Scuola Internazionale Superiore di Studi Avanzati – Trieste
Physics area
PhD course in Astroparticle Physics

DOCTORAL THESIS

Quantum Aspects of Metric-Affine Gravity

Candidate:
Oleg Igorevich Melichev

Advisor:
Prof. Roberto Percacci

Academic Year 2021-2022

MA PER SEGUIR VIRTUTE E CANOSCENZA

SISSA

To the memory of my grandmother Rimma

Foreword

Gravitational theories with independent metric and affine connection, which are commonly referred to as Metric-Affine gravity theories (MAGs), remain of interest as alternative formulations or extensions of General Relativity (GR). In general, they may possess a nonvanishing torsion, nonmetricity, curvature, or any combination thereof. Torsion, the antisymmetric part of the affine connection, was first used by A. Einstein to formulate what is nowadays known as the Teleparallel Equivalent of General Relativity (TEGR) [1–4]. In this theory curvature and nonmetricity tensors are assumed to vanish a priori, and when one solves this constraint for the connection the usual GR dynamics can be recovered. In GR on the other hand, torsion and nonmetricity tensors are assumed to vanish a priori.

One can also construct a theory of gravity based entirely on nonmetricity, with vanishing torsion and curvature (STEGR). Another possibility consists of having both torsion and nonmetricity, with vanishing curvature (GTEGR). In all three cases we mentioned, Einstein equations can be recovered, and the requirement of achieving that fixes the action uniquely. These facts are appreciated as the Geometrical Trinity of Gravity [4].

Recently, a substantial amount of work has been done on phenomenological applications of various versions of MAG and its subclasses (such as Einstein-Cartan and Teleparallel theories) [5–15]. Indeed, when a specific form of classical action is chosen, one can study the dynamics and deviations from GR. MAGs have a large potential to resolve multiple cosmological and gravitational conundrums, such as black hole and Big Bang singularity problems [5, 16], describe Dark Energy and/or Dark Matter [5]. It is of interest to understand whether or not it is possible to arrive at such a version of MAG that would remain viable when quantum corrections are taken into account. In this thesis, we endeavour to pursue this direction.

Another motivation for studying MAGs is that, in many ways and more than GR, they resemble the theories of the other fundamental interactions. When considered in this context they are called "gauge theories of gravity", where one tries to apply ideas and tools of Yang-Mills theories to gravity. There exists a hope that this enhanced similarity is a step towards a possible unification [17]. We refer to [18] for a useful collection of references, covering also the history of the subject.

In this thesis, we adopt a modest approach to quantum gravity, in which we assume that our models are unlikely to be valid above the Planck scale, and have to be replaced by a yet unknown theory, different from those based on the conservative approach to quantisation.

In most cases existing in the literature, the lowest order Lagrangian is considered, which leads to torsion and nonmetricity being non-dynamical. In order to conduct a systematic study of MAGs with propagating torsion and nonmetricity, one needs to consider operators of order four in mass dimension. In this thesis, we consider classical and quantum properties of a general theory with propagating torsion and nonmetricity fields. In particular, we will perform several computations of local one-loop counterterms. We will also discuss the spectrum of MAG and the effect of field redefinitions.

Computing gravitational loop corrections is notoriously difficult. Furthermore, depending on the choice of the action, metric gravity is either power-counting nonrenormalisable [19] (for the lowest-order Hilbert–Einstein term) or yields an apparent violation of unitarity by the presence of ghosts [20] (when higher-order curvature terms are considered). Nevertheless, there remains a hope that a unitary renormalisable theory of gravity can be found within the quantum field theory domain [21–24].

A popular direction to explore is Asymptotic Safety, a conjecture that implies that when the renormalisation group (RG) running of the couplings is under control even at arbitrarily high energies (usually due to the existence of one or several fixed points) the theory can be renormalisable nonperturbatively [25–27] That means that conventional perturbative renormalisability is abandoned, but theory remains predictive. Gravity with non-propagating torsion has been also studied in this context and nonperturbative RG flow has been found [28, 29].

On the other side, one may ask whether renormalizability is necessary in the first place. In the framework of Effective Field Theory (EFT), one can assume that there exists a physical cutoff scale (for example, the Planck mass), above which the theory is not applicable. Effective action (EA) is then seen as an expansion in mass dimensions, and higher-order contributions are suppressed by powers of the cutoff. Despite being non-renormalisable, metric gravity as an EFT is self-consistent and predictive, for quantum corrections affecting low-energy observables can be unambiguously calculated [30]. Computation of one-loop EA is relevant regardless of whether gravity will eventually be understood as an EFT or instead a local, unitary and renormalisable theory will be found. We will assume that perturbation theory is applicable, at least in some energy region.

Another issue worth mentioning is the unitarity problem. Metric-affine gravity, as well as the usual metric gravity, is in general plagued with ghosts, which are solutions with negative kinetic energy. At the classical level, they would render theory unstable. However, many authors have claimed that at the quantum level the apparent violation of unitarity does not actually occur [31–37]. Alternatively, one can impose additional symmetries, so that the ghosts do not enter the spectrum [38]. Assuming certain relations between the couplings may also solve the problem [39–42], however, one has to be careful and introduce them in such a way that would not be spoiled by quantum corrections. On the other hand, from the EFT perspective, ghosts do not represent an issue if the masses thereof are larger than the cutoff scale. In this work, when computing the RG flow, we will ignore the ghost problem, assuming that we work in a situation when they are absent or harmless for a reason which is yet to be understood.

A systematic study of gravity with propagating torsion dates back at least to [39, 43]. A comprehensive study of independent Lagrangian contributions has been performed in [44, 45]. Studies of particular cases and issues of unitarity can be found in [40, 46–51]. A first computation of one-loop divergences in gravity with propagating torsion on the flat background was performed in [52]. Non-perturbative beta functions for MAG with non-propagating torsion and nonmetricity were found in [28, 29].

In this work, we will look at the case when kinetic terms for torsion and nonmetricity are also added to the action. Since the connection has a dimension of mass, they naturally come as operators of dimension four. A major difficulty with such operators is that they often have nonminimal structure. We will call operator minimal if all its derivatives are contracted with each other, and nonminimal otherwise. In general, at order 4 in mass dimension, MAG has 11 minimal and 27 nonminimal contributions of the type $(\nabla T)^2$, $(\nabla Q)^2$ and $\nabla T \nabla Q$. Unless prohibited by symmetry or can be reabsorbed into field redefinitions, all these terms must be taken into consideration. Furthermore, additional terms of types RT^2, $T^2\nabla T$, T^4, etc. can be considered as some kind of complicated potential, for they do not contribute to the flat space 2-point function. In this work, we will mostly disregard this potential.

This thesis is organised into three major directions. In the first direction, MAG is considered from the Effective Field Theory (EFT) perspective. Independent even-parity contributions to the Lagrangian of mass dimension up to 4 have been classified. The focus is given to those, which can contribute to the flat space 2-point function. The flat space kinetic operator was decomposed into spin projectors. The particle content of MAG was discussed in detail. It is shown that any metric theory of gravity has a teleparallel equivalent. This direction occupies chapters 2,3 and 4.

The second direction is devoted to the covariant quantisation of Einstein gravity and Unimodular gravity. It is known that Einstein and Unimodular theories of gravity are equivalent at the classical level. A quantisation procedure that keeps the divergences in these two formulations equal is presented, thus proving a formal quantum equivalence between them. The one-loop divergences have been computed. It is shown that any distinction between Unimodular and Einstein gravities can be merely attributed to different quantisation procedures. This direction occupies chapter 6.

The third direction is devoted to the renormalisation of MAG. An important consistency criterion of a quantum theory is our ability to keep the RG flow of its couplings under control. The off-diagonal heat kernel technique was used to compute the logarithmically divergent part of the one-loop effective action of a Poincare gauge theory. It is proven that off-shell renormalisability requires adding to the usual Poincare gauge action terms of the form $(\nabla T)^2$. This direction occupies chapters 7 and 8.2

Chapter 1 does not contain original results. Chapters 2, 3, 4, 5, 6, 7, 8 do contain original results.

Acknowledgements

First of all, I would like to thank my supervisor Prof. Roberto Percacci for his guidance, support and a multitude of helpful discussions during the last four years. I am grateful to Kevin Falls and Alessio Baldazzi, with whom I have been collaborating the most and from whom I have learnt a lot. I would also like to thank Gustavo Pazzini de Brito and Antonio Duarte Pereira, collaboration with whom has been of great significance. I am thankful to Christian Steinwachs, Yegor Zenkevich and Yannick Kluth for multiple helpful discussions of the heat kernel technique and the renormalisation group.

I am indebted to all my friends who supported me on this journey: Alessandro Granelli, Katya Kravtsova, Natalia Nazarova, Elena Lukzen, Maddalena Premuti Bonetta, Youness Diouane, Dmitrii Rachenkov, Sara Folchini and others. A special thank goes to Pavel Novichkov, who inspired me to learn computer algebra early on and without whom this journey would have been very different. I thank Miguel Vanvlasselaer, Lotte ter Haar and José Guillermo Lara Delgado with whom I had the pleasure to share an office for many years, and the country of Italy, for kind hospitality.

Notation and conventions

We use standard GR notation for the Levi-Civita connection and standard Yang-Mills notation for the dynamically independent connection, as in the following table

	coefficients	covariant derivative	curvature
LC connection	$\Gamma_\mu{}^\rho{}_\sigma$	∇_μ	$R_{\mu\nu}{}^\rho{}_\sigma$
Independent connection	$A_\mu{}^\rho{}_\sigma$	D_μ	$F_{\mu\nu}{}^\rho{}_\sigma$

We will use same symbol for a given geometrical object in any frame, thus for example $A_\mu{}^\rho{}_\sigma$ are the connection coefficients in a coordinate frame and $A_\mu{}^a{}_b$ are the connection coefficients in a frame (1.1). The action of the covariant derivative on a tensor of rank (n, m) is

$$D_\mu T^{\alpha_1\ldots\alpha_n}_{\beta_1\ldots\beta_m} = \partial_\mu T^{\alpha_1\ldots\alpha_n}_{\beta_1\ldots\beta_m} + \sum_{i=1}^{n} A_\mu{}^{\alpha_i}{}_\sigma T^{\alpha_1\ldots\sigma\ldots\alpha_n}_{\beta_1\ldots\beta_m} - \sum_{j=1}^{m} A_\mu{}^\sigma{}_{\beta_j} T^{\alpha_1\ldots\alpha_n}_{\beta_1\ldots\sigma\ldots\beta_m}$$

The torsion tensor is antisymmetric in the first and the last indices:

$$T_\mu{}^\rho{}_\nu = T_{[\mu}{}^\rho{}_{\nu]} \ ,$$

whereas the nonmetricity tensor is symmetric in the last two indices:

$$Q_{\lambda\mu\nu} = Q_{\lambda(\mu\nu)} \ .$$

In order to identify more easily expressions involving the same tensors with indices contracted in different ways, it proves convenient to use the following notation. Given a tensor ϕ_{abc}, we define

$$\begin{aligned}
&\mathrm{tr}_{(12)}\phi_c \equiv \phi^{(12)}_c = \phi_a{}^a{}_c \ ,\\
&\mathrm{tr}_{(13)}\phi^b \equiv \phi^{(13)b} = \phi_a{}^{ba} \ , \text{etc.}\\
&\mathrm{div}_{(1)}\phi^b{}_c = \nabla_a\phi^{ab}{}_c \ ,\\
&\mathrm{div}_{(2)}\phi_{ac} = \nabla_b\phi_a{}^b{}_c \ , \ \text{etc.}\\
&\mathrm{div}_{(23)}\phi_c = \nabla_a\nabla_b\phi^{ab}{}_c \ , \ \text{etc.}\\
&\mathrm{trdiv}_{(1)}\phi = \mathrm{div}_{(1)}\phi^a{}_a \ ,\\
&\mathrm{div}\,\mathrm{tr}_{(12)}\phi = \nabla_a\mathrm{tr}_{(12)}\phi^a \ , \ \text{etc.}
\end{aligned}$$

Note that with the LC connection $\mathrm{div}\,\mathrm{tr}_{(12)}\phi = \mathrm{trdiv}_{(3)}\phi$, etc.

When the divergence is calculated with the independent dynamical connection A, it will be written as "Div". In this case, one has to be more careful about raising and lowering indices, because the covariant derivative of the metric may not be zero. Then one has to make conventions, for example, $\mathrm{Div}_{(1)}\phi^b{}_c = D_a(g^{ad}\phi_d{}^b{}_c)$ or $\mathrm{Div}_{(1)}\phi^b{}_c = g^{ad}D_a\phi_d{}^b{}_c$.

$$\sqrt{g} = \sqrt{|det(g_{\mu\nu})|}$$

List of abbreviations

EA - Effective action

MAG - Metric-Affine gravity

PGT - Poincaré gauge theorie

RG - renormalisation group

GR - General Relativity

EFT - Effective Field Theory

LC - Levi-Civita (connection or corresponding covariant derivative)

QFT - quantum field theory

4DG, HDG - Quadratic Four-Derivative gravity (High-Derivative gravity)

UV - ultraviolet

dof's - degrees of freedom

FRG - Functional Renormalisation Group

Contents

Abstract

We discuss theories of gravity with independent metric and affine connection. We count the parity-even Lagrangian terms of dimension up to four and give explicit bases for the independent terms that contribute to the two-point function. We then give the decomposition of the linearised action on a complete basis of spin projectors and consider various subclasses of MAGs. We show that teleparallel theories can be dynamically equivalent to any metric theory of gravity and give the particle content of those whose Lagrangian contains only dimension-two terms. We point out the existence of a class of MAGs whose EOMs do not admit propagating degrees of freedom. Finally, we construct simple MAGs that contain only a massless graviton and a state of spin/parity 2^- or 3^-. As a side result, we write the relativistic wave equation for a spin/parity 2^- state. Additionally, we perform an irreducible decomposition of torsion and nonmetricity with respect to the group of permutations and show how the basis of independent terms in the classical action can be rewritten via decomposed fields.

Poincaré gauge theories are a class of metric-affine theories with a metric-compatible (i.e. Lorentz) connection and with an action quadratic in curvature and torsion. We show by an explicit one-loop calculation that this class of theories is not closed under renormalisation off-shell. This statement extends to more general classes of metric-affine theories. We, therefore, generalise them to include other necessary terms. We discuss how their spectrum can be affected by quantum corrections. We prove that at the perturbative level, all local counterterms that may affect the flat-space propagator can be reabsorbed into appropriate invertible field redefinitions.

We formally prove the existence of a quantisation procedure that makes the path integral of a general diffeomorphism-invariant theory of gravity, with fixed total spacetime volume, equivalent to that of its unimodular version. This is achieved by means of a partial gauge fixing of diffeomorphisms together with a careful definition of the unimodular measure. The statement holds also in the presence of matter. As an explicit example, we consider scalar-tensor theories and compute the corresponding logarithmic divergences in both settings. In spite of significant differences in the coupling of the scalar field to gravity, the results are equivalent for all couplings, including non-minimal ones.

Chapter 1

Introduction

1.1 Gauge group

Any consistent theory must give the same physical predictions independently of field redefinitions. Such redefinitions may or may not be local or linear. It is only required that they are invertible. Geometrically speaking, physical theories are defined on a manifold M and fields belong to a vector bundle E over M. Field redefinitions are then invertible maps between sections of different vector bundles. Observables are always independent of such transformations.

When talking about (quantum) field theory, there exists an important class of local field redefinitions that keep the Lagrangian unchanged. They are called gauge symmetries and characterise the degeneracy of a Lagrangian system. For many modern physical theories gauge symmetries are necessary to make the Lagrangian local. However, in certain situations, one still has some sort of freedom to change the gauge group. In the context of gravity, the group that is used normally is the group space-time diffeomorphisms, which we will refer to as the diffeomorphisms (*Diff*). At least at low energies, gravitational interaction must be carried by a massless particle of spin 2 for it to always be an attractive force. It leads to the diffeomorphism group being a necessity. However, there exists a possibility to shrink the gauge algebra to only include transverse shifts, while still keeping the Lagrangian local. In this case, the determinant of the metric is kept unchanged. This theory of gravity is called Unimodular and the corresponding group is called the special diffeomorphism group (*SDiff*).

On the other hand, one can also enlarge the gauge group. In fact, as we will argue in the following, the way in which GR is defined already requires a larger local symmetry group $\mathbb{R}^{1,3} \rtimes GL(4)$ [21]. The caveat here is that in the metric formulation (as opposed to the vierbein formulation) the subgroup $GL(4)$ is trivially realised. To see that, let us look at how the gravitational field is defined. All fundamental interactions, such as the strong interactions described by a Yang-Mills field, are defined on vector bundles over a four-dimensional manifold M. A special characteristic of the gravitational field that distinguishes it from other interactions is that it belongs to a vector bundle with fibres in $\mathbf{R}^4$ that is isomorphic to the tangent bundle TM. This automatically implies that we can define the following:

- a metric on E which is called fibre metric γ,
- a linear connection in E, $A_\mu{}^a{}_b$,
- a linear isomorphism between the fibre bundle and the tangent bundle, which is called a

soldering form or frame field $\theta^a{}_\mu$.

Hereafter we adopt a distinction in the notation of indices so that the Greek ones always enumerate coordinates on the tangent bundle and the Latin ones – on the fibre bundle. Metric γ belongs to the coset space $GL(4)/O(1,3)$. The fibre connection can be seen as a Yang-Mills field that however lies in an adjoint representation of a noncompact group $GL(4)$. Alternatively one can define soldering using arbitrary bases $\{e_a\}$ in the tangent spaces and $\{e^a\}$ in the cotangent spaces. Given a coordinate system x^μ, they are related to the coordinate bases by

$$e_a = \theta_a{}^\mu \partial_\mu \ , \quad e^a = dx^\mu \theta^{-1}{}_\mu{}^a \ . \tag{1.1}$$

Then, we can construct a metric and connection on the tangent bundle TM as:

$$g_{\mu\nu} = \theta^a{}_\mu \, \theta^b{}_\nu \, \gamma_{ab} \ , \tag{1.2}$$

$$A_\lambda{}^\mu{}_\nu = \theta_a{}^\mu A_\lambda{}^a{}_b \theta^b{}_\nu + \theta_a{}^\mu \partial_\lambda \theta^a{}_\nu \ . \tag{1.3}$$

Hereafter we abbreviate the inverse of soldering (coframe field) as

$$\theta^\mu{}_a = \theta^{-1\mu}{}_a = \gamma_{ab}\theta^a{}_\nu g^{\nu\mu} \ . \tag{1.4}$$

The existence of soldering as a direct consequence of isomorphic mapping between E and TM is gravity's biggest peculiarity and curse. To see that, let us move on to the definition of field strength and have a first glimpse at the equations of motion. We define the curvature of the fibre bundle as:

$$F_{\mu\nu}{}^a{}_b = \partial_\mu A_\nu{}^a{}_b - \partial_\nu A_\mu{}^a{}_b + A_\mu{}^a{}_c A_\nu{}^c{}_b - A_\nu{}^a{}_c A_\mu{}^c{}_b \ , \tag{1.5}$$

which represents the strength of the connection field. Then, the standard Yang-Mills action will have the following form:

$$S = \frac{1}{4g^2} \int d^4x \ F_{\mu\nu}{}^a{}_b F_{\rho\lambda}{}^c{}_d g^{\mu\rho} g^{\nu\lambda} \gamma_{ac} \gamma^{bd} \ . \tag{1.6}$$

Such action leads to different degrees of freedom having the opposite signs before the kinetic terms. We see that, in addition to (1.6), there are different ways to write the action by contracting the indices in a different way. In fact, there are 16 such contributions (more on counting independent terms later). Furthermore, owing to the existence of the soldering, the term linear in curvature is also allowed:

$$S = \int d^4x \ F_{\mu\nu}{}^a{}_b \theta_a{}^\mu \theta^b{}_\rho g^{\rho\nu} \ . \tag{1.7}$$

It is called the Palatini action. We will discuss the difference between the dynamics that it would lead to and the usual dynamics of GR later on.

Alongside with curvature, we introduce two other independent characteristics of the fibre bundle. Torsion is defined as the exterior covariant derivative of the soldering:

$$T_\mu{}^a{}_\nu = \partial_\mu \theta^a{}_\nu - \partial_\nu \theta^a{}_\mu + A_\mu{}^a{}_b \, \theta^b{}_\nu - A_\nu{}^a{}_b \, \theta^b{}_\mu \ . \tag{1.8}$$

And nonmetricity is defined as

$$Q_{\lambda ab} = -\nabla_\lambda \gamma_{ab} = -\partial_\lambda \gamma_{ab} + A_\lambda{}^c{}_a \, \gamma_{cb} + A_\lambda{}^c{}_b \, \gamma_{ac} \ . \tag{1.9}$$

Given the soldering, any tensorial index can be moved from the fibre bundle to the tangent bundle and vice versa. For example,

$$F_{\mu\nu}{}^{\alpha}{}_{\beta} = F_{\mu\nu}{}^{a}{}_{b}\theta_{a}{}^{\alpha}\theta^{b}{}_{\beta} \tag{1.10}$$

is the curvature of the tangent bundle.

Let us now come back to the discussion of the gauge group. An arbitrary non-degenerate $\Lambda^{a}{}_{b}(x)$ describes a local changes of frame $e'_a(x) = e_b(x)\Lambda^{a}{}_{b}(x)$, which is independent of the diffeomorphisms $x'(x)$. The action of these transformations on the fields is given by

$$\begin{aligned} \theta^{a}{}_{\mu}(x) \mapsto& \theta'^{a}{}_{\mu}(x') = \Lambda^{-1a}{}_{b}(x)\,\theta^{b}{}_{\nu}(x)\frac{\partial x^{\nu}}{\partial x'^{\mu}} \ , \\ \gamma_{ab}(x) \mapsto& \gamma'_{ab}(x') = \Lambda^{c}{}_{a}(x)\,\Lambda^{d}{}_{b}(x)\,\gamma_{cd}(x) \ , \\ A_{\mu}{}^{a}{}_{b}(x) \mapsto& A'_{\mu}{}^{a}{}_{b}(x') = \frac{\partial x^{\nu}}{\partial x'^{\mu}}\left[\Lambda^{-1a}{}_{c}(x)A_{\nu}{}^{c}{}_{d}(x)\Lambda^{d}{}_{b}(x) + \Lambda^{-1a}{}_{c}(x)\partial_{\nu}\Lambda^{c}{}_{b}(x)\right] \ . \end{aligned} \tag{1.11}$$

There are two common ways to fix this gauge:

- *Metric gauge*, in which we demand

  $$\theta_{a}{}^{\mu} = \delta^{\mu}_{a} \ . \tag{1.12}$$

  After that one can just stop making any distinction between tangent and fibre bundle indices. This is the way in which gravity is normally described. Torsion then becomes a purely algebraic object:

  $$T_{\mu}{}^{\rho}{}_{\nu} = A_{\mu}{}^{\rho}{}_{\nu} - A_{\nu}{}^{\rho}{}_{\mu} \tag{1.13}$$

  whereas nonmetricity involves a derivative of g.

- In *vierbein gauge* we instead demand

  $$g_{ab} = \eta_{ab} \ . \tag{1.14}$$

  Then (1.2) becomes the defining relation for the tetrad (vierbein) and the connection in this case is called the "spin connection". [1] In this gauge the nonmetricity is a purely algebraic object:

  $$Q_{cab} = A_{cab} + A_{cba} \tag{1.15}$$

  whereas torsion still involves a derivative of θ.

In the following we will also adopt more elaborate gauges, however, in the majority of this thesis we will stick to the metric gauge, which will be implicitly assumed when formulae are written with Greek indices only, unless otherwise stated.

In GR one *a priori* assumes that torsion and nonmetricity vanish everywhere:

$$Q_{\mu\nu\rho} = T_{\mu\nu\rho} = 0 \ . \tag{1.16}$$

This implies that the theory propagates only metric degrees of freedom. In the following sections, we will explain why such constraints are unnecessary from the conceptual point of view, for they can be enforced *automatically* as a consequence of mass suppression of the additional degrees of freedom, that happens everywhere except a measure zero subset of the theory space. This implies that the correct dynamics describing gravity at high energies may differ from the standard spin-2 dynamics of GR and may involve independent connection degrees of freedom.

[1] We stick to the convention that the components of the same geometrical object in different bases should not be given different names.

1.2 Gravity in EFT perspective

According to conventional wisdom, general relativity does not make sense as a quantum theory. This statement originates from futile attempts to treat gravity in a fully renormalisable way. If one tries to "unify" gravity with other interactions within the QFT framework, demanding that the resulting theory is applicable to arbitrarily high energies, renormalisability is a necessary requirement for consistency. On the other hand, there exists a natural gravitational energy scale - the Planck mass, demanding the applicability of a "quantum theory of gravity", or quite possibly quantum theory at all, above which may seem to be a somewhat naïve goal. Instead, we will adopt the view in which we expect our theories to be applicable only up to some cutoff scale Λ. It can be comparable to the Planck mass or be much higher. We demand though that Λ does not enter in final expressions for any observable quantities. In other words, for any physical quantity its change coming from variation of Λ within a certain finite range must be zero.

The fundamental mathematical object in this perspective is the *Wilsonian* effective action. Let us suppose that we can decompose quantum fields in "light" and "heavy" components, and only the former ones can propagate below the cutoff scale and be described by our theory, while the latter ones can only enter as residual quantum corrections to the couplings. They can be seen as integrated out virtual heavy particles, if the conventional notion of a particle is applicable above Λ, or some other corrections of unknown nature. The Wilsonian effective action is defined as the part of the one-particle-irreducible quantum effective action that remains after all "heavy" components are integrated out. The idea that such an operation should be attainable *in principle* stems from a simple philosophical conjecture that low-energy physics should be tractable without any knowledge of higher-energy physics. It would be very surprising if low-energy degrees of freedom of GR cannot at all be treated quantum mechanically.

Inside the Wilsonian EA, terms are usually arranged according to their mass dimension. Given a heavy mass scale, both momenta and the values of fields within the applicability range are expected to be small compared to it.[2] This leads to terms of higher dimensions being suppressed compared to the low dimensional ones. Even staying within classical framework, dimensional considerations provide a systematic way to arrange action contributions. We write down the terms up to mass dimension four as

$$S_{HDG} = \frac{1}{2}\int d^4x \,\left[2\Lambda - m_P^2 R + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu} + \theta E\right] \,. \tag{1.17}$$

Here $E = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\lambda}R^{\mu\nu\rho\lambda}$ is the Euler (Gauß–Bonnet) invariant that does not contribute to the equations of motion, and m_P is the Planck mass. Another convenient way to represent the same action is

$$S_{HDG} = -\frac{1}{2}\int d^4x \left[2\Lambda - m_0^2 R + \frac{1}{\xi}R^2 + \frac{1}{2\lambda}C_{\mu\nu\rho\lambda}C^{\mu\nu\rho\lambda} + \frac{1}{\rho}E\right] \,, \tag{1.18}$$

where

$$C_{\mu\nu}{}^{\rho\lambda} = R_{\mu\nu}{}^{\rho\lambda} - 2\delta_{[\mu}{}^{[\rho}R_{\nu]}{}^{\sigma]} + \frac{1}{3}R\delta_{[\mu}{}^{\rho}\delta_{\nu]}{}^{\sigma} \tag{1.19}$$

[2] In principle, the values of the fields do not have to be small; but if the gradients are small everywhere it should be possible to normalise the fields in such a way that makes them small.

is the Weyl tensor. This form provides more clarity because different particles are contained in terms. The R^2 term propagates an additional scalar, whereas the Weyl tensor squared term propagates a massive ghost. Graviton, however, is contained in all of them. This theory admits perturbative treatment when $\lambda \ll 1$. It is obvious that in the limit $R \ll m_P$ gravity is described by the Hilbert–Einstein action because additional contributions are strongly suppressed. Similar logic applies to the Wilsonian EA. In that case, quantum corrections coming from heavy modes can affect the behaviour of the couplings. If such effects were found to be of the same order, that would mean that it is impossible to obtain any knowledge about the quantum behaviour of gravity at energies below the Planck mass without a full understanding of trans-Planckian physics. We instead adopt a more optimistic narrative that, in similarity with all other known physical systems, the opposite is the case. It is possible that the field theory still makes sense above this cutoff by some non-perturbative effect [27, 53], but in this thesis, we will restrict our attention to sub-Planckian physics.

The theory of gravity with action (1.17) will be referred to as four-derivative gravity (4DG) (a.k.a. quadratic gravity) and has been studied, independently of the EFT framework, for a long time. It is known to be renormalisable [20] asymptotically free [54, 55] (for the right signs of the couplings) and to contain ghosts, but the ghosts are massive and do not appear at sub-Planckian energies. In fact, there exists a close analogy between 4DG and chiral perturbation theory. The latter describes strong interactions at energies below the chiral symmetry breaking scale. Chiral field, belonging to the coset space $U \in (SU(2)_L \times SU(2)_R)/SU(2)_V$, can be expressed as

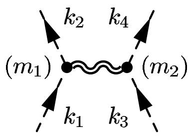
$$U = exp(\pi/f_\pi) \;, \tag{1.20}$$

where π is the pion field and f_π is the pion decay constant. Then the chirar action is

$$S = \int dx \left[\frac{f_\pi^2}{4} tr(U^{-1}\partial U)^2 + \ell_1 tr((U^{-1}\partial U)^2)^2 + \ell_2 tr((U^{-1}\partial U)^2)^2 + O(\partial^6) \right] \tag{1.21}$$

As we will discuss later on, the metric field in $\mathbb{R}^4$ belongs to the coset $g \in GL(4)/O(1,3)$, and when the action (1.17) is very similar to the chiral action. The only difference is owing to the cosmological constant, and when the unimodular version of the action (1.17) is considered, it becomes exactly equivalent to (1.21). Chiral theory is in great agreement with experimental data, telling us that gravity in the EFT framework can also be predictive.

GR has been regarded as an EFT with a range of validity that goes from macroscopic scales to the Planck scale [56] (for review see [30, 57, 58]). Similar principles have been also applied to various modified gravity theories [59–62]. Even though the theory based on covariant quantisation of the Hilbert–Einstein action is not perturbatively renormalisable, quantum corrections affecting low-energy observables can be unambiguously calculated [30, 56]. Indeed, let us look at a simple process of gravitational scattering. At the tree level, we have a diagram which gives rise to the Newtonian potential



$$\sim \frac{Gm_1m_2}{\mathrm{q}^2}$$

$$V(r) = \int \frac{d^3q}{(2\pi)^3} \frac{Gm_1m_2}{\mathrm{q}^2} e^{i\mathrm{qr}} = -\frac{Gm_1m_2}{r} \tag{1.22}$$

When loop corrections are taken into account, one obtains the following correction [56]:

$$V(r) = -\frac{Gm_1 m_2}{r}\left[1 + 3\frac{G(m_1+m_2)}{rc^2} + \frac{41}{10\pi}\frac{G\hbar}{r^2c^3} + \dots\right] \ . \tag{1.23}$$

It is not affected by UV divergences, completely unambiguous. It also agrees with all experimental data, just because of its tininess.

On the other hand, the potential can also receive additional corrections from the higher order terms, such as (1.18). The result, obtained in [63], looks like

$$V(r) = -\frac{Gm_1 m_2}{r}\left[1 - \frac{3}{4}e^{-rm_{spin-2}} + \frac{1}{3}e^{-rm_{spin-0}}\right] \ , \tag{1.24}$$

where m_{spin-2} and m_{spin-0} are masses of the additional to graviton propagating particles. Looking at the action (1.17) as a truncation of the effective action and comparing (1.23) and (1.24) we see that quantum corrections, that decrease as a power law, are superior to the classical corrections coming from using a better truncation, that decrease exponentially. Moreover, when $\lambda \sim \xi \sim 1$ we have $m_{spin-2} \sim m_{spin-2} \sim m_P$ and these classical corrections can be neglected at all reasonable scales.

Now let us look at a more general context of independent affine connection. When we relax the condition (1.16) many more terms can appear in the action. In this section, we will only discuss their effects at a qualitative level. Writing terms of mass dimension up to two allowed by symmetries we get:

$$S = \frac{1}{2}\int d^4x \ \left[2\Lambda - m_0 F + m_{\cdot}T_{\dots}T^{\cdots} + m_{\cdot}Q_{\dots}Q^{\cdots} + m_{\cdot}T_{\dots}Q^{\cdots}\right] \ . \tag{1.25}$$

This action leads to both torsion and nonmetricity being frozen, unless in a measure zero subset of the theory space where some of the mass contributions cancel out. For example, the Palatini action, which comprises the first two terms in (1.25), leaves the projective mode of torsion unfixed. This means that it enters as a gauge degree of freedom and can be easily removed or reabsorbed into a newly defined connection. However, when coupling with matter is considered, it can lead to different physics that the one of GR.

