psi_x = \mathrm{Op}^{BW}\big(\mathrm{i}\eta_x\xi - \tfrac{1}{2}\eta_{xx}\big)\psi + \mathrm{Op}^{BW}\big(\mathrm{i}\psi_x\xi - \tfrac{1}{2}\psi_{xx}\big)\eta + R(\eta)\psi + R'(\psi)\eta \tag{4.5.23}$$

where $R(\eta), R'(\psi)$ are smoothing homogeneous operators in $\widetilde{\mathcal{R}}_1^{-\varrho}$. Finally, using (4.5.17), (4.5.23), Proposition 4.2.15, and exploiting the explicit form (4.5.13) of the function V, we conclude that (4.5.22) is equal to

$$(4.5.20) = \mathrm{Op}^{BW}(-B^2|\xi|+\breve{c}_0)\eta + \mathrm{Op}^{BW}(B|\xi|+\mathrm{i}B\eta_x\xi+\breve{d}_0)\psi + R(\eta,\psi)\psi + R'(\eta,\psi)\eta \tag{4.5.24}$$

where $\breve{c}_0, \breve{d}_0$ are symbols in $\Sigma\Gamma^0_{K,0,1}[r,N]$ and $R(\eta,\psi), R'(\eta,\psi)$ are smoothing operators in $\Sigma\mathcal{R}^{-\varrho}_{K,0,1}[r,N]$.

Next we paralinearize the capillary term

$$\kappa\partial_x\Big[\frac{\eta_x}{(1+\eta_x^2)^{1/2}}\Big] = \kappa\partial_x F(\eta_x)\,, \quad F(t) := \frac{t}{(1+t^2)^{1/2}}\,.$$

The Bony paralinearization formula for the composition (Lemma 3.19 in [27]) and Proposition 4.2.14 imply

$$\begin{aligned}\partial_x F(\eta_x) &= \mathrm{Op}^{BW}(\mathrm{i}\xi)\mathrm{Op}^{BW}\big(F'(\eta_x)\big)\eta_x + R(\eta)\eta = \mathrm{Op}^{BW}\Big(-(1+\eta_x^2)^{-\frac{3}{2}}\xi^2 + c'_0\Big)\eta + R(\eta)\eta \\ &= -D^2\eta - \mathrm{Op}^{BW}\big(\mathtt{f}(\eta;x)\xi^2 + c'_0\big)\eta + R(\eta)\eta\end{aligned} \tag{4.5.25}$$

where c'_0 is symbol in $\Sigma\Gamma^0_{K,0,1}[r,N]$, the function $\mathtt{f}(\eta;x) := (1+\eta_x^2(x))^{-\frac{3}{2}} - 1$ belongs to $\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,2}[r,N]$ and $R(\eta)$ is a smoothing operator in $\Sigma\mathcal{R}^{-\varrho}_{K,0,1}[r,N]$.

Next, by Lemma 4.2.12 and Proposition 4.2.14 we get

$$\eta\psi_x = \mathrm{Op}^{BW}\big(\mathrm{i}\eta\xi - \tfrac{1}{2}\eta_x\big)\psi + \mathrm{Op}^{BW}(\psi_x)\eta + R(\eta)\psi + R'(\psi)\eta \tag{4.5.26}$$

where $R(\eta)$, $R'(\psi)$ are homogeneous smoothing operators in $\widetilde{\mathcal{R}}_1^{-\varrho}$.

Finally using (4.5.17), Propositions 4.2.14 and 4.2.15-(i), and that $\partial_x^{-1} = \mathrm{Op}^{\mathrm{BW}}\left(\frac{1}{\mathrm{i}\xi}\right)$ we get

$$\partial_x^{-1} G(\eta)\psi = G(0)\partial_x^{-1}\psi + \mathrm{Op}^{\mathrm{BW}}(c_{-2})\psi + \mathrm{Op}^{\mathrm{BW}}(d_0')\eta + R(\eta,\psi)\eta + R'(\eta,\psi)\psi \tag{4.5.27}$$

where c_{-2} is a symbol in $\Sigma\Gamma^{-2}_{K,0,1}[r,N]$, the symbol d_0' is in $\Sigma\Gamma^{0}_{K,0,1}[r,N]$ and R, R' are smoothing operators in $\Sigma\mathcal{R}^{-\varrho}_{K,0,1}[r,N]$.

In conclusion, collecting (4.5.19), (4.5.24), (4.5.25), (4.5.26), (4.5.27) and using the explicit form of V in (4.5.13) we deduce the second equation (4.5.16) with symbols $c_0 := \breve{c}_0 - \kappa c_0' + \gamma\psi_x + \gamma d_0'$ and $d_0 := \frac{1}{2}\psi_{xx} + \breve{d}_0 - \frac{\gamma}{2}\eta_x + \gamma c_{-2}$ in $\Sigma\Gamma^{0}_{K,0,1}[r,N]$. Since (4.5.16) is a real equation we may assume that c_0 and d_0 satisfy (4.2.27) arguing as for the first equation. □

Remark 4.5.2. The symbols a_0, c_0, d_0 in (4.5.15)–(4.5.16) can be explicitly computed in terms of V and B (e.g. see [29]). On the other hand the symbol $b_{-1}(\eta;x,\xi)$ is expected to be of order $-\infty$ but in [27] it has been estimated as a symbol of order -1 only.

We write (4.5.15)-(4.5.16) as the system

$$\begin{aligned}\partial_t \begin{pmatrix}\eta\\ \psi\end{pmatrix} &= \begin{pmatrix}0 & G(0)\\ -(g+\kappa D^2) & \gamma G(0)\partial_x^{-1}\end{pmatrix}\begin{pmatrix}\eta\\ \psi\end{pmatrix} \\ &\quad + \mathrm{Op}^{\mathrm{BW}}\left(\begin{bmatrix}-B|\xi| - \mathrm{i}V_\gamma\xi & 0\\ -\kappa \mathtt{f}\xi^2 - B^2|\xi| & B|\xi| - \mathrm{i}V_\gamma\xi\end{bmatrix} + \begin{bmatrix}a_0 & b_{-1}\\ c_0 & d_0\end{bmatrix}\right)\begin{pmatrix}\eta\\ \psi\end{pmatrix} + R(\eta,\psi)\begin{pmatrix}\eta\\ \psi\end{pmatrix}.\end{aligned} \tag{4.5.28}$$

Wahlén coordinates. We now transform system (4.5.28) in the Wahlén coordinates (η,ζ) defined in (4.5.2).

Lemma 4.5.3. (Water-waves equations in Wahlén variables) *Let $N \in \mathbb{N}_0$ and $\varrho \geq 0$. For any $K \in \mathbb{N}_0$ there exist $s_0, r > 0$ such that, if $(\eta,\psi) \in B^K_{s_0}(I;r)$ solves (4.1.2) then $(\eta,\zeta) = \mathcal{W}^{-1}(\eta,\psi)$ defined in (4.5.2) solves*

$$\begin{aligned}\partial_t \begin{pmatrix}\eta\\ \zeta\end{pmatrix} &= \begin{pmatrix}\frac{\gamma}{2}G(0)\partial_x^{-1} & G(0)\\ -(g+\kappa D^2 + \frac{\gamma^2}{4}G(0)D^{-2}) & \frac{\gamma}{2}G(0)\partial_x^{-1}\end{pmatrix}\begin{pmatrix}\eta\\ \zeta\end{pmatrix} \\ &\quad + \mathrm{Op}^{\mathrm{BW}}\left(\begin{bmatrix}-B^{(1)}(\eta,\zeta;x)|\xi| - \mathrm{i}V^{(1)}(\eta,\zeta;x)\xi & 0\\ -\kappa\mathtt{f}(\eta;x)\xi^2 - [B^{(1)}(\eta,\zeta;x)]^2|\xi| & B^{(1)}(\eta,\zeta;x)|\xi| - \mathrm{i}V^{(1)}(\eta,\zeta;x)\xi\end{bmatrix}\right)\begin{pmatrix}\eta\\ \zeta\end{pmatrix} \\ &\quad + \mathrm{Op}^{\mathrm{BW}}\left(\begin{bmatrix}a_0^{(1)}(\eta,\zeta;x,\xi) & b_{-1}(\eta;x,\xi)\\ c_0^{(1)}(\eta,\zeta;x,\xi) & d_0^{(1)}(\eta,\zeta;x,\xi)\end{bmatrix}\right)\begin{pmatrix}\eta\\ \zeta\end{pmatrix} + R(\eta,\zeta)\begin{pmatrix}\eta\\ \zeta\end{pmatrix}\end{aligned} \tag{4.5.29}$$

where

- *$B^{(1)}(\eta,\zeta;x) := B(\mathcal{W}(\eta,\zeta);x)$ and $V^{(1)}(\eta,\zeta;x) := V_\gamma(\mathcal{W}(\eta,\zeta);x)$ are functions in $\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[r,N]$, $\mathtt{f}(\eta;x)$ is the function in $\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,2}[r,N]$ defined in Lemma 4.5.1;*

- *$a_0^{(1)}(\eta,\zeta;x,\xi)$, $c_0^{(1)}(\eta,\zeta;x,\xi)$, $d_0^{(1)}(\eta,\zeta;x,\xi)$ are symbols in $\Sigma\Gamma^0_{K,0,1}[r,N]$ satisfying (4.2.27), and the symbol $b_{-1}(\eta;x,\xi)$ in $\Sigma\Gamma^{-1}_{K,0,1}[r,N]$ is defined in Lemma 4.5.1;*

- *$R(\eta,\zeta)$ is a matrix of real smoothing operators in $\Sigma\mathcal{R}^{-\varrho}_{K,0,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$.*

Moreover system (4.5.29) is the Hamiltonian *system (4.5.3).*

Proof. By (4.5.2) and (4.5.28) one has

$$\partial_t \begin{pmatrix} \eta \\ \zeta \end{pmatrix} = \mathcal{W}^{-1} \begin{pmatrix} 0 & G(0) \\ -(g+\kappa D^2) & \gamma G(0)\partial_x^{-1} \end{pmatrix} \mathcal{W} \begin{pmatrix} \eta \\ \zeta \end{pmatrix} \tag{4.5.30}$$

$$+ \mathcal{W}^{-1} \mathrm{Op}^{\mathrm{BW}} \left(\begin{bmatrix} -B|\xi| - \mathrm{i} V_\gamma \xi & 0 \\ -\kappa \mathtt{f}(\eta)\xi^2 - B^2|\xi| & B|\xi| - \mathrm{i} V_\gamma \xi \end{bmatrix} \right) \mathcal{W} \begin{pmatrix} \eta \\ \zeta \end{pmatrix} \tag{4.5.31}$$

$$+ \mathcal{W}^{-1} \mathrm{Op}^{\mathrm{BW}} \left(\begin{bmatrix} a_0 & b_{-1} \\ c_0 & d_0 \end{bmatrix} \right) \mathcal{W} \begin{pmatrix} \eta \\ \zeta \end{pmatrix} + R(\eta,\zeta) \begin{pmatrix} \eta \\ \zeta \end{pmatrix} \tag{4.5.32}$$

where $R(\eta,\zeta)$ is a matrix of smoothing operators in $\Sigma\mathcal{R}^{-\varrho}_{K,0,1}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$. We now compute the above conjugated operators applying the transformation rule

$$\mathcal{W}^{-1} \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} \mathcal{W} = \begin{pmatrix} \mathcal{A} + \frac{\gamma}{2}\mathcal{B}\partial_x^{-1} & \mathcal{B} \\ \mathcal{C} - \frac{\gamma}{2}\partial_x^{-1}\mathcal{A} - \frac{\gamma^2}{4}\partial_x^{-1}\mathcal{B}\partial_x^{-1} + \frac{\gamma}{2}\mathcal{D}\partial_x^{-1} & \mathcal{D} - \frac{\gamma}{2}\partial_x^{-1}\mathcal{B} \end{pmatrix}. \tag{4.5.33}$$

The operator in the right hand side of (4.5.30) is given in (4.5.4). Then by (4.5.33) and Proposition 4.2.14,

$$(4.5.31) = \mathrm{Op}^{\mathrm{BW}} \left(\begin{bmatrix} -B|\xi| - \mathrm{i} V_\gamma \xi & 0 \\ -\kappa \mathtt{f}(\eta)\xi^2 - B^2|\xi| + \breve{c}_0 & B|\xi| - \mathrm{i} V_\gamma \xi \end{bmatrix} \right) + R(\eta,\zeta), \tag{4.5.34}$$

where the symbol $\breve{c}_0 := \frac{\gamma}{2}\frac{1}{\mathrm{i}\xi} \#_\varrho [B|\xi| + \mathrm{i} V_\gamma \xi] + \frac{\gamma}{2}[B|\xi| - \mathrm{i} V_\gamma \xi] \#_\varrho \frac{1}{\mathrm{i}\xi}$ belongs to $\Sigma\Gamma^0_{K,0,1}[r,N]$ and $R(\eta,\zeta)$ is matrix of smoothing operators in $\Sigma\mathcal{R}^{-\varrho}_{K,0,1}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$.

Finally, by (4.5.33) and Proposition 4.2.14, we deduce that

$$(4.5.32) = \mathrm{Op}^{\mathrm{BW}} \left(\begin{bmatrix} a_0' & b_{-1} \\ c_0' & d_0' \end{bmatrix} \right) + R(\eta,\zeta), \tag{4.5.35}$$

where a_0', c_0', d_0' are symbols in $\Sigma\Gamma^0_{K,0,1}[r,N]$ and $R(\eta,\zeta)$ are smoothing operators in $\Sigma\mathcal{R}^{-\varrho}_{K,0,1}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$. In conclusion, by (4.5.4), (4.5.34), (4.5.35), we deduce that system (4.5.30)-(4.5.32) has the form (4.5.29) with symbols $a_0^{(1)} := a_0'$, $c_0^{(1)} := \breve{c}_0 + c_0'$ and $d_0^{(1)} := d_0'$ evaluated at $(\eta,\psi) = \mathcal{W}(\eta,\zeta)$ which belong to $\Sigma\Gamma^0_{K,0,1}[r,N]$. Since the Wahlén transformation is a real map, we may assume that $a_0^{(1)}, c_0^{(1)}, d_0^{(1)}$ satisfy (4.2.27) arguing as in the previous lemma. □

Remark 4.5.4. The first two matrices of para-differential operators in (4.5.29) have the linear Hamiltonian structure (4.3.9)-(4.3.10). We do not claim that the third matrix of para-differential operators in (4.5.29) has the linear Hamiltonian structure (4.3.9)-(4.3.10). Nevertheless in Lemma 4.5.5 we shall recover the complex linear Hamiltonian structure of $J_c \mathrm{Op}^{\mathrm{BW}}(A_0^{(2)})$, up to homogeneity N, thanks to the abstract Lemma 4.3.20.

Complex coordinates. We now diagonalize the linear part of the system (4.5.29) at $(\eta,\zeta) = (0,0)$ introducing the complex variables

$$U := \begin{pmatrix} u \\ \overline{u} \end{pmatrix} = \mathcal{M}^{-1} \begin{pmatrix} \eta \\ \zeta \end{pmatrix}, \quad \mathcal{M}^{-1} : \dot{H}^{s+\frac14}(\mathbb{T},\mathbb{R}) \times \dot{H}^{s-\frac14}(\mathbb{T},\mathbb{R}) \to \dot{H}^s_{\mathbb{R}}(\mathbb{T},\mathbb{C}^2), \ \forall s \in \mathbb{R}, \tag{4.5.36}$$

where $\mathcal{M}$ is the matrix of Fourier multipliers defined in (4.5.6).

Lemma 4.5.5. (Hamiltonian formulation of the water waves in complex coordinates) *Let $N \in \mathbb{N}_0$ and $\varrho \geq 0$. For any $K \in \mathbb{N}_0$ there exist $s_0, r > 0$ such that, if $(\eta, \psi) \in B^K_{s_0}(I; r)$ is a solution of (4.5.29) then U defined in (4.5.36) solves*

$$\begin{aligned} \partial_t U = J_c \mathrm{Op}^{\mathrm{BW}}\Big(A_{\frac{3}{2}}(U;x)\omega(\xi)\Big)U + \frac{\gamma}{2}G(0)\partial_x^{-1}U \\ + J_c \mathrm{Op}^{\mathrm{BW}}\Big(A_1(U;x,\xi) + A_{\frac{1}{2}}(U;x,\xi) + A_0^{(2)}(U;x,\xi)\Big)U + R(U)U \end{aligned} \tag{4.5.37}$$

where J_c is the Poisson tensor defined in (4.1.22) and

- *$\omega(\xi) \in \widetilde{\Gamma}_0^{\frac{3}{2}}$ is the symbol in (4.5.9);*
- *$A_{\frac{3}{2}}(U;x) \in \Sigma\mathcal{F}^{\mathbb{R}}_{K,0,0}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$ is the matrix of real functions*

$$A_{\frac{3}{2}}(U;x) := \begin{pmatrix} f(U;x) & 1 + f(U;x) \\ 1 + f(U;x) & f(U;x) \end{pmatrix}, \tag{4.5.38}$$

  where $f(U;x) := \frac{1}{2}\mathtt{f}(\mathcal{M}U;x)$ belongs to $\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,2}[r,N]$. Note that $J_c\mathrm{Op}^{\mathrm{BW}}(A_{\frac{3}{2}}\omega(\xi))$ is linearly Hamiltonian according to Definition 4.3.2;

- *$A_1(U;x,\xi) \in \Sigma\Gamma^1_{K,0,1}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$ is the matrix of symbols*

$$A_1(U;x,\xi) := \begin{pmatrix} \mathrm{i}B^{(2)}(U;x)|\xi| & -V^{(2)}(U;x)\xi \\ V^{(2)}(U;x)\xi & -\mathrm{i}B^{(2)}(U;x)|\xi| \end{pmatrix} \tag{4.5.39}$$

  where $B^{(2)}(U;x) := B^{(1)}(\mathcal{M}U;x)$ and $V^{(2)}(U;x) := V^{(1)}(\mathcal{M}U;x)$ are real functions in $\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[r,N]$. Note that $J_c\mathrm{Op}^{\mathrm{BW}}(A_1)$ is linearly Hamiltonian;

- *$A_{\frac{1}{2}}(U;x,\xi) \in \Sigma\Gamma^{\frac{1}{2}}_{K,0,2}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$ is the symmetric matrix of symbols*

$$A_{\frac{1}{2}}(U;x,\xi) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}[B^{(2)}(U;x)]^2|\xi|M^2(\xi) \tag{4.5.40}$$

  where $M(\xi) \in \widetilde{\Gamma}_0^{-\frac{1}{4}}$ is the symbol of the Fourier multiplier $M(D)$ in (4.5.7). Note that $J_c\mathrm{Op}^{\mathrm{BW}}(A_{\frac{1}{2}})$ is linearly Hamiltonian;

- *$A_0^{(2)}(U;x,\xi)$ is a matrix of symbols in $\Sigma\Gamma^0_{K,0,1}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$ and the operator $J_c\mathrm{Op}^{\mathrm{BW}}(A_0^{(2)})$ is linearly Hamiltonian up to homogeneity N according to Definition 4.3.6;*
- *$R(U)$ is a real-to-real matrix of smoothing operators in $\Sigma\mathcal{R}^{-\varrho}_{K,0,1}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$.*

Moreover system (4.5.37) is Hamiltonian in the complex sense*, i.e. has the form (4.3.18).*

Proof. We begin by noting that the operator $\mathcal{M}$ in the change of coordinates (4.5.36) has the form (cfr. (4.5.6))

$$\mathcal{M} = \breve{\mathcal{M}} \circ \mathcal{C}, \qquad \breve{\mathcal{M}} := \begin{pmatrix} M(D) & 0 \\ 0 & M(D)^{-1} \end{pmatrix}$$

where $M(D)$ is the Fourier multiplier in (4.5.7) and $\mathcal{C}$ the matrix in (4.3.11). The operator $\breve{\mathcal{M}}$ is symplectic whereas under the change of variables $\mathcal{C}$ a real Hamiltonian system in standard Darboux form (4.5.3) assumes the standard complex form (4.3.18), see the paragraph at page 145. Therefore U solves a system which is Hamiltonian in the complex sense.

Since (η,ζ) solves (4.5.29), the complex variable U in (4.5.36) solves

$$\partial_t U = \mathcal{M}^{-1}\begin{pmatrix} \frac{\gamma}{2}G(0)\partial_x^{-1} & G(0) \\ -(g+\kappa D^2+\frac{\gamma^2}{4}G(0)D^{-2}) & \frac{\gamma}{2}G(0)\partial_x^{-1}\end{pmatrix}\mathcal{M}U \tag{4.5.41}$$

$$+\mathcal{M}^{-1}\mathrm{Op}^{BW}\left(\begin{bmatrix} -B^{(1)}|\xi|-\mathrm{i}V^{(1)}\xi & 0 \\ -\kappa\mathtt{f}(\eta)\xi^2-[B^{(1)}]^2|\xi| & B^{(1)}|\xi|-\mathrm{i}V^{(1)}\xi\end{bmatrix}\right)\mathcal{M}U \tag{4.5.42}$$

$$+\mathcal{M}^{-1}\mathrm{Op}^{BW}\left(\begin{bmatrix} a_0^{(1)} & b_{-1} \\ c_0^{(1)} & d_0^{(1)}\end{bmatrix}\right)\mathcal{M}U+R(U)U \tag{4.5.43}$$

where $R(U)$ is a real-to-real matrix of smoothing operators in $\Sigma\mathcal{R}^{-\varrho}_{K,0,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$.

The operator in the right hand side of (4.5.41) is computed in (4.5.10)-(4.5.11). In order to compute the conjugated operators in (4.5.42)-(4.5.43), we apply the following transformation rule, where we denote by $M := M(D)$ the Fourier multiplier in (4.5.7) (which satisfies $M(D)=\overline{M}(D)$),

$$\mathcal{M}^{-1}\begin{pmatrix}\mathcal{A}_1 & \mathcal{A}_2\\ \mathcal{A}_3 & \mathcal{A}_4\end{pmatrix}\mathcal{M} \tag{4.5.44}$$
$$\overset{(4.5.6)}{=}\frac{1}{2}\begin{pmatrix} M^{-1}\mathcal{A}_1M+M\mathcal{A}_4M^{-1}+\mathrm{i}M\mathcal{A}_3M-\mathrm{i}M^{-1}\mathcal{A}_2M^{-1} & M^{-1}\mathcal{A}_1M-M\mathcal{A}_4M^{-1}+\mathrm{i}M\mathcal{A}_3M+\mathrm{i}M^{-1}\mathcal{A}_2M^{-1}\\ M^{-1}\mathcal{A}_1M-M\mathcal{A}_4M^{-1}-\mathrm{i}M\mathcal{A}_3M-\mathrm{i}M^{-1}\mathcal{A}_2M^{-1} & M^{-1}\mathcal{A}_1M+M\mathcal{A}_4M^{-1}-\mathrm{i}M\mathcal{A}_3M+\mathrm{i}M^{-1}\mathcal{A}_2M^{-1}\end{pmatrix}.$$

Using (4.5.44) and Proposition 4.2.14 we get that

$$(4.5.42)=\frac{1}{2}\mathrm{Op}^{BW}\left(\begin{bmatrix} a_1 & b_1\\ \overline{b}_1^{\vee} & \overline{a}_1^{\vee}\end{bmatrix}\right)+R(U) \tag{4.5.45}$$

where a_1,b_1 are the symbols

$$\begin{aligned} a_1 &:= M^{-1}(\xi)\#_\varrho\big(-B^{(1)}|\xi|-\mathrm{i}V^{(1)}\xi\big)\#_\varrho M(\xi)+M(\xi)\#_\varrho(B^{(1)}|\xi|-\mathrm{i}V^{(1)}\xi)\#_\varrho M^{-1}(\xi)\\ &\quad+\mathrm{i}M(\xi)\#_\varrho\big(-\kappa\mathtt{f}(\eta)\xi^2-[B^{(1)}]^2|\xi|\big)\#_\varrho M(\xi)\\ b_1 &:= M^{-1}(\xi)\#_\varrho\big(-B^{(1)}|\xi|-\mathrm{i}V^{(1)}\xi\big)\#_\varrho M(\xi)-M(\xi)\#_\varrho(B^{(1)}|\xi|-\mathrm{i}V^{(1)}\xi)\#_\varrho M^{-1}(\xi)\\ &\quad+\mathrm{i}M(\xi)\#_\varrho\big(-\kappa\mathtt{f}(\eta)\xi^2-[B^{(1)}]^2|\xi|\big)\#_\varrho M(\xi)\end{aligned}$$

and $R(U)$ is a real-to-real matrix of smoothing operators in $\Sigma\mathcal{R}^{-\varrho}_{K,0,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$. Noting that

$$\omega(\xi)-\sqrt{\kappa}|\xi|^{\frac{3}{2}}\in\widetilde{\Gamma}_0^{-\frac{1}{2}},\quad M(\xi)-\kappa^{-\frac{1}{4}}|\xi|^{-\frac{1}{4}}\in\widetilde{\Gamma}_0^{-\frac{9}{4}},\quad M^{-1}(\xi)-\kappa^{\frac{1}{4}}|\xi|^{\frac{1}{4}}\in\widetilde{\Gamma}_0^{-\frac{7}{4}}, \tag{4.5.46}$$

so that $\kappa\xi^2M^2(\xi)-\omega(\xi)\in\widetilde{\Gamma}_0^{-\frac{1}{2}}$, we deduce, using also the remarks after Definition 4.2.13, that

$$a_1=-2\mathrm{i}V^{(1)}\xi-\mathrm{i}\mathtt{f}(\eta)\omega(\xi)-\mathrm{i}[B^{(1)}]^2|\xi|M^2(\xi)+a_0'\quad\text{with}\quad a_0'\in\Sigma\Gamma^0_{K,0,1}[r,N], \tag{4.5.47}$$
$$b_1=-2B^{(1)}|\xi|-\mathrm{i}\mathtt{f}(\eta)\omega(\xi)-\mathrm{i}[B^{(1)}]^2|\xi|M^2(\xi)+b_0'\quad\text{with}\quad b_0'\in\Sigma\Gamma^0_{K,0,1}[r,N]. \tag{4.5.48}$$

Finally, noting that $M^{-1}(\xi)\#_\varrho b_{-1}\#_\varrho M^{-1}(\xi)$ belongs to $\Sigma\Gamma^{-\frac{1}{2}}_{K,0,1}[r,N]$, we deduce that

$$(4.5.43)=\mathrm{Op}^{BW}(A_0')+R(U) \tag{4.5.49}$$

where A_0' is a real-to-real matrix of symbols in $\Sigma\Gamma^0_{K,0,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ and $R(U)$ is a real-to-real matrix of smoothing operators in $\Sigma\mathcal{R}^{-\varrho}_{K,0,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$.

In conclusion, by (4.5.10)-(4.5.11), (4.5.45), (4.5.47), (4.5.48), (4.5.49), computing the symbols at $(\eta,\zeta)=\mathcal{M}U$, we deduce that system (4.5.41)-(4.5.43) has the form (4.5.37). Note that the matrices of para-differential operators $J_c\mathrm{Op}^{\mathrm{BW}}(A_{\frac{3}{2}}\omega(\xi))$, $J_c\mathrm{Op}^{\mathrm{BW}}(A_1)$, $J_c\mathrm{Op}^{\mathrm{BW}}(A_{\frac{1}{2}})$ in (4.5.38), (4.5.39), (4.5.40) are linearly Hamiltonian according to (4.3.22), whereas $J_c\mathrm{Op}^{\mathrm{BW}}(A_0^{(2)})$ might not be. Thanks to Lemma 4.3.20 we replace each homogeneous component of $A_{\frac{3}{2}}\omega(\xi)+A_1+A_{\frac{1}{2}}+A_0^{(2)}$ with its symmetrized version, by adding another smoothing operator. Since the symbols with positive orders are unchanged we obtain a new operator $J_c\mathrm{Op}^{\mathrm{BW}}(A_0^{(2)})$ (that we denote in the same way) which is linearly Hamiltonian up to homogeneity N. □

4.6 Block-diagonalization and reduction to constant coefficients

In this section we perform several transformations in order to symmetrize and reduce system (4.5.37) to constant coefficients up to smoothing remainders. In particular we will prove the following:

Proposition 4.6.1. (Reduction to constant coefficients up to smoothing operators) *Let $N\in\mathbb{N}_0$ and $\varrho>3(N+1)$. Then there exists $\underline{K}':=\underline{K}'(\varrho)>0$ such that for any $K\geq\underline{K}'$ there are $s_0>0$, $r>0$ such that for any solution $U\in B^K_{s_0,\mathbb{R}}(I;r)$ of (4.5.37), there exists a real-to-real invertible matrix of spectrally localized maps $\mathbf{B}(U;t)$ such that $\mathbf{B}(U;t)-\mathrm{Id}\in\Sigma\mathcal{S}^{\frac{3}{2}(N+1)}_{K,\underline{K}'-1,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ and the following holds true:*

(i) **Boundedness:** *$\mathbf{B}(U;t)$ and $\mathbf{B}(U;t)^{-1}$ are non–homogeneous maps in $\mathcal{S}^0_{K,\underline{K}'-1,0}[r]\otimes\mathcal{M}_2(\mathbb{C})$, cfr. (4.2.72) with $m=N=0$.*

(ii) **Linear symplecticity:** *The map $\mathbf{B}(U;t)$ is linearly symplectic up to homogeneity N, according to Definition 4.3.7.*

(iii) **Conjugation:** *If U solves (4.5.37) then $W:=\mathbf{B}(U;t)U$ solves*

$$\partial_t W=\mathrm{Op}^{BW}_{\mathtt{vec}}\Big(\mathrm{i}\mathtt{m}_{\frac{3}{2}}(U;t,\xi)\Big)W+R(U;t)W \tag{4.6.1}$$

(recall notation (4.2.24)) where

$$\mathtt{m}_{\frac{3}{2}}(U;t,\xi):=-\Big[(1+\zeta(U))\omega(\xi)+\frac{\gamma}{2}\frac{\mathtt{G}(\xi)}{\xi}+\mathtt{V}(U;t)\xi+\mathtt{b}_{\frac{1}{2}}(U;t)|\xi|^{\frac{1}{2}}+\mathtt{b}_0(U;t,\xi)\Big] \tag{4.6.2}$$

and

- *$\omega(\xi)\in\widetilde{\Gamma}^{\frac{3}{2}}_0$ is the Fourier multiplier defined in (4.5.9);*
- *$\zeta(U)$ is a real function in $\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,2}[r,N]$ independent of x;*
- *$\mathtt{V}(U;t)$ is a real function in $\Sigma\mathcal{F}^{\mathbb{R}}_{K,1,2}[r,N]$ independent of x;*
- *$\mathtt{b}_{\frac{1}{2}}(U;t)$ is a real function in $\Sigma\mathcal{F}^{\mathbb{R}}_{K,2,2}[r,N]$ independent of x;*
- *$\mathtt{b}_0(U;t,\xi)$ is a symbol in $\Sigma\Gamma^0_{K,\underline{K}',2}[r,N]$ independent of x and its imaginary part $\mathrm{Im}\,\mathtt{b}_0(U;t,\xi)$ is in $\Gamma^0_{K,\underline{K}',N+1}[r]$;*
- *$R(U;t)$ is a real-to-real matrix of smoothing operators in $\Sigma\mathcal{R}^{-\varrho+3(N+1)}_{K,\underline{K}',1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$.*

Remark 4.6.2. The symbol $\mathtt{m}_{\frac{3}{2}}(U;t,\xi)$ in (4.6.2) is real valued except for the term $\mathtt{b}_0(U;t,\xi)$ whose imaginary part has order 0 and homogeneity at least $N+1$. Hence system (4.6.1) fulfills energy estimates in $\dot{H}^s(\mathbb{T},\mathbb{C}^2)$, of the type (4.8.15) with $N=0$.

Remark 4.6.3. One can choose $\underline{K}'(\varrho) \geq 3\varrho - 8(N+1)+1$.

The rest of Section 4.6 is devoted to the proof of Proposition 4.6.1. We shall use constantly the identities

$$\omega(\xi) = \sqrt{\kappa}|\xi|^{\frac{3}{2}} + \widetilde{\Gamma}_0^{-\frac{1}{2}}, \quad \omega'(\xi) = \frac{3}{2}\sqrt{\kappa}|\xi|^{\frac{1}{2}} \operatorname{sign}\xi + \widetilde{\Gamma}_0^{-\frac{3}{2}}. \tag{4.6.3}$$

4.6.1 A complex good unknown of Alinhac

In this section we introduce a complex version of the good unknown of Alinhac, whose goal is to diagonalize the matrix of para-differential operators of order 1 in (4.5.37) and remove the para-differential operators of order $\frac{1}{2}$. The complex good unknown that we use coincides at principal order with $\mathcal{M}^{-1}\mathcal{G}_A\mathcal{M}$ where $\mathcal{G}_A$ is the classical good unknown of Alinhac in (4.1.17) and $\mathcal{M}$ is the change of variables in (4.5.6).

Lemma 4.6.4. *Let $N \in \mathbb{N}_0$ and $\varrho > 0$. Then for any $K \in \mathbb{N}$ there are $s_0 > 0$, $r > 0$ such that for any solution $U \in B^K_{s_0,\mathbb{R}}(I;r)$ of (4.5.37), there exists a real-to-real invertible matrix of spectrally localized maps $\mathcal{G}(U)$ satisfying $\mathcal{G}(U) - \mathrm{Id} \in \Sigma\mathcal{S}^0_{K,0,1}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$ and the following holds true:*

(i) **Boundedness:** *$\mathcal{G}(U)$ and its inverse are non–homogeneous maps in $\mathcal{S}^0_{K,0,0}[r] \otimes \mathcal{M}_2(\mathbb{C})$.*

(ii) **Linear symplecticity:** *The map $\mathcal{G}(U)$ is linearly symplectic according to Definition 4.3.3;*

(iii) **Conjugation:** *If U solves (4.5.37) then $V_0 := \mathcal{G}(U)U$ solves*

$$\begin{aligned}\partial_t V_0 = J_c \mathrm{Op}^{\mathrm{BW}}\left(A_{\frac{3}{2}}(U;x)\omega(\xi)\right)V_0 + \frac{\gamma}{2}G(0)\partial_x^{-1}V_0 \\ + \mathrm{Op}^{BW}_{\mathtt{vec}}\left(-\mathrm{i}V^{(2)}(U;x)\xi\right)V_0 + J_c\mathrm{Op}^{\mathrm{BW}}\left(A_0^{(3)}(U;t,x,\xi)\right)V_0 + R(U)V_0\end{aligned} \tag{4.6.4}$$

where

- *the matrix of real functions $A_{\frac{3}{2}}(U;x) \in \Sigma\mathcal{F}^{\mathbb{R}}_{K,0,0}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$ and the real function $V^{(2)}(U;x) \in \Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[r,N]$ are defined in Lemma 4.5.5;*
- *$\omega(\xi) \in \widetilde{\Gamma}_0^{\frac{3}{2}}$ is the symbol defined in (4.5.9);*
- *The matrix of symbols $A_0^{(3)}(U;t,x,\xi)$ belongs to $\Sigma\Gamma^0_{K,1,1}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$ and the operator $J_c\mathrm{Op}^{\mathrm{BW}}(A_0^{(3)})$ is linearly Hamiltonian up to homogeneity N according to Definition 4.3.6;*
- *$R(U)$ is a real-to-real matrix of smoothing operators in $\Sigma\mathcal{R}^{-\varrho}_{K,0,1}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$.*

Proof. We define $\mathcal{G}(U)$ to be the real-to-real map

$$\mathcal{G}(U) := \mathrm{Id} - \frac{\mathrm{i}}{2}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \mathrm{Op}^{\mathrm{BW}}\left(B^{(2)}(U;x)M^2(\xi)\right) \tag{4.6.5}$$

where $B^{(2)}(U;x)$ is the function in $\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[r,N]$ defined in (4.5.39) and $M(\xi) \in \widetilde{\Gamma}_0^{-\frac{1}{4}}$ is the symbol of the Fourier multiplier $M(D)$ defined in (4.5.7). Its inverse and transpose are given by

$$\begin{aligned}\mathcal{G}(U)^{-1} &= \mathrm{Id} + \frac{\mathrm{i}}{2}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \mathrm{Op}^{\mathrm{BW}}\left(B^{(2)}(U;x)M^2(\xi)\right), \\ \mathcal{G}(U)^{\top} &= \mathrm{Id} - \frac{\mathrm{i}}{2}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \mathrm{Op}^{\mathrm{BW}}\left(B^{(2)}(U;x)M^2(\xi)\right).\end{aligned} \tag{4.6.6}$$

By the fourth bullet below Definition 4.2.16 the matrices of para-differential operators $\mathcal{G}(U)^{\pm 1} - \mathrm{Id}$ belong to $\Sigma\mathcal{S}^0_{K,0,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ and item (i) follows. Also (ii) follows by a direct computation using the explicit expressions in (4.6.5) and (4.6.6).

Let us prove item (iii). Since U solves (4.5.37) the variable $V_0 := \mathcal{G}(U)U$ solves

$$\partial_t V_0 = \mathcal{G}(U)\Big[J_c\mathrm{Op}^{BW}\Big(A_{\frac32}\omega(\xi) + A_1 + A_{\frac12} + A_0^{(2)}\Big) + \frac{\gamma}{2}G(0)\partial_x^{-1}\Big]\mathcal{G}(U)^{-1}V_0 + (\partial_t\mathcal{G}(U))\mathcal{G}(U)^{-1}V_0$$
$$+\,\mathcal{G}(U)R(U)\mathcal{G}(U)^{-1}V_0\,. \tag{4.6.7}$$

We now expand each of the above operators. By (4.6.5), the form of J_c in (4.1.22), (4.6.6), the symbolic calculus Proposition 4.2.14, writing $J_cA_{\frac32}\omega(\xi) = -\mathrm{i}\big(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\big)\omega(\xi) - \mathrm{i}\big(\begin{smallmatrix}1 & 1\\ -1 & -1\end{smallmatrix}\big)f(U)\omega(\xi)$ (see (4.5.38)), and since $\big(\begin{smallmatrix}1 & 1\\ -1 & -1\end{smallmatrix}\big)^2 = 0$, after a lengthy computation we obtain that the first term in (4.6.7) is

$$\begin{aligned}
&\mathcal{G}(U)J_c\mathrm{Op}^{BW}\Big(A_{\frac32}\,\omega(\xi)\Big)\mathcal{G}^{-1}(U) = J_c\mathrm{Op}^{BW}\Big(A_{\frac32}\,\omega(\xi)\Big)\\
&\quad+\frac12\mathrm{Op}^{BW}\left(\begin{bmatrix}\omega(\xi)\#_\varrho B^{(2)}M^2(\xi) - B^{(2)}M^2(\xi)\#_\varrho\omega(\xi) & \omega(\xi)\#_\varrho B^{(2)}M^2(\xi) + B^{(2)}M^2(\xi)\#_\varrho\omega(\xi)\\ \omega(\xi)\#_\varrho B^{(2)}M^2(\xi) + B^{(2)}M^2(\xi)\#_\varrho\omega(\xi) & \omega(\xi)\#_\varrho B^{(2)}M^2(\xi) - B^{(2)}M^2(\xi)\#_\varrho\omega(\xi)\end{bmatrix}\right)\\
&\quad-\frac{\mathrm{i}}{2}\begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix}\mathrm{Op}^{BW}\Big(B^{(2)}M^2(\xi)\#_\varrho\omega(\xi)\#_\varrho B^{(2)}M^2(\xi)\Big) + R(U)\\
&= J_c\mathrm{Op}^{BW}\Big(A_{\frac32}\,\omega(\xi)\Big) + \mathrm{Op}^{BW}\left(\begin{bmatrix}0 & B^{(2)}|\xi|\\ B^{(2)}|\xi| & 0\end{bmatrix}\right)\\
&\quad-\frac{\mathrm{i}}{2}\begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix}\mathrm{Op}^{BW}\Big([B^{(2)}]^2|\xi|M^2(\xi)\Big) + J_c\mathrm{Op}^{BW}(A_0) + R(U)\,,
\end{aligned}\tag{4.6.8}$$

where A_0 is a matrix of symbols in $\Sigma\Gamma^0_{K,0,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ and $R(U)$ is a matrix of smoothing operators in $\Sigma\mathcal{R}^{-\varrho}_{K,0,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$. In the last passage to get (4.6.8) we also used that $M^2(\xi)\omega(\xi) = \mathtt{G}(\xi) = |\xi| + \widetilde{\Gamma}^{-\varrho}_0$, for any $\varrho \geq 0$, cfr. (4.5.8), (4.5.5).

Next using the explicit form (4.5.39) of A_1 we get, arguing similarly,

$$\begin{aligned}
&\mathcal{G}(U)J_c\mathrm{Op}^{BW}(A_1)\mathcal{G}(U)^{-1}\\
&\qquad= J_c\mathrm{Op}^{BW}(A_1) + \mathrm{i}\begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix}\mathrm{Op}^{BW}\Big([B^{(2)}]^2|\xi|M^2(\xi)\Big) + J_c\mathrm{Op}^{BW}\big(A_0'\big) + R(U)
\end{aligned}\tag{4.6.9}$$

where A_0' is a matrix of symbols in $\Sigma\Gamma^0_{K,0,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ and $R(U)$ is a matrix of smoothing operators in $\Sigma\mathcal{R}^{-\varrho}_{K,0,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$.

Moreover, using the form (4.5.40) of $A_{\frac12}$ we get

$$\mathcal{G}(U)J_c\mathrm{Op}^{BW}\Big(A_{\frac12}\Big)\mathcal{G}(U)^{-1} = J_c\mathrm{Op}^{BW}\Big(A_{\frac12}\Big) = -\frac{\mathrm{i}}{2}\begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix}\mathrm{Op}^{BW}\Big([B^{(2)}]^2|\xi|M^2(\xi)\Big)\,. \tag{4.6.10}$$

Then, since $A_0^{(2)}$ is a matrix of symbols of order zero, by Proposition 4.2.14 we have

$$\mathcal{G}(U)\Big[J_c\mathrm{Op}^{BW}\Big(A_0^{(2)}\Big) + \frac{\gamma}{2}G(0)\partial_x^{-1}\Big]\mathcal{G}(U)^{-1} = \frac{\gamma}{2}G(0)\partial_x^{-1} + J_c\mathrm{Op}^{BW}(A_0'') + R(U)\,, \tag{4.6.11}$$

for a matrix of symbols A_0'' in $\Sigma\Gamma^0_{K,0,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ and smoothing operators $R(U)$ in $\Sigma\mathcal{R}^{-\varrho}_{K,0,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$. Next by (4.6.5)-(4.6.6)

$$(\partial_t\mathcal{G}(U))\mathcal{G}(U)^{-1} = J_c\mathrm{Op}^{BW}\Big(A_{-\frac12}\Big) + R(U)\,,\quad A_{-\frac12} := \frac12\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\mathrm{Op}^{BW}\Big(\partial_t B^{(2)}M^2(\xi)\Big) \tag{4.6.12}$$

where, in view of the last bullets at the end of Section 4.2.1 and Proposition 4.2.15-(iv), $A_{-\frac{1}{2}}$ is a matrix of symbols in $\Sigma\Gamma^{-\frac{1}{2}}_{K,1,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ and $R(U)$ is a matrix of smoothing operator in $\Sigma\mathcal{R}^{-\varrho}_{K,0,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$.

Finally by Proposition 4.2.19 we have that $\mathcal{G}(U)R(U)\mathcal{G}(U)^{-1}$ is a matrix of smoothing operators in $\Sigma\mathcal{R}^{-\varrho}_{K,0,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$, being $\mathcal{G}(U)$ a spectrally localized map.

Note that, using the expression (4.5.39), the sum of the terms of order 1 which are in (4.6.8) and (4.6.9) is

$$J_c\mathrm{Op}^{\mathrm{BW}}(A_1)+\mathrm{Op}^{\mathrm{BW}}\left(\begin{bmatrix}0 & B^{(2)}|\xi|\\ B^{(2)}|\xi| & 0\end{bmatrix}\right)=\mathrm{Op}^{BW}_{\mathtt{vec}}\left(-\mathrm{i}V^{(2)}(U;x)\xi\right). \tag{4.6.13}$$

Note also that the sum of terms which are of order $\frac{1}{2}$ in (4.6.8), (4.6.9) and (4.6.10) equals zero.

In conclusion, by (4.6.8), (4.6.9) (4.6.10), (4.6.11), (4.6.12) and (4.6.13) we obtain that system (4.6.7) has the form (4.6.4) with $A_0^{(3)}:=A_0+A_0'+A_0''+A_{-\frac{1}{2}}$ in $\Sigma\Gamma^0_{K,1,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$. Note that the para-differential operators of positive order in (4.6.4) are linearly Hamiltonian, whereas $J_c\mathrm{Op}^{\mathrm{BW}}(A_0^{(3)})$ might not be. However the operator in the first line of (4.6.7) is a spectrally localized map which is linearly Hamiltonian up to homogeneity N by Lemma 4.3.9, with a para-differential structure as in (4.3.66). Then by Lemma 4.3.21 we replace each homogeneous component of $A_{\frac{3}{2}}\omega(\xi)+\left[\begin{smallmatrix}0 & -V^{(2)}\xi\\ V^{(2)}\xi & 0\end{smallmatrix}\right]+A_0^{(3)}$ with its symmetrized version, by adding another smoothing operator. Since the symbols with positive orders are unchanged we obtain a new operator $J_c\mathrm{Op}^{\mathrm{BW}}(A_0^{(3)})$ (that we denote in the same way) which is linearly Hamiltonian up to homogeneity N. ☐

4.6.2 Block-Diagonalization at highest order

In this section we diagonalize the operator $J_c\mathrm{Op}^{\mathrm{BW}}\big(A_{\frac{3}{2}}(U;x)\omega(\xi)\big)$ in (4.6.4) where $A_{\frac{3}{2}}$ is the matrix defined in (4.5.38). Note that the eigenvalues of the matrix

$$J_cA_{\frac{3}{2}}(U;x)=\mathrm{i}\begin{bmatrix}-(1+f(U;x)) & -f(U;x)\\ f(U;x) & 1+f(U;x)\end{bmatrix}, \tag{4.6.14}$$

where $f(U;x)$ is the real function defined in (4.5.38), are $\pm\mathrm{i}\lambda(U;x)$ with

$$\lambda(U;x):=\sqrt{(1+f(U;x))^2-f(U;x)^2}=\sqrt{1+2f(U;x)}. \tag{4.6.15}$$

Since the function $f(U;x)$ is in $\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,2}[r,N]$, for any $U\in B^K_{s_0,\mathbb{R}}(I;r)$ with $r>0$ small enough it results that $|f(U;x)|\leq\frac{1}{4}$, the function $\lambda(U;x)-1$ belongs to $\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,2}[r,N]$ and

$$\lambda(U;x)\geq\tfrac{\sqrt{2}}{2}>0,\quad\forall x\in\mathbb{T}.$$

Actually the function $\lambda(U;x)$ is real valued also for not small U, see Remark 4.6.6.

A matrix which diagonalizes (4.6.14) is

$$\begin{gathered}F(U;x):=\begin{pmatrix}h(U;x) & g(U;x)\\ g(U;x) & h(U;x)\end{pmatrix}\\ h:=\frac{1+f+\lambda}{\sqrt{(1+f+\lambda)^2-f^2}},\quad g:=\frac{-f}{\sqrt{(1+f+\lambda)^2-f^2}}.\end{gathered} \tag{4.6.16}$$

Note that $F(U;x)$ is well defined since $(1+f+\lambda)^2-f^2=(1+2f+\lambda)(1+\lambda)\geq\frac{1}{2}$. Moreover the matrix $F(U;x)$ is symplectic, i.e.

$$\det F=h^2-g^2=1. \tag{4.6.17}$$

The inverse of $F(U;x)$ is the symplectic and symmetric matrix

$$F(U;x)^{-1} := \begin{pmatrix} h(U;x) & -g(U;x) \\ -g(U;x) & h(U;x) \end{pmatrix} . \tag{4.6.18}$$

Moreover $F(U;x) - \mathrm{Id}$ is a matrix of real functions in $\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$ and

$$F(U;x)^{-1}\, J_c A_{\frac{3}{2}}(U;x) F(U;x) = \begin{bmatrix} -\mathrm{i}\lambda(U;x) & 0 \\ 0 & \mathrm{i}\lambda(U;x) \end{bmatrix} , \tag{4.6.19}$$

which amounts to $(h^2+g^2)(1+f) + 2hgf = \lambda$ and $2hg(1+f) + (h^2+g^2)f = 0$.

Lemma 4.6.5. *Let $N \in \mathbb{N}_0$ and $\varrho > 0$. Then for any $K \in \mathbb{N}$ there are $s_0 > 0$, $r > 0$ such that for any solution $U \in B^K_{s_0,\mathbb{R}}(I;r)$ of (4.5.37), there exists a real-to-real invertible matrix of spectrally localized maps $\Psi_1(U)$ satisfying $\Psi_1(U) - \mathrm{Id} \in \Sigma\mathcal{S}^0_{K,0,1}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$ and the following holds true:*

(i) **Boundedness:** *The operator $\Psi_1(U)$ and its inverse are non–homogeneous maps in $\mathcal{S}^0_{K,0,0}[r] \otimes \mathcal{M}_2(\mathbb{C})$;*

(ii) **Linear symplecticity:** *The map $\Psi_1(U)$ is linearly symplectic according to Definition 4.3.3;*

(iii) **Conjugation:** *If V_0 solves (4.6.4) then $V_1 := \Psi_1(U)V_0$ solves the system*

$$\begin{aligned} \partial_t V_1 = \mathrm{Op}^{BW}_{\mathtt{vec}} \Big(-\mathrm{i}\lambda(U;x)\omega(\xi) - \mathrm{i}V^{(2)}(U;x)\xi \Big) V_1 + \frac{\gamma}{2} G(0)\partial_x^{-1} V_1 \\ + J_c \mathrm{Op}^{\mathrm{BW}} \Big(A_0^{(4)}(U;t,x,\xi) \Big) V_1 + R(U;t)V_1 \end{aligned} \tag{4.6.20}$$

where

- *the function $\lambda(U;x) \in \Sigma\mathcal{F}^{\mathbb{R}}_{K,0,0}[r,N]$, defined in (4.6.15), fulfills $\lambda(U;x) - 1 \in \Sigma\mathcal{F}^{\mathbb{R}}_{K,0,2}[r,N]$;*
- *the Fourier multiplier $\omega(\xi) \in \widetilde{\Gamma}^{\frac{3}{2}}_0$ is defined in (4.5.9);*
- *the real function $V^{(2)}(U;x) \in \Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[r,N]$ is defined in Lemma 4.5.5;*
- *the matrix of symbols $A_0^{(4)}(U;t,x,\xi)$ belongs to $\Sigma\Gamma^0_{K,1,1}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$ and the operator $J_c \mathrm{Op}^{\mathrm{BW}}(A_0^{(4)})$ is linearly Hamiltonian up to homogeneity N;*
- *$R(U;t)$ is a real-to-real matrix of smoothing operators in $\Sigma\mathcal{R}^{-\varrho}_{K,1,1}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$.*

Proof. By Lemma 3.11 of [29], there exists a real valued function $m(U;x) \in \Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[r,N]$ (actually $m(U;x) := -\log(h(U;x) + g(U;x))$) such that the time 1 flow $\Psi_1(U) := \Psi^\tau(U)|_{\tau=1}$ of

$$\begin{cases} \partial_\tau \Psi^\tau(U) = J_c\, \mathrm{Op}^{\mathrm{BW}}(M(U;x))\Psi^\tau(U) \\ \Psi^0(U) = \mathrm{Id}\,, \end{cases} \qquad M(U;x) := \begin{bmatrix} -\mathrm{i}m(U;x) & 0 \\ 0 & \mathrm{i}m(U;x) \end{bmatrix} ,$$

fulfills

$$\Psi_1(U) = \mathrm{Op}^{\mathrm{BW}}\big(F^{-1}(U;x)\big) + R(U)\,, \quad \Psi_1(U)^{-1} = \mathrm{Op}^{\mathrm{BW}}(F(U;x)) + R'(U)\,, \tag{4.6.21}$$

where the matrix of functions $F(U;x)$ is defined in (4.6.16) and $R(U), R'(U)$ are matrices of smoothing operators in $\Sigma\mathcal{R}^{-\varrho}_{K,0,1} \otimes \mathcal{M}_2(\mathbb{C})$.

Since the operator $J_c\mathrm{Op}^{\mathrm{BW}}(M(U;x))$ is linearly Hamiltonian according to Definition 4.3.2, Lemma 4.3.16 guarantees that $\Psi_1(U)$ is invertible, linearly symplectic and $\Psi_1(U)^{\pm 1} - \mathrm{Id}$ belong to $\Sigma\mathcal{S}^0_{K,0,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$.

Since V_0 solves (4.6.4) then the variable $V_1 = \Psi_1(U)V_0$ solves

$$\partial_t V_1 = \Psi_1(U)\Big[J_c\mathrm{Op}^{\mathrm{BW}}\Big(A_{\frac{3}{2}}\omega(\xi)\Big) + \frac{\gamma}{2}G(0)\partial_x^{-1} + \mathrm{Op}^{BW}_{\mathtt{vec}}\Big(-\mathrm{i}V^{(2)}\xi\Big) + J_c\mathrm{Op}^{\mathrm{BW}}\Big(A_0^{(3)}\Big)\Big]\Psi_1(U)^{-1}V_1 \\ + (\partial_t\Psi_1(U))\Psi_1(U)^{-1}V_1 + \Psi_1(U)R(U)\Psi_1(U)^{-1}V_1\,. \tag{4.6.22}$$

Next we compute each term in (4.6.22). We begin with $J_c\mathrm{Op}^{\mathrm{BW}}(A_{\frac{3}{2}}\omega(\xi))$. Using (4.6.21), Proposition 4.2.19-(i), Proposition 4.2.15-(i) and the explicit form of $A_{\frac{3}{2}}$ in (4.5.38), one computes

$$\begin{aligned}\Psi_1(U)J_c\mathrm{Op}^{\mathrm{BW}}\Big(A_{\frac{3}{2}}\omega(\xi)\Big)\Psi_1(U)^{-1} &= \mathrm{Op}^{\mathrm{BW}}\big(F^{-1}\big)J_c\mathrm{Op}^{\mathrm{BW}}\Big(A_{\frac{3}{2}}\omega(\xi)\Big)\mathrm{Op}^{\mathrm{BW}}(F) + R(U)\\ &= \mathrm{Op}^{\mathrm{BW}}\left(\begin{bmatrix} a_{\frac{3}{2}} & b_{-\frac{1}{2}}\\ \overline{b_{-\frac{1}{2}}}^{\vee} & \overline{a_{\frac{3}{2}}}^{\vee}\end{bmatrix}\right) + R(U)\end{aligned} \tag{4.6.23}$$

where $R(U)$ is a real-to-real matrix of smoothing operators in $\Sigma\mathcal{R}^{-\varrho}_{K,0,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ and using (4.6.14), (4.6.16), (4.6.18), Proposition 4.2.14, we have

$$\begin{aligned} a_{\frac{3}{2}} &:= -\mathrm{i}\big[h\#_\varrho(1+f)\omega\#_\varrho h + g\#_\varrho(1+f)\omega\#_\varrho g + h\#_\varrho f\omega\#_\varrho g + g\#_\varrho f\omega\#_\varrho h\big]\\ b_{-\frac{1}{2}} &:= -\mathrm{i}\big[h\#_\varrho(1+f)\omega\#_\varrho g + g\#_\varrho(1+f)\omega\#_\varrho h + h\#_\varrho f\omega\#_\varrho h + g\#_\varrho f\omega\#_\varrho g\big]\,.\end{aligned}$$

The real-to-real structure of the symbols in (4.6.23) follows also by the last bullet after Definition 4.2.13. By the symbolic calculus rule (4.2.53), the second and third bullets after Definition 4.2.13 and (4.6.19) one has

$$\begin{aligned} a_{\frac{3}{2}} &= -\mathrm{i}\big[(h^2+g^2)(1+f) + 2hgf\big]\,\omega + a_{-\frac{1}{2}} = -\mathrm{i}\lambda\omega + \breve{a}_{-\frac{1}{2}}\\ b_{-\frac{1}{2}} &= -\mathrm{i}\big[2hg(1+f) + (h^2+g^2)f\big]\omega + \breve{b}_{-\frac{1}{2}} = \breve{b}_{-\frac{1}{2}}\end{aligned}$$

with symbols $\breve{a}_{-\frac{1}{2}}, \breve{b}_{-\frac{1}{2}}$ in $\Sigma\Gamma^{-\frac{1}{2}}_{K,0,1}[r,N]$. Then we obtain

$$\mathrm{Op}^{\mathrm{BW}}\left(\begin{bmatrix} a_{\frac{3}{2}} & b_{-\frac{1}{2}}\\ \overline{b_{-\frac{1}{2}}}^{\vee} & \overline{a_{\frac{3}{2}}}^{\vee}\end{bmatrix}\right) = \mathrm{Op}^{BW}_{\mathtt{vec}}(-\mathrm{i}\lambda\omega) + J_c\mathrm{Op}^{\mathrm{BW}}\Big(A_{-\frac{1}{2}}\Big) \tag{4.6.24}$$

where $A_{-\frac{1}{2}}$ is a real-to-real matrix of symbols in $\Sigma\Gamma^{-\frac{1}{2}}_{K,0,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$.

Proceeding similarly one finds that

$$\Psi_1(U)\mathrm{Op}^{BW}_{\mathtt{vec}}\Big(-\mathrm{i}V^{(2)}\xi\Big)\Psi_1(U)^{-1} = \mathrm{Op}^{\mathrm{BW}}\left(\begin{bmatrix} a_1 & b_0\\ \overline{b}_0^{\vee} & \overline{a}_1^{\vee}\end{bmatrix}\right) + R(U) \tag{4.6.25}$$

where $R(U)$ is a real-to-real matrix of smoothing operators in $\Sigma\mathcal{R}^{-\varrho}_{K,0,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ and

$$\begin{aligned} a_1 &:= h\#_\varrho(-\mathrm{i}V^{(2)}\xi)\#_\varrho h - g\#_\varrho(-\mathrm{i}V^{(2)}\xi)\#_\varrho g = (h^2-g^2)(-\mathrm{i}V^{(2)}\xi) \overset{(4.6.17)}{=} -\mathrm{i}V^{(2)}\xi\\ b_0 &:= h\#_\varrho(-\mathrm{i}V^{(2)}\xi)\#_\varrho g - g\#_\varrho(-\mathrm{i}V^{(2)}\xi)\#_\varrho h \in \mathcal{F}^{\mathbb{R}}_{K,0,1}[r,N]\,.\end{aligned} \tag{4.6.26}$$

In addition, using (4.6.21), the last bullets at the end of Section 4.2.1, Proposition 4.2.15-(iv) we obtain that

$$\begin{aligned}\Psi_1(U)\left[\frac{\gamma}{2}G(0)\partial_x^{-1} + J_c\,\mathrm{Op}^{\mathrm{BW}}\left(A_0^{(3)}\right)\right]\Psi_1(U)^{-1} + (\partial_t\Psi_1(U))\Psi_1(U)^{-1}\\ = \frac{\gamma}{2}G(0)\partial_x^{-1} + J_c\mathrm{Op}^{\mathrm{BW}}\left(A_0'\right) + R(U;t)\end{aligned} \tag{4.6.27}$$

where A_0' is a real-to-real matrix of symbols in $\Sigma\Gamma^0_{K,1,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ and $R(U;t)$ is a matrix of real-to-real smoothing operators in $\Sigma\mathcal{R}^{-\varrho}_{K,1,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$.

Finally, by Proposition 4.2.19, $\Psi_1(U)R(U)\Psi_1(U)^{-1}$ is a matrix of smoothing operators in $\Sigma\mathcal{R}^{-\varrho}_{K,0,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$.

In conclusion, by (4.6.23)-(4.6.24), (4.6.25)-(4.6.26) and (4.6.27) we deduce that system (4.6.22) has the form (4.6.20) with a matrix of symbols $A_0^{(4)} := A_{-\frac{1}{2}} + J_c\begin{bmatrix}0 & b_0\\ b_0 & 0\end{bmatrix} + A_0'$ in $\Sigma\Gamma^0_{K,1,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$. Note that the para-differential operators of positive order in (4.6.20) are linearly Hamiltonian, whereas $J_c\mathrm{Op}^{BW}(A_0^{(4)})$ might not be. However the sum of the operators in the first line of (4.6.22) plus $(\partial_t\Psi_1(U))\Psi_1(U)^{-1}$ is a spectrally localized map which is a linearly Hamiltonian operator up to homogeneity N by Lemma 4.3.9, with a para-differential structure as in (4.3.66). Then by Lemma 4.3.21 we can replace each homogeneous component of $A_0^{(4)}$ with its symmetrized version obtaining that $J_c\mathrm{Op}^{BW}(A_0^{(4)})$ is linearly Hamiltonian up to homogeneity N, by adding another smoothing operator. □

Remark 4.6.6. In view of Lemmata 4.5.5 and 4.5.1 the function $\lambda(U;x)$ in (4.6.15) is equal to $(1+\eta_x^2)^{-\frac{3}{4}}$. Therefore the symbols $\pm\mathrm{i}\lambda(U;x)\omega(\xi)$ are elliptic also for not small data and system (4.6.20) is hyperbolic at order $\frac{3}{2}$. This is the well known fact that, in presence of capillarity, there is no need of the Taylor sign condition for the local well-posedness.

4.6.3 Reduction to constant coefficients of the highest order

In this section we perform a linearly symplectic change of variable which reduces the highest order para-differential operator $\mathrm{Op}^{BW}_{\mathtt{vec}}(-\mathrm{i}\lambda(U;x)\omega(\xi))$ in (4.6.20) to constant coefficients.

Lemma 4.6.7 (Reduction of the highest order). *Let $N\in\mathbb{N}_0$ and $\varrho > 2(N+1)$. Then for any $K\in\mathbb{N}$ there are $s_0>0$, $r>0$ such that for any solution $U\in B^K_{s_0,\mathbb{R}}(I;r)$ of (4.5.37), there exists a real-to-real invertible matrix of spectrally localized maps $\Psi_2(U)$ satisfying $\Psi_2(U)-\mathrm{Id}\in\Sigma\mathcal{S}^{N+1}_{K,0,2}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ and the following holds true:*

(i) **Boundedness:** *The linear map $\Psi_2(U)$ and its inverse are non–homogeneous maps in $\mathcal{S}^0_{K,0,0}[r]\otimes\mathcal{M}_2(\mathbb{C})$;*

(ii) **Linear symplecticity:** *The map $\Psi_2(U)$ is linearly symplectic according to Definition 4.3.3;*

(iii) **Conjugation:** *If V_1 solves (4.6.20) then $V_2 := \Psi_2(U)V_1$ solves the system*

$$\begin{aligned}\partial_t V_2 = \mathrm{Op}^{BW}_{\mathtt{vec}}\Big(-\mathrm{i}(1+\zeta(U))\omega(\xi) - \mathrm{i}V^{(3)}(U;t,x)\xi\Big)V_2 + \frac{\gamma}{2}G(0)\partial_x^{-1}V_2\\ + J_c\mathrm{Op}^{\mathrm{BW}}\Big(A_0^{(5)}(U;t,x,\xi)\Big)V_2 + R(U;t)V_2\end{aligned} \tag{4.6.28}$$

where

- *$\zeta(U)$ is a x-independent function in $\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,2}[r,N]$ and $\omega(\xi)$ is defined in (4.5.9);*
- *$V^{(3)}(U;t,x)$ is a real valued function in $\Sigma\mathcal{F}^{\mathbb{R}}_{K,1,1}[r,N]$;*

- *The matrix of symbols* $A_0^{(5)}(U;t,x,\xi)$ *belongs to* $\Sigma\Gamma^0_{K,1,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ *and the para-differential operator* $J_c\mathrm{Op}^{\mathrm{BW}}(A_0^{(5)})$ *is linearly Hamiltonian up to homogeneity* N*;*
- $R(U;t)$ *is a real-to-real matrix of smoothing operators in* $\Sigma\mathcal{R}^{-\varrho+2(N+1)}_{K,1,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$*.*

Proof. We define the map $\Psi_2(U)$ as the time 1 flow $\Psi_2(U):=\Psi^\tau(U)|_{\tau=1}$ of

$$\partial_\tau\Psi^\tau(U)=J_c\,\mathrm{Op}^{\mathrm{BW}}(B(\tau,U;x,\xi))\Psi^\tau(U),\qquad \Psi^0(U)=\mathrm{Id},$$

where

$$B(\tau,U;x,\xi):=\begin{pmatrix}0 & b(\tau,U;x,\xi)\\ b(\tau,U;x,-\xi) & 0\end{pmatrix},\quad b(\tau,U;x,\xi):=\frac{\beta(U;x)}{1+\tau\beta_x(U;x)}\xi,$$

and the function $\beta(U;x)$ in $\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,2}[r,N]$ has to be determined. As $\beta(U;x)$ is real valued, the operator $J_c\,\mathrm{Op}^{\mathrm{BW}}(B(\tau,U;\cdot))$ is linearly Hamiltonian. Thus Lemma 4.3.16, applied with $m=1$, guarantees that $\Psi_2(U)$ is a spectrally localized map in $\mathcal{S}^0_{K,0,0}[r]\otimes\mathcal{M}_2(\mathbb{C})$, it is linearly symplectic and $\Psi_2(U)^{\pm}-\mathrm{Id}$ belongs to $\Sigma\mathcal{S}^{N+1}_{K,0,2}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$. Note that the diagonal operator $J_c\mathrm{Op}^{\mathrm{BW}}(B)=\mathrm{Op}^{\mathrm{BW}}(\mathrm{i}b(\tau,U;x,\xi))\mathrm{Id}$ is a multiple of the identity and hence the flow $\Psi_2(U)$ acts as a scalar operator.

Since V_1 solves (4.6.20), then the variable $V_2=\Psi_2(U)V_1$ solves

$$\begin{aligned}\partial_t V_2=\Psi_2(U)\Big[\mathrm{Op}^{BW}_{\mathtt{vec}}\Big(-\mathrm{i}\lambda\omega(\xi)-\mathrm{i}V^{(2)}\xi\Big)+\frac{\gamma}{2}G(0)\partial_x^{-1}+J_c\mathrm{Op}^{\mathrm{BW}}\Big(A_0^{(4)}\Big)\Big]\Psi_2(U)^{-1}V_2\\ +(\partial_t\Psi_2(U))\circ\Psi_2(U)^{-1}V_2+\Psi_2(U)R(U;t)\Psi_2(U)^{-1}V_2.\end{aligned}\tag{4.6.29}$$

We now compute each term in (4.6.29). By Lemma 3.21 of [27], the diffeomorphism $\Phi_U:x\mapsto x+\beta(U;x)$ of $\mathbb{T}$ is invertible with inverse $\Phi_U^{-1}: y\mapsto y+\breve\beta(U;y)$ and $\breve\beta(U;y)$ belongs to $\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,2}[r,N]$. By Theorem 3.27 of [27] one has

$$\begin{aligned}&\Psi_2(U)\,\mathrm{Op}^{BW}_{\mathtt{vec}}(-\mathrm{i}\,\lambda(U;x)\omega(\xi))\,\Psi_2(U)^{-1}\\ &=\mathrm{Op}^{BW}_{\mathtt{vec}}\Big(\Big[-\mathrm{i}\,\lambda(U;y)\,\omega\Big(\xi\big(1+\breve\beta_y(U;y)\big)\Big)\Big]\Big|_{y=\Phi_U(x)}\Big)+J_c\mathrm{Op}^{\mathrm{BW}}\Big(A_{-\frac12}\Big)+R(U)\end{aligned}\tag{4.6.30}$$

with a diagonal matrix of symbols $A_{-\frac12}$ in $\Sigma\Gamma^{-\frac12}_{K,0,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$, and a diagonal matrix of smoothing operators $R(U)$ in $\Sigma\mathcal{R}^{-\varrho+\frac32}_{K,0,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$. Note that $\omega\big(\xi(1+\breve\beta_y(U;y))\big)$ is a symbol in $\Sigma\Gamma^{\frac32}_{K,0,0}[r,N]$ by Lemma 3.23 of [27].

Now we choose $\breve\beta$ in such a way that the principal symbol in (4.6.30) is x-independent. Since, by (4.5.46), $\omega_{-\frac12}(\xi):=\omega(\xi)-\sqrt{\kappa}|\xi|^{\frac32}$ is a Fourier multiplier in $\widetilde\Gamma_0^{-\frac12}$ we get

$$\lambda(U;y)\omega\Big(\xi(1+\breve\beta_y(U;y))\Big)=\lambda(U;y)\sqrt{\kappa}|\xi|^{\frac32}\big|1+\breve\beta_y(U;y)\big|^{\frac32}+\lambda(U;y)\omega_{-\frac12}\big(\xi(1+\breve\beta_y(U;y))\big)\tag{4.6.31}$$

and we select $\breve\beta(U;\cdot)$ so that

$$\lambda(U;y)\,\big|1+\breve\beta_y(U;y)\big|^{\frac32}=1+\zeta(U)\tag{4.6.32}$$

with a y-independent function $\zeta(U)$. In order to fulfill (4.6.32) we define the functions

$$\zeta(U):=\Big(\frac{1}{2\pi}\int_{\mathbb{T}}\lambda(U;y)^{-\frac23}\mathrm{d}y\Big)^{-\frac32}-1,\quad \breve\beta(U;y):=\partial_y^{-1}\Big[\Big(\frac{1+\zeta(U)}{\lambda(U;y)}\Big)^{\frac23}-1\Big]$$

which belong to $\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,2}[r,N]$. By (4.6.32) and since $\omega(\xi)-\sqrt{\kappa}|\xi|^{\frac{3}{2}}\in\widetilde{\Gamma}_0^{-\frac{1}{2}}$, the expression (4.6.31) becomes

$$\begin{aligned}\lambda(U;y)\omega\Big(\xi(1+\breve{\beta}_y(U;y))\Big) &= \sqrt{\kappa}|\xi|^{\frac{3}{2}}(1+\zeta(U))+\lambda(U;y)\omega_{-\frac{1}{2}}\big(\xi(1+\breve{\beta}_y(U;y))\big)\\ &= \omega(\xi)(1+\zeta(U))+a_{-\frac{1}{2}}\end{aligned}$$

where $a_{-\frac{1}{2}}$ is a real valued symbol in $\Sigma\Gamma^{-\frac{1}{2}}_{K,0,1}[r,N]$. Note that we used that $\omega_{-\frac{1}{2}}\big(\xi(1+\breve{\beta}_y(U;y))\big)\lambda(U;y)-\omega_{-\frac{1}{2}}(\xi)$ is a symbol in $\Sigma\Gamma^{-\frac{1}{2}}_{K,0,1}[r,N]$. In conclusion (4.6.30) is

$$\Psi_2(U)\mathrm{Op}^{BW}_{\mathtt{vec}}(-\mathrm{i}\lambda\omega(\xi))\Psi_2(U)^{-1}=\mathrm{Op}^{BW}_{\mathtt{vec}}(-\mathrm{i}(1+\zeta(U))\omega(\xi))+J_c\mathrm{Op}^{\mathrm{BW}}\Big(A_{-\frac{1}{2}}\Big)+R(U) \quad (4.6.33)$$

where $A_{-\frac{1}{2}}$ is a diagonal matrix of symbols in $\Sigma\Gamma^{-\frac{1}{2}}_{K,0,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ and $R(U)$ is a diagonal matrix of smoothing operators in $\Sigma\mathcal{R}^{-\varrho+\frac{3}{2}}_{K,0,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$.

We now compute the other terms in (4.6.29). Again by Theorem 3.27 of [27] (and Lemma A.4 of [29])

$$\Psi_2(U)\,\mathrm{Op}^{BW}_{\mathtt{vec}}\Big(-\mathrm{i}V^{(2)}\xi\Big)\,\Psi_2(U)^{-1}=\mathrm{Op}^{BW}_{\mathtt{vec}}\Big(-\mathrm{i}\breve{V}(U;x)\xi\Big)+R(U) \quad (4.6.34)$$

where $\breve{V}(U;x)$ is a real function in $\Sigma\mathcal{F}^{\mathbb{R}}_{K,0,1}[r,N]$, and $R(U)$ is a diagonal matrix of smoothing operators in $\Sigma\mathcal{R}^{-\varrho+1}_{K,0,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$. In addition, again by Theorem 3.27 of [27]

$$\Psi_2(U)\left[\frac{\gamma}{2}G(0)\partial_x^{-1}+J_c\mathrm{Op}^{\mathrm{BW}}\Big(A_0^{(4)}\Big)\right]\Psi_2(U)^{-1}=\frac{\gamma}{2}G(0)\partial_x^{-1}+J_c\mathrm{Op}^{\mathrm{BW}}(A_0)+R(U) \quad (4.6.35)$$

where A_0 is a real-to-real matrix of symbols in $\Sigma\Gamma^0_{K,0,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ and $R(U)\in\Sigma\mathcal{R}^{-\varrho}_{K,0,1}[r,N]$. Moreover, by Lemma A.5 of [29]

$$(\partial_t\Psi_2(U))\circ\Psi_2(U)^{-1}=\mathrm{Op}^{BW}_{\mathtt{vec}}(-\mathrm{i}g(U;t,x)\xi)+R(U;t) \quad (4.6.36)$$

where $g(U;t,x)$ is a real function in $\Sigma\mathcal{F}^{\mathbb{R}}_{K,1,1}[r,N]$ and $R(U;t)$ is a matrix of real-to-real smoothing operators in $\Sigma\mathcal{R}^{-\varrho+1}_{K,1,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$. Finally $\Psi_2(U)R(U;t)\Psi_2(U)^{-1}$ in (4.6.29) is a matrix of real-to-real smoothing operators in $\Sigma\mathcal{R}^{-\varrho+2(N+1)}_{K,1,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$, by Proposition 4.2.19.

In conclusion, by (4.6.33), (4.6.34), (4.6.35), (4.6.36), we deduce that system (4.6.29) has the form (4.6.28) with $V^{(3)}:=\breve{V}+g$ and $A_0^{(5)}:=A_{-\frac{1}{2}}+A_0$. Note that the para-differential operators of positive order in (4.6.28) are linearly Hamiltonian, whereas $J_c\mathrm{Op}^{\mathrm{BW}}(A_0^{(5)})$ might not be. However the sum of the operators in the first line of (4.6.29) and $(\partial_t\Psi_2(U))\Psi_2(U)^{-1}$ is a spectrally localized map which is linearly Hamiltonian up to homogeneity N by Lemma 4.3.9, with a para-differential structure as in (4.3.66). Then by Lemma 4.3.21 we can replace each homogeneous component of $A_0^{(5)}$ with its symmetrized version obtaining that $J_c\mathrm{Op}^{\mathrm{BW}}(A_0^{(5)})$ is linearly Hamiltonian up to homogeneity N, by adding another smoothing operator. □

4.6.4 Block-Diagonalization up to smoothing operators

The goal of this section is to block-diagonalize system (4.6.28) up to smoothing remainders.

Lemma 4.6.8. *Let $N \in \mathbb{N}_0$ and $\varrho > 2(N+1)$. Then for any $n \in \mathbb{N}_0$ there is $K' := K'(n) \geq 0$ (one can choose $K' = n$) such that for all $K \geq K'+1$ there are $s_0 > 0$, $r > 0$ such that for any solution $U \in B^K_{s_0,\mathbb{R}}(I;r)$ of (4.5.37), there exists a real-to-real invertible matrix of spectrally localized maps $\Phi_n(U)$ satisfying $\Phi_n(U) - \mathrm{Id} \in \Sigma\mathcal{S}^0_{K,K',1}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$ and the following holds true:*

(i) **Boundedness:** *Each $\Phi_n(U)$ and its inverse are non–homogeneous maps in $\mathcal{S}^0_{K,K',0}[r] \otimes \mathcal{M}_2(\mathbb{C})$;*

(ii) **Linear symplecticity:** *The map $\Phi_n(U)$ is linearly symplectic up to homogeneity N according to Definition 4.3.7;*

(iii) **Conjugation:** *If V_2 solves (4.6.28) then $V_{n+2} := \Phi_n(U)V_2$ solves*

$$\begin{aligned}\partial_t V_{n+2} = \mathrm{Op}^{BW}_{\mathtt{vec}}\left(-\mathrm{i}\left[(1+\zeta(U))\omega(\xi) + \frac{\gamma}{2}\frac{\mathtt{G}(\xi)}{\xi} + V^{(3)}(U;t,x)\xi + a_0^{(n)}(U;t,x,\xi)\right]\right)V_{n+2} \\ + J_c\mathrm{Op}^{\mathrm{BW}}\left(A_{-n}(U;t,x,\xi)\right)V_{n+2} + R(U;t)V_{n+2}\end{aligned} \tag{4.6.37}$$

where

- *the Fourier multiplier $\omega(\xi)$ is defined in (4.5.9), the x-independent real function $\zeta(U) \in \Sigma\mathcal{F}^{\mathbb{R}}_{K,0,2}[r,N]$ and the real function $V^{(3)}(U;t,x) \in \Sigma\mathcal{F}^{\mathbb{R}}_{K,1,1}[r,N]$ are defined in Lemma 4.6.7;*
- *$a_0^{(n)}(U;t,x,\xi)$ is a symbol in $\Sigma\Gamma^0_{K,K',1}[r,N]$ and $\mathrm{Im}\, a_0^{(n)}(U;t,x,\xi)$ belongs to $\Gamma^0_{K,K',N+1}[r]$;*
- *The matrix of symbols $A_{-n}(U;t,x,\xi)$ belongs to $\Sigma\Gamma^{-n}_{K,K'+1,1}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$ and $J_c\mathrm{Op}^{\mathrm{BW}}(A_{-n})$ is a linearly Hamiltonian operator up to homogeneity N;*
- *$R(U;t)$ is a real-to-real matrix of smoothing operators in $\Sigma\mathcal{R}^{-\varrho+2(N+1)}_{K,K'+1,1}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$.*

Proof. We prove the thesis by induction on $n \in \mathbb{N}_0$.
<u>Case $n = 0$.</u> It follows by (4.6.28) with $a_0^{(0)} := 0$, $A_0 := A_0^{(5)}$, $K' = 0$ and $\Phi_0(U) := \mathrm{Id}$.
<u>Case $n \rightsquigarrow n+1$.</u> Suppose (4.6.37) holds. We perform a transformation to push the off diagonal part of $J_c\mathrm{Op}^{\mathrm{BW}}(A_{-n})$ to lower order. We write the real-to-real matrix J_cA_{-n} as

$$J_cA_{-n} = J_c\begin{pmatrix}-\mathrm{i}\overline{b}^{\vee}_{-n} & -\overline{a}^{\vee}_{-n} \\ -a_{-n} & \mathrm{i}b_{-n}\end{pmatrix} = \begin{bmatrix}\mathrm{i}a_{-n} & b_{-n} \\ \overline{b}^{\vee}_{-n} & -\mathrm{i}\overline{a}^{\vee}_{-n}\end{bmatrix}, \qquad a_{-n}, b_{-n} \in \Sigma\Gamma^{-n}_{K,K'+1,1}[r,N] \tag{4.6.38}$$

where, since $J_c\mathrm{Op}^{\mathrm{BW}}(A_{-n})$ is linearly Hamiltonian up to homogeneity N, by (4.3.31) we have

$$\mathrm{Im}\, a_{-n} \in \Gamma^{-n}_{K,K'+1,N+1}[r], \qquad b_{-n} - b^{\vee}_{-n} \in \Gamma^{-n}_{K,K'+1,N+1}[r]. \tag{4.6.39}$$

Denote by $\Phi_{F^{(n)}}(U) := \Phi^{\tau}_{F^{(n)}}(U;t)|_{\tau=1}$ the time 1-flow of

$$\begin{cases}\partial_\tau \Phi^{\tau}_{F^{(n)}}(U) = \mathrm{Op}^{\mathrm{BW}}\left(F^{(n)}(U)\right)\Phi^{\tau}_{F^{(n)}}(U) \\ \Phi^0_{F^{(n)}}(U) = \mathrm{Id},\end{cases} \qquad F^{(n)}(U) := \begin{bmatrix}0 & f_{-n-\frac{3}{2}} \\ \overline{f}^{\vee}_{-n-\frac{3}{2}} & 0\end{bmatrix}, \tag{4.6.40}$$

where, see (4.6.38),

$$f_{-n-\frac{3}{2}}(U;t,x,\xi) := -\frac{b_{-n}(U;t,x,\xi)}{2\mathrm{i}\omega(\xi)(1+\zeta(U))} \in \Sigma\Gamma^{-n-\frac{3}{2}}_{K,K'+1,1}[r,N]. \tag{4.6.41}$$

By (4.6.39), (4.6.41), the symbol $f_{-n-\frac{3}{2}} - f^{\vee}_{-n-\frac{3}{2}}$ is in $\Gamma^{-n-\frac{3}{2}}_{K,K'+1,N+1}[r]$ and therefore $\mathrm{Op}^{\mathrm{BW}}\left(F^{(n)}(U)\right)$ is a linearly Hamiltonian operator up to homogeneity N. Lemma 4.3.16 implies that $\Phi_{F^{(n)}}(U)$ is invertible, linearly symplectic up to homogeneity N and $\Phi_{F^{(n)}}(U)^{\pm 1} - \mathrm{Id}$ belong to $\Sigma\mathcal{S}^0_{K,K'+1,1}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$.

If V_{n+2} fulfills (4.6.37), the variable $V_{n+3} := \Phi_{F^{(n)}}(U)V_{n+2}$ solves

$$\partial_t V_{n+3} = \Phi_{F^{(n)}}(U)\left[\mathrm{Op}^{BW}_{\mathtt{vec}}\left(\mathtt{d}^{(n)}_{\frac{3}{2}}\right) + J_c\mathrm{Op}^{\mathrm{BW}}(A_{-n})\right]\Phi_{F^{(n)}}(U)^{-1}V_{n+3} \tag{4.6.42}$$

$$+ (\partial_t\Phi_{F^{(n)}}(U))\Phi_{F^{(n)}}(U)^{-1}V_{n+3} + \Phi_{F^{(n)}}(U)R(U;t)\Phi_{F^{(n)}}(U)^{-1}V_{n+3} \tag{4.6.43}$$

where, to shorten notation, we denoted

$$\mathtt{d}^{(n)}_{\frac{3}{2}}(U;t,x,\xi) := -\mathrm{i}\Big((1+\zeta(U))\omega(\xi) + \frac{\gamma}{2}\frac{\mathtt{G}(\xi)}{\xi} + V^{(3)}(U;t,x)\xi + a_0^{(n)}(U;t,x,\xi)\Big)\,. \tag{4.6.44}$$

We first expand (4.6.42). The Lie expansion formula (see e.g. Lemma A.1 of [29]) says that for any operator $M(U)$, setting $\mathbf{\Phi}(U) := \Phi_{F^{(n)}}(U)$, $\mathbf{F} := \mathrm{Op}^{\mathrm{BW}}\big(F^{(n)}(U)\big)$ and $\mathrm{Ad}_{\mathbf{F}}[M] := [\mathbf{F}, M]$, one has

$$\mathbf{\Phi}(U)M(U)(\mathbf{\Phi}(U))^{-1} = M + \sum_{q=1}^{L}\frac{1}{q!}\mathrm{Ad}^q_{\mathbf{F}}[M] + \frac{1}{L!}\int_0^1(1-\tau)^L\mathbf{\Phi}^\tau(U)\mathrm{Ad}^{L+1}_{\mathbf{F}}[M](\mathbf{\Phi}^\tau(U))^{-1}\mathrm{d}\tau\,. \tag{4.6.45}$$

We apply this formula with $L := L(\varrho) \geq (\varrho - n)/(n+\frac{3}{2})$ (in this way the integral remainder above is a smoothing operator in $\mathcal{R}^{-\varrho}_{K,K'+1,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$), and by symbolic calculus in Proposition 4.2.14, (4.6.38), (4.6.40), (4.6.41) and formula (4.6.3) we find

$$\begin{aligned}(4.6.42) &= \mathrm{Op}^{BW}_{\mathtt{vec}}\left(\mathtt{d}^{(n)}_{\frac{3}{2}} + \mathrm{i}a_{-n}\right)V_{n+3}\\ &+ \mathrm{Op}^{\mathrm{BW}}\left(\begin{bmatrix} 0 & \big[(\overline{\mathtt{d}}^{(n)}_{\frac{3}{2}})^\vee - \mathtt{d}^{(n)}_{\frac{3}{2}}\big]f_{-n-\frac{3}{2}} + b_{-n} \\ \big[\mathtt{d}^{(n)}_{\frac{3}{2}} - (\overline{\mathtt{d}}^{(n)}_{\frac{3}{2}})^\vee\big]\overline{f}^\vee_{-n-\frac{3}{2}} + \overline{b_{-n}}^\vee & 0\end{bmatrix}\right)V_{n+3}\\ &+ J_c\mathrm{Op}^{\mathrm{BW}}\left(A'_{-(n+1)}\right)V_{n+3} + R(U;t)\,V_{n+3}\end{aligned} \tag{4.6.46}$$

with a real-to-real matrix of symbols $A'_{-(n+1)}$ in $\Sigma\Gamma^{-n-1}_{K,K'+1,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ and a matrix of smoothing operators $R(U;t)$ in $\Sigma\mathcal{R}^{-\varrho}_{K,K'+1,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ (we also used Lemma 4.3.16 and Proposition 4.2.19 to estimate the Taylor remainder in the Lie expansion formula). By (4.6.44), (4.6.41) and since $a_0^{(n)}$ is of order 0, we have

$$\big[(\overline{\mathtt{d}}^{(n)}_{\frac{3}{2}})^\vee - \mathtt{d}^{(n)}_{\frac{3}{2}}\big]f_{-n-\frac{3}{2}} + b_{-n} =: b_{-n-\frac{3}{2}} \in \Sigma\Gamma^{-n-\frac{3}{2}}_{K,K'+1,1}[r,N]\,. \tag{4.6.47}$$

We pass to the first term in (4.6.43). Using the Lie expansion (cfr. Lemma A.1 of [29])

$$\begin{aligned}(\partial_t\mathbf{\Phi}(U))(\mathbf{\Phi}(U))^{-1} &= \partial_t\mathbf{F} + \sum_{q=2}^{L}\frac{1}{q!}\mathrm{Ad}^{q-1}_{\mathbf{F}}[\partial_t\mathbf{F}]\\ &\quad + \frac{1}{L!}\int_0^1(1-\tau)^L\mathbf{\Phi}^\tau(U)\mathrm{Ad}^L_{\mathbf{F}}[\partial_t\mathbf{F}](\mathbf{\Phi}^\tau(U))^{-1}\mathrm{d}\tau\end{aligned} \tag{4.6.48}$$

with the same L as above, the last bullets at the end of Section 4.2.1, Proposition 4.2.15-(iv) and (4.6.41) we get

$$(\partial_t\Phi_{F^{(n)}}(U))\Phi_{F^{(n)}}(U)^{-1} = J_c\mathrm{Op}^{\mathrm{BW}}(Q(U;t,x,\xi)) + R(U;t) \tag{4.6.49}$$

with a real-to-real matrix of symbols $Q(U;t,x,\xi)$ in $\Sigma\Gamma^{-n-\frac{3}{2}}_{K,K'+2,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ and a matrix of smoothing operators $R(U;t)$ in $\Sigma\mathcal{R}^{-\varrho}_{K,K'+2,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$.

Thanks to (i) and (ii) of Proposition 4.2.19, the operator $\Phi_{F^{(n)}}(U)R(U;t)\Phi_{F^{(n)}}(U)^{-1}$ in (4.6.43) is a smoothing operator in $\Sigma\mathcal{R}^{-\varrho+2(N+1)}_{K,K'+1,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ as well as $R(U;t)$. In conclusion, by (4.6.46), (4.6.47), (4.6.49), the system in (4.6.42)–(4.6.43) has the form

$$\partial_t V_{n+3} = \mathrm{Op}^{BW}_{\mathtt{vec}}\left(\mathtt{d}^{(n)}_{\frac{3}{2}} + \mathrm{i}a_{-n}\right)V_{n+3} + J_c\mathrm{Op}^{\mathrm{BW}}(A_{-n-1})V_{n+3} + R(U;t)V_{n+3} \tag{4.6.50}$$

where the matrix of symbols A_{-n-1} in $\Sigma\Gamma^{-n-1}_{K,K'+2,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ is given by $A_{-n-1} := A'_{-(n+1)} + J_c\begin{bmatrix} 0 & b_{-n-\frac{3}{2}} \\ \bar{b}^{\vee}_{-n-\frac{3}{2}} & 0 \end{bmatrix} + Q$ and $R(U;t)$ a matrix of smoothing operators in $\Sigma\mathcal{R}^{-\varrho+2(N+1)}_{K,K'+2,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$. By Lemma 4.3.21, we replace each homogeneous component of A_{-n-1} with its symmetrized version obtaining that $J_c\mathrm{Op}^{\mathrm{BW}}(A_{-n-1})$ is linearly Hamiltonian up to homogeneity N, by adding another smoothing operator.

In conclusion, by (4.6.44), system (4.6.50) has the form (4.6.37) at step $n+1$ with $a_0^{(n+1)} := a_0^{(n)} - a_{-n}$ and $K'(n+1) := K'(n)+1$. Note that the imaginary part $\operatorname{Im} a_0^{(n+1)}$ is in $\Gamma^0_{K,K',N+1}[r]$ by the inductive assumption and (4.6.39).

Finally we define $\Phi_{n+1}(U) := \Phi_{F^{(n)}}(U)\circ\Phi_n(U)$. The claimed properties of $\Phi_n(U)$ follow by the analogous ones of each $\Phi_{F^{(n)}}(U)$ and Proposition 4.2.19. □

For any $\varrho > 2(N+1)$ we now choose in Lemma 4.6.8 a number of iterative steps $n := n_1(\varrho)$ such that $n_1 \geq \varrho - 2(N+1)$, so that we can incorporate $J_c\mathrm{Op}^{\mathrm{BW}}(A_{-n}(U))$ in the smoothing remainder $R(U)$ in $\Sigma\mathcal{R}^{-\varrho+2(N+1)}_{K,K'+1,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$. Thus, denoting $Z := V_{n+2}$, we write system (4.6.37) as

$$\partial_t Z = \mathrm{Op}^{BW}_{\mathtt{vec}}\left(-\mathrm{i}\left[(1+\zeta(U))\omega(\xi) + \frac{\gamma}{2}\frac{\mathtt{G}(\xi)}{\xi} + V^{(3)}(U;t,x)\xi + a_0(U;t,x,\xi)\right]\right)Z + R(U;t)Z \tag{4.6.51}$$

where the symbol $a_0(U;t,x,\xi) := a_0^{(n_1)}(U;t,x,\xi)$ is given in (4.6.37) with $n = n_1(\varrho)$. Thus $a_0(U;t,x,\xi)$ belongs to $\Sigma\Gamma^0_{K,K',1}[r,N]$ and its imaginary part $\operatorname{Im} a_0(U;t,x,\xi)$ in $\Gamma^0_{K,K',N+1}[r]$ with $K' = n_1(\varrho)$.

4.6.5 Reduction to constant coefficients up to smoothing operators

The goal of this section is to reduce the symbol in the para-differential operator in (4.6.51) to an x-independent one, up to smoothing operators.