On the other hand, if all mass contributions are included (as they should be in the framework of EFT) torsion and nonmetricity degrees of freedom generically stop propagating. That means that they can enter the equations of motion only as external fields. If there are no sources of such fields, the resulting theory automatically becomes equivalent to GR. A more interesting situation happens when one includes terms of higher mass dimension. Alongside with R^2, $R_{\mu\nu}R^{\mu\nu}$ we now have contributions of the following types:

$$\nabla_{\cdot}T_{\dots}\nabla^{\cdot}T^{\cdots} \ , \quad \nabla_{\cdot}Q_{\dots}\nabla^{\cdot}Q^{\cdots} \ , \quad \nabla_{\cdot}T_{\dots}\nabla^{\cdot}Q^{\cdots} \ , \tag{1.26}$$

which represent themselves kinetic terms for torsion and nonmetricity fields, mixing contributions of the types

$$R_{\dots\cdot}\nabla^{\cdot}T^{\cdots} \ , \quad R_{\dots\cdot}\nabla^{\cdot}Q^{\cdots} \ , \tag{1.27}$$

and various interaction terms such as

$$R_{\dots\cdot}T^{\cdot\cdot}_{\cdot}T^{\cdot\cdot}_{\cdot} \ , \quad \nabla_{\cdot}T_{\dots\cdot}T^{\cdot\cdot}_{\cdot}T^{\cdot\cdot}_{\cdot} \ , \quad Q_{\dots}Q_{\dots}Q^{\cdots}Q^{\cdots} \ , \tag{1.28}$$

etc. Merely the construction of a basis for such contributions represents a formidable challenge which we will address in chapter 3. We expect that torsion and nonmetricity will give corrections to (1.23). Now, we shall build some more apparatus to deal with such complexity and discuss the dynamics.

1.3 Low– and high–energy dynamics of MAG

Before discussing how the usual dynamics of GR can be recovered at low energies, let us introduce an important notion of the distorsion tensor. We will argue that, excluding a measure zero subspace of the it is at low energies helps to recover It is known, that on any Riemann-Cartan manifold one can define a unique connection which is torsionless and metric compatible. It is called the Levi-Civita connection, and its components in the coordinate basis are represented by the Christoffel symbol:

$$\Gamma_{abc} = \frac{1}{2}\left(E_{acb} + E_{cab} - E_{bac}\right) - \frac{1}{2}\left(f_{abc} + f_{cab} - f_{bca}\right) \ , \tag{1.29}$$

where

$$\begin{aligned} E_{cab} =& \theta_c{}^\lambda\, \partial_\lambda g_{ab} \ , \\ f_{bc}{}^a =& \left(\theta_b{}^\mu\, \partial_\mu \theta_c{}^\lambda - \theta_c{}^\mu\, \partial_\mu \theta_b{}^\lambda\right) \theta^a{}_\lambda \ . \end{aligned} \tag{1.30}$$

Note that E and f are not tensors (f are the structure functions of the frame fields). The curvature of the LC connection is the Riemann tensor:

$$R_{\rho\sigma}{}^\mu{}_\nu = \partial_\rho \Gamma_\sigma{}^\mu{}_\nu - \partial_\sigma \Gamma_\rho{}^\mu{}_\nu + \Gamma_\rho{}^\mu{}_\lambda \Gamma_\sigma{}^\lambda{}_\nu - \Gamma_\sigma{}^\mu{}_\lambda \Gamma_\rho{}^\lambda{}_\nu \ . \tag{1.31}$$

Note that our notation of curvature (1.5) is different from the one that is mostly used in gravitational literature. The reason for our chose is to demonstrate in an explicit manner the similarity between MAG and Yang-Mills theories. The last two indices in (1.5) correspond to a single group index. We preserved the same spirit in (1.31), however, due to the symmetries of the Riemann tensor, the Yang-Mills-inspired notation does not differ from the standard one.

A generic connection in the tangent bundle can be decomposed into

$$A_\mu{}^a{}_b = \Gamma_\mu{}^a{}_b + \phi_\mu{}^a{}_b \ , \tag{1.32}$$

where ϕ is a proper tensor called distorsion (following [64]). In general, it has no symmetry properties. Indices are raised and lowered with $g_{\mu\nu}$. From (1.8) and (1.9) one finds

$$T_{\alpha\beta\gamma} = \phi_{\alpha\beta\gamma} - \phi_{\gamma\beta\alpha} \ , \quad Q_{\alpha\beta\gamma} = \phi_{\alpha\beta\gamma} + \phi_{\alpha\gamma\beta} \ . \tag{1.33}$$

These relations can be inverted, to give the distortion as a function of torsion and nonmetricity. In fact we can write

$$\phi_{\alpha\beta\gamma} = L_{\alpha\beta\gamma} + K_{\alpha\beta\gamma} \ , \tag{1.34}$$

where

$$\begin{aligned} L_{\alpha\beta\gamma} &= \frac{1}{2}\left(Q_{\alpha\beta\gamma} + Q_{\gamma\beta\alpha} - Q_{\beta\alpha\gamma}\right) \ , \\ K_{\alpha\beta\gamma} &= \frac{1}{2}\left(T_{\alpha\beta\gamma} + T_{\beta\alpha\gamma} - T_{\alpha\gamma\beta}\right) \ . \end{aligned} \tag{1.35}$$

Note that the tensor $K_{\alpha\beta\gamma}$, called the contortion, is antisymmetric in the second and third index (whereas T is antisymmetric in the first and third). The tensor $L_{\alpha\beta\gamma}$, that does not seem to have a commonly accepted name, is symmetric in the first and third index (whereas Q is symmetric in the second and third index).

Notice that (1.33) can then also be written as

$$T_{\alpha\beta\gamma} = K_{\alpha\beta\gamma} - K_{\gamma\beta\alpha} \ , \quad Q_{\alpha\beta\gamma} = L_{\alpha\beta\gamma} + L_{\alpha\gamma\beta} \ , \tag{1.36}$$

so L contains all the nonmetricity and K contains all the torsion. Another way of saying this is that $\Gamma + K$ is torsion-free and $\Gamma + L$ is metric. We shall actually not use the tensors K and L in the following and prefer to express everything either in terms of ϕ or of T and Q.

We denote $F_{\mu\nu}{}^{\rho}{}_{\sigma}$ the curvature tensor of $A_{\mu}{}^{\rho}{}_{\sigma}$, and $R_{\mu\nu}{}^{\rho}{}_{\sigma}$ the curvature tensor of $\Gamma_{\mu}{}^{\rho}{}_{\sigma}$. They are related as follows:

$$F_{\mu\nu}{}^{\alpha}{}_{\beta} = R_{\mu\nu}{}^{\alpha}{}_{\beta} + \nabla_{\mu}\phi_{\nu}{}^{\alpha}{}_{\beta} - \nabla_{\nu}\phi_{\mu}{}^{\alpha}{}_{\beta} + \phi_{\mu}{}^{\alpha}{}_{\gamma}\phi_{\nu}{}^{\gamma}{}_{\beta} - \phi_{\nu}{}^{\alpha}{}_{\gamma}\phi_{\mu}{}^{\gamma}{}_{\beta} \ . \tag{1.37}$$

In general, F is only antisymmetric in the first two indices. It has three independent contractions: the Ricci-like tensors

$$F^{(13)}_{\mu\nu} = F_{\lambda\mu}{}^{\lambda}{}_{\nu} \ , \quad F^{(14)}_{\mu\nu} = g^{\alpha\beta}F_{\alpha\mu\nu\beta}$$

that do not have symmetry properties in general, and the antisymmetric tensor

$$F^{(34)}_{\mu\nu} = F_{\mu\nu}{}^{\lambda}{}_{\lambda} \ .$$

The analog of the Ricci scalar for the connection $A_{\mu}{}^{\alpha}{}_{\beta}$ is the unique contraction $F_{\mu\nu}{}^{\mu\nu}$, which, up to total derivatives, can be written as

$$F_{\mu\nu}{}^{\mu\nu} = R + \phi_{\mu}{}^{\mu}{}_{\gamma}\phi_{\nu}{}^{\gamma\nu} - \phi_{\nu\mu\gamma}\phi^{\mu\gamma\nu} \ . \tag{1.38}$$

This can be reexpressed in terms of non-metricity and torsion as

$$\begin{aligned} F_{\mu\nu}{}^{\mu\nu} &= R + \frac{1}{4}T_{\alpha\beta\gamma}T^{\alpha\beta\gamma} + \frac{1}{2}T_{\alpha\beta\gamma}T^{\alpha\gamma\beta} - \mathrm{tr}_{(12)}T_{\alpha}\mathrm{tr}_{(12)}T^{\alpha} \\ &+ \frac{1}{4}Q_{\alpha\beta\gamma}Q^{\alpha\beta\gamma} - \frac{1}{2}Q_{\alpha\beta\gamma}Q^{\beta\alpha\gamma} - \frac{1}{4}\mathrm{tr}_{(23)}Q_{\alpha}\mathrm{tr}_{(23)}Q^{\alpha} + \frac{1}{2}\mathrm{tr}_{(12)}Q_{\alpha}\mathrm{tr}_{(23)}Q^{\alpha} \\ &- Q_{\alpha\beta\gamma}T^{\alpha\beta\gamma} - \mathrm{tr}_{(23)}Q_{\alpha}\mathrm{tr}_{(12)}T^{\alpha} + \mathrm{tr}_{(12)}Q_{\alpha}\mathrm{tr}_{(12)}T^{\alpha} \ . \end{aligned} \tag{1.39}$$

1.4 Mass hierarchy

In the following, we shall assume that the cutoff is Planck mass. The main conclusions will be valid if the cutoff differs from it by a few orders of magnitude. There are two possible scenarios. The most natural one, from the EFT point of view, is that all the masses that arise in the theory are comparable to the Planck mass. In this case the only physical particle in the MAG would be the graviton, and the EFT would be very similar to the metric EFT of gravity already discussed in the literature. All the massive states would already be “integrated out” and would only contribute tiny effects through quantum loops. This scenario is somewhat dull, however even in this case MAGs have a greater explanatory power than metric EFTs of gravity, because the vanishing of torsion and nonmetricity can be shown to be generic consequences of the dynamics at low energy, whereas in the metric theories, it has to be postulated.

A more interesting scenario would occur if some of the massive states are much lighter than the Planck scale, so there would be an energy interval where these massive states could exist as physical particles. There is no difficulty in arranging this at the level of the Lagrangian parameters, but this scenario would give rise to various issues. The first is maintaining the mass hierarchy in the presence of loop corrections would likely entail some degree of fine-tuning. The second and more important issue is related to the fact that tree-level unitarity could be violated already at energies much below the Planck scale. This has been discussed for higher spin fields in [65], and MAGs are generally higher-spin theories, because the connection is a three-index field and generally contains a spin-3 degree of freedom. Third, it is in general difficult to find Lagrangians for MAG that do not contain pathological features such as ghosts or tachyons [38–43, 50, 66].

From this point of view, MAGs are similar to 4DG. There has been recently a revived interest in possible mechanisms to avoid these issues in 4DG [31–37], and there have been some first steps to carry them over also to MAGs [67, 68]. If these ideas are successful, one potential consequence is that the spectrum of MAG may be very different from what a naive tree-level analysis would indicate. [3] In this thesis we shall not venture so far, but it is important to keep in mind that all our statements may be subject to important changes when quantum corrections are taken into account.

Regardless of which of these two scenarios one considers, masses of torsion and nonmetricity are naturally expected to be large compared to the energy scales that are currently accessible at collider experiments. Therefore, MAG action leads to standard GR dynamics at low energies. As energies grow, one expects corrections to become important. They will come of the following types. First, there will be quantum corrections from graviton dynamics, analogous to (1.23). Second, there will be classical corrections due to the dynamics of independent connection. And third, there will be quantum corrections from the connection dynamics. As we saw in section (1.2), quantum corrections in metric theories of gravity, though suppressed by Planck mass, decrees slower with distance then higher order classical correction are therefore are of especial interest.

1.5 Loop Divergences

In absence of experiments, a useful criterion for the validity of a given theory is, of course, its mathematical self-consistency. For a quantum field theory, an important consistency criterion is the finiteness of RG flow trajectories. If some RG trajectories go into a fixed point in the UV limit (or into a closed circle etc.), a theory is called asymptotically safe. There exist indications that Einstein gravity may have an interacting UV fixed point [70–72]. A particular case of asymptotic safety occurs when a subset of the trajectories go into a free UV fixed point is called asymptotic freedom. As we mentioned earlier, it has been proven that this scenario is realised in Quadratic gravity [54, 55], alongside with the best-known example of Quantum Chromodynamics. It may happen instead that all the trajectories go into high but finite values of couplings within the applicability regime of the theory. Such behaviour indicates that perturbative treatment is not applicable, but it is still possible that the theory would still make sense nonperturbatively. Another case may happen when all the trajectories go into infinitely high values of couplings. That would mean that the theory makes sense only at much lower energies, with respect to those at which it happens. Such a scenario takes place, for example, in quantum electrodynamics, which exhibits a Landau pole. However, since it appears only at energies that are much higher than the Planck mass, it is generally agreed not to be an issue.

[3] It is even possible that no bosonic field propagates above the Planck mass, a statement that has sometimes been made in the context of noncommutative geometry [69].

In the following, we endeavour to learn whether the presence of non-vanishing, propagating torsion and/or nonmetricity can improve or spoil the behaviour of the gravitational RG trajectories. We shall perform perturbative computations of RG flow in certain subsets of the MAG theory space and see what sort of counterterms are needed to cancel the logarithmic UV divergences at one-loop order.

To that end, we employ the heat kernel technique [73, 74], for review see [75], which is a formal way to treat functional traces and determinants of local pseudo-differential operators (including but not necessarily Laplace-type). Consider a theory of a set of fields φ with arbitrary whole spins with a kinetic operator

$$\Delta = -\Box + E, \tag{1.40}$$

where $\Box = -g^{\mu\nu}\nabla_\mu\nabla_\nu$ whereas E does not contain derivatives and is usually referred to as the endomorphism. This represents the simplest kinetic operator of minimal type. The heat kernel is defined as a solution to the heat equation

$$\left(\frac{d}{ds} + \Delta_x\right) H(x, x'; s) = 0 \tag{1.41}$$

with the initial condition $H(x, x', 0) = \delta(x, x')1$, or formally equivalently as

$$H(x, x'; s) = \langle x'|H(s)|x\rangle, \tag{1.42}$$

where

$$H(s) = e^{-s\Delta}\,. \tag{1.43}$$

This allows us to formally express the propagator

$$\frac{1}{\Delta} = \int_0^\infty ds\ H(s). \tag{1.44}$$

Trace of the heat kernel means taking the coincidence limit $x' \to x$, taking the trace on the internal structure of fields and integration over the spacetime

$$Tr\ H(s) = \int d^dx\sqrt{g}\ tr\ H(x, x; s) \tag{1.45}$$

The one-loop EA relates to the trace of the heat kernel:

$$\Gamma_{1-\text{loop}} = S + \frac{1}{2}\mathrm{Tr}\log\Delta = S + \frac{1}{2}\int_0^\infty \frac{ds}{s}\mathrm{Tr}H(s). \tag{1.46}$$

The integrand can be expressed as asymptotic series for the small values of s:

$$\mathrm{Tr}\ H(s) = \frac{1}{(4\pi s)^{d/2}}\sum_{n\geq 0}\int d^dx\sqrt{g}s^n\,\mathrm{tr}\,a_n\,, \tag{1.47}$$

where $a_n(x)$ are local functions of the curvatures, torsion and their covariant derivatives. The dimension of a_n grows with n as $[a_n] = 2n$.

$$\begin{aligned}
\overline{a_0\,(\Delta)} &= 1,\\
\overline{a_1\,(\Delta)} &= \frac{1}{6}R - E,\\
\overline{a_2\,(\Delta)} &= \frac{1}{180}\left(R_{\mu\nu\rho\lambda}R^{\mu\nu\rho\lambda} - R_{\mu\nu}R^{\mu\nu} + \frac{5}{2}R^2 + 6\Box R\right)\\
&\quad + \frac{1}{12}\Omega_{\mu\nu}\Omega^{\mu\nu} + \frac{1}{2}E^2 - \frac{1}{6}RE - \frac{1}{6}\Box E\ ,
\end{aligned} \tag{1.48}$$

where Ω is the curvature of the field space:

$$[\nabla_\mu, \nabla_\nu]\,\varphi = \Omega_{\mu\nu}\varphi\ . \tag{1.49}$$

If one is interested in the divergent part it is sufficient to only look at the first three terms in the expansion. Specifically, logarithmic divergence comes from the terms of dimension 4, and therefore, in the case of minimal kinetic operator (1.40) the logarithmically divergent part of the one-loop EA is

$$\Gamma^{log.div.}_{1-loop} = -\frac{1}{2}\frac{1}{(4\pi)^2} log\left(\frac{\Lambda^2}{\mu^2}\right)\int d^4x\sqrt{g}\ tr\ a_2. \tag{1.50}$$

An elegant way to extend this method for nonminimal operators was introduced in [76] and is called the "generalised Schwinger-DeWitt technique" or the "Off-diagonal heat kernel technique". It is based on the early-time asymptotic expansion of the kernel of the heat equation. Application of this procedure in the context of non-perturbative calculation with Functional Renormalisation Group (FRG) is also called the Universal RG Machine [77, 78]. Let us consider a theory with kinetic operator F which contains derivatives which are non contracted with each other. At one-loop level, the EA has the form

$$\Gamma_{1-loop} = S + \frac{1}{2} Tr\ log\ F. \tag{1.51}$$

Let us define a new kinetic operator F_λ:

$$F_\lambda = F_m + \lambda N. \tag{1.52}$$

Where λ is a newly introduced parameter. Let us assume the following:
1) $F_m = F|_{\lambda=0}$ is minimal,
2) $F_{\lambda=1} = F$,
3) F_λ is invertible for all $\lambda \in [0,1]$.
Then, differentiating (1.51) by λ,

$$\frac{d}{d\lambda}\Gamma_{1-loop} = \frac{1}{2}\ Tr\ F^{-1}_\lambda \cdot \frac{dF_\lambda}{d\lambda} \tag{1.53}$$

and then by integrating back we get

$$\Gamma_{1-loop}[F] = \frac{1}{2} Tr\ log\ \tilde{F} + \frac{1}{2}\int_0^1 d\lambda\ Tr\left[F^{-1}_\lambda \cdot N\right]. \tag{1.54}$$

The inverse of the kinetic operator can be found first in flat space, and then by working out appropriate corrections, one can solve it for any given order in background quantities (more on that in chapters 7 and 8.2 and Appendix B).Then, all derivatives that are contracted with each other must be commuted to the very right (or very left) to form boxes. Furthermore, there will be additional contributions from commutators. For an arbitrary operator X and function f, using the Laplace transform, one can derive the following formula [79]:

$$[X, f(\Delta)] = \sum_{n=1}^{\infty}\frac{1}{n!}(-1)^{n-1}\left[\, X\ ,\ \Delta\,\right]_n f^{(n)}(\Delta) \tag{1.55}$$

After cumbersome algebraic manipulations, the second term will be expressed as the following traces

$$\mathrm{Tr}[\nabla_{\mu_1} \dots \nabla_{\mu_n} f(\Delta)], \tag{1.56}$$

and then after taking the Laplace transform of this expression, one can relate it to the off-diagonal heat kernel traces (see app C.1):

$$H_{\mu_1 \dots \mu_n}(x,s) = Tr[\nabla_{\mu_1} \dots \nabla_{\mu_n} e^{-s\Delta}] \;, \tag{1.57}$$

that allow us to compute the second term in (1.54).

Chapter 2

Classical MAG

This chapter is devoted to study of Metric-Affine gravity, although motivated by quantum theoretical considerations of section 1.2, but still classical in their nature. We will further discuss structure of MAG, identify several subclasses, build bases of independent Lagrangian contributions, describe their spectrum. We will also provide two examples of MAGs that propagate only a massless graviton and a state of spin/parity 2^- or 3^-. Additionally, we will show that any metric theory of gravity has a teleparallel equivalent.

2.1 Equivalent forms

It appears from the discussion in the previous sections that any MAG can be described in two equivalent ways, depending on what connection is used to write covariant derivatives.

- if the connection $A_\mu{}^\alpha{}_\beta$ is used to write the covariant derivatives, the Lagrangian will be a combination of curvature tensors $F_{\alpha\beta\gamma\delta}$, their covariant derivatives, the tensors T, Q and their covariant derivatives $D_\mu T_{\alpha\beta\gamma}$, $D_\mu Q_{\alpha\beta\gamma}$. In this form, the theory is very similar to a Yang-Mills theory. We will call this **"the Cartan form"** of MAG.

- if the LC connection $\Gamma_\mu{}^\alpha{}_\beta$ is used to write the covariant derivatives, the Lagrangian will be a combination of the Riemann tensor $R_{\alpha\beta\gamma\delta}$ and its covariant derivatives, the distortion $\phi_{\alpha\beta\gamma}$ and its covariant derivatives $\nabla_\mu\phi_{\alpha\beta\gamma}$, (or equivalently T, Q and their covariant derivatives). In this form, MAG looks like ordinary metric gravity coupled to a peculiar matter field. We will call this **"the Einstein form"** of MAG.

Using equation (1.32), any action for a MAG in Cartan form can be rewritten in Einstein form

$$S_C(g, A) = S_c(g, \Gamma + \phi) = S_E(g, \phi) \ . \tag{2.1}$$

We see that the transformation from Cartan to Einstein form is just a change of field variables. [1] The two forms of the theory are physically equivalent.

Because of this choice, and of the possibility of using different frames (either general or natural or orthonormal), the same MAG can be presented in several ways, that may not be

[1]A choice of variables in field theory is sometimes called a "frame". Thus we could also speak of "Cartan frame" and "Einstein frame". We prefer not to do so, in order to avoid confusion with the Einstein frame of conformal geometry, and more importantly because we are already using the term "frame" in its more standard meaning of linear basis in the tangent space.

immediately recognisable. It is thus important to distinguish statements that depend upon the gauge (i.e. the choice of frame) or on the choice of field variables and have no physical content from physical statements that do not depend on these choices.

One such aspect is the number of derivatives, which in the EFT approach is often used to assess the relative importance of different terms in the Lagrangian. In the Einstein form of MAG, the independent fields are the metric $g_{\mu\nu}$ and the distortion $\phi_\rho{}^\mu{}_\nu$. The torsion and non-metricity tensors are algebraic linear combinations of the distortion and can themselves be taken as independent dynamical variables. Thus for example, a term like T^2 has no derivatives and counts as a mass term, while a term like $(\nabla T)^2$ has two derivatives and counts as an ordinary kinetic term.

In the Cartan form of MAG, the status of torsion and non-metricity depends on the choice of basis, i.e. on the gauge. In a general linear basis, they are the covariant derivatives of the fundamental dynamical variables θ and g. Thus terms like T^2 or Q^2 have two derivatives, while $(\nabla T)^2$ or $(\nabla Q)^2$ have four derivatives. Things will look different if we use special frames. In coordinate frames, a term like T^2 has no derivatives and $(DT)^2$ has two derivatives but Q^2 has two derivatives and $(DQ)^2$ has four derivatives. Conversely, in an orthonormal frame Q^2 has no derivatives and $(DQ)^2$ has two derivatives but T^2 has two derivatives and $(DT)^2$ has four derivatives. Obviously the physics cannot change. In particular, the physical propagating degrees of freedom must be the same in all these different versions of the theory. We see that the number of derivatives depends on the choice of field variables, and on the choice of gauge. This highlights that the derivative expansion is not a useful approach in MAG. When we regard MAG as an EFT, we shall therefore classify the terms in the Lagrangian according to their canonical dimension. [2]

2.2 Basic classification of MAGs

Even in its simplest form (using coordinate bases), a general MAG contains 74 component functions and, as we shall discuss later, its Lagrangian has hundreds of free parameters. There are two ways in which one can reduce this complexity. One is to impose additional gauge invariances, on top of diffeomorphisms. These gauge invariances have two effects: they make some field components unphysical, and they constrain the form of the Lagrangian, reducing the number of free parameters. We shall discuss in Section 4.3 some examples of gauge invariances. It is important that such symmetries should be present at the full nonlinear level, because in this case one could hope that they persist when quantum corrections are taken into account. Accidental symmetries that may be present at linearised level but not in the full theory, will generally be broken by quantum effects.

The other way is to impose kinematical constraints on the fields. There are very many ways of doing this, but here we shall discuss only the most basic possibilities, which are suggested by the discussion in the previous sections: we will say that a MAG is **symmetric** if ϕ_{abc} is symmetric in a, c, **antisymmetric** if ϕ_{abc} is antisymmetric in b, c, or **general** if ϕ_{abc} has no symmetry property. [3] Then, from (1.33) we see the following:

[2] It is worth emphasising that similar, though somewhat simpler, considerations apply also to EFT's containing Yang-Mills fields.

[3] A three-index tensor that is simultaneously symmetric in one pair of indices and antisymmetric in another is zero. Thus a MAG that is simultaneously symmetric and antisymmetric is not a MAG - it does not have an

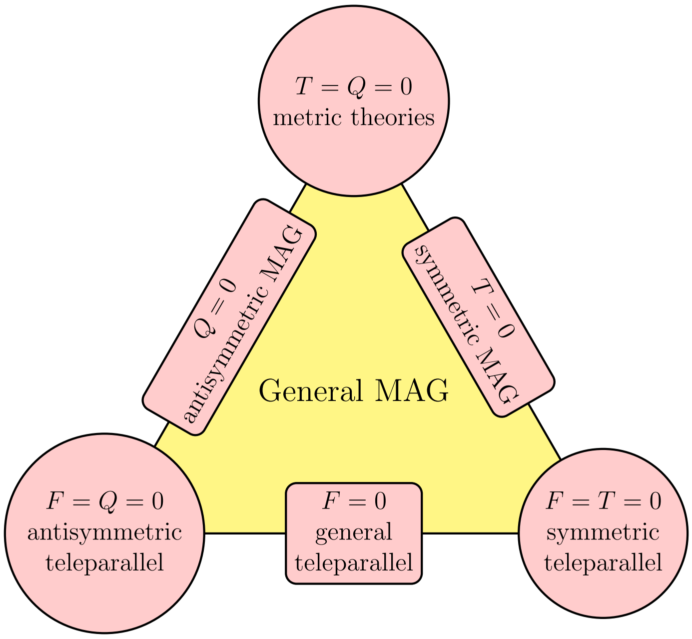


Figure 2.1: The MAGic triangle. The interior of the triangle represents general MAGs, the sides MAGs with one kinematical constraint, the vertices MAGs with two kinematical constraints. This figure had been used in [80, 81] as a representation of the relation between GR and its teleparallel equivalents.

- **"Antisymmetric MAG"**. In this case $Q = 0$, so the connection is metric-compatible. These may also be called "metric MAGs", but we will refrain from doing so in order not to confuse them with metric theories of gravity (where the only variable is the metric).
- **"Symmetric MAG"**. In this case $T = 0$, so this type of theory can be equivalently characterized as being torsion-free.
- **"General MAG"**. In this case both T and Q are generally nonzero.

More restrictive kinematical constraints could consist in assuming that torsion or nonmetricity are of a special form, for example $T_{\alpha\beta\gamma} = v^\delta \epsilon_{\alpha\beta\gamma\delta}$ (this example arises in supergravity) or $Q_{\lambda\mu\nu} = b_\lambda g_{\mu\nu}$ (as in Weyl's theory). Another interesting class of MAGs are the teleparallel theories, where one imposes $F_{\alpha\beta\gamma\delta} = 0$. We emphasise that at this stage these are just kinematical restrictions on the theory, without implications for the dynamics.

According to the presence or absence of kinematical constraints, MAGs can be arranged in a triangle, as in Fig.2.1. The theories in the top vertex are formulated in terms of the metric (and possibly a frame field, but this is just a different gauge choice) and the connection is the LC one. The geometry they use is Riemannian geometry. These are the metric theories of gravity. GR is the metric theory of gravity whose Lagrangian contains at most two derivatives of the metric, but there are infinitely many more complicated ones, containing higher powers of curvature.

independent connection.

The base of the triangle contains the teleparallel theories. Historically the first and still best-known example is the Weitzenböck theory, or antisymmetric teleparallel theory, that contains only torsion and resides in the bottom left corner. Slightly less well-known are teleparallel theories constructed only with nonmetricity, which occupy the right corner [80, 82–84] General teleparallel theories, filling the base of the triangle, have only been discussed more recently [81, 85]. We shall discuss them further in the next section.

For many purposes, it is enough to consider theories that contain only torsion or only nonmetricity. These simplified models correspond to the sides of the triangle. They have fewer fields (34 and 50, respectively, when one uses coordinate frames) and correspondingly fewer terms in the action. In the following, we will discuss these cases separately and then proceed with the general case. We will construct bases of independent terms, compute kinetic coefficients and discuss particular cases. But prior to that, we digress to discuss teleparallel theories, specifically, we raise and answer the question of whether for any metric theory of gravity, there exists a teleparallel equivalent.

2.3 Universality of teleparallelism

At the dynamical level, it is known how to formulate actions for any teleparallel geometry that yield equations that are equivalent to Einstein's equations ("teleparallel equivalents of GR"). Their Lagrangian is

$$\mathbb{T} = \frac{1}{4}T_{\alpha\beta\gamma}T^{\alpha\beta\gamma} + \frac{1}{2}T_{\alpha\beta\gamma}T^{\alpha\gamma\beta} - \mathrm{tr}_{(12)}T_\alpha \mathrm{tr}_{(12)}T^\alpha \tag{2.2}$$

for the antisymmetric case,

$$\mathbb{Q} = \frac{1}{4}Q_{\alpha\beta\gamma}Q^{\alpha\beta\gamma} - \frac{1}{2}Q_{\alpha\beta\gamma}Q^{\beta\alpha\gamma} - \frac{1}{4}\mathrm{tr}_{(23)}Q_\alpha \mathrm{tr}_{(23)}Q^\alpha + \frac{1}{2}\mathrm{tr}_{(23)}Q_\alpha \mathrm{tr}_{(12)}Q^\alpha \tag{2.3}$$

for the symmetric case and

$$\mathbb{G} = \mathbb{T} + \mathbb{Q} - Q_{\alpha\beta\gamma}T^{\alpha\beta\gamma} - \mathrm{tr}_{(23)}Q_\alpha \mathrm{tr}_{(12)}T^\alpha + \mathrm{tr}_{(12)}Q_\alpha \mathrm{tr}_{(12)}T^\alpha \ . \tag{2.4}$$

for the general case. These combinations differ from the Hilbert term only by a total derivative, as is seen from (1.39). More general teleparallel theories with actions of the form $f(\mathbb{T})$ or $f(\mathbb{Q})$ have also been studied in some detail. They are in some sense analogous to the Lagrangians for metric theories of the form $f(R)$, but not equivalent to them.

It is an interesting question, whether any metric theory of gravity has a teleparallel equivalent. We can answer this question in the affirmative. To begin with, let us consider a general action for a metric theory of gravity that contains only powers of undifferentiated curvature tensors:

$$S_M(g) = \int d^4x\sqrt{-g}\,\mathcal{L}(g_{\mu\nu}, R_{\mu\nu}{}^\rho{}_\sigma) \ . \tag{2.5}$$

While ultimately everything only depends on the metric, we have separated the dependence of the Lagrangian on the Riemann tensor and on the metric, which is used to contract all indices.