Lemma 4.6.9. *Let $N\in\mathbb{N}_0$ and $\varrho > 3(N+1)$. Then for any $n\in\mathbb{N}_0$ there is $K'' := K''(\varrho,n) > 0$ (one can choose $K'' = K'(n_1(\varrho)) + n$) such that for all $K \geq K''+1$ there are $s_0 > 0$, $r > 0$ such that for any solution $U \in B^K_{s_0,\mathbb{R}}(I;r)$ of (4.5.37), there exists a real-to-real invertible matrix of spectrally localized maps $\mathcal{F}_n(U)$ satisfying $\mathcal{F}_n(U) - \mathrm{Id} \in \Sigma\mathcal{S}^{(N+1)/2}_{K,K'',1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ and the following holds true:*

(i) **Boundedness:** *Each $\mathcal{F}_n(U)$ and its inverse are non–homogeneous maps in $\mathcal{S}^0_{K,K'',0}[r]\otimes\mathcal{M}_2(\mathbb{C})$;*

(ii) **Linear symplecticity:** *The map $\mathcal{F}_n(U)$ is linearly symplectic up to homogeneity N according to Definition 4.3.7.*

(iii) **Conjugation:** *If Z solves (4.6.51) then $Z_n := \mathcal{F}_n(U)Z$ solves*

$$\partial_t Z_n = \mathrm{Op}^{BW}_{\mathtt{vec}}\left(\mathrm{i}\mathtt{m}^{(n)}_{\frac{3}{2}}(U;t,\xi) + \mathrm{i}a_{-\frac{n}{2}}(U;t,x,\xi)\right)Z_n + R(U;t)Z_n \tag{4.6.52}$$

with the x–independent symbol

$$\mathtt{m}^{(n)}_{\frac{3}{2}}(U;t,\xi) := -\left[(1+\zeta(U))\omega(\xi) + \frac{\gamma}{2}\frac{\mathtt{G}(\xi)}{\xi} + \mathtt{V}(U;t)\xi + \mathtt{b}_{\frac{1}{2}}(U;t)|\xi|^{\frac{1}{2}}\right] + \mathtt{b}^{(n)}_0(U;t,\xi) \tag{4.6.53}$$

where

- *the x–independent function $\zeta(U) \in \Sigma\mathcal{F}^{\mathbb{R}}_{K,0,2}[r,N]$ is defined in Lemma 4.6.7 and $\omega(\xi)$ in (4.5.9);*
- *the function $\mathtt{V}(U;t) \in \Sigma\mathcal{F}^{\mathbb{R}}_{K,1,2}[r,N]$ is x-independent;*
- *the function $\mathtt{b}_{\frac12}(U;t) \in \Sigma\mathcal{F}^{\mathbb{R}}_{K,2,2}[r,N]$ is x-independent;*
- *the symbol $\mathtt{b}_0^{(n)}(U;t,\xi) \in \Sigma\Gamma^0_{K,K'',2}[r,N]$ is x–independent and its imaginary part $\operatorname{Im}\mathtt{b}_0^{(n)}(U;t,\xi)$ is in $\Gamma^0_{K,K'',N+1}[r]$;*
- *the symbol $a_{-\frac{n}{2}}(U;t,x,\xi)$ belongs to $\Sigma\Gamma^{-\frac{n}{2}}_{K,K''+1,1}[r,N]$ and its imaginary part $\operatorname{Im} a_{-\frac{n}{2}}(U;t,x,\xi)$ is in $\Gamma^{-\frac{n}{2}}_{K,K''+1,N+1}[r]$;*
- *$R(U;t)$ is a real-to-real matrix of smoothing operators in $\Sigma\mathcal{R}^{-\varrho+3(N+1)}_{K,K''+1,1}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$.*

Proof. We transform the equation (4.6.51) for the variable Z.

Case $n = 0$. **Reduction to constant coefficients of order 1.** We first reduce to constant coefficients the transport term of order 1 in (4.6.51). Let $\Phi_{\beta_{\frac12}}(U) := \Phi^\tau_{\beta_{\frac12}}(U;t)|_{\tau=1}$ be the time 1-flow of

$$\partial_\tau \Phi^\tau_{\beta_{\frac12}}(U) = \operatorname{Op}^{BW}_{\mathtt{vec}}\Big(-\mathrm{i}\beta_{\frac12}(U;t,x)|\xi|^{\frac12}\Big)\Phi^\tau_{\beta_{\frac12}}(U), \qquad \Phi^0_{\beta_{\frac12}}(U) = \mathrm{Id}\,,$$

where $\beta_{\frac12}$ is the real function in $\Sigma\mathcal{F}^{\mathbb{R}}_{K,1,1}[r,N]$ defined by

$$\begin{aligned} \beta_{\frac12}(U;t,x) &:= \frac{2}{3\sqrt{\kappa}(1+\zeta(U))}\,\partial_x^{-1}\Big[\mathtt{V}(U;t) - V^{(3)}(U;t,x)\Big] \\ \mathtt{V}(U;t) &:= \frac{1}{2\pi}\int_{\mathbb{T}} V^{(3)}(U;t,x)\,\mathrm{d}x\,. \end{aligned} \tag{4.6.54}$$

Note that the real x-independent function $\mathtt{V}(U;t)$ is in $\Sigma\mathcal{F}^{\mathbb{R}}_{K,1,2}[r,N]$ thanks to Remark 4.2.2 (it could be also directly verified that the linear component in U of the space average of $V^{(3)}$ vanishes).

By (4.3.22) the operator $\operatorname{Op}^{BW}_{\mathtt{vec}}\Big(-\mathrm{i}\beta_{\frac12}|\xi|^{\frac12}\Big)$ is linearly Hamiltonian. By Lemma 4.3.16, the flow $\Phi_{\beta_{\frac12}}(U)$ is a diagonal matrix of spectrally localized maps in $\mathcal{S}^0_{K,1,0}[r] \otimes \mathcal{M}_2(\mathbb{C})$ with its inverse, it is linearly symplectic and $\Phi_{\beta_{\frac12}}(U)^{\pm1} - \mathrm{Id}$ belong to $\Sigma\mathcal{S}^{(N+1)/2}_{K,1,1}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$.

If Z solves equation (4.6.51), then the variable $\breve{Z} := \Phi_{\beta_{\frac12}}(U)Z$ satisfies

$$\begin{aligned} \partial_t \breve{Z} = {} & \Phi_{\beta_{\frac12}}(U)\operatorname{Op}^{BW}_{\mathtt{vec}}\Big(-\mathrm{i}\Big[(1+\zeta(U))\omega(\xi) + \frac{\gamma}{2}\frac{\mathtt{G}(\xi)}{\xi} + V^{(3)}\xi + a_0\Big]\Big)\Phi_{\beta_{\frac12}}(U)^{-1}\breve{Z} \\ & + (\partial_t\Phi_{\beta_{\frac12}}(U))\Phi_{\beta_{\frac12}}(U)^{-1}\breve{Z} + \Phi_{\beta_{\frac12}}(U)R(U;t)\Phi_{\beta_{\frac12}}(U)^{-1}\breve{Z}\,. \end{aligned}$$

Using the Lie expansions in (4.6.45), (4.6.48) with $\mathbf{\Phi} := \Phi_{\beta_{\frac12}}$, $\mathbf{F} := \operatorname{Op}^{BW}_{\mathtt{vec}}\Big(-\mathrm{i}\beta_{\frac12}|\xi|^{\frac12}\Big)$, $L := L(\varrho) \geq 2(\varrho+1)$ (so the integral remainders in the Lie expansions are smoothing operators in $\mathcal{R}^{-\varrho}_{K,K',1}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$), Proposition 4.2.13 and (4.6.3) we obtain

$$\begin{aligned} \partial_t\breve{Z} = {} & \operatorname{Op}^{BW}_{\mathtt{vec}}\Big(-\mathrm{i}\Big[(1+\zeta(U))\omega(\xi) + \frac{\gamma}{2}\frac{\mathtt{G}(\xi)}{\xi}\Big]\Big)\breve{Z} \\ & + \operatorname{Op}^{BW}_{\mathtt{vec}}\Big(-\mathrm{i}\Big[V^{(3)} + \frac32\sqrt{\kappa}(\beta_{\frac12})_x(1+\zeta(U))\Big]\xi\Big)\breve{Z} \\ & + \operatorname{Op}^{BW}_{\mathtt{vec}}\Big(-\mathrm{i}b_{\frac12}(U;t,x)|\xi|^{\frac12} - \mathrm{i}a_0^{(1)}(U;t,x,\xi)\Big)\breve{Z} + R'(U;t)\breve{Z} + \Phi_{\beta_{\frac12}}(U)R(U;t)\Phi_{\beta_{\frac12}}(U)^{-1}\breve{Z} \end{aligned} \tag{4.6.55}$$

where $R'(U;t)$ belongs to $\Sigma\mathcal{R}^{-\varrho}_{K,K',1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ and

$$b_{\frac12}(U;t,x) := -\frac12\beta_{\frac12}V^{(3)}_x + (\beta_{\frac12})_x V^{(3)} - \frac34\sqrt{\kappa}(1+\zeta(U))\left(\frac12\beta_{\frac12}(\beta_{\frac12})_{xx} + (\beta_{\frac12})^2_x\right) + \partial_t\beta_{\frac12} \tag{4.6.56}$$

is a real valued function in $\Sigma\mathcal{F}^{\mathbb{R}}_{K,2,1}[r,N]$ (use also the last bullets at the end of Section 4.2.1 and Proposition 4.2.15-(iv)), and we collect in $a_0^{(1)}(U;t,x,\xi)$ all the symbols in $\Sigma\Gamma^0_{K,K',1}[r,N]$. Finally by Proposition 4.2.19 we deduce that $\Phi_{\beta_{\frac12}}(U)R(U;t)\Phi_{\beta_{\frac12}}(U)^{-1}$ is in $\Sigma\mathcal{R}^{-\varrho+3(N+1)}_{K,K',1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$. By (4.6.54) the first order term in (4.6.55) is constant coefficient, namely

$$V^{(3)}(U;t,x) + \frac32\sqrt{\kappa}(\beta_{\frac12})_x(U;t,x)(1+\zeta(U)) = \mathtt{V}(U;t)\,,$$

and (4.6.55) reduces to

$$\partial_t\breve{Z} = \mathrm{Op}^{BW}_{\mathtt{vec}}\left(-\mathrm{i}\left[(1+\zeta(U))\omega(\xi) + \frac{\gamma}{2}\frac{\mathtt{G}(\xi)}{\xi} + \mathtt{V}(U;t)\xi + b_{\frac12}|\xi|^{\frac12} + a_0^{(1)}\right]\right)\breve{Z} + R(U;t)\breve{Z}\,. \tag{4.6.57}$$

The para-differential operators of positive order in (4.6.57) are linearly Hamiltonian, whereas $\mathrm{Op}^{\mathrm{BW}}_{\mathtt{vec}}(-\mathrm{i}a_0^{(1)})$ might not be. By the usual argument, we replace each homogeneous component of $a_0^{(1)}$ with its symmetrized version obtaining that $\mathrm{Op}^{\mathrm{BW}}_{\mathtt{vec}}(-\mathrm{i}a_0^{(1)})$ is linearly Hamiltonian up to homogeneity N, by adding another smoothing operator.

Reduction to constant coefficients of order $\frac12$. The next step is to put to constant coefficients the symbol $-\mathrm{i}b_{\frac12}|\xi|^{\frac12}$ in system (4.6.57). Let $\Phi_{\beta_0}(U) := \Phi^{\tau}_{\beta_0}(U;t)|_{\tau=1}$ be the time 1-flow of

$$\partial_\tau\Phi^\tau_{\beta_0}(U) = \mathrm{Op}^{BW}_{\mathtt{vec}}\left(\mathrm{i}\beta_0(U;t,x)\,\mathrm{sign}\,\xi\right)\Phi^\tau_{\beta_0}(U), \qquad \Phi^0_{\beta_0}(U) = \mathrm{Id}\,,$$

where the real function β_0 in $\Sigma\mathcal{F}^{\mathbb{R}}_{K,2,1}[r,N]$ is

$$\begin{aligned}\beta_0(U;t,x) &= \frac{2}{3\sqrt{\kappa}(1+\zeta(U))}\partial_x^{-1}\left(b_{\frac12}(U;t,x) - \mathtt{b}_{\frac12}(U;t)\right)\,,\\ \mathtt{b}_{\frac12}(U;t) &:= \frac{1}{2\pi}\int_{\mathbb{T}} b_{\frac12}(U;t,x)\,\mathrm{d}x\,.\end{aligned} \tag{4.6.58}$$

Note that the real x-independent function $\mathtt{b}_{\frac12}(U;t)$ is in $\Sigma\mathcal{F}^{\mathbb{R}}_{K,2,2}[r,N]$ thanks to Remark 4.2.2 (it also follows by (4.6.56) since its linear component in U comes from $\partial_t\beta_{\frac12}$ which has zero average, see (4.6.54)).

By (4.3.22), the operator $\mathrm{Op}^{BW}_{\mathtt{vec}}(\mathrm{i}\beta_0\,\mathrm{sign}\,\xi)$ is linearly Hamiltonian. Hence by Lemma 4.3.16, $\Phi_{\beta_0}(U)$ is a diagonal matrix of spectrally localized maps in $\mathcal{S}^0_{K,2,0}[r]\otimes\mathcal{M}_2(\mathbb{C})$ with its inverse, it is linearly symplectic and $\Phi_{\beta_0}(U)^{\pm1} - \mathrm{Id}$ belong to $\Sigma\mathcal{S}^0_{K,2,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$.

If $\breve{Z}$ solves (4.6.57) then the variable $Z_0 := \Phi_{\beta_0}(U)\breve{Z}$ solves

$$\begin{aligned}\partial_t Z_0 = {}& \Phi_{\beta_0}(U)\,\mathrm{Op}^{BW}_{\mathtt{vec}}\left(-\mathrm{i}\left[(1+\zeta(U))\omega(\xi) + \frac{\gamma}{2}\frac{\mathtt{G}(\xi)}{\xi} + \mathtt{V}(U;t)\xi\right]\right)\Phi_{\beta_0}(U)^{-1}Z_0\\ &+ \Phi_{\beta_0(U)}\,\mathrm{Op}^{BW}_{\mathtt{vec}}\left(-\mathrm{i}b_{\frac12}|\xi|^{\frac12} - \mathrm{i}a_0^{(1)}\right)\Phi_{\beta_0}(U)^{-1}Z_0\\ &+ (\partial_t\Phi_{\beta_0}(U))\Phi^{-1}_{\beta_0}(U)Z_0 + \Phi_{\beta_0}(U)R(U;t)\Phi_{\beta_0}(U)^{-1}Z_0\,.\end{aligned}$$

Using the Lie expansions in (4.6.45), (4.6.48) with $\mathbf{\Phi} := \Phi_{\beta_0}$, $\mathbf{F} := {\rm Op}^{BW}_{\tt vec}({\rm i}\beta_0 {\rm sign}(\xi))$, $L := L(\varrho)$ large enough so that the integral remainders in the Lie expansions are ϱ-smoothing operators, and (4.6.3) we obtain

$$\begin{aligned}\partial_t Z_0 &= {\rm Op}^{BW}_{\tt vec}\Big(-{\rm i}\Big[(1+\zeta(U))\omega(\xi)+\frac{\gamma}{2}\frac{{\tt G}(\xi)}{\xi}+{\tt V}(U;t)\xi\Big]\Big)Z_0 \\ &\quad + {\rm Op}^{BW}_{\tt vec}\Big({\rm i}\Big(\frac{3}{2}\sqrt{\kappa}(\beta_0)_x(1+\zeta(U))-b_{\frac12}\Big)|\xi|^{\frac12}+{\rm i}a_0^{(2)}\Big)Z_0+R(U;t)Z_0\end{aligned} \tag{4.6.59}$$

where $a_0^{(2)}$ is a symbol in $\Sigma\Gamma^0_{K,K',1}[r,N]$ and $R(U;t)$ is a real-to-real matrix of smoothing operators in $\Sigma\mathcal{R}^{-\varrho+3(N+1)}_{K,K',1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$. By (4.6.58) the symbol of order $\frac12$ in (4.6.59) is constant coefficient, namely

$$\frac{3}{2}\sqrt{\kappa}(\beta_0)_x(1+\zeta(U))-b_{\frac12}=-{\tt b}_{\frac12}(U;t)\,,$$

and system (4.6.59) reduces to

$$\begin{aligned}\partial_t Z_0 &= {\rm Op}^{BW}_{\tt vec}\Big(-{\rm i}\Big[(1+\zeta(U))\omega(\xi)+\frac{\gamma}{2}\frac{{\tt G}(\xi)}{\xi}+{\tt V}(U;t)\xi+{\tt b}_{\frac12}(U;t)|\xi|^{\frac12}\Big]+{\rm i}a_0^{(2)}\Big)Z_0 \\ &\quad + R(U;t)Z_0\,.\end{aligned} \tag{4.6.60}$$

By Lemma 4.3.21, we replace each homogeneous component of ${\rm Op}^{\rm BW}_{\tt vec}({\rm i}a_0^{(2)})$ so that it becomes linearly Hamiltonian up to homogeneity N, i.e. by (4.3.31), it results that ${\rm Im}\,a_0^{(2)}$ belongs to $\Sigma\Gamma^0_{K,K',N+1}[r,N]$.

So far we have shown that (4.6.60) becomes (4.6.52) with $n=0$, putting $a_0 := a_0^{(2)}$ (which we consider as a symbol in $\Sigma\Gamma^0_{K,K''+1,1}[r,N]$) and ${\tt b}_0^{(0)} := 0$. We put $\mathcal{F}_0(U) := \Phi_{\beta_0}(U)\Phi_{\beta_{\frac12}}(U)$.

Case $n \rightsquigarrow n+1$. The proof is by induction on n. Suppose that Z_n is a solution of system (4.6.52). Let $\Phi_{F_n}(U) := \Phi^\tau_{F_n}(U;t)|_{\tau=1}$ be the time 1-flow

$$\partial_\tau\Phi^\tau_{F_n}(U)={\rm Op}^{BW}_{\tt vec}\Big({\rm i}\beta_{-\frac{n}{2}-\frac12}(U;t,x,\xi)\Big)\Phi^\tau_{F_n}(U)\,,\quad \Phi^0_{F_n}(U)={\rm Id}\,,$$

where

$$\begin{aligned}\beta_{-\frac{n}{2}-\frac12}(U;t,x,\xi) &:= -\frac{2\,{\rm sign}\,\xi}{3\sqrt{\kappa}(1+\zeta(U))|\xi|^{\frac12}}\partial_x^{-1}\Big(a_{-\frac{n}{2}}(U;t,x,\xi)-{\tt a}_{-\frac{n}{2}}(U;t)\Big)\,, \\ {\tt a}_{-\frac{n}{2}}(U;t) &:= \frac{1}{2\pi}\int_{\mathbb{T}} a_{-\frac{n}{2}}(U;t,x,\xi){\rm d}x\,.\end{aligned} \tag{4.6.61}$$

By the inductive assumption, the symbol $a_{-\frac{n}{2}}$ belongs to $\Sigma\Gamma^{-\frac{n}{2}}_{K,K''+1,1}[r,N]$ and has imaginary part in $\Gamma^{-\frac{n}{2}}_{K,K''+1,N+1}[r]$. Then the x-independent symbol ${\tt a}_{-\frac{n}{2}}$ belongs to $\Sigma\Gamma^{-\frac{n}{2}}_{K,K''+1,2}[r,N]$ thanks to Remark 4.2.2 and ${\rm Im}\,{\tt a}_{-\frac{n}{2}}\in\Gamma^{-\frac{n}{2}}_{K,K''+1,N+1}[r]$. It follows that the symbol $\beta_{-\frac{n}{2}-\frac12}$ belongs to $\Sigma\Gamma^{-\frac{n}{2}-\frac12}_{K,K''+1,1}[r,N]$ and has imaginary part in $\Gamma^{-\frac{n}{2}-\frac12}_{K,K''+1,N+1}[r]$.

Therefore by (4.3.31) the operator ${\rm Op}^{\rm BW}_{\tt vec}({\rm i}\beta_{-\frac{n}{2}-\frac12})$ is linearly Hamiltonian up to homogeneity N. By Lemma 4.3.16, the flow $\Phi_{F_n}(U)$ is invertible, linearly symplectic up to homogeneity N and $\Phi_{F_n}(U)^{\pm1}-{\rm Id}$ belong to $\Sigma\mathcal{S}^0_{K,K''+1,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$.

If Z_n solves (4.6.52) then the variable $Z_{n+1} := \Phi_{F_n}(U)Z_n$ solves

$$\partial_t Z_{n+1} = \Phi_{F_n}(U)\,\mathrm{Op}^{BW}_{\mathtt{vec}}\Big(\mathrm{i}\mathtt{m}^{(n)}_{\frac{3}{2}}(U;t,\xi) + \mathrm{i}a_{-\frac{n}{2}}\Big)\Phi_{F_n}(U)^{-1}Z_{n+1} \\ + (\partial_t\Phi_{F_n}(U))\Phi_{F_n}(U)^{-1}Z_n + \Phi_{F_n}(U)R(U;t)\Phi_{F_n}(U)^{-1}Z_{n+1}\,.$$

Using the Lie expansions in (4.6.45), (4.6.48) with $\mathbf{\Phi} := \Phi_{F_n}$, $\mathbf{F} := \mathrm{Op}^{BW}_{\mathtt{vec}}\Big(\mathrm{i}\beta_{-\frac{n}{2}-\frac{1}{2}}\Big)$ with ($L := L(\varrho)$ large enough), the last bullets at the end of Section 4.2.1 and Proposition 4.2.15-(iv), (4.6.53), (4.6.3), we obtain that

$$\begin{aligned}\partial_t Z_{n+1} &= \mathrm{Op}^{BW}_{\mathtt{vec}}\Big(\mathrm{i}\mathtt{m}^{(n)}_{\frac{3}{2}} + \mathrm{i}\Big[\frac{3}{2}\sqrt{\kappa}(\beta_{-\frac{n}{2}-\frac{1}{2}})_x(1+\zeta(U))|\xi|^{\frac{1}{2}}\mathrm{sign}\,\xi + a_{-\frac{n}{2}}\Big]\Big)Z_{n+1} \\ &\quad + \mathrm{Op}^{BW}_{\mathtt{vec}}\Big(\mathrm{i}a_{-\frac{n}{2}-\frac{1}{2}}\Big)Z_{n+1} + R(U;t)Z_{n+1} \\ &\overset{(4.6.61)}{=} \mathrm{Op}^{BW}_{\mathtt{vec}}\Big(\mathrm{i}\mathtt{m}^{(n)}_{\frac{3}{2}} + \mathrm{i}\mathtt{a}_{-\frac{n}{2}} + \mathrm{i}a_{-\frac{n}{2}-\frac{1}{2}}\Big)Z_{n+1} + R(U;t)Z_{n+1}\end{aligned} \tag{4.6.62}$$

where we collect in $a_{-\frac{n}{2}-\frac{1}{2}}$ all the symbols in $\Sigma\Gamma^{-\frac{n}{2}-\frac{1}{2}}_{K,K''+2,1}[r,N]$, and $R(U;t)$ is a smoothing operator in $\Sigma\mathcal{R}^{-\varrho+3(N+1)}_{K,K''+2,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$. By Lemma 4.3.21, we replace each homogeneous component of $\mathrm{Op}^{\mathrm{BW}}_{\mathtt{vec}}(\mathrm{i}a_{-\frac{n}{2}-\frac{1}{2}})$ so that it is linearly Hamiltonian up to homogeneity N; which, by (4.3.31), is equivalent to assume that the imaginary part $\mathrm{Im}\,a_{-\frac{n}{2}-\frac{1}{2}}$ is a symbol in $\Gamma^{-\frac{n}{2}-\frac{1}{2}}_{K,K''+2,N+1}[r]$.

System (4.6.62) has the form (4.6.52) at step $n+1$ with $\mathtt{b}^{(n+1)}_0 := \mathtt{b}^{(n)}_0 + \mathtt{a}_{-\frac{n}{2}}$ (hence $\mathtt{m}^{(n+1)}_{\frac{3}{2}} := \mathtt{m}^{(n)}_{\frac{3}{2}} + \mathtt{a}_{-\frac{n}{2}}$) and $K''(n+1) := K''(n)+1$.

The thesis follows with $\mathcal{F}_{n+1}(U) := \Phi_{F_n}(U)\mathcal{F}_n(U)$. The proof of Lemma 4.6.9 is complete. □

Proof of Proposition 4.6.1. We now choose in Lemma 4.6.9 a number $n := n_2(\varrho)$ of iterative steps satisfying $n_2(\varrho) \geq 2(\varrho-3(N+1))$ so that we incorporate $\mathrm{Op}^{\mathrm{BW}}_{\mathtt{vec}}(\mathrm{i}a_{-\frac{n}{2}})$ in the smoothing remainder $R(U;t)$, which belongs to $\Sigma\mathcal{R}^{-\varrho+3(N+1)}_{K,K''+1,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ with $K'' = n_1(\varrho)+n_2(\varrho) \geq 3\varrho-8(N+1)$, with $n_1(\varrho)$ fixed above (4.6.51). Denoting $W := Z_n$, system (4.6.52) has the form (4.6.1) with $\mathtt{b}_0(U;t,\xi) := -\mathtt{b}^{(n_2)}_0(U;t,\xi)$ in (4.6.2) and taking as $\underline{K}' := K''+1 = n_1(\varrho)+n_2(\varrho)+1$ (this proves Remark 4.6.3). The variable W can be written as $W = \mathbf{B}(U;t)U$ where

$$\mathbf{B}(U;t) := \mathcal{F}_{n_2}(U)\circ\Phi_{n_1}(U)\circ\Psi_2(U)\circ\Psi_1(U)\circ\mathcal{G}(U)$$

and $\mathcal{G}(U)$ is the map of Lemma 4.6.4, $\Psi_1(U)$ is the map of Lemma 4.6.5, $\Psi_2(U)$ is the map of Lemma 4.6.7, $\Phi_{n_1}(U)$ is the map of Lemma 4.6.8 with number of steps $n_1 := n_1(\varrho)$ and $\mathcal{F}_{n_2}(U)$ is the map of Lemma 4.6.9 with number of steps $n_2 := n_2(\varrho)$. Since

$$\mathcal{G}(U)-\mathrm{Id},\quad \Psi_1(U)-\mathrm{Id}\in\Sigma\mathcal{S}^0_{K,0,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C}),\quad \Phi_{n_1}(U)-\mathrm{Id}\in\Sigma\mathcal{S}^0_{K,K',1}[r,N]\otimes\mathcal{M}_2(\mathbb{C}) \\ \Psi_2(U)-\mathrm{Id}\in\Sigma\mathcal{S}^{N+1}_{K,0,2}[r,N]\otimes\mathcal{M}_2(\mathbb{C}),\quad \mathcal{F}_{n_2}(U)-\mathrm{Id}\in\Sigma\mathcal{S}^{(N+1)/2}_{K,K'',1}[r,N]\otimes\mathcal{M}_2(\mathbb{C}),$$

we deduce by Proposition 4.2.19 that $\mathbf{B}(U;t)-\mathrm{Id}$ is a real-to-real matrix of spectrally localized maps in $\Sigma\mathcal{S}^{\frac{3}{2}(N+1)}_{K,\underline{K}'-1,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$. In addition $\mathbf{B}(U;t)$ is a spectrally localized map in $\mathcal{S}^0_{K,\underline{K}'-1,0}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ with its inverse, as each map $\mathcal{F}_{n_2}(U)$, $\Phi_{n_1}(U)$, $\Psi_2(U)$, $\Psi_1(U)$, $\mathcal{G}(U)$ separately. Finally $\mathbf{B}(U;t)$ is linearly symplectic up to homogeneity N, being the composition of linearly symplectic maps up to homogeneity N. This completes the proof of Proposition 4.6.1.

4.7 Hamiltonian Birkhoff normal form

The main result of this section is Proposition 4.7.12 which transforms the water waves equations in Hamiltonian Birkhoff normal form. This is required to ensure that the life span of the solutions is of order ε^{-N-1} with $N \in \mathbb{N}$. So from now on we take $N \in \mathbb{N}$.

In Proposition 4.6.1 we have conjugated the water waves Hamiltonian system (4.5.37) into (4.6.1), by applying the transformation $W = \mathbf{B}(U;t)U$ which is just linearly symplectic up to homogeneity N. Thus the transformed system (4.6.1) is not Hamiltonian anymore. The first goal of this section is to construct a nearby transformation which is symplectic up to homogeneity N, according to Definition 4.3.11, thus obtaining a Hamiltonian system up to homogeneity N, according to Definition 4.3.10.

4.7.1 Hamiltonian correction up to homogeneity N

We first prove the following abstract result, which is a direct consequence of Theorem 4.4.1.

Theorem 4.7.1. *Let $p, N \in \mathbb{N}$ with $p \leq N$, $K, K' \in \mathbb{N}_0$ with $K' + 1 \leq K$, $r > 0$. Let $Z = \mathbf{M}_0(U;t)U$ with $\mathbf{M}_0(U;t) \in \mathcal{M}^0_{K,K',0}[r] \otimes \mathcal{M}_2(\mathbb{C})$ as in (4.3.35). Assume that $Z(t)$ solves a Hamiltonian system up to homogeneity N, according to Definition 4.3.10. Consider*

$$\Phi(Z) := \mathbf{B}(Z;t)Z \tag{4.7.1}$$

where

- $\mathbf{B}(Z;t) - \mathrm{Id}$ *is a matrix of spectrally localized maps in*

$$\mathbf{B}(Z;t) - \mathrm{Id} \in \begin{cases} \Sigma\mathcal{S}_{K,K',p}[r,N] \otimes \mathcal{M}_2(\mathbb{C}) & \text{if} \quad \mathbf{M}_0(U;t) = \mathrm{Id}\,, \\ \Sigma\mathcal{S}_{K,0,p}[\breve{r},N] \otimes \mathcal{M}_2(\mathbb{C})\,, \forall \breve{r} > 0 & \text{otherwise}\,. \end{cases} \tag{4.7.2}$$

- $\mathbf{B}(Z;t)$ *is linearly symplectic up to homogeneity N, according to Definition 4.3.7.*

Then there exists a real-to-real matrix of pluri–homogeneous smoothing operators $R_{\leq N}(\cdot)$ in $\Sigma_p^N \widetilde{\mathcal{R}}^{-\varrho}_q \otimes \mathcal{M}_2(\mathbb{C})$, for any $\varrho > 0$, such that the non-linear map

$$Z_+ := \big(\mathrm{Id} + R_{\leq N}(\Phi(Z))\big)\Phi(Z)$$

is symplectic up to homogeneity N (Definition 4.3.11) and thus Z_+ solves a system which is Hamiltonian up to homogeneity N.

Proof. We decompose $\mathbf{B}(Z;t) = \mathbf{B}_{\leq N}(Z) + \mathbf{B}_{>N}(Z;t)$ where $\mathbf{B}_{\leq N}(Z) := \mathcal{P}_{\leq N}[\mathbf{B}(Z;t)]$. Note that $\mathbf{B}_{\leq N}(Z) - \mathrm{Id}$ is in $\Sigma_p^N \widetilde{\mathcal{S}}_q \otimes \mathcal{M}_2(\mathbb{C})$ and $\mathbf{B}_{>N}(Z;t)$ is in $\mathcal{S}_{K,K',N+1}[r] \otimes \mathcal{M}_2(\mathbb{C})$. Since $\mathbf{B}(Z;t)$ is linearly symplectic up to homogeneity N, its pluri-homogeneous component $\mathbf{B}_{\leq N}(Z)$ is linearly symplectic up to homogeneity N as well. Then Theorem 4.4.1 applied to $\Phi_{\leq N}(Z) := \mathbf{B}_{\leq N}(Z)Z$ implies the existence of pluri-homogeneous smoothing operators $R_{\leq N}(\cdot)$ in $\Sigma_p^N \widetilde{\mathcal{R}}^{-\varrho}_q \otimes \mathcal{M}_2(\mathbb{C})$ for any $\varrho \geq 0$, such that the nonlinear map $\mathcal{D}_N(Z) := \big(\mathrm{Id} + R_{\leq N}(\Phi_{\leq N}(Z))\big)\Phi_{\leq N}(Z)$ is *symplectic up to homogeneity N*. We then write

$$\mathcal{D}(Z;t) := \big(\mathrm{Id} + R_{\leq N}(\Phi(Z))\big)\Phi(Z) = \mathcal{D}_N(Z) + M_{>N}(Z;t)Z$$

where, using Proposition 4.2.15-(ii) and the first bullet after Definition 4.2.5,

$$M_{>N}(Z;t) \in \begin{cases} \mathcal{M}_{K,K',N+1}[r] \otimes \mathcal{M}_2(\mathbb{C}) & \text{if} \quad \mathbf{M}_0(U;t) = \mathrm{Id}\,, \\ \mathcal{M}_{K,0,N+1}[\breve{r}] \otimes \mathcal{M}_2(\mathbb{C})\,, \forall \breve{r} > 0 & \text{otherwise}\,, \end{cases}$$

showing that $\mathcal{D}(Z;t)$ is symplectic up to homogeneity N as well. Then Lemma 4.3.15 implies the thesis. □

The first application of Theorem 4.7.1 is to provide a symplectic correction of the map $\Phi(U) := \mathbf{B}(U;t)U$ of Proposition 4.6.1 and to conjugate the Hamiltonian system (4.5.37) into system (4.7.4), which is *Hamiltonian up to homogeneity* N.

Proposition 4.7.2 (Hamiltonian reduction up to smoothing operators). *Let $N \in \mathbb{N}$ and $\varrho > c(N) := 3(N+1) + \frac{3}{2}(N+1)^3$. Then for any $K \geq \underline{K}'$ (fixed in Proposition 4.6.1) there is $s_0 > 0, r > 0$, such that for any solution $U \in B^K_{s_0,\mathbb{R}}(I;r)$ of (4.5.37), there exists a real-to-real matrix of pluri–homogeneous smoothing operators $R(U)$ in $\Sigma_1^N \widetilde{\mathcal{R}}_q^{-\varrho'} \otimes \mathcal{M}_2(\mathbb{C})$ for any $\varrho' \geq 0$, such that defining*

$$Z_0 := \big(\mathrm{Id} + R(\Phi(U))\big)\Phi(U), \quad \Phi(U) := \mathbf{B}(U;t)U\,, \tag{4.7.3}$$

where $\mathbf{B}(U;t)$ is the real-to-real matrix of spectrally localized maps defined in Proposition 4.6.1, the following holds true:

(i) **Symplecticity:** *The non-linear map in (4.7.3) is symplectic up to homogeneity N according to Definition 4.3.11.*

(ii) **Conjugation:** *the variable Z_0 solves the* Hamiltonian system up to homogeneity N *(cfr. Definition 4.3.10)*

$$\begin{aligned}\partial_t Z_0 = -\mathrm{i}\boldsymbol{\Omega}(D)Z_0 + \mathrm{Op}^{BW}_{\mathtt{vec}}\Big(-\mathrm{i}(\mathtt{m}_{\frac{3}{2}})_{\leq N}(Z_0;\xi) - \mathrm{i}(\mathtt{m}_{\frac{3}{2}})_{>N}(U;t,\xi)\Big)Z_0 \\ + R_{\leq N}(Z_0)Z_0 + R_{>N}(U;t)U\end{aligned} \tag{4.7.4}$$

where

- *$\boldsymbol{\Omega}(D)$ is the diagonal matrix of Fourier multipliers defined in (4.5.10);*
- *$(\mathtt{m}_{\frac{3}{2}})_{\leq N}(Z_0;\xi)$ is a real valued symbol, independent of x, in $\Sigma_2^N \widetilde{\Gamma}_q^{\frac{3}{2}}$;*
- *$(\mathtt{m}_{\frac{3}{2}})_{>N}(U;t,\xi)$ is a non–homogeneous symbol, independent of x, in $\Gamma^{\frac{3}{2}}_{K,\underline{K}',N+1}[r]$ with imaginary part $\mathrm{Im}\,(\mathtt{m}_{\frac{3}{2}})_{>N}(U;t,\xi)$ in $\Gamma^0_{K,\underline{K}',N+1}[r]$;*
- *$R_{\leq N}(Z_0)$ is a real-to-real matrix of smoothing operators in $\Sigma_1^N \widetilde{\mathcal{R}}_q^{-\varrho+c(N)} \otimes \mathcal{M}_2(\mathbb{C})$;*
- *$R_{>N}(U;t)$ is a real-to-real matrix of non–homogeneous smoothing operators in $\mathcal{R}^{-\varrho+c(N)}_{K,\underline{K}',N+1}[r] \otimes \mathcal{M}_2(\mathbb{C})$.*

(iii) **Boundedness:** *The variable $Z_0 = \mathbf{M}_0(U;t)U$ with $\mathbf{M}_0(U;t) \in \mathcal{M}^0_{K,\underline{K}'-1,0}[r] \otimes \mathcal{M}_2(\mathbb{C})$ and for any $s \geq s_0$, for all $0 < r < r_0(s)$ small enough, for any $U \in B^K_{s_0}(I;r) \cap C^K_{*\mathbb{R}}(I;\dot{H}^s(\mathbb{T},\mathbb{C}^2))$, there is a constant $C := C_{s,K} > 0$ such that, for all $k = 0,\ldots,K-\underline{K}'$,*

$$C^{-1}\|U\|_{k,s} \leq \|Z_0\|_{k,s} \leq C\|U\|_{k,s}\,. \tag{4.7.5}$$

Proof. We construct the symplectic corrector to the map $W := \Phi(U) = \mathbf{B}(U;t)U$ of Proposition 4.6.1 by Theorem 4.7.1. Let us check its assumptions. By Lemma 4.5.5, the function U solves the Hamiltonian system (4.5.37). By Proposition 4.6.1, $\mathbf{B}(U;t) - \mathrm{Id}$ is a spectrally localized map in $\Sigma\mathcal{S}^{\frac{3}{2}(N+1)}_{K,\underline{K}'-1,1}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$ and $\mathbf{B}(U;t)$ is linearly symplectic up to homogeneity N. So Theorem 4.7.1 (in the case $\mathbf{M}_0(U;t) = \mathrm{Id}$) implies the existence of a matrix of pluri–homogeneous smoothing operators $R(W)$ in $\Sigma_1^N\widetilde{\mathcal{R}}_q^{-\varrho'} \otimes \mathcal{M}_2(\mathbb{C})$, for any $\varrho' \geq 0$, such that the variable

$$Z_0 := (\mathrm{Id} + R(\Phi(U)))\Phi(U) = (\mathrm{Id} + R(W))W \tag{4.7.6}$$

solves a system which is Hamiltonian up to homogeneity N. We now prove that such system has the form (4.7.4). We will compute it by transforming system (4.6.1) solved by $W(t)$ under the change of variable $Z_0 := (\mathrm{Id}+R(W))W$, exploiting that $R(W)$ is a pluri-homogeneous smoothing operator. We first substitute the variable U with the variable W in the homogeneous components up to degree N of both the symbols and the smoothing operator in (4.6.1). We first use Lemma A.0.1 (with $\mathbf{M}_0(U;t) = \mathrm{Id}$ and $p = 1$) to construct an approximate inverse of $W = \Phi(U)$, getting

$$U = \Psi_{\leq N}(W) + M_{>N}(U;t)U\,, \qquad \Psi_{\leq N}(W) = W + \breve{S}_{\leq N}(W)W\,, \tag{4.7.7}$$

where $\breve{S}_{\leq N}(W) \in \Sigma_1^N \widetilde{\mathcal{S}}_q^{\frac{3}{2}(N+1)N} \otimes \mathcal{M}_2(\mathbb{C})$ and $M_{>N}(U;t)$ is a matrix of operators in $\mathcal{M}^{\frac{3}{2}(N+1)^2}_{K,\underline{K}',N+1}[r] \otimes \mathcal{M}_2(\mathbb{C})$. Next we substitute (4.7.7) in the homogeneous components of order $\leq N$ in system (4.6.1) of

$$\mathrm{Op}^{BW}_{\mathtt{vec}}\Big(\mathtt{im}_{\frac{3}{2}}(U;t,\xi)\Big) = \mathrm{Op}^{BW}_{\mathtt{vec}}\Big(\mathrm{i}(\mathtt{m}_{\frac{3}{2}})_{\leq N}(U) + \mathrm{i}(\mathtt{m}_{\frac{3}{2}})_{>N}(U;t,\xi)\Big)\,, \quad R(U;t) = R_{\leq N}(U) + R_{>N}(U;t)\,,$$

and substitute $W = \mathbf{B}(U;t)U$ in the term $R_{>N}(U;t)W$. By (4.6.1), (4.6.2), Lemma A.0.2 (with $Z \rightsquigarrow U$, $m' \rightsquigarrow \frac{3}{2}$, $m \rightsquigarrow \frac{3}{2}(N+1)^2$ and $\varrho \rightsquigarrow \varrho - 3(N+1)$) and Proposition 4.2.15 (i) we obtain

$$\begin{aligned} \partial_t W = -\mathrm{i}\boldsymbol{\Omega}(D)W + \mathrm{Op}^{BW}_{\mathtt{vec}}\Big(-\mathrm{i}(\widetilde{\mathtt{m}}_{\frac{3}{2}})_{\leq N}(W;\xi) - \mathrm{i}(\widetilde{\mathtt{m}}_{\frac{3}{2}})_{>N}(U;t,\xi)\Big)W \\ + \widetilde{R}_{\leq N}(W)W + \widetilde{R}_{>N}(U;t)U \end{aligned} \tag{4.7.8}$$

where

- $(\widetilde{\mathtt{m}}_{\frac{3}{2}})_{\leq N}(W;\xi)$ is a real valued symbol, independent of x, in $\Sigma_2^N \widetilde{\Gamma}_q^{\frac{3}{2}}$;
- $(\widetilde{\mathtt{m}}_{\frac{3}{2}})_{>N}(U;t,\xi)$ is a non–homogeneous symbol, independent of x, in $\Gamma^{\frac{3}{2}}_{K,\underline{K}',N+1}[r]$ given by the sum of the old non-homogeneous symbol $\mathcal{P}_{\geq N+1}(-\mathtt{m}_{\frac{3}{2}}(U;t,\xi))$ in (4.6.1)-(4.6.2) and a purely real correction coming from formula (A.0.7) (cfr. $a^+_{>N}$) hence its imaginary part $\mathrm{Im}\,(\widetilde{\mathtt{m}}_{\frac{3}{2}})_{>N}(U;t,\xi)$ is in $\Gamma^0_{K,\underline{K}',N+1}[r]$;
- $\widetilde{R}_{\leq N}(W)$ is a matrix of pluri-homogeneous smoothing operators in $\Sigma_1^N \widetilde{\mathcal{R}}_q^{-\varrho+c(N)} \otimes \mathcal{M}_2(\mathbb{C})$ with $c(N) = 3(N+1) + \frac{3}{2}(N+1)^3$;
- $\widetilde{R}_{>N}(U;t)$ is a matrix of non–homogeneous smoothing operators in $\mathcal{R}^{-\varrho+c(N)}_{K,\underline{K}',N+1}[r] \otimes \mathcal{M}_2(\mathbb{C})$.

We finally conjugate system (4.7.8) under the change of variable $Z_0 = W + R(W)W$ defined in (4.7.6). Note that system (4.7.8) fulfills Assumption **(A)** at page 215 with $p \rightsquigarrow 1$, with $W(t)$ replacing $Z(t)$, $\mathbf{M}_0(U;t) \rightsquigarrow \mathbf{B}(U;t)$ and $\varrho \rightsquigarrow \varrho - c(N)$. Then we apply Lemma A.0.5 with the smoothing perturbation of the identity defined in (4.7.6) (choosing also $\varrho' := \varrho + \frac{3}{2}$) and we deduce that $Z(t)$ satisfies system (4.7.4). Item (iii) follows from (4.7.3), the fact that $\mathbf{B}(U;t) \in \mathcal{S}^0_{K,\underline{K}'-1,0}[r] \otimes \mathcal{M}_2(\mathbb{C})$ (Proposition 4.6.1 (i)), the fact that $\mathrm{Id} + R(Z) \in \mathcal{M}^0_{K,0,0}[\breve{r}] \otimes \mathcal{M}_2(\mathbb{C})$ for any $\breve{r} > 0$ (by Lemma 4.2.8 and since $R(Z)$ is pluri-homogeneous) and by Proposition 4.2.15 items (iii) (with $K' \rightsquigarrow \underline{K}' - 1$) and (i). Finally estimate (4.7.5) follows combining also (4.2.40) and the estimate below (4.2.72) for $\mathbf{B}(U;t)$ and $\mathbf{B}(U;t)^{-1}$. □

4.7.2 Super action preserving symbols and Hamiltonians

In this section we define the special class of "super–action preserving" SAP homogeneous symbols and Hamiltonians which will appear in the Birkhoff normal form reduction of the next Section 4.7.3.

Definition 4.7.3. (SAP **multi-index**) *A multi-index* $(\alpha,\beta) \in \mathbb{N}_0^{\mathbb{Z}\setminus\{0\}} \times \mathbb{N}_0^{\mathbb{Z}\setminus\{0\}}$ *is* super action preserving *if*

$$\alpha_n + \alpha_{-n} = \beta_n + \beta_{-n}\,, \qquad \forall n \in \mathbb{N}\,. \tag{4.7.9}$$

A super action preserving multi-index (α,β) satisfies $|\alpha| = |\beta|$ where $|\alpha| := \sum_{j\in\mathbb{Z}\setminus\{0\}} \alpha_j$. If a multi-index $(\alpha,\beta) \in \mathbb{N}_0^{\mathbb{Z}\setminus\{0\}} \times \mathbb{N}_0^{\mathbb{Z}\setminus\{0\}}$ is not super action preserving, then the set

$$\mathfrak{N}(\alpha,\beta) := \Big\{ n \in \mathbb{N} : \ \alpha_n + \alpha_{-n} - \beta_n - \beta_{-n} \neq 0 \Big\} \tag{4.7.10}$$

is not empty and, since $\mathfrak{N}(\alpha,\beta) \subset \{n \in \mathbb{N} : \alpha_n + \alpha_{-n} + \beta_n + \beta_{-n} \neq 0\}$, its cardinality satisfies

$$|\mathfrak{N}(\alpha,\beta)| \le |\alpha + \beta| = |\alpha| + |\beta|. \tag{4.7.11}$$

Definition 4.7.4. (SAP **monomial**) *Let $p \in \mathbb{N}$. Given $(\vec{\jmath},\vec{\sigma}) = (j_a,\sigma_a)_{a=1,\dots,p} \in (\mathbb{Z}\setminus\{0\})^p \times \{\pm\}^p$ we define the multi-index $(\alpha,\beta) \in \mathbb{N}_0^{\mathbb{Z}\setminus\{0\}} \times \mathbb{N}_0^{\mathbb{Z}\setminus\{0\}}$ with components, for any $k \in \mathbb{Z}\setminus\{0\}$,*

$$\begin{aligned} \alpha_k(\vec{\jmath},\vec{\sigma}) &:= \#\big\{a = 1,\dots,p : (j_a,\sigma_a) = (k,+)\big\}, \\ \beta_k(\vec{\jmath},\vec{\sigma}) &:= \#\big\{a = 1,\dots,p : (j_a,\sigma_a) = (k,-)\big\}. \end{aligned} \tag{4.7.12}$$

We say that a monomial of the form $z_{\vec{\jmath}}^{\vec{\sigma}} = z_{j_1}^{\sigma_1} \dots z_{j_p}^{\sigma_p}$ is super-action preserving if the associated multi-index $(\alpha,\beta) = (\alpha(\vec{\jmath},\vec{\sigma}),\beta(\vec{\jmath},\vec{\sigma}))$ is super-action preserving according to Definition 4.7.3.