The EOM is obtained from the variation

$$\delta S_M = \int d^4x\sqrt{-g}\left[\frac{1}{2}\mathcal{L}\, g^{\alpha\beta}\delta g_{\alpha\beta} - \frac{\partial\mathcal{L}}{\partial g^{\mu\nu}}g^{\mu\alpha}g^{\nu\beta}\delta g_{\alpha\beta} + Z^{\mu\nu}{}_\rho{}^\sigma\delta R_{\mu\nu}{}^\rho{}_\sigma\right] \tag{2.6}$$

where $Z^{\mu\nu}{}_{\rho}{}^{\sigma} = \frac{\partial \mathcal{L}}{\partial R_{\mu\nu}{}^{\rho}{}_{\sigma}}$. Thus the EOM is

$$\frac{1}{2}\mathcal{L}\, g^{\alpha\beta} - \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} g^{\mu\alpha} g^{\nu\beta} + \left(\delta_{\sigma}^{(\alpha}\delta_{[\mu}^{\beta)}\nabla^{\rho}\nabla_{\nu]} - g^{\rho(\alpha}\delta_{\sigma}^{\beta)}\nabla_{[\mu}\nabla_{\nu]} - g^{\rho(\alpha}\nabla_{\sigma}\delta_{[\mu}^{\beta)}\nabla_{\nu]}\right) Z^{\mu\nu}{}_{\rho}{}^{\sigma} = 0 \ . \tag{2.7}$$

For a teleparallel theory, $F_{\mu\nu}{}^{\rho}{}_{\sigma} = 0$ so equation (1.37) implies that

$$R_{\mu\nu}{}^{\alpha}{}_{\beta} = -P_{\mu\nu}{}^{\alpha}{}_{\beta} \tag{2.8}$$

where

$$P_{\mu\nu}{}^{\alpha}{}_{\beta} = \nabla_{\mu}\phi_{\nu}{}^{\alpha}{}_{\beta} - \nabla_{\nu}\phi_{\mu}{}^{\alpha}{}_{\beta} + \phi_{\mu}{}^{\alpha}{}_{\gamma}\phi_{\nu}{}^{\gamma}{}_{\beta} - \phi_{\nu}{}^{\alpha}{}_{\gamma}\phi_{\mu}{}^{\gamma}{}_{\beta} \ .$$

Now consider the following action for a teleparallel theory in Einstein form:

$$S_T(g,\phi) = \int d^4x\sqrt{-g}\, \mathcal{L}(g_{\mu\nu}, -P_{\mu\nu}{}^{\rho}{}_{\sigma}) \ . \tag{2.9}$$

where $\mathcal{L}$ is the same as in S_M.

The constraint $F_{\mu\nu}{}^{\rho}{}_{\sigma} = 0$ also implies that

$$A_{\nu}{}^{\rho}{}_{\sigma} = \left(\Lambda^{-1}\right)^{\rho}{}_{\alpha}\partial_{\nu}\Lambda^{\alpha}{}_{\sigma} \ , \tag{2.10}$$

which in turn implies

$$\phi_{\nu}{}^{\rho}{}_{\sigma} = \left(\Lambda^{-1}\right)^{\rho}{}_{\alpha}\partial_{\nu}\Lambda^{\alpha}{}_{\sigma} - \Gamma_{\nu}{}^{\rho}{}_{\sigma} \ . \tag{2.11}$$

Inserting in (2.9) we obtain a new unconstrained action $S'_T(g_{\mu\nu}, \Lambda^{\alpha}{}_{\beta})$. Now Λ is a pure gauge degree of freedom and its EOM is empty, as follows from the observation that due to (2.8), P does not depend on Λ. The only nontrivial equation follows from the variation of the metric:

$$\delta S_T = \int d^4x\sqrt{-g}\left[\frac{1}{2}\mathcal{L}\, g^{\alpha\beta}\delta g_{\alpha\beta} - \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} g^{\mu\alpha} g^{\nu\beta}\delta g_{\alpha\beta} + W^{\mu\nu}{}_{\rho}{}^{\sigma}\delta P_{\mu\nu}{}^{\rho}{}_{\sigma}\right] \tag{2.12}$$

where $W^{\mu\nu}{}_{\rho}{}^{\sigma} = \frac{\partial \mathcal{L}}{\partial P_{\mu\nu}{}^{\rho}{}_{\sigma}}$. But

$$W^{\mu\nu}{}_{\rho}{}^{\sigma} = -Z^{\mu\nu}{}_{\rho}{}^{\sigma}\Big|_{R\to -P}$$

and

$$\delta P_{\mu\nu}{}^{\rho}{}_{\sigma} = -\delta R_{\mu\nu}{}^{\rho}{}_{\sigma} \ ,$$

so the EOM of this teleparallel theory is the same as the one of the original metric theory.

Let us now come to the more general case when the action contains also up to n-times differentiated Riemann tensors:

$$S_M(g) = \int d^4x\sqrt{-g}\, \mathcal{L}(g_{\mu\nu}, R_{\mu\nu}{}^{\rho}{}_{\sigma}, \nabla_{\alpha}R_{\mu\nu}{}^{\rho}{}_{\sigma}, \ldots, \nabla_{\alpha_1}\cdots\nabla_{\alpha_n}R_{\mu\nu}{}^{\rho}{}_{\sigma}) \ . \tag{2.13}$$

In this case the variation will contain n additional terms:

$$\sum_{i=1}^{n} Z_i^{\alpha_1\ldots\alpha_i\mu\nu}{}_{\rho}{}^{\sigma}\delta(\nabla_{\alpha_1}\cdots\nabla_{\alpha_i}R_{\mu\nu}{}^{\rho}{}_{\sigma}) \ ,$$

where $Z_i^{\alpha_1\ldots\alpha_i\mu\nu}{}_{\rho}{}^{\sigma} = \frac{\partial \mathcal{L}}{\partial(\nabla_{\alpha_1}\cdots\nabla_{\alpha_i}R_{\mu\nu}{}^{\rho}{}_{\sigma})}$. The teleparallel equivalent action is

$$S_T(g,\phi) = \int d^4x\sqrt{-g}\, \mathcal{L}(g_{\mu\nu}, -P_{\mu\nu}{}^{\rho}{}_{\sigma}, -\nabla_{\alpha}P_{\mu\nu}{}^{\rho}{}_{\sigma}, \ldots, -\nabla_{\alpha_1}\cdots\nabla_{\alpha_n}P_{\mu\nu}{}^{\rho}{}_{\sigma}) \ . \tag{2.14}$$

Following the same argument as above, based on the constraint (2.8), the EOMs of this theory are the same as those of the original metric theory.

Chapter 3

Lagrangians

3.1 General structure of the action

As discussed in Section 1.2, terms within the Wilsonian EA are ordered based on canonical dimension, with the understanding that terms of lower dimension are generally more important at low energy. We shall now discuss the possible Lagrangians for MAG containing terms of dimension two and four. In the first overview, we will entirely omit all indices and only consider the structures that can appear. This is useful to understand the relation between the Cartan and Einstein forms of the Lagrangian, in an uncluttered environment. In the rest of this section, we shall count, and in part enumerate, all the structures.

We start from the Cartan form of the theory. The covariant field strengths are the curvature F, of mass dimension two, the torsion T and non-metricity Q, both of mass dimension one. The scalars of dimension two that can be formed with these ingredients are either linear in F or quadratic in T and Q. These terms will appear in the action with coefficients of dimension two. The scalars of dimension four are of the forms F^2 or FDT/FDQ or quadratic in DT/DQ, or cubic in T/Q with one derivative, or quartic in T/Q. All these terms appear in the action with dimensionless coefficients.

In order not to introduce too many different symbols, we shall use a slightly cumbersome but helpful notation, where all the dimension-two couplings are called a and all dimensionless ones are called c, and the type of term they multiply is indicated by a superscript in brackets. Once indices are reinstated, different couplings of the same type will be distinguished by a subscript. Thus, ignoring all numerical factors and signs, we write the Lagrangian in the schematic form

$$\begin{aligned}\mathcal{L}_C = {} & a^F F + a^{TT} TT + a^{TQ} TQ + a^{QQ} QQ \\ & + c^{FF} FF + c^{FT} FDT + c^{FQ} FDQ + c^{TT} (DT)^2 + c^{TQ} DTDQ + c^{QQ} (DQ)^2 \\ & + c^{FTT} FTT + c^{FTQ} FTQ + c^{FQQ} FQQ \\ & + c^{TTT} TTDT + \ldots + c^{QQQ} QQDQ \\ & + c^{TTTT} TTTT + \ldots + c^{QQQQ} QQQQ \ ,\end{aligned} \tag{3.1}$$

where the ellipses stand for cubic and quartic terms involving different powers of T and Q.

The action in Einstein form is related to the action in Cartan form by (2.1). In practice the transformations achieved by using $D = \nabla + \phi$ and equations (1.37) and (1.33), that we can write schematically as

$$F \sim R + \nabla\phi + \phi\phi \ , \quad T \sim \phi \ , \quad Q \sim \phi \ .$$

One then obtains the Lagrangian in Einstein form

$$\begin{aligned}\mathcal{L}_E = & \, m^R R + m^{\phi\phi}\phi\phi \\ & + b^{RR} RR + b^{R\phi} R\nabla\phi + b^{\phi\phi}(\nabla\phi)^2 \\ & + b^{R\phi\phi} R\phi\phi + b^{\phi\phi\phi}\phi\phi\nabla\phi + b^{\phi\phi\phi\phi}\phi\phi\phi\phi \ ,\end{aligned} \tag{3.2}$$

where the dimension-two couplings are now called m and the dimensionless ones are called b. This is the most general Lagrangian for the Einstein form of MAG, involving terms of dimension two and four.

At this point one can use (1.34) and (1.35) to reexpress ϕ in terms of T and Q. The Lagrangian then looks again more similar to (3.1), but there is a difference: in (3.1), T and Q have to be thought of as depending on A and g, whereas here they have to be treated as independent variables. To distinguish the two Lagrangians, in $\mathcal{L}_E$ the coefficients will be called b^{RT}, b^{RQ}, b^{TT} etc.

In this thesis, we will be interested mainly in the linearisation of the theory around flat space. We observe that in this approximation only the first two lines of (3.1) and (3.2) contribute to the propagator, while all the other terms are interactions. Also, we note that whereas the dependence on the metric is nonpolynomial, as usual, the dependence on distortion is at most quartic.

Many terms in (3.1) and (3.2) are dependent. There are two types of relations between different terms. Relations of the first type are obtained from Bianchi identities. For the independent dynamical connection, they read

$$F_{[\alpha\beta}{}^{\gamma}{}_{\delta]} - D_{[\alpha}T_{\beta}{}^{\gamma}{}_{\delta]} - T_{[\alpha}{}^{\epsilon}{}_{\beta|}T_{\epsilon}{}^{\gamma}{}_{|\delta]} = 0 \ , \tag{3.3}$$

$$D_{[\alpha}F_{\beta\gamma]}{}^{\delta}{}_{\epsilon} + T_{[\alpha}{}^{\eta}{}_{\beta|}F_{\eta|\gamma]}{}^{\delta}{}_{\epsilon} = 0 \ . \tag{3.4}$$

The Bianchi identities of the LC connections are the same, except that the torsion terms are missing. Relations of the second type come from a simple observation that although curvature, torsion and nonmetricity are independent, their products are related as long as we are only concerned with propagator contributions as explained above. Schematically, we represent this relation as

$$F^2 = R^2 + R\nabla\phi + (\nabla\phi)^2 \ , \tag{3.5}$$

where terms not contributing to flat propagator are displayed. Such and similar relations for FDT and FDQ allow us to express certain terms in Cartan formulation via other terms.

Counting independent terms turns out to be far easier in the Einstein point of view, where we use the variables (g, ϕ) and only relations of the first type have to be considered. We, therefore, start from this case. We will loosely refer to scalar monomials in the fields which appear in the Lagrangian as "invariants". In the Einstein form of the theory, they will be denoted

$$H_i^{X,Y} \quad \text{where} \quad X, Y \in \{R, T, Q\}$$

and i is an index labelling different monomials. We shall discuss first the antisymmetric MAG, which is simplest, then the symmetric MAG and finally the general case.

The count of the possible terms in the Lagrangian in the Cartan form of the theory is more tricky. The invariants that can appear in the Lagrangian in Cartan form are denoted

$$L_i^{X,Y} \ , \quad \text{where} \quad X,Y \in \{F,T,Q\} \ ,$$

to distinguish them from the H_i^{XY} of the Einstein form of the theory. In what follows we ignore the cosmological constant and start from dimension two terms of general MAG, moving then to dimension four while discussing Antisymmetric and Symmetric MAG separately for clarity.

3.2 Dimension-two terms

Let us look more carefully at the dimension-two part of the Lagrangian. In the Cartan form, it is

$$\mathcal{L}_C^{(2)} = -\frac{1}{2}\left[-a^F F + \sum_{i=1}^{3} a_i^{TT} M_i^{TT} + \sum_{i=1}^{3} a_i^{TQ} M_i^{TQ} + \sum_{i=1}^{5} a_i^{QQ} M_i^{QQ}\right] \ , \tag{3.6}$$

where $F = F_{\mu\nu}{}^{\mu\nu}$ is the unique scalar that can be constructed from the curvature and $a^F = m_P^2$, where m_P is the Planck mass. This will be referred to as the Palatini term. The other scalars are

$$\begin{aligned}
&M_1^{TT} = T^{\mu\rho\nu}T_{\mu\rho\nu} \ , \quad M_2^{TT} = T^{\mu\rho\nu}T_{\mu\nu\rho} \ , \quad M_3^{TT} = \mathrm{tr}_{(12)}T^{\mu}\mathrm{tr}_{(12)}T_{\mu} \ , \\
&M_1^{QQ} = Q^{\rho\mu\nu}Q_{\rho\mu\nu} \ , \quad M_2^{QQ} = Q^{\rho\mu\nu}Q_{\nu\mu\rho} \ , \\
&\quad \mathrm{M}_3^{QQ} = \mathrm{tr}_{(23)}Q^{\mu}\mathrm{tr}_{(23)}Q_{\mu} \ , \quad M_4^{QQ} = \mathrm{tr}_{(12)}Q^{\mu}\mathrm{tr}_{(12)}Q_{\mu} \ , \quad M_5^{QQ} = \mathrm{tr}_{(23)}Q^{\mu}\mathrm{tr}_{(12)}Q_{\mu} \ , \\
&\quad \mathrm{M}_1^{TQ} = T^{\mu\rho\nu}Q_{\mu\rho\nu} \ , \quad M_2^{TQ} = \mathrm{tr}_{(12)}T^{\mu}\mathrm{tr}_{(23)}Q_{\mu} \ , \quad M_3^{TQ} = \mathrm{tr}_{(12)}T^{\mu}\mathrm{tr}_{(12)}Q_{\mu} \ .
\end{aligned} \tag{3.7}$$

Going from the Cartan to the Einstein form, as discussed in the previous subsection, yields

$$\mathcal{L}_E^{(2)} = -\frac{1}{2}\left[-m^R R + \sum_{i=1}^{11} m_i^{\phi\phi} M_i^{\phi\phi}\right] \ , \tag{3.8}$$

where

$$\begin{aligned}
&M_1^{\phi\phi} = \phi_{\mu\nu\rho}\phi^{\mu\nu\rho} \ , \ M_2^{\phi\phi} = \phi_{\mu\nu\rho}\phi^{\mu\rho\nu} \ , \ M_3^{\phi\phi} = \phi_{\mu\nu\rho}\phi^{\rho\nu\mu} \ , \ M_4^{\phi\phi} = \phi_{\mu\nu\rho}\phi^{\nu\mu\rho} \ , \ M_5^{\phi\phi} = \phi_{\mu\nu\rho}\phi^{\nu\rho\mu} \ , \\
&M_6^{\phi\phi} = \mathrm{tr}_{(12)}\phi_{\mu}\mathrm{tr}_{(12)}\phi^{\mu} \ , \quad M_7^{\phi\phi} = \mathrm{tr}_{(13)}\phi_{\mu}\mathrm{tr}_{(13)}\phi^{\mu} \ , \quad M_8^{\phi\phi} = \mathrm{tr}_{(23)}\phi_{\mu}\mathrm{tr}_{(23)}\phi^{\mu} \ , \\
&M_9^{\phi\phi} = \mathrm{tr}_{(12)}\phi_{\mu}\mathrm{tr}_{(13)}\phi^{\mu} \ , \quad M_{10}^{\phi\phi} = \mathrm{tr}_{(12)}\phi_{\mu}\mathrm{tr}_{(23)}\phi^{\mu} \ , \quad M_{11}^{\phi\phi} = \mathrm{tr}_{(13)}\phi_{\mu}\mathrm{tr}_{(23)}\phi^{\mu} \ .
\end{aligned} \tag{3.9}$$

The first term is now the Hilbert Lagrangian and the rest are mass terms for ϕ. The correspondence between the parameters m_i and a_i is

$$\begin{aligned}
&m^R = a^F, \quad m_1^{\phi\phi} = 2a_1^{TT} + 2a_1^{QQ} + a_1^{TQ}, \quad m_2^{\phi\phi} = a_2^{TT} + 2a_1^{QQ} + a_1^{TQ}, \\
&m_3^{\phi\phi} = -2a_1^{TT} + a_2^{QQ} - a_1^{TQ}, \quad m_4^{\phi\phi} = a_2^{TT} + a_2^{QQ}, \quad m_5^{\phi\phi} = a^F - 2a_2^{TT} + 2a_2^{QQ} - a_1^{TQ}, \\
&m_6^{\phi\phi} = a_3^{TT} + a_4^{QQ} + a_3^{TQ}, \quad m_7^{\phi\phi} = a_4^{QQ}, \quad m_8^{\phi\phi} = a_3^{TT} + 4a_3^{QQ} - 2a_2^{TQ}, \\
&m_9^{\phi\phi} = -a^F + 2a_4^{QQ} + a_3^{TQ}, \quad m_{10}^{\phi\phi} = -2a_3^{TT} + 2a_5^{QQ} + 2a_2^{TQ} - a_3^{TQ}, \quad m_{11}^{\phi\phi} = 2a_5^{QQ} - a_3^{TQ}.
\end{aligned} \tag{3.10}$$

The inverse map is given in Appendix D.2.1, Equation (D.2.1). Reexpressing ϕ in terms of T and Q we obtain

$$\mathcal{L}_E^{(2)} = -\frac{1}{2}\left[-m^R R + \sum_{i=1}^{3} m_i^{TT} M_i^{TT} + \sum_{i=1}^{3} m_i^{TQ} M_i^{TQ} + \sum_{i=1}^{5} m_i^{QQ} M_i^{QQ}\right] , \tag{3.11}$$

where T, Q are now independent variables and

$$\begin{aligned}
&m_1^{TT} = a_1^{TT} - \frac{1}{4}a^F, \quad m_2^{TT} = a_2^{TT} - \frac{1}{2}a^F, \quad m_3^{TT} = a_3^{TT} + a^F,\\
&m_1^{QQ} = a_1^{QQ} - \frac{1}{4}a^F, \quad m_2^{QQ} = a_2^{QQ} + \frac{1}{2}a^F, \quad m_3^{QQ} = a_3^{QQ} + \frac{1}{4}a^F,\\
&m_4^{QQ} = a_4^{QQ}, \quad m_5^{QQ} = a_5^{QQ} - \frac{1}{2}a^F,\\
&m_1^{TQ} = a_1^{TQ} + a^F, \quad m_2^{TQ} = a_2^{TQ} + a^F, \quad m_3^{TQ} = a_3^{TQ} - a^F.
\end{aligned} \tag{3.12}$$

These formulae can be specialised to antisymmetric and symmetric MAG, simply setting $Q = 0$ and $T = 0$, respectively. In the following sections, we will move on to consider dimension four terms.

3.3 Antisymmetric MAG

3.3.1 Einstein form

We start from the subclass of antisymmetric MAGs, taking g and T as basic variables. This case has been considered in [44], where the potential terms of the types T^4 and others were also thoroughly considered. The numbers of independent terms or each type are

R^2	$(\nabla T)^2$	$R\,\nabla T$	$R\,T^2$	$T^2\nabla T$	T^4	Total
3	9	2	14	31	33	92

Let us list explicitly the terms of the first three columns, that are relevant for the flat space propagators. We have three RR terms

$$H_1^{RR} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \ , \quad H_2^{RR} = R_{\mu\nu}R^{\mu\nu} \ , \quad H_3^{RR} = R^2 \ , \tag{3.13}$$

nine $(\nabla T)^2$ terms

$$\begin{aligned}
H_1^{TT} &= \nabla^\alpha T^{\beta\gamma\delta}\nabla_\alpha T_{\beta\gamma\delta} \ , & H_2^{TT} &= \nabla^\alpha T^{\beta\gamma\delta}\nabla_\alpha T_{\beta\delta\gamma} \ ,\\
H_3^{TT} &= \nabla^\alpha \mathrm{tr}_{(12)}T^\beta \nabla_\alpha \mathrm{tr}_{(12)}T_\beta \ , & &\\
H_4^{TT} &= \mathrm{div}_{(1)}T^{\alpha\beta}\mathrm{div}_{(1)}T_{\alpha\beta} \ , & H_5^{TT} &= \mathrm{div}_{(1)}T^{\alpha\beta}\mathrm{div}_{(1)}T_{\beta\alpha} \ ,\\
H_6^{TT} &= \mathrm{div}_{(2)}T^{\alpha\beta}\mathrm{div}_{(2)}T_{\alpha\beta} \ , & H_7^{TT} &= \mathrm{div}_{(1)}T^{\alpha\beta}\mathrm{div}_{(2)}T_{\alpha\beta} \ ,\\
H_8^{TT} &= \mathrm{div}_{(2)}T^{\alpha\beta}\nabla_\alpha \mathrm{tr}_{(12)}T_\beta \ , & H_9^{TT} &= (\mathrm{trdiv}_{(1)}T)^2 \ ,
\end{aligned} \tag{3.14}$$

and just considering the independent contractions one has five $R\nabla T$-type terms

$$\begin{aligned}
&H_1^{RT} = R^{\alpha\beta\gamma\delta}\nabla_\alpha T_{\beta\gamma\delta} \ , \quad H_2^{RT} = R^{\alpha\gamma\beta\delta}\nabla_\alpha T_{\beta\gamma\delta} \ ,\\
&H_3^{RT} = R^{\beta\gamma}\mathrm{div}_{(1)}T_{\beta\gamma} \ , \quad H_4^{RT} = R^{\alpha\beta}\nabla_\alpha \mathrm{tr}_{(12)}T_\beta \ , \quad H_5^{RT} = R\,\mathrm{trdiv}_{(1)}T \ .
\end{aligned} \tag{3.15}$$

However, these invariants are not all independent. Indeed we note that contracting the first (algebraic) Bianchi identity with ∇T we obtain the relation

$$H_2^{RT} = 2H_1^{RT} \ , \tag{3.16}$$

while using the second Bianchi identity, contracted with T, and integrating by parts we obtain the relations

$$\begin{aligned} H_3^{RT} &= H_1^{RT} \ , \\ H_5^{RT} &= -2H_4^{RT} \ . \end{aligned} \tag{3.17}$$

A possible choice consists of keeping

$$\{H_3^{RT} \ , \ H_5^{RT}\} \tag{3.18}$$

as independent invariants of type $R\nabla T$. Thus, there are $3+9+2=14$ independent terms quadratic in the fields.

In the table, we also give the number of interaction terms. We have determined these numbers using the function `AllContractions` of the *xTras* package for *Mathematica*. [1] For the RTT terms, this gives 18 different contractions, but the first Bianchi identity, contracted with TT, gives 4 relations between these terms, leading to 14. For $TT\nabla T$, `AllContractions` gives 46 terms, but there are 15 total derivative terms of this type, so the number of independent ones is 31. [2]

3.3.2 Cartan form

We shall begin by listing all the terms that can appear in the first three terms of (3.1).

FF terms:

$$\begin{aligned}
&L_1^{FF} = F^{\mu\nu\rho\sigma}F_{\mu\nu\rho\sigma} \ , \quad L_2^{FF} = F^{\mu\nu\rho\sigma}F_{\mu\nu\sigma\rho} \ , \quad L_3^{FF} = F^{\mu\nu\rho\sigma}F_{\rho\sigma\mu\nu} \ , \\
&L_4^{FF} = F^{\mu\nu\rho\sigma}F_{\mu\rho\nu\sigma} \ , \quad L_5^{FF} = F^{\mu\nu\rho\sigma}F_{\mu\sigma\nu\rho} \ , \quad L_6^{FF} = F^{\mu\nu\rho\sigma}F_{\mu\sigma\rho\nu} \ , \\
&L_7^{FF} = F^{(13)\mu\nu}F^{(13)}_{\mu\nu} \ , \quad L_8^{FF} = F^{(13)\mu\nu}F^{(13)}_{\nu\mu} \ , \\
&L_9^{FF} = F^{(14)\mu\nu}F^{(14)}_{\mu\nu} \ , \quad L_{10}^{FF} = F^{(14)\mu\nu}F^{(14)}_{\nu\mu} \ , \\
&L_{11}^{FF} = F^{(13)\mu\nu}F^{(14)}_{\mu\nu} \ , \quad L_{12}^{FF} = F^{(13)\mu\nu}F^{(14)}_{\nu\mu} \ , \\
&L_{13}^{FF} = F^{(34)\mu\nu}F^{(34)}_{\mu\nu} \ , \quad L_{14}^{FF} = F^{(34)\mu\nu}F^{(13)}_{\mu\nu} \ , \quad L_{15}^{FF} = F^{(34)\mu\nu}F^{(14)}_{\mu\nu} \ , \\
&L_{16}^{FF} = F^2 \ .
\end{aligned} \tag{3.19}$$

[1] While this counting may still be possible by hand in this case, it becomes practically impossible for general MAG.

[2] The number of total derivative terms can be determined by applying `AllContractions` to $qTTT$, where q^μ is any vector (it can be thought of as the momentum).

FDT terms:

$$
\begin{array}{lll}
L^{FT}_1 = F^{\mu\nu\rho\sigma} D_\mu T_{\nu\rho\sigma} \ , & L^{FT}_2 = F^{\mu\nu\rho\sigma} D_\mu T_{\nu\sigma\rho} \ , & L^{FT}_3 = F^{\mu\nu\rho\sigma} D_\mu T_{\rho\nu\sigma} \ , \\
L^{FT}_4 = F^{\mu\nu\rho\sigma} D_\rho T_{\mu\nu\sigma} \ , & L^{FT}_5 = F^{\mu\nu\rho\sigma} D_\rho T_{\mu\sigma\nu} \ , & L^{FT}_6 = F^{\mu\nu\rho\sigma} D_\sigma T_{\mu\nu\rho} \ , \\
L^{FT}_7 = F^{\mu\nu\rho\sigma} D_\sigma T_{\mu\rho\nu} \ , & & \\
L^{FT}_8 = F^{(13)\mu\nu} D_\mu \mathrm{tr}_{(12)} T_\nu \ , & L^{FT}_9 = F^{(13)\mu\nu} D_\nu \mathrm{tr}_{(12)} T_\mu \ , & \\
L^{FT}_{10} = F^{(14)\mu\nu} D_\mu \mathrm{tr}_{(12)} T_\nu \ , & L^{FT}_{11} = F^{(14)\mu\nu} D_\nu \mathrm{tr}_{(12)} T_\mu \ , & L^{FT}_{12} = F^{(34)\mu\nu} D_\mu \mathrm{tr}_{(12)} T_\nu \ , \\
L^{FT}_{13} = F^{(13)\mu\nu} \mathrm{Div}_{(1)} T_{\mu\nu} \ , & L^{FT}_{14} = F^{(13)\mu\nu} \mathrm{Div}_{(1)} T_{\nu\mu} \ , & \\
L^{FT}_{15} = F^{(14)\mu\nu} \mathrm{Div}_{(1)} T_{\mu\nu} \ , & L^{FT}_{16} = F^{(14)\mu\nu} \mathrm{Div}_{(1)} T_{\nu\mu} \ , & \\
L^{FT}_{17} = F^{(13)\mu\nu} \mathrm{Div}_{(2)} T_{\mu\nu} \ , & L^{FT}_{18} = F^{(14)\mu\nu} \mathrm{Div}_{(2)} T_{\mu\nu} \ , & \\
L^{FT}_{19} = F^{(34)\mu\nu} \mathrm{Div}_{(1)} T_{\mu\nu} \ , & L^{FT}_{20} = F^{(34)\mu\nu} \mathrm{Div}_{(2)} T_{\mu\nu} \ , & L^{FT}_{21} = F\, \mathrm{trDiv}_{(1)} T \ .
\end{array}
\tag{3.20}
$$

$(DT)^2$ terms:

$$
\begin{array}{ll}
L^{TT}_1 = D^\alpha T^{\beta\gamma\delta} D_\alpha T_{\beta\gamma\delta} \ , & L^{TT}_2 = D^\alpha T^{\beta\gamma\delta} D_\alpha T_{\beta\delta\gamma} \ , \\
L^{TT}_3 = D^\alpha \mathrm{tr}_{(12)} T^\beta D_\alpha \mathrm{tr}_{(12)} T_\beta \ , & \\
L^{TT}_4 = \mathrm{Div}_{(1)} T^{\alpha\beta} \mathrm{Div}_{(1)} T_{\alpha\beta} \ , & L^{TT}_5 = \mathrm{Div}_{(1)} T^{\alpha\beta} \mathrm{Div}_{(1)} T_{\beta\alpha} \ , \\
L^{TT}_6 = \mathrm{Div}_{(2)} T^{\alpha\beta} \mathrm{Div}_{(2)} T_{\alpha\beta} \ , & L^{TT}_7 = \mathrm{Div}_{(1)} T^{\alpha\beta} \mathrm{Div}_{(2)} T_{\alpha\beta} \ , \\
L^{TT}_8 = \mathrm{Div}_{(2)} T^{\alpha\beta} D_\alpha \mathrm{tr}_{(12)} T_\beta \ , & L^{TT}_9 = (\mathrm{trDiv}_{(1)} T)^2 \ .
\end{array}
\tag{3.21}
$$

We observe that whereas the 38 terms L^{TT}, L^{QQ}, L^{TQ} in (3.21, 3.46, 3.74) are in one-to-one correspondence with the terms H^{TT}, H^{QQ}, H^{TQ} in (3.14, 3.41, 3.72), there are many more terms of type FF, FDT, FDQ than RR, $R\nabla T$, $R\nabla Q$. This is due to the fact that the curvature tensor F has less symmetries than the Riemann tensor. This also means that there will also be many more relations. Our goal now will be to uncover these relations, exhibit a basis of invariants and construct the map between the couplings in the Cartan basis and those in the previously established Einstein basis.

Concerning the cubic and quartic interaction terms, we shall not attempt to count them here, as this would be overly complicated. However, we know that ultimately they will be in one-to-one correspondence with those of the Einstein formulation, that have been counted in the previous sections.

Since in antisymmetric MAG F is antisymmetric in both pairs of indices, there are fewer independent terms than in general MAG. We keep L^{FF}_i with $i = 1, 3, 4, 7, 8, 16$, while

$$
\begin{array}{l}
L^{FF}_2 = -L^{FF}_1 \ , \quad L^{FF}_5 = -L^{FF}_4 \ , \quad L^{FF}_6 = L^{FF}_4 \ , \\
L^{FF}_9 = L^{FF}_7 \ , \quad L^{FF}_{10} = L^{FF}_8 \ , \quad L^{FF}_{11} = -L^{FF}_7 \ , \quad L^{FF}_{12} = -L^{FF}_8 \ , \\
L^{FF}_{13} = L^{FF}_{14} = L^{FF}_{15} = 0 \ .
\end{array}
\tag{3.22}
$$

We keep all the terms L^{TT}. They are the same as the invariants of type $(\nabla T)^2$, except for the replacement of ∇ by D. We keep L^{FT}_i with $i = 1, 3, 4, 5, 8, 9, 13, 14, 17, 21$, while

$$
\begin{array}{l}
L^{FT}_2 = -L^{FT}_1 \ , \quad L^{FT}_6 = -L^{FT}_4 \ , \quad L^{FT}_7 = -L^{FT}_5 \ , \\
L^{FT}_{10} = -L^{FT}_8 \ , \quad L^{FT}_{11} = -L^{FT}_9 \ , \quad L^{FT}_{15} = -L^{FT}_{13} \ , \quad L^{FT}_{16} = -L^{FT}_{14} \ , \\
L^{FT}_{18} = -L^{FT}_{17} \ , \quad L^{FT}_{12} = L^{FT}_{19} = L^{FT}_{20} = 0 \ .
\end{array}
\tag{3.23}
$$

We now have 25 quadratic terms, compared to the 14 quadratic terms in the Einstein form of antisymmetric MAG.

There are several additional relations. Multiplying (3.3) by F, we obtain, up to interaction term of the form FTT,

$$\begin{aligned} L_8^{FT} - L_9^{FT} - L_{17}^{FT} &= -L_7^{FF} + L_8^{FF} \ , \\ -2L_1^{FT} + L_5^{FT} &= -L_1^{FF} + 2L_4^{FF} \ , \\ L_3^{FT} - 2L_4^{FT} &= -L_3^{FF} + 2L_4^{FF} \ , \end{aligned} \tag{3.24}$$

and multiplying (3.4) by T (and using integrations by parts) gives, again up to terms of the form FTT,

$$\begin{aligned} L_5^{FT} - 2L_{14}^{FT} &= 0 \ , \\ -L_4^{FT} + L_{13}^{FT} - L_{17}^{FT} &= 0 \ , \\ 2L_9^{FT} + L_{21}^{FT} &= 0 \ . \end{aligned} \tag{3.25}$$

Furthermore, multiplying (3.3) by ∇T gives, up to terms cubic in T,

$$\begin{aligned} L_{17}^{FT} &= 1/2L_6^{TT} - L_8^{TT} \ , \\ L_{13}^{FT} - L_{14}^{FT} &= L_7^{TT} - L_8^{TT} \ , \\ L_8^{FT} - L_9^{FT} &= -L_3^{TT} + L_8^{TT} + L_9^{TT} \ , \\ 2L_4^{FT} - L_5^{FT} &= -L_6^{TT} + 2L_7^{TT} \ , \\ L_1^{FT} - L_3^{FT} + L_4^{FT} &= -L_2^{TT} + L_5^{TT} + L_7^{TT} \ , \\ 2L_1^{FT} - L_5^{FT} &= L_1^{TT} - 2L_4^{TT} \ . \end{aligned} \tag{3.26}$$

Altogether we have obtained 12 relations, of which 11 turn out to be linearly independent. Therefore we can eliminate 22 out of the 36 invariants listed in (3.19, 3.21, 3.20), and we remain with 14 independent quadratic invariants, exactly as in the counting in the Einstein form.