We now introduce the subset $\mathfrak{S}_p$ of the indexes of $\mathfrak{T}_p$ defined in (4.2.10) composed by super-action preserving indexes

$$\mathfrak{S}_p := \Big\{ (\vec{\jmath},\vec{\sigma}) \in \mathfrak{T}_p : (\alpha(\vec{\jmath},\vec{\sigma}),\beta(\vec{\jmath},\vec{\sigma})) \in \mathbb{N}_0^{\mathbb{Z}\setminus\{0\}} \times \mathbb{N}_0^{\mathbb{Z}\setminus\{0\}} \text{ in (4.7.12) are super action preserving} \Big\}. \tag{4.7.13}$$

We remark that the multi-index (α,β) associated to $(\vec{\jmath},\vec{\sigma}) \in (\mathbb{Z}\setminus\{0\} \times \{\pm\})^p$ as in (4.7.12) satisfies $|\alpha+\beta| = p$ and

$$z_{\vec{\jmath}}^{\vec{\sigma}} = z^{\alpha}\overline{z}^{\beta} := \prod_{j\in\mathbb{Z}\setminus\{0\}} z_j^{\alpha_j}\overline{z_j}^{\beta_j} = \prod_{n\in\mathbb{N}} z_n^{\alpha_n} z_{-n}^{\alpha_{-n}} \overline{z_n}^{\beta_n} \overline{z_{-n}}^{\beta_{-n}}. \tag{4.7.14}$$

It turns out

$$\vec{\sigma}\cdot\Omega_{\vec{\jmath}}(\kappa) = \sigma_1\Omega_{j_1}(\kappa) + \dots + \sigma_p\Omega_{j_p}(\kappa) = (\alpha-\beta)\cdot\vec{\Omega}(\kappa) = \sum_{k\in\mathbb{Z}\setminus\{0\}} (\alpha_k - \beta_k)\Omega_k(\kappa), \tag{4.7.15}$$

where we denote

$$\vec{\Omega}(\kappa) := \{\Omega_j(\kappa)\}_{j\in\mathbb{Z}\setminus\{0\}}, \quad \Omega_{\vec{\jmath}}(\kappa) := (\Omega_{j_1}(\kappa),\dots,\Omega_{j_p}(\kappa)). \tag{4.7.16}$$

Remark 4.7.5. In view of (4.7.14) and (4.7.9) a super action monomial has either the integrable form $|z_{j_1}|^2 \dots |z_{j_m}|^2$ or the one described in (4.1.14) (with not necessarily distinct indexes $j_1,\dots,j_m$).

Remark 4.7.6. If the monomial $z_{\vec{\jmath}}^{\vec{\sigma}}$ is super–action preserving then, for any $j \in \mathbb{Z}\setminus\{0\}$, the monomial $z_{\vec{\jmath}}^{\vec{\sigma}} z_j \overline{z}_j$ is super-action preserving as well.

For any $n \in \mathbb{N}$ we define the *super action*

$$J_n := |z_n|^2 + |z_{-n}|^2. \tag{4.7.17}$$

Lemma 4.7.7. *The Poisson bracket between a monomial $z_{\vec{\jmath}}^{\vec{\sigma}}$ and a super-action J_n, $n \in \mathbb{N}$, defined in (4.7.17), is*

$$\{z_{\vec{\jmath}}^{\vec{\sigma}}, J_n\} = \mathrm{i}\big(\beta_n + \beta_{-n} - \alpha_n - \alpha_{-n}\big) z_{\vec{\jmath}}^{\vec{\sigma}}, \tag{4.7.18}$$

where $(\alpha,\beta) = (\alpha(\vec{\jmath},\vec{\sigma}),\beta(\vec{\jmath},\vec{\sigma}))$ is the multi-index defined in (4.7.12). In particular a super action preserving monomial $z_{\vec{\jmath}}^{\vec{\sigma}}$ (according to Definition 4.7.4) Poisson commutes with any super action J_n, $n \in \mathbb{N}$.

Proof. We write the monomial $z_{\vec{j}}^{\vec{\sigma}} = z^\alpha \overline{z}^\beta$ as in (4.7.14). Then, for any $n \in \mathbb{N}$ and $j, k \in \mathbb{Z} \setminus \{0\}$, one has

$$\begin{aligned} &\partial_{z_j}(z_{\vec{j}}^{\vec{\sigma}}) = \alpha_j z_j^{\alpha_j - 1} \overline{z_j}^{\beta_j} \prod_{k \neq j} z_k^{\alpha_k} \overline{z_k}^{\beta_k}, \quad \partial_{\overline{z_j}}(z_{\vec{j}}^{\vec{\sigma}}) = \beta_j z_j^{\alpha_j} \overline{z_j}^{\beta_j - 1} \prod_{k \neq j} z_k^{\alpha_k} \overline{z_k}^{\beta_k}, \\ &\partial_{z_j} J_n = \begin{cases} \overline{z_j} & j = \pm n \\ 0 & j \neq \pm n, \end{cases} \quad \partial_{\overline{z_j}} J_n = \begin{cases} z_j & j = \pm n \\ 0 & j \neq \pm n. \end{cases} \end{aligned} \tag{4.7.19}$$

Then by (4.3.29) and (4.7.19) we deduce (4.7.18). □

We now define a super action preserving Hamiltonian.

Definition 4.7.8. **(SAP Hamiltonian)** *Let $p \in \mathbb{N}_0$. A $(p+2)$–homogeneous super action preserving Hamiltonian $H_{p+2}^{(\mathtt{SAP})}(Z)$ is a real function of the form*

$$H_{p+2}^{(\mathtt{SAP})}(Z) = \frac{1}{p+2} \sum_{(\vec{\jmath}_{p+2}, \vec{\sigma}_{p+2}) \in \mathfrak{S}_{p+2}} H_{\vec{\jmath}_{p+2}}^{\vec{\sigma}_{p+2}} z_{\vec{\jmath}_{p+2}}^{\vec{\sigma}_{p+2}}$$

where $\mathfrak{S}_{p+2}$ is defined as in (4.7.13). A pluri-homogeneous super action preserving Hamiltonian is a finite sum of homogeneous super action preserving Hamiltonians. A Hamiltonian vector field is super action preserving if it is generated by a super action preserving Hamiltonian.

We now define a super action preserving symbol.

Definition 4.7.9. **(SAP symbol)** *Let $p \in \mathbb{N}_0$ and $m \in \mathbb{R}$. For $p \geq 1$ a real valued, p–homogeneous* super action preserving symbol *of order m is a symbol $\mathtt{m}_p^{(\mathtt{SAP})}(Z;\xi)$ in $\widetilde{\Gamma}_p^m$, independent of x, of the form*

$$\mathtt{m}_p^{(\mathtt{SAP})}(Z;\xi) = \sum_{(\vec{\jmath}_p, \vec{\sigma}_p) \in \mathfrak{S}_p} M_{\vec{\jmath}_p}^{\vec{\sigma}_p}(\xi) z_{\vec{\jmath}_p}^{\vec{\sigma}_p}. \tag{4.7.20}$$

For $p = 0$ we say that any symbol in $\widetilde{\Gamma}_0^m$ is super action preserving. A pluri-homogeneous super action preserving symbol is a finite sum of homogeneous super action preserving symbols.

Remark 4.7.10. A super action preserving symbol has even degree p of homogeneity. Indeed, if $z_{\vec{\jmath}_p}^{\vec{\sigma}_p}$ is super- action preserving then (α, β) defined in (4.7.12) satisfies $|\alpha| = |\beta|$ and $p = |\alpha + \beta| = 2|\alpha|$ is even.

Given a super action preserving symbol we associate a super action preserving Hamiltonian according to the following lemma.

Lemma 4.7.11. *Let $p \in \mathbb{N}_0$, $m \in \mathbb{R}$. If $(\mathtt{m}^{(\mathtt{SAP})})_p(Z;\xi)$ is a p–homogeneous super action preserving symbol in $\widetilde{\Gamma}_p^m$ according to Definition 4.7.9 then*

$$H_{p+2}^{(\mathtt{SAP})}(Z) := \mathrm{Re} \left\langle \mathrm{Op}^{BW} \Big((\mathtt{m}^{(\mathtt{SAP})})_p(Z;\xi) \Big) z, \overline{z} \right\rangle_{\dot{L}_r^2}$$

is a $(p+2)$–homogeneous super action preserving Hamiltonian according to Definition 4.7.8.

Proof. By the expression (4.7.20) and (4.2.23) we have

$$\int_{\mathbb{T}} \mathrm{Op}^{BW} \Big(\mathtt{m}_p^{(\mathtt{SAP})}(Z;\xi) \Big) \Pi_0^\perp z \cdot \Pi_0^\perp \overline{z} \, \mathrm{d}x = \sum_{j \in \mathbb{Z} \setminus \{0\}} \sum_{(\vec{\jmath}_p, \vec{\sigma}_p) \in \mathfrak{S}_p} \chi_p(\vec{\jmath}_p, j) M_{\vec{\jmath}_p}^{\vec{\sigma}_p}(j) z_{\vec{\jmath}_p}^{\vec{\sigma}_p} z_j \overline{z_j}$$

where $\mathfrak{S}_p$ is defined in (4.7.13). Then Remark 4.7.6 implies the thesis. For $p = 0$ the Hamiltonian $H_2^{(\mathtt{SAP})}(Z)$ is a series of integrable monomials $z_j \overline{z_j}$. The proof of the lemma is complete. □

4.7.3 Birkhoff normal form reduction

In this section we finally transform system (4.7.4) into its Hamiltonian Birkhoff normal form, up to homogeneity N.

Proposition 4.7.12. (Hamiltonian Birkhoff normal form) *Let $N \in \mathbb{N}$. Assume that, for any value of the gravity $g > 0$, vorticity $\gamma \in \mathbb{R}$ and depth $\mathtt{h} \in (0, +\infty]$, the surface tension coefficient κ is outside the zero measure set $\mathcal{K} \subset (0, +\infty)$ defined in Theorem B.0.1.*

Then there exists $\underline{\varrho}$ (depending on N) such that, for any $\varrho \geq \underline{\varrho}$, for any $K \geq \underline{K}'(\varrho)$ (defined in Proposition 4.6.1), there exists $\underline{s}_0 > 0$ such that, for any $s \geq \underline{s}_0$ there is $\underline{r}_0 := \underline{r}_0(s) > 0$ such that for all $0 < r < \underline{r}_0(s)$ small enough, and any solution $U \in B^K_{\underline{s}_0}(I;r) \cap C^K_{*\mathbb{R}}(I;\dot{H}^s(\mathbb{T};\mathbb{C}^2))$ of the water waves system (4.5.37), there exists a non–linear map $\mathcal{F}_{\mathtt{nf}}(Z_0)$ such that:

(i) **Simpleticity:** *$\mathcal{F}_{\mathtt{nf}}(Z_0)$ is symplectic up to homogeneity N (Definition 4.3.11);*

(ii) **Conjugation:** *If Z_0 solves the system (4.7.4) then the variable $Z := \mathcal{F}_{\mathtt{nf}}(Z_0)$ solves the* Hamiltonian system up to homogeneity N *(cfr. Definition 4.3.10)*

$$\begin{aligned} \partial_t Z = -\mathrm{i}\boldsymbol{\Omega}(D)Z + J_c \nabla H^{(\mathtt{SAP})}_{\frac{3}{2}}(Z) + J_c \nabla H^{(\mathtt{SAP})}_{-\varrho}(Z) \\ + \mathrm{Op}^{BW}_{\mathtt{vec}}\Big(-\mathrm{i}(\mathtt{m}_{\frac{3}{2}})_{>N}(U;t,\xi)\Big)Z + R_{>N}(U;t)U \end{aligned} \tag{4.7.21}$$

where

- *$H^{(\mathtt{SAP})}_{\frac{3}{2}}(Z)$ is the super action preserving Hamiltonian*

$$\mathrm{Re}\left\langle \mathrm{Op}^{BW}\left((\mathtt{m}^{(\mathtt{SAP})}_{\frac{3}{2}})_{\leq N}(Z;\xi)\right)z, \overline{z}\right\rangle_{\dot{L}^2_r}$$

  with a pluri homogeneous super action preserving symbol $(\mathtt{m}^{(\mathtt{SAP})}_{\frac{3}{2}})_{\leq N}(Z;\xi)$ in $\Sigma_2^N \widetilde{\Gamma}^{\frac{3}{2}}_q$, according to Definition 4.7.9;

- *$J_c \nabla H^{(\mathtt{SAP})}_{-\varrho}(Z)$ is a super action preserving, Hamiltonian, smoothing vector field in $\Sigma_3^{N+1}\widetilde{\mathfrak{X}}^{-\varrho+\underline{\varrho}}_q$ (see Definitions 4.2.25 and 4.7.8);*

- *$(\mathtt{m}_{\frac{3}{2}})_{>N}(U;t,\xi)$ is a non–homogeneous symbol in $\Gamma^{\frac{3}{2}}_{K,\underline{K}',N+1}[r]$ with imaginary part $\mathrm{Im}\,(\mathtt{m}_{\frac{3}{2}})_{>N}(U;t,\xi)$ in $\Gamma^0_{K,\underline{K}',N+1}[r]$;*

- *$R_{>N}(U;t)$ is a real-to-real matrix of non–homogeneous smoothing operators in $\mathcal{R}^{-\varrho+\underline{\varrho}}_{K,\underline{K}',N+1}[r] \otimes \mathcal{M}_2(\mathbb{C})$.*

(iii) **Boundedness:** *there exists $C := C_{s,K} > 0$ such that for all $0 \leq k \leq K$ and any $Z_0 \in B^K_{\underline{s}_0}(I;r) \cap C^K_{*\mathbb{R}}(I;\dot{H}^s(\mathbb{T},\mathbb{C}^2))$ one has*

$$C^{-1}\|Z_0\|_{k,s} \leq \|\mathcal{F}_{\mathtt{nf}}(Z_0)\|_{k,s} \leq C\|Z_0\|_{k,s}\,. \tag{4.7.22}$$

and

$$C^{-1}\|U(t)\|_{\dot{H}^s} \leq \|Z(t)\|_{\dot{H}^s} \leq C\|U(t)\|_{\dot{H}^s}\,, \quad \forall t \in I\,. \tag{4.7.23}$$

Proof. We divide the proof in N steps. At the p-th step, $1 \le p \le N$, we reduce the p–homogeneous component of the Hamiltonian vector field which appears in the equation to its super action preserving part, up to higher homogeneity terms.

Step 1: Elimination of the quadratic smoothing remainder in equation (4.7.4).

The x-independent symbol $(\mathtt{m}_{\frac{3}{2}})_{\le N}(Z_0;\xi)$ in (4.7.4) belongs to $\Sigma_2^N \widetilde{\Gamma}_q^{\frac{3}{2}}$ and the only quadratic component of the vector field in (4.7.4) is $R_1(Z_0)Z_0$ where

$$R_1(Z_0) := \mathcal{P}_1[R_{\le N}(Z_0)] \in \widetilde{\mathcal{R}}_1^{-\varrho+c(N)} \otimes \mathcal{M}_2(\mathbb{C}) \,. \tag{4.7.24}$$

Since system (4.7.4) is Hamiltonian up to homogeneity N, $R_1(Z_0)Z_0$ is a Hamiltonian vector field in $\widetilde{\mathfrak{X}}_2^{-\varrho+c(N)}$ that we expand in Fourier coordinates as in (4.2.122)

$$\big(R_1(Z_0)Z_0\big)_k^\sigma = \sum_{(j_1,j_2,k,\sigma_1,\sigma_2,-\sigma)\in\mathfrak{T}_3} X_{j_1,j_2,k}^{\sigma_1,\sigma_2,\sigma}(z_0)_{j_1}^{\sigma_1}(z_0)_{j_2}^{\sigma_2} \,. \tag{4.7.25}$$

In order to remove $R_1(Z_0)Z_0$ from equation (4.7.4) we perform the change of variable $Z_1 = \mathtt{F}_{\le N}^{(1)}(Z_0)$ where $\mathtt{F}_{\le N}^{(1)}(Z_0)$ is the time 1-approximate flow, given by Lemma 4.2.28, generated by the smoothing vector field

$$\big(G_1(Z_0)Z_0\big)_k^\sigma = \sum_{(j_1,j_2,k,\sigma_1,\sigma_2,-\sigma)\in\mathfrak{T}_3} G_{j_1,j_2,k}^{\sigma_1,\sigma_2,\sigma}(z_0)_{j_1}^{\sigma_1}(z_0)_{j_2}^{\sigma_2} \tag{4.7.26}$$

with

$$G_{j_1,j_2,k}^{\sigma_1,\sigma_2,\sigma} := \begin{cases} 0 & \text{if } (j_1,j_2,k,\sigma_1,\sigma_2,-\sigma)\notin\mathfrak{T}_3 \\ \dfrac{X_{j_1,j_2,k}^{\sigma_1,\sigma_2,\sigma}}{\mathrm{i}\big(\sigma_1\Omega_{j_1}(\kappa)+\sigma_2\Omega_{j_2}(\kappa)-\sigma\Omega_k(\kappa)\big)} & \text{if } (j_1,j_2,k,\sigma_1,\sigma_2,-\sigma)\in\mathfrak{T}_3 \,. \end{cases} \tag{4.7.27}$$

Lemma 4.7.13. *Let $\kappa\in(0,+\infty)\setminus\mathcal{K}$. Then the vector field $G_1(Z_0)Z_0$ in (4.7.26), (4.7.27) is a well defined Hamiltonian vector field in $\widetilde{\mathfrak{X}}_2^{-\varrho'}$ with $\varrho' := \varrho - c(N) - \tau$ and where τ is defined in Theorem B.0.1.*

Proof. We claim that for any $\kappa\in(0,+\infty)\setminus\mathcal{K}$ there exist $\tau,\nu>0$ such that

$$|\sigma_1\Omega_{j_1}(\kappa)+\sigma_2\Omega_{j_2}(\kappa)-\sigma\Omega_k(\kappa)| > \frac{\nu}{\max\{|j_1|,|j_2|,|k|\}^\tau} \,, \quad \forall (j_1,j_2,k,\sigma_1,\sigma_2,-\sigma)\in\mathfrak{T}_3 \,. \tag{4.7.28}$$

Indeed, to any $(j_1,j_2,k,\sigma_1,\sigma_2,-\sigma)$ we associate the multi–index (α,β) as in (4.7.12) whose length is $|\alpha+\beta|=3$ and satisfies $\sigma_1\Omega_{j_1}(\kappa)+\sigma_2\Omega_{j_2}(\kappa)-\sigma\Omega_k(\kappa) = (\alpha-\beta)\cdot\vec{\Omega}(\kappa)$ by (4.7.15). Having length 3, by Remark 4.7.10, the multi-index (α,β) is not super–action preserving and therefore Theorem B.0.1 implies (4.7.28). In view of (4.7.28) the coefficients in (4.7.27) are well defined.

Next we show that $G_1(Z_0)Z_0$ is a vector field in $\widetilde{\mathfrak{X}}_2^{-\varrho'}$. As $R_1(Z_0)Z_0$ belongs to $\widetilde{\mathfrak{X}}_2^{-\varrho+c(N)}$, by Lemma 4.2.26 the coefficients $X_{j_1,j_2,k}^{\sigma_1,\sigma_2,\sigma}$ in (4.7.25) satisfy the symmetric and reality properties (4.2.124), (4.2.125) and the estimate: for some $\mu\ge 0$, $C>0$,

$$|X_{j_1,j_2,k}^{\sigma_1,\sigma_2,\sigma}| \le C\frac{\max_2\{|j_1|,|j_2|\}^\mu}{\max\{|j_1|,|j_2|\}^{\varrho-c(N)}} \,, \qquad \forall (j_1,j_2,k,\sigma_1,\sigma_2,-\sigma)\in\mathfrak{T}_3 \,. \tag{4.7.29}$$

Hence also the coefficients $G_{j_1,j_2,k}^{\sigma_1,\sigma_2,\sigma}$ in (4.7.27) fulfill the symmetric, reality properties (4.2.124), (4.2.125) as well as $X_{j_1,j_2,k}^{\sigma_1,\sigma_2,\sigma}$. Moreover, using (4.7.29), (4.7.28) and the momentum relation $\sigma k = \sigma_1 j_1 + \sigma_2 j_2$, they also satisfy

$$|G_{j_1,j_2,k}^{\sigma_1,\sigma_2,\sigma}| \le C\frac{\max_2\{|j_1|,|j_2|\}^\mu}{\max\{|j_1|,|j_2|\}^{\varrho-c(N)-\tau}}$$

for a new constant $C > 0$ (depending on ν). Then Lemma 4.2.26 implies that $G_1(Z_0)Z_0$ belongs to $\widetilde{\mathfrak{X}}_2^{-\varrho'}$ with $\varrho' := \varrho - c(N) - \tau$.

Finally we show that $G_1(Z_0)Z_0$ is Hamiltonian. Recall that $R_1(Z_0)Z_0$ in (4.7.25) is a Hamiltonian vector field whose Hamiltonian function $H_{R_1}(Z_0)$ is, thanks to Lemma 4.3.5,

$$H_{R_1}(Z_0) = \frac{1}{3} \sum_{(j_1,j_2,j_3,\sigma_1,\sigma_2,\sigma_3)\in\mathfrak{T}_3} [H_{R_1}]^{\sigma_1,\sigma_2,\sigma_3}_{j_1,j_2,j_3} (z_0)^{\sigma_1}_{j_1} (z_0)^{\sigma_2}_{j_2} (z_0)^{\sigma_3}_{j_3}\,, \qquad [H_{R_1}]^{\sigma_1,\sigma_2,\sigma_3}_{j_1,j_2,j_3} := -\mathrm{i}\sigma_3 X^{\sigma_1,\sigma_2,-\sigma_3}_{j_1,j_2,j_3}\,. \tag{4.7.30}$$

Then the coefficients defined for $\sigma_1 j_1 + \sigma_2 j_2 + \sigma_3 j_3 = 0$ by

$$[H_{G_1}]^{\sigma_1,\sigma_2,\sigma_3}_{j_1,j_2,j_3} := -\mathrm{i}\sigma_3 G^{\sigma_1,\sigma_2,-\sigma_3}_{j_1,j_2,j_3} \overset{(4.7.27),(4.7.30)}{=} \frac{[H_{R_1}]^{\sigma_1,\sigma_2,\sigma_3}_{j_1,j_2,j_3}}{\mathrm{i}\big(\sigma_1\Omega_{j_1}(\kappa) + \sigma_2\Omega_{j_2}(\kappa) + \sigma_3\Omega_{j_3}(\kappa)\big)} \tag{4.7.31}$$

satisfy the symmetric, reality properties (4.2.135), (4.2.134) as well as the coefficients $[H_{R_1}]^{\sigma_1,\sigma_2,\sigma}_{j_1,j_2,j_3}$. Then Lemma 4.3.5 implies that $G_1(Z_0)Z_0$ is the Hamiltonian vector field generated by the Hamiltonian H_{G_1} with coefficients defined in (4.7.31). □

We now conjugate system (4.7.4) by the approximate time 1-flow $\mathtt{F}^{(1)}_{\leq N}(Z_0)$ generated by $G_1(Z)Z$ provided by Lemma 4.2.28, which has the form

$$Z_1 := \mathtt{F}^{(1)}_{\leq N}(Z_0) = Z_0 + F_{\leq N}(Z_0)Z_0\,, \quad F_{\leq N}(Z_0) \in \Sigma_1^N \widetilde{\mathcal{R}}^{-\varrho'}_q \otimes \mathcal{M}_2(\mathbb{C})\,. \tag{4.7.32}$$

Since $G_1(Z)Z$ is a Hamiltonian vector field, by Lemma 4.3.14 the approximate flow $\mathtt{F}^{(1)}_{\leq N}$ is symplectic up to homogeneity N. Applying Lemma 4.3.15 (with $Z \rightsquigarrow Z_0$, $W \rightsquigarrow Z_1$ and $\mathbf{M}_0(U;t) \in \mathcal{M}^0_{K,\underline{K}'-1,0}[r] \otimes \mathcal{M}_2(\mathbb{C})$), we obtain that the variable Z_1 solves a system which is Hamiltonian up to homogeneity N. We compute it using Lemma A.0.5. Its assumption **(A)** at page 215 holds since Z_0 solves (4.7.4) (with $a_{\leq N} = -(\mathtt{m}_{\frac{3}{2}})_{\leq N}$, $K' = \underline{K}'$ and $\varrho \rightsquigarrow \varrho - c(N)$). Then Lemma A.0.5 (with $W \rightsquigarrow Z_1$, $p = 1$ and $\varrho' = \varrho - c(N) - \tau$) implies that the variable Z_1 solves

$$\begin{aligned} \partial_t Z_1 = -\mathrm{i}\boldsymbol{\Omega}(D)Z_1 + \mathrm{Op}^{BW}_{\mathtt{vec}}\Big(-\mathrm{i}(\mathtt{m}_{\frac{3}{2}})^+_{\leq N}(Z_1;\xi) - \mathrm{i}(\mathtt{m}_{\frac{3}{2}})^+_{>N}(U;t,\xi)\Big)Z_1 \\ + [R_1(Z_1) + G_1^+(Z_1)]Z_1 + R^+_{\geq 2}(Z_1)Z_1 + R^+_{>N}(U;t)U \end{aligned} \tag{4.7.33}$$

where

- $(\mathtt{m}_{\frac{3}{2}})^+_{\leq N}(Z_1;\xi)$ is a real valued symbol, independent of x, in $\Sigma_2^N \widetilde{\Gamma}^{\frac{3}{2}}_q$;
- $(\mathtt{m}_{\frac{3}{2}})^+_{>N}(U;t,\xi)$ is a non-homogeneous real valued symbol, independent of x, in $\Gamma^{\frac{3}{2}}_{K,\underline{K}',N+1}[r]$ with imaginary part $\mathrm{Im}(\mathtt{m}_{\frac{3}{2}})_{>N}(U;t,\xi)$ in $\Gamma^0_{K,\underline{K}',N+1}[r]$;
- $R_1(Z_1)$ is defined in (4.7.24) and $G_1^+(Z_1)Z_1 \in \widetilde{\mathfrak{X}}_2^{-\varrho'+\frac{3}{2}}$ has Fourier expansion, by (A.0.51) and (4.7.26),

$$(G_1^+(Z_1)Z_1)^\sigma_k = \sum_{(j_1,j_2,k,\sigma_1,\sigma_2,-\sigma)\in\mathfrak{T}_3} -\mathrm{i}\big(\sigma_1\Omega_{j_1}(\kappa) + \sigma_2\Omega_{j_2}(\kappa) - \sigma\Omega_k(\kappa)\big) G^{\sigma_1,\sigma_2,\sigma}_{j_1,j_2,k} (z_1)^{\sigma_1}_{j_1} (z_1)^{\sigma_2}_{j_2}; \tag{4.7.34}$$

- $R^+_{\geq 2}(Z_1)$ is a matrix of pluri-homogeneous smoothing operators in $\Sigma_2^N \widetilde{\mathcal{R}}^{-\varrho+\underline{\varrho}(2)}_q \otimes \mathcal{M}_2(\mathbb{C})$ where

$$\underline{\varrho}(2) := c(N) + \tau + \frac{3}{2}\,; \tag{4.7.35}$$

- $R^+_{>N}(U;t)$ is a matrix of non–homogeneous smoothing operators in $\mathcal{R}^{-\varrho+\underline{\varrho}(2)}_{K,\underline{K}',N+1}[r] \otimes \mathcal{M}_2(\mathbb{C})$.

By (4.7.25), (4.7.34), (4.7.27) we have

$$R_1(Z_1)Z_1 + G_1^+(Z_1)Z_1 = 0\,. \tag{4.7.36}$$

Step $p \geq 2$**:** We claim the following inductive statements hold true. Let Z_0 solve (4.7.4). Then for any $p \geq 2$

(S0)$_p$ *There is a transformation* $\mathcal{F}^{(p-1)}_{\leq N}(Z_0)$*, symplectic up to homogeneity* N*, fulfilling* (iii) *of Proposition 4.7.12 (with* $C = 2 \times 8^{p-2}$ *in (4.7.22)) such that the variable* $Z_{p-1} = \mathcal{F}^{(p-1)}_{\leq N}(Z_0)$ *has the form* $Z_{p-1} = \mathbf{M}_0^{(p-1)}(U;t)U$ *with* $\mathbf{M}_0^{(p-1)}(U;t) \in \mathcal{M}^0_{K,\underline{K}'-1,0}[r] \otimes \mathcal{M}_2(\mathbb{C})$ *and solves the system*

$$\begin{aligned}
\partial_t Z_{p-1} = &-\mathrm{i}\boldsymbol{\Omega}(D)Z_{p-1} + J_c\nabla\big(H^{(\mathtt{SAP})}_{\frac32}\big)_{\leq p+1}(Z_{p-1}) + J_c\nabla\big(H^{(\mathtt{SAP})}_{-\varrho}\big)_{\leq p+1}(Z_{p-1})\\
&+ \mathrm{Op}^{BW}_{\mathtt{vec}}\Big(-\mathrm{i}(\mathtt{m}_{\frac32})_p(Z_{p-1};\xi) - \mathrm{i}(\mathtt{m}_{\frac32})_{\geq p+1}(Z_{p-1};\xi)\Big)Z_{p-1} + R_{\geq p}(Z_{p-1})Z_{p-1}\\
&+ \mathrm{Op}^{BW}_{\mathtt{vec}}\Big(-\mathrm{i}(\mathtt{m}_{\frac32})_{>N}(U;t,\xi)\Big)Z_{p-1} + R_{>N}(U;t)U
\end{aligned} \tag{4.7.37}$$

where
(S1)$_p$ $\big(H^{(\mathtt{SAP})}_{\frac32}\big)_{\leq p+1}(Z_{p-1})$ *is the real valued Hamiltonian*

$$\big(H^{(\mathtt{SAP})}_{\frac32}\big)_{\leq p+1}(Z_{p-1}) := \mathrm{Re}\Big\langle \mathrm{Op}^{\mathrm{BW}}\Big((\mathtt{m}^{(\mathtt{SAP})}_{\frac32})_{\leq p-1}(Z_{p-1};\xi)\Big)z_{p-1}, \overline{z}_{p-1}\Big\rangle_{\dot L^2_r} \tag{4.7.38}$$

with a super action preserving symbol $(\mathtt{m}^{(\mathtt{SAP})}_{\frac32})_{\leq p-1}(Z_{p-1};\xi)$ *in* $\Sigma_2^{p-1}\widetilde\Gamma_q^{\frac32}$ *(see Definition 4.7.9); its Hamiltonian vector field is given by*

$$J_c\nabla\big(H^{(\mathtt{SAP})}_{\frac32}\big)_{\leq p+1}(Z_{p-1}) = \mathrm{Op}^{BW}_{\mathtt{vec}}\Big(-\mathrm{i}(\mathtt{m}^{(\mathtt{SAP})}_{\frac32})_{\leq p-1}(Z_{p-1};\xi)\Big)Z_{p-1} + R_{\leq p-1}(Z_{p-1})Z_{p-1} \tag{4.7.39}$$

with $R_{\leq p-1}(Z_{p-1}) \in \Sigma_2^{p-1}\widetilde{\mathcal{R}}_q^{-\varrho'} \otimes \mathcal{M}_2(\mathbb{C})$ *for any* $\varrho' > 0$ *(see Lemma 4.3.19).*
(S2)$_p$ $J_c\nabla\big(H^{(\mathtt{SAP})}_{-\varrho}\big)_{\leq p+1}(Z_{p-1})$ *is a super action preserving, Hamiltonian, smoothing vector field in* $\Sigma_3^p\widetilde{\mathfrak{X}}_q^{-\varrho+\underline{\varrho}(p)}$*, where*

$$\underline{\varrho}(1) := c(N)\,, \qquad \underline{\varrho}(p) := \underline{\varrho}(p-1) + \tau + \tfrac32\,, \;\; p \geq 2\,; \tag{4.7.40}$$

(S3)$_p$ $(\mathtt{m}_{\frac32})_p(Z_{p-1};\xi)$ *and* $(\mathtt{m}_{\frac32})_{\geq p+1}(Z_{p-1};\xi)$ *are real valued symbols, independent of* x*, respectively in* $\widetilde\Gamma_p^{\frac32}$ *and* $\Sigma^N_{p+1}\widetilde\Gamma_q^{\frac32}$*;*
(S4)$_p$ $R_{\geq p}(Z_{p-1})$ *is a smoothing operator in* $\Sigma_p^N\widetilde{\mathcal{R}}_q^{-\varrho+\underline{\varrho}(p)} \otimes \mathcal{M}_2(\mathbb{C})$*;*
(S5)$_p$ $(\mathtt{m}_{\frac32})_{>N}(U;t,\xi)$ *is a non–homogeneous symbol in* $\Gamma^{\frac32}_{K,\underline{K}',N+1}[r]$ *with imaginary part* $\mathrm{Im}\,(\mathtt{m}_{\frac32})_{>N}(U;t,\xi)$ *in* $\Gamma^0_{K,\underline{K}',N+1}[r]$*;*
(S6)$_p$ $R_{>N}(U;t)$ *is a matrix of non–homogeneous smoothing operators in* $\mathcal{R}^{-\varrho+\underline{\varrho}(p)}_{K,\underline{K}',N+1}[r] \otimes \mathcal{M}_2(\mathbb{C})$*;*
(S7)$_p$ *the system (4.7.37) is Hamiltonian up to homogeneity* N*.*

Note that for $p = N+1$, system (4.7.37) has the claimed form in (4.7.21) with $Z \equiv Z_N$, Hamiltonians $H^{(\mathtt{SAP})}_{\frac32} := \big(H^{(\mathtt{SAP})}_{\frac32}\big)_{\leq N+2}$, $H^{(\mathtt{SAP})}_{-\varrho} := \big(H^{(\mathtt{SAP})}_{-\varrho}\big)_{\leq N+2}$ and $\underline{\varrho} := \underline{\varrho}(N+1)$, thus proving Proposition 4.7.12. We now prove the inductive statements **(S0)**$_p$-**(S7)**$_p$.

Initialization: case $p=2$. We set $\mathcal{F}^{(1)}_{\le N} := \mathtt{F}^{(1)}_{\le N}$ defined in (4.7.32) which is symplectic up to homogeneity N. Thanks to (4.2.40), the non–linear map $\mathcal{F}^{(1)}_{\le N}$ satisfies (iii) of Proposition 4.7.12. The system (4.7.33) with $R_1(Z_1)Z_1 + G_1^+(Z_1)Z_1 = 0$ is (4.7.37) with Hamiltonians $\big(H^{(\mathtt{SAP})}_{\frac32}\big)_{\le 3} = \big(H^{(\mathtt{SAP})}_{-\varrho}\big)_{\le 3} = 0$, and symbols $(\mathtt{m}_{\frac32})_2 = \mathcal{P}_2[(\mathtt{m}_{\frac32})^+_{\le N}]$, $(\mathtt{m}_{\frac32})_{\ge 3} = \mathcal{P}_{\ge 3}[(\mathtt{m}_{\frac32})^+_{\le N}]$ and $(\mathtt{m}_{\frac32})_{>N} = (\mathtt{m}_{\frac32})^+_{>N}$. Furthermore $Z_1 = \mathbf{M}^{(1)}_0(U;t)U$ with $\mathbf{M}^{(1)}_0(U;t) \in \mathcal{M}^0_{K,\underline{K}'-1,0}[r]\otimes\mathcal{M}_2(\mathbb{C})$ because the map in (4.7.32) has the form $\mathtt{F}^{(1)}_{\le N}(Z_0) = \breve{\mathbf{M}}_0(Z_0)Z_0$ with $\breve{\mathbf{M}}_0(Z_0)\in\mathcal{M}^0_{K,0,0}[r]\otimes\mathcal{M}_2(\mathbb{C})$ thanks to Lemma 4.2.8, Proposition 4.7.2-(iii) and Proposition 4.2.15 (iii) (with $K' \rightsquigarrow \underline{K}'-1$). Thus **(S0)**$_2$-**(S7)**$_2$ are satisfied.

Iteration: reduction of the p–homogenous symbol. Suppose **(S0)**$_p$-**(S7)**$_p$ hold true. The goal of this step is to reduce the real valued, x-independent, p-homogenous symbol $-\mathrm{i}(\mathtt{m}_{\frac32})_p(Z_{p-1};\xi) \in \widetilde{\Gamma}^{\frac32}_p$ in (4.7.37). We Fourier expand as in (4.2.19)

$$(\mathtt{m}_{\frac32})_p(Z_{p-1};\xi) = \sum_{(\vec{\jmath}_p,\vec{\sigma}_p)\in\mathfrak{T}_p} \mathtt{m}^{\vec{\sigma}_p}_{\vec{\jmath}_p}(\xi)\,(z_{p-1})^{\vec{\sigma}_p}_{\vec{\jmath}_p}\,, \quad \overline{\mathtt{m}^{-\vec{\sigma}_p}_{\vec{\jmath}_p}}(\xi) = \mathtt{m}^{\vec{\sigma}_p}_{\vec{\jmath}_p}(\xi)\,. \tag{4.7.41}$$

to its super action preserving normal form. We conjugate (4.7.37) under the change of variable

$$W := \Phi_p(Z_{p-1}) := \mathcal{G}^1_{g_p}(Z_{p-1})Z_{p-1} \tag{4.7.42}$$

where $\mathcal{G}^1_{g_p}(Z_{p-1})$ is the time 1-linear flow generated by $\mathrm{Op}^{BW}_{\mathtt{vec}}(\mathrm{i}g_p)$ as in (4.3.56), where g_p is the Fourier multiplier

$$g_p(Z_{p-1};\xi) := \sum_{(\vec{\jmath}_p,\vec{\sigma}_p)\in\mathfrak{T}_p} G^{\vec{\sigma}_p}_{\vec{\jmath}_p}(\xi)(z_{p-1})^{\vec{\sigma}_p}_{\vec{\jmath}_p} \in \widetilde{\Gamma}^{\frac32}_p \tag{4.7.43}$$

with coefficients

$$G^{\vec{\sigma}_p}_{\vec{\jmath}_p}(\xi) := \begin{cases} 0 & \text{if } (\vec{\jmath}_p,\vec{\sigma}_p)\in\mathfrak{S}_p \\ \dfrac{\mathtt{m}^{\vec{\sigma}_p}_{\vec{\jmath}_p}(\xi)}{-\mathrm{i}\vec{\sigma}_p\cdot\Omega_{\vec{\jmath}_p}(\kappa)} & \text{if } (\vec{\jmath}_p,\vec{\sigma}_p)\notin\mathfrak{S}_p\,, \end{cases} \tag{4.7.44}$$

where the super action set $\mathfrak{S}_p$ is defined in (4.7.13) and $\Omega_{\vec{\jmath}_p}(\kappa)$ is the frequency vector in (4.7.16).

Lemma 4.7.14. *Let $\kappa \in (0,+\infty)\setminus\mathcal{K}$. The function $g_p(Z_{p-1};\xi)$ in (4.7.43), (4.7.44) is a well defined, x-independent, real valued, p-homogeneous symbol in $\widetilde{\Gamma}^{\frac32}_p$.*

Proof. We claim that for any $\kappa\in(0,+\infty)\setminus\mathcal{K}$ there exist $\tau,\nu>0$ such that

$$|\vec{\sigma}_p\cdot\Omega_{\vec{\jmath}_p}(\kappa)| > \frac{\nu}{\max(|j_1|,\dots,|j_p|)^\tau}\,, \qquad \forall(\vec{\jmath}_p,\vec{\sigma}_p)\notin\mathfrak{S}_p\,. \tag{4.7.45}$$

Indeed, to any $(\vec{\jmath}_p,\vec{\sigma}_p)$ we associate the multi–index (α,β) as in (4.7.12) whose length is $|\alpha+\beta| = p$ and satisfies $\vec{\sigma}_p\cdot\Omega_{\vec{\jmath}_p}(\kappa) = (\alpha-\beta)\cdot\vec{\Omega}(\kappa)$ by (4.7.15). Recalling (4.7.13), the vector $(\vec{\jmath}_p,\vec{\sigma}_p)\notin\mathfrak{S}_p$ if and only if (α,β) is not super action-preserving and therefore Theorem B.0.1 implies (4.7.45). Note also that, by Remark 4.7.10, if p is odd, there are not super-action preserving indexes, i.e. $\mathfrak{S}_p=\emptyset$.

In view of (4.7.45) the coefficients in (4.7.44) are well defined and, since the coefficients $\mathtt{m}^{\vec{\sigma}_p}_{\vec{\jmath}_p}(\xi)$ of the symbol $(\mathtt{m}_{\frac32})_p$ fulfill (4.2.20) (with $m=\frac32$), then the coefficients $G^{\vec{\sigma}_p}_{\vec{\jmath}_p}(\xi)$ in (4.7.44) satisfy (4.2.20) as well (with μ replaced by $\mu+\tau$), implying that the Fourier multiplier g_p in (4.7.43) belongs to $\widetilde{\Gamma}^{\frac32}_p$. Finally g_p is real because the coefficients $G^{\vec{\sigma}_p}_{\vec{\jmath}_p}(\xi)$ in (4.7.44) satisfy (4.2.21) as $\mathtt{m}^{\vec{\sigma}_p}_{\vec{\jmath}_p}(\xi)$. □

By Lemma 4.3.17 the flow map (4.7.42) is well defined and, by (4.3.57) for $\|Z_{p-1}\|_{k,s_0} < r < r_0(s,K)$ small enough,

$$2^{-1}\|Z_{p-1}\|_{k,s} \leq \|\Phi_p(Z_{p-1})\|_{k,s} \leq 2\|Z_{p-1}\|_{k,s}\,, \quad \forall k = 0,\ldots,K\,. \tag{4.7.46}$$

In order to transform (4.7.37) under the change of variable (4.7.42) we use Lemma A.0.4. Its assumption **(A)** at page 215 holds since Z_{p-1} solves (4.7.37) which, in view of (4.7.39) and $\mathbf{(S2)}_p$, has the form (A.0.21) (with $Z \equiv Z_{p-1}$, $a_{\leq N} \equiv -(\mathtt{m}^{(\mathtt{SAP})}_{\frac32})_{\leq p-1} - (\mathtt{m}_{\frac32})_p - (\mathtt{m}_{\frac32})_{\geq p+1}$, $\varrho \rightsquigarrow \varrho - \underline{\varrho}(p)$ and $K' \equiv \underline{K}'$).

Then Lemma A.0.4 implies that the variable W defined in (4.7.42) solves

$$\begin{aligned}\partial_t W = &-\mathrm{i}\boldsymbol{\Omega}(D)W + J_c\nabla\big(H^{(\mathtt{SAP})}_{\frac32}\big)_{\leq p+1}(W) + J_c\nabla\big(H^{(\mathtt{SAP})}_{-\varrho}\big)_{\leq p+1}(W)\\ &+ \mathrm{Op}^{BW}_{\mathtt{vec}}\Big(-\mathrm{i}[(\mathtt{m}_{\frac32})_p(W;\xi) - g_p^+(W;\xi)] - \mathrm{i}(\mathtt{m}_{\frac32})^+_{\geq p+1}(W;\xi)\Big)W + R^+_{\geq p}(W)W\\ &+ \mathrm{Op}^{BW}_{\mathtt{vec}}\Big(-\mathrm{i}(\mathtt{m}_{\frac32})^+_{>N}(U;t,\xi)\Big)W + R^+_{>N}(U;t)U\end{aligned} \tag{4.7.47}$$

where
- $g_p^+(W;\xi) \in \widetilde{\Gamma}^{\frac32}_p$ is given in (A.0.27);
- $(\mathtt{m}_{\frac32})^+_{\geq p+1}(W;\xi)$ is a real valued symbol, independent of x, in $\Sigma^N_{p+1}\widetilde{\Gamma}^{\frac32}_q$;
- $(\mathtt{m}_{\frac32})^+_{>N}(U;t,\xi)$ is a non-homogeneous symbol independent of x in $\Gamma^{\frac32}_{K,\underline{K}',N+1}[r]$ with imaginary part $\mathrm{Im}\,(\mathtt{m}_{\frac32})^+_{>N}(U;t,\xi)$ in $\Gamma^0_{K,\underline{K}',N+1}[r]$;
- $R^+_{\geq p}(W)$ is a matrix of pluri–homogeneous smoothing operators in $\Sigma^N_p\widetilde{\mathcal{R}}^{-\varrho+\underline{\varrho}(p)+c(N,p)}_q \otimes \mathcal{M}_2(\mathbb{C})$ for a certain $c(N,p) \geq 0$;
- $R^+_{>N}(U;t)$ is a matrix of non–homogeneous smoothing operators in $\mathcal{R}^{-\varrho+\underline{\varrho}(p)+c(N,p)}_{K,\underline{K}',N+1}[r] \otimes \mathcal{M}_2(\mathbb{C})$.

Note that the Hamiltonian part of degree of homogeneity $\leq p$ in (4.7.47) has been unchanged with respect to (4.7.37), thanks to the first identity in (A.0.26) and (A.0.28). In view of (A.0.27), (4.7.41), (4.7.44), the symbol of homogeneity p in (4.7.47) reduces to its super action component

$$(\mathtt{m}_{\frac32})_p(W;\xi) - g_p^+(W;\xi) = (\mathtt{m}^{(\mathtt{SAP})}_{\frac32})_p(W;\xi) := \sum_{(\vec{\jmath}_p,\vec{\sigma}_p)\in\mathfrak{S}_p} \mathtt{m}^{\vec{\sigma}_p}_{\vec{\jmath}_p}(\xi)\, w^{\vec{\sigma}_p}_{\vec{\jmath}_p}$$

where the super action set $\mathfrak{S}_p$ is defined in (4.7.13), and then (4.7.47) becomes

$$\begin{aligned}\partial_t W = &-\mathrm{i}\boldsymbol{\Omega}(D)W + J_c\nabla\big(H^{(\mathtt{SAP})}_{\frac32}\big)_{\leq p+1}(W) + J_c\nabla\big(H^{(\mathtt{SAP})}_{-\varrho}\big)_{\leq p+1}(W)\\ &+ \mathrm{Op}^{BW}_{\mathtt{vec}}\Big(-\mathrm{i}(\mathtt{m}^{(\mathtt{SAP})}_{\frac32})_p(W;\xi) - \mathrm{i}(\mathtt{m}_{\frac32})^+_{\geq p+1}(W;\xi)\Big)W + R^+_{\geq p}(W)W\\ &+ \mathrm{Op}^{BW}_{\mathtt{vec}}\Big(-\mathrm{i}(\mathtt{m}_{\frac32})^+_{>N}(U;t,\xi)\Big)W + R^+_{>N}(U;t)U\,.\end{aligned} \tag{4.7.48}$$

We now observe that, by Lemma 4.3.19,

$$\mathrm{Op}^{BW}_{\mathtt{vec}}\Big(-\mathrm{i}(\mathtt{m}^{(\mathtt{SAP})}_{\frac32})_p(W;\xi)\Big)W = J_c\nabla\big(H^{(\mathtt{SAP})}_{\frac32}\big)_{p+2}(W) + R'_p(W)W \tag{4.7.49}$$

with the Hamiltonian

$$\big(H^{(\mathtt{SAP})}_{\frac32}\big)_{p+2}(W) := \mathrm{Re}\Big\langle \mathrm{Op}^{BW}\Big((\mathtt{m}^{(\mathtt{SAP})}_{\frac32})_p(W;\xi)\Big)w, \overline{w}\Big\rangle_{\dot{L}^2_r}\,, \tag{4.7.50}$$

which is super action preserving by Lemma 4.7.11, and a matrix of smoothing operators $R'_p(W)$ in $\widetilde{\mathcal{R}}_p^{-\varrho'} \otimes \mathcal{M}_2(\mathbb{C})$ for any $\varrho' \geq 0$. Therefore (4.7.48) becomes

$$\begin{aligned}\partial_t W = & - \mathrm{i}\boldsymbol{\Omega}(D)W + J_c\nabla\big(H_{\frac{3}{2}}^{(\mathtt{SAP})}\big)_{\leq p+2}(W) + J_c\nabla\big(H_{-\varrho}^{(\mathtt{SAP})}\big)_{\leq p+1}(W) \\ & + \mathrm{Op}^{BW}_{\mathtt{vec}}\Big(-\mathrm{i}(\mathtt{m}_{\frac{3}{2}})^+_{\geq p+1}(W;\xi)\Big)W + [R^+_{\geq p}(W) + R'_p(W)]W \\ & + \mathrm{Op}^{BW}_{\mathtt{vec}}\Big(-\mathrm{i}(\mathtt{m}_{\frac{3}{2}})^+_{>N}(U;t,\xi)\Big)W + R^+_{>N}(U;t)U\end{aligned} \tag{4.7.51}$$

where (see (4.7.38), (4.7.50))

$$\big(H_{\frac{3}{2}}^{(\mathtt{SAP})}\big)_{\leq p+2} := \big(H_{\frac{3}{2}}^{(\mathtt{SAP})}\big)_{\leq p+1} + \big(H_{\frac{3}{2}}^{(\mathtt{SAP})}\big)_{p+2}\,. \tag{4.7.52}$$

Note that the new system (4.7.51) is not Hamiltonian up to homogeneity N (unlike system (4.7.37) for Z_{p-1}), since the map $\Phi_p(Z_{p-1}) = \mathcal{G}^1_{g_p}(Z_{p-1})Z_{p-1}$ in (4.7.42) is not symplectic up to homogeneity N. By Lemma 4.3.17 we only know that $\mathcal{G}^1_{g_p}(Z_{p-1})$ is linearly symplectic. We now apply Theorem 4.7.1 to find a correction of $\Phi_p(Z_{p-1})$ which is symplectic up to homogeneity N. By Lemma 4.3.17, the map $\Phi_p(Z_{p-1})$ satisfies the assumptions of Theorem 4.7.1 (with $Z \rightsquigarrow Z_{p-1}$, $\mathbf{B}(Z;t) \rightsquigarrow \mathcal{G}^1_{g_p}(Z_{p-1})$ and using the inductive assumption $Z_{p-1} = \mathbf{M}_0^{(p-1)}(U;t)U$ with $\mathbf{M}_0^{(p-1)}(U;t) \in \mathcal{M}^0_{K,\underline{K}'-1,0}[r] \otimes \mathcal{M}_2(\mathbb{C})$). Therefore Theorem 4.7.1 implies the existence of a matrix of pluri–homogeneous smoothing operators $R^{(p)}_{\leq N}(W)$ in $\Sigma_p^N \widetilde{\mathcal{R}}_q^{-\varrho-\frac{3}{2}} \otimes \mathcal{M}_2(\mathbb{C})$ (the thesis holds for any $\varrho > 0$ and we take $\varrho \rightsquigarrow \varrho + \frac{3}{2}$) such that the variable

$$V := \mathcal{C}^{(p)}_N(W) := \big(\mathrm{Id} + R^{(p)}_{\leq N}(W)\big)W = \big(\mathrm{Id} + R^{(p)}_{\leq N}(\Phi_p(Z_{p-1}))\big)\Phi_p(Z_{p-1}) \tag{4.7.53}$$

is symplectic up to homogeneity N, thus solves a system which is Hamiltonian up to homogeneity N. By (4.7.53) one has

$$V = \breve{\mathbf{M}}_0(U;t)U, \quad \breve{\mathbf{M}}_0(U;t) \in \mathcal{M}^0_{K,\underline{K}'-1,0}[r] \otimes \mathcal{M}_2(\mathbb{C}) \tag{4.7.54}$$

using that $\mathrm{Id}+R^{(p)}_{\leq N}(W)$ belongs to $\mathcal{M}^0_{K,0,0}[r]\otimes\mathcal{M}_2(\mathbb{C})$ (by Lemma 4.2.8), since $\Phi_p(Z_{p-1}) = \mathcal{G}^1_{g_p}(Z_{p-1})Z_{p-1}$ with $\mathcal{G}^1_{g_p} \in \mathcal{M}^0_{K,0,0}[r] \otimes \mathcal{M}_2(\mathbb{C})$ (Lemma 4.3.17 (i)), the inductive assumption $Z_{p-1} = \mathbf{M}_0^{(p-1)}(U;t)U$ with $\mathbf{M}_0^{(p-1)}(U;t) \in \mathcal{M}^0_{K,\underline{K}'-1,0}[r] \otimes \mathcal{M}_2(\mathbb{C})$ and Proposition 4.2.15 (iii).

Moreover, regarding $R^{(p)}_{\leq N}(W)$ as a non-homogeneous smoothing operator in $\mathcal{R}^{-\varrho-\frac{3}{2}}_{K,0,p}[r] \otimes \mathcal{M}_2(\mathbb{C})$ for any $r > 0$ (see Lemma 4.2.8), estimate (4.2.40) implies, for $0 < r < r_0(s,K)$ small, the bound

$$2^{-1}\|W\|_{k,s} \leq \|\mathcal{C}^{(p)}_N(W)\|_{k,s} \leq 2\|W\|_{k,s}\,, \quad \forall k = 0,\dots,K\,. \tag{4.7.55}$$

We compute the new system satisfied by V in (4.7.53) using Lemma A.0.5. Its assumption **(A)** at page 215 holds (with $K' = \underline{K}'$ and $\varrho \rightsquigarrow \varrho - \underline{\varrho}(p) - c(N,p)$) since W solves (4.7.51) and (4.7.52), (4.7.49), (4.7.39), $(\mathbf{S2})_p$. Then Lemma A.0.5 implies that the variable V solves

$$\begin{aligned}\partial_t V = & - \mathrm{i}\boldsymbol{\Omega}(D)V + J_c\nabla\big(H_{\frac{3}{2}}^{(\mathtt{SAP})}\big)_{\leq p+2}(V) + J_c\nabla\big(H_{-\varrho}^{(\mathtt{SAP})}\big)_{\leq p+1}(V) \\ & + \mathrm{Op}^{BW}_{\mathtt{vec}}\Big(-\mathrm{i}\widetilde{(\mathtt{m}_{\frac{3}{2}})}_{\geq p+1}(V;\xi)\Big)V + \widetilde{R}_{\geq p}(V)V \\ & + \mathrm{Op}^{BW}_{\mathtt{vec}}\Big(-\mathrm{i}\widetilde{(\mathtt{m}_{\frac{3}{2}})}_{>N}(U;t,\xi)\Big)V + \widetilde{R}_{>N}(U;t)U\end{aligned} \tag{4.7.56}$$

where

- $\widetilde{(\mathtt{m}_{\frac{3}{2}})}_{\geq p+1}(V;\xi)$ is a real valued symbol, independent of x, in $\Sigma_{p+1}^{N}\widetilde{\Gamma}_{q}^{\frac{3}{2}}$;
- $\widetilde{(\mathtt{m}_{\frac{3}{2}})}_{>N}(U;t,\xi)$ is a non-homogeneous symbol independent of x in $\Gamma^{\frac{3}{2}}_{K,\underline{K}',N+1}[r]$ with imaginary part $\mathrm{Im}\,\widetilde{(\mathtt{m}_{\frac{3}{2}})}_{>N}(U;t,\xi)$ in $\Gamma^{0}_{K,\underline{K}',N+1}[r]$;
- $\widetilde{R}_{\geq p}(V)$ is a matrix of pluri–homogeneous smoothing operators in $\Sigma_p^N\widetilde{\mathcal{R}}_q^{-\varrho+\underline{\varrho}(p)+c(N,p)}\otimes\mathcal{M}_2(\mathbb{C})$;
- $\widetilde{R}_{>N}(U;t)$ is a matrix of non–homogeneous smoothing operators in $\mathcal{R}^{-\varrho+\underline{\varrho}(p)+c(N,p)}_{K,\underline{K}',N+1}[r]\otimes\mathcal{M}_2(\mathbb{C})$.

Note that in (4.7.56) the pluri-homogeneous components up to order $p+1$ of the symbol and up to order $p-1$ of the smoothing operators are unchanged with respect to (4.7.51), whereas the homogeneous part of order p of the smoothing remainder have been corrected by a new smoothing operator in $\widetilde{\mathcal{R}}_p^{-\varrho+\underline{\varrho}(p)+c(N,p)}\otimes\mathcal{M}_2(\mathbb{C})$, see (A.0.48).

Since system (4.7.56) is Hamiltonian up to homogeneity N (unlike (4.7.51)), we have in particular that

$$-\,\mathrm{i}\boldsymbol{\Omega}(D)V+J_c\nabla\Big(\big(H^{(\mathtt{SAP})}_{\frac{3}{2}}\big)_{\leq p+2}(V)+\big(H^{(\mathtt{SAP})}_{-\varrho}\big)_{\leq p+1}(V)\Big)+\mathcal{P}_p[\widetilde{R}_{\geq p}(V)]V \tag{4.7.57}$$

is a pluri-homogeneous Hamiltonian vector field.

Iteration: reduction of the p–homogeneous smoothing remainder. The goal of this step is to reduce the smoothing homogenous vector field $\widetilde{R}_p(V)V:=\mathcal{P}_p[\widetilde{R}_{\geq p}(V)]V$ in (4.7.57), which belongs to $\widetilde{\mathfrak{X}}_{p+1}^{-\varrho+\underline{\varrho}(p)+c(N,p)}$, to its super action preserving normal form. By (4.7.57) we deduce, by difference, that $\widetilde{R}_p(V)V$ is Hamiltonian. We expand $\widetilde{R}_p(V)V$ in Fourier coordinates as in (4.2.122)

$$\big(\widetilde{R}_p(V)V\big)_k^{\sigma}=\sum_{(\vec{\jmath}_{p+1},k,\vec{\sigma}_{p+1},-\sigma)\in\mathfrak{T}_{p+2}}\widetilde{X}^{\vec{\sigma}_{p+1},\sigma}_{\vec{\jmath}_{p+1},k}\,v^{\vec{\sigma}_{p+1}}_{\vec{\jmath}_{p+1}}\,. \tag{4.7.58}$$

In order to reduce $\widetilde{R}_p(V)V$ to its super action preserving part we transform (4.7.56) under the change of variable $Z_p:=\mathtt{F}^{(p)}_{\leq N}(V)$ where $\mathtt{F}^{(p)}_{\leq N}(V)$ is the time 1-approximate flow, given by Lemma 4.2.28, generated by the smoothing vector field

$$\big(G_p(V)V\big)_k^{\sigma}=\sum_{(\vec{\jmath}_{p+1},k,\vec{\sigma}_{p+1},-\sigma)\in\mathfrak{T}_{p+2}}G^{\vec{\sigma}_{p+1},\sigma}_{\vec{\jmath}_{p+1},k}\,v^{\vec{\sigma}_{p+1}}_{\vec{\jmath}_{p+1}} \tag{4.7.59}$$

with

$$G^{\vec{\sigma}_{p+1},\sigma}_{\vec{\jmath}_{p+1},k}:=\begin{cases}0 & \text{if } (\vec{\jmath}_{p+1},k,\vec{\sigma}_{p+1},-\sigma)\in\mathfrak{S}_{p+2}\\[4pt] \dfrac{\widetilde{X}^{\vec{\sigma}_{p+1},\sigma}_{\vec{\jmath}_{p+1},k}}{\mathrm{i}(\vec{\sigma}_{p+1}\cdot\Omega_{\vec{\jmath}_{p+1}}(\kappa)-\sigma\Omega_k(\kappa))} & \text{if } (\vec{\jmath}_{p+1},k,\vec{\sigma}_{p+1},-\sigma)\notin\mathfrak{S}_{p+2}\end{cases} \tag{4.7.60}$$

where the super action set $\mathfrak{S}_{p+2}$ is defined in (4.7.13) (with p replaced by $p+2$).

Lemma 4.7.15. *Let $\kappa\in(0,+\infty)\setminus\mathcal{K}$. The vector field $G_p(V)V$ in (4.7.59), (4.7.60) is a well defined Hamiltonian vector field in $\widetilde{\mathfrak{X}}^{-\varrho'}_{p+1}$ with $\varrho':=\varrho-\underline{\varrho}(p)-c(N,p)-\tau$ and where τ is defined in Theorem B.0.1.*

Proof. We claim that for any $\kappa\in(0,+\infty)\setminus\mathcal{K}$ there exist $\tau,\nu>0$ such that

$$|\vec{\sigma}_{p+1}\cdot\Omega_{\vec{\jmath}_{p+1}}(\kappa)-\sigma\Omega_k(\kappa)|>\frac{\nu}{\max(|j_1|,\dots,|j_{p+1}|,|k|)^{\tau}}\,,\qquad\forall(\vec{\jmath}_{p+1},k,\vec{\sigma}_{p+1},-\sigma)\notin\mathfrak{S}_{p+2}\,. \tag{4.7.61}$$

Indeed, to any $(\vec{\jmath}_{p+1}, k, \vec{\sigma}_{p+1}, -\sigma)$ we associate the multi–index (α, β) as in (4.7.12) whose length is $|\alpha + \beta| = p + 2$ and satisfies $\vec{\sigma}_{p+1} \cdot \Omega_{\vec{\jmath}_{p+1}}(\kappa) - \sigma \Omega_k(\kappa) = (\alpha - \beta) \cdot \vec{\Omega}(\kappa)$ by (4.7.15). Recalling (4.7.13), the vector $(\vec{\jmath}_{p+1}, k, \vec{\sigma}_{p+1}, -\sigma) \notin \mathfrak{S}_{p+2}$ if and only if (α, β) is not super action-preserving and therefore Theorem B.0.1 implies (4.7.61). Note also that, by Remark 4.7.10, if p is odd, there are not super-action preserving indexes, i.e. $\mathfrak{S}_{p+2} = \emptyset$. In view of (4.7.61) the coefficients in (4.7.60) are well defined.

Next we show that $G_p(V)V$ is a vector field in $\widetilde{\mathfrak{X}}^{-\varrho'}_{p+1}$. As $\widetilde{R}_p(V)V$ belongs to $\widetilde{\mathfrak{X}}^{-\varrho+\underline{\varrho}(p)+c(N,p)}_{p+1}$, by Lemma 4.2.26 the coefficients $\widetilde{X}^{\vec{\sigma}_{p+1},\sigma}_{\vec{\jmath}_{p+1},k}$ in (4.7.58) satisfy the symmetric and reality properties (4.2.124), (4.2.125) and the estimate: for some $\mu \geq 0$, $C > 0$,

$$|\widetilde{X}^{\vec{\sigma}_{p+1},\sigma}_{\vec{\jmath}_{p+1},k}| \leq C \frac{\max_2\{|j_1|, \dots, |j_{p+1}|\}^{\mu}}{\max\{|j_1|, \dots, |j_{p+1}|\}^{\varrho - \underline{\varrho}(p) - c(N,p)}}, \quad \forall (\vec{\jmath}_{p+1}, k, \vec{\sigma}_{p+1}, -\sigma) \in \mathfrak{T}_{p+2}. \tag{4.7.62}$$

Hence also the coefficients $G^{\vec{\sigma}_{p+1},\sigma}_{\vec{\jmath}_{p+1},k}$ in (4.7.60) satisfy the symmetric and reality properties (4.2.124), (4.2.125) as $\widetilde{X}^{\vec{\sigma}_{p+1},\sigma}_{\vec{\jmath}_{p+1},k}$. Moreover, using (4.7.62), (4.7.61) and the momentum relation $\sigma k = \vec{\sigma}_{p+1} \cdot \vec{\jmath}_{p+1}$ they also satisfy

$$|G^{\vec{\sigma}_{p+1},\sigma}_{\vec{\jmath}_{p+1},k}| \leq C \frac{\max_2\{|j_1|, \dots, |j_{p+1}|\}^{\mu}}{\max\{|j_1|, \dots, |j_{p+1}|\}^{\varrho - \underline{\varrho}(p) - c(N,p) - \tau}}$$

for a new constant $C > 0$ (depending on ν). Then Lemma 4.2.26 implies that $G_p(V)V$ belongs to $\widetilde{\mathfrak{X}}^{-\varrho'}_{p+1}$ with $\varrho' := \varrho - \underline{\varrho}(p) - c(N,p) - \tau$.

Finally we show that $G_p(V)V$ is Hamiltonian. Recall that $\widetilde{R}_p(V)V$ in (4.7.58) is a Hamiltonian vector field whose Hamiltonian function $H_{\widetilde{R}_p}(V)$ is, thanks to Lemma 4.3.5,

$$H_{\widetilde{R}_p}(V) = \tfrac{1}{p+2} \sum_{(\vec{\jmath}_{p+2}, \vec{\sigma}_{p+2}) \in \mathfrak{T}_{p+2}} [H_{\widetilde{R}_p}]^{\vec{\sigma}_{p+2}}_{\vec{\jmath}_{p+2}} v^{\vec{\sigma}_{p+2}}_{\vec{\jmath}_{p+2}}, \qquad [H_{\widetilde{R}_p}]^{\sigma_1, \dots, \sigma_{p+2}}_{j_1, \dots, j_{p+2}} := -\mathrm{i}\sigma_{p+2} \widetilde{X}^{\vec{\sigma}_{p+1}, -\sigma_{p+2}}_{\vec{\jmath}_{p+1}, j_{p+2}}. \tag{4.7.63}$$

Then the coefficients defined for $\vec{\sigma}_{p+2} \cdot \vec{\jmath}_{p+2} = 0$ by

$$[H_{G_p}]^{\vec{\sigma}_{p+2}}_{\vec{\jmath}_{p+2}} := -\mathrm{i}\sigma_{p+2} G^{\sigma_1, \dots, \sigma_{p+1}, -\sigma_{p+2}}_{j_1, \dots, j_{p+1}, j_{p+2}} \overset{(4.7.60),(4.7.63)}{=} \frac{[H_{\widetilde{R}_p}]^{\vec{\sigma}_{p+2}}_{\vec{\jmath}_{p+2}}}{\mathrm{i}\big(\vec{\sigma}_{p+1} \cdot \Omega_{\vec{\jmath}_{p+1}}(\kappa) + \sigma_{p+2} \Omega_{j_{p+2}}(\kappa)\big)} \tag{4.7.64}$$

satisfy the symmetric and reality properties (4.2.135), (4.2.134) as well as the coefficients $[H_{\widetilde{R}_p}]^{\vec{\sigma}_{p+2}}_{\vec{\jmath}_{p+2}}$. Then Lemma 4.3.5 implies that $G_p(V)V$ is the Hamiltonian vector field generated by the Hamiltonian H_{G_p} with coefficients defined in (4.7.64). □

We now conjugate system (4.7.56) by the approximate time 1-flow $\mathtt{F}^{(p)}_{\leq N}(V)$ generated by $G_p(V)V$ provided by Lemma 4.2.28, which has the form

$$Z_p = \mathtt{F}^{(p)}_{\leq N}(V) = V + F^{(p)}_{\leq N}(V)V, \quad F^{(p)}_{\leq N}(V) \in \Sigma_p^N \widetilde{\mathcal{R}}^{-\varrho'}_q \otimes \mathcal{M}_2(\mathbb{C}). \tag{4.7.65}$$

Since $G_p(Z)Z$ is a Hamiltonian vector field, by Lemma 4.3.14 the approximate flow $\mathtt{F}^{(p)}_{\leq N}$ is symplectic up to homogeneity N. Applying Lemma 4.3.15 (with $Z \rightsquigarrow V$, $W \rightsquigarrow Z_p$ and by (4.7.54)), the variable Z_p solves a system which is Hamiltonian up to homogeneity N. We compute it using Lemma A.0.5 (with $W \rightsquigarrow Z_p$ and $Z \rightsquigarrow V$). Its assumption **(A)** at page 215 holds (with $K' = \underline{K}'$ and $\varrho \rightsquigarrow \varrho - \underline{\varrho}(p) - c(N,p)$)

since V solves (4.7.56) and (4.7.52), (4.7.49), (4.7.39), **(S2)**$_p$. Lemma A.0.5 implies that the variable Z_p in (4.7.65) solves (see in particular (A.0.50))

$$\begin{aligned}\partial_t Z_p = &- \mathrm{i}\boldsymbol{\Omega}(D)Z_p + J_c\nabla\big(H^{(\mathtt{SAP})}_{\frac32}\big)_{\le p+2}(Z_p) + J_c\nabla\big(H^{(\mathtt{SAP})}_{-\varrho}\big)_{\le p+1}(Z_p)\\ &+ \mathrm{Op}^{BW}_{\mathtt{vec}}\Big(-\mathrm{i}(\breve{\mathtt{m}}_{\frac32})_{\ge p+1}(Z_p;\xi)\Big)Z_p + \Big[\widetilde{R}_p(Z_p) + G_p^+(Z_p)\Big]Z_p + R_{\ge p+1}(Z_p)Z_p\\ &+ \mathrm{Op}^{BW}_{\mathtt{vec}}\Big(-\mathrm{i}(\breve{\mathtt{m}}_{\frac32})_{>N}(U;t,\xi)\Big)Z_p + R_{>N}(U;t)U\end{aligned} \tag{4.7.66}$$

where the part homogeneous up to order p of the symbol and up to order $p-1$ of the smoothing operators are unchanged with respect to (4.7.56), whereas

- $(\breve{\mathtt{m}}_{\frac32})_{\ge p+1}(V;\xi)$ is a real valued symbol, independent of x, in $\Sigma_{p+1}^N\widetilde{\Gamma}_q^{\frac32}$;
- $(\breve{\mathtt{m}}_{\frac32})_{>N}(U;t,\xi)$ is a non-homogeneous symbol independent of x in $\Gamma^{\frac32}_{K,\underline{K}',N+1}[r]$ with imaginary part $\mathrm{Im}\,\widetilde{(\mathtt{m}_{\frac32})}_{>N}(U;t,\xi)$ in $\Gamma^0_{K,\underline{K}',N+1}[r]$;
- $G_p^+(Z_p)Z_p$ is $p+1$-homogeneous smoothing vector field in $\widetilde{\mathfrak{X}}^{-\varrho'+\frac32}_{p+1}$ with Fourier expansion (see (A.0.51))

$$(G_p^+(Z_p)Z_p)_k^\sigma = \sum_{(\vec{\jmath}_{p+1},k,\vec{\sigma}_{p+1},-\sigma)\in\mathfrak{T}_{p+2}} -\mathrm{i}\big(\vec{\sigma}_{p+1}\cdot\vec{\Omega}_{\vec{\jmath}_{p+1}}(\kappa) - \sigma\Omega_k(\kappa)\big)G^{\vec{\sigma}_{p+1},\sigma}_{\vec{\jmath}_{p+1},k}(z_p)^{\vec{\sigma}_{p+1}}_{\vec{\jmath}_{p+1}}\,; \tag{4.7.67}$$

- $R_{\ge p+1}(Z_p)$ is a matrix of pluri–homogeneous smoothing operators in $\Sigma^N_{p+1}\widetilde{\mathcal{R}}_q^{-\varrho+\underline{\varrho}(p+1)}\otimes\mathcal{M}_2(\mathbb{C})$ where

$$\underline{\varrho}(p+1) = \underline{\varrho}(p) + c(N,p) + \tau + \tfrac32\,; \tag{4.7.68}$$

- $R_{>N}(U;t)$ is a matrix of non–homogeneous smoothing operators in $\mathcal{R}^{-\varrho+\underline{\varrho}(p+1)}_{K,\underline{K}',N+1}[r]\otimes\mathcal{M}_2(\mathbb{C})$.

By (4.7.58), (4.7.60), (4.7.67), the smoothing operators of homogeneity p in (4.7.66) reduce to

$$\widetilde{R}_p(Z_p)Z_p + G_p^+(Z_p)Z_p = R_p^{(\mathtt{SAP})}(Z_p)Z_p \tag{4.7.69}$$

where $R_p^{(\mathtt{SAP})}(Z_p)Z_p$ is the vector field

$$(R_p^{(\mathtt{SAP})}(Z_p)Z_p)_k^\sigma := \sum_{(\vec{\jmath}_{p+1},k,\vec{\sigma}_{p+1},-\sigma)\in\mathfrak{S}_{p+2}} \widetilde{X}^{\vec{\sigma}_{p+1},\sigma}_{\vec{\jmath}_{p+1},k}(z_p)^{\vec{\sigma}_{p+1}}_{\vec{\jmath}_{p+1}} \in \widetilde{\mathfrak{X}}^{-\varrho+\underline{\varrho}(p)+c(N,p)}_{p+1}\,,$$

that, in view of (4.7.63), is generated by the super action preserving Hamiltonian (cfr. Definition 4.7.8)

$$H^{(\mathtt{SAP})}_{\widetilde{R}_p}(Z_p) := \frac{1}{p+2}\sum_{(\vec{\jmath}_{p+2},\vec{\sigma}_{p+2})\in\mathfrak{S}_{p+2}} [H_{\widetilde{R}_p}]^{\vec{\sigma}_{p+2}}_{\vec{\jmath}_{p+2}}(z_p)^{\vec{\sigma}_{p+2}}_{\vec{\jmath}_{p+2}}\,.$$

By (4.7.69) and $R_p^{(\mathtt{SAP})}(Z_p)Z_p = J_c\nabla H^{(\mathtt{SAP})}_{\widetilde{R}_p}(Z_p)$, system (4.7.66) becomes

$$\begin{aligned}\partial_t Z_p = &-\mathrm{i}\boldsymbol{\Omega}(D)Z_p + J_c\nabla\big(H^{(\mathtt{SAP})}_{\frac32}\big)_{\le p+2}(Z_p) + J_c\nabla\big(H^{(\mathtt{SAP})}_{-\varrho}\big)_{\le p+2}(Z_p)\\ &+ \mathrm{Op}^{BW}_{\mathtt{vec}}\Big(-\mathrm{i}(\mathtt{m}_{\frac32})_{p+1}(Z_p;\xi) - \mathrm{i}(\mathtt{m}_{\frac32})_{\ge p+2}(Z_p;\xi)\Big)Z_p + R_{\ge p+1}(Z_p)Z_p\\ &+ \mathrm{Op}^{BW}_{\mathtt{vec}}\Big(-\mathrm{i}(\mathtt{m}_{\frac32})_{>N}(U;t,\xi)\Big)Z_p + R_{>N}(U;t)U\end{aligned} \tag{4.7.70}$$

with $\big(H^{(\mathtt{SAP})}_{\frac{3}{2}}\big)_{\leq p+2}$ defined in (4.7.52) (see also (4.7.49)-(4.7.50)),

$$\big(H^{(\mathtt{SAP})}_{-\varrho}\big)_{\leq p+2}(Z_p) := \big(H^{(\mathtt{SAP})}_{-\varrho}\big)_{\leq p+1}(Z_p) + H^{(\mathtt{SAP})}_{\widetilde{R}_p}(Z_p)\,, \tag{4.7.71}$$

and x-independent real symbols

$$\big(\mathtt{m}_{\frac{3}{2}}\big)_{p+1}(Z_p;\xi) := \mathcal{P}_{p+1}\big[\big(\breve{\mathtt{m}}_{\frac{3}{2}}\big)_{\geq p+1}(Z_p;\xi)\big] \in \widetilde{\Gamma}^{\frac{3}{2}}_{p+1}$$
$$\big(\mathtt{m}_{\frac{3}{2}}\big)_{\geq p+2}(Z_p;\xi) := \mathcal{P}_{\geq p+2}\big[\big(\breve{\mathtt{m}}_{\frac{3}{2}}\big)_{\geq p+1}(Z_p;\xi)\big] \in \Sigma^N_{p+2}\widetilde{\Gamma}^{\frac{3}{2}}_q\,.$$

System (4.7.70) has therefore the form (4.7.37) at the step $p+1$ with

$$Z_p := \mathcal{F}^{(p)}_{\leq N}(Z_0) := \mathtt{F}^{(p)}_{\leq N} \circ \mathcal{C}^{(p)}_N \circ \Phi_p \circ \mathcal{F}^{(p-1)}_{\leq N}(Z_0)\,, \tag{4.7.72}$$

see (4.7.42), (4.7.53), (4.7.65). The map $\mathcal{F}^{(p)}_{\leq N}$ is symplectic up to homogeneity N as $\mathcal{F}^{(p-1)}_{\leq N}$, because $\mathcal{C}^{(p)}_N \circ \Phi_p$ is symplectic up to homogeneity N (cfr. (4.7.53)) as well as the time 1-approximate flow $\mathtt{F}^{(p)}_{\leq N}$ generated by the smoothing Hamiltonian vector field $G_p(V)V$ (cfr. Lemma 4.7.15) by Lemma 4.3.14. In addition the map $\mathcal{F}^{(p)}_{\leq N}(Z_0)$ satisfies (4.7.22) (with p-dependent constants) because of the inductive assumption, (4.7.46), (4.7.55) and (4.2.40). Furthermore $Z_p = \mathbf{M}^{(p)}_0(U;t)U$ with $\mathbf{M}^{(p)}_0(U;t) \in \mathcal{M}^0_{K,\underline{K}'-1,0}[r] \otimes \mathcal{M}_2(\mathbb{C})$. This follows from (4.7.72) using that $\mathrm{Id} + F^{(p)}_{\leq N}$ and $\mathrm{Id} + R^{(p)}_{\leq N}$ belong to $\mathcal{M}^0_{K,0,0}[r] \otimes \mathcal{M}_2(\mathbb{C})$ (recall identities (4.7.65), (4.7.53) and use Lemma 4.2.8), since $\Phi_p(Z_{p-1}) = \mathcal{G}^1_{g_p}(Z_{p-1})Z_{p-1}$ with $\mathcal{G}^1_{g_p} \in \mathcal{M}^0_{K,0,0}[r] \otimes \mathcal{M}_2(\mathbb{C})$ (Lemma 4.3.17 (i)), the inductive assumption $\mathcal{F}^{(p-1)}_{\leq N}(Z_0) = \mathbf{M}^{(p-1)}_0(U;t)U$ with $\mathbf{M}^{(p-1)}_0(U;t) \in \mathcal{M}^0_{K,\underline{K}'-1,0}[r] \otimes \mathcal{M}_2(\mathbb{C})$ and Proposition 4.2.15 (iii). The proof of $(\mathbf{S0})_{p+1}$ is complete.

System (4.7.70) satisfies also $(\mathbf{S1})_{p+1}$–$(\mathbf{S6})_{p+1}$ with $\underline{\varrho}(p+1)$ defined in (4.7.68) by (4.7.52), (4.7.49)-(4.7.50) and (4.7.71). $(\mathbf{S7})_{p+1}$ follows by Lemma 4.3.15 because $\mathcal{F}^{(p)}_{\leq N}$ is symplectic up to homogeneity N. This concludes the proof of the inductive step.

Finally (4.7.23) follows by (4.7.22) and (4.7.5) (where C denote different constants). □

Remark 4.7.16. (Integrability of fourth and six order Hamiltonian Birkhoff normal form) The SAP Hamiltonian monomials (Definition 4.7.8) of degree 4 are integrable. Indeed, a-priori they are either the integrable ones $|z_{j_1}|^2|z_{j_2}|^2$ or (i) $|z_{j_1}|^2 z_{j_2}\overline{z_{-j_2}}$ or (ii) $z_{j_1}\overline{z_{-j_1}}z_{j_2}\overline{z_{-j_2}}$. The momentum condition implies in case (i) that $j_2 = 0$, which is not allowed. In case (ii) it yields $j_1 + j_2 = 0$ and so $z_{j_1}\overline{z_{-j_1}}z_{j_2}\overline{z_{-j_2}} = z_{j_1}\overline{z_{j_2}}z_{j_2}\overline{z_{j_1}}$ is integrable. The SAP Hamiltonian monomials of degree 6 may contain the not integrable monomials

$$(i)\ |z_{j_1}|^2 z_{j_2}\overline{z_{-j_2}}z_{j_3}\overline{z_{-j_3}}\,, \quad (ii)\ |z_{j_1}|^2|z_{j_2}|^2 z_{j_3}\overline{z_{-j_3}}\,, \quad (iii)\ z_{j_1}\overline{z_{-j_1}}z_{j_2}\overline{z_{-j_2}}z_{j_3}\overline{z_{-j_3}}\,.$$

By momentum conservation, a monomial of the form (i) is integrable (as in case (ii) above) and a monomial of the form (ii) has $j_3 = 0$ thus it is not allowed. The monomials (iii) turn out to be, for $\gamma \neq 0$, $\mathtt{h} = +\infty$, Birkhoff non-resonant, namely

$$\begin{aligned} &\big|\Omega_{j_1}(\kappa) - \Omega_{-j_1}(\kappa) + \Omega_{j_2}(\kappa) - \Omega_{-j_2}(\kappa) + \Omega_{j_3}(\kappa) - \Omega_{-j_3}(\kappa)\big| \\ &= \gamma\big|\mathrm{sign}(j_1) + \mathrm{sign}(j_2) + \mathrm{sign}(j_3)\big| \geq \gamma\,. \end{aligned} \tag{4.7.73}$$

Therefore they might be eliminated, obtaining an integrable normal form Hamiltonian at the degree 6. The same holds in finite depth exploiting also the momentum restriction $j_1+j_2+j_3 = 0$ and that $\gamma\big(\tanh(\mathtt{h}j_1) + \tanh(\mathtt{h}j_2) + \tanh(\mathtt{h}j_3)\big) \neq 0$ by the concavity of $y \mapsto \tanh(\mathtt{h}y)$ for $y > 0$. Note that for $|j_1|, |j_2|, |j_3| \geq M$ large enough we have a uniform lower bound as in (4.7.73). In conclusion the fourth and six order Hamiltonian Birkhoff normal form of the water waves equations (4.1.2) is integrable.

4.8 Energy estimate and proof of the main theorem

The Hamiltonian equation

$$\partial_t Z = -\mathrm{i}\boldsymbol{\Omega}(D)Z + J_c \nabla H^{(\mathtt{SAP})}_{\frac{3}{2}}(Z) + J_c \nabla H^{(\mathtt{SAP})}_{-\varrho}(Z)\,, \tag{4.8.1}$$

obtained by (4.7.21) neglecting the symbol and the smoothing operator of homogeneity larger than N, preserves the Sobolev norms. Equation (4.8.1) can be also written as the Hamiltonian PDE

$$\partial_t Z = J_c \nabla H^{(\mathtt{SAP})}(Z) \tag{4.8.2}$$

where $H^{(\mathtt{SAP})}(Z)$ is the super–action preserving Hamiltonian (cfr. Definition 4.7.8)

$$H^{(\mathtt{SAP})}(Z) := H^{(2)}(Z) + H^{(\mathtt{SAP})}_{\frac{3}{2}}(Z) + H^{(\mathtt{SAP})}_{-\varrho}(Z)\,, \quad H^{(2)}(Z) := \sum_{j\in\mathbb{Z}\setminus\{0\}} \Omega_j(\kappa)|z_j|^2\,. \tag{4.8.3}$$

Actually the following more precise result holds.

Lemma 4.8.1. **(Super-actions)** *The super–actions $J_n(Z) = |z_n|^2 + |z_{-n}|^2$, defined in* (4.7.17)*, for any $n \in \mathbb{N}$, are prime integrals of the Hamiltonian equation* (4.8.1)*. In particular the Sobolev norm*

$$\|Z\|^2_{\dot H^s} = \sum_{j\in\mathbb{Z}\setminus\{0\}} |j|^{2s}|z_j|^2 = \sum_{n\in\mathbb{N}} n^{2s} J_n(Z)$$

is constant along the flow of (4.8.1).

Proof. By (4.8.2), (4.3.29) and Lemma 4.7.7 we have

$$\frac{\mathrm{d}}{\mathrm{d}t} J_n(Z) = \mathrm{d}_Z J_n(Z)\left[J_c \nabla H^{(\mathtt{SAP})}(Z)\right] = \{J_n(Z), H^{(\mathtt{SAP})}(Z)\} = 0$$

since the Hamiltonian $H^{(\mathtt{SAP})}(Z)$ contains only super action preserving monomials. □

By the previous lemma, in order to derive an energy estimate for the solutions of (4.7.21), and thus for a solution U of (4.5.37), we have to estimate the non-homogeneous term in (4.7.21). We need the following lemma.

Lemma 4.8.2. *Let $K \in \mathbb{N}$. Then there is $s_0 > 0$ such that, for any $s \geq s_0$, for all $0 < r \leq \overline{r}(s)$ small, if U belongs to $B^0_{s_0}(I;r) \cap C^0_{*\mathbb{R}}(I;\dot H^s(\mathbb{T},\mathbb{C}^2))$ and solves* (4.5.37)*, then U belongs to $C^K_{*\mathbb{R}}(I;\dot H^s(\mathbb{T},\mathbb{C}^2))$ and $\forall 0 \leq k \leq K$ there is a constant $\overline{C}_{s,k} \geq 1$ such that*

$$\|\partial_t^k U(t)\|_{\dot H^{s-\frac{3}{2}k}} \leq \overline{C}_{s,k}\|U(t)\|_{\dot H^s}\,, \quad \forall t \in I\,. \tag{4.8.4}$$

In particular the norm $\|U(t)\|_{K,s}$ defined in (4.2.2) *is equivalent to the norm $\|U(t)\|_{\dot H^s}$.*

Proof. We argue by induction proving that for any $0 \leq k \leq K$, there are $s_0, \overline{r}_k > 0$ such that if $U \in B^0_{s_0}(I;\overline{r}_k) \cap C^0_{*\mathbb{R}}(I;\dot H^s(\mathbb{T},\mathbb{C}^2))$ solves (4.5.37), then $U \in C^k_{*\mathbb{R}}(I;\dot H^s(\mathbb{T};\mathbb{C}^2))$ with estimate (4.8.4). For $k = 0$ the estimate (4.8.4) is trivial. Then assume (4.8.4) holds true for $0,\dots,k-1 \leq K-1$. Next we write (4.5.37) as $\partial_t U = \mathrm{Op}^{BW}(A(U;x;\xi))U + R(U)U$ where, by Lemma 4.5.5, the smoothing operator $R(U)$ is in $\Sigma\mathcal{R}^{-\varrho}_{k-1,0,1}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$ and the matrix of symbols

$$A(U;x,\xi) := J_c A_{\frac{3}{2}}(U;x)\omega(\xi) + \frac{\gamma}{2}\frac{\mathtt{G}(\xi)}{\mathrm{i}\xi}\mathbb{1} + J_c\big[A_1(U;x,\xi) + A_{\frac{1}{2}}(U;x,\xi) + A^{(2)}_0(U;x,\xi)\big]$$

belongs to $\Sigma\Gamma^{\frac{3}{2}}_{k-1,0,0}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ and so, by the fourth bullet after Definition 4.2.16, $\mathrm{Op}^{\mathrm{BW}}(A(U;x,\xi))$ belongs to $\Sigma\mathcal{S}^{\frac{3}{2}}_{k-1,0,0}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$. Let $s_0 := s_0(k) > 0$ given in Definitions 4.2.5, 4.2.16. By the inductive estimate (4.8.4) up to $k-1$, we have that $U\in B^{k-1}_{s_0}(I,\overline{C}'_{s_0,k}r)$ with $\overline{C}'_{s_0,k} := \sum_{j=0}^{k-1}\overline{C}_{s_0,j}$. Then, for any $s\geq s_0$, there is $\overline{r}_k := \overline{r}(s,k) > 0$ such that if $0<r<\overline{r}_k$ the operator $\mathrm{Op}^{\mathrm{BW}}(A(U;x,\xi))$ fulfills estimate below (4.2.72) (with $K'=0$, $m=\frac{3}{2}$), and $R(U)$ estimate below (4.2.34) (with $K'=0$, $m=0$) so that

$$\begin{aligned}\|\partial_t^k U\|_{\dot H^{s-\frac{3}{2}k}} &\leq \|\partial_t^{k-1}(\mathrm{Op}^{\mathrm{BW}}(A(U;x,\xi))U)\|_{\dot H^{s-\frac{3}{2}(k-1)-\frac{3}{2}}} + \|\partial_t^{k-1}(R(U)U)\|_{\dot H^{s-\frac{3}{2}(k-1)-\frac{3}{2}}}\\ &\leq C_{s,k}\|U\|_{k-1,s} = C_{s,k}\sum_{j=0}^{k-1}\|\partial_t^j U\|_{\dot H^{s-\frac{3}{2}j}} \leq \overline{C}_{s,k}\|U\|_{\dot H^s}\end{aligned} \tag{4.8.5}$$

by the inductive hypothesis (4.8.4) for $j=0,\dots,k-1$ and setting $\overline{C}_{s,k} := C_{s,k}\sum_{j=0}^{k-1}\overline{C}_{s,j}$. This proves (4.8.4) at step k. We finally fix $\overline{r}(s) := \min(\overline{r}_1,\dots,\overline{r}_K)$ proving the lemma. □

Proof of Theorem 4.1.1

We deal only with the case $N\in\mathbb{N}$, since the cubic energy estimate (4.8.15) in case $N=0$ follows directly from Proposition 4.6.1 (see also Remark 4.6.2), yielding the local time of existence of order ε^{-1}.

So from now on we consider $N\in\mathbb{N}$. For any value of the gravity $g>0$, depth $\mathtt{h}\in(0,+\infty]$ and vorticity $\gamma\in\mathbb{R}$, let $\mathcal{K}\subset(0,+\infty)$ be the zero measure set defined in Theorem B.0.1. Assume that the surface tension coefficient κ belongs to the complementary set $(0,+\infty)\setminus\mathcal{K}$. Let $\underline{\varrho}>0$ be the constant given by Proposition 4.7.12.

- **Choice of the parameters:** *From now on we fix in Proposition 4.7.12 the parameters*

$$\varrho := \underline{\varrho}, \quad K := \underline{K}'(\varrho)\in\mathbb{N}, \tag{4.8.6}$$

where $\underline{K}'(\varrho)$ is defined in Proposition 4.6.1. Thus Proposition 4.7.12 provides $\underline{s}_0>0$ and for any

$$s\geq \underline{s}_0 \quad \textit{we fix} \quad 0<r<\min(\underline{r}_0(s),\overline{r}(s)), \tag{4.8.7}$$

where $\underline{r}_0(s)>0$ is given by Proposition 4.7.12 and $\overline{r}(s)>0$ is given by Lemma 4.8.2. Let $U(t)$ be a solution of system (4.5.37) in $B^K_{\underline{s}_0}(I;r)\cap C^K_{*\mathbb{R}}(I;\dot H^s(\mathbb{T},\mathbb{C}^2))$.

In order to prove Theorem 4.1.1 we have to provide energy estimates of the non-homogenous "vector field" in (4.7.21):

$$X_{\geq N+2}(U,Z) := \begin{pmatrix} X^+_{\geq N+2}(U,Z)\\ \overline{X^+_{\geq N+2}(U,Z)}\end{pmatrix} := \mathrm{Op}^{BW}_{\mathtt{vec}}\left(-\mathrm{i}(\mathtt{m}_{\frac{3}{2}})_{>N}(U;t,\xi)\right)Z + R_{>N}(U;t)U\,. \tag{4.8.8}$$

The following lemma holds since the imaginary part of the x–independent symbol $(\mathtt{m}_{\frac{3}{2}})_{>N}(U;t,\xi)$ has order zero and because, with the choice of $\varrho=\underline{\varrho}$ in (4.8.6), the remainder $R_{>N}(U;t)$ in (4.8.8) belongs to $\mathcal{R}^0_{K,\underline{K}',N+1}[r]\otimes\mathcal{M}_2(\mathbb{C})$.

Lemma 4.8.3. (Energy estimate) *The non-homogenous vector field $X_{\geq N+2}(U,Z)$ defined in (4.8.8), where $(\mathtt{m}_{\frac{3}{2}})_{>N}$ and $R_{>N}(U;t)$ are defined in Proposition 4.7.12 with parameters given in (4.8.6), (4.8.7), satisfies, for any $t\in I$, the energy estimate*

$$\mathrm{Re}\int_{\mathbb{T}}|D|^s X^+_{\geq N+2}(U,Z)\cdot\overline{|D|^s z}\,\mathrm{d}x \leq C_{s,K}\|Z\|^{N+3}_{\dot H^s}\,. \tag{4.8.9}$$

Proof. Since $(\mathtt{m}_{\frac{3}{2}})_{>N}(U;t,\xi)$ is x–independent and has imaginary part $\mathrm{Im}\,(\mathtt{m}_{\frac{3}{2}})_{>N}$ in $\Gamma^0_{K,\underline{K}',N+1}[r]$ (cfr. Proposition 4.7.12) then $\mathrm{Op}^{BW}\big(-\mathrm{i}(\mathtt{m}_{\frac{3}{2}})_{>N}\big)$ commutes with the derivatives $|D|^s$ and, by (4.2.72) (with $k=0$, $m=0$, $s=0$ and $K=\underline{K}'$),

$$
\begin{aligned}
&\mathrm{Re}\int_{\mathbb{T}}|D|^s\mathrm{Op}^{BW}\Big(-\mathrm{i}(\mathtt{m}_{\frac{3}{2}})_{>N}(U;t,\xi)\Big)z\cdot|D|^s\overline{z}\,\mathrm{d}x\\
&=\mathrm{Re}\int_{\mathbb{T}}\mathrm{Op}^{BW}\Big(\mathrm{Im}\,(\mathtt{m}_{\frac{3}{2}})_{>N}(U;t,\xi)\Big)|D|^s z\cdot|D|^s\overline{z}\,\mathrm{d}x\lesssim\|U\|^{N+1}_{\underline{K}',s_0}\|Z\|^2_{\dot H^s}\\
&\overset{\text{Lemma 4.8.2,(4.7.23)}}{\lesssim_{s,K}}\|Z\|^{N+3}_{\dot H^s}.
\end{aligned}
\tag{4.8.10}
$$

We consider now the contribution coming from the smoothing operator $R_{>N}(U;t)$ in $\mathcal{R}^0_{K,\underline{K}',N+1}[r]\otimes\mathcal{M}_2(\mathbb{C})$. Using (4.2.34) (with $k=0$, $m=0$), Lemma 4.8.2 and (4.7.23) we get

$$
\mathrm{Re}\int_{\mathbb{T}}|D|^s\big(R_{>N}(U;t)U\big)^+\cdot|D|^s\overline{z}\,\mathrm{d}x\lesssim_s\|U\|^{N+1}_{\underline{K}',s_0}\|U\|_{\underline{K}',s}\|Z\|_{\dot H^s}\lesssim_{s,K}\|Z\|^{N+3}_{\dot H^s}.
\tag{4.8.11}
$$

The estimate (4.8.9) follows by (4.8.10) and (4.8.11). □

We now prove the following key bootstrap result. By time reversibility we may, without loss of generality, consider only positive times $t>0$.