There are many ways of solving these relations, but we shall consider here only two. The first is to retain all the nine L^{TT} terms, plus

$$\{L_1^{FF} \ , \ L_7^{FF} \ , \ L_{16}^{FF}\} \quad \text{and} \quad \{L_{13}^{FT} \ , \ L_{21}^{FT}\} \ , \tag{3.27}$$

which is in one-to-one correspondence with (3.13) and (3.18). Thus, the elements of this basis are in one-to-one correspondence with the elements of the basis in the Einstein form, from which they are obtained just by replacing $R \to F$ and $\nabla \to D$. The remaining invariants are given in Equation (D.1.1) in Appendix D.1.1.

Due to the geometrical meaning of the curvature, when we use the independent connection A, it seems desirable to keep all terms that contain F, and instead remove others. We can choose as a basis the six L^{FF} invariants $\{L_1^{FF} \ , \ L_3^{FF} \ , \ L_4^{FF} \ , \ L_7^{FF} \ , \ L_8^{FF} \ , L_{16}^{FF}\}$, plus

$$\{L_1^{TT} \ , \ L_2^{TT} \ , \ L_3^{TT} \ , \ L_5^{TT}\} \quad \text{and} \quad \{L_1^{FT} \ , \ L_8^{FT} \ , \ L_9^{FT} \ , \ L_{13}^{FT}\} \ . \tag{3.28}$$

The remaining invariants are given in Equation (D.1.1) in Appendix D.1.1. In the next section, we will consider another form of the same theory, rewritten via decomposed variables.

3.3.3 Decomposition of Torsion

The torsion field can be decomposed into representations of the Lorentz group

$$\begin{aligned}
&T_\mu = T_\mu{}^\alpha{}_\alpha, \\
&\check{T}_\mu = \epsilon_{\mu\nu\rho\lambda} T^{\nu\rho\lambda}, \\
&\kappa_{\alpha\beta\gamma} = T_{\alpha\beta\gamma} - T_{[\alpha\beta\gamma]} - \frac{1}{6} g_{[\alpha\beta} T_{\gamma]} = \frac{2}{3} T_{\alpha\beta\gamma} + \frac{1}{3} T_{\alpha\gamma\beta} + \frac{1}{3} T_{\beta\alpha\gamma} + \frac{1}{3} g_{\alpha\gamma} T_\beta - \frac{1}{3} g_{\alpha\beta} T_\gamma,
\end{aligned} \tag{3.29}$$

where T is a vector, $\check{T}$ is an axial vector and κ is the pure tensorial part of torsion that satisfies

$$\kappa_\mu{}^\alpha{}_\alpha \equiv 0, \quad \epsilon^{\mu\nu\rho\lambda} \kappa_{\nu\rho\lambda} \equiv 0 \tag{3.30}$$

and $\eta^{\mu\nu\rho\lambda}$ is the Levi-Civita tensor:

$$\eta^{\mu\nu\rho\lambda} = \sqrt{g}\, \epsilon^{\mu\nu\rho\lambda}, \tag{3.31}$$

where ϵ is the Levi-Civita symbol. It is sometimes convenient to use condensed notations:

$$T_\mu T^\mu \equiv T^2, \quad \check{T}_\mu \check{T}^\mu \equiv \check{T}^2, \quad \kappa_{\alpha\beta\gamma} \kappa^{\alpha\beta\gamma} \equiv \kappa^2 \ . \tag{3.32}$$

Then,

$$\begin{aligned}
T^{\alpha\beta\gamma} T_{\alpha\beta\gamma} &= \frac{2}{3} T^2 - \frac{1}{6} \check{T}^2 + \kappa^2 \\
T^{\alpha\beta\gamma} T_{\alpha\gamma\beta} &= \frac{1}{3} T^2 + \frac{1}{6} \check{T}^2 + \frac{1}{2} \kappa^2 \ .
\end{aligned} \tag{3.33}$$

Here we display some useful consequences of (3.30) which are straightforward to prove:

$$\begin{aligned}
&\kappa_{\lambda\mu\nu} \kappa^{\lambda\nu\mu} = \frac{1}{2} \kappa^2, \\
&\kappa^{\lambda\mu\nu} \kappa_{\mu\rho\nu} = \frac{1}{2} \kappa^{\mu\lambda\nu} \kappa_{\mu\rho\nu}, \\
&\kappa^{\lambda\mu\nu} \kappa_{\rho\mu\nu} - \kappa^{\lambda\mu\nu} \kappa_{\rho\nu\mu} - \kappa_\mu{}^\lambda{}_\nu \kappa_{\mu\rho\nu} = 0, \\
&\eta_{\alpha\beta\delta\zeta} \check{T}^\alpha (\nabla^\zeta \nabla_\gamma \kappa^{\beta\gamma\delta}) - 2 \eta_{\alpha\gamma\delta\zeta} \check{T}^\alpha (\nabla^\zeta \nabla_\beta \kappa^{\beta\gamma\delta}) = 0.
\end{aligned} \tag{3.34}$$

These identities also hold through derivatives, for example,

$$\kappa_{\lambda\mu\nu} \Box \kappa^{\lambda\nu\mu} = \frac{1}{2} \kappa_{\lambda\mu\nu} \Box \kappa^{\lambda\mu\nu}. \tag{3.35}$$

From the second Bianchi identity for any vector T, we have

$$R^{\mu\nu} \nabla_\mu T_\nu = \frac{1}{2} R \nabla_\mu T^\mu. \tag{3.36}$$

We rewrite the action in terms of the decomposed fields as

$$\begin{aligned}
S_T = S_{HDG} - \frac{1}{2} \int d^4x \, \big[& m_1 T_\mu T^\mu + m_2 \check{T}_\mu \check{T}^\mu + m_3 \kappa_{\mu\nu\rho} \kappa^{\mu\nu\rho} + r_1 R \, \nabla_\mu T^\mu + r_2 R_{\mu\nu} \nabla_\alpha \kappa^\alpha{}_{\rho\lambda} \\
& + d_1 T_\mu \Box T^\mu + d_2 T_\mu \nabla_\mu \nabla^\nu T^\nu + d_3 \check{T}_\mu \Box \check{T}^\mu + d_4 \check{T}_\mu \nabla_\mu \nabla^\nu \check{T}^\nu + d_5 \kappa_{\mu\nu\rho} \Box \kappa^{\mu\nu\rho} + d_6 \kappa_{\mu\rho\lambda} \nabla^\mu \nabla_\nu \kappa^{\nu\rho\lambda} \\
& + d_7 \kappa_{\mu\rho\lambda} \nabla^\mu \nabla_\nu \kappa^{\nu\lambda\rho} + d_8 T_\rho \nabla_\mu \nabla_\nu \kappa^{\rho\mu\nu} + d_9 \eta_{\mu\nu\rho\lambda} T^\mu \nabla^\lambda \nabla_\sigma \kappa^{\sigma\nu\rho} + \ldots \big] \ .
\end{aligned} \tag{3.37}$$

where S_{HDG} is expressed in (1.17). and the dots, as usual, represent terms that do not contribute to the flat space propagator.

3.3.4 Field redefinitions

As was discussed in [86] there exists freedom to redefine torsion tensor as

$$T_{\alpha\beta\gamma} \to \alpha_1 \; T_{\alpha\beta\gamma} + \alpha_2 \; T_{\alpha\gamma\beta} + \alpha_3 \; g_{\alpha\beta} T_\gamma \tag{3.38}$$

This linear transformation is invertible if

$$\alpha_1^2 \neq \alpha_2^2, \quad \alpha_1 + \alpha_2 + \alpha_3 \neq 0. \tag{3.39}$$

We can use such redefinition to remove two terms from the Lagrangian, for example, H_5^{TT} and H_9^{TT}. We achieve this with

$$\alpha_1 = 0, \quad \frac{\alpha_3}{\alpha_2} = \frac{b_8}{b_7 - b_8}. \tag{3.40}$$

After that, there still remains freedom to perform an overall scaling of the torsion. Alternatively, working in the decomposed variables (3.29) one can remove the terms proportional to d_8 and d_9 from (3.37). This way the three components of torsion would be decoupled from each other (at the linearised level). The reabsorbed terms will reappear as loop corrections. In order to make them absent along the RG trajectory one can allow the coefficients α_1, α_2 and α_3 to depend on the cutoff. Removing the term $R^{\mu\nu}\nabla_\mu T_\nu$ requires redefining the curvature (and therefore the metric). Later we will discuss a possibility to consider more general nonlinear redefinitions and those involving derivatives [87].

3.4 Symmetric MAG

3.4.1 Einstein form

For symmetric (torsion-free) theories, one can take g and Q as fundamental variables. Then, the counting of dimension-four terms is as follows:

R^2	$(\nabla Q)^2$	$R\,\nabla Q$	$R\,Q^2$	$Q^2\nabla Q$	Q^4	Total
3	16	4	22	59	69	173

The quadratic invariants are the three R^2 terms already listed in (3.13), plus the following $(\nabla Q)^2$ terms

$$\begin{aligned}
&H_1^{QQ} = \nabla^\alpha Q^{\beta\gamma\delta}\,\nabla_\alpha Q_{\beta\gamma\delta} \;, && H_2^{QQ} = \nabla^\alpha Q^{\beta\gamma\delta}\,\nabla_\alpha Q_{\gamma\beta\delta} \;,\\
&H_3^{QQ} = \nabla^\alpha \mathrm{tr}_{(12)} Q^\beta\,\nabla_\alpha \mathrm{tr}_{(12)} Q_\beta \;, && H_4^{QQ} = \nabla^\alpha \mathrm{tr}_{(23)} Q^\beta\,\nabla_\alpha \mathrm{tr}_{(23)} Q_\beta \;,\\
&H_5^{QQ} = \nabla^\alpha \mathrm{tr}_{(12)} Q^\beta\,\nabla_\alpha \mathrm{tr}_{(23)} Q_\beta \;, && \\
&H_6^{QQ} = \mathrm{div}_{(1)} Q^{\alpha\beta}\,\mathrm{div}_{(1)} Q_{\alpha\beta} \;, && H_7^{QQ} = \mathrm{div}_{(2)} Q^{\alpha\beta}\,\mathrm{div}_{(2)} Q_{\alpha\beta} \;,\\
&H_8^{QQ} = \mathrm{div}_{(2)} Q^{\alpha\beta}\,\mathrm{div}_{(2)} Q_{\beta\alpha} \;, && H_9^{QQ} = \mathrm{div}_{(1)} Q^{\alpha\beta}\,\mathrm{div}_{(2)} Q_{\alpha\beta} \;,\\
&H_{10}^{QQ} = \mathrm{div}_{(2)} Q^{\alpha\beta}\nabla_\alpha \mathrm{tr}_{(12)} Q_\beta \;, && H_{11}^{QQ} = \mathrm{div}_{(2)} Q^{\alpha\beta}\nabla_\alpha \mathrm{tr}_{(23)} Q_\beta \;,\\
&H_{12}^{QQ} = \mathrm{div}_{(2)} Q^{\alpha\beta}\nabla_\beta \mathrm{tr}_{(12)} Q_\alpha \;, && H_{13}^{QQ} = \mathrm{div}_{(2)} Q^{\alpha\beta}\nabla_\beta \mathrm{tr}_{(23)} Q_\alpha \;,\\
&H_{14}^{QQ} = (\mathrm{trdiv}_{(1)} Q)^2 \;, && H_{15}^{QQ} = (\mathrm{trdiv}_{(2)} Q)^2 \;,\\
&H_{16}^{QQ} = \mathrm{trdiv}_{(1)} Q\,\mathrm{trdiv}_{(2)} Q \;, &&
\end{aligned} \tag{3.41}$$

and the $R\nabla Q$ terms

$$\begin{array}{lll} H_1^{RQ} = R^{\alpha\gamma\beta\delta}\nabla_\alpha Q_{\beta\gamma\delta} \ , & H_2^{RQ} = R^{\alpha\beta}\nabla_\alpha \mathrm{tr}_{(12)}Q_\beta \ , & H_3^{RQ} = R^{\alpha\beta}\nabla_\beta \mathrm{tr}_{(23)}Q_\alpha \ , \\ H_4^{RQ} = R^{\alpha\beta}\,\mathrm{div}_{(1)}Q_{\alpha\beta} \ , & H_5^{RQ} = R^{\alpha\beta}\,\mathrm{div}_{(2)}Q_{\alpha\beta} \ , & \\ H_6^{RQ} = R\,\mathrm{trdiv}_{(1)}Q \ , & H_7^{RQ} = R\,\mathrm{trdiv}_{(2)}Q \ . & \end{array} \tag{3.42}$$

Once again, not all these invariants are independent. We note that using the second Bianchi identity contracted with Q, and allowing integrations by parts, we obtain three relations

$$\begin{aligned} H_1^{RQ} &= H_4^{RQ} - H_5^{RQ} \ , \\ 2H_2^{RQ} &= H_7^{RQ} \ , \\ 2H_3^{RQ} &= H_6^{RQ} \ . \end{aligned} \tag{3.43}$$

For example, we can solve for H_1^{RQ}, H_2^{RQ}, H_3^{RQ} and keep

$$\{H_4^{RQ} \ , \ H_5^{RQ} \ , \ H_6^{RQ} \ , \ H_7^{RQ}\} \tag{3.44}$$

as independent invariants. There are therefore $3+16+4 = 23$ independent invariants quadratic in the fields.

The numbers of cubic and quartic interaction terms are determined as in the previous subsection. `AllContractions` gives 23 terms of the type RQQ, and the first Bianchi identity contracted with QQ gives one relation between them, bringing the number of independent terms of this type to 22. For $QQ\nabla Q$ terms, `AllContractions` gives 95 terms, but 36 of them are total derivatives, so the number of independent ones is 59.

3.4.2 Cartan form

Now we proceed with our consideration of Symmetric MAG in geometrically motivated Cartan form. In addition to (3.19), we have the following contributions of FDQ type:

$$\begin{array}{ll} L_1^{FQ} = F^{\mu\nu\rho\sigma}D_\mu Q_{\nu\rho\sigma} \ , & L_2^{FQ} = F^{\mu\nu\rho\sigma}D_\nu Q_{\rho\sigma\mu} \ , \\ L_3^{FQ} = F^{\mu\nu\rho\sigma}D_\nu Q_{\sigma\rho\mu} \ , & L_4^{FQ} = F^{\mu\nu\rho\sigma}D_\rho Q_{\mu\nu\sigma} \ , \\ L_5^{FQ} = F^{\mu\nu\rho\sigma}D_\sigma Q_{\mu\nu\rho} \ , & L_6^{FQ} = F^{(13)\mu\nu}D_\mu \mathrm{tr}_{(12)}Q_\nu \ , \\ L_7^{FQ} = F^{(13)\mu\nu}D_\nu \mathrm{tr}_{(12)}Q_\mu \ , & L_8^{FQ} = F^{(13)\mu\nu}D_\mu \mathrm{tr}_{(23)}Q_\nu \ , \\ L_9^{FQ} = F^{(13)\mu\nu}D_\nu \mathrm{tr}_{(23)}Q_\mu \ , & L_{10}^{FQ} = F^{(14)\mu\nu}D_\mu \mathrm{tr}_{(12)}Q_\nu \ , \\ L_{11}^{FQ} = F^{(14)\mu\nu}D_\nu \mathrm{tr}_{(12)}Q_\mu \ , & L_{12}^{FQ} = F^{(14)\mu\nu}D_\mu \mathrm{tr}_{(23)}Q_\nu \ , \\ L_{13}^{FQ} = F^{(14)\mu\nu}D_\nu \mathrm{tr}_{(23)}Q_\mu \ , & L_{14}^{FQ} = F^{(34)\mu\nu}D_\mu \mathrm{tr}_{(12)}Q_\nu \ , \\ L_{15}^{FQ} = F^{(34)\mu\nu}D_\mu \mathrm{tr}_{(23)}Q_\nu \ , & L_{16}^{FQ} = F^{(13)\mu\nu}\,\mathrm{Div}_{(1)}Q_{\mu\nu} \ , \\ L_{17}^{FQ} = F^{(14)\mu\nu}\,\mathrm{Div}_{(1)}Q_{\mu\nu} \ , & L_{18}^{FQ} = F^{(13)\mu\nu}\,\mathrm{Div}_{(2)}Q_{\mu\nu} \ , \\ L_{19}^{FQ} = F^{(13)\mu\nu}\,\mathrm{Div}_{(2)}Q_{\nu\mu} \ , & L_{20}^{FQ} = F^{(14)\mu\nu}\,\mathrm{Div}_{(2)}Q_{\mu\nu} \ , \\ L_{21}^{FQ} = F^{(14)\mu\nu}\,\mathrm{Div}_{(2)}Q_{\nu\mu} \ , & L_{22}^{FQ} = F^{(34)\mu\nu}\,\mathrm{Div}_{(2)}Q_{\mu\nu} \ , \\ L_{23}^{FQ} = F\,\mathrm{trDiv}_{(1)}Q \ , & L_{24}^{FQ} = F\,\mathrm{trDiv}_{(2)}Q \ . \end{array} \tag{3.45}$$

and of $(DQ)^2$ type:

$$
\begin{array}{llll}
L_1^{QQ} & = D^\alpha Q^{\beta\gamma\delta}\, D_\alpha Q_{\beta\gamma\delta} \ , & L_2^{QQ} & = D^\alpha Q^{\beta\gamma\delta}\, D_\alpha Q_{\gamma\beta\delta} \ , \\
L_3^{QQ} & = D^\alpha \mathrm{tr}_{(12)} Q^\beta\, D_\alpha \mathrm{tr}_{(12)} Q_\beta \ , & L_4^{QQ} & = D^\alpha \mathrm{tr}_{(23)} Q^\beta\, D_\alpha \mathrm{tr}_{(23)} Q_\beta \ , \\
L_5^{QQ} & = D^\alpha \mathrm{tr}_{(12)} Q^\beta\, D_\alpha \mathrm{tr}_{(23)} Q_\beta \ , & & \\
L_6^{QQ} & = \mathrm{Div}_{(1)} Q^{\alpha\beta}\, \mathrm{Div}_{(1)} Q_{\alpha\beta} \ , & L_7^{QQ} & = \mathrm{Div}_{(2)} Q^{\alpha\beta}\, \mathrm{Div}_{(2)} Q_{\alpha\beta} \ , \\
L_8^{QQ} & = \mathrm{Div}_{(2)} Q^{\alpha\beta}\, \mathrm{Div}_{(2)} Q_{\beta\alpha} \ , & L_9^{QQ} & = \mathrm{Div}_{(1)} Q^{\alpha\beta}\, \mathrm{Div}_{(2)} Q_{\alpha\beta} \ , \\
L_{10}^{QQ} & = \mathrm{Div}_{(2)} Q^{\alpha\beta}\, D_\alpha \mathrm{tr}_{(12)} Q_\beta \ , & L_{11}^{QQ} & = \mathrm{Div}_{(2)} Q^{\alpha\beta}\, D_\alpha \mathrm{tr}_{(23)} Q_\beta \ , \\
L_{12}^{QQ} & = \mathrm{Div}_{(2)} Q^{\alpha\beta}\, D_\beta \mathrm{tr}_{(12)} Q_\alpha \ , & L_{13}^{QQ} & = \mathrm{Div}_{(3)} Q^{\alpha\beta}\, D_\beta \mathrm{tr}_{(23)} Q_\alpha \ , \\
L_{14}^{QQ} & = (\mathrm{tr}\mathrm{Div}_{(1)} Q)^2 \ , & L_{15}^{QQ} & = (\mathrm{tr}\mathrm{Div}_{(2)} Q)^2 \ , \\
L_{16}^{QQ} & = \mathrm{tr}\mathrm{Div}_{(1)} Q\, \mathrm{tr}\mathrm{Div}_{(2)} Q \ , & &
\end{array}
\tag{3.46}
$$

In symmetric (torsion-free) MAG, the curvature tensor is only antisymmetric in the first pair of indices, but the first Bianchi identity (3.3) leads to six independent relations

$$
\begin{aligned}
L_1^{FF} - 2L_6^{FF} &= 0 \ , \\
L_2^{FF} - 2L_5^{FF} &= 0 \ , \\
L_3^{FF} - L_4^{FF} + L_5^{FF} &= 0 \ , \\
L_{13}^{FF} + 2L_{14}^{FF} &= 0 \ , \\
L_7^{FF} - L_8^{FF} + L_{14}^{FF} &= 0 \ , \\
L_{11}^{FF} - L_{12}^{FF} + L_{15}^{FF} &= 0 \ .
\end{aligned}
\tag{3.47}
$$

This reduces the number of independent curvature squared terms to 10. We keep the invariants L_i^{FF} with $i = 1, 2, 3, 7, 8, 9, 10, 11, 12, 16$ and solve for the others:

$$
\begin{array}{llll}
L_4^{FF} & = 1/2 L_2^{FF} + L_3^{FF} \ , & L_5^{FF} & = 1/2 L_2^{FF} \ , \\
L_6^{FF} & = 1/2 L_1^{FF} \ , & L_{13}^{FF} & = 2(L_7^{FF} - L_8^{FF}) \ , \\
L_{14}^{FF} & = -L_7^{FF} + L_8^{FF} \ , & L_{15}^{FF} & = -L_{11}^{FF} + L_{12}^{FF} \ .
\end{array}
\tag{3.48}
$$

Multiplying (3.3) by DQ we obtain, up to interaction terms, the relations

$$
\begin{aligned}
L_{18}^{FQ} - L_{19}^{FQ} + L_{22}^{FQ} &= 0 \ , \\
L_6^{FQ} - L_7^{FQ} + L_{14}^{FQ} &= 0 \ , \\
L_8^{FQ} - L_9^{FQ} + L_{15}^{FQ} &= 0 \ , \\
L_1^{FQ} + L_3^{FQ} + L_5^{FQ} &= 0 \ ,
\end{aligned}
\tag{3.49}
$$

and multiplying (3.4) by Q we obtain, again up to interaction terms, the relations

$$
\begin{aligned}
L_7^{FQ} - L_{11}^{FQ} - L_{24}^{FQ} &= 0 \ , \\
L_9^{FQ} - L_{13}^{FQ} - L_{23}^{FQ} &= 0 \ , \\
L_5^{FQ} - L_{17}^{FQ} + L_{20}^{FQ} &= 0 \ , \\
L_4^{FQ} - L_{16}^{FQ} + L_{18}^{FQ} &= 0 \ .
\end{aligned}
\tag{3.50}
$$

In this case, the Bianchi identities are not enough to uncover all the relations and we have to resort to another method. We can use (1.37) in the FF terms; this will give among other things RR terms, $R\nabla T$ and $R\nabla Q$. We can look for linear combinations of the FF terms such that these terms involving R in the r.h.s. cancel out. In this way, up to cubic and quartic terms, we will relate FF terms to $(DT)^2$ terms etc. From the FF terms we obtain

$$\begin{aligned}
L_1^{FF} + L_2^{FF} &= L_1^{QQ} - L_6^{QQ} \ , \\
2(L_1^{FF} - L_3^{FF}) &= 3L_1^{QQ} - 2L_2^{QQ} - 3L_6^{QQ} - 2L_7^{QQ} + 4L_9^{QQ} \ , \\
4(L_7^{FF} - L_8^{FF}) &= L_4^{QQ} - L_{14}^{QQ} \ , \\
4(L_9^{FF} - L_{10}^{FF}) &= 4L_3^{QQ} + L_4^{QQ} - 4L_5^{QQ} + 4L_7^{QQ} - 4L_8^{QQ} - 8L_{10}^{QQ} \\
&\quad + 4L_{11}^{QQ} + 8L_{12}^{QQ} - 4L_{13}^{QQ} - L_{14}^{QQ} - 4L_{15}^{QQ} + 4L_{16}^{QQ} \ , \\
4(L_{11}^{FF} - L_{12}^{FF}) &= -L_4^{QQ} + 2L_5^{QQ} - 2L_{11}^{QQ} + 2L_{13}^{QQ} + L_{14}^{QQ} - 2L_{16}^{QQ} \ , \\
L_7^{FF} + L_8^{FF} + L_9^{FF} + & \\
L_{10}^{FF} + 2L_{11}^{FF} + 2L_{12}^{FF} &= L_3^{QQ} + L_7^{QQ} + L_8^{QQ} - 2L_{10}^{QQ} - 2L_{12}^{QQ} + L_{15}^{QQ} \ .
\end{aligned} \tag{3.51}$$

Operating in a similar way on the FDQ terms we obtain

$$\begin{aligned}
2L_1^{FQ} &= L_1^{QQ} - L_6^{QQ} \ , \\
L_2^{FQ} + L_3^{FQ} &= -L_2^{QQ} + L_9^{QQ} \ , \\
2(L_2^{FQ} + L_4^{FQ}) &= L_1^{QQ} - 2L_2^{QQ} - L_6^{QQ} - 2L_7^{QQ} + 4L_9^{QQ} \ , \\
2(L_2^{FQ} - L_5^{FQ}) &= L_1^{QQ} - 2L_2^{QQ} - L_6^{QQ} + 2L_9^{QQ} \ , \\
2(L_6^{FQ} - L_7^{FQ}) &= -L_5^{QQ} + L_{16}^{QQ} \ , \\
L_6^{FQ} + L_{10}^{FQ} &= -L_3^{QQ} + L_{10}^{QQ} \ , \\
2(L_{10}^{FQ} - L_{11}^{FQ}) &= -2L_3^{QQ} + L_5^{QQ} + 2L_{10}^{QQ} - 2L_{12}^{QQ} + 2L_{15}^{QQ} - L_{16}^{QQ} \ , \\
2(L_8^{FQ} - L_9^{FQ}) &= -L_4^{QQ} + L_{14}^{QQ} \ , \\
L_8^{FQ} + L_{12}^{FQ} &= -L_5^{QQ} + L_{11}^{QQ} \ , \\
2(L_{12}^{FQ} - L_{13}^{FQ}) &= L_4^{QQ} - 2L_5^{QQ} + 2L_{11}^{QQ} - 2L_{13}^{QQ} - L_{14}^{QQ} + 2L_{16}^{QQ} \ , \\
2L_{14}^{FQ} &= L_5^{QQ} - L_{16}^{QQ} \ , \\
2L_{15}^{FQ} &= L_4^{QQ} - L_{14}^{QQ} \ , \\
L_{16}^{FQ} + L_{17}^{FQ} &= L_9^{QQ} - L_{10}^{QQ} \ , \\
2(L_{18}^{FQ} - L_{19}^{FQ}) &= -L_{11}^{QQ} + L_{13}^{QQ} \ , \\
L_{18}^{FQ} + L_{20}^{FQ} &= L_7^{QQ} - L_{10}^{QQ} \ , \\
2(L_{20}^{FQ} - L_{21}^{FQ}) &= 2L_7^{QQ} - 2L_8^{QQ} - 2L_{10}^{QQ} + L_{11}^{QQ} + 2L_{12}^{QQ} \ , \\
2L_{22}^{FQ} &= L_{11}^{QQ} - L_{13}^{QQ} \ .
\end{aligned} \tag{3.52}$$

We need one additional relation involving both FF and FDQ:

$$2(L_{10}^{FF} + L_{12}^{FF} - L_6^{FQ} + L_{18}^{FQ}) = L_5^{QQ} + 2L_8^{QQ} - L_{11}^{QQ} - 4L_{12}^{QQ} + L_{13}^{QQ} + 2L_{15}^{QQ} - L_{16}^{QQ} \ . \tag{3.53}$$

These 24 relations are all linearly independent, but they are not independent when one takes them together with the 8 relations (3.49, 3.50) coming from the Bianchi identities. In fact, the system of all 32 relations has rank 27. This means that we have 50-27=23 independent invariants, in agreement with the counting of the previous section.

We can choose as an independent set the relations (3.51, 3.52, 3.53), plus the first three relations in (3.50). There are many ways of solving these relations, but we shall consider here only two. The first is to retain all the 16 L^{QQ} terms, plus

$$\{L_1^{FF}\ ,\ L_7^{FF}\ ,\ L_{16}^{FF}\} \quad \text{and} \quad \{L_{16}^{FQ}\ ,\ L_{18}^{FQ}\ ,\ L_{23}^{FQ}\ ,\ L_{24}^{FQ}\}\ , \tag{3.54}$$

which is in one-to-one correspondence with the sum of (3.13) and (3.44). The remaining invariants are given in equation A.13 of [86].

As in the antisymmetric case, we can also keep in the basis the ten L^{FF} invariants

$$\{L_1^{FF}\ ,\ L_2^{FF}\ ,\ L_3^{FF}\ ,\ L_7^{FF}\ ,\ L_8^{FF}\ ,\ L_9^{FF}\ ,\ L_{10}^{FF}\ ,\ L_{11}^{FF}\ ,\ L_{12}^{FF}\ ,L_{16}^{FF}\}$$

plus

$$\{L_1^{QQ}\ ,\ L_{10}^{QQ}\ ,\ L_{11}^{QQ}\ ,\ L_{12}^{QQ}\ ,\ L_{14}^{QQ}\} \quad \text{and} \quad \{L_{10}^{FQ}\ ,\ L_{11}^{FQ}\ ,\ L_{12}^{FQ}\ ,\ L_{14}^{FQ}\ ,\ L_{16}^{FQ}\ ,\ L_{17}^{FQ}\ ,\ L_{18}^{FQ}\ ,\ L_{23}^{FQ}\}\ . \tag{3.55}$$

The remaining invariants are given in Equation (D.1.2) in Appendix D.1.2.