Proposition 4.8.4. (Bootstrap) *For any $s\geq\underline{s}_0$ there exist $c_0:=c_0(s,K)\in(0,1)$, such that for any solution $U(t)\in C^K_{*\mathbb{R}}(I;\dot H^s(\mathbb{T};\mathbb{C}^2))$ of (4.5.37) fulfilling, for some $0<\varepsilon_1<\min(\underline{r}_0(s),\overline{r}(s))$,*

$$
\|U(0)\|_{\dot H^s}\leq c_0\varepsilon_1\,,\qquad\sup_{t\in[0,T]}\sum_{k=0}^K\|\partial_t^kU(t)\|_{\dot H^{s-\frac{3}{2}k}}\leq\varepsilon_1\,,\qquad T\leq c_0\varepsilon_1^{-N-1}\,,
\tag{4.8.12}
$$

then we have the improved bound

$$
\sup_{t\in[0,T]}\sum_{k=0}^K\|\partial_t^kU(t)\|_{\dot H^{s-\frac{3}{2}k}}\leq\frac{\varepsilon_1}{2}\,.
\tag{4.8.13}
$$

Proof. By (4.8.12) we have that $U\in B^K_s(I;\varepsilon_1)$ with $I:=[-T,T]$. By Proposition 4.7.12 (applied with $r\rightsquigarrow\varepsilon_1<\underline{r}_0(s)$), the variable

$$
Z(t):=\mathcal{F}_{\mathtt{nf}}(Z_0(t)),\quad Z_0(t):=\big(\mathrm{Id}+R(\mathbf{B}(U(t);t)U(t))\big)\mathbf{B}(U(t);t)U(t)\,,
$$

where $\mathbf{B}(U;t)$ is defined in Proposition 4.6.1 and the smoothing operator $R(\cdot)$ is defined in (4.7.3), solves (4.7.21) and has a Sobolev norm equivalent to that of $U(t)$, see (4.7.23). Lemmata 4.8.1 and 4.8.3 (using also $\varepsilon_1<\min(\underline{r}_0(s),\overline{r}(s))$) imply that the solution $Z(t)$ of system (4.7.21) satisfies the energy estimate

$$
\frac{\mathrm{d}}{\mathrm{d}t}\|Z(t)\|^2_{\dot H^s}\leq C_{s,K}\|Z\|^{N+3}_{\dot H^s}
\tag{4.8.14}
$$

and therefore for all $0\leq t\leq T$, by (4.7.23),

$$
\begin{aligned}
\|U(t)\|^2_{\dot H^s}\leq C_{s,K}\|Z(t)\|^2_{\dot H^s}&\overset{(4.8.14)}{\leq}C_{s,K}\Big(\|Z(0)\|^2_{\dot H^s}+\int_0^t\|Z(\tau)\|^{N+3}_{\dot H^s}\,\mathrm{d}\tau\Big)\\
&\overset{(4.7.23)}{\leq}C'_{s,K}\|U(0)\|^2_{\dot H^s}+C'_{s,K}\int_0^t\|U(\tau)\|^{N+3}_{\dot H^s}\,\mathrm{d}\tau\,.
\end{aligned}
\tag{4.8.15}
$$

Then, by the a priori assumption (4.8.12) we deduce that, for all $0 \le t \le T \le c_0 \varepsilon_1^{-(N+1)}$,

$$\|U(t)\|_{\dot{H}^s}^2 \le C'_{s,K} c_0^2 \varepsilon_1^2 + C'_{s,K} T \varepsilon_1^{N+3} \le \varepsilon_1^2 (C'_{s,K} c_0^2 + C'_{s,K} c_0) . \tag{4.8.16}$$

The desired conclusion (4.8.13) on the norms $C_t^k H_x^{s-\frac{3}{2}k}$ follows by Lemma 4.8.2, (4.8.16), choosing c_0 small enough depending on s and K. □

Proof of Theorem 4.1.1 concluded.

Step 1: Local existence. By the local existence theory, there exist $r_{\rm loc}, s_{\rm loc} > 0$ such that for any $s \ge s_{\rm loc}$ there are $T_{\rm loc} > 0, C_{\rm loc} \ge 1$ such that, any initial datum $(\eta_0, \psi_0) \in H_0^{s+\frac14}(\mathbb{T}) \times \dot{H}^{s-\frac14}$ with

$$\|\eta_0\|_{H_0^{s+\frac14}} + \|\psi_0\|_{\dot{H}^{s-\frac14}} \le r_{\rm loc}$$

there exists a unique classical solution $(\eta(t), \psi(t))$ in $C^0\big([-T_{\rm loc}, T_{\rm loc}], H_0^{s+\frac14}(\mathbb{T}) \times \dot{H}^{s-\frac14}(\mathbb{T})\big)$ of (4.1.2) satisfying the initial condition $\eta(0) = \eta_0, \psi(0) = \psi_0$ and

$$\sup_{t \in [-T_{\rm loc}, T_{\rm loc}]} \big(\|\eta(t)\|_{H_0^{s+\frac14}} + \|\psi(t)\|_{\dot{H}^{s-\frac14}}\big) \le C_{\rm loc}\big(\|\eta_0\|_{H_0^{s+\frac14}} + \|\psi_0\|_{\dot{H}^{s-\frac14}}\big) . \tag{4.8.17}$$

Remark 4.8.5. The local existence result can be derived arguing as in [70, 32]. One could also extend the proof of [2] to the case of constant vorticity.

Step 2: Complex variables. System (4.5.37) and the water waves equations (4.1.2) are equivalent under the linear change of variables

$$U = \begin{pmatrix} u \\ \overline{u} \end{pmatrix} = \mathcal{M}^{-1} \circ \mathcal{W}^{-1} \begin{pmatrix} \eta \\ \psi \end{pmatrix}, \quad u = \frac{1}{\sqrt{2}} M(D)^{-1} \eta + \frac{\mathrm{i}}{\sqrt{2}} M(D) \zeta , \quad \zeta := \psi - \frac{\gamma}{2} \partial_x^{-1} \eta , \tag{4.8.18}$$

defined in (4.5.2), (4.5.6). In view of (4.8.18) we have the equivalence of the norms: for some $\mathtt{C} := \mathtt{C}(\mathtt{h}, \gamma, \kappa) \ge 1$, any t

$$\mathtt{C}^{-1} \|U(t)\|_{\dot{H}^s} \le \|\eta(t)\|_{H_0^{s+\frac14}} + \|\psi(t)\|_{\dot{H}^{s-\frac14}} \le \mathtt{C} \|U(t)\|_{\dot{H}^s} . \tag{4.8.19}$$

Step 3: Bootstrap argument. Consider the local solution of (4.1.2) with initial datum (η_0, ψ_0) satisfying

$$\|\eta_0\|_{H_0^{s+\frac14}} + \|\psi_0\|_{\dot{H}^{s-\frac14}} \le \varepsilon \le \varepsilon_0 \tag{4.8.20}$$

where

$$\varepsilon_0 := \min\Big(\frac{\underline{r_0}(s)}{\breve{C}_{s,K}}, \frac{\overline{r}(s)}{\breve{C}_{s,K}}, r_{\rm loc}\Big), \quad \breve{C}_{s,K} := \max\Big(\frac{\mathtt{C}}{c_0(s,K)}, K \overline{C}_{s,K} \mathtt{C} C_{\rm loc}\Big) . \tag{4.8.21}$$

By (4.8.19), (4.8.20), (4.8.21), Lemma 4.8.2 (with $r \rightsquigarrow \mathtt{C} C_{\rm loc} \varepsilon_0 < \overline{r}(s)$) and (4.8.17) we deduce that

$$\|U(0)\|_{\dot{H}^s} \le \mathtt{C}\varepsilon, \quad \|\partial_t^k U(t)\|_{\dot{H}^{s-\frac32 k}} \le \overline{C}_{s,K} \mathtt{C} C_{\rm loc} \varepsilon , \quad \forall 0 < t < T_{\rm loc} , \, \forall k = 0, \dots, K . \tag{4.8.22}$$

By (4.8.22) and (4.8.21), the solution $U(t)$ of (4.5.37) satisfies, for any $0 < \varepsilon < \varepsilon_0$, the smallness condition

$$\|U(0)\|_{\dot{H}^s} \le c_0 \varepsilon_1 , \qquad \sup_{t \in [0, T_{\rm loc}]} \sum_{k=0}^{K} \|\partial_t^k U(t)\|_{\dot{H}^{s-\frac32 k}} \le \varepsilon_1 \quad \text{with} \quad \varepsilon_1 := \breve{C}_{s,K} \varepsilon ,$$

which is (4.8.12) with $T = T_{\rm loc}$. Proposition 4.8.4 and a standard bootstrap argument imply that the maximal time $T_{\max} \ge T_{\rm loc}$ of existence of the solution $U(t)$ is larger than $T_\varepsilon := c_0 \varepsilon_1^{-N-1}$ and $\|U(t)\|_{\dot{H}^s} \le \varepsilon_1$ for any $0 \le t < T_\varepsilon$. By (4.8.18) this proves that the solution $(\eta, \psi)(t) = \mathcal{W}\mathcal{M}U(t)$ of the water waves equations (4.1.2) satisfies (4.1.7) and (4.1.8) with $c := c_0 \breve{C}_{s,K}^{-N-1}$ and $C := \mathtt{C}\breve{C}_{s,K}$.

Appendix A

Technical lemmata

We collect in this Appendix important results used along Chapter 4 about how para-differential equations are conjugated under the flow of an unbounded Fourier multiplier (Lemma A.0.4) and an approximate flow generated by a smoothing vector field (Lemma A.0.5).

The following result about the approximate inverse is a consequence of Lemma 4.2.24.

Lemma A.0.1. *Assume* (4.3.35) *with* U *in* $B^K_{s_0,\mathbb{R}}(I;r)$, $K, K' \in \mathbb{N}_0$ *with* $K' \leq K$, $r > 0$. *Let* $p, N \in \mathbb{N}$, $p \leq N$, $m \geq 0$, *and consider*

$$W := \Phi(Z) = Z + S(Z;t)Z \tag{A.0.1}$$

where

$$S(Z;t) \in \begin{cases} \Sigma\mathcal{S}^m_{K,K',p}[r,N] \otimes \mathcal{M}_2(\mathbb{C}) & \text{if} \quad \mathbf{M}_0(U;t) = \mathrm{Id}\,, \\ \Sigma\mathcal{S}^m_{K,0,p}[\breve{r},N] \otimes \mathcal{M}_2(\mathbb{C})\,, \forall \breve{r} > 0 & \text{otherwise}\,. \end{cases} \tag{A.0.2}$$

Then we may write

$$Z = \Psi_{\leq N}(W) + M_{>N}(U;t)U \quad \text{with} \quad \Psi_{\leq N}(W) := W + \breve{S}_{\leq N}(W)W\,, \tag{A.0.3}$$

where

- $\breve{S}_{\leq N}(W)$ *is a matrix of pluri-homogeneous spectrally localized maps in* $\Sigma_p^N \widetilde{\mathcal{S}}_q^{m(N-p+1)} \otimes \mathcal{M}_2(\mathbb{C})$;
- $M_{>N}(U;t)$ *is a matrix of non-homogeneous operators in* $\mathcal{M}^{m(N-p+2)}_{K,K',N+1}[r] \otimes \mathcal{M}_2(\mathbb{C})$.

Proof. By Lemma 4.2.24 there exists an approximate inverse up to homogeneity N of the pluri-homogeneous nonlinear map (obtained by (A.0.1))

$$\Phi_{\leq N}(Z) := Z + \mathcal{P}_{\leq N}[S(Z;t)]Z$$

having the form

$$\Psi_{\leq N}(W) = W + \breve{S}_{\leq N}(W)W \quad \text{with} \quad \breve{S}_{\leq N}(W) = \sum_{q=p}^{N} \breve{S}_q(W)\,, \quad \breve{S}_q(W) \in \widetilde{\mathcal{S}}_q^{m(N-p+1)} \otimes \mathcal{M}_2(\mathbb{C})\,.$$

Applying $\Psi_{\le N}$ to (A.0.1), writing $\Phi(Z) = \Phi_{\le N}(Z) + S_{>N}(Z;t)Z$ with $S_{>N}(Z;t) := S(Z;t) - \mathcal{P}_{\le N}[S(Z;t)]$, we get

$$\begin{aligned}\Psi_{\le N}(W) &= \Psi_{\le N}\big(\Phi_{\le N}(Z) + S_{>N}(Z;t)Z\big)\\ &= \Psi_{\le N}\big(\Phi_{\le N}(Z)\big) + \underbrace{\int_0^1 \mathrm{d}_W\Psi_{\le N}\left(\Phi_{\le N}(Z) + \tau S_{>N}(Z;t)Z\right)\left[S_{>N}(Z;t)Z\right]\mathrm{d}\tau}_{=:S''_{>N}(Z;t)Z}\\ &\overset{(4.2.111)}{=} Z + S'_{>N}(Z)Z + S''_{>N}(Z;t)Z \end{aligned} \tag{A.0.4}$$

where $S'_{>N}(Z) \in \Sigma_{N+1}\widetilde{\mathcal{S}}_q^{m(N-p+2)} \otimes \mathcal{M}_2(\mathbb{C})$ by Lemma 4.2.24-(i) and, according to (A.0.2), by Proposition 4.2.19-(v)

$$S''_{>N}(Z;t) \in \begin{cases} \mathcal{S}^{m(N-p+2)}_{K,K',N+1}[r] \otimes \mathcal{M}_2(\mathbb{C}) & \text{if} \quad \mathbf{M}_0(U;t) = \mathrm{Id}\,,\\ \mathcal{S}^{m(N-p+2)}_{K,0,N+1}[\breve{r}] \otimes \mathcal{M}_2(\mathbb{C})\,, \forall \breve{r} > 0 & \text{otherwise}\,. \end{cases} \tag{A.0.5}$$

Finally we substitute $Z = \mathbf{M}_0(U;t)U$ where $\mathbf{M}_0(U;t)$ is in $\mathcal{M}^0_{K,K',0}[r] \otimes \mathcal{M}_2(\mathbb{C})$ (cfr. (4.3.35)) in the non–homogeneous term $S'_{>N}(Z)Z$ and $S''_{>N}(Z;t)Z$ in (A.0.4)-(A.0.5) and using (iii) and (i) Proposition 4.2.15 we deduce (A.0.3) and that $M_{>N}(U;t) \in \mathcal{M}^{m(N-p+2)}_{K,K',N+1}[r] \otimes \mathcal{M}_2(\mathbb{C})$. □

We provide a lemma concerning how para-differential and smoothing operators change by substituting in the 'internal" variables a close to the identity map.

Lemma A.0.2. *Assume $W = \mathbf{M}_0(U;t)U$ with $\mathbf{M}_0(U;t) \in \mathcal{M}^0_{K,K',0}[r] \otimes \mathcal{M}_2(\mathbb{C})$, $U \in B^K_{s_0,\mathbb{R}}(I;r)$ for some $r, s_0 > 0$ and $0 \le K' \le K$. Let $p, N \in \mathbb{N}$ with $p \le N$, $m \in \mathbb{R}$ and consider a nonlinear map*

$$Z = \Psi_{\le N}(W) + M_{>N}(U;t)U \quad \textit{where} \quad \Psi_{\le N}(W) = W + M_{\le N}(W)W \tag{A.0.6}$$

with
- *$M_{\le N}(W)$ is a matrix of pluri-homogeneous m-operators in $\Sigma_p^N \widetilde{\mathcal{M}}^m_q \otimes \mathcal{M}_2(\mathbb{C})$;*
- *$M_{>N}(U;t)$ is a matrix of non homogeneous m-operators in $\mathcal{M}^m_{K,K',N+1}[r] \otimes \mathcal{M}_2(\mathbb{C})$.*

Then

(i) *(**Symbols**) if $a_{\le N}(Z;\xi)$ is a pluri-homogeneous real-valued symbol, independent of x, in $\Sigma_2^N\widetilde{\Gamma}^{m'}_q$, $m' \in \mathbb{R}$, then*

$$\begin{aligned}&\mathrm{Op}^{\mathrm{BW}}\left(a_{\le N}(Z;\xi)\right)_{|Z=\Psi_{\le N}(W)+M_{>N}(U;t)U}\\ &= \mathrm{Op}^{\mathrm{BW}}\Big(a^+_{\le N}(W;\xi) + a^+_{>N}(U;t,\xi)\Big) + R^+_{\le N}(W) + R^+_{>N}(U;t)\,,\end{aligned} \tag{A.0.7}$$

where
- *$a^+_{\le N}(W;\xi)$ is a pluri-homogeneous real-valued symbol independent of x in $\Sigma_2^N\widetilde{\Gamma}^{m'}_q$ such that*

$$\mathcal{P}_{\le p+1}(a^+_{\le N}(W;\xi)) = \mathcal{P}_{\le p+1}(a_{\le N}(W;\xi))\,; \tag{A.0.8}$$

- *$a^+_{>N}(U;t,\xi)$ is a non-homogeneous real valued symbol independent of x in $\Gamma^{m'}_{K,K',N+1}[r]$;*
- *$R^+_{\le N}(W)$ is a pluri-homogeneous smoothing operator in $\Sigma_{p+2}^N\widetilde{\mathcal{R}}^{-\varrho}_q \otimes \mathcal{M}_2(\mathbb{C})$ for any $\varrho \ge 0$;*
- *$R^+_{>N}(U;t)$ is a non-homogeneous smoothing operator in $\mathcal{R}^{-\varrho}_{K,K',N+1}[r]$ for any $\varrho \ge 0$.*

(ii) **(Smoothing operators)** *If $R_{\le N}(Z)$ is a pluri-homogeneous smoothing operator in $\Sigma_1^N \widetilde{\mathcal{R}}_q^{-\varrho}$, for some $\varrho \ge 0$, then*

$$\begin{aligned} (R_{\le N}(Z)Z)_{|Z=\Psi_{\le N}(W)+M_{>N}(U;t)U} &= R^+_{\le N}(W)W + R^+_{>N}(U;t)U\,, \\ R_{\le N}(Z)_{|Z=\Psi_{\le N}(W)+M_{>N}(U;t)U} &= R'_{\le N}(W) + R'_{>N}(U;t) \end{aligned} \tag{A.0.9}$$

where

- *$R^+_{\le N}(W)$ and $R'_{\le N}(W)$ are pluri-homogeneous smoothing operators in $\Sigma_1^N \widetilde{\mathcal{R}}_q^{-\varrho+(N+1)\max(m,0)}$ such that*

$$\mathcal{P}_{\le p}(R^+_{\le N}(W)) = \mathcal{P}_{\le p}(R'_{\le N}(W)) = \mathcal{P}_{\le p}(R_{\le N}(W))\,; \tag{A.0.10}$$

- *$R^+_{>N}(U;t)$ and $R'_{>N}(U;t)$ are non-homogeneous smoothing operators in $\mathcal{R}^{-\varrho+(N+1)\max(m,0)}_{K,K',N+1}[r]$.*

If $\varrho \le (N+1)\max(m,0)$ we regard $R^+_{\le N}$ and $R^+_{>N}$ as operators in $\Sigma_1^N \widetilde{\mathcal{M}}_q^{-\varrho+(N+1)\max(m,0)}$ and respectively $\mathcal{M}^{-\varrho+(N+1)\max(m,0)}_{K,K',N+1}[r]$.

Remark A.0.3. The previous lemma is stated for x independent symbols (since it is used in this case) but it holds also for a general symbol.

Proof. **Proof of (i):** We expand by multilinearity the operator in (A.0.7). We denote the homogeneous components of $M_{\le N}(W)$ in (A.0.6) as $M_\ell(W) := \mathcal{P}_\ell(M_{\le N}(W))$ for $\ell = p,\dots,N$ and $M_0(W) := \mathrm{Id}$. Note that

$$(M_\ell(W)W)_{|W=\mathbf{M}_0(U;t)U} = M_\ell^{(\mathbf{M}_0)}(U;t)U$$

where $M_\ell^{(\mathbf{M}_0)}(U;t) := M_\ell(\mathbf{M}_0(U;t)U)\mathbf{M}_0(U;t)$ belongs to $\mathcal{M}^m_{K,K',\ell}[r] \otimes \mathcal{M}_2(\mathbb{C})$ for $\ell \in \{0,p,\dots,N\}$ thanks to (i) and (ii) of Proposition 4.2.15. Then by multilinearity decompose the operator in (A.0.7) as

$$\begin{aligned} &\mathrm{Op}^{\mathrm{BW}}\left(a_{\le N}(Z;\xi)\right)_{|Z=\Psi_{\le N}(W)+M_{>N}(U;t)U} \\ &= \sum_{\mathtt{a}=2}^N \sum_{q=2}^N \sum_{\substack{\ell_1,\dots,\ell_q \in \{0,p,\dots,N\} \\ \ell_1+\dots+\ell_q+q=\mathtt{a}}} \mathrm{Op}^{\mathrm{BW}}\left(a_q(Z_1,\dots,Z_q;\xi)\right)_{|Z_1=M_{\ell_1}(W)W,\dots,Z_q=M_{\ell_q}(W)W} && \text{(A.0.11)} \\ &+ \sum_{q=2}^N \sum_{\substack{\ell_1,\dots,\ell_q \in \{0,p,\dots N\} \\ \ell_1+\dots+\ell_q+q \ge N+1}} \mathrm{Op}^{\mathrm{BW}}\left(a_q(Z_1,\dots,Z_q;\xi)\right)_{|Z_1=M_{\ell_1}^{(\mathbf{M}_0)}(U;t)U,\dots,Z_q=M_{\ell_q}^{(\mathbf{M}_0)}(U;t)U} && \text{(A.0.12)} \\ &+ \sum_{q=2}^N \sum_{n=0}^{q-1} \sum_{\ell_1,\dots,\ell_n \in \{0,p,\dots N\}} \binom{q}{n} \mathrm{Op}^{\mathrm{BW}}\left(a_q(Z_1,\dots,Z_q;\xi)\right)_{\Big| \substack{Z_1=M_{\ell_1}^{(\mathbf{M}_0)}(U;t)U,\dots,Z_n=M_{\ell_n}^{(\mathbf{M}_0)}(U;t)U \\ Z_{n+1}=\dots=Z_q=M_{>N}(U;t)U}} . && \text{(A.0.13)} \end{aligned}$$

By (iv) of Proposition 4.2.15 we have that

$$\text{(A.0.11)} = \mathrm{Op}^{\mathrm{BW}}\left(a^+_{\le N}(W)\right) + R^+_{\le N}(W) \quad \text{with} \quad R^+_{\le N}(W) \in \Sigma_2^N \widetilde{\mathcal{R}}_q^{-\varrho},\ \forall \varrho \ge 0\,, \tag{A.0.14}$$

where $a^+_{\le N}(W) = \sum_{\mathtt{a}=2}^N a^+_{\mathtt{a}}(W;\xi)$ and

$$a^+_{\mathtt{a}}(W;\xi) := \sum_{q=2}^N \sum_{\substack{\ell_1,\dots,\ell_q \in \{0,p,\dots,N\} \\ \ell_1+\dots+\ell_q+q=\mathtt{a}}} a_q(M_{\ell_1}(W)W,\dots,M_{\ell_q}(W)W;\xi) \tag{A.0.15}$$

belongs to $\widetilde{\Gamma}^{m'}_{\mathtt{a}}$. For $\mathtt{a} \le p+1$ the sum in (A.0.11) and (A.0.15) reduces to the indices $q = \mathtt{a}$, $\ell_1 = \cdots = \ell_{\mathtt{a}} = 0$. As a consequence $a^+_{\mathtt{a}}(W;\xi) = a_{\mathtt{a}}(W;\xi)$ for $\mathtt{a} = 2,\ldots,p+1$, proving (A.0.8). For the same reason the remainder $R^+_{\le N}(W)$ in (A.0.14) actually belongs to $\Sigma_{p+2}^N \widetilde{\mathcal{R}}_q^{-\varrho}$.

Now we consider the non-homogeneous terms which arise from lines (A.0.12) and (A.0.13). Thanks to (iv) of Proposition 4.2.15 we have that

$$(\text{A.0.12}) + (\text{A.0.13}) = \mathrm{Op}^{\mathrm{BW}}\left(a^+_{>N}(U;t,\xi)\right) + R^+_{>N}(U;t)$$

where $R^+_{>N}(U;t)$ is a smoothing operator in $\mathcal{R}^{-\varrho}_{K,K',N+1}[r]$ for any $\varrho \ge 0$, and $a^+_{>N}(U;t,\xi)$ is a non–homogeneous symbol in $\Gamma^{m'}_{K,K',N+1}[r]$ which is real valued and x-independent as well as $a_{\le N}$.

Proof of (ii): Proceeding in similarly to (i) we expand the left hand side of (A.0.9) as

$$\begin{aligned}
&(R_{\le N}(Z)Z)_{|Z=\Psi_{\le N}(W)+M_{>N}(U;t)U} \\
&= \sum_{\mathtt{a}=1}^N \sum_{q=1}^N \sum_{\substack{\ell_1,\ldots,\ell_{q+1}\in\{0,p,\ldots,N\} \\ \ell_1+\cdots+\ell_{q+1}+q=\mathtt{a}}} R_q(M_{\ell_1}(W)W,\ldots,M_{\ell_q}(W)W)M_{\ell_{q+1}}(W)W && \text{(A.0.16)} \\
&+ \sum_{q=1}^N \sum_{\substack{\ell_1,\ldots,\ell_q\in\{0,p,\ldots N\} \\ \ell_1+\cdots+\ell_q+q\ge N+1}} R_q(M^{(\mathbf{M}_0)}_{\ell_1}(U;t)U,\ldots,M^{(\mathbf{M}_0)}_{\ell_q}(U;t)U)M^{(\mathbf{M}_0)}_{\ell_{q+1}}(U;t)U && \text{(A.0.17)} \\
&+ \sum_{q=1}^N \sum_{n=0}^{q-1} \sum_{\ell_1,\ldots,\ell_{n+1}\in\{0,p,\ldots N\}} \binom{q}{n} R_q(Z_1,\ldots,Z_q)M^{(\mathbf{M}_0)}_{\ell_{n+1}}(U;t)U_{\Big| \substack{Z_1=M^{(\mathbf{M}_0)}_{\ell_1}(U;t)U,\ldots,Z_n=M^{(\mathbf{M}_0)}_{\ell_n}(U;t)U \\ Z_{n+1}=\cdots=Z_q=M_{>N}(U;t)U}} && \text{(A.0.18)} \\
&+ \sum_{q=1}^N \sum_{n=0}^{q-1} \sum_{\ell_1,\ldots,\ell_n\in\{0,p,\ldots N\}} \binom{q}{n} R_q(Z_1,\ldots,Z_q)M_{>N}(U;t)U_{\Big| \substack{Z_1=M^{(\mathbf{M}_0)}_{\ell_1}(U;t)U,\ldots,Z_n=M^{(\mathbf{M}_0)}_{\ell_n}(U;t)U \\ Z_{n+1}=\cdots=Z_q=M_{>N}(U;t)U}}. && \text{(A.0.19)}
\end{aligned}$$

Thanks to (ii) of Proposition 4.2.15 (with $m \rightsquigarrow -\varrho$, $m_\ell \rightsquigarrow m$), the term in (A.0.16) can be written as $R^+_{\le N}(W)W$ where $R^+_{\le N}(W)$ is a pluri–homogeneous smoothing operator in $\Sigma_1^N \widetilde{\mathcal{R}}_q^{-\varrho+(N+1)\max(m,0)}$, moreover $R^+_{\le N}(W) = \sum_{\mathtt{a}=1}^N R^+_{\mathtt{a}}(W)$ with

$$R^+_{\mathtt{a}}(W) := \sum_{q=1}^N \sum_{\substack{\ell_1,\ldots,\ell_{q+1}\in\{0,p,\ldots,N\} \\ \ell_1+\cdots+\ell_{q+1}+q=\mathtt{a}}} R_q(M_{\ell_1}(W)W,\ldots,M_{\ell_q}(W)W)M_{\ell_{q+1}}(W). \tag{A.0.20}$$

For $\mathtt{a} \le p$ the sum in (A.0.20) reduces to the indices $q = \mathtt{a}$, $\ell_1 = \cdots = \ell_{\mathtt{a}+1} = 0$. As a consequence $R^+_{\mathtt{a}}(W) = R_{\mathtt{a}}(W)$ for $\mathtt{a} = 1,\ldots,p$, proving (A.0.10). Applying again Proposition 4.2.15 (ii) we get that the terms in (A.0.17)–(A.0.19) can be written as $R^+_{>N}(U;t)U$ where $R^+_{>N}(U;t)$ is a non–homogenous smoothing operator in $\mathcal{R}^{-\varrho+(N+1)\max(m,0)}_{K,K',N+1}[r]$. This concludes the proof of the first identity in (A.0.9). The second one follows with the same analysis, without the need of substitute the last variable. □

Conjugation lemmata. The following conjugation Lemmata A.0.4 and A.0.5 are used in the nonlinear Hamiltonian Birkhoff normal form reduction performed in Section 4.7.

The following hypothesis shall be assumed in both Lemmata A.0.4 and A.0.5:

Assumption (A): *Assume* $Z := \mathbf{M}_0(U;t)U$ *where* $\mathbf{M}_0(U;t) \in \mathcal{M}^0_{K,K',0}[r] \otimes \mathcal{M}_2(\mathbb{C})$, $U \in B^K_{s_0,\mathbb{R}}(I;r)$ *for some* $r, s_0 > 0$ *and* $0 \le K' \le K$. *Let* $N \in \mathbb{N}$ *and assume that* Z *solves the system*

$$\partial_t Z = -\mathrm{i}\boldsymbol{\Omega}(D)Z + \mathrm{Op}^{BW}_{\mathtt{vec}}\left(\mathrm{i}a_{\le N}(Z;\xi) + \mathrm{i}a_{>N}(U;t,\xi)\right)Z + R_{\le N}(Z)Z + R_{>N}(U;t)U \tag{A.0.21}$$

where $\boldsymbol{\Omega}(D)$ *is the diagonal matrix of Fourier multiplier operators defined in* (4.5.10) *and*

- $a_{\le N}(Z;\xi)$ *is a real valued pluri-homogenous symbol, independent of* x, *in* $\Sigma_2^N \widetilde{\Gamma}_q^{\frac{3}{2}}$;
- $a_{>N}(U;t,\xi)$ *is a non-homogenous symbol, independent of* x, *in* $\Gamma^{\frac{3}{2}}_{K,K',N+1}[r]$ *with imaginary part* $\mathrm{Im}\, a_{>N}(U;t,\xi)$ *in* $\Gamma^0_{K,K',N+1}[r]$;
- $R_{\le N}(Z)$ *is a real-to-real matrix of pluri-homogeneous smoothing operators in* $\Sigma_1^N \widetilde{\mathcal{R}}_q^{-\varrho} \otimes \mathcal{M}_2(\mathbb{C})$;
- $R_{>N}(U;t)$ *is a real-to-real matrix of non-homogeneous smoothing operators in* $\mathcal{R}^{-\varrho}_{K,K',N+1}[r] \otimes \mathcal{M}_2(\mathbb{C})$.

We also write system (A.0.21) in the form

$$\partial_t Z = -\mathrm{i}\boldsymbol{\Omega}(D)Z + M_{\le N}(Z)Z + M_{>N}(U;t)U \tag{A.0.22}$$

where $M_{\le N}(Z)$ are $\frac{3}{2}$-operators in $\Sigma_1^N \widetilde{\mathcal{M}}_q^{\frac{3}{2}} \otimes \mathcal{M}_2(\mathbb{C})$ and $M_{>N}(U;t)$ is in $\mathcal{M}^{\frac{3}{2}}_{K,K',N+1}[r] \otimes \mathcal{M}_2(\mathbb{C})$ by the fourth remark below Definition 4.2.16.

Lemma A.0.4 (Conjugation under the flow of a Fourier multiplier). *Assume* **(A)** *at page 215. Let* $g_p(Z;\xi)$ *be a p–homogeneous real symbol independent of* x *in* $\widetilde{\Gamma}_p^{\frac{3}{2}}$, $p \ge 2$, *that we expand as*

$$g_p(Z;\xi) = \sum_{(\vec{\jmath}_p,\vec{\sigma}_p)\in\mathfrak{T}_p} G^{\vec{\sigma}_p}_{\vec{\jmath}_p}(\xi) z^{\vec{\sigma}_p}_{\vec{\jmath}_p}\,, \quad \overline{G^{-\vec{\sigma}_p}_{\vec{\jmath}_p}}(\xi) = G^{\vec{\sigma}_p}_{\vec{\jmath}_p}(\xi) \in \mathbb{C} \tag{A.0.23}$$

and denote by $\mathcal{G}_{g_p}(Z) := \mathcal{G}^1_{g_p}(Z)$ *the time* 1*-flow defined in* (4.3.56) *generated by* $\mathrm{Op}^{BW}_{\mathtt{vec}}(\mathrm{i}g_p(Z;\xi))$. *If* $Z(t)$ *solves system* (A.0.21), *then the variable*

$$W := \mathcal{G}_{g_p}(Z)Z \tag{A.0.24}$$

solves the system

$$\partial_t W = -\mathrm{i}\boldsymbol{\Omega}(D)W + \mathrm{Op}^{BW}_{\mathtt{vec}}\Big(\mathrm{i}a^+_{\le N}(W;\xi) + \mathrm{i}a^+_{>N}(U;t,\xi)\Big)W + R^+_{\le N}(W)W + R^+_{>N}(U;t)U \tag{A.0.25}$$

where

- $a^+_{\le N}(W;\xi)$ *is a real valued pluri-homogenous symbol, independent of* x, *in* $\Sigma_2^N \widetilde{\Gamma}_q^{\frac{3}{2}}$, *with components*

$$\begin{aligned} \mathcal{P}_{\le p-1}[a^+_{\le N}(W;\xi)] &= \mathcal{P}_{\le p-1}[a_{\le N}(W;\xi)]\,, \\ \mathcal{P}_p\left[a^+_{\le N}(W;\xi)\right] &= \mathcal{P}_p\left[a_{\le N}(W;\xi)\right] + g^+_p(W;\xi) \end{aligned} \tag{A.0.26}$$

where $g^+_p(W;\xi) \in \widetilde{\Gamma}_p^{\frac{3}{2}}$ *is the real,* x*-independent symbol*

$$g^+_p(W;\xi) := \sum_{(\vec{\jmath}_p,\vec{\sigma}_p)\in\mathfrak{T}_p} -\mathrm{i}\big(\vec{\sigma}_p \cdot \Omega_{\vec{\jmath}_p}(\kappa)\big) G^{\vec{\sigma}_p}_{\vec{\jmath}_p}(\xi) w^{\vec{\sigma}_p}_{\vec{\jmath}_p}; \tag{A.0.27}$$

- $a^+_{>N}(U;t,\xi)$ *is a non-homogeneous symbol, independent of* x, *in* $\Gamma^{\frac{3}{2}}_{K,K',N+1}[r]$ *with imaginary part* $\operatorname{Im} a^+_{>N}(U;t,\xi)$ *belonging to* $\Gamma^0_{K,K',N+1}[r]$*;*

- $R^+_{\leq N}(W)$ *is a real-to-real matrix of pluri–homogeneous smoothing operators in* $\Sigma_1^N \widetilde{\mathcal{R}}_q^{-\varrho+c(N,p)} \otimes \mathcal{M}_2(\mathbb{C})$ *for some* $c(N,p)>0$ *(depending only on* N,p*) and fulfilling*

$$\mathcal{P}_{\leq p}[R^+_{\leq N}(W)] = \mathcal{P}_{\leq p}[R_{\leq N}(W)]; \tag{A.0.28}$$

- $R^+_{>N}(U;t)$ *is a real-to-real matrix of non–homogeneous smoothing operators in* $\mathcal{R}^{-\varrho+c(N,p)}_{K,K',N+1}[r] \otimes \mathcal{M}_2(\mathbb{C})$.

Proof. Since $Z(t)$ solves (A.0.21) then differentiating (A.0.24) we get

$$\begin{aligned}
\partial_t W = \mathcal{G}_{g_p}(Z)\big[-\mathrm{i}\boldsymbol{\Omega}(D) + \mathrm{Op}^{BW}_{\mathtt{vec}}\left(\mathrm{i}a_{\leq N}(Z;\xi) + \mathrm{i}a_{>N}(U;t,\xi)\right)\big]\mathcal{G}_{g_p}(Z)^{-1}W \qquad &(A.0.29)\\
+ \mathcal{G}_{g_p}(Z)\, R_{\leq N}(Z)Z + \mathcal{G}_{g_p}(Z) R_{>N}(U;t)U \qquad &(A.0.30)\\
+ (\partial_t \mathcal{G}_{g_p}(Z))\mathcal{G}_{g_p}(Z)^{-1}W. \qquad &(A.0.31)
\end{aligned}$$

We now compute (A.0.29)–(A.0.31) separately. As $\mathcal{G}_{g_p}(Z)$ is the time 1-flow of a Fourier multiplier it commutes with every Fourier multiplier, and (A.0.29) is equal to

$$\text{(A.0.29)} = -\mathrm{i}\boldsymbol{\Omega}(D)W + \mathrm{Op}^{BW}_{\mathtt{vec}}\left(\mathrm{i}a_{\leq N}(Z;\xi) + \mathrm{i}a_{>N}(U;t,\xi)\right)W\,. \tag{A.0.32}$$

Now we write the symbol $a_{\leq N}(Z;\xi)$ in terms of W. By Lemma 4.3.17 (iii) we have that $\mathcal{G}_{g_p}(Z) - \mathrm{Id}$ is a matrix of spectrally localized maps in $\Sigma\mathcal{S}^{\frac{3}{2}(N+1)}_{K,0,p}[\breve{r},N] \otimes \mathcal{M}_2(\mathbb{C})$ for any $\breve{r}>0$. By Assumption **(A)** we have $Z = \mathbf{M}_0(U;t)U$ with $\mathbf{M}_0(U;t) \in \mathcal{M}^0_{K,K',0}[r] \otimes \mathcal{M}_2(\mathbb{C})$. Lemma A.0.1 (with $S(Z;t) = \mathcal{G}_{g_p}(Z) - \mathrm{Id}$) provides an approximate inverse of $W = \mathcal{G}_{g_p}(Z)Z$ of the form

$$Z = \Psi_{\leq N}(W) + M_{>N}(U;t)U, \quad \Psi_{\leq N}(W) := W + \breve{S}_{\leq N}(W)W \tag{A.0.33}$$

where $\breve{S}_{\leq N}(W)$ is a matrix of spectrally localized maps in $\Sigma_p^N \widetilde{\mathcal{S}}_q^{\frac{3}{2}(N+1)(N-p+1)} \otimes \mathcal{M}_2(\mathbb{C})$ and $M_{>N}(U;t)$ is in $\mathcal{M}^{\frac{3}{2}(N+1)(N-p+2)}_{K,K',N+1}[r] \otimes \mathcal{M}_2(\mathbb{C})$. The map (A.0.33) has the form (A.0.6), so by Lemma A.0.2 (i) (with $m' = 3/2$) we obtain

$$\begin{aligned}
&\mathrm{Op}^{BW}_{\mathtt{vec}}\left(\mathrm{i}a_{\leq N}(Z;\xi)\right)_{|Z=\Psi_{\leq N}(W)+M_{>N}(U;t)U}\\
&= \mathrm{Op}^{BW}_{\mathtt{vec}}\left(\mathrm{i}a'_{\leq N}(W;\xi) + \mathrm{i}a'_{>N}(U;t,\xi)\right) + R'_{\leq N}(W) + R'_{>N}(U;t)\,,
\end{aligned} \tag{A.0.34}$$

where

- $a'_{\leq N}(W;\xi)$ is a real valued pluri-homogenous symbol independent of x in $\Sigma_2^N \widetilde{\Gamma}_q^{\frac{3}{2}}$ with

$$\mathcal{P}_{\leq p+1}(a'_{\leq N}(W;\xi)) = \mathcal{P}_{\leq p+1}(a_{\leq N}(W;\xi)); \tag{A.0.35}$$

- $a'_{>N}(U;t,\xi)$ is a non-homogenous real valued symbol independent of x in $\Gamma^{\frac{3}{2}}_{K,K',N+1}[r]$;
- $R'_{\leq N}(W)$ is a real-to-real matrix of pluri-homogeneous smoothing operators in $\Sigma_{p+2}^N \widetilde{\mathcal{R}}_q^{-\varrho} \otimes \mathcal{M}_2(\mathbb{C})$ for any $\varrho \geq 0$;
- $R'_{>N}(U;t)$ is a real-to-real matrix of non-homogeneous smoothing operators in $\mathcal{R}^{-\varrho}_{K,K',N+1}[r] \otimes \mathcal{M}_2(\mathbb{C})$ for any $\varrho \geq 0$.

We now consider the terms in (A.0.30). Since $\mathcal{G}_{g_p}(Z) - \mathrm{Id}$ belongs to $\Sigma\mathcal{S}^{\frac{3}{2}(N+1)}_{K,0,p}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$ by Lemma 4.3.17, Proposition 4.2.19-(i) implies that

$$\mathcal{G}_{g_p}(Z) R_{\leq N}(Z) Z = R'_{\leq N}(Z)Z + R'_{>N}(Z;t)Z \tag{A.0.36}$$

where $R'_{\leq N}(Z)$ is in $\Sigma_1^N \widetilde{\mathcal{R}}_q^{-\varrho+\frac{3}{2}(N+1)} \otimes \mathcal{M}_2(\mathbb{C})$ with

$$\mathcal{P}_{\leq p}[R'_{\leq N}(Z)] = \mathcal{P}_{\leq p}[R_{\leq N}(Z)] \tag{A.0.37}$$

and $R'_{>N}(Z;t)$ is in $\mathcal{R}^{-\varrho+\frac{3}{2}(N+1)}_{K,0,N+1}[\breve{r}] \otimes \mathcal{M}_2(\mathbb{C})$ for any $\breve{r} > 0$. We now substitute $Z = \Psi_{\leq N}(W) + M_{>N}(U;t)U$ (cfr. (A.0.33)) in the homogeneous components of (A.0.36) and $Z = \mathbf{M}_0(U;t)U$ in the non-homogeneous ones of (A.0.36) and (A.0.30). So using Lemma A.0.2 (with $m \rightsquigarrow \frac{3}{2}(N+1)(N-p+2)$, $\varrho \rightsquigarrow \varrho - \frac{3}{2}(N+1)$), (i) and (iii) of Proposition 4.2.15, (i) and (iv) of Proposition 4.2.19 we get

$$\begin{aligned}(\text{A.0.30}) = {} & (R'_{\leq N}(Z)Z)_{|Z=\Psi_{\leq N}(W)+M_{>N}(U;t)U} + R'_{>N}(\mathbf{M}_0(U;t)U;t)\mathbf{M}_0(U;t)U \\ & + \mathcal{G}_{g_p}(\mathbf{M}_0(U;t)U)R_{>N}(U;t)U = R''_{\leq N}(W)W + R''_{>N}(U;t)U\,,\end{aligned} \tag{A.0.38}$$

where

• $R''_{\leq N}(W)$ is a pluri-homogeneous smoothing operator in $\Sigma_1^N \widetilde{\mathcal{R}}_q^{-\varrho+c(N,p)} \otimes \mathcal{M}_2(\mathbb{C})$ with $c(N,p) = \frac{3}{2}(N+1) + \frac{3}{2}(N+1)^2(N-p+2)$ and

$$\mathcal{P}_{\leq p}[R''_{\leq N}(W)] \overset{(\text{A.0.10})}{=} \mathcal{P}_{\leq p}[R'_{\leq N}(W)] \overset{(\text{A.0.37})}{=} \mathcal{P}_{\leq p}[R_{\leq N}(W)]\,; \tag{A.0.39}$$

• $R''_{>N}(U;t)$ is a non–homogeneous smoothing operator in $\mathcal{R}^{-\varrho+c(N,p)}_{K,K',N+1}[r] \otimes \mathcal{M}_2(\mathbb{C})$.

We finally consider the last term (A.0.31). Using that $\mathcal{G}_{g_p}(Z)$ commutes with every Fourier multiplier we get

$$(\text{A.0.31}) = \mathrm{Op}^{BW}_{\mathtt{vec}}\,(\mathrm{i}\partial_t g_p(Z;\xi))W\,. \tag{A.0.40}$$

So, using (A.0.22), (A.0.23) and the identity $(-\mathrm{i}\boldsymbol{\Omega}(D)Z)_j^\sigma = -\mathrm{i}\sigma\Omega_j(\kappa)z_j^\sigma$, we obtain

$$\begin{aligned}\partial_t g_p(Z;\xi) = {} & \sum_{a=1}^{p} g_p(\overbrace{Z,\dots,Z}^{a-\text{times}}, -\mathrm{i}\boldsymbol{\Omega}(D)Z, Z,\dots,Z;\xi) + p\, g_p(M_{\leq N}(Z)Z, Z,\dots,Z;\xi) \\ & + p\, g_p(M_{>N}(U;t)U, Z,\dots,Z;\xi)_{|Z=\mathbf{M}_0(U;t)U} \\ = {} & g_p^+(Z;\xi) + p\, g_p(\mathcal{P}_{\leq N-p}(M_{\leq N})(Z)Z, Z,\dots,Z;\xi) \\ & + p\, g_p(\mathcal{P}_{>N-p}(M_{\leq N})(Z)Z, Z,\dots,Z;\xi)_{|Z=\mathbf{M}_0(U;t)U} + p\, g_p(M_{>N}(U;t)U, Z,\dots,Z;\xi)_{|Z=\mathbf{M}_0(U;t)U} \\ = {} & g_p^+(Z;\xi) + g'_{\geq p+1}(Z;\xi) + g'_{>N}(U;t,\xi)\end{aligned} \tag{A.0.41}$$

where g_p^+ is the real valued symbol in $\widetilde{\Gamma}_p^{\frac{3}{2}}$ in (A.0.27), the real valued pluri-homogeneous symbol $g'_{\geq p+1}$ is in $\Sigma_{p+1}^N \widetilde{\Gamma}_q^{\frac{3}{2}}$ thanks to (iv) of Proposition 4.2.15 and the real valued non-homogeneous symbol $g'_{>N}$ is in $\Gamma^{\frac{3}{2}}_{K,K',N+1}[r]$ using also (ii) of Proposition 4.2.15. Then by (A.0.40), (A.0.41) and using the second part of (iv) of Proposition 4.2.15 we obtain

$$(\text{A.0.31}) = \mathrm{Op}^{BW}_{\mathtt{vec}}\left(\mathrm{i}g_p^+(Z;\xi) + \mathrm{i}g'_{\geq p+1}(Z;\xi) + \mathrm{i}g'_{>N}(U;t,\xi)\right)W + R'_{\geq p+1}(Z)W + R'_{>N}(U;t)U \tag{A.0.42}$$

where $R'_{\geq p+1}$ is a matrix of pluri-homogeneous smoothing operators in $\Sigma_{p+1}^N \widetilde{\mathcal{R}}_q^{-\varrho} \otimes \mathcal{M}_2(\mathbb{C})$ and $R'_{>N}(U;t)$ belongs to $\mathcal{R}^{-\varrho}_{K,K',N+1}[r] \otimes \mathcal{M}_2(\mathbb{C})$. Then we substitute the variable Z in (A.0.31) using (A.0.33) and Lemma A.0.2, to obtain

$$(\text{A.0.31}) = \mathrm{Op}^{BW}_{\mathtt{vec}}\Big(\mathrm{i}g_p^+(W;\xi) + \mathrm{i}g^+_{\geq p+1}(W;\xi) + \mathrm{i}g^+_{>N}(U;t,\xi)\Big)W + R'''_{\leq N}(W)W + R'''_{>N}(U;t)U \tag{A.0.43}$$

where

- $g_p^+(W;\xi)$ is the homogeneous symbol in (A.0.27);
- $g^+_{\geq(p+1)}(W;\xi)$ is a pluri-homogeneous real valued symbol in $\Sigma^N_{p+1}\widetilde{\Gamma}^{\frac{3}{2}}_q$;
- $g^+_{>N}(U;t,\xi)$ is a non-homogeneous real valued symbol in $\Gamma^{\frac{3}{2}}_{K,K',N+1}[r]$;
- $R'''_{\leq N}(W)$ is a matrix of pluri-homogeneous operators in $\Sigma^N_{p+1}\widetilde{\mathcal{R}}_q^{-\varrho+c(N,p)} \otimes \mathcal{M}_2(\mathbb{C})$;
- $R'''_{>N}(U;t)$ is a matrix of smoothing operators in $\mathcal{R}^{-\varrho+c(N,p)}_{K,K',N+1}[r] \otimes \mathcal{M}_2(\mathbb{C})$.

In conclusion, combining (A.0.32), (A.0.34), (A.0.39) and (A.0.43) we deduce (A.0.25) with $a^+_{\leq N} = a'_{\leq N} + g^+_p + g^+_{\geq p+1}$ and $a^+_{>N} := a_{>N} + a'_{>N} + g^+_{>N}$ which has imaginary part equal to the one of $a_{>N}$ belonging to $\Gamma^0_{K,K',N+1}[r]$. Moreover (A.0.26) follows by (A.0.35) and (A.0.28) by (A.0.38). □

The following lemma describes how a system is conjugated under a smoothing perturbation of the identity.

Lemma A.0.5 (Conjugation under a smoothing perturbation of the identity). *Assume* **(A)** *at page 215. Let $F_{\leq N}(Z)$ be a real-to-real matrix of pluri-homogeneous smoothing operators in $\Sigma^N_p \widetilde{\mathcal{R}}_q^{-\varrho'} \otimes \mathcal{M}_2(\mathbb{C})$ for some $\varrho' \geq 0$. If $Z(t)$ solves (A.0.21) then the variable*

$$W := \mathcal{F}_{\leq N}(Z) := Z + F_{\leq N}(Z)Z \tag{A.0.44}$$

solves

$$\partial_t W = -\mathrm{i}\boldsymbol{\Omega}(D)W + \mathrm{Op}^{BW}_{\mathtt{vec}}\Big(\mathrm{i}a^+_{\leq N}(W;\xi) + \mathrm{i}a^+_{>N}(U;t,\xi)\Big)W + R^+_{\leq N}(W)W + R^+_{>N}(U;t)U \tag{A.0.45}$$

where

- *$a^+_{\leq N}(W;\xi)$ is a real valued pluri-homogenous symbol, independent of x, in $\Sigma^N_2\widetilde{\Gamma}^{\frac{3}{2}}_q$, with components*

$$\mathcal{P}_{\leq p+1}[a^+_{\leq N}(W;\xi)] = \mathcal{P}_{\leq p+1}[a_{\leq N}(W;\xi)]; \tag{A.0.46}$$

- *$a^+_{>N}(U;t,\xi)$ is a non-homogeneous symbol, independent of x, in $\Gamma^{\frac{3}{2}}_{K,K',N+1}[r]$ with imaginary part $\mathrm{Im}\, a^+_{>N}(U;t,\xi)$ belonging to $\Gamma^0_{K,K',N+1}[r]$;*
- *$R^+_{\leq N}(W)$ is a real-to-real matrix of pluri–homogeneous smoothing operators in $\Sigma^N_1\widetilde{\mathcal{R}}_q^{-\varrho_*} \otimes \mathcal{M}_2(\mathbb{C})$, $\varrho_* := \min(\varrho, \varrho' - \frac{3}{2})$ ($\varrho \geq 0$ is the smoothing order in Assumption* **(A)** *at page 215), with components*

$$\mathcal{P}_{\leq p-1}[R^+_{\leq N}(W)] = \mathcal{P}_{\leq p-1}[R_{\leq N}(W)]\,, \tag{A.0.47}$$

and, denoting $F_p(W) := \mathcal{P}_p(F_{\leq N}(W))$ in $\widetilde{\mathcal{R}}_p^{-\varrho'} \otimes \mathcal{M}_2(\mathbb{C})$, one has

$$\mathcal{P}_p[R^+_{\leq N}(W)] = \mathcal{P}_p[R_{\leq N}(W)] + \mathrm{d}_W\big(F_p(W)W\big)[-\mathrm{i}\boldsymbol{\Omega}(D)] + \mathrm{i}\boldsymbol{\Omega}(D)F_p(W); \tag{A.0.48}$$

- $R^+_{>N}(U;t)$ *is a real-to-real matrix of non–homogeneous smoothing operators in* $\mathcal{R}^{-\varrho_*}_{K,K',N+1}[r] \otimes \mathcal{M}_2(\mathbb{C})$.

In addition, if $\mathcal{F}_{\leq N}(Z)$ *in* (A.0.44) *is the approximate time* 1*-flow (given by Lemma* 4.2.28*) of a vector field* $G_p(Z)Z$*, where* $G_p(Z) \in \widetilde{\mathcal{R}}^{-\varrho'}_p \otimes \mathcal{M}_2(\mathbb{C})$ *has Fourier expansion*

$$\big(G_p(Z)Z\big)^\sigma_k = \sum_{(\vec{\jmath}_{p+1},k,\vec{\sigma}_{p+1},-\sigma)\in\mathfrak{T}_{p+2}} G^{\vec{\sigma}_{p+1},\sigma}_{\vec{\jmath}_{p+1},k}\, z^{\vec{\sigma}_{p+1}}_{\vec{\jmath}_{p+1}}\,, \tag{A.0.49}$$

then (A.0.48) *reduces to*

$$\mathcal{P}_p[R^+_{\leq N}(W)] = \mathcal{P}_p[R_{\leq N}(W)] + G^+_p(W) \tag{A.0.50}$$

where $G^+_p(W) \in \widetilde{\mathcal{R}}^{-\varrho'+\frac{3}{2}}_p \otimes \mathcal{M}_2(\mathbb{C})$ *is the smoothing operator with Fourier expansion*

$$(G^+_p(W)W)^\sigma_k = \sum_{(\vec{\jmath}_{p+1},k,\vec{\sigma}_{p+1},-\sigma)\in\mathfrak{T}_{p+2}} -\mathrm{i}\big(\vec{\sigma}_{p+1}\cdot\Omega_{\vec{\jmath}_{p+1}}(\kappa) - \sigma\Omega_k(\kappa)\big) G^{\vec{\sigma}_{p+1},\sigma}_{\vec{\jmath}_{p+1},k}\, w^{\vec{\sigma}_{p+1}}_{\vec{\jmath}_{p+1}}\,. \tag{A.0.51}$$

Proof. Since $Z(t)$ solves (A.0.21) then differentiating (A.0.44) we get

$$\begin{aligned}
\partial_t W = -\mathrm{i}\boldsymbol{\Omega}(D)Z + \mathrm{Op}^{BW}_{\mathtt{vec}}\,(\mathrm{i}a_{\leq N}(Z;\xi) + \mathrm{i}a_{>N}(U;t,\xi))Z \qquad & (\text{A.0.52})\\
+ R_{\leq N}(Z)Z + \mathrm{d}_Z(F_{\leq N}(Z)Z)[-\mathrm{i}\boldsymbol{\Omega}(D)Z] \qquad & (\text{A.0.53})\\
+ \mathrm{d}_Z(F_{\leq N}(Z)Z)[\mathrm{Op}^{BW}_{\mathtt{vec}}\,(\mathrm{i}a_{\leq N}(Z;\xi))Z] + \mathrm{d}_Z(F_{\leq N}(Z)Z)[R_{\leq N}(Z)Z] \qquad & (\text{A.0.54})\\
+ R_{>N}(U;t)U + \mathrm{d}_Z(F_{\leq N}(Z)Z)\mathrm{Op}^{BW}_{\mathtt{vec}}\,(\mathrm{i}a_{>N}(U;t,\xi))Z + \mathrm{d}_Z(F_{\leq N}(Z)Z)[R_{>N}(U;t)U]\,. \qquad & (\text{A.0.55})
\end{aligned}$$

Note that, by the first remark below Definition 4.2.5, $\mathrm{d}_Z(F_{\leq N}(Z)Z)$ are pluri-homogeneous smoothing operators in $\Sigma^N_p\widetilde{\mathcal{R}}^{-\varrho'}_q \otimes \mathcal{M}_2(\mathbb{C})$. We proceed to analyze the various lines. In (A.0.52) we substitute $Z = W - F_{\leq N}(Z)Z$ and use Proposition 4.2.19 (i) to get

$$\begin{aligned}
(\text{A.0.52}) = &-\mathrm{i}\boldsymbol{\Omega}(D)W + \mathrm{Op}^{BW}_{\mathtt{vec}}\,(\mathrm{i}a_{\leq N}(Z;\xi) + \mathrm{i}a_{>N}(U;t,\xi))W\\
&+ \mathrm{i}\boldsymbol{\Omega}(D)F_p(Z)Z + R_{\geq p+1}(Z)Z + R_{>N}(U;t)U
\end{aligned} \tag{A.0.56}$$

with smoothing operators $R_{\geq p+1}(Z)$ in $\Sigma^N_{p+1}\widetilde{\mathcal{R}}^{-\varrho'+\frac{3}{2}}_q \otimes \mathcal{M}_2(\mathbb{C})$ and $R'_{>N}(U;t)$ in $\mathcal{R}^{-\varrho'+\frac{3}{2}}_{K,K',N+1}[r] \otimes \mathcal{M}_2(\mathbb{C})$. Note that to obtain (A.0.56) we also substituted $Z = \mathbf{M}_0(U;t)U$ with $\mathbf{M}_0(U;t) \in \mathcal{M}^0_{K,K',0}[r] \otimes \mathcal{M}_2(\mathbb{C})$ in the smoothing operators of homogeneity $\geq N+1$ using also (i)–(ii) of Proposition 4.2.15. From now on we will do this consistently.

We consider now lines (A.0.53), (A.0.54). Using Proposition 4.2.15 (i)–(ii) and Proposition 4.2.19 (i) we get

$$\begin{aligned}
(\text{A.0.53}) + (\text{A.0.54}) = &R_{\leq p-1}(Z)Z + R_p(Z)Z + \mathrm{d}_Z\big(F_p(Z)Z\big)[-\mathrm{i}\boldsymbol{\Omega}(D)Z]\\
&+ R'_{\geq p+1}(Z)Z + R'_{>N+1}(U;t)U
\end{aligned} \tag{A.0.57}$$

with

$$R_{\leq p-1}(Z) := \mathcal{P}_{\leq p-1}[R_{\leq N}(Z)]\,, \quad R_p(Z) := \mathcal{P}_p[R_{\leq N}(Z)]\,,$$

and smoothing operators $R'_{\geq p+1}(Z)$ in $\Sigma^N_{p+1}\widetilde{\mathcal{R}}^{-\varrho_*}_q \otimes \mathcal{M}_2(\mathbb{C})$ and $R'_{>N+1}(U;t)$ in $\mathcal{R}^{-\varrho_*}_{K,K',N+1}[r] \otimes \mathcal{M}_2(\mathbb{C})$, where $\varrho_* := \min(\varrho, \varrho' - \frac{3}{2})$.

Next consider (A.0.55). Substituting $Z = \mathbf{M}_0(U;t)U$ and using Proposition 4.2.15 (i)–(ii) we get

$$(\text{A.0.55}) = R''_{>N}(U;t)U \quad \text{with} \quad R''_{>N}(U;t) \in \mathcal{R}^{-\varrho_*}_{K,K',N+1}[r] \otimes \mathcal{M}_2(\mathbb{C})\,. \tag{A.0.58}$$

Collecting (A.0.56), (A.0.57) and (A.0.58) we have obtained that (A.0.52)-(A.0.55) is the system

$$
\begin{aligned}
\partial_t W = & -\mathrm{i}\boldsymbol{\Omega}(D)W + \mathrm{Op}^{BW}_{\mathtt{vec}}\left(\mathrm{i}a_{\le N}(Z;\xi) + \mathrm{i}a_{>N}(U;t,\xi)\right)W \\
& + R_{\le p-1}(Z)Z + R_p(Z)Z + \mathrm{d}_Z\big(F_p(Z)Z\big)[-\mathrm{i}\boldsymbol{\Omega}(D)Z] + \mathrm{i}\boldsymbol{\Omega}(D)F_p(Z)Z \\
& + R'''_{\ge p+1}(Z)Z + R'''_{>N}(U;t)U
\end{aligned}
\tag{A.0.59}
$$

with smoothing operators $R'''_{\ge p+1}(Z)$ in $\Sigma_{p+1}^N \widetilde{\mathcal{R}}^{-\varrho_*}_q \otimes \mathcal{M}_2(\mathbb{C})$ and $R'''_{>N+1}(U;t)$ in $\mathcal{R}^{-\varrho_*}_{K,K',N+1}[r] \otimes \mathcal{M}_2(\mathbb{C})$. Finally we replace the variable Z with the variable W in (A.0.59) by means of an approximate inverse of $W = \mathcal{F}_{\le N}(Z)$ in (A.0.44). Lemma 4.2.24 implies the existence of an approximate inverse

$$
\Phi_{\le N}(W) := W + \breve{F}_{\le N}(W)W\,, \quad \breve{F}_{\le N}(W) \in \Sigma_p^N \widetilde{\mathcal{R}}^{-\varrho'}_q \otimes \mathcal{M}_2(\mathbb{C})
$$

of the map $\mathcal{F}_{\le N}(Z)$ in (A.0.44). Then, applying $\Phi_{\le N}$ to (A.0.44), we get $Z = \Phi_{\le N}(W) + \widetilde{R}_{>N}(Z)Z$ where $\widetilde{R}_{>N}(Z)$ belongs to $\Sigma_{N+1}\widetilde{\mathcal{R}}^{-\varrho'}_q \otimes \mathcal{M}_2(\mathbb{C})$, and substituting $Z = \mathbf{M}_0(U;t)U$ in the pluri-homogeneous high–homogeneity term $\widetilde{R}_{>N}(Z)Z$ and using (ii) of Proposition 4.2.15 we get

$$
Z = \Phi_{\le N}(W) + \breve{R}_{>N}(U;t)U \quad \text{where} \quad \breve{R}_{>N}(U;t) \in \mathcal{R}^{-\varrho'}_{K,K',N+1}[r] \otimes \mathcal{M}_2(\mathbb{C})\,. \tag{A.0.60}
$$

Finally we substitute (A.0.60) in (A.0.59) and, using Lemma A.0.2 (ii), we deduce

$$
\begin{aligned}
\partial_t W = & -\mathrm{i}\boldsymbol{\Omega}(D)W + \mathrm{Op}^{BW}_{\mathtt{vec}}\Big(\mathrm{i}a^+_{\le N}(W;\xi) + \mathrm{i}a^+_{>N}(U;t,\xi)\Big)W \\
& + R_{\le p-1}(W)W + R_p(W)W + \mathrm{d}_W\big(F_p(W)W\big)[-\mathrm{i}\boldsymbol{\Omega}(D)W] + \mathrm{i}\boldsymbol{\Omega}(D)F_p(W)W \\
& + R^+_{\ge p+1}(W)W + R^+_{>N}(U;t)U
\end{aligned}
$$

which gives (A.0.45) and the properties below. Note that $a^+_{>N}(U;t,\xi)$ is given by the old non-homogeneous symbol $a_{>N}(U;t,\xi)$ and a purely real correction coming from formula (A.0.7). Hence the imaginary part $\mathrm{Im}\, a^+_{>N}(U;t,\xi) = \mathrm{Im}\, a_{>N}(U;t,\xi)$ belongs to $\Gamma^0_{K,K',N+1}[r]$.

Let us prove the last part of the lemma. By (4.2.129) and since $G_p(Z)$ is τ-independent, $F_p(Z) = G_p(Z)$. Then the correction term in (A.0.48) is $G^+_p(W) := \mathrm{d}_W\big(G_p(W)W\big)[-\mathrm{i}\boldsymbol{\Omega}(D)] + \mathrm{i}\boldsymbol{\Omega}(D)G_p(W)$, which has the Fourier expansion (A.0.51) by (A.0.49) and the identity $(-\mathrm{i}\boldsymbol{\Omega}(D)Z)^\sigma_j = -\mathrm{i}\sigma\Omega_j(\kappa)z^\sigma_j$. Note that the smoothing properties of $G^+_p(W)W$ can also be directly verified by the characterization of Lemma 4.2.9. □

Appendix B

Non–resonance conditions

The goal of this section is to prove that the linear frequencies $\vec{\Omega}(\kappa)$, defined in (4.7.16) and (4.5.12), satisfy, for any value of the gravity g, vorticity γ and depth $\mathtt{h}$, the following non-resonance properties, except a zero measure set of surface tension coefficients κ.

Theorem B.0.1. (Non-resonance) *Let $M \in \mathbb{N}$. For any $g > 0$, $\mathtt{h} \in (0, +\infty]$ and $\gamma \in \mathbb{R}$, there exists a zero measure set $\mathcal{K} \subset (0, +\infty)$ such that, for any compact interval $[\kappa_1, \kappa_2] \subset (0, +\infty)$ there is $\tau > 0$ and, for any $\kappa \in [\kappa_1, \kappa_2] \setminus \mathcal{K}$ the following holds: there is a positive constant $\nu > 0$ such that for any multi-index $(\alpha, \beta) \in \mathbb{N}_0^{\mathbb{Z}\setminus\{0\}} \times \mathbb{N}_0^{\mathbb{Z}\setminus\{0\}}$ of length $|\alpha + \beta| \leq M$, which is not super action preserving (cfr. Definition 4.7.3), it results*

$$|\vec{\Omega}(\kappa) \cdot (\alpha - \beta)| > \frac{\nu}{\Big(\max\limits_{j \in \text{supp}(\alpha\cup\beta)} |j|\Big)^{\tau}} \tag{B.0.1}$$

where $\text{supp}(\alpha \cup \beta) := \{j \in \mathbb{Z} \setminus \{0\} \colon \alpha_j + \beta_j \neq 0\}$.

Theorem B.0.1 extends Proposition 8.1 in [27], which is valid only in the irrotational case $\gamma = 0$ and in finite depth. Theorem B.0.1 follows by the next result where we fix a compact interval of surface tension coefficients putting $\mathcal{K} := \cap_{\nu>0} \mathcal{K}_\nu$.

Proposition B.0.2. *Let $M \in \mathbb{N}$ and fix a compact interval $\mathcal{I} := [\kappa_1, \kappa_2]$ with $0 < \kappa_1 < \kappa_2$. Then there exist $\nu_M, \tau, \delta > 0$ such that for any $\nu \in (0, \nu_M)$, there is a set $\mathcal{K}_\nu \subset \mathcal{I}$ of measure $O(\nu^\delta)$ such that for any $\kappa \in \mathcal{I} \setminus \mathcal{K}_\nu$ the following holds: for any multi-index $(\alpha, \beta) \in \mathbb{N}_0^{\mathbb{Z}\setminus\{0\}} \times \mathbb{N}_0^{\mathbb{Z}\setminus\{0\}}$ of length $|\alpha + \beta| \leq M$, which is not super action preserving (cfr. Definition 4.7.3), one has*

$$|\vec{\Omega}(\kappa) \cdot (\alpha - \beta)| > \frac{\nu}{\Big(\max\limits_{j \in \text{supp}(\alpha\cup\beta)} |j|\Big)^{\tau}}\,. \tag{B.0.2}$$

The proof makes use of Delort-Szeftel Theorem 5.1 of [60] about measure estimates for sublevels of subanalytic functions, whose statement is the following.

Theorem B.0.3 (Delort-Szeftel)**.** *Let X be a closed ball $\overline{B(0, r_0)} \subset \mathbb{R}^d$ and Y a compact interval of $\mathbb{R}$. Let $f \colon X \times Y \to \mathbb{R}$ be a continuous subanalytic function, $\rho : X \to \mathbb{R}$ a real analytic function, $\rho \not\equiv 0$. Assume*

(H1) f is real analytic on $\{x \in X \colon \rho(x) \neq 0\} \times Y$;

(H2) for all $x_0 \in X$ with $\rho(x_0) \neq 0$, the equation $f(x_0, y) = 0$ has only finitely many solutions $y \in Y$.

Then there are $N_0 \in \mathbb{N}$, $\alpha_0 > 0$, $\delta > 0$, $C > 0$, *such that for any* $\alpha \in (0, \alpha_0]$, *any* $N \in \mathbb{N}$, $N \geq N_0$, *any* x *with* $\rho(x) \neq 0$,

$$\operatorname{meas}\big\{y \in Y : \quad |f(x,y)| \leq \alpha |\rho(x)|^N \big\} \leq C\alpha^\delta |\rho(x)|^{N\delta} .$$

We shall first prove Proposition B.0.2 for deep water, in Section B.1, and then, for any finite depth, in Section B.2.

B.1 Deep-water case

In the deep water case $\mathtt{h} = +\infty$, by (4.5.12) and (4.5.5), the linear frequencies are

$$\Omega_j(\kappa) = \omega_j(\kappa) + \frac{\gamma}{2}\operatorname{sign}(j) , \quad \omega_j(\kappa) = \sqrt{(\kappa j^2 + g)|j| + \frac{\gamma^2}{4}} . \tag{B.1.1}$$

In this case Proposition B.0.2 is a consequence of the following result.

Proposition B.1.1. *Let* $\mathcal{I} = [\kappa_1, \kappa_2]$ *and consider two integers* $\mathtt{A}, \mathtt{M} \in \mathbb{N}$. *Then there exist* $\alpha_0, \tau, \delta > 0$ *(depending on* $\mathtt{A}, \mathtt{M}$*) such that for any* $\alpha \in (0, \alpha_0)$, *there is a set* $\mathcal{K}_\alpha \subset \mathcal{I}$ *of measure* $O(\alpha^\delta)$, *such that for any* $\kappa \in \mathcal{I} \setminus \mathcal{K}_\alpha$ *the following holds: for any* $1 \leq n_1 < \ldots < n_{\mathtt{A}}$, *any* $\vec{c} := (c_0, c_1, \ldots, c_{\mathtt{A}}) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})^{\mathtt{A}}$, *with* $|\vec{c}|_\infty := \max_{a=0,\ldots,\mathtt{A}} |c_a| \leq \mathtt{M}$, *one has*

$$\Big|\sum_{a=1}^{\mathtt{A}} c_a \omega_{n_a}(\kappa) + \frac{\gamma}{2} c_0\Big| > \frac{\alpha}{(\sum_{a=1}^{\mathtt{A}} n_a)^\tau} . \tag{B.1.2}$$

Before proving Proposition B.1.1 we deduce as a corollary Proposition B.0.2 when $\mathtt{h} = +\infty$.

Proof of Proposition B.0.2. For any multi-index $(\alpha, \beta) \in \mathbb{N}_0^{\mathbb{Z}\setminus\{0\}} \times \mathbb{N}_0^{\mathbb{Z}\setminus\{0\}}$ with length $|\alpha+\beta| \leq M$, using that $\omega_j(\kappa)$ in (B.1.1) is even in j, we get

$$\begin{aligned}
\vec{\Omega}(\kappa) \cdot (\alpha - \beta) &= \sum_{j \in \mathbb{Z}\setminus\{0\}} \omega_j(\kappa)(\alpha_j - \beta_j) + \sum_{j \in \mathbb{Z}\setminus\{0\}} \frac{\gamma}{2}\operatorname{sign}(j)(\alpha_j - \beta_j) \\
&= \sum_{n>0} \omega_n(\kappa)(\alpha_n + \alpha_{-n} - \beta_n - \beta_{-n}) + \frac{\gamma}{2}\sum_{n>0}(\alpha_n - \alpha_{-n} - \beta_n + \beta_{-n}) \\
&= \sum_{n \in \mathfrak{N}(\alpha,\beta)} \omega_n(\kappa)(\alpha_n + \alpha_{-n} - \beta_n - \beta_{-n}) + \frac{\gamma}{2} c_0
\end{aligned} \tag{B.1.3}$$

where $\mathfrak{N}(\alpha, \beta)$ is the set defined in (4.7.10) and $c_0 := \sum_{n>0} \alpha_n - \alpha_{-n} - \beta_n + \beta_{-n} \in \mathbb{Z}$. Since (α, β) is not super action preserving (cfr. Definition 4.7.3) then $\mathfrak{N}(\alpha, \beta)$ is not empty. The cardinality $\mathtt{A} := |\mathfrak{N}(\alpha, \beta)|$ satisfies $1 \leq \mathtt{A} \leq M$. Denoting by $1 \leq n_1 < \ldots < n_{\mathtt{A}}$ the distinct elements of $\mathfrak{N}(\alpha, \beta)$, and the integer numbers

$$c_a := \alpha_{n_a} + \alpha_{-n_a} - \beta_{n_a} - \beta_{-n_a} \in \mathbb{Z} \setminus \{0\} , \quad \forall a = 1, \ldots, \mathtt{A} ,$$

we deduce by (B.1.3) that

$$\vec{\Omega}(\kappa) \cdot (\alpha - \beta) = \sum_{a=1}^{\mathtt{A}} \omega_{n_a}(\kappa) c_a + \frac{\gamma}{2} c_0 .$$

By the definition of $\mathfrak{N}(\alpha, \beta)$, each integer $c_a \neq 0$, $a = 1, \ldots, \mathtt{A}$, and $|c_a| \leq |\alpha| + |\beta| \leq M$. Similarly $|c_0| \leq M$. Applying Proposition B.1.1 with $\mathtt{M} = M$ we deduce (B.0.2) with $\nu := \frac{\alpha}{M^\tau}$. □

The rest of the section is devoted to the proof of Proposition B.1.1.

Proof of Proposition B.1.1. For any $\vec{n} := (n_1, \dots, n_{\mathtt{A}}) \in \mathbb{N}^{\mathtt{A}}$ with $1 \le n_1 < \dots < n_{\mathtt{A}}$, we denote

$$x_0(\vec{n}) := \frac{1}{\sum_{a=1}^{\mathtt{A}} n_a}, \qquad x_a(\vec{n}) := x_0(\vec{n})\sqrt{n_a}, \quad \forall a = 1, \dots, \mathtt{A}. \tag{B.1.4}$$

Clearly

$$0 < \tfrac{1}{\sum_{a=1}^{\mathtt{A}} n_a} \le x_a(\vec{n}) \le 1, \qquad \forall a = 0, \dots, \mathtt{A}. \tag{B.1.5}$$

If (B.1.2) holds, then multiplying it by $x_0(\vec{n})^3$, one gets that the inequalities

$$\Big|\sum_{a=1}^{\mathtt{A}} c_a \sqrt{\kappa x_a^6 + g x_a^2 x_0^4 + \frac{\gamma^2}{4} x_0^6} + \frac{\gamma}{2} c_0 x_0^3\Big| \ge \alpha\, x_0^{\tau+3} \tag{B.1.6}$$

hold at any $x_a = x_a(\vec{n})$, $a = 0, \dots, \mathtt{A}$, defined in (B.1.4). This suggests to define the function

$$\lambda(y, x_0, \kappa) := \sqrt{\kappa y^6 + g y^2 x_0^4 + \frac{\gamma^2}{4} x_0^6}, \tag{B.1.7}$$

and, for $\vec{c} = (c_0, c_1, \dots, c_{\mathtt{A}}) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})^{\mathtt{A}}$,

$$f_{\vec{c}} \colon [-1,1]^{\mathtt{A}+1} \times \mathcal{I} \to \mathbb{R}, \quad f_{\vec{c}}(x, \kappa) := \sum_{a=1}^{\mathtt{A}} c_a\, \lambda(x_a, x_0, \kappa) + \frac{\gamma}{2} c_0 x_0^3 \tag{B.1.8}$$

where $x := (x_0, x_1, \dots, x_{\mathtt{A}})$.

We estimate the sublevels of $\kappa \mapsto f_{\vec{c}}(x, \kappa)$ using Theorem B.0.3. Let us verify its assumptions. The set $X := [-1,1]^{\mathtt{A}+1}$ is a closed ball in $\mathbb{R}^{\mathtt{A}+1}$. The function $f_{\vec{c}} : X \times \mathcal{I} \to \mathbb{R}$ is continuous and subanalytic. Then we define the non-zero real analytic function

$$\rho(x) := \prod_{a=0}^{\mathtt{A}} x_a \prod_{1 \le a < b \le \mathtt{A}} (x_a^2 - x_b^2). \tag{B.1.9}$$

We observe that $\rho(x)$ evaluated at $x(\vec{n}) := (x_0(\vec{n}), \dots, x_{\mathtt{A}}(\vec{n}))$, defined in (B.1.4), satisfies

$$\Big(\sum_{a=1}^{\mathtt{A}} n_a\Big)^{-\tau_1} \le |\rho(x(\vec{n}))| \lesssim_{\mathtt{A}} \Big(\sum_{a=1}^{\mathtt{A}} n_a\Big)^{-1} \tag{B.1.10}$$

with $\tau_1 := \mathtt{A} + 1 + 2\binom{\mathtt{A}}{2}$, as follows by (B.1.5), (B.1.9) and the assumption that the n_a's are all distinct, thus $|n_a - n_b| \ge 1$, for any $a \ne b$.

We show now that the assumptions (H1) and (H2) of Theorem B.0.3 hold true.

Verification of (H1). If $\rho(x) \ne 0$ then, by (B.1.9),

$$x_a \ne 0,\ \forall 0 \le a \le \mathtt{A}, \qquad \text{and} \qquad x_a^2 \ne x_b^2,\ \forall 1 \le a < b \le \mathtt{A}. \tag{B.1.11}$$

In particular on the set $\{x \in X : \rho(x) \ne 0\} \times Y$ the function $\lambda(x_a, x_0, \kappa)$ in (B.1.7) in real analytic and thus the function $f_{\vec{c}}(x, \kappa)$ in (B.1.8) is real analytic.

Verification of (H2). The fact that, for any $x \in X$ such that $\rho(x) \ne 0$, the *analytic* function $\kappa \mapsto f_{\vec{c}}(x, \kappa)$ possesses only a finite number of zeros on the interval $\mathcal{I}$, is a consequence of the next lemma.

Lemma B.1.2. *For any $x \in X$ such that $\rho(x) \neq 0$, the function $\kappa \mapsto f_{\vec{c}}(x,\kappa)$ is not identically zero in $\mathcal{I}$.*

Proof. We argue by contradiction, assuming that there exists $x = (x_a)_{0 \le a \le \mathtt{A}} \in X$ with $\rho(x) \neq 0$ such that $f_{\vec{c}}(x,\kappa) = 0$ for any κ in the interval $\mathcal{I}$. Then the function $\kappa \mapsto f_{\vec{c}}(x,\kappa)$ is identically zero also on the larger domain of analyticity $(-\frac{\gamma^2}{4}x_0^6, +\infty)$. Note that $x_0^2 > 0$ because $\rho(x) \neq 0$, cfr. (B.1.11). In particular, for any $l \in \mathbb{N}$, all the derivatives $\partial_\kappa^l f_{\vec{c}}(x,\kappa) \equiv 0$ are zero in the interval $(-\frac{\gamma^2}{4}x_0^6, +\infty)$.

Now we compute such derivatives at $\kappa = 0$ by differentiating (B.1.8). The derivatives of the function $\lambda(y, x_0, \kappa)$ defined in (B.1.7) are given by, for suitable constants $C_l \neq 0$,

$$\partial_\kappa^l \lambda(y, x_0, \kappa) = C_l y^{6l} \lambda(y, x_0, \kappa)^{1-2l}, \quad \forall l \in \mathbb{N}.$$

Thus we obtain

$$\partial_\kappa^l \lambda(y, x_0, \kappa)|_{\kappa=0} = C_l \, \mu(y, x_0)^l \lambda(y, x_0, 0) \qquad \text{where} \qquad \mu(y, x_0) := \frac{y^6}{g y^2 x_0^4 + \frac{\gamma^2}{4} x_0^6}, \tag{B.1.12}$$

and, recalling (B.1.8),

$$\partial_\kappa^l f_{\vec{c}}(x,\kappa)|_{\kappa=0} = C_l \sum_{a=1}^{\mathtt{A}} c_a \mu(x_a, x_0)^l \lambda(x_a, x_0, 0), \quad \forall l \in \mathbb{N}.$$

As a consequence, the conditions $\partial_\kappa^l f_{\vec{c}}(x,\kappa)|_{\kappa=0} = 0$ for any $l = 1, \ldots, \mathtt{A}$, imply that

$$A(x)\vec{\mathfrak{c}} = 0 \tag{B.1.13}$$

where $A(x)$ is the $\mathtt{A} \times \mathtt{A}$-matrix

$$A(x) := \begin{pmatrix} \mu(x_1,x_0)\lambda(x_1,x_0,0) & \cdots & \mu(x_{\mathtt{A}},x_0)\lambda(x_{\mathtt{A}},x_0,0) \\ \mu(x_1,x_0)^2\lambda(x_1,x_0,0) & \cdots & \mu(x_{\mathtt{A}},x_0)^2\lambda(x_{\mathtt{A}},x_0,0) \\ \vdots & \ddots & \vdots \\ \mu(x_1,x_0)^{\mathtt{A}}\lambda(x_1,x_0,0) & \ldots & \mu(x_{\mathtt{A}},x_0)^{\mathtt{A}}\lambda(x_{\mathtt{A}},x_0,0) \end{pmatrix} \quad \text{and} \quad \vec{\mathfrak{c}} := \begin{bmatrix} c_1 \\ \vdots \\ c_{\mathtt{A}} \end{bmatrix} \in (\mathbb{Z} \setminus \{0\})^{\mathtt{A}}.$$

Since the vector $\vec{\mathfrak{c}} \neq 0$, we deduce by (B.1.13) that the matrix $A(x)$ has zero determinant. On the other hand, by the multi-linearity of the determinant,

$$\begin{aligned} \det A(x) &= \prod_{a=1}^{\mathtt{A}} \mu(x_a,x_0)\lambda(x_a,x_0,0) \det \begin{pmatrix} 1 & \cdots & 1 \\ \mu(x_1,x_0) & \cdots & \mu(x_{\mathtt{A}},x_0) \\ \vdots & \ddots & \vdots \\ \mu(x_1,x_0)^{\mathtt{A}-1} & \ldots & \mu(x_{\mathtt{A}},x_0)^{\mathtt{A}-1} \end{pmatrix} \\ &= \prod_{a=1}^{\mathtt{A}} \mu(x_a,x_0)\lambda(x_a,x_0,0) \prod_{1 \le a < b \le \mathtt{A}} (\mu(x_a,x_0) - \mu(x_b,x_0)) \end{aligned} \tag{B.1.14}$$

by a Vandermonde determinant. The condition $\rho(x) \neq 0$ implies, by (B.1.11), (B.1.12) and (B.1.7), that

$$\prod_{a=1}^{\mathtt{A}} \mu(x_a,x_0)\lambda(x_a,x_0,0) \neq 0,$$

and, in view of (B.1.14), the determinant $\det A(x) = 0$ if and only if $\mu(x_a, x_0) = \mu(x_b, x_0)$ for some $1 \le a < b \le \mathtt{A}$. Since the function $y \to \mu(y, x_0)$ in (B.1.12) is even and strictly monotone on the two intervals $(0, +\infty)$ and $(-\infty, 0)$, it follows that

$$\mu(x_a, x_0) = \mu(x_b, x_0) \quad \Rightarrow \quad |x_a| = |x_b|\,.$$

This contradicts $\rho(x) \neq 0$, see (B.1.11). The lemma is proved. □

We have verified assumptions (H1) and (H2) of Theorem B.0.3. We thus conclude that there are $N_0 \in \mathbb{N}$, $\alpha_0, \delta, C > 0$, such that for any $\alpha \in (0, \alpha_0]$, any $N \in \mathbb{N}$, $N \ge N_0$, any $x \in X$ with $\rho(x) \neq 0$,

$$\operatorname{meas}\big\{\kappa \in \mathcal{I}\colon \quad |f_{\vec{c}}(x,\kappa)| \le \alpha |\rho(x)|^N\big\} \le C\alpha^\delta\,|\rho(x)|^{N\delta}\,. \tag{B.1.15}$$

For any $\vec{n} = (n_1, \ldots, n_{\mathtt{A}}) \in \mathbb{N}^{\mathtt{A}}$ with $1 \le n_1 < \ldots < n_{\mathtt{A}}$, we consider the set

$$\mathcal{B}_{\vec{c},\vec{n}}(\alpha, N) := \big\{\kappa \in \mathcal{I}\colon \quad |f_{\vec{c}}(x(\vec{n}),\kappa)| \le \alpha |\rho(x(\vec{n}))|^N\big\} \tag{B.1.16}$$

where $x(\vec{n}) = (x_a(\vec{n}))_{a=1,\ldots,\mathtt{A}} \in X$ is defined in (B.1.4). By (B.1.10) we get $\rho(x(\vec{n})) \neq 0$. Then (B.1.15) yields

$$\operatorname{meas} \mathcal{B}_{\vec{c},\vec{n}}(\alpha, N) \le C\alpha^\delta |\rho(x(\vec{n}))|^{N\delta}. \tag{B.1.17}$$

Consider the set

$$\mathcal{K}(\alpha, N) := \bigcup_{\substack{\vec{n}=(n_1,\ldots,n_{\mathtt{A}})\in\mathbb{N}^{\mathtt{A}},\,1\le n_1<\ldots<n_{\mathtt{A}} \\ \vec{c}\in\mathbb{Z}\times(\mathbb{Z}\setminus\{0\})^{\mathtt{A}},\,|\vec{c}|_\infty\le\mathtt{M}}} \mathcal{B}_{\vec{c},\vec{n}}(\alpha, N) \subset \mathcal{I}\,. \tag{B.1.18}$$

By (B.1.17) and (B.1.10) it results

$$\operatorname{meas} \mathcal{K}(\alpha, N) \le C(\mathtt{A},\mathtt{M})\alpha^\delta \sum_{n_1,\ldots,n_{\mathtt{A}}\in\mathbb{N}} \frac{1}{(\sum_{a=1}^{\mathtt{A}} n_a)^{N\delta}} \le C'(\mathtt{A},\mathtt{M})\alpha^\delta \tag{B.1.19}$$

for some finite constant $C'(\mathtt{A},\mathtt{M}) < +\infty$, provided $N\delta > \mathtt{A}$. We fix

$$\underline{N} := [\mathtt{A}\delta^{-1}] + 1 \qquad \text{and} \qquad \mathcal{K}_\alpha := \mathcal{K}(\alpha, \underline{N})$$

whose measure satisfies $|\mathcal{K}_\alpha| \lesssim_{\mathtt{A},\mathtt{M}} \alpha^\delta$ by (B.1.19). For any $\kappa \in \mathcal{I} \setminus \mathcal{K}_\alpha$, for any $\vec{n} = (n_1, \ldots, n_{\mathtt{A}})$ with $1 \le n_1 < \ldots < n_{\mathtt{A}}$, for any $\vec{c} \in \mathbb{Z} \times (\mathbb{Z}\setminus\{0\})^{\mathtt{A}}$ with $|\vec{c}|_\infty \le \mathtt{M}$, one has, by (B.1.18) and (B.1.16),

$$|f_{\vec{c}}(x(\vec{n}),\kappa)| > \alpha|\rho(x(\vec{n}))|^{\underline{N}} \overset{\text{(B.1.10)}}{\ge} \frac{\alpha}{\big(\sum_{a=1}^{\mathtt{A}} n_a\big)^{\tau_1 \underline{N}}}\,. \tag{B.1.20}$$

Recalling the definition of $f_{\vec{c}}$ in (B.1.8), (B.1.7) and $x_0(\vec{n})$ in (B.1.4), the lower bound (B.1.20) implies (B.1.2) with $\tau := \tau_1 \underline{N} - 3$, cfr. (B.1.6). □

B.2 Finite depth case

We consider now the finite depth case $0 < \mathtt{h} < +\infty$ where the frequencies are, by (4.5.12) and (4.5.5),

$$\Omega_j(\kappa) = \omega_j(\kappa) + \frac{\gamma}{2}\tanh(\mathtt{h}j)\,, \quad \omega_j(\kappa) = \sqrt{j\tanh(\mathtt{h}j)\Big(\kappa j^2 + g + \frac{\gamma^2}{4}\frac{\tanh(\mathtt{h}j)}{j}\Big)}\,. \tag{B.2.1}$$

In this case Proposition B.0.2 is a consequence of the following result.

Proposition B.2.1. *Let $\mathcal{I} = [\kappa_1, \kappa_2]$ and consider $\mathtt{A}, \mathtt{M} \in \mathbb{N}$ and $\mathtt{B} \in \mathbb{N}_0$. Then there exist $\alpha_0, \tau, \delta > 0$ (depending on $\mathtt{A}, \mathtt{B}, \mathtt{M}$) such that for any $\alpha \in (0, \alpha_0)$, there is a set $\mathcal{K}_\alpha \subset \mathcal{I}$ of measure $O(\alpha^\delta)$, such that for any $\kappa \in \mathcal{I} \setminus \mathcal{K}_\alpha$ the following holds: for any $1 \leq n_1 \leq \ldots \leq n_{\mathtt{A}}$ any $\vec{m} = (m_1, \ldots, m_{\mathtt{B}}) \in \mathbb{N}^{\mathtt{B}}$, any $\vec{c} := (c_1, \ldots, c_{\mathtt{A}}) \in (\mathbb{Z} \setminus \{0\})^{\mathtt{A}}$ with $|\vec{c}|_\infty \leq \mathtt{M}$ and $\vec{d} = (d_1, \ldots, d_{\mathtt{B}}) \in (\mathbb{Z} \setminus \{0\})^{\mathtt{B}}$ with $|\vec{d}|_\infty \leq \mathtt{M}$, one has*

$$\Big| \sum_{a=1}^{\mathtt{A}} c_a \omega_{n_a}(\kappa) + \frac{\gamma}{2} \sum_{b=1}^{\mathtt{B}} d_b \tanh(\mathtt{h} m_b) \Big| > \frac{\alpha}{(\sum_{a=1}^{\mathtt{A}} n_a + \sum_{b=1}^{\mathtt{B}} m_b)^\tau} . \tag{B.2.2}$$

If $\mathtt{B} = 0$, by definition, the sums in (B.2.2) in the index b are empty and the vectors $\vec{m}, \vec{d}$ are not present.

Before proving Proposition B.2.1 we deduce Proposition B.0.2 in the case of finite depth $\mathtt{h}$.