3.4.3 Decomposition of nonmetricity

Primer Decomposition

In the nonmetricity sector, we first separate the two traces

$$\begin{aligned} Q_\mu &\equiv Q_\mu^{(23)} = Q_\mu{}^\alpha{}_\alpha, \\ \hat{Q}_\mu &\equiv Q_\mu^{(12)} = Q^\alpha{}_{\alpha\mu}\ . \end{aligned} \tag{3.56}$$

Then,

$$Q_{\alpha\beta\gamma} = \frac{1}{18}\left[g_{\beta\gamma}\left(5Q_\alpha - 2\hat{Q}_\alpha\right) + g_{\alpha\beta}\left(4\hat{Q}_\gamma - Q_\gamma\right) + g_{\alpha\gamma}\left(4\hat{Q}_\beta - Q_\beta\right)\right] + q_{\alpha\beta\gamma}\ , \tag{3.57}$$

where the remaining part is denoted as q. We introduce again condensed notations

$$Q_\mu Q^\mu \equiv Q^2, \quad \hat{Q}_\mu\hat{Q}^\mu \equiv \hat{Q}^2, \quad q_{\mu\nu\rho}q^{\mu\nu\rho} \equiv q^2. \tag{3.58}$$

and we have, for example,

$$Q_{\lambda\mu\nu}Q^{\lambda\mu\nu} = \frac{5}{18}Q^2 + \frac{4}{9}\hat{Q}^2 - \frac{2}{9}Q_\lambda\hat{Q}^\lambda + q^2. \tag{3.59}$$

One also has to consider relations that hold up to our non-interacting truncation such as

$$q^{\alpha\beta\gamma}\nabla^\delta\nabla_\gamma q_{\delta\alpha\beta} - q^{\alpha\beta\gamma}\nabla_\alpha\nabla^\delta q_{\beta\gamma\delta} \simeq 0. \tag{3.60}$$

Finer Decomposition

We can further decompose q to the totally symmetric part and the rest:

$$q_{\mu\nu\rho} = s_{\mu\nu\rho} + u_{\mu\nu\rho}, \quad s_{\mu\nu\rho} = q_{(\mu\nu\rho)}. \tag{3.61}$$

Then the field T, $\check{T}$, κ, Q, $\hat{Q}$, s, u are irreducible representations of the Lorentz group. There are multiple relations that follow from the conditions $s_{\mu\nu\rho} = s_{(\mu\nu\rho)}$ and $u_{(\mu\nu\rho)} = 0$. We will display a few most relevant ones:

$$\begin{aligned} &u_{\mu\nu}{}^{\rho} + u_{\nu\mu}{}^{\rho} + u_{\rho}{}^{\mu\nu} = 0 \ , \\ &s^{\alpha\beta\gamma} u_{\alpha\beta\gamma} = 0 \ , \\ &s^{\alpha\gamma\delta}(u_{\beta\alpha\gamma} + 2u_{\alpha\beta\gamma}) = 0 \ , \\ &u^{\alpha\beta\gamma} u_{\beta\alpha\gamma} = -\frac{1}{2} u^2 \end{aligned} \tag{3.62}$$

There are also some mystically appearing identities. Notice that

$$\hat{Q}^{\alpha} u_{\alpha\beta\gamma} + 2\hat{Q}^{\alpha} u_{\beta\alpha\gamma} \neq 0 \ . \tag{3.63}$$

However,

$$\hat{Q}^{\alpha} \nabla^{\gamma} \nabla^{\beta} u_{\alpha\beta\gamma} + 2\hat{Q}^{\alpha} \nabla^{\gamma} \nabla^{\beta} u_{\beta\alpha\gamma} \simeq 0 \ . \tag{3.64}$$

After implementing these and a considerable number of other identities which are long to write down but however relatively easy to obtain, we arrive at the following results for the action of the Symmetric MAG:

$$\begin{aligned} S_Q = S_{HDG} - \frac{1}{2}\int d^4x \Big[& m_4 Q_{\alpha} Q^{\alpha} + m_5 \hat{Q}_{\alpha} \hat{Q}^{\alpha} + m_6 \hat{Q}^{\alpha} Q_{\alpha} + m_7 s_{\mu\nu\rho} s^{\mu\nu\rho} + m_8 u_{\mu\nu\rho} u^{\mu\nu\rho} \\ & + r_3 R \nabla_{\alpha} Q^{\alpha} + r_4 R \nabla_{\alpha} \hat{Q}^{\alpha} + r_5 R_{\mu\nu} \nabla_{\rho} s^{\rho\mu\nu} + r_6 R_{\mu\nu} \nabla_{\rho} u^{\rho\mu\nu} \\ & + d_{10} Q_{\alpha} \Box Q^{\alpha} + d_{11} Q^{\alpha} \nabla_{\alpha} \nabla_{\beta} Q^{\beta} + d_{12} \hat{Q}_{\alpha} \Box \hat{Q}^{\alpha} + d_{13} \hat{Q}^{\alpha} \nabla_{\alpha} \nabla_{\beta} \hat{Q}^{\beta} + d_{14} \hat{Q}_{\alpha} \Box Q^{\alpha} \\ & + d_{15} \hat{Q}^{\alpha} \nabla_{\alpha} \nabla_{\beta} Q^{\beta} + d_{16} s_{\mu\nu\rho} \Box s^{\mu\nu\rho} + d_{17} s^{\mu}{}_{\alpha\beta} \nabla_{\mu} \nabla_{\nu} s^{\nu\alpha\beta} + d_{18} u_{\mu\nu\rho} \Box u^{\mu\nu\rho} \\ & + d_{19} u^{\mu}{}_{\alpha\beta} \nabla_{\mu} \nabla_{\nu} u^{\nu\alpha\beta} + d_{20} u_{\alpha\beta}{}^{\mu} \nabla_{\mu} \nabla_{\nu} u^{\alpha\beta\nu} + d_{21} u^{\mu}{}_{\alpha\beta} \nabla_{\mu} \nabla_{\nu} s^{\nu\alpha\beta} + d_{22} Q_{\rho} \nabla_{\mu} \nabla_{\nu} s^{\rho\mu\nu} \\ & + d_{23} \hat{Q}_{\rho} \nabla_{\mu} \nabla_{\nu} s^{\rho\mu\nu} + d_{24} Q_{\rho} \nabla_{\mu} \nabla_{\nu} u^{\rho\mu\nu} + d_{25} \hat{Q}_{\rho} \nabla_{\mu} \nabla_{\nu} u^{\rho\mu\nu} \Big] \ , \end{aligned} \tag{3.65}$$

where S_{HDG} is expressed in (1.17).

3.5 General MAG

3.5.1 Einstein form

In the general case, the counting is simpler if we use ϕ as a variable, rather than T and Q. Then we have

R^2	$(\nabla\phi)^2$	$R\,\nabla\phi$	$R\,\phi^2$	$\phi^2\nabla\phi$	ϕ^4	Total
3	38	6	56	315	504	922

The list of the $(\nabla\phi)^2$ terms is [3]

$$
\begin{array}{lll}
H^{\phi\phi}_1 = \nabla^\alpha\phi^{\beta\gamma\delta}\nabla_\alpha\phi_{\beta\gamma\delta} \ , & H^{\phi\phi}_2 = \nabla^\alpha\phi^{\beta\gamma\delta}\nabla_\alpha\phi_{\beta\delta\gamma} \ , & H^{\phi\phi}_3 = \nabla^\alpha\phi^{\beta\gamma\delta}\nabla_\alpha\phi_{\delta\gamma\beta} \ , \\
H^{\phi\phi}_4 = \nabla^\alpha\phi^{\beta\gamma\delta}\nabla_\alpha\phi_{\gamma\beta\delta} \ , & H^{\phi\phi}_5 = \nabla^\alpha\phi^{\beta\gamma\delta}\nabla_\alpha\phi_{\delta\beta\gamma} \ , & \\
H^{\phi\phi}_6 = \nabla^\alpha \mathrm{tr}_{(12)}\phi^\beta\nabla_\alpha \mathrm{tr}_{(12)}\phi_\beta \ , & H^{\phi\phi}_7 = \nabla^\alpha \mathrm{tr}_{(13)}\phi^\beta\nabla_\alpha \mathrm{tr}_{(13)}\phi_\beta \ , & H^{\phi\phi}_8 = \nabla^\alpha \mathrm{tr}_{(23)}\phi^\beta\nabla_\alpha \mathrm{tr}_{(23)}\phi_\beta \ , \\
H^{\phi\phi}_9 = \nabla^\alpha \mathrm{tr}_{(12)}\phi^\beta\nabla_\alpha \mathrm{tr}_{(13)}\phi_\beta \ , & H^{\phi\phi}_{10} = \nabla^\alpha \mathrm{tr}_{(12)}\phi^\beta\nabla_\alpha \mathrm{tr}_{(23)}\phi_\beta \ , & H^{\phi\phi}_{11} = \nabla^\alpha \mathrm{tr}_{(13)}\phi^\beta\nabla_\alpha \mathrm{tr}_{(23)}\phi_\beta \ , \\
H^{\phi\phi}_{12} = \mathrm{div}_{(1)}\phi^{\alpha\beta}\mathrm{div}_{(1)}\phi_{\alpha\beta} \ , & H^{\phi\phi}_{13} = \mathrm{div}_{(1)}\phi^{\alpha\beta}\mathrm{div}_{(1)}\phi_{\beta\alpha} \ , & \\
H^{\phi\phi}_{14} = \mathrm{div}_{(2)}\phi^{\alpha\beta}\mathrm{div}_{(2)}\phi_{\alpha\beta} \ , & H^{\phi\phi}_{15} = \mathrm{div}_{(2)}\phi^{\alpha\beta}\mathrm{div}_{(2)}\phi_{\beta\alpha} \ , & \\
H^{\phi\phi}_{16} = \mathrm{div}_{(3)}\phi^{\alpha\beta}\mathrm{div}_{(3)}\phi_{\alpha\beta} \ , & H^{\phi\phi}_{17} = \mathrm{div}_{(3)}\phi^{\alpha\beta}\mathrm{div}_{(3)}\phi_{\beta\alpha} \ , & \\
H^{\phi\phi}_{18} = \mathrm{div}_{(1)}\phi^{\alpha\beta}\mathrm{div}_{(2)}\phi_{\alpha\beta} \ , & H^{\phi\phi}_{19} = \mathrm{div}_{(1)}\phi^{\alpha\beta}\mathrm{div}_{(2)}\phi_{\beta\alpha} \ , & \\
H^{\phi\phi}_{20} = \mathrm{div}_{(1)}\phi^{\alpha\beta}\mathrm{div}_{(3)}\phi_{\alpha\beta} \ , & H^{\phi\phi}_{21} = \mathrm{div}_{(1)}\phi^{\alpha\beta}\mathrm{div}_{(3)}\phi_{\beta\alpha} \ , & \\
H^{\phi\phi}_{22} = \mathrm{div}_{(2)}\phi^{\alpha\beta}\mathrm{div}_{(3)}\phi_{\alpha\beta} \ , & H^{\phi\phi}_{23} = \mathrm{div}_{(2)}\phi^{\alpha\beta}\mathrm{div}_{(3)}\phi_{\beta\alpha} \ , & \\
H^{\phi\phi}_{24} = \mathrm{div}_{(1)}\phi^{\alpha\beta}\nabla_\alpha \mathrm{tr}_{(12)}\phi_\beta \ , & H^{\phi\phi}_{25} = \mathrm{div}_{(1)}\phi^{\alpha\beta}\nabla_\alpha \mathrm{tr}_{(13)}\phi_\beta \ , & H^{\phi\phi}_{26} = \mathrm{div}_{(1)}\phi^{\alpha\beta}\nabla_\alpha \mathrm{tr}_{(23)}\phi_\beta \ , \\
H^{\phi\phi}_{27} = \mathrm{div}_{(3)}\phi^{\alpha\beta}\nabla_\alpha \mathrm{tr}_{(12)}\phi_\beta \ , & H^{\phi\phi}_{28} = \mathrm{div}_{(3)}\phi^{\alpha\beta}\nabla_\alpha \mathrm{tr}_{(13)}\phi_\beta \ , & H^{\phi\phi}_{29} = \mathrm{div}_{(3)}\phi^{\alpha\beta}\nabla_\alpha \mathrm{tr}_{(23)}\phi_\beta \ , \\
H^{\phi\phi}_{30} = \mathrm{div}_{(2)}\phi^{\alpha\beta}\nabla_\beta \mathrm{tr}_{(12)}\phi_\alpha \ , & H^{\phi\phi}_{31} = \mathrm{div}_{(2)}\phi^{\alpha\beta}\nabla_\beta \mathrm{tr}_{(13)}\phi_\alpha \ , & H^{\phi\phi}_{32} = \mathrm{div}_{(2)}\phi^{\alpha\beta}\nabla_\beta \mathrm{tr}_{(23)}\phi_\alpha \ , \\
H^{\phi\phi}_{33} = (\mathrm{trdiv}_{(1)}\phi)^2 \ , & H^{\phi\phi}_{34} = (\mathrm{trdiv}_{(2)}\phi)^2 \ , & H^{\phi\phi}_{35} = (\mathrm{trdiv}_{(3)}\phi)^2 \ , \\
H^{\phi\phi}_{36} = \mathrm{trdiv}_{(1)}\phi\,\mathrm{trdiv}_{(2)}\phi \ , & H^{\phi\phi}_{37} = \mathrm{trdiv}_{(1)}\phi\,\mathrm{trdiv}_{(3)}\phi \ , & H^{\phi\phi}_{38} = \mathrm{trdiv}_{(2)}\phi\,\mathrm{trdiv}_{(3)}\phi \ .
\end{array}
\tag{3.66}
$$

Note that the contraction of indices in the terms $H^{\phi\phi}_{30}$ - $H^{\phi\phi}_{32}$ is different from the order in the preceding six terms. This is necessary to make them independent. In fact, another way of writing those nine terms is

$$
\begin{array}{lll}
H^{\phi\phi}_{24} = -\mathrm{div}_{(12)}\phi^\alpha \mathrm{tr}_{(12)}\phi_\alpha \ , & H^{\phi\phi}_{25} = -\mathrm{div}_{(12)}\phi^\alpha \mathrm{tr}_{(13)}\phi_\alpha \ , & H^{\phi\phi}_{26} = -\mathrm{div}_{(12)}\phi^\alpha \mathrm{tr}_{(23)}\phi_\alpha \ , \\
H^{\phi\phi}_{27} = -\mathrm{div}_{(13)}\phi^\alpha \mathrm{tr}_{(12)}\phi_\alpha \ , & H^{\phi\phi}_{28} = -\mathrm{div}_{(13)}\phi^\alpha \mathrm{tr}_{(13)}\phi_\alpha \ , & H^{\phi\phi}_{29} = -\mathrm{div}_{(13)}\phi^\alpha \mathrm{tr}_{(23)}\phi_\alpha \ , \\
H^{\phi\phi}_{30} = -\mathrm{div}_{(23)}\phi^\alpha \mathrm{tr}_{(12)}\phi_\alpha \ , & H^{\phi\phi}_{31} = -\mathrm{div}_{(23)}\phi^\alpha \mathrm{tr}_{(13)}\phi_\alpha \ , & H^{\phi\phi}_{32} = -\mathrm{div}_{(23)}\phi^\alpha \mathrm{tr}_{(23)}\phi_\alpha \ .
\end{array}
\tag{3.67}
$$

The $R\nabla\phi$ terms are

$$
\begin{array}{lll}
H^{R\phi}_1 = R^{\alpha\beta\gamma\delta}\nabla_\alpha\phi_{\beta\gamma\delta} \ , & H^{R\phi}_2 = R^{\alpha\beta\gamma\delta}\nabla_\delta\phi_{\alpha\gamma\beta} \ , & H^{R\phi}_3 = R^{\alpha\beta\gamma\delta}\nabla_\delta\phi_{\alpha\beta\gamma} \ , \\
H^{R\phi}_4 = R^{\alpha\beta}\nabla_\alpha \mathrm{tr}_{(12)}\phi_\beta \ , & H^{R\phi}_5 = R^{\alpha\beta}\nabla_\alpha \mathrm{tr}_{(13)}\phi_\beta \ , & H^{R\phi}_6 = R^{\alpha\beta}\nabla_\alpha \mathrm{tr}_{(23)}\phi_\beta \ , \\
H^{R\phi}_7 = R^{\alpha\beta}\mathrm{div}_{(1)}\phi_{\alpha\beta} \ , & H^{R\phi}_8 = R^{\alpha\beta}\mathrm{div}_{(2)}\phi_{\alpha\beta} \ , & H^{R\phi}_9 = R^{\alpha\beta}\mathrm{div}_{(3)}\phi_{\alpha\beta} \ , \\
H^{R\phi}_{10} = R\,\mathrm{trdiv}_{(1)}\phi \ , & H^{R\phi}_{11} = R\,\mathrm{trdiv}_{(2)}\phi \ , & H^{R\phi}_{12} = R\,\mathrm{trdiv}_{(3)}\phi \ .
\end{array}
\tag{3.68}
$$

Using the first Bianchi identity for ∇ and contracting with $\nabla\phi$ we obtain the relation

$$H^{R\phi}_1 + H^{R\phi}_2 - H^{R\phi}_3 = 0 \ , \tag{3.69}$$

Contracting the second Bianchi identity with ϕ and using integrations by parts, one finds:

$$
\begin{aligned}
H^{R\phi}_1 - H^{R\phi}_8 + H^{R\phi}_9 &= 0 \ , \\
H^{R\phi}_2 + H^{R\phi}_7 - H^{R\phi}_9 &= 0 \ , \\
H^{R\phi}_3 + H^{R\phi}_7 - H^{R\phi}_8 &= 0 \ , \\
2H^{R\phi}_4 - H^{R\phi}_{12} &= 0 \ , \\
2H^{R\phi}_5 - H^{R\phi}_{11} &= 0 \ , \\
2H^{R\phi}_6 - H^{R\phi}_{10} &= 0 \ .
\end{aligned}
\tag{3.70}
$$

[3]note that up to terms of the form $R\nabla\phi$, $\mathrm{div}_{(1)}\phi^{\alpha\beta}\nabla_\beta \mathrm{tr}_{(12)}\phi_\alpha = \mathrm{div}_{(3)}\phi^{\alpha\beta}\nabla_\alpha \mathrm{tr}_{(12)}\phi_\beta$ etc.

A linear combination of the first three is equivalent to (3.69), so there are six independent relations. Using these we can eliminate six invariants, bringing the number of $R\nabla\phi$ terms from 12 to 6, as indicated in the table. For example, we can solve for $H_1^{R\phi}$, $H_2^{R\phi}$, $H_3^{R\phi}$, $H_4^{R\phi}$, $H_5^{R\phi}$, $H_6^{R\phi}$ and keep

$$\{H_7^{R\phi}\ ,\ H_8^{R\phi}\ ,\ H_9^{R\phi}\ ,\ H_{10}^{R\phi}\ ,\ H_{11}^{R\phi}\ ,\ H_{12}^{R\phi}\} \tag{3.71}$$

as independent invariants. There are therefore $3+38+6 = 47$ independent invariants quadratic in the fields.

In the following, we will mostly use T and Q as independent fields instead of ϕ. Then the kinetic terms for these fields would be given by (3.14, 3.41) and by the following $\nabla T \nabla Q$ terms:

$$\begin{aligned}
H_1^{TQ} &= \nabla^\alpha T^{\beta\gamma\delta}\,\nabla_\alpha Q_{\beta\gamma\delta}\ , & H_2^{TQ} &= \nabla^\alpha \mathrm{tr}_{(12)}T^\beta\,\nabla_\alpha \mathrm{tr}_{(12)}Q_\beta\ ,\\
H_3^{TQ} &= \nabla^\alpha \mathrm{tr}_{(12)}T^\beta\,\nabla_\alpha \mathrm{tr}_{(23)}Q_\beta\ , & &\\
H_4^{TQ} &= \mathrm{div}_{(1)}T^{\alpha\beta}\,\mathrm{div}_{(1)}Q_{\alpha\beta}\ , & H_5^{TQ} &= \mathrm{div}_{(2)}T^{\alpha\beta}\,\mathrm{div}_{(2)}Q_{\alpha\beta}\ ,\\
H_6^{TQ} &= \mathrm{div}_{(1)}T^{\alpha\beta}\,\mathrm{div}_{(2)}Q_{\alpha\beta}\ , & H_7^{TQ} &= \mathrm{div}_{(1)}T^{\alpha\beta}\,\mathrm{div}_{(2)}Q_{\beta\alpha}\ ,\\
H_8^{TQ} &= \mathrm{div}_{(2)}T^{\alpha\beta}\nabla_\alpha \mathrm{tr}_{(12)}Q_\beta\ , & H_9^{TQ} &= \mathrm{div}_{(2)}T^{\alpha\beta}\nabla_\alpha \mathrm{tr}_{(23)}Q_\beta\ ,\\
H_{10}^{TQ} &= \mathrm{div}_{(2)}Q^{\alpha\beta}\nabla_\alpha \mathrm{tr}_{(12)}T_\beta\ , & H_{11}^{TQ} &= \mathrm{div}_{(2)}Q^{\alpha\beta}\nabla_\beta \mathrm{tr}_{(12)}T_\alpha\ ,\\
H_{12}^{TQ} &= \mathrm{trdiv}_{(1)}T\,\mathrm{trdiv}_{(1)}Q\ , & H_{13}^{TQ} &= \mathrm{trdiv}_{(1)}T\,\mathrm{trdiv}_{(2)}Q\ .
\end{aligned} \tag{3.72}$$

We count 9 $(\nabla T)^2$ terms, 16 $(\nabla Q)^2$ terms and 13 $\nabla T\nabla Q$ terms. In total they amount to 38 terms, that can be used interchangeably with the 38 $(\nabla\phi)^2$ terms listed above. A basis for the quadratic terms is given by these 38 terms, plus the three R^2 terms, plus

$$\{H_3^{RT}\ ,\ H_5^{RT}\ ,\ H_4^{RQ}\ ,\ H_5^{RQ}\ ,\ H_6^{RQ}\ ,\ H_7^{RQ}\}\ , \tag{3.73}$$

(which is the union of (3.18) and (3.44)), for a total 47 terms.

For the cubic interactions, `AllContractions` gives 65 terms of the type $R\phi\phi$, but the first Bianchi identity, contracted with $\phi\phi$, yields 9 relations between them, so that the number of independent ones is 56. [4] `AllContractions` also gives 483 terms of the form $\phi\phi\nabla\phi$, out of which 168 are total derivatives, so the number of independent ones is 315. The numbers are obviously the same if one uses T and Q as variables.

3.5.2 Cartan Form

DTDQ terms

$$\begin{aligned}
L_1^{TQ} &= D^\alpha T^{\beta\gamma\delta}\,D_\alpha Q_{\beta\gamma\delta}\ , & L_2^{TQ} &= D^\alpha \mathrm{tr}_{(12)}T^\beta\,D_\alpha \mathrm{tr}_{(12)}Q_\beta\ ,\\
L_3^{TQ} &= D^\alpha \mathrm{tr}_{(12)}T^\beta\,D_\alpha \mathrm{tr}_{(23)}Q_\beta\ , & &\\
L_4^{TQ} &= \mathrm{Div}_{(1)}T^{\alpha\beta}\,\mathrm{Div}_{(1)}Q_{\alpha\beta}\ , & L_5^{TQ} &= \mathrm{Div}_{(2)}T^{\alpha\beta}\,\mathrm{Div}_{(2)}Q_{\alpha\beta}\ ,\\
L_6^{TQ} &= \mathrm{Div}_{(1)}T^{\alpha\beta}\,\mathrm{Div}_{(2)}Q_{\alpha\beta}\ , & L_7^{TQ} &= \mathrm{Div}_{(1)}T^{\alpha\beta}\,\mathrm{Div}_{(2)}Q_{\beta\alpha}\ ,\\
L_8^{TQ} &= \mathrm{Div}_{(2)}T^{\alpha\beta}D_\alpha \mathrm{tr}_{(12)}Q_\beta\ , & L_9^{TQ} &= \mathrm{Div}_{(2)}T^{\alpha\beta}D_\alpha \mathrm{tr}_{(23)}Q_\beta\ ,\\
L_{10}^{TQ} &= \mathrm{Div}_{(2)}Q^{\alpha\beta}D_\alpha \mathrm{tr}_{(12)}T_\beta\ , & L_{11}^{TQ} &= \mathrm{Div}_{(2)}Q^{\alpha\beta}D_\beta \mathrm{tr}_{(12)}T_\alpha\ ,\\
L_{12}^{TQ} &= \mathrm{trDiv}_{(1)}T\,\mathrm{trDiv}_{(1)}Q\ , & L_{13}^{TQ} &= \mathrm{trDiv}_{(1)}T\,\mathrm{trDiv}_{(2)}Q\ .
\end{aligned} \tag{3.74}$$

[4] The nine relations can be most easily counted in terms of $R\phi\phi$, but they are equivalent to the 4 relations that we have already mentioned for the RTT terms, one relation already mentioned for the RQQ terms and four additional ones for the RTQ terms.

We have listed in (3.19),(3.20), (3.21),(3.45), (3.46), (3.74), 16 terms of type FF, 38 terms of type $D(T/Q)^2$ and 45 terms of type $FD(T/Q)$. We thus have 99 quadratic terms, compared to the 47 ones of the Einstein form of the theory. We now look for linear relations between these terms. As in the previous sections, these relations hold up to terms cubic and quartic in F, T, Q.

Multiplying the first Bianchi identity by F we get

$$\begin{aligned}
L_1^{FF} - 2L_6^{FF} &= 2L_1^{FT} + L_7^{FT} \ , \\
L_2^{FF} - 2L_5^{FF} &= 2L_2^{FT} + L_5^{FT} \ , \\
L_3^{FF} - L_4^{FF} + L_5^{FF} &= -L_3^{FT} + L_4^{FT} - L_6^{FT} \ , \\
L_7^{FF} - L_8^{FF} + L_{14}^{FF} &= -L_8^{FT} + L_9^{FT} + L_{17}^{FT} \ , \\
L_{11}^{FF} - L_{12}^{FF} + L_{15}^{FF} &= -L_{10}^{FT} + L_{11}^{FT} + L_{18}^{FT} \ , \\
L_{13}^{FF} + 2L_{14}^{FF} &= -2L_{12}^{FT} + L_{20}^{FT} \ ,
\end{aligned} \tag{3.75}$$

while multiplying it by DT we get

$$\begin{aligned}
2L_1^{FT} + L_7^{FT} &= L_1^{TT} - 2L_4^{TT} \ , \\
L_2^{FT} + L_3^{FT} + L_6^{FT} &= L_2^{TT} - L_5^{TT} - L_7^{TT} \ , \\
2L_4^{FT} - L_5^{FT} &= -L_6^{TT} + 2L_7^{TT} \ , \\
L_8^{FT} - L_9^{FT} + L_{12}^{FT} &= -L_3^{TT} + L_8^{TT} + L_9^{TT} \ , \\
L_{13}^{FT} - L_{14}^{FT} + L_{19}^{FT} &= L_7^{TT} - L_8^{TT} \ , \\
2L_{17}^{FT} + L_{20}^{FT} &= L_6^{TT} - 2L_8^{TT} \ ,
\end{aligned} \tag{3.76}$$

and multiplying it by DQ we get

$$\begin{aligned}
L_1^{FQ} + L_3^{FQ} + L_5^{FQ} &= L_1^{TQ} - L_4^{TQ} + L_7^{TQ} \ , \\
L_6^{FQ} - L_7^{FQ} + L_{14}^{FQ} &= -L_2^{TQ} + L_8^{TQ} - L_{13}^{TQ} \ , \\
L_8^{FQ} - L_9^{FQ} + L_{15}^{FQ} &= -L_3^{TQ} + L_9^{TQ} - L_{12}^{TQ} \ , \\
L_{18}^{FQ} - L_{19}^{FQ} + L_{22}^{FQ} &= L_5^{TQ} - L_{10}^{TQ} + L_{11}^{TQ} \ .
\end{aligned} \tag{3.77}$$

Multiplying the second Bianchi identity by T we get

$$\begin{aligned}
L_5^{FT} - 2L_{14}^{FT} &= 0 \ , \\
L_7^{FT} - 2L_{16}^{FT} &= 0 \ , \\
L_4^{FT} - L_{13}^{FT} + L_{17}^{FT} &= 0 \ , \\
L_6^{FT} - L_{15}^{FT} + L_{18}^{FT} &= 0 \ , \\
L_9^{FT} - L_{11}^{FT} + L_{21}^{FT} &= 0 \ , \\
2L_{19}^{FT} - L_{20}^{FT} &= 0 \ ,
\end{aligned} \tag{3.78}$$

and multiplying it by Q we get

$$\begin{aligned}
L_4^{FQ} - L_{16}^{FQ} + L_{18}^{FQ} &= 0 \ , \\
L_5^{FQ} - L_{17}^{FQ} + L_{20}^{FQ} &= 0 \ , \\
L_7^{FQ} - L_{11}^{FQ} - L_{24}^{FQ} &= 0 \ , \\
L_9^{FQ} - L_{13}^{FQ} - L_{23}^{FQ} &= 0 \ .
\end{aligned} \tag{3.79}$$

In total these are 26 relations, of which 25 are independent. [5] One can obtain an independent set by eliminating for example the fifth relation in (3.78).

As in the case of symmetric MAG, the Bianchi identities do not exhaust the set of linear relations between the invariants. The additional ones can be obtained by the same method that we used for symmetric MAGs, namely using (1.37) and eliminating terms of the form R^2, $R\nabla T$ and $R\nabla Q$ from the right hand side. This gives many additional relations that are listed in Appendix D.1. Considering also these, we have altogether a system of 70 relations of which 52 are independent. Since the initial number of invariants is 99, we remain with 47 independent invariants, in agreement with the counting in the Einstein form of the theory.

We can now exhibit two bases. The first consists of all the 38 L^{TT}, L^{QQ} and L^{TQ} terms, plus

$$\{L_1^{FF}\ ,\ L_7^{FF}\ ,\ L_{16}^{FF}\} \quad \text{and} \quad \{L_{13}^{FT}\ ,\ L_{21}^{FT}\} \quad \text{and} \quad \{L_{16}^{FQ}\ ,\ L_{18}^{FQ}\ ,\ L_{23}^{FQ}\ ,\ L_{24}^{FQ}\}\ , \tag{3.80}$$

which is the sum of (3.27) and (3.54), and thus is in one-to-one correspondence with the sum of (3.13) and (3.73). The remaining invariants are given in Equations (D.1.3-D.1.3-D.1.3) of Appendix D.1.3.

As before, we can also choose as a basis all the 16 L^{FF} invariants in (3.19) plus

$$\begin{aligned}
&\{L_1^{TT}\ ,\ L_2^{TT}\ ,\ L_3^{TT}\ ,\ L_5^{TT}\}\\
&\{L_1^{QQ}\ ,\ L_{10}^{QQ}\ ,\ L_{11}^{QQ}\ ,\ L_{12}^{QQ}\ ,\ L_{14}^{QQ}\}\\
&\{L_1^{TQ}, L_{10}^{TQ}, L_{11}^{TQ}, L_{12}^{TQ}\}\\
&\{L_1^{FT}\ ,\ L_8^{FT}\ ,\ L_9^{FT},\ L_{12}^{FT}\ ,\ L_{13}^{FT}\ ,\ L_{14}^{FT}\ ,\ L_{15}^{FT}\ ,\ L_{18}^{FT}\ ,\ L_{21}^{FT}\}\\
&\{L_{10}^{FQ}\ ,\ L_{11}^{FQ}\ ,\ L_{12}^{FQ}\ ,\ L_{14}^{FQ}\ ,\ L_{16}^{FQ}\ ,\ L_{17}^{FQ}\ ,\ L_{18}^{FQ}\ ,\ L_{19}^{FQ}\ ,\ L_{23}^{FQ}\}\ .
\end{aligned} \tag{3.81}$$

The remaining invariants can be expressed as linear combination of these. Explicit formulas are given in Equations (D.1.3-D.1.3-D.1.3) in Appendix D.1.3.

We observe that the bases given for antisymmetric and symmetric MAGs can be obtained from these by dropping the terms that contain Q and T respectively. In the case of the first basis this is enough. In the case of the second basis, one has to further eliminate certain terms of type FF, FDT or FDQ.

3.5.3 Decomposition of the mixed sector

We list some of the additional identities that we will need

$$\begin{aligned}
&\kappa^{\alpha\beta\gamma}q_{\alpha\beta\delta} - \kappa^{\alpha\beta\gamma}q_{\beta\alpha\delta} - \kappa^{\alpha\gamma\beta}q_{\alpha\beta\delta} = 0,\\
&u^{\alpha\beta\gamma}\left(2\nabla_\gamma s_{\alpha\beta}{}^{\delta} + \nabla_\alpha s_{\beta\gamma}{}^{\delta}\right) = 0,\\
&\left(u^{\alpha\beta\gamma} + 2u^{\gamma\alpha\beta}\right)\nabla_\gamma\nabla_\beta T_\alpha \simeq 0.
\end{aligned} \tag{3.82}$$

[5] Twice the first of (3.75), minus the second minus the fourth, minus the second of (3.78), minus twice the third plus the fifth, is identically zero.

Finally, the resulting decomposed action of general MAG becomes:

$$
\begin{aligned}
S_{TQ} = S_{HDG} + S_T + S_Q - \frac{1}{2}\int d^4x \Big[& m_9 Q^\alpha T_\alpha + m_{10}\hat{Q}^\alpha T_\alpha + m_{11} u^{\rho\mu\nu}\kappa_{\rho\mu\nu} \\
& + d_{26} T_\alpha \Box Q^\alpha + d_{27} T^\alpha \nabla_\alpha \nabla_\beta Q^\beta + d_{28} T_\alpha \Box \hat{Q}^\alpha + d_{29} T^\alpha \nabla_\alpha \nabla_\beta \hat{Q}^\beta + d_{30} Q_\alpha \nabla_\beta \nabla_\gamma \kappa^{\alpha\beta\gamma} \\
& + d_{31} \hat{Q}_\alpha \nabla_\beta \nabla_\gamma \kappa^{\alpha\beta\gamma} + d_{32} T_\alpha \nabla_\beta \nabla_\gamma s^{\alpha\beta\gamma} + d_{33} T_\alpha \nabla_\beta \nabla_\gamma u^{\alpha\beta\gamma} + d_{34} \eta_{\alpha\beta\gamma\delta} \check{T}^\alpha \nabla^\beta \nabla_\zeta u^{\gamma\delta\zeta} \\
& + d_{35} \kappa^\mu{}_{\alpha\beta} \nabla_\mu \nabla_\nu s^{\nu\alpha\beta} + d_{36} \kappa_{\alpha\beta\gamma} \Box u^{\alpha\beta\gamma} + d_{37} \kappa^\mu{}_{\alpha\beta} \nabla_\mu \nabla_\nu u^{\nu\alpha\beta} + d_{38} \kappa_{\alpha\beta}{}^\mu \nabla_\mu \nabla_\nu u^{\alpha\beta\nu} \Big]
\end{aligned} \tag{3.83}
$$

3.5.4 Linear field redefinitions in general MAG

Next, we consider redefinitions of the distortion. It is enough to consider redefinitions that are linear in the distortion and either ultralocal (i.e. do not contain derivatives) or contain two derivatives. The former map mass terms to mass terms and kinetic terms to kinetic terms; the latter map mass terms to kinetic terms. More complicated redefinitions will only affect the interaction terms. The linear ultralocal redefinitions of ϕ are

$$
\begin{aligned}
\delta\phi_{\alpha\beta\gamma} = {} & \alpha_1 \phi_{\alpha\beta\gamma} + \alpha_2 \phi_{\beta\gamma\alpha} + \alpha_3 \phi_{\gamma\alpha\beta} + \alpha_4 \phi_{\alpha\gamma\beta} + \alpha_5 \phi_{\gamma\beta\alpha} + \alpha_6 \phi_{\beta\alpha\gamma} \\
& + g_{\alpha\beta} \left(\alpha_7 \mathrm{tr}_{(12)}\phi_\gamma + \alpha_8 \mathrm{tr}_{(13)}\phi_\gamma + \alpha_9 \mathrm{tr}_{(23)}\phi_\gamma \right) \\
& + g_{\alpha\gamma} \left(\alpha_{10} \mathrm{tr}_{(12)}\phi_\beta + \alpha_{11} \mathrm{tr}_{(13)}\phi_\beta + \alpha_{12} \mathrm{tr}_{(23)}\phi_\beta \right) \\
& + g_{\beta\gamma} \left(\alpha_{13} \mathrm{tr}_{(12)}\phi_\alpha + \alpha_{14} \mathrm{tr}_{(13)}\phi_\alpha + \alpha_{15} \mathrm{tr}_{(23)}\phi_\alpha \right) \ .
\end{aligned} \tag{3.84}
$$

Using such field redefinition, one can remove 14 terms from the Lagrangian.

3.6 Maps

For certain purposes it is useful to have the map between the coefficients of the Lagrangian in the Cartan form and in the Einstein form. This has already been discussed in the case of the terms of dimension two. For the terms of dimension four, we shall limit ourselves to the transformation of the 47 quadratic terms. The procedure has already been described in sect.3.1. Inserting (1.37) in (3.1), a straightforward calculation leads to a Lagrangian of the form (3.2), whose b coefficients are functions of the original c coefficients. These linear relations are given in Appendix D.2.2. We furthermore provide a map between Einstein and decomposed forms therein.

3.7 Conclusions

Leaving aside the cosmological term, and the possibility that distortion may contain a massless state, the dynamics of MAGs at very low energies (by which we mean energies below all the masses that are present in the theory) is dominated by the 12 dimension-two terms. These comprise the Palatini term and terms quadratic in distortion (or equivalently in torsion and nonmetricity). In this regime the theory behaves like simple Palatini theory: the equations of motion generically imply that the connection has to be equal to the LC connection. Thus, unless the distortion contains some massless state, at sufficiently low energy the EFT of MAG becomes indistinguishable form the EFT of the metric theory of gravity. If the masses of the distortion (or equivalently of torsion and nonmetricity) are much lower that the Planck mass, which we assume to be the UV cutoff for this EFT, there will be a regime where distortion could propagate. For this one has to consider also the dimension-four terms, of which, in a general MAG, there are 934.

Already listing bases of independent terms requires considerable work. We have restricted our attention mainly to the terms of dimension 4 that are quadratic in (R, T, Q) (in the Einstein form) or (F, T, Q) (in the Cartan form). These are the only terms that contribute to the propagator in flat space. We found that there are 47 independent invariants, that have to be picked among 53 invariants in the Einstein form of the theory and 99 invariants in the Cartan form. Listing the independent terms in the Lagrangian implies a choice of basis and we have given two examples of such bases, one containing all terms quadratic in (T, Q), plus more, and one containing all terms quadratic in F, plus more.

Chapter 4

Propagators in flat space

4.1 Linearised action

We consider the linearisation of the action around Minkowski space

$$g_{\mu\nu} = \eta_{\mu\nu} \ , \quad A_\rho{}^\mu{}_\nu = 0 \ , \quad \phi_\rho{}^\mu{}_\nu = 0 \ . \tag{4.1}$$

The terms in the Lagrangian that contribute at quadratic order in the fluctuation fields are those that are quadratic in F, T and Q, including also covariant derivatives of T and Q. In the Cartan form, these are

$$\begin{aligned}\mathcal{L}_C = -\frac{1}{2}\Big[&- a^F F + \sum_i a_i^{TT} M_i^{TT} + \sum_i a_i^{TQ} M_i^{TQ} + \sum_i a_i^{QQ} M_i^{QQ} \\ &+\sum_i c_i^{FF} L_i^{FF} + \sum_i c_i^{FT} L_i^{FT} + \sum_i c_i^{FQ} L_i^{FQ} + \sum_i c_i^{TT} L_i^{TT} + \sum_i c_i^{TQ} L_i^{TQ} + \sum_i c_i^{QQ} L_i^{QQ} \Big] \ ,\end{aligned} \tag{4.2}$$

where the first line contains all the dimension-two terms and the second contains the dimension-four terms. We do not specify the ranges of the sums, because they depend upon the choice of basis. In the Einstein form, the terms contributing to the two-point function are

$$\begin{aligned}\mathcal{L}_E = -\frac{1}{2}\Big[&- m^R R + \sum_i m_i^{TT} M_i^{TT} + \sum_i m_i^{TQ} M_i^{TQ} + \sum_i m_i^{QQ} M_i^{QQ} \\ &+\sum_i b_i^{RR} H_i^{RR} + \sum_i b_i^{RT} H_i^{RT} + \sum_i b_i^{RQ} H_i^{RQ} + \sum_i b_i^{TT} H_i^{TT} + \sum_i b_i^{TQ} H_i^{TQ} + \sum_i b_i^{QQ} H_i^{QQ} \Big] \ .\end{aligned} \tag{4.3}$$

The metric fluctuation field is $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$. Since the VEV of A (and ϕ) is zero, we shall not use a different symbol for its fluctuation and identify it with A. By Poincare invariance, the quadratic Lagrangian in the Cartan form of the theory, takes the form

$$S^{(2)} = \frac{1}{2}\int \frac{d^4q}{(2\pi)^4} \left(A^{\lambda\mu\nu}\, \mathcal{O}_{(C)}^{(AA)}{}_{\lambda\mu\nu}{}^{\tau\rho\sigma}\, A_{\tau\rho\sigma} + 2A^{\lambda\mu\nu}\, \mathcal{O}_{(C)}^{(Ah)}{}_{\lambda\mu\nu}{}^{\rho\sigma}\, h_{\rho\sigma} + h^{\mu\nu}\, \mathcal{O}_{(C)}^{(hh)}{}_{\mu\nu}{}^{\rho\sigma}\, h_{\rho\sigma} \right) \ , \tag{4.4}$$

where, after Fourier transforming, $\mathcal{O}$ is constructed only with the metric $\eta_{\mu\nu}$ and with momentum q^μ. Similarly, in the Einstein form of the theory one obtains

$$S^{(2)} = \frac{1}{2}\int \frac{d^4q}{(2\pi)^4} \left(\phi^{\lambda\mu\nu}\, \mathcal{O}_{(E)}^{(\phi\phi)}{}_{\lambda\mu\nu}{}^{\tau\rho\sigma}\, \phi_{\tau\rho\sigma} + 2\phi^{\lambda\mu\nu}\, \mathcal{O}_{(E)}^{(\phi h)}{}_{\lambda\mu\nu}{}^{\rho\sigma}\, h_{\rho\sigma} + h^{\mu\nu}\, \mathcal{O}_{(E)}^{(hh)}{}_{\mu\nu}{}^{\rho\sigma}\, h_{\rho\sigma} \right) \ . \tag{4.5}$$

From (1.32) one finds that

$$A_{\lambda\mu\nu} = \phi_{\lambda\mu\nu} + J_{\lambda\mu\nu}{}^{\rho\sigma} h_{\rho\sigma} \ , \tag{4.6}$$

where

$$J_{\lambda\mu\nu}{}^{\rho\sigma} = \frac{i}{2}\left(q_\lambda \delta^\rho_\mu \delta^\sigma_\nu + q_\nu \delta^\rho_\lambda \delta^\sigma_\mu - q_\mu \delta^\rho_\lambda \delta^\sigma_\nu\right) \ .$$

Then we obtain the following relations between the linearised operators in the Cartan and Einstein formulations:

$$\begin{aligned}
\mathcal{O}^{(\phi\phi)}_{(E)\ \lambda\mu\nu}{}^{\tau\rho\sigma} &= \mathcal{O}^{(AA)}_{(C)\ \lambda\mu\nu}{}^{\tau\rho\sigma} \ , \\
\mathcal{O}^{(\phi h)}_{(E)\ \lambda\mu\nu}{}^{\rho\sigma} &= \mathcal{O}^{(Ah)}_{(C)\ \lambda\mu\nu}{}^{\rho\sigma} + \mathcal{O}^{(AA)}_{(C)\ \lambda\mu\nu}{}^{\tau\alpha\beta} J_{\tau\alpha\beta}{}^{\rho\sigma} \ , \\
\mathcal{O}^{(hh)}_{(E)\ \mu\nu}{}^{\rho\sigma} &= \mathcal{O}^{(hh)}_{(C)\ \mu\nu}{}^{\rho\sigma} + 2J^{\lambda\gamma\delta}{}_{\mu\nu}\mathcal{O}^{(Ah)}_{(C)\ \lambda\gamma\delta}{}^{\rho\sigma} + J^{\lambda\gamma\delta}{}_{\mu\nu}\mathcal{O}^{(AA)}_{(C)\ \lambda\gamma\delta}{}^{\tau\alpha\beta} J_{\tau\alpha\beta}{}^{\rho\sigma} \ .
\end{aligned} \tag{4.7}$$

4.2 Spin projectors

In the analysis of the spectrum of operators acting on multi-index fields in flat space, it is very convenient to use spin-projection operators, which can be used to decompose the fields in their irreducible components under the three-dimensional rotation group [88–90]. This is familiar in the case of vectors and two-index tensors: a vector can be decomposed in its transverse and longitudinal components; a two index tensor can be decomposed into its symmetric and antisymmetric components, and each of these can be further decomposed in its transverse and longitudinal parts in each index. This gives rise to representations of $O(3)$ labelled by spin and parity, and listed in the following table:

	s	a
TT	2^+_4, 0^+_5	1^+_4
TL	1^-_7	1^-_8
LL	0^+_6	-

Table 4.1: $SO(3)$ spin content of projection operators for a two-index tensor in $d = 4$ (s=symmetric, a=antisymmetric).

Here the subscript distinguishes different instances of the same representation. These representations arise as perturbations of the tetrad. If one works only with the metric, the antisymmetric parts can be dropped.

We will need the projectors of spin up to three, which were obtained in [41, 91]and have a quite complicated form. We show them explicitly for tensors of rank up to two. The analogous decomposition for a three-index tensor is given in the following table, that is explained in more detail in [41] For a vector

$$L_{\mu\nu} = \frac{q_\mu q_\nu}{q^2} \ , \qquad T_{\mu\nu} = \eta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \ . \tag{4.8}$$

For symmetric 2-rank tensors

$$
\begin{aligned}
P_s\left(2^+\right)_{\mu\nu}{}^{\rho\sigma} &= T^{(\rho}_{(\mu}T^{\rho)}_{\nu)} - \frac{1}{d-1}T_{\mu\nu}T^{\rho\sigma}\,,\\
P_s\left(1^-\right)_{\mu\nu}{}^{\rho\sigma} &= 2T^{(\rho}_{(\mu}L^{\rho)}_{\nu)}\,,\\
P_s\left(0^+,ss\right)_{\mu\nu}{}^{\rho\sigma} &= \frac{1}{d-1}T_{\mu\nu}T^{\rho\sigma}\,,\\
P_s\left(0^+,ww\right)_{\mu\nu}{}^{\rho\sigma} &= L_{\mu\nu}L^{\rho\sigma}\,,\\
P_s\left(0^+,sw\right)_{\mu\nu}{}^{\rho\sigma} &= \frac{1}{\sqrt{d-1}}T_{\mu\nu}L^{\rho\sigma}\,,\\
P_s\left(0^+,ws\right)_{\mu\nu}{}^{\rho\sigma} &= \frac{1}{\sqrt{d-1}}L_{\mu\nu}T^{\rho\sigma}\,,
\end{aligned}
\tag{4.9}
$$

such that

$$
P_s\left(2^+\right)_{\mu\nu}{}^{\rho\sigma} + P_s\left(1^-\right)_{\mu\nu}{}^{\rho\sigma} + P_s\left(0^+,ss\right)_{\mu\nu}{}^{\rho\sigma} + P_s\left(0^+,ww\right)_{\mu\nu}{}^{\rho\sigma} = \delta^{(\rho}_{(\mu}\delta^{\sigma)}_{\nu)}\,. \tag{4.10}
$$

Diagonal terms of projectors between symmetric 2-rank tensors and vectors

$$
\begin{aligned}
P_{sv}\left(1^-\right)_{\mu\nu}{}^{\rho} &= \frac{\sqrt{2}}{|q|}q_{(\mu}T^{\rho}_{\nu)}\,,\\
P_{sv}\left(0^+,sv\right)_{\mu\nu}{}^{\rho} &= \frac{1}{\sqrt{d-1}}\frac{q^\rho}{|q|}T_{\mu\nu}\,,\\
P_{sv}\left(0^+,wv\right)_{\mu\nu}{}^{\rho} &= \frac{q^\rho}{|q|}L_{\mu\nu}\,,
\end{aligned}
\tag{4.11}
$$

For antisymmetric 2-rank tensors

$$
\begin{aligned}
P_a\left(1^+\right)_{\mu\nu}{}^{\rho\sigma} &= T^{[\rho}_{[\mu}T^{\rho]}_{\nu]}\,,\\
P_a\left(1^-\right)_{\mu\nu}{}^{\rho\sigma} &= 2T^{[\rho}_{[\mu}L^{\rho]}_{\nu]}\,,
\end{aligned}
\tag{4.12}
$$

such that

$$
P_a\left(1^+\right)_{\mu\nu}{}^{\rho\sigma} + P_a\left(1^-\right)_{\mu\nu}{}^{\rho\sigma} = \delta^{[\rho}_{[\mu}\delta^{\sigma]}_{\nu]}\,. \tag{4.13}
$$

Diagonal terms of projectors between antisymmetric 2-rank tensors and symmetric 2-rank tensors

$$
\begin{aligned}
P_{sa}\left(1^+\right)_{\mu\nu}{}^{\rho\sigma} &= 2T^{[\rho}_{(\mu}L^{\rho]}_{\nu)}\,,\\
P_{as}\left(1^+\right)_{\mu\nu}{}^{\rho\sigma} &= 2T^{(\rho}_{[\mu}L^{\rho)}_{\nu]}\,.
\end{aligned}
\tag{4.14}
$$

Diagonal terms of projectors between antisymmetric 2-rank tensors and vectors

$$
P_{av}\left(1^-\right)_{\mu\nu}{}^{\rho} = \frac{\sqrt{2}}{|q|}q_{[\mu}T^{\rho}_{\nu]}\,. \tag{4.15}
$$

The result for tensors of rank three is presented in the table 4.2. In antisymmetric or symmetric MAG, only the last two or the first two columns appear, respectively. For anti-symmetric tensors, the spin projectors were given in [39, 40, 43] and used to study ghost- and

	ts	hs	ha	ta
TTT	3^-, 1^-_1	2^-_1, 1^-_2	2^-_2, 1^-_3	0^-
$TTL+TLT+LTT$	2^+_1, 0^+_1	-	-	1^+_3
$\frac{3}{2}LTT$	-	2^+_2, 0^+_2	1^+_2,	-
$TTL+TLT-\frac{1}{2}LTT$	-	1^+_1	2^+_3, 0^+_3	-
$TLL+LTL+LLT$	1^-_4	1^-_5	1^-_6	-
LLL	0^+_4	-	-	-

Table 4.2: $SO(3)$ spin content of projection operators for a three-index tensor in $d = 4$. (ts/ta=totally (anti)symmetric; hs/ha=hook (anti)symmetric

tachyon-free theories that do not have accidental symmetries (i.e. symmetries that are present at the linearised level but not in the full nonlinear theory). The general case where accidental symmetries are present has been discussed in [42]. The spin projectors for general three-tensors have been given in [41, 91].

For each representation $J^{\mathcal{P}}_i$ there is a projector denoted $P_{ii}(J^{\mathcal{P}})$. In addition, for each pair of representations with the same spin-parity, labelled by i, j, there is an intertwining operator $P_{ij}(J^{\mathcal{P}})$. We collectively refer to all the projectors and intertwiners as the "spin-projectors".

Using these spin projectors, the quadratic action can be rewritten in the form

$$S^{(2)} = \frac{1}{2}\int \frac{d^4q}{(2\pi)^4} \sum_{JPij} \Phi(-q)\cdot a_{ij}(J^{\mathcal{P}})\, P_{ij}(J^{\mathcal{P}})\cdot\Phi(q)\ , \tag{4.16}$$

where $\Phi = (A, h)$ in Cartan form and $\Phi = (\phi, h)$ in Einstein form, the dot implies contraction of all indices as appropriate and $a_{ij}(J^P)$ are matrices of coefficients. For example, the $A - A$ part of (4.4) is

$$\frac{1}{2}\int \frac{d^4q}{(2\pi)^4} \sum_{JPij} a_{ij}(J^{\mathcal{P}})\, A^{\lambda\mu\nu} P_{ij}(J^{\mathcal{P}})_{\lambda\mu\nu}{}^{\tau\rho\sigma} A_{\tau\rho\sigma}\ ,$$

with the sums running over all the representations listed in the preceding table.

4.3 Gauge invariances

As mentioned in Section 2.2, one way of reducing the complexity of MAG is to introduce additional gauge invariances. These will eliminate degrees of freedom and at the same time constrain the form of the Lagrangian. One could try to analyze systematically all possible such transformations, for example one could classify them as having a scalar, vector or tensor parameter. As we shall note, such a general analysis would contain a large number of arbitrary parameters. Here we shall content ourselves to only mention a few important examples.

Diffeomorphisms

The action of MAG, when written in a coordinate basis, is in general invariant only under diffeomorphisms

$$\begin{aligned} g'_{\mu\nu}(x') &= \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x) \ , \\ A'_\mu{}^\alpha{}_\beta(x') &= \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial x'^\alpha}{\partial x^\gamma}\frac{\partial x^\delta}{\partial x'^\beta} A_\nu{}^\gamma{}_\delta(x) + \frac{\partial x'^\alpha}{\partial x^\gamma}\frac{\partial^2 x^\gamma}{\partial x'^\mu \partial x'^\beta} \ . \end{aligned} \tag{4.17}$$

For an infinitesimal transformation $x'^\mu = x^\mu - \xi^\mu(x)$ the transformation is given by the Lie derivatives, plus an inhomogeneous term for the connection:

$$\delta g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu} \ , \qquad \delta A_\rho{}^\mu{}_\nu = \mathcal{L}_\xi A_\rho{}^\mu{}_\nu + \partial_\rho \partial_\nu \xi^\mu \ , \tag{4.18}$$

where $\mathcal{L}_\xi A_\rho{}^\mu{}_\nu = \xi^\lambda \partial_\lambda A_\rho{}^\mu{}_\nu + A_\lambda{}^\mu{}_\nu \partial_\rho \xi^\lambda - A_\rho{}^\lambda{}_\nu \partial_\lambda \xi^\mu + A_\rho{}^\mu{}_\lambda \partial_\rho \xi^\lambda$. On a flat background $A = 0$ and the Lie derivative term is absent.

Invariance under diffeomorphisms lowers by one the rank of the coefficient matrices $a(1^-)$ and $a(0^+)$. (This is because the transformation parameter ξ_μ can be decomposed as a three-scalar and a three-vector). This is particularly clear in the Einstein form of the theory, where diffeomorphism invariance implies

$$a(1^-)_{i7} = a(1^-)_{7i} = 0 \ , \qquad a(0^+)_{i6} = a(0^+)_{6i} = 0 \ . \tag{4.19}$$

Vector transformations of A

Certain classes of MAGs are invariant under additional transformations of the connection, parametrised by a vector $\lambda_\mu(x)$:

$$\delta_1 A_\mu{}^\rho{}_\nu = \lambda_\mu \delta^\rho_\nu \ , \qquad \delta_1 g_{\mu\nu} = 0 \ , \tag{4.20}$$

$$\delta_2 A_\mu{}^\rho{}_\nu = \lambda^\rho g_{\mu\nu} \ , \qquad \delta_2 g_{\mu\nu} = 0 \ , \tag{4.21}$$

$$\delta_3 A_\mu{}^\rho{}_\nu = \delta^\rho_\mu \lambda_\nu \ , \qquad \delta_3 g_{\mu\nu} = 0 \ . \tag{4.22}$$

The first of these is the projective transformation. In order to spell out the conditions for invariance of the action, it is easier to work in the Einstein formulation. Since the metric (and therefore the Christoffel coefficients) transforms trivially, the transformations of $\phi_\mu{}^\rho{}_\nu$ are the same as those of $A_\mu{}^\rho{}_\nu$ given above. The conditions on the kinetic coefficients for invariance of the Lagrangian, are listed in Appendix B of [86] (see also [92] for earlier related work). We note that these transformations could also be present in arbitrary linear combinations, each yielding different conditions on the coefficients.

Each one of these invariances, when present, lowers by one the rank of the coefficient matrices $a(1^-)$ and $a(0^+)$.

Weyl transformations

By definition, Weyl transformations are local rescalings of the metric:

$$\delta g_{\mu\nu} = 2\omega g_{\mu\nu} \ . \tag{4.23}$$

This implies that the LC connection transforms as:

$$\delta \Gamma_\mu{}^\rho{}_\nu = \partial_\mu \omega \delta^\rho_\nu + \partial_\nu \omega \delta^\rho_\mu - g^{\rho\tau}\partial_\tau \omega g_{\mu\nu} \ . \tag{4.24}$$

If we now consider the decomposition (1.32), we see that there are infinitely many ways of splitting this transformation between A and ϕ. We consider here only

$$\delta A_\mu{}^\rho{}_\nu = 0 \qquad \delta\phi_\mu{}^\rho{}_\nu = -\partial_\mu\omega\delta^\rho_\nu - \partial_\nu\omega\delta^\rho_\mu + g^{\rho\tau}\partial_\tau\omega g_{\mu\nu} \ , \tag{4.25}$$

which is the usual way in which Weyl transformations are realized on Yang-Mills fields.

The action (4.3) is invariant under this transformation if all the dimension 2 terms are absent and, additionally, the following relations hold:

$$\begin{aligned}
&4b^{RR}_1 + 2b^{RR}_2 + b^{RQ}_1 + b^{RQ}_2 + 4b^{RQ}_3 + b^{RQ}_5 = 0 \ ,\\
&6b^{RR}_1 + 6b^{RR}_2 + 18b^{RR}_3 - b^{RQ}_1 + 3b^{RQ}_2 + 2b^{RQ}_4 + 3b^{RQ}_5 + 6b^{RQ}_7 - 8b^{QQ}_1 + 2b^{QQ}_2 + 2b^{QQ}_3\\
&\qquad - 32b^{QQ}_4 - 8b^{QQ}_6 + 2b^{QQ}_7 + 2b^{QQ}_8 + 2b^{QQ}_9 + 2b^{QQ}_{10} + 2b^{QQ}_{12} - 32b^{QQ}_{14} + 2b^{QQ}_{15} = 0 \ ,\\
&b^{RR}_2 + 6b^{RR}_3 + b^{RQ}_4 + 4b^{RQ}_6 + b^{RQ}_7 = 0 \ ,\\
&b^{RT}_1 + 2b^{RT}_2 + b^{RT}_3 - 3b^{RT}_4 + 6b^{RT}_5 + 2b^{TQ}_1 - 2b^{TQ}_2 - 8b^{TQ}_3 + 2b^{TQ}_4\\
&\qquad - 2b^{TQ}_{10} - 2b^{TQ}_{11} + 8b^{TQ}_{12} + 2b^{TQ}_{13} = 0 \ ,\\
&b^{RQ}_1 - 3b^{RQ}_2 - b^{RQ}_5 - 6b^{RQ}_7 - 4b^{QQ}_2 - 4b^{QQ}_3 - 8b^{QQ}_5 - 2b^{QQ}_9 - 2b^{QQ}_{10}\\
&\qquad - 2b^{QQ}_{12} - 4b^{QQ}_{15} - 8b^{QQ}_{16} = 0 \ ,\\
&b^{RQ}_4 + b^{RQ}_5 + 2b^{QQ}_7 + 2b^{QQ}_8 + b^{QQ}_9 + b^{QQ}_{10} + 4b^{QQ}_{11} + b^{QQ}_{12} + 4b^{QQ}_{13} = 0 \ .
\end{aligned} \tag{4.26}$$

As a check we observe that if all the coefficients of type b^{RT}, b^{RQ}, b^{QQ} and b^{TQ} are zero, the remaining relations imply that the R^2 terms appear in the combination

$$b^{RR}_1\left(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2\right) = b^{RR}_1 C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} \ ,$$

which is the square of the Weyl tensor. A much more thorough analysis of Weyl symmetry in the context of MAG can be found in [93].

4.4 MAGs with dimension-two terms only

In this section we discuss, at linearised level, the case of theories without dimension-4 operators in the Lagrangian. In an EFT, the dimension-two terms will be the dominant ones at very low energy.

Consider again Fig.1. At the top vertex of the triangle ($Q = T = 0$) one has Riemannian geometry, and the only invariant of dimension two is the Hilbert action. At linearised level we get the Fierz-Pauli action

$$S^{(2)} = \frac{m^R}{2}\int\frac{d^4q}{(2\pi)^4}\left(-\frac{1}{4}q^2h_{\mu\nu}h^{\mu\nu} + \frac{1}{2}q_\mu q_\lambda h^{\mu\nu}h^\lambda_\nu - \frac{1}{2}q_\mu q_\nu h^{\mu\nu}h + \frac{1}{4}q^2h^2\right) . \tag{4.27}$$

In the interior of the triangle we have the generalised Palatini action (3.6). The generalisation consists of the following. In the "standard" Palatini approach, the action is just $a^F F$. When varied, this is not enough to constrain the connection completely. One can either assume $T = 0$ and obtain $Q = 0$ as an equation, or assume $Q = 0$ and obtain $T = 0$ as an equation. Thus, the standard Palatini action works on the left and right sides of the triangle, but not in the interior. This is due to the fact that the Palatini action is invariant under the projective transformations (4.20). The addition of the other terms in (3.6), which is only natural from the point of view of EFT, generically breaks projective invariance and fixes this problem. In the Einstein form, the action becomes (3.8) (or (3.11)), which consists just of the Hilbert action for g and a mass term for the distortion (or equivalently torsion and nonmetricity). Generically, this mass term will be non-degenerate and the EOM will imply that distortion vanishes. Thus the theory is dynamically equivalent to GR, on shell. We note that the addition to the standard Palatini action of torsion-squared terms in antisymmetric MAG or nonmetricity-squared terms in symmetric MAG, will generically not change the EOMs. Still, these terms are expected to be present when we think of MAG as an EFT.

We now turn to the bottom of the triangle, which does not follow the generic behaviour of the interior. We first look at the left and right corners, then at the bottom edge. The following analyses will be carried out in the Cartan version of the theory.

4.4.1 Antisymmetric teleparallel theory

This is also known as Weitzenböck theory. We have $F = Q = 0$, so the action must be quadratic in torsion

$$S = -\frac{1}{2}\int d^4x\sqrt{|g|}\left[a_1^{TT}T_{\mu\rho\nu}T^{\mu\rho\nu} + a_2^{TT}T_{\mu\rho\nu}T^{\mu\nu\rho} + a_3^{TT}\mathrm{tr}_{(12)}T_\mu \mathrm{tr}_{(12)}T^\mu\right] . \tag{4.28}$$

The condition $F = 0$ implies (2.10). When the theory is linearised around flat space, this becomes $A_\mu{}^\rho{}_\nu = \partial_\mu\lambda^\rho{}_\nu$, where $\Lambda^\rho{}_\sigma = \delta^\rho_\sigma + \lambda^\rho{}_\sigma$. The condition $Q = 0$ implies for the metric fluctuation that $A_{\mu\rho\nu} + A_{\mu\nu\rho} = \partial_\mu h_{\rho\nu}$. Putting these conditions together we have

$$A_{\mu\rho\nu} = \frac{1}{2}\partial_\mu h_{\rho\nu} + \partial_\mu\Omega_{\rho\nu} \ , \tag{4.29}$$

where Ω is the antisymmetric part of λ. So the action of the linearised theory becomes

$$\begin{aligned} S = -\frac{1}{2}\int\frac{d^4q}{(2\pi)^4}\Bigg[&-\frac{(2a_1^{TT}+a_2^{TT})}{4}q^2h_{\mu\nu}h^{\mu\nu} + \frac{(2a_1^{TT}+a_2^{TT}-a_3^{TT})}{4}q_\mu q_\lambda h^{\mu\nu}h^\lambda_\nu \\ &+ \frac{a_3^{TT}}{2}q_\mu q_\nu h^{\mu\nu}h - \frac{a_3^{TT}}{4}q^2h^2 - (2a_1^{TT}-a_2^{TT})q^2\Omega^{\mu\nu}\Omega_{\mu\nu} \\ &+ (2a_1^{TT}-3a_2^{TT}-a_3^{TT})q_\mu q_\lambda\Omega^{\mu\nu}\Omega^\lambda{}_\nu - (2a_1^{TT}+a_2^{TT}+a_3^{TT})q_\mu q_\lambda\Omega^{\mu\nu}h^\lambda_\nu\Bigg] . \end{aligned} \tag{4.30}$$

The linearised action can then be written in a form analogous to (4.16):

$$\begin{aligned} S = \frac{1}{2}\sum_{P,i,j}\int\frac{d^4q}{(2\pi)^4}\,(\Omega(-q)\ h(-q))\cdot\begin{pmatrix} a_{ij}^{\Omega\Omega}P_{ij}^{\Omega\Omega} & a_{ij}^{\Omega h}P_{ij}^{\Omega h} \\ a_{ij}^{h\Omega}P_{ij}^{h\Omega} & a_{ij}^{hh}P_{ij}^{hh}\end{pmatrix}\cdot\begin{pmatrix}\Omega(q)\\ h(q)\end{pmatrix} \\ + \int\frac{d^4q}{(2\pi)^4}\left\{\sigma(-q)\cdot h(q) + \tau(-q)\cdot\Omega(q)\right\} , \end{aligned} \tag{4.31}$$

where

$$a\left(2^+\right) = \frac{(2a_1^{TT} + a_2^{TT})}{4}q^2\,, \tag{4.32a}$$

$$a\left(1^+\right) = \left(2a_1^{TT} - a_2^{TT}\right)q^2\,, \tag{4.32b}$$

$$a\left(1^-\right) = \frac{a_4^{TT}}{8}q^2\begin{pmatrix}4 & -2\\ -2 & 1\end{pmatrix}\,, \tag{4.32c}$$

$$a\left(0^+\right) = \frac{(a_4^{TT} + 2a_3^{TT})}{4}q^2\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\,, \tag{4.32d}$$

where

$$a_4^{TT} \equiv 2a_1^{TT} + a_2^{TT} + a_3^{TT}\,. \tag{4.33}$$

In the 1^- sector, the order of the rows and columns is (Ω, h). Note that the matrices $a\left(1^-\right)$ and $a\left(0^+\right)$ have rank 1 because of the diffeomorphism invariance. We fix the gauge by removing the second row and column. At the linearised level the diffeomorphism transformation reads

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu\,, \tag{4.34a}$$

$$\Omega_{\mu\nu} \to \Omega_{\mu\nu} - \frac{1}{2}\partial_\mu\xi_\nu + \frac{1}{2}\partial_\nu\xi_\mu\,, \tag{4.34b}$$

where the transformation of Ω follows from those of A and h and formula (4.29) So the sources satisfy the following constraint

$$-2q^\mu\sigma_{\mu\nu} + q^\mu\tau_{\mu\nu} = 0\,. \tag{4.35}$$

The saturated propagator is

$$\begin{aligned}\Pi = &-\frac{1}{2}\int\frac{d^4q}{(2\pi)^4}\left\{\frac{4}{(2a_1^{TT} + a_2^{TT})q^2}\left[\sigma_{\mu\nu}\sigma^{\mu\nu} - \frac{a_3^{TT}}{2a_1^{TT} + a_2^{TT} + 3a_3^{TT}}\left(\sigma^\mu_\mu\right)^2\right]\right.\\ &\left.+\frac{1}{(2a_1^{TT} - a_2^{TT})q^2}\left[\tau_{\mu\nu}\tau^{\mu\nu} - \frac{4(a_2^{TT\,2} + 2a_1^{TT}a_2^{TT} + 2a_1^{TT}a_3^{TT})}{(2a_1^{TT} + a_2^{TT})(2a_1^{TT} + a_2^{TT} + a_3^{TT})}\frac{q^\mu q^\nu}{q^2}\tau_{\mu\rho}\tau_\nu{}^\rho\right]\right\}.\end{aligned} \tag{4.36}$$

Making the following redefinitions

$$\tilde{\sigma}_{\mu\nu} \equiv \sigma_{\mu\nu} + C\,\sigma^\rho_\rho\,\eta_{\mu\nu}\,, \tag{4.37a}$$

$$\tau_{\mu\nu} \equiv -\frac{i}{q^2}\left(q_\mu\chi_\nu - q_\nu\chi_\mu\right) + \tilde{\tau}_{\mu\nu} \qquad \text{with } q^\mu\chi_\mu = q^\mu\tilde{\tau}_{\mu\nu} = 0\,, \tag{4.37b}$$

and adjusting the parameter C , we can reduce the saturated propagator to the following form

$$\begin{aligned}\Pi = -\frac{1}{2}\int\frac{d^4q}{(2\pi)^4}&\left[\frac{4}{(2a_1^{TT} + a_2^{TT})q^2}\left(\tilde{\sigma}_{\mu\nu}\tilde{\sigma}^{\mu\nu} - \frac{1}{2}\left(\tilde{\sigma}^\mu_\mu\right)^2\right)\right.\\ &\left.+\frac{1}{(2a_1^{TT} - a_2^{TT})q^2}\tilde{\tau}_{\mu\nu}\tilde{\tau}^{\mu\nu} - \frac{2(2a_1^{TT} + a_2^{TT} - a_3^{TT})}{(2a_1^{TT} + a_2^{TT})(2a_1^{TT} + a_2^{TT} + a_3^{TT})q^4}\chi_\mu\chi^\mu\right].\end{aligned} \tag{4.38}$$

In the first term we recognize the usual graviton, in the second one we have a massless spin 1^+ state and in the last a dipole ghost with spin 1^-.

The latter is pathological and in order to eliminate it, we have to impose

$$a_4^{TT} = 0\,. \tag{4.39}$$

With this constraint, we recover linearised GR together with a spin 1^+ particle. If we impose that $2a_1^{TT} - a_2^{TT} > 0$ its propagator takes the proper form. In this theory there are two different gauge invariances: the previously mentioned diffeomorphisms, and

$$\Omega_{\mu\nu} \to \Omega_{\mu\nu} + \partial_\mu \chi_\nu - \partial_\nu \chi_\mu \,. \tag{4.40}$$

The additional degree of freedom can be removed by imposing

$$2a_1^{TT} - a_2^{TT} = 0 \;, \tag{4.41}$$

in which case Ω disappears from the action (it is a pure gauge degree of freedom) and the rest reduces to the antisymmetric teleparallel equivalent of the Hilbert action, (2.2).

4.4.2 Symmetric teleparallel theory

Now we have $F = T = 0$, so the action is a generic quadratic combination of non-metricity:

$$\begin{aligned} S = -\frac{1}{2}\int d^4x\sqrt{|g|}\,\Big[&a_1^{QQ} Q_{\rho\mu\nu}Q^{\rho\mu\nu} + a_2^{QQ} Q_{\rho\mu\nu}Q^{\mu\rho\nu} \\ &+a_3^{QQ}\mathrm{tr}_{(23)}Q_\mu \mathrm{tr}_{(23)}Q^\mu + a_4^{QQ}\mathrm{tr}_{(12)}Q_\mu \mathrm{tr}_{(12)}Q^\mu + a_5^{QQ}\mathrm{tr}_{(23)}Q_\mu \mathrm{tr}_{(12)}Q^\mu\Big] \,. \end{aligned} \tag{4.42}$$

As in the antisymmetric case, in the linearised theory $F = 0$ implies $A_\mu{}^\rho{}_\nu = \partial_\mu \lambda^\rho{}_\nu$. The condition $T = 0$ implies that $A_\mu{}^\rho{}_\nu = A_\nu{}^\rho{}_\mu$. Putting these conditions together we have

$$A_\mu{}^\rho{}_\nu = \partial_\mu\partial_\nu u^\rho \,. \tag{4.43}$$

Substituting $Q_{\rho\mu\nu} = -\partial_\rho h_{\mu\nu} + \partial_\rho\partial_\mu u_\nu + \partial_\rho\partial_\nu u_\mu$ and linearizing, the action becomes

$$\begin{aligned} S = -\frac{1}{2}\int \frac{d^4q}{(2\pi)^4}\Big[&- a_1^{QQ}q^2 h_{\mu\nu}h^{\mu\nu} - (a_2^{QQ} + a_4^{QQ})q_\mu q_\lambda h^{\mu\nu}h^\lambda_\nu - a_5^{QQ}q_\mu q_\nu h^{\mu\nu}h - a_3^{QQ}q^2h^2 \\ &+ (2a_1^{QQ} + a_2^{QQ} + a_4^{QQ})q^4 u_\lambda u^\lambda + (2a_1^{QQ} + 3a_2^{QQ} + 4a_3^{QQ} + 3a_4^{QQ} + 4a_5^{QQ})q^2 q_\mu q_\nu u^\mu u^\nu \\ &- 2i(2a_1^{QQ} + a_2^{QQ} + a_4^{QQ})q^2 q_\mu u_\nu h^{\mu\nu} - 2i(a_2^{QQ} + a_4^{QQ} + a_5^{QQ})q_\lambda q_\mu q_\nu u^\lambda h^{\mu\nu} \\ &- 2i(2a_3^{QQ} + a_5^{QQ})q^2 q_\lambda u^\lambda h\Big] \,. \end{aligned} \tag{4.44}$$

For a generic choice of coefficients the linearised action is

$$\begin{aligned} S = \frac{1}{2}\sum_{P,i,j}\int \frac{d^4q}{(2\pi)^4}\,(u(-q)\;\, h(-q))\cdot \begin{pmatrix} a_{ij}^{uu}P_{ij}^{uu} & a_{ij}^{uh}P_{ij}^{uh} \\ a_{ij}^{hu}P_{ij}^{hu} & a_{ij}^{hh}P_{ij}^{hh}\end{pmatrix}\cdot\begin{pmatrix}u(q)\\ h(q)\end{pmatrix} \\ + \int \frac{d^4q}{(2\pi)^4}\left\{\sigma(-q)\cdot h(q) + \tau(-q)\cdot u(q)\right\} \,, \end{aligned} \tag{4.45}$$

where

$$a\left(2^+\right) = a_1^{QQ}q^2 \,, \tag{4.46a}$$

$$a(1^-) = \frac{1}{2}\left(2a_1^{QQ} + a_6^{QQ}\right)q^2\,\begin{pmatrix} -2\,q^2 & i\sqrt{2}\,|q| \\ i\sqrt{2}\,|q| & 1\end{pmatrix} \,, \tag{4.46b}$$

$$a(0^+) = q^2\,\begin{pmatrix} -4\,a_7^{QQ}\,q^2 & i\sqrt{3}(2a_3^{QQ} + a_5^{QQ})|q| & 2i\,a_7^{QQ}\,|q| \\ i\sqrt{3}(2a_3^{QQ} + a_5^{QQ})|q| & (a_1^{QQ} + 3a_3^{QQ}) & \sqrt{3}(2a_3^{QQ} + a_5^{QQ})/2 \\ 2i\,a_7^{QQ}\,|q| & \sqrt{3}(2a_3^{QQ} + a_5^{QQ})/2 & a_7^{QQ}\end{pmatrix} \,, \tag{4.46c}$$

where the rows/columns of $a(1^-)$ refer to u, h, in this order, those of $a(0^+)$ to u, h, h. We defined

$$a_6^{QQ} \equiv a_2^{QQ} + a_4^{QQ} \,, \tag{4.47a}$$
$$a_7^{QQ} \equiv a_1^{QQ} + a_2^{QQ} + a_3^{QQ} + a_4^{QQ} + a_5^{QQ} \,. \tag{4.47b}$$

Note that the matrix $a\,(1^-)$ has rank 1 and $a\,(0^+)$ has rank 2 because of the diffeomorphism invariance.

At the linearised level the diffeomorphism transformation reads

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \,, \tag{4.48a}$$
$$u_\mu \to u_\mu + \xi_\mu \,, \tag{4.48b}$$

so the sources satisfy the following constraint

$$-2iq^\mu \sigma_{\mu\nu} + \tau_\nu = 0 \,. \tag{4.49}$$

The saturated propagator is

$$\begin{aligned} \Pi = -\frac{1}{2} \int \frac{d^4q}{(2\pi)^4} \Bigg\{ & \frac{1}{a_1^{QQ} q^2} \left[\sigma_{\mu\nu}\sigma^{\mu\nu} + (\ldots) \left(\sigma^\mu_\mu\right)^2 \right] - \frac{i}{2} \frac{(\ldots)}{q^4} q^\mu \tau_\mu \sigma^\nu_\nu \\ & + \frac{a_6^{QQ}}{2a_4(2a_1^{QQ} + a_6^{QQ})q^4} \left(\tau_\mu \tau^\mu + (\ldots) \frac{q^\mu q^\nu}{q^2} \tau_\mu \tau_\nu \right) \Bigg\} \,, \end{aligned} \tag{4.50}$$

where the ellipses stand for complicated combinations of couplings whose explicit form is not very relevant. Making the redefinitions

$$\tilde{\sigma}_{\mu\nu} \equiv \sigma_{\mu\nu} - \frac{iA}{q^2} \left(q_\mu \tau_\nu + q_\nu \tau_\mu \right) + C\, \sigma^\rho_\rho \, \eta_{\mu\nu} \,, \tag{4.51a}$$
$$\tau_\mu \equiv -\frac{i}{q^2} q_\mu j + \tilde{\tau}_\mu \qquad\qquad \text{with } \; q^\mu \tilde{\tau}_\mu = 0 \,, \tag{4.51b}$$

and adjusting the parameters (A, C) , we can reduce the saturated propagator to the form

$$\Pi = -\frac{1}{2} \int \frac{d^4q}{(2\pi)^4} \left\{ \frac{1}{a_1^{QQ} q^2} \left[\tilde{\sigma}_{\mu\nu} \tilde{\sigma}^{\mu\nu} - \frac{1}{2} \left(\tilde{\sigma}^\mu_\mu \right)^2 \right] + \frac{(\ldots)}{q^4} \tau^\mu \tilde{\tau}^\nu + \frac{(\ldots)}{q^6} j^2 \right\} \,. \tag{4.52}$$

These dipole and tripole ghosts can be eliminated imposing the conditions

$$2a_1^{QQ} + a_6^{QQ} = 0 \,, \tag{4.53a}$$
$$a_6^{QQ} + a_5^{QQ} = 0 \,, \tag{4.53b}$$
$$2a_3^{QQ} + a_5^{QQ} = 0 \,, \tag{4.53c}$$

leaving us just with the standard graviton saturated propagator. With these constraints, u becomes a pure gauge and we recover the symmetric teleparallel equivalent of GR (2.3). This is in agreement with the results of [83, 84].

4.4.3 General teleparallel theory

We now only assume $F = 0$. The action is

$$
\begin{aligned}
S = -\frac{1}{2}\int d^4x\sqrt{|g|}\Big[& a_1^{TT}T_{\mu\rho\nu}T^{\mu\rho\nu} + a_2^{TT}T_{\mu\rho\nu}T^{\mu\nu\rho} + a_3^{TT}\mathrm{tr}_{(12)}T_\mu \mathrm{tr}_{(12)}T^\mu \\
& + a_1^{QQ}Q_{\rho\mu\nu}Q^{\rho\mu\nu} + a_2^{QQ}Q_{\rho\mu\nu}Q^{\mu\rho\nu} \\
& + a_3^{QQ}\mathrm{tr}_{(23)}Q_\mu \mathrm{tr}_{(23)}Q^\mu + a_4^{QQ}\mathrm{tr}_{(12)}Q_\mu \mathrm{tr}_{(12)}Q^\mu + a_5^{QQ}\mathrm{tr}_{(23)}Q_\mu \mathrm{tr}_{(12)}Q^\mu \\
& + a_1^{TQ}T_{\mu\rho\nu}Q^{\mu\rho\nu} + a_2^{TQ}\mathrm{tr}_{(12)}T_\mu \mathrm{tr}_{(23)}Q^\mu + a_3^{TQ}\mathrm{tr}_{(12)}T_\mu \mathrm{tr}_{(12)}Q^\mu\Big] .
\end{aligned}
\tag{4.54}
$$

As in the previous cases, in the linearised theory $F = 0$ implies $A_\mu{}^\rho{}_\nu = \partial_\mu\lambda^\rho{}_\nu$, but now both the symmetric part H and the antisymmetric part Ω of λ have to be treated as dynamical fields.

So the action becomes

$$
\begin{aligned}
S = -\frac{1}{2}\int \frac{d^4q}{(2\pi)^4}\Big(& -a_1^{QQ}q^2h_{\mu\nu}h^{\mu\nu} - (a_2^{QQ}+a_4^{QQ})q_\mu q_\lambda h^{\mu\nu}h^\lambda_\nu - a_5^{QQ}q_\mu q_\nu h^{\mu\nu}h \\
& -a_3^{QQ}q^2h^2 - (2a_1^{TT}+a_2^{TT}+4a_1^{QQ}+2a_1^{TQ})q^2H_{\mu\nu}H^{\mu\nu} \\
& +(2a_1^{TT}+a_2^{TT}-a_3^{TT}-4a_2^{QQ}-4a_4^{QQ}+2a_1^{TQ}-2a_3^{TQ})q_\mu q_\lambda H^{\mu\nu}H^\lambda_\nu \\
& +2(a_3^{TT}-2a_5^{QQ}-a_2^{TQ}+a_3^{TQ})q_\mu q_\nu H^{\mu\nu}H - (a_3^{TT}+4a_3^{QQ}-2a_2^{TQ})q^2H^2 \\
& -(2a_1^{TT}-a_2^{TT})q^2\Omega_{\mu\nu}\Omega^{\mu\nu} + (2a_1^{TT}-3a_2^{TT}-a_3^{TT})q_\mu q_\lambda\Omega^{\mu\nu}\Omega^\lambda_\nu \\
& +(4a_1^{QQ}+a_1^{TQ})q^2H_{\mu\nu}h^{\mu\nu} + (4a_2^{QQ}+4a_4^{QQ}-a_1^{TQ}+a_3^{TQ})q_\mu q_\lambda H^{\mu\nu}h^\lambda_\nu \\
& +(2a_5^{QQ}+a_2^{TQ})q_\mu q_\nu H^{\mu\nu}h + (2a_5^{QQ}-a_3^{TQ})q_\mu q_\nu Hh^{\mu\nu} + (4a_3^{QQ}-a_2^{TQ})q^2Hh \\
& +(a_1^{TQ}+a_3^{TQ})q_\mu q_\lambda\Omega^{\mu\nu}h^\lambda_\nu - 2(2a_1^{TT}+a_2^{TT}+a_3^{TT}+a_1^{TQ}+a_3^{TQ})q_\mu q_\lambda\Omega^{\mu\nu}H^\lambda_\nu\Big) .
\end{aligned}
\tag{4.55}
$$

For a generic choice, we write

$$
\begin{aligned}
S = \frac{1}{2}\sum_{P,i,j}\int \frac{d^4q}{(2\pi)^4}\,(\Omega\ H\ h)\cdot & \begin{pmatrix} a_{ij}^{\Omega\Omega}P_{ij}^{\Omega\Omega} & a_{ij}^{\Omega H}P_{ij}^{\Omega H} & a_{ij}^{\Omega h}P_{ij}^{\Omega h} \\ a_{ij}^{H\Omega}P_{ij}^{H\Omega} & a_{ij}^{HH}P_{ij}^{HH} & a_{ij}^{Hh}P_{ij}^{Hh} \\ a_{ij}^{h\Omega}P_{ij}^{h\Omega} & a_{ij}^{hH}P_{ij}^{hH} & a_{ij}^{hh}P_{ij}^{hh} \end{pmatrix}\cdot\begin{pmatrix}\Omega \\ H \\ h\end{pmatrix} \\
& + \int \frac{d^4q}{(2\pi)^4}\left\{\sigma\cdot h + \Sigma\cdot H + \tau\cdot\Omega\right\} ,
\end{aligned}
\tag{4.56}
$$

where

$$a\left(2^{+}\right)=q^{2}\begin{pmatrix}(2a_1^{TT}+a_2^{TT}+4a_1^{QQ}+2a_1^{TQ}) & -(4a_1^{QQ}+a_1^{TQ})/2\\ -(4a_1^{QQ}+a_1^{TQ})/2 & a_1^{QQ}\end{pmatrix}, \tag{4.57a}$$

$$a\left(1^{+}\right)=\left(2a_1^{TT}-a_2^{TT}\right)q^{2}, \tag{4.57b}$$

$$a\left(1^{-}\right)=q^{2}\begin{pmatrix}a_4^{TT}/2 & -(a_4^{TT}+a_1^{TQ}+a_3^{TQ})/2 & (a_1^{TQ}+a_3^{TQ})/4\\ -(a_4^{TT}+a_1^{TQ}+a_3^{TQ})/2 & \left(a_4^{TT}+a_5^{TQ}+a_1^{TQ}+a_3^{TQ}\right)/2 & -a_5^{TQ}/4\\ (a_1^{TQ}+a_3^{TQ})/4 & -a_5^{TQ}/4 & \left(2a_1^{QQ}+a_6^{QQ}\right)/2\end{pmatrix}, \tag{4.57c}$$

$$a\left(0^{+}\right)=q^{2}\begin{pmatrix}a_4^{TQ} & \sqrt{3}\,a_6^{TQ} & -a_7^{TQ}/2 & -\sqrt{3}\,a_6^{TQ}/2\\ \sqrt{3}\,a_6^{TQ} & 4\,a_7^{QQ} & -\sqrt{3}(2a_3^{QQ}+a_5^{QQ}) & -2\,a_7^{QQ}\\ -a_7^{TQ}/2 & -\sqrt{3}(2a_3^{QQ}+a_5^{QQ}) & (a_1^{QQ}+3a_3^{QQ}) & \sqrt{3}(2a_3^{QQ}+a_5^{QQ})/2\\ -\sqrt{3}\,a_6^{TQ}/2 & -2\,a_7^{QQ} & \sqrt{3}(2a_3^{QQ}+a_5^{QQ})/2 & a_7^{QQ}\end{pmatrix}, \tag{4.57d}$$

where the rows/columns of $a(2^{+})$ refer to H and h (in this order), those of $a(1^{-})$ to Ω, H, h, those of $a(0^{+})$ to H, H, h, h, and we defined

$$a_4^{TQ}\equiv a_4^{TT}+2a_3^{TT}+4a_1^{QQ}+12a_3^{QQ}+2a_1^{TQ}-6a_2^{TQ}, \tag{4.58a}$$

$$a_5^{TQ}\equiv 8a_1^{QQ}+4a_6^{QQ}+a_1^{TQ}+a_3^{TQ}, \tag{4.58b}$$

$$a_6^{TQ}\equiv 4a_3^{QQ}+2a_5^{QQ}-a_2^{TQ}-a_3^{TQ}, \tag{4.58c}$$

$$a_7^{TQ}\equiv 4a_1^{QQ}+12a_3^{QQ}+a_1^{TQ}-3a_2^{TQ}. \tag{4.58d}$$

As usual the matrix $a\left(1^{-}\right)$ has rank 2 and $a\left(0^{+}\right)$ has rank 3 because of diffeomorphism invariance. At the linearised level, the diffeomorphism transformation reads

$$h_{\mu\nu}\to h_{\mu\nu}+\partial_\mu\xi_\nu+\partial_\nu\xi_\mu, \tag{4.59a}$$

$$H_\mu\to H_{\mu\nu}+\frac{1}{2}\partial_\mu\xi_\nu+\frac{1}{2}\partial_\nu\xi_\mu, \tag{4.59b}$$

$$\Omega_{\mu\nu}\to\Omega_{\mu\nu}-\frac{1}{2}\partial_\mu\xi_\nu+\frac{1}{2}\partial_\nu\xi_\mu, \tag{4.59c}$$

so the sources satisfy the following constraint

$$2q^\mu\sigma_{\mu\nu}+q^\mu\Sigma_{\mu\nu}-q^\mu\tau_{\mu\nu}=0. \tag{4.60}$$

The saturated propagator is

$$\begin{aligned}\Pi=-\frac{1}{2}\int\frac{d^4q}{(2\pi)^4}\Bigg\{&\frac{(\dots)}{q^2}\left[\sigma_{\mu\nu}\sigma^{\mu\nu}+(\dots)\left(\sigma^\mu_\mu\right)^2\right]+\frac{(\dots)}{q^2}\left[\Sigma_{\mu\nu}\Sigma^{\mu\nu}+(\dots)\left(\Sigma^\mu_\mu\right)^2\right]\\ &+\frac{q^\mu q^\nu}{q^4}\left[(\dots)\Sigma_{\mu\nu}\Sigma^\rho_\rho+(\dots)\Sigma_{\mu\rho}\Sigma_\nu{}^\rho+(\dots)\frac{q^\rho q^\lambda}{q^2}\Sigma_{\mu\nu}\Sigma_{\rho\lambda}\right]\\ &+\frac{(\dots)}{q^2}\left[\tau_{\mu\nu}\tau^{\mu\nu}+(\dots)\frac{q^\mu q^\nu}{q^2}\tau_{\mu\rho}\tau_\nu{}^\rho\right]+(\dots)\frac{q^\mu q^\nu}{q^2}\Sigma_{\mu\rho}\tau_\nu{}^\rho\\ &+\frac{(\dots)}{q^2}\left[\Sigma_{\mu\nu}\sigma^{\mu\nu}+(\dots)\frac{q^\mu q^\nu}{q^2}\Sigma_{\mu\nu}\sigma^\rho_\rho+(\dots)\Sigma^\mu_\mu\,\sigma^\nu_\nu\right]\Bigg\}.\end{aligned} \tag{4.61}$$

Making the following redefinitions

$$\tilde{\sigma}_{\mu\nu} \equiv \sigma_{\mu\nu} + A\,\Sigma_{\mu\nu} + \left(C\,\sigma^{\rho}_{\rho} + D\,\Sigma^{\rho}_{\rho}\right)\,\eta_{\mu\nu}\,, \tag{4.62a}$$

$$\tilde{\Sigma}_{\mu\nu} \equiv \Sigma_{\mu\nu} + B\,\sigma_{\mu\nu} + \left(E\,\Sigma^{\rho}_{\rho} + F\,\sigma^{\rho}_{\rho}\right)\,\eta_{\mu\nu}\,, \tag{4.62b}$$

and adjusting the parameters (A, B, C, D, E) , we can reduce the saturated propagator to the following form

$$\Pi = -\frac{1}{2}\int \frac{d^4q}{(2\pi)^4}\left\{\frac{(\ldots)}{q^2}\left[\tilde{\sigma}_{\mu\nu}\tilde{\sigma}^{\mu\nu} - \frac{1}{2}\left(\tilde{\sigma}^{\mu}_{\mu}\right)^2\right] + \frac{(\ldots)}{q^2}\left[\tilde{\Sigma}_{\mu\nu}\tilde{\Sigma}^{\mu\nu} + (\ldots)\left(\tilde{\Sigma}^{\mu}_{\mu}\right)^2\right]\right.$$
$$+\frac{q^{\mu}q^{\nu}}{q^4}\left[(\ldots)\tilde{\Sigma}_{\mu\nu}\tilde{\Sigma}^{\rho}_{\rho} + (\ldots)\tilde{\Sigma}_{\mu\rho}\tilde{\Sigma}_{\nu}{}^{\rho} + (\ldots)\frac{q^{\rho}q^{\lambda}}{q^2}\tilde{\Sigma}_{\mu\nu}\tilde{\Sigma}_{\rho\lambda}\right] \tag{4.63}$$
$$\left.+\frac{(\ldots)}{q^2}\left[\tau_{\mu\nu}\tau^{\mu\nu} + (\ldots)\frac{q^{\mu}q^{\nu}}{q^2}\tau_{\mu\rho}\tau_{\nu}{}^{\rho}\right]\right\}\,.$$

Now that we have decoupled the sources, we decompose

$$\tilde{\Sigma}_{\mu\nu} \equiv \tilde{\Sigma}^{T}_{\mu\nu} - \frac{i}{q^2}\left(q_{\mu}\kappa_{\nu} + q_{\nu}\kappa_{\mu}\right) + \frac{1}{q^2}\left(L_{\mu\nu}j_1 + T_{\mu\nu}j_2\right) \quad \text{with } q^{\mu}\kappa_{\mu} = q^{\mu}\tilde{\Sigma}^{T}_{\mu\nu} = 0\,, \tag{4.64a}$$

$$\tau_{\mu\nu} \equiv -\frac{i}{q^2}\left(q_{\mu}\upsilon_{\nu} - q_{\nu}\upsilon_{\mu}\right) + \tilde{\tau}_{\mu\nu} \quad \text{with } q^{\mu}\upsilon_{\mu} = q^{\mu}\tilde{\tau}_{\mu\nu} = 0\,, \tag{4.64b}$$

and adjusting the parameter F, the saturated propagator becomes

$$\Pi = -\frac{1}{2}\int \frac{d^4q}{(2\pi)^4}\left\{\frac{(\ldots)}{q^2}\left[\tilde{\sigma}_{\mu\nu}\tilde{\sigma}^{\mu\nu} - \frac{1}{2}\left(\tilde{\sigma}^{\mu}_{\mu}\right)^2\right] + \frac{(\ldots)}{q^2}\left[\tilde{\Sigma}^{T}{}_{\mu\nu}\tilde{\Sigma}^{T\mu\nu} - \frac{1}{2}\left(\tilde{\Sigma}^{T\mu}{}_{\mu}\right)^2\right]\right.$$
$$\left.+\frac{(\ldots)}{q^2}\tilde{\tau}_{\mu\nu}\tilde{\tau}^{\mu\nu} + \frac{(\ldots)}{q^4}\kappa_{\mu}\kappa^{\mu} + \frac{(\ldots)}{q^4}\upsilon_{\mu}\upsilon^{\mu} + \frac{(\ldots)}{q^6}J\cdot M\cdot J\right\}\,, \tag{4.65}$$

where $J = (j_1, j_2)$. The first term gives the GR contribution, the second one another massless spin 2^+, the third is a massless 1^+ state, the remaining ones are two spin 1^- dipole ghosts and two 0^+ tripole ghosts. The last four are pathological and must be eliminated. This can be achieved by adjusting the coefficients so that the various terms $(\ldots)$ diverge (this is equivalent to setting to zero some terms in the a-matrices). In the process new gauge invariances appear.

The dipole ghost υ_{μ} coming from Ω, can be eliminated imposing (4.39) and

$$a^{TQ}_{1} + a^{TQ}_{3} = 0\,. \tag{4.66}$$

In this way the following gauge invariance appears

$$h_{\mu\nu} \to h_{\mu\nu}\,, \qquad H_{\mu\nu} \to H_{\mu\nu}\,, \tag{4.67a}$$

$$\Omega_{\mu\nu} \to \Omega_{\mu\nu} + \partial_{\mu}\chi_{\nu} - \partial_{\nu}\chi_{\mu}\,. \tag{4.67b}$$

Instead to eliminate κ_{μ} and J we impose the constraints (4.53), and

$$a^{TQ}_{1} - a^{TQ}_{2} = 0\,. \tag{4.68}$$

This amounts to imposing separate "Diff-invariance" on h and H, i.e.

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \,, \tag{4.69a}$$
$$H_{\mu\nu} \to H_{\mu\nu} + \partial_\mu \Xi_\nu + \partial_\nu \Xi_\mu \,, \tag{4.69b}$$
$$\Omega_{\mu\nu} \to \Omega_{\mu\nu} \,. \tag{4.69c}$$

Using these constraints, we find a well defined theory containing two massless particles with spin 2^+ and one with spin 1^+ with three different gauge invariances. In such a theory the graviton is a combination of h and H. Then if we want to decouple h from H, we have to impose

$$4a_1^{QQ} + a_1^{TQ} = 0 \,. \tag{4.70}$$

At this point if we want to have a single massless graviton we have to kill the (non-pathological) degrees of freedom 1^+ and 2^+. From the coefficient matrices, this is achieved by imposing (4.41) and

$$2a_1^{TT} + a_2^{TT} + 4a_1^{QQ} + 2a_1^{TQ} = 0 \ . \tag{4.71}$$

Imposing relations (4.39,4.41, 4.53,4.68,4.70,4.71), the unique solution is the choice $a_1^{TT} = -\frac{1}{4}m^R$, $a_2^{TT} = -\frac{1}{2}m^R$, $a_3^{TT} = m^R$, $a_1^{QQ} = -\frac{1}{4}m^R$, $a_2^{QQ}+a_4^{QQ} = \frac{1}{2}m^R$, $a_3^{QQ} = \frac{1}{4}m^R$, $a_5^{QQ} = -\frac{1}{2}m^R$, $a_1^{TQ} = m^R$, $a_2^{TQ} = m^R$, $a_3^{TQ} = -m^R$, which reproduce the general teleparallel equivalent of GR (2.4). This analysis agrees with the findings of [85].

Chapter 5

Examples of MAG

As discussed in chapter 3, in general metric-affine gravity, even when including terms of dimension up to four, one encounters 934 independent terms, with 59 of them contributing to the flat propagator. Such a theory, initially motivated by the absence of additional assumptions and geometrical beauty, depends on a huge number of free parameters and hence suffers from the lack of predictivity. In what follows we consider special cases of MAG, being motivated mainly by the reasons of simplicity. First, we will consider two cases when states of spin/parity 2^- or 3^- are propagated in addition to the graviton, and then we will discuss a peculiar case of a theory that appears to be MAG from Cartan point of view, but in fact, is just GR.

5.1 DIY MAGs

The spin-projector formalism has been used to look for MAGs that are free of ghosts and tachyons [38–43, 66]. The general procedure has been to impose conditions on the kinetic coefficients and see what kind of particles the theory describes. Here we would like to use a different approach: to decide a priori what particles we want and then construct a MAG that has the right propagator for those particles. This goes as follows: we know the correct forms of the propagators for particles of any spin/parity. These are listed in Appendix D of [86]. At the linearised level, one can write down a kinetic term that gives the correct propagators for the desired states, and nothing else. Then one can turn this kinetic term into a full nonlinear Lagrangian for a MAG in Einstein form by the simple procedure of minimal coupling. The Lagrangian obtained in this way is highly non-unique: the order of the covariant derivatives is arbitrary and all the cubic and quartic terms are absent. Nevertheless, this is a MAG that has the desired propagators. As a subsequent step one can try to add the cubic and quartic terms, and, if necessary, adjust the ordering of the derivatives at the cost of adding terms of the form $R\phi\phi$. We note that this procedure will work if we remain in the context of the general Lagrangians of the Chapter 3. This is because the general linearised kinetic term for MAG has 47 free parameters, corresponding to the 47 independent terms of a general Lagrangian. It would not work in general for the Lagrangians that only have dimension-four terms of the form F^2, that depend altogether on 28 free parameters.

In this section we will give two examples of this construction. Being a three-index tensor, distortion can carry any of the states listed in Table 4.2. From the point of view of particle physics, it may seem redundant to use distortion to describe a particle of spin $0^\pm$, $1^\pm$ or 2^+, because all these particles can be described by tensor fields of lower rank. The only states that do require a three index tensor have spin 2^- and 3^-. We will therefore analyze here these two cases at the linearised level. We stress that the MAGs constructed in this way can only be said to be consistent at the linearised level. We do not make any claim as to their consistency when interactions are turned on.

5.1.1 Simple MAG with a 2^- state

We start from a general MAG. A look at Table 4.2 shows that there are two possible d.o.f.'s with spin 2^-: 2^-_1 being hook-symmetric and 2^-_2 being hook-antisymmetric (recall that in this context we refer here to symmetry or antisymmetry in the last two indices). The free Lagrangians for a spin/parity 2^- state carried by an antisymmetric or symmetric tensor, and the corresponding propagators, are given in Appendix D of [86]. Here we show how to recover those linearised Lagrangians from MAGs.

We will use the coefficient matrices for the theory in the Einstein form, for which the last row and column are identically zero as a result of diffeomorphism invariance. In order to remove the unwanted propagating dof's we impose various conditions on the coefficient matrices. Demanding that the matrices for spins 3^-, 1^+ and 0^- have no terms proportional to q^2 leads to the constraints:

$$\begin{aligned} b_2^{TT} &= b_1^{TT}\ , & b_5^{TT} &= b_1^{TT} + b_4^{TT}, \\ b_6^{TT} &= -b_1^{TT}\ , & b_7^{TT} &= -2b_1^{TT}\ , \\ b_2^{QQ} &= -b_1^{QQ}\ , & b_8^{QQ} &= 3b_1^{QQ} + b_7^{QQ}\ , \\ b_5^{TQ} &= -b_1^{TQ}\ , & b_7^{TQ} &= b_1^{TQ} + b_6^{TQ}\ . \end{aligned} \tag{5.1}$$

Next, in the sectors 2^+ and 0^+ we demand that the mixed a-h terms vanish and that all the other terms, except for those corresponding to the standard graviton, have no q^2 terms. This leads to

$$\begin{aligned} b_4^{TT} &= -2b_1^{TT} + 2b_2^{TT} + b_7^{TT} + 2b_1^{QQ} + b_7^{QQ} + b_1^{TQ} + b_5^{TQ}, \\ b_6^{TT} &= -b_1^{TT} - b_1^{QQ} - 1/2b_7^{QQ} - 1/2b_1^{TQ} - 1/2b_5^{TQ}, \qquad b_9^{TT} = -b_3^{TT}, \\ b_6^{QQ} &= 3b_1^{TT} - 3/2b_2^{TT} + 1/2b_5^{TT} + 1/2b_7^{TT} - 2b_1^{QQ} - b_2^{QQ}, \\ b_{12}^{QQ} &= -b_{10}^{QQ}, \quad b_{13}^{QQ} = -b_{11}^{QQ}, \quad b_{14}^{QQ} = -b_4^{QQ}\ , \quad b_{15}^{QQ} = -b_3^{QQ}\ , \quad b_{16}^{QQ} = -b_5^{QQ}\ , \\ b_4^{TQ} &= -6b_1^{TT} + 2b_2^{TT} - 2b_5^{TT} - b_7^{TT} + 2b_1^{QQ} + 2b_2^{QQ} - b_1^{TQ}, \\ b_6^{TQ} &= -4b_1^{TT} + 2b_2^{TT} - b_7^{TT} + 4b_1^{QQ} + 2b_2^{QQ} - b_9^{QQ}, \\ b_7^{TQ} &= 2b_2^{TT} + b_7^{TT} + 4b_1^{QQ} + 2b_7^{QQ} + 2b_1^{TQ} + b_5^{TQ}, \\ b_{11}^{TQ} &= -b_{10}^{TQ} \qquad b_{12}^{TQ} = b_3^{TQ}\ , \qquad b_{13}^{TQ} = b_2^{TQ}, \end{aligned} \tag{5.2}$$

and further six relations for the $R\nabla T$ and $R\nabla Q$ that, together withe Bianchi identities, remove all the terms of this type.

Then we impose that the spin 2^- and 1^- are properly related, as discussed in Appendix

D.6 of [86]. This leads to

$$
\begin{aligned}
&b_2^{TT} = 2b_1^{TT} + 1/3 b_3^{TT} \ , \qquad b_8^{TT} = -2b_3^{TT} \ , \\
&b_2^{QQ} = -2b_1^{QQ} - b_3^{QQ} \ , \quad b_4^{QQ} = b_3^{QQ} \ , \\
&b_5^{QQ} = b_{10}^{QQ} = -b_{11}^{QQ} = -2b_3^{QQ} \ , \\
&b_3^{TQ} = -b_2^{TQ} = b_8^{TQ} = -b_9^{TQ} = b_{10}^{TQ} = 2b_1^{TT} + 2/3 b_3^{TT} + 2b_1^{QQ} + 2b_3^{QQ} + b_1^{TQ} \ .
\end{aligned}
\tag{5.3}
$$

The same requirement for the mass parameters implies

$$
\begin{aligned}
&m_3^{TT} = -2m_1^{TT} - m_2^{TT} \ , \qquad m_4^{QQ} = 2m_1^{QQ} - m_2^{QQ} + 4m_3^{QQ} \ , \\
&m_5^{QQ} = 4m_1^{QQ} - 2m_2^{QQ} + 4m_3^{QQ} \ , \quad m_2^{TQ} = -m_3^{TQ} = m_1^{TQ} \ .
\end{aligned}
\tag{5.4}
$$

The coefficient matrices now depend only on b_1^{TT}, b_1^{QQ}, b_1^{TQ} and on the mass parameters m_1^{TT}, m_2^{TT}, m_1^{QQ}, m_2^{QQ}, m_3^{QQ}, m_1^{TQ}. In particular the sectors 2^+_{44} and 0^+_{55} have the right form to propagate a massless graviton. Similarly the matrix for the 2^- and the submatrix 1^-_{22}, 1^-_{23}, 1^-_{32}, 1^-_{33} describe two mixed spin 2^- dof's. All the remaining components are either zero on shell (if the mass is nonzero) or a gauge dof (if the mass is zero). In particular the matrix $a(2^-)$ is

$$
\begin{aligned}
a_{11}(2^-) &= \frac{1}{2}\Big(3(3b_1^{TT} + 4b_1^{QQ} + 2b_1^{TQ})(-q^2) \\
&\quad +(-6m_1^{TT} - 3m_2^{TT} - 8m_1^{QQ} + 4m_2^{QQ} - 6m_1^{TQ})\Big) \ , \\
a_{12}(2^-) &= \frac{\sqrt{3}}{2}((3b_1^{TT} + b_1^{TQ})(-q^2) - (2m_1^{TT} + m_2^{TT} + m_1^{TQ})) \ , \\
a_{22}(2^-) &= \frac{1}{2}\left(3b_1^{TT}(-q^2) + (-2m_1^{TT} - m_2^{TT})\right) \ ,
\end{aligned}
\tag{5.5}
$$

and the submatrix 1^-_{22}, 1^-_{23}, 1^-_{32}, 1^-_{33} is the same up to the sign. Then, there is the graviton contribution inside $a(2^+)_{44}$ and $a(0^+)_{55}$, with the correct proportionality discussed in Appendix D.5 of [86]. Finally, except for the entries constraint by the diffeomorphism invariance, i.e. (4.19), all the other entries are just mass terms.

The two spin 2^- dof's are generically mixed. The mixing can be eliminated by assuming

$$
b_1^{TQ} = -3b_1^{TT} \qquad \text{and} \qquad m_1^{TQ} = -2m_1^{TT} - m_2^{TT} \ .
$$

To avoid ghosts we must assume that $4b_1^{QQ} > 3b_1^{TT}$ and $b_1^{TT} > 0$. In particular this condition can be satisfied by both dofs.

In order to propagate only the hook-antisymmetric component 2^-_2 , discussed in Section D.6.1 of [86], we have to set

$$
b_1^{QQ} = 3/4 b_1^{TT}
$$

and then we must assume $b_1^{TT} > 0$. The mass squared term is proportional to $(2m_1^{TT} + m_2^{TT})$. In such a theory, the kinetic term for the state 2^- in the Lagrangian involves terms $\nabla T \nabla T$, $\nabla Q \nabla Q$ and $\nabla T \nabla Q$. [1]

In order to propagate only the hook-symmetric component 2^-_1, discussed in Section D.6.2 of [86], we have to set

$$
b_1^{TT} = 0
$$

and assume $b_1^{QQ} > 0$. The mass squared term is proportional to $(6m_1^{TT} + 3m_2^{TT} - 8m_1^{QQ} + 4m_2^{QQ})$. In such a theory, the kinetic term for the state 2^- in the Lagrangian involves only terms $\nabla Q \nabla Q$.

[1] This complication could be avoided by adopting another definition of hook (anti)symmetry.

5.1.2 Simple MAG with a 3^- state

We can remove all the terms proportional to q^2 in the coefficient matrices, except for those that propagate the massless 3^- and 2^+ d.o.f.'s. This gives linear equations for the coefficients that are solved by

$$\begin{aligned}
&b_i^{TT} = 0 \quad \text{for} \quad i = 1,2,3,4,5,6,7,8,9 \\
&b_i^{TQ} = 0 \quad \text{for} \quad i = 1,2,3,4,5,6,7,8,9,10,11,12,13 \\
&b_2^{QQ} = 2b_1^{QQ}\ ,\ b_3^{QQ} = -4b_1^{QQ}\ ,\ b_4^{QQ} = -b_1^{QQ}\ ,\ b_5^{QQ} = -4b_1^{QQ}\ ,\ b_6^{QQ} = -b_1^{QQ}\ , \\
&b_7^{QQ} = -2b_1^{QQ}\ ,\ b_8^{QQ} = -2b_1^{QQ}\ ,\ b_9^{QQ} = -4b_1^{QQ}\ ,\ b_{10}^{QQ} = 8b_1^{QQ}\ , \\
&b_{11}^{QQ} = b_{12}^{QQ} = 4b_1^{QQ}\ ,\ b_{13}^{QQ} = 2b_1^{QQ}\ ,\ b_{15}^{QQ} = b_{16}^{QQ} = 4b_{14}^{QQ}\ ,
\end{aligned} \tag{5.6}$$

and further six relations for the $R\nabla T$ and $R\nabla Q$ that, together with the Bianchi identities, remove all the terms of this type. This puts to zero the matrices $a(2^-)$, $a(1^+)$ and $a(0^-)$. Further requiring that the ratio of the coefficients of q^2 in $a(3^-)$ and $a_{11}(0^+)$ be equal to $-9/2$, due to the requirements discussed in Section D.7 of [86], fixes $b_{14}^{QQ} = -1/2b_1^{QQ}$. Then, the remaining coefficient matrices are

$$a(3^-) = 12b_1^{QQ}(-q^2)\ , \tag{5.7a}$$

$$a(2^+) = (-q^2)\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{m^R}{4} \end{pmatrix}\ , \tag{5.7b}$$

$$a(1^-) = (-q^2)\begin{pmatrix} -48b_1^{QQ} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\ , \tag{5.7c}$$

$$a(0^+) = (-q^2)\begin{pmatrix} -54b_1^{QQ} & 0 & 0 & -18b_1^{QQ} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -18b_1^{QQ} & 0 & 0 & -6b_1^{QQ} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2}m^R & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\ . \tag{5.7d}$$

Note that the first 4×4 block in $a(0^+)$ has rank one. All the degrees of freedom are pure gauge, except for the desired 2^+ and 3^-. In such a theory, the kinetic term for the state 3^- in the Lagrangian involves only terms $\nabla Q \nabla Q$. We have chosen $b_1^{RR} = b_2^{RR} = b_3^{RR} = 0$, so the graviton propagator is as in GR.

Some comments are in order at this point. The subject of higher spin theories is a thorny one. Normally it is approached in a bottom-up fashion, starting from a free theory in flat space and then trying to construct interactions. In the process, one encounters numerous difficulties. Here we have started from a ready-made nonlinear theory (MAG) and tried to arrange its parameters so that at the linearised level it reproduces the known free spin-3 Lagrangian. With our choice of coefficients, the $\phi\phi$ part of the linearised action (4.5) is

$$\begin{aligned}S^{(2)} = -2b_1^{QQ}\int\frac{d^4q}{(2\pi)^4}\Big[&q^2\Big(\frac{1}{4}Q_{\alpha\beta\gamma}Q^{\alpha\beta\gamma}+\frac{1}{2}Q_{\alpha\beta\gamma}Q^{\beta\alpha\gamma}\\&-\mathrm{tr}_{(12)}Q_\alpha\mathrm{tr}_{(12)}Q^\alpha-\mathrm{tr}_{(12)}Q_\alpha\mathrm{tr}_{(23)}Q^\alpha-\frac{1}{4}\mathrm{tr}_{(23)}Q_\alpha\mathrm{tr}_{(23)}Q^\alpha\Big)\\&-\frac{1}{4}\mathrm{div}_{(1)}Q_{\alpha\beta}\mathrm{div}_{(1)}Q^{\alpha\beta}-\mathrm{div}_{(1)}Q_{\alpha\beta}\mathrm{div}_{(2)}Q^{\alpha\beta}-\mathrm{div}_{(2)}Q_{(\alpha\beta)}\mathrm{div}_{(2)}Q^{(\alpha\beta)}\\&-\frac{1}{8}\mathrm{trdiv}_{(1)}Q_{\alpha\beta}\mathrm{trdiv}_{(1)}Q^{\alpha\beta}-\frac{1}{2}\mathrm{trdiv}_{(1)}Q_{\alpha\beta}\mathrm{trdiv}_{(2)}Q^{\alpha\beta}-\frac{1}{2}\mathrm{trdiv}_{(2)}Q_{\alpha\beta}\mathrm{trdiv}_{(2)}Q^{\alpha\beta}\\&+\mathrm{div}_{(12)}Q_\alpha\mathrm{tr}_{(23)}Q^\alpha+\frac{1}{2}\mathrm{div}_{(23)}Q_\alpha\mathrm{tr}_{(23)}Q^\alpha+2\mathrm{div}_{(12)}Q_\alpha\mathrm{tr}_{(12)}Q^\alpha+\mathrm{div}_{(23)}Q_\alpha\mathrm{tr}_{(12)}Q^\alpha\Big]\,,\end{aligned} \tag{5.8}$$

where now $\mathrm{div}_{(1)}Q_{\alpha\beta} = iq^\lambda Q_{\lambda\alpha\beta}$ etc.. The standard description of the spin-3 particle is by a totally symmetric 3-tensor. Thus in the formula above we replace $Q_{\alpha\beta\gamma}$ by $S_{\alpha\beta\gamma} = Q_{(\alpha\beta\gamma)}$, and set $b_1^{QQ} = 1/3$ to obtain

$$\begin{aligned}S^{(2)} = \frac{1}{2}\int\frac{d^4q}{(2\pi)^4}\Big[&-q^2S_{\alpha\beta\gamma}S^{\alpha\beta\gamma}+3q^2\mathrm{tr}_{(12)}S_\alpha\mathrm{tr}_{(12)}S^\alpha\\&+3\mathrm{div}_{(1)}S_{\alpha\beta}\mathrm{div}_{(1)}S^{\alpha\beta}+\frac{3}{2}(\mathrm{trdiv}_{(1)}S)^2-6\mathrm{div}_{(12)}S_\alpha\mathrm{tr}_{(12)}S^\alpha\Big]\,.\end{aligned} \tag{5.9}$$

This is indeed the Fronsdal Lagrangian that correctly describes a free massless spin-3 particle [94]. However, this is only a very limited success. The "higher spin symmetry" $\delta S_{\alpha\beta\gamma} = \partial_{(\alpha}\Lambda_{\beta\gamma)}$, that is a necessary invariance of a higher spin theory, is only an accidental symmetry here. More details on these issues, the relation of this approach to earlier attempts to embed higher-spin theory in MAG [66, 95] and a discussion of the massive case will be given elsewhere.

5.2 MAGs without propagation

There are classes of MAGs that look perfectly normal when presented in the Cartan form, but have no propagating degrees of freedom. [2] The initial step towards these theories is the observation that known ghost- and tachyon-free MAGs, when presented in Einstein form, do not contain terms quadratic in curvature [41]. This is reasonable, insofar as 4DG is known to contain ghosts or tachyons.

However, we can now demand more: in the notation of equation (3.2), suppose that $m^R = 0$, $b^{RR} = 0$ and $b^{R\phi} = 0$. This means that the Hilbert term is absent, as well as the terms quadratic in curvature and mixed terms of the form $R\nabla\phi$. The first two lines of the Lagrangian can therefore be written in the form

$$\phi_{\alpha\beta\gamma}\left(K^{\alpha\beta\gamma|\rho\sigma|\lambda\mu\nu}\nabla_\rho\nabla_\sigma+\mathcal{M}^{\alpha\beta\gamma|\lambda\mu\nu}\right)\phi_{\lambda\mu\nu}\ , \tag{5.10}$$

where K and $\mathcal{M}$ are tensors constructed exclusively with the metric. The remaining terms do not contribute to the propagator in flat space, but only to interactions. For simplicity we shall

[2] This observation came up in discussions with E. Sezgin.

ignore them in the subsequent discussion, but they do not change the conclusions. When the Lagrangian is linearised, it gives a kinetic operator of the form (5.10), where all the metrics are Minkowski metrics and all covariant derivatives are replaced by partial derivatives.

If we were just considering this as a theory of a field ϕ propagating in a fixed background metric, it would have, in general, propagating degrees of freedom obeying the field equation

$$\left(K^{\alpha\beta\gamma|\rho\sigma|\lambda\mu\nu}\nabla_\rho\nabla_\sigma + \mathcal{M}^{\alpha\beta\gamma|\lambda\mu\nu}\right)\phi_{\lambda\mu\nu}\ . \tag{5.11}$$

However, in a MAG we have to satisfy also the equation for the metric, which in the basence of matter simply says that the energy-momentum tensor of ϕ has to vanish. Since plane waves carry nonzero energy and momentum, it is already clear that this will forbid normal propagation. To see this more explicitly, write the Lagrangian as

$$\phi_{\alpha\beta\gamma}\mathcal{O}^{\alpha\beta\gamma|\lambda\mu\nu}\phi_{\lambda\mu\nu}\ . \tag{5.12}$$

In flat space one can Fourier transform and write

$$\mathcal{O}^{\alpha\beta\gamma|\lambda\mu\nu} = -K^{\alpha\beta\gamma|\rho\sigma|\lambda\mu\nu}q_\rho q_\sigma + \mathcal{M}^{\alpha\beta\gamma|\lambda\mu\nu}\ .$$

The energy-momentum tensor is

$$T^{\rho\sigma} = \frac{2}{\sqrt{-g}}\phi_{\alpha\beta\gamma}\frac{\partial(\sqrt{-g}\,\mathcal{O}^{\alpha\beta\gamma|\lambda\mu\nu})}{\partial g_{\rho\sigma}}\,\phi_{\lambda\mu\nu}\,. \tag{5.13}$$

The operator $\mathcal{O}$ has zero modes corresponding to infinitesimal coordinate transformations, but generically there will be no others. When this is the case, demanding $T^{\rho\sigma} = 0$ implies that ϕ can be at most a coordinate transform of zero.

Let us observe that while the absence of terms containing the curvature $R_{\alpha\beta\gamma\delta}$ (and its contractions) is immediately conspicuous in the Einstein form, it is not in the Cartan form. We can now ask, in the Cartan form of MAG, what choices of coefficients will produce a theory of this type. From (D.21) we see that the vanishing of the R^2 terms implies

$$\begin{aligned} c_1^{FF} - c_2^{FF} + c_3^{FF} + 1/2(c_4^{FF} - c_5^{FF} + c_6^{FF}) &= 0\,, \\ c_7^{FF} + c_8^{FF} + c_9^{FF} + c_{10}^{FF} - c_{11}^{FF} - c_{12}^{FF} &= 0\,, \\ c_{16}^{FF} &= 0\,, \end{aligned} \tag{5.14}$$

and from (D.22), the vanishing of the terms $R\nabla(T/Q)$ implies

$$\begin{aligned} 2(4c_1^{FF} - 4c_2^{FF} + 4c_3^{FF} + 2c_4^{FF} - 2c_5^{FF} + 2c_6^{FF} + c_7^{FF} + c_8^{FF} + c_9^{FF} + c_{10}^{FF} - c_{11}^{FF} - c_{12}^{FF}) &= 0\ , \\ -c_7^{FF} - c_8^{FF} - c_9^{FF} - c_{10}^{FF} + c_{11}^{FF} + c_{12}^{FF} + 4c_{16}^{FF} &= 0\ , \\ 2(-2c_1^{FF} + 2c_2^{FF} - 2c_3^{FF} - c_4^{FF} + c_5^{FF} - c_6^{FF} + c_7^{FF} + c_8^{FF}) - c_{11}^{FF} - c_{12}^{FF} &= 0\ , \\ -3c_7^{FF} - 3c_8^{FF} - c_9^{FF} - c_{10}^{FF} + 2c_{11}^{FF} + 2c_{12}^{FF} &= 0\ , \\ c_9^{FF} + c_{10}^{FF} - c_{11}^{FF}/2 - c_{12}^{FF}/2 - 2c_{16}^{FF} &= 0\ , \\ 1/2(-c_7^{FF} - c_8^{FF} - c_9^{FF} - c_{10}^{FF} + c_{11}^{FF} + c_{12}^{FF} + 4c_{16}^{FF}) &= 0\ . \end{aligned} \tag{5.15}$$

Furthermore, it is also important to notice that this phenomenon will not be apparent in the linearised form of the theory: the energy-momentum tensor is quadratic in ϕ and the linearised EOM for the metric on a flat background will just be $0 = 0$. Instead, the linearised theory will contain some accidental symmetry.

Probably the simplest and most illuminating example is the action where we retain only $c_{13}^{FF} = 2$, all the others being zero:

$$\mathcal{L} = -F^{(34)}_{\mu\nu} F^{(34)\mu\nu} \ .$$

Using (1.37),

$$F^{(34)}_{\mu\nu} = \frac{1}{2}\left(\nabla_\mu \mathrm{tr}_{(23)} Q_\nu - \nabla_\nu \mathrm{tr}_{(23)} Q_\mu\right) \ .$$

Thus, in spite of appearances, this is just a free Maxwell field coupled to a metric that does not have a kinetic term. There is an EOM stating that the electromagnetic energy-momentum tensor is zero, which implies that $F_{\mu\nu} = 0$. On the other hand, if we study this theory with the methods of Section 4, we find that all coefficient matrices are zero except for $a(1^-)$, that has rank one. All the nonzero rows/columns are proportional to q^2 and choosing a gauge appropriately one would erroneously conclude that the theory contains a free massless spin one particle.

A less trivial example is obtained by setting all coefficients to zero except $c_2^{FF} = c_1^{FF} = 2$:

$$\mathcal{L} = -F_{\mu\nu(\rho\sigma)} F^{\mu\nu(\rho\sigma)} \, .$$

In this case the linearised analysis seems to indicate several propagating (and interacting) particles, but this conclusion is false in the full nonlinear theory.

Chapter 6

Quantum GR and Unimodular gravity

In this chapter, we build up our understanding of UV divergences appearing in metric theories of gravity. But first, we digress to discuss path integrals in Einstein and Unimodular gravity and address the question of classical quantum equivalence between these theories.

6.1 Classical dynamics of Unimodular gravity

We start this section by discussing one of the best-known problems of modern theoretical physics, the cosmological constant problem. It arises when one attempts to reconcile two following assumptions: first, that the observed acceleration of the universe is due to the constant term in gravitational Lagrangian and second, that the energy density generated by vacuum fluctuations depends quartically on a cutoff. These assumptions lead to a huge apparent discrepancy between the "predicted" and observed values of spacetime curvature. However, neither of these assumptions is based on strong ground. Firstly, it may also be the case that there is no cosmic acceleration, and the observed apparent "acceleration" is an artefact of the homogeneous Friedmann–Lemaître–Robertson–Walker (FLRW) spacial metric which may not be applicable to the current inhomogeneous state of the Universe [96, 97]. Secondly, assuming that the acceleration is real, it may be due to some dynamical mechanism, possibly related to the independent connection [98]. As we have mentioned earlier, the observed cosmic acceleration can be explained by some hidden gravitational dynamics, for example, the dynamics of independent connection. Thirdly, power divergences appear in some regularisation schemes, such as momenta cutoff, but do not appear in others, such as dimensional regularisation. Fourthly, observable quantities cannot depend on the cutoff.

If gravity is only "renormalisable" in the sense explained in section 1.2, meaning that all quantum divergences can be removed by the addition of local counterterms with only a finite number of them being not suppressed by the Planck mass, the cosmological constant, which does affect the dynamics at low energies, must not receive cutoff-dependant corrections.

On the other hand, if gravity is understood as an EFT, the CC will receive finite contributions from integrating out heavy degrees of freedom, and generally, one would expect the Wilson coefficient before the dimension zero term to be of order one, which is in huge disagreement with experimental data. In the following, we will discuss how this aspect of the CC problem is related to the gauge group choice.

Unimodular gravity is defined as a metric theory with the following condition on the determinant of the metric being imposed:

$$\sqrt{|g|} = \omega(x) \ , \tag{6.1}$$

where ω is a scalar function. Here one immediately notices that due to the local diffeomorphic symmetry, one can impose (6.1) locally everywhere as a gauge condition. The subtlety comes from the fact that (6.1) can be integrated over the full space-time volume, with the result being a genuine physical constraint:

$$V = \int d^d x \sqrt{|g|} = \int d^d x \ \omega \ . \tag{6.2}$$

Since the determinant of the metric is fixed, no fields couple to the constant vacuum energy, which therefore does not gravitate or contribute to spacetime curvature. The unimodular condition can also be imposed as a Lagrange multiplier by the addition of the following term:

$$\frac{\Lambda}{8\pi G}\left(V - \int d^4 x \sqrt{|g|}\right) \tag{6.3}$$

where Λ has to be thought of as a Lagrange multiplier enforcing that the spacetime volume is equal to V[1] Therefore, Unimodular gravity has exactly one degree of freedom less than GR (not one per spacetime point). It is natural to expect that this distinction will affect the infrared properties of the theory, but not its behaviour at short scales. In this thesis, we shall not discuss the global, large-scale properties and when we say "GR" we shall implicitly mean "GR with fixed total volume".

Let us look at these theories from the point of view of gauge symmetry.

The transformation of the metric determinant under the general diffeomorphisms $x^\mu \to x^\mu + \epsilon^\mu(x)$ is given by

$$\delta_\epsilon \sqrt{|g|} = \frac{1}{2}\sqrt{|g|} g^{\mu\nu} \delta g_{\mu\nu} = \sqrt{|g|} \nabla_\mu \epsilon^\mu \tag{6.4}$$

Therefore, the Special diffeomorphism group (*SDiff*), [2] the local gauge symmetry of Unimodular gravity, is generated by transverse vectors:

$$\nabla_\mu \epsilon^\mu = 0 \ . \tag{6.5}$$

The action of Unimodular gravity is much alike with (1.17) and its lowest energy term is the Hilbert–Einstein one:

$$S_{UG}(g) = Z_N \int d^d x \ \omega \ R, \quad Z_N = \frac{1}{16\pi G} \ . \tag{6.6}$$

In the rest of this section, we will show that this theory is equivalent to GR. Equations of motion obtained by extremising the action (6.6) are

$$-\tilde{E}_{\mu\nu} + \frac{1}{d} g_{\mu\nu} \tilde{E} = \frac{1}{2}\left(\tilde{T}_{\mu\nu} - \frac{1}{d} g_{\mu\nu} \tilde{T}\right) \tag{6.7}$$

[1] This point of view has been used by Hawking in Euclidean quantum gravity, where he interpreted the resulting partition function as the "volume canonical ensemble", see [99]

[2] In the recent literature, the group is often referred to as *TDiff*, where T stands for "transverse".

where

$$\tilde{E}_{\mu\nu} = \frac{1}{\omega}\frac{\delta S^g_{UG}}{\delta g^{\mu\nu}} \tag{6.8}$$

is the analogue of the Einstein tensor and

$$\tilde{T}_{\mu\nu} = \frac{2}{\omega}\frac{\delta S^m_{UG}}{\delta g^{\mu\nu}} \ , \tag{6.9}$$

is the energy-momentum tensor, which is not however covariantly conserved. Instead,

$$\nabla_\mu \tilde{T}^{\mu\nu} = \nabla^\nu \Sigma \neq 0 \ , \tag{6.10}$$

where § is some scalar. We can define another conserved energy-momentum tensor

$$T_{\mu\nu} = \tilde{T}_{\mu\nu} + g_{\mu\nu}\mathcal{L}_m \ . \tag{6.11}$$

Then,

$$-\tilde{E}_{\mu\nu} + \frac{1}{d}g_{\mu\nu}\tilde{E} = \frac{1}{2}\left(T_{\mu\nu} - \frac{1}{d}g_{\mu\nu}T\right) \ . \tag{6.12}$$

The *SDiff*-invariance implies “generalised Bianchi identity”:

$$\nabla_\mu \tilde{E}^{\mu\nu} + \frac{1}{2}\nabla^\nu \mathcal{L}_g = 0 \tag{6.13}$$

Acting with a derivative on both sides of (6.12) and using (6.13) we obtain the equation

$$\nabla^\nu\left(\frac{1}{2}\mathcal{L}_g + \frac{1}{d}\tilde{E} + \frac{1}{2d}T\right) = 0 \ , \tag{6.14}$$

that can be integrated to get

$$\frac{1}{2}\mathcal{L}_g + \frac{1}{d}\tilde{E} + \frac{1}{2d}T = Z_N\Lambda_1 \ . \tag{6.15}$$

Using (6.12) we finally obtain

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda_1 g_{\mu\nu} = \frac{1}{2Z_N}T_{\mu\nu} \ . \tag{6.16}$$

This is exactly what we would have obtained by extremising the *Diff*-invariant Hilbert–Einstein action. The difference however is that the cosmological constant enters (6.16) as an integration constant, rather than follows from the action. This confirms that Unimodular gravity is equivalent to GR, except for one degree of freedom.

6.2 Path integrals in Einstein and Unimodular gravity

It is then natural to ask whether this “almost equivalence” holds also for the quantum versions of the theories. In recent years, there appeared in the literature conflicting statements about the equivalence, or lack thereof, between GR and UG at the quantum level, see, e.g., [100–124]. We believe that some of these contradictions may be just due to different quantisation procedures. In this work, we prove in general, based on formal path integral arguments, that there exists a quantisation procedure that preserves the "almost equivalence" between these theories. The proof goes through for any *Diff*-invariant action and in this sense extends beyond ordinary GR. Of course, there may be other definitions of the quantum theories that break the equivalence, but in the absence of other independent arguments in their favour, we think that the one we describe here is more natural. Our argument is in the same spirit as the one presented in [103, 122] and extends the results of [106, 107, 125] beyond one-loop order. We should remark that both GR and UG are not renormalisable in perturbation theory and the formal path integrals should be ultraviolet (UV) regularised. Our formal proof relies on the use of the background field method, but we leave the parameterisation of the metric, i.e., the way that we split the full metric in the background and fluctuating parts, generic. Hence, this also extends previous results [106, 107] which made explicit use of the so-called exponential split of the metric to impose the unimodularity condition [123].

Our proof of equivalence is given initially for pure gravity and one may again worry that as soon as matter degrees of freedom are introduced, the equivalence would fall apart. This is due to the different vertex structures. In GR, the determinant of the metric produces infinitely many vertices between gravitons and matter fields that are absent in UG. Hence, Feynman rules are different in the two settings and one might expect that it is very unlikely that in the computation of an observable, miraculous cancellations lead to equivalent results. Yet, there are results in the literature explicitly showing that this happens, see, e.g., [108, 109, 126]. In fact, we shall see that our formal proof of equivalence extends also to the case when matter is present.

The starting point of our analysis is the (Euclidean)[3] path integral defined by a gravitational action $S_{\text{Diff}}(g_{\mu\nu})$, $g_{\mu\nu} = g_{\mu\nu}(\bar{g}; h)$ being the metric, $\bar{g}_{\mu\nu}$ a fixed background metric and $h_{\mu\nu}$ the fluctuating field which is integrated over. The fluctuating field $h_{\mu\nu}$ does not need to be small, i.e., a perturbation around $\bar{g}_{\mu\nu}$. Moreover, the split of the full metric $g_{\mu\nu}$ in background and fluctuating parts is also general, not being restricted to the standard additive (linear) split. The action is assumed to be invariant under diffeomorphisms (but it is not restricted to be the Hilbert–Einstein action), and so is the functional measure $\mathcal{D}h_{\mu\nu}$. Formally, the path integral is expressed as

$$\mathcal{Z}_{\text{Diff}} = \int \frac{\mathcal{D}h_{\mu\nu}}{V_{\text{Diff}}} \mathrm{e}^{-S_{\text{Diff}}[g(\bar{g};h)]} \,. \tag{6.17}$$

The factor V_{Diff} stands for the volume of the diffeomorphism group.

In most practical calculations within a continuum quantum-field theoretic setting, a gauge-fixing term must be introduced in (6.17). This is typically achieved by the Faddeev-Popov procedure. The redundancy is generated by vector fields ϵ^μ which can be decomposed as

$$\epsilon^\mu = \epsilon^\mu_{\mathrm{T}} + \nabla^\mu \phi \,, \tag{6.18}$$

with $\nabla_\mu \epsilon^\mu_{\mathrm{T}} = 0$ and ∇_μ the covariant derivative defined with respect to the metric $g_{\mu\nu}$. The transverse vectorfields $\epsilon^\mu_{\mathrm{T}}$ generate the group *SDiff* of special (volume-preserving) diffeomorphisms.

Instead of introducing a single gauge-fixing condition for the entire group of diffeomorphism, we