Proof of Proposition B.0.2. Let $(\alpha, \beta) \in \mathbb{N}_0^{\mathbb{Z}\setminus\{0\}} \times \mathbb{N}_0^{\mathbb{Z}\setminus\{0\}}$ be a multi-index with length $|\alpha + \beta| \leq M$, which is not super action preserving (cfr. Definition 4.7.3). Let $\mathtt{A} := |\mathfrak{N}(\alpha, \beta)| \geq 1$ and denote by $1 \leq n_1 < \ldots < n_{\mathtt{A}}$ the elements of $\mathfrak{N}(\alpha, \beta)$ defined in (4.7.10). We also consider the set $\mathfrak{M}(\alpha, \beta) := \{n \in \mathbb{N} \colon \alpha_n - \alpha_{-n} - \beta_n + \beta_{-n} \neq 0\}$, which could be empty. We denote by $\mathtt{B} := |\mathfrak{M}(\alpha, \beta)|$ its cardinality, which could be zero, and $1 \leq m_1 < \cdots < m_{\mathtt{B}}$ its distinct elements, if any. Note that $1 \leq \mathtt{A} \leq M$ and $0 \leq \mathtt{B} \leq M$. By (B.2.1) and since $\omega_j(\kappa)$ is even in j, we get

$$\begin{aligned}
\vec{\Omega}(\kappa) \cdot (\alpha - \beta) &= \sum_{j \in \mathbb{Z}\setminus\{0\}} \omega_j(\kappa)(\alpha_j - \beta_j) + \sum_{j \in \mathbb{Z}\setminus\{0\}} \frac{\gamma}{2} \tanh(\mathtt{h} j)(\alpha_j - \beta_j) \\
&= \sum_{n>0} \omega_n(\kappa)(\alpha_n + \alpha_{-n} - \beta_n - \beta_{-n}) + \frac{\gamma}{2} \sum_{n>0} \tanh(\mathtt{h} n)(\alpha_n - \alpha_{-n} - \beta_n + \beta_{-n}) \\
&= \sum_{n \in \mathfrak{N}(\alpha,\beta)} \omega_n(\kappa)(\alpha_n + \alpha_{-n} - \beta_n - \beta_{-n}) + \frac{\gamma}{2} \sum_{n \in \mathfrak{M}(\alpha,\beta)} \tanh(\mathtt{h} n)(\alpha_n - \alpha_{-n} - \beta_n + \beta_{-n}) \\
&= \sum_{a=1}^{\mathtt{A}} \omega_{n_a}(\kappa) c_a + \frac{\gamma}{2} \sum_{b=1}^{\mathtt{B}} d_b \tanh(\mathtt{h} m_b)
\end{aligned} \tag{B.2.3}$$

having defined

$$c_a := \alpha_{n_a} + \alpha_{-n_a} - \beta_{n_a} - \beta_{-n_a} , \qquad d_b := \alpha_{m_b} - \alpha_{-m_b} - \beta_{m_b} + \beta_{-m_b} .$$

By the definition of $\mathfrak{N}(\alpha, \beta)$, each $c_a \in \mathbb{Z} \setminus \{0\}$ and $|c_a| \leq |\alpha| + |\beta| \leq M$ for any $a = 1, \ldots, \mathtt{A}$. If $\mathfrak{M}(\alpha, \beta)$ is empty then $\mathtt{B} = 0$, and the second sum in (B.2.3) in the index b is not present. On the other hand, if $\mathtt{B} \geq 1$, by the definition of $\mathfrak{M}(\alpha, \beta)$, each $d_b \in \mathbb{Z} \setminus \{0\}$ and $|d_b| \leq |\alpha| + |\beta| \leq M$ for any $b = 1, \ldots, \mathtt{B}$. Applying in both cases Proposition B.2.1 with $\mathtt{M} = M$ we deduce (B.0.2) with $\nu := \frac{\alpha}{(2M)^\tau}$. □

Proof of Proposition B.2.1. We write the proof in the case $\mathtt{B} \geq 1$. In the case $\mathtt{B} = 0$ the same argument works. For any $\vec{n} := (n_1, \ldots, n_{\mathtt{A}}) \in \mathbb{N}^{\mathtt{A}}$ with $1 \leq n_1 < \ldots < n_{\mathtt{A}}$ and $\vec{m} := (m_1, \ldots, m_{\mathtt{B}}) \in \mathbb{N}^{\mathtt{B}}$ we define

$$\begin{aligned}
&x_0(\vec{n}, \vec{m}) := \frac{1}{\sum_{a=1}^{\mathtt{A}} n_a + \sum_{b=1}^{\mathtt{B}} m_b} , \qquad x_a(\vec{n}, \vec{m}) := x_0(\vec{n}, \vec{m}) \sqrt{n_a} , \ \forall a = 1, \ldots, \mathtt{A} \\
&t_a(\vec{n}) := \sqrt{\tanh(\mathtt{h} n_a)} , \ \ \forall a = 1, \ldots, \mathtt{A} , \quad t_{\mathtt{A}+b}(\vec{m}) := \sqrt{\tanh(\mathtt{h} m_b)} , \ \ \forall b = 1, \ldots, \mathtt{B} .
\end{aligned} \tag{B.2.4}$$

Clearly

$$\begin{aligned}
0 < \tfrac{1}{\sum_{a=1}^{\mathtt{A}} n_a + \sum_{b=1}^{\mathtt{B}} m_b} \leq x_a(\vec{n}, \vec{m}) \leq 1 , \qquad &\sqrt{\tanh(\mathtt{h})} \leq t_a(\vec{n}) \leq 1 , \qquad \forall a = 0, \ldots, \mathtt{A} , \\
&\sqrt{\tanh(\mathtt{h})} \leq t_{\mathtt{A}+b}(\vec{m}) \leq 1 , \quad \forall b = 1, \ldots, \mathtt{B} .
\end{aligned} \tag{B.2.5}$$

If (B.2.2) holds, then multiplying it by $x_0(\vec{n},\vec{m})^3$, one gets, recalling (B.2.1), that the inequalities

$$\Big|\sum_{a=1}^{\mathtt{A}} c_a t_a \sqrt{\kappa x_a^6 + g x_a^2 x_0^4 + \frac{\gamma^2}{4} t_a^2 x_0^6} + \frac{\gamma}{2}\sum_{b=1}^{\mathtt{B}} d_b t_{\mathtt{A}+b}^2 x_0^3\Big| \geq \alpha\, x_0^{\tau+3} \tag{B.2.6}$$

hold at any $x_0 = x_0(\vec{n},\vec{m})$, $x_a = x_a(\vec{n},\vec{m})$, $t_a = t_a(\vec{n})$, $a = 1,\ldots,\mathtt{A}$ and $t_{\mathtt{A}+b} = t_{\mathtt{A}+b}(\vec{m})$, $b = 1,\ldots,\mathtt{B}$, defined in (B.2.4). This suggests to define the function

$$\lambda(y,s,x_0,\kappa) := \sqrt{\kappa y^6 + g y^2 x_0^4 + \frac{\gamma^2}{4} s^2 x_0^6}\,, \tag{B.2.7}$$

and, for $\vec{c} := (c_1,\ldots,c_{\mathtt{A}}) \in (\mathbb{Z}\setminus\{0\})^{\mathtt{A}}$ and $\vec{d} = (d_1,\ldots,d_{\mathtt{B}}) \in (\mathbb{Z}\setminus\{0\})^{\mathtt{B}}$,

$$f_{\vec{c},\vec{d}}\colon [-1,1]^{2\mathtt{A}+\mathtt{B}+1}\times\mathcal{I}\to\mathbb{R}, \quad f_{\vec{c},\vec{d}}(x,t,\kappa) := \sum_{a=1}^{\mathtt{A}} c_a t_a\, \lambda(x_a,t_a,x_0,\kappa) + \frac{\gamma}{2}\sum_{b=1}^{\mathtt{B}} d_b t_{\mathtt{A}+b}^2 x_0^3\,, \tag{B.2.8}$$

with variables $x = (x_0,\ldots,x_{\mathtt{A}})$ and $t = (t_1,\ldots,t_{\mathtt{A}+\mathtt{B}})$.

We estimate the sublevels of $\kappa \mapsto f_{\vec{c},\vec{d}}(x,t,\kappa)$ using Theorem B.0.3. The set $X := [-1,1]^{2\mathtt{A}+\mathtt{B}+1}$ is a closed ball of $\mathbb{R}^{2\mathtt{A}+\mathtt{B}+1}$. The function $f_{\vec{c},\vec{d}}\colon X\times\mathcal{I}\to\mathbb{R}$ is continuous and subanalytic. Then we define the non-zero real analytic function

$$\rho(x,t) := x_0 \prod_{a=1}^{\mathtt{A}} x_a t_a \prod_{1\leq a<b\leq \mathtt{A}} \Big[\big(g x_a^2 x_0^4 + \frac{\gamma^2}{4} t_a^2 x_0^6\big) x_b^6 - \big(g x_b^2 x_0^4 + \frac{\gamma^2}{4} t_b^2 x_0^6\big) x_a^6\Big]\,. \tag{B.2.9}$$

We observe the following lemma.

Lemma B.2.2. *There exist positive constants $c(\mathtt{A}) := c(\mathtt{A},g,\gamma,h)$, $C(\mathtt{A}) := C(\mathtt{A},g,\gamma) > 0$ such that, for any*

$$x(\vec{n},\vec{m}) := (x_a(\vec{n},\vec{m}))_{a=0,\ldots,\mathtt{A}}\,, \quad t(\vec{n},\vec{m}) := (t_1(\vec{n}),\ldots,t_{\mathtt{A}}(\vec{n}),t_{\mathtt{A}+1}(\vec{m}),\ldots,t_{\mathtt{A}+\mathtt{B}}(\vec{m}))\,, \tag{B.2.10}$$

defined by (B.2.4)*, it results*

$$c(\mathtt{A})\Big(\sum_{a=1}^{\mathtt{A}} n_a + \sum_{b=1}^{\mathtt{B}} m_b\Big)^{-\tau_2} \leq |\rho(x(\vec{n},\vec{m}),t(\vec{n},\vec{m}))| \leq C(\mathtt{A})\Big(\sum_{a=1}^{\mathtt{A}} n_a + \sum_{b=1}^{\mathtt{B}} m_b\Big)^{-1} \tag{B.2.11}$$

with $\tau_2 := \mathtt{A} + 1 + 12\binom{\mathtt{A}}{2}$.

Proof. The upper bound (B.2.11) directly follows by (B.2.5). The lower bound (B.2.11) follows by (B.2.5) and the fact that, since $1 \leq n_1 < \ldots < n_{\mathtt{A}}$ are all distinct,

$$\begin{aligned}
&\Big|(g x_a^2 x_0^4 + \frac{\gamma^2}{4} t_a^2 x_0^6) x_b^6 - (g x_b^2 x_0^4 + \frac{\gamma^2}{4} t_b^2 x_0^6)) x_a^6\Big| \\
&= x_0^{12}(\vec{n},\vec{m})(g n_a + \frac{\gamma^2}{4}\tanh(\mathtt{h} n_a))(g n_b + \frac{\gamma^2}{4}\tanh(\mathtt{h} n_b))\Big|\frac{n_a^3}{g n_a + \frac{\gamma^2}{4}\tanh(\mathtt{h} n_a)} - \frac{n_b^3}{g n_b + \frac{\gamma^2}{4}\tanh(\mathtt{h} n_b)}\Big| \\
&\geq x_0(\vec{n},\vec{m})^{12} g^2 \min_{y\geq 1}\Big(\frac{\mathrm{d}}{\mathrm{d}y}\frac{y^3}{g y + \frac{\gamma^2}{4}\tanh(\mathtt{h} y)}\Big)|n_a - n_b| \geq c\, x_0(\vec{n},\vec{m})^{12}
\end{aligned}$$

for some constant $c > 0$, having used in the last passage that, for any $y \geq 1$, $g > 0$, $\gamma \in \mathbb{R}$ and $h > 0$,

$$\frac{\mathrm{d}}{\mathrm{d}y}\frac{y^3}{gy + \frac{\gamma^2}{4}\tanh(\mathtt{h}y)} = \frac{4y^2(8gy + 3\gamma^2\tanh(\mathtt{h}y) - \gamma^2\mathtt{h}y\,\mathrm{sech}^2(\mathtt{h}y))}{(4gy + \gamma^2\tanh(\mathtt{h}y))^2}$$
$$\geq \frac{2y^2}{16g^2y^2 + \gamma^4}\left(8g + \gamma^2(3\tanh(\mathtt{h}y) - \mathtt{h}y\,\mathrm{sech}^2(\mathtt{h}y))\right) \geq \frac{2}{16g^2 + \gamma^4}8g$$

using that the function $3\tanh(y) - y\,\mathrm{sech}^2(y)$ is positive for $y > 0$. □

We show now that the assumptions (H1) and (H2) of Theorem B.0.3 hold true.
Verification of (H1). By (B.2.9), if $\rho(x, t) \neq 0$ then, by (B.2.9),

$$x_a \neq 0\,,\ \forall a = 0, \ldots, \mathtt{A}\,, \quad t_a \neq 0\,,\ \forall a = 1, \ldots, \mathtt{A}\,. \tag{B.2.12}$$

In particular on the set $\{(x, t) \in X : \rho(x, t) \neq 0\} \times Y$ the function $f_{\vec{c},\vec{d}}$ in (B.2.8) is real analytic.
Verification of (H2). For any (x, t) such that $\rho(x, t) \neq 0$, the analytic function $\kappa \mapsto f_{\vec{c},\vec{d}}(x, t, \kappa)$ possesses only a finite number of zeros as a consequence of the next lemma.

Lemma B.2.3. *For any (x, t) such that $\rho(x, t) \neq 0$, the analytic function $\kappa \mapsto f_{\vec{c},\vec{d}}(x, t, \kappa)$ is not identically zero in $\mathcal{I}$.*

Proof. Assume by contradiction that there exists

$$(x, t) \in X\,,\ x = (x_a)_{0\leq a\leq \mathtt{A}}\,,\ t = (t_a)_{0\leq a\leq \mathtt{A}+\mathtt{B}} \quad \text{with} \quad \rho(x, t) \neq 0 \quad \text{such that} \quad f_{\vec{c},\vec{d}}(x, t, \kappa) = 0$$

for any κ in the interval $\mathcal{I}$. Then, by analyticity, the function $\kappa \mapsto f_{\vec{c},\vec{d}}(x, t, \kappa)$ is identically zero also on the larger interval $(-\delta, +\infty)$ where $\delta := \min_{1\leq a\leq \mathtt{A}} t_a^2 \frac{\gamma^2}{4} x_0^6 > 0$. Note that $x_0^2 > 0$ by (B.2.12). In particular, for any $l \in \mathbb{N}$, all the derivatives $\partial_\kappa^l f_{\vec{c},\vec{d}}(x, t, \kappa) \equiv 0$ are zero in the interval $\kappa \in (-\delta, +\infty)$.

We now compute such derivatives at $\kappa = 0$ differentiating (B.2.8). The derivatives of the function $\lambda(y, s, x_0, \kappa)$ defined in (B.2.7) are given by, for suitable constants $C_l \neq 0$,

$$\partial_\kappa^l \lambda(y, s, x_0, \kappa) = C_l y^{6l}\,\lambda(y, s, x_0, \kappa)^{1-2l}\,, \quad \forall l \in \mathbb{N}\,.$$

Thus we obtain

$$\partial_\kappa^l \lambda(y, s, x_0, \kappa)|_{\kappa=0} = C_l\,\mu(y, s, x_0)^l\,\lambda(y, s, x_0, 0) \quad \text{where} \quad \mu(y, s, x_0) := \frac{y^6}{gy^2x_0^4 + \frac{\gamma^2}{4}s^2x_0^6}\,, \tag{B.2.13}$$

and, recalling (B.2.8),

$$\partial_\kappa^l f_{\vec{c},\vec{d}}(x, t, \kappa)|_{\kappa=0} = C_l \sum_{a=1}^{\mathtt{A}} c_a t_a \mu(x_a, t_a, x_0)^l\,\lambda(x_a, t_a, x_0, 0)\,, \quad \forall l \in \mathbb{N}\,.$$

As a consequence, the conditions $\partial_\kappa^l f_{\vec{c},\vec{d}}(x, t, \kappa)|_{\kappa=0} = 0$ for any $l = 1, \ldots, \mathtt{A}$ imply that

$$A(x, t)\vec{r} = 0 \tag{B.2.14}$$

where $A(x, t)$ is the $\mathtt{A} \times \mathtt{A}$ matrix

$$A(x, t) := \begin{pmatrix} \mu(x_1, t_1, x_0)\lambda(x_1, t_1, x_0, 0) & \cdots & \mu(x_{\mathtt{A}}, t_{\mathtt{A}}, x_0)\lambda(x_{\mathtt{A}}, t_{\mathtt{A}}, x_0, 0) \\ \mu(x_1, t_1, x_0)^2\lambda(x_1, t_1, x_0, 0) & \cdots & \mu(x_{\mathtt{A}}, t_{\mathtt{A}}, x_0)^2\lambda(x_{\mathtt{A}}, t_{\mathtt{A}}, x_0, 0) \\ \vdots & \ddots & \vdots \\ \mu(x_1, t_1, x_0)^{\mathtt{A}}\lambda(x_1, t_1, x_0, 0) & \ldots & \mu(x_{\mathtt{A}}, t_{\mathtt{A}}, x_0)^{\mathtt{A}}\lambda(x_{\mathtt{A}}, t_{\mathtt{A}}, x_0, 0) \end{pmatrix}$$

and $\vec{r}$ is the vector

$$\vec{r} := \begin{bmatrix} r_1 \\ \vdots \\ r_{\mathtt{A}} \end{bmatrix}, \quad r_a := c_a t_a \in \mathbb{Z}\setminus\{0\}, \ \forall a = 1,\ldots,\mathtt{A},$$

because by assumption each $c_a \neq 0$ and (B.2.12) holds. Since $\vec{r} \neq 0$, we deduce by (B.2.14) that the matrix $A(x,t)$ has zero determinant. On the other hand, by the multilinearity of the determinant,

$$\begin{aligned}\det A(x,t) &= \prod_{a=1}^{\mathtt{A}} \mu(x_a,t_a,x_0)\lambda(x_a,t_a,x_0,0)\det\begin{pmatrix} 1 & \cdots & 1 \\ \mu(x_1,t_1,x_0) & \cdots & \mu(x_{\mathtt{A}},t_{\mathtt{A}},x_0) \\ \vdots & \ddots & \vdots \\ \mu(x_1,t_1,x_0)^{\mathtt{A}-1} & \ldots & \mu(x_{\mathtt{A}},t_{\mathtt{A}},x_0)^{\mathtt{A}-1}\end{pmatrix} \\ &= \prod_{a=1}^{\mathtt{A}} \mu(x_a,t_a,x_0)\lambda(x_a,t_a,x_0,0) \prod_{1\le a<b\le \mathtt{A}} (\mu(x_a,t_a,x_0) - \mu(x_b,t_b,x_0)).\end{aligned} \tag{B.2.15}$$

The condition $\rho(x,t) \neq 0$ implies, by (B.2.12), (B.2.13) and (B.2.7), that

$$\prod_{a=1}^{\mathtt{A}} \mu(x_a,t_a,x_0)\lambda(x_a,t_a,x_0,0) \neq 0,$$

and, in view of (B.2.15), the determinant $\det A(x,t) = 0$ if only if $\mu(x_a,t_a,x_0) = \mu(x_b,t_b,x_0)$ for some $1 \le a < b \le \mathtt{A}$. By the definition of the function $(y,s,x_0) \to \mu(y,s,x_0)$ in (B.2.13) it follows that

$$\mu(x_a,t_a,x_0) = \mu(x_b,t_b,x_0) \quad \Rightarrow \quad \big(g x_a^2 x_0^4 + \frac{\gamma^2}{4} t_a^2 x_0^6\big)x_b^6 - \big(g x_b^2 x_0^4 + \frac{\gamma^2}{4} t_b^2 x_0^6\big)x_a^6 = 0.$$

In view of (B.2.9) this contradicts $\rho(x,t) \neq 0$. ☐

We have verified assumptions (H1) and (H2) of Theorem B.0.3. We thus conclude that there are $N_0 \in \mathbb{N}$, $\alpha_0, \delta, C > 0$, such that for any $\alpha \in (0,\alpha_0]$, any $N \in \mathbb{N}$, $N \ge N_0$, any $(x,t) \in X$ with $\rho(x,t) \neq 0$,

$$\operatorname{meas}\big\{\kappa \in \mathcal{I}\colon \quad |f_{\vec{c},\vec{d}}(x,t,\kappa)| \le \alpha|\rho(x,t)|^N\big\} \le C\alpha^\delta |\rho(x,t)|^{N\delta}. \tag{B.2.16}$$

For $\vec{n} = (n_1,\ldots,n_{\mathtt{A}}) \in \mathbb{N}^{\mathtt{A}}$ with $1 \le n_1 < \ldots < n_{\mathtt{A}}$ and $\vec{m} = (m_1,\ldots,m_{\mathtt{B}}) \in \mathbb{N}^{\mathtt{B}}$ we consider the set

$$\mathcal{B}_{\vec{c},\vec{d},\vec{m},\vec{n}}(\alpha,N) := \Big\{\kappa \in \mathcal{I}\colon \quad |f_{\vec{c},\vec{d}}(x(\vec{n},\vec{m}),t(\vec{n},\vec{m}),\kappa)| \le \alpha|\rho(x(\vec{n},\vec{m}),t(\vec{n},\vec{m}))|^N\Big\} \tag{B.2.17}$$

where $x(\vec{n},\vec{m})$ and $t(\vec{n},\vec{m})$ are defined in (B.2.10). By (B.2.11) we deduce that $\rho(x(\vec{n},\vec{m}),t(\vec{n},\vec{m})) \neq 0$, and (B.2.16) implies that

$$\operatorname{meas}\mathcal{B}_{\vec{c},\vec{d},\vec{m},\vec{n}}(\alpha,N) \le C\alpha^\delta|\rho(x(\vec{n},\vec{m}),t(\vec{n},\vec{m}))|^{N\delta}. \tag{B.2.18}$$

Consider the set

$$\mathcal{K}(\alpha,N) := \bigcup_{\substack{\vec{n}=(n_1,\ldots,n_{\mathtt{A}})\in\mathbb{N}^{\mathtt{A}},\, 1\le n_1<\ldots<n_{\mathtt{A}} \\ \vec{m}=(m_1,\ldots m_{\mathtt{B}})\in\mathbb{N}^{\mathtt{B}} \\ \vec{c}\in(\mathbb{Z}\setminus\{0\})^{\mathtt{A}},\, |\vec{c}|_\infty\le \mathtt{M} \\ \vec{d}\in(\mathbb{Z}\setminus\{0\})^{\mathtt{B}},\, |\vec{d}|_\infty\le\mathtt{M}}} \mathcal{B}_{\vec{c},\vec{d},\vec{m},\vec{n}}(\alpha,N) \subset \mathcal{I}. \tag{B.2.19}$$

By (B.2.18) and (B.2.11) one gets

$$\operatorname{meas}\mathcal{K}(\alpha, N) \le C(\mathtt{A},\mathtt{M})\alpha^{\delta} \sum_{\substack{n_1,\ldots,n_{\mathtt{A}}\in\mathbb{N} \\ m_1,\ldots,m_{\mathtt{B}}\in\mathbb{N}}} \frac{1}{(\sum_{a=1}^{\mathtt{A}} n_a + \sum_{b=1}^{\mathtt{B}} m_b)^{N\delta}} \le C'(\mathtt{A},\mathtt{M})\alpha^{\delta} \tag{B.2.20}$$

for some finite constant $C'(\mathtt{A},\mathtt{M}) < +\infty$, provided $N\delta > \mathtt{A}+\mathtt{B}$. We fix

$$\underline{N} := [(\mathtt{A}+\mathtt{B})\delta^{-1}] + 1 \quad \text{and define} \quad \mathcal{K}_{\alpha} := \mathcal{K}(\alpha, \underline{N})$$

whose measure satisfies $|\mathcal{K}_{\alpha}| = O(\alpha^{\delta})$, in view of (B.2.20).

In conclusion, for any $\kappa \in \mathcal{I} \setminus \mathcal{K}_{\alpha}$, for any $\vec{n} = (n_1, \ldots, n_{\mathtt{A}})$ with $1 \le n_1 < \ldots < n_{\mathtt{A}}$ and $\vec{m} \in \mathbb{N}^{\mathtt{B}}$, any $\vec{c} \in (\mathbb{Z} \setminus \{0\})^{\mathtt{A}}$ with $|\vec{c}|_{\infty} \le \mathtt{M}$ and $\vec{d} \in (\mathbb{Z} \setminus \{0\})^{\mathtt{B}}$ with $|\vec{d}|_{\infty} \le \mathtt{M}$, it results, by (B.2.19) and (B.2.17), that

$$|f_{\vec{c},\vec{d}}(x(\vec{n},\vec{m}), t(\vec{n},\vec{m}), \kappa)| > \alpha |\rho(x(\vec{n},\vec{m}), t(\vec{n},\vec{m}))|^{\underline{N}} \overset{(B.2.11)}{\ge} \frac{c(\mathtt{A})\alpha}{\left(\sum_{a=1}^{\mathtt{A}} n_a + \sum_{b=1}^{\mathtt{B}} m_b\right)^{\tau_2 \underline{N}}} . \tag{B.2.21}$$

Recalling the definition of $f_{\vec{c},\vec{d}}$ in (B.2.8) and $x_0(\vec{n},\vec{m})$ in (B.2.4), the lower bound (B.2.21) implies (B.2.2) with $\tau := \tau_2\underline{N} - 3$ (cfr. (B.2.6)) and redenoting $\alpha c(\mathtt{A}) \rightsquigarrow \alpha$. $\square$

Appendix C

Derivation of water waves equation with vorticity

The water waves in the domain $\mathcal{D}_\eta$ defined in (4.1.1) is described by the free surface $\eta(t,x)$ and the velocity field $(u(t,x,y), v(t,x,y))$. The equation of motions are the mass conservation and Euler's equations

$$\begin{cases} \operatorname{div} \vec{u} = 0 \\ \partial_t \vec{u} + \vec{u} \cdot \nabla \vec{u} = -\nabla P - g \mathbf{e}_y \end{cases} \tag{C.0.1}$$

where $P(t,x,y)$ denotes the pressure and g the gravity. They read in components, denoting $\vec{u} := \begin{pmatrix} u \\ v \end{pmatrix}$, as

$$\begin{cases} u_x + v_y = 0 \\ u_t + uu_x + vu_y = -P_x \\ v_t + uv_x + vv_y = -P_y - g\,. \end{cases} \qquad \text{in } \mathcal{D}_\eta \tag{C.0.2}$$

The boundary conditions are

$$\begin{cases} v = \eta_t + u\eta_x & \text{at } y = \eta(t,x) \\ v \to 0 & \text{for } y \to -\mathtt{h} \\ P = P_0 - \kappa \left(\dfrac{\eta_x}{\sqrt{1+\eta_x^2}} \right)_x & \text{at } y = \eta(t,x)\,. \end{cases} \tag{C.0.3}$$

The second condition in (C.0.3) at the bottom is equivalent to

$$\begin{cases} v(t,x,-\mathtt{h}) = 0\,, & \text{if } \mathtt{h} \text{ is finite}\,, \\ \lim_{y \to -\infty} v(t,x,y) = 0 & \text{if } \mathtt{h} = +\infty\,. \end{cases}$$

Taking the rotor of the Euler equation (C.0.1) we obtain that vorticity

$$\operatorname{rot} \vec{u} := \omega := v_x - u_y$$

evolves according to the Helmotz equation

$$\partial_t \omega + (u\partial_x + v\partial_y)\omega = 0\,. \tag{C.0.4}$$

We assume that the vorticity of the vector field $\vec{u}$ is constant

$$\omega := v_x - u_y = \gamma\,. \tag{C.0.5}$$

Remark C.0.1. Notice that by (C.0.4), if initially the vorticity $\omega_{|t=0} = \gamma$ is constant then $\omega = \gamma$ at any time t.

Moving frame. We can regard these equations in a frame moving horizontally with an arbitrary constant speed c. The new variables

$$\begin{aligned} \widetilde{u}(t,x,y) &:= u(t,x+ct,y) - c \\ \widetilde{v}(t,x,y) &:= v(t,x+ct,y) \\ \widetilde{\eta}(t,x) &:= \eta(t,x+ct) \\ \widetilde{P}(t,x,y) &:= P(t,x+ct,y) \end{aligned} \tag{C.0.6}$$

satisfy the same equations (C.0.1)-(C.0.3). This means that we can always add an arbitrary constant c to the horizontal component of the velocity field.

Lemma C.0.2. $\int_0^{2\pi} v(t,x,y)\,\mathrm{d}x = 0$ *for all* t *and* $y < -1$.

Proof. Notice that, by the divergence free condition $u_x + v_y = 0$ in (C.0.2) we have

$$\partial_y \int_0^{2\pi} v(t,x,y)\,\mathrm{d}x = \int_0^{2\pi} -u_x(t,x,y)\,\mathrm{d}x = 0$$

by the 2π-periodicity of u. Hence

$$\int_0^{2\pi} v(t,x,y)\,\mathrm{d}x = \lim_{y\to -\mathtt{h}} \int_0^{2\pi} v(t,x,y)\,\mathrm{d}x = 0$$

by the second boundary condition in (C.0.3). □

By Lemma C.0.3 below we have that there exists a constant $\mathtt{c}(t)$ and potential $\Phi(t,x,y)$, 2π-periodic in x, such that

$$\begin{aligned} u(t,x,y) &= \mathtt{c} - \gamma y + \Phi_x(t,x,y) \\ v(t,x,y) &= \Phi_y(t,x,y) \end{aligned} \tag{C.0.7}$$

where

$$\mathtt{c} := \frac{1}{2\pi}\int_0^{2\pi} u(t,x,y)\mathrm{d}x + \gamma y\,, \quad \forall y < -1\,. \tag{C.0.8}$$

Notice that $\mathtt{c}$ is independent of $y < -1$. Actually $\mathtt{c}$ is constant in t.

Lemma C.0.3. $\partial_t \mathtt{c} = 0$.

Proof. By differentiating (C.0.8) and using the second equation in (C.0.2) we get

$$\begin{aligned} \partial_t \mathtt{c}(t) &:= \frac{1}{2\pi}\int_0^{2\pi} \partial_t u(t,x,y)\mathrm{d}x = \frac{1}{2\pi}\int_0^{2\pi} -\frac{1}{2}(u^2)_x - v u_y - P_x \mathrm{d}x \\ &= \frac{1}{2\pi}\int_0^{2\pi} -v(v_x - \gamma)\mathrm{d}x \overset{(C.0.5)}{=} \frac{\gamma}{2\pi}\int_0^{2\pi} v\mathrm{d}x = 0 \end{aligned}$$

by Lemma C.0.2. □

Thus, in view of (C.0.6) about the moving frame, i.e. substituting u, v, η, P with $\widetilde{u}, \widetilde{v}, \widetilde{\eta}, \widetilde{P}$ with $c = \mathtt{c}$ we can always assume that

$$\begin{cases} u(t,x,y) = -\gamma y + \Phi_x(t,x,y) \\ v(t,x,y) = \Phi_y(t,x,y) . \end{cases} \tag{C.0.9}$$

By (C.0.7) and since $\vec{u}$ is divergence free, it follows that

$$\Delta\Phi(t,x,y) = 0 . \tag{C.0.10}$$

We can also express the boundary conditions (C.0.3) in terms of Φ, getting

$$\begin{cases} \eta_t = \Phi_y - \Phi_x \eta_x + \gamma \eta \eta_x & \text{at } y = \eta(t,x) \\ \Phi_y \to 0 & \text{for } y \to -\mathtt{h} . \end{cases} \tag{C.0.11}$$

We define the trace at the boundary

$$\psi(t,x) = \Phi(t;x,y)|_{y=\eta} = \Phi(t;x,\eta(t,x)) . \tag{C.0.12}$$

In such a way, given η, ψ, the function Φ is recovered by solving the Dirichlet problem

$$\begin{cases} \Delta\Phi = 0 & \text{in } \mathcal{D}_\eta \\ \Phi = \psi & \text{at } y = \eta(t,x) \\ \Phi_y = 0 & \text{at } y = -\mathtt{h}. \end{cases} \tag{C.0.13}$$

Defining the Dirichlet-Neumann operator $G(\eta)\psi$ as

$$G(\eta)\psi := \sqrt{1+\eta_x^2}\,(\partial_{\vec{n}}\Phi)|_{y=\eta(t,x)} = (-\Phi_x\eta_x + \Phi_y)|_{y=\eta(t,x)} , \tag{C.0.14}$$

we get from (C.0.11) that

$$\eta_t = G(\eta)\psi + \gamma\eta\eta_x \tag{C.0.15}$$

which is the first equation in (4.1.2).

Remark C.0.4. We have that

$$G(\eta)[1] = 0 , \quad \int_{\mathbb{T}} G(\eta)[\psi]\,\mathrm{d}x = 0 .$$

Lemma C.0.5. (Stream function) *There exists Ψ on $\mathcal{D}_\eta$ such that*

$$u = \Psi_y , \quad v = -\Psi_x , \tag{C.0.16}$$

and therefore $\widetilde{\Psi} = \Psi + \frac{\gamma y^2}{2}$ solves

$$\Phi_x = \widetilde{\Psi}_y = u + \gamma y , \quad \Phi_y = -\widetilde{\Psi}_x = v . \tag{C.0.17}$$

Proof. The existence of Ψ defined in $\mathcal{D}_{\eta,h}$ and satisfying (C.0.16) follows from the classical Helmholtz decomposition of the irrotational vector field $\begin{pmatrix} -v \\ u \end{pmatrix}$. □

Remark C.0.6. Notice that the fluid particles evolve according to the time-dependent Hamiltonian system

$$\begin{cases} \dot{x} = u = \Psi_y = \partial_y(\widetilde{\Psi} - \frac{\gamma}{2}y^2) \\ \dot{y} = v = -\Psi_x = -\partial_x(\widetilde{\Psi} - \frac{\gamma}{2}y^2) . \end{cases}$$

To deduce the second equation of water waves, we start again with the Euler equation and use the vectorial identity

$$\vec{u}\cdot\nabla\vec{u} = \nabla\left(\frac{|u|^2}{2}\right) - \vec{u}\wedge \mathrm{rot}\vec{u}$$

to write the second equation of (C.0.1) as

$$\partial_t\vec{u} + \nabla\left(\frac{|u|^2}{2}\right) - \vec{u}\wedge\mathrm{rot}\vec{u} = -\nabla(P+gy)\,. \tag{C.0.18}$$

Now we use that, by (C.0.9), the velocity field $\vec{u} = \begin{pmatrix}-\gamma y\\ 0\end{pmatrix} + \nabla\Phi$, that, by (C.0.16), $|\vec{u}|^2 = |\nabla\Psi|^2$ and, finally that, by (C.0.5) and (C.0.16),

$$\vec{u}\wedge\mathrm{rot}\vec{u} = \gamma\begin{pmatrix}v\\ -u\end{pmatrix} = -\gamma\nabla\Psi\,,$$

to write (C.0.18) in terms of Φ and Ψ as

$$\partial_t\left(\nabla\Phi + \begin{pmatrix}-\gamma y\\ 0\end{pmatrix}\right) + \nabla\left(\frac{|\nabla\Psi|^2}{2}\right) + \gamma\nabla\Psi + \nabla(P+gy) = 0\,.$$

Therefore in the time dependent fluid domain we have that

$$\partial_t\Phi + \frac{|\nabla\Psi|^2}{2} + \gamma\Psi + P + gy = C(t) \tag{C.0.19}$$

for some $C(t)$, which determines the pressure in the fluid. This generalizes Bernoulli theorem for fluids with constant vorticity.

Evaluating (C.0.19) at the free surface, and imposing the last dynamic condition in (C.0.3) we obtain that

$$\partial_t\Phi + \frac{|\nabla\Psi|^2}{2} + \gamma\Psi - \kappa\left(\frac{\eta_x}{\sqrt{1+\eta_x^2}}\right)_x + g\eta = c(t) \quad \text{at } y=\eta(t,x)\,, \tag{C.0.20}$$

where $c(t) = C(t) - P_0$.

Finally we write this equation in terms of ψ and η only. We use the following two preliminary lemma. Given a 2π-periodic function $f(x)$ with zero average we define $g := \partial_x^{-1} f$ the unique 2π-periodic function with zero average such that $\partial_x g = f$.

Lemma C.0.7. *There is $c_0(t)$ such that*

$$\Psi(t,x,\eta(t,x)) = -\frac{\gamma}{2}\eta^2 - \partial_x^{-1}G(\eta)\psi + c_0(t)\,. \tag{C.0.21}$$

Proof. We have

$$\begin{aligned}\frac{d}{dx}\left(\Psi(t,x,\eta(t,x)) + \frac{\gamma}{2}\eta^2\right) &= \Psi_x(t,x,\eta(t,x)) + \Psi_y(t,x,\eta(t,x))\eta_x + \gamma\eta\eta_x\\ &= -\Phi_y(t,x,\eta(t,x)) + (\Phi_x(t,x,\eta(t,x)) - \gamma\eta(t,x))\,\eta_x + \gamma\eta\eta_x\\ &= -G(\eta)\psi\end{aligned}$$

implying (C.0.21). □

Remark C.0.8. The previous computation gives another proof that $\int_{\mathbb{T}} G(\eta)\psi \, \mathrm{d}x = 0$.

Inverting

$$\psi_x = \Phi_x + \Phi_y \eta_x \,, \quad G(\eta)\psi = \Phi_y - \Phi_x \eta_x \,, \quad \text{at } y = \eta(t,x)\,, \tag{C.0.22}$$

see (C.0.12) and (C.0.14), we get

$$\begin{cases} \Phi_x(x,\eta(x)) = \dfrac{\psi_x - \eta_x G(\eta)\psi}{1+\eta_x^2} \\ \Phi_y(x,\eta(x)) = \dfrac{\psi_x \eta_x + G(\eta)\psi}{1+\eta_x^2} \,. \end{cases} \tag{C.0.23}$$

By Lemma C.0.5 we have that, at $y = \eta$,

$$\frac{|\nabla\Psi|^2}{2} = \frac{(\Phi_x - \gamma\eta)^2 + \Phi_y^2}{2} = \gamma^2 \frac{\eta^2}{2} + \frac{\Phi_x^2 - 2\gamma\Phi_x\eta + \Phi_y^2}{2} \,. \tag{C.0.24}$$

Differentiating (C.0.12) we have, at $y = \eta(t,x)$,

$$\begin{aligned} \partial_t \psi &= \Phi_t + \Phi_y \eta_t \\ &\overset{(C.0.20),(C.0.15)}{=} -\frac{|\nabla\Psi|^2}{2} - \gamma\Psi + \kappa \left(\frac{\eta_x}{\sqrt{1+\eta_x^2}} \right)_x - g\eta + c(t) + \Phi_y (G(\eta)\psi + \gamma\eta\eta_x) \,. \end{aligned} \tag{C.0.25}$$

We now expand

$$\begin{aligned} &-\frac{|\nabla\Psi|^2}{2} - \gamma\Psi + \Phi_y(G(\eta)\psi + \gamma\eta\eta_x) \\ &\overset{(C.0.24),(C.0.21)}{=} -\gamma^2\frac{\eta^2}{2} - \frac{\Phi_x^2 - 2\gamma\Phi_x\eta + \Phi_y^2}{2} + \frac{\gamma^2}{2}\eta^2 + \gamma\partial_x^{-1}G(\eta)\psi + \Phi_y(G(\eta)\psi + \gamma\eta\eta_x) - \gamma c_0(t) \\ &= -\frac{\Phi_x^2}{2} - \frac{\Phi_y^2}{2} + \Phi_y G(\eta)\psi + \gamma\eta(\Phi_y\eta_x + \Phi_x) + \gamma\partial_x^{-1}G(\eta)\psi - \gamma c_0(t) \\ &\overset{(C.0.22)}{=} -\frac{\Phi_x^2}{2} - \frac{1}{2}\big(\Phi_y - G(\eta)\psi\big)^2 + \frac{1}{2}(G(\eta)\psi)^2 + \gamma\eta\psi_x + \gamma\partial_x^{-1}G(\eta)\psi - \gamma c_0(t) \\ &\overset{(C.0.22)}{=} -\frac{\Phi_x^2(1+\eta_x^2)}{2} + \frac{1}{2}(G(\eta)\psi)^2 + \gamma\eta\psi_x + \gamma\partial_x^{-1}G(\eta)\psi - \gamma c_0(t) \,. \end{aligned} \tag{C.0.26}$$

Finally, using (C.0.23), we check the identity

$$-\frac{\Phi_x^2(1+\eta_x^2)}{2} + \frac{1}{2}(G(\eta)\psi)^2 = -\frac{\psi_x^2}{2} + \frac{(\eta_x\psi_x + G(\eta)\psi)^2}{2(1+\eta_x^2)} \tag{C.0.27}$$

and, collecting, (C.0.25), (C.0.26), (C.0.27)

$$\psi_t = -g\eta - \frac{\psi_x^2}{2} + \frac{(\eta_x\psi_x + G(\eta)\psi)^2}{2(1+\eta_x^2)} + \kappa\left(\frac{\eta_x}{\sqrt{1+\eta_x^2}}\right)_x + \gamma\eta\psi_x + \gamma\partial_x^{-1}G(\eta)\psi + \tilde{c}(t)\,,$$

which is the second equation in (4.1.2) with ψ in the homogeneous space $\dot{H}^s$.

Remark that

$$\int_{\mathbb{T}} \eta(x)\,\mathrm{d}x$$

is a prime integral of (C.0.15). For simplicity we fix $\int_{\mathbb{T}} \eta \, \mathrm{d}x = 0$.

Remark C.0.9. Given initial data (η_0, ψ_0) we solve $(\eta(t,x), \psi(t,x))$ solving (4.1.2), then we define $\Phi(t,x,y)$ solving (C.0.13), then we define $\vec{u} = (u,v)(t,x,y)$ according to (C.0.9), which is divergence free and has vorticity γ. We notice that the boundary condition $v(t,x,-\mathtt{h}) = 0$ and so we define $\Psi(t,x,y)$ according to Lemma C.0.16. Finally (C.0.19) defines the pressure inside $\mathcal{D}_\eta$ so that the Euler equation (C.0.1) holds in $\mathcal{D}_\eta$. The first and the third kinematic and dynamics boundary conditions in (C.0.3) follow by the equations (4.1.2).

C.1 Hamiltonian formulation

We now prove that the equations (4.1.2), defined on $H_0^1 \times \dot{H}^1$ are Hamiltonian. We denote for clarity $[\psi]$ an element of $\dot{H}^1$, remind that $\psi_1 \sim \psi_2$ if and only if $\psi_1 - \psi_2 = c$. We define the symplectic form

$$\mathcal{W}\left(\begin{pmatrix}\eta_1\\ [\psi_1]\end{pmatrix}, \begin{pmatrix}\eta_2\\ [\psi_2]\end{pmatrix}\right) := \left\langle \begin{pmatrix}\gamma\partial_x^{-1} & -\mathrm{Id}\\ \mathrm{Id} & 0\end{pmatrix}\begin{pmatrix}\eta_1\\ [\psi_1]\end{pmatrix}, \begin{pmatrix}\eta_2\\ [\psi_2]\end{pmatrix}\right\rangle_{L^2}$$
$$= \langle \gamma\partial_x^{-1}\eta_1 - \psi_1, \eta_2\rangle_{L^2} + \langle \eta_1, \psi_2\rangle_{L^2} \tag{C.1.1}$$

and since η_1, η_2 have zero average it is well defined and non-degenerate on $H_0^1 \times \dot{H}^1$.

We consider the Hamiltonian

$$H(\eta,\psi) = \frac{1}{2}\int_{\mathbb{T}} (\psi G(\eta)\psi + g\eta^2)\mathrm{d}x + \kappa\int_{\mathbb{T}} \sqrt{1+\eta_x^2}\,\mathrm{d}x + \frac{\gamma}{2}\int_{\mathbb{T}} (-\psi_x\eta^2 + \frac{\gamma}{3}\eta^3)\,\mathrm{d}x\,.$$

which is well defined on $H_0^1 \times \dot{H}^1$ since $G(\eta)[1] = 0$ and $\int_{\mathbb{T}} G(\eta)\psi\,\mathrm{d}x = 0$. The associated Hamiltonian vector field is defined by the identity

$$dH(u)[\widehat{u}] = \mathcal{W}(X_H(u), \widehat{u})\,, \quad \forall u := \begin{pmatrix}\eta\\ [\psi]\end{pmatrix}, \quad \widehat{u} := \begin{pmatrix}\widehat{\eta}\\ [\widehat{\psi}]\end{pmatrix}.$$

We have[1]

$$\begin{aligned} dH(u)[\widehat{u}] &= \langle \nabla_\eta H, \widehat{\eta}\rangle_{L^2} + \left\langle \nabla_\psi H, \widehat{\psi}\right\rangle_{L^2} \\ &= \left\langle g\eta + \frac{\psi_x^2}{2} - \frac{(\eta_x\psi_x + G(\eta)\psi)^2}{2(1+\eta_x^2)} - \kappa\left(\frac{\eta_x}{\sqrt{1+\eta_x^2}}\right)_x - \gamma\psi_x\eta + \frac{\gamma^2}{2}\eta^2, \widehat{\eta}\right\rangle_{L^2} \\ &\quad + \left\langle G(\eta)\psi + \gamma\eta\eta_x, \widehat{\psi}\right\rangle_{L^2} \end{aligned}$$

and

$$\begin{aligned} \nabla_\psi H &= G(\eta)\psi + \gamma\eta\eta_x \in H_0^1(\mathbb{T}) \\ \nabla_\eta H &= \Big[\frac{\psi_x^2}{2} - \frac{(\eta_x\psi_x + G(\eta)\psi)^2}{2(1+\eta_x^2)} + g\eta - \kappa\left(\frac{\eta_x}{\sqrt{1+\eta_x^2}}\right)_x - \gamma\psi_x\eta + \frac{\gamma^2}{2}\eta^2\Big] \in \dot{H}^1\,. \end{aligned}$$

[1] Setting $K(\eta,\psi) := \frac{1}{2}\int_{\mathbb{T}} \psi G(\eta)\psi\,\mathrm{d}x$ we have, using the shape-derivative formula,

$$\begin{aligned} \nabla_\eta K &= -\frac{1}{2}\Phi_y G(\eta)\psi + \frac{1}{2}\Phi_x\psi_x\,, \quad \text{at } y = \eta(x) \\ &\overset{(C.0.23)}{=} \frac{\psi_x^2}{2} - \frac{(\eta_x\psi_x + G(\eta)\psi)^2}{2(1+\eta_x^2)} \end{aligned}$$

Comparing with (C.1.1) we see that the Hamiltonian vector field $X_H(\eta, [\psi]) \in H_0^1(\mathbb{T}) \times \dot{H}^1$ is

$$X_H \begin{pmatrix} \widehat{\eta} \\ [\widehat{\psi}] \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & \gamma\partial_x^{-1} \end{pmatrix} \begin{pmatrix} \nabla_\eta H \\ \nabla_\psi H \end{pmatrix}$$

which is system (4.1.2).

If we define the symplectic form

$$J_\gamma = \begin{pmatrix} 0 & \mathrm{Id} \\ \mathrm{Id} & \gamma\partial_x^{-1} \end{pmatrix}$$

we have

$$\partial_t \begin{pmatrix} \eta \\ \psi \end{pmatrix} = J_\gamma \begin{pmatrix} \nabla_\eta H \\ \nabla_\psi H \end{pmatrix}$$

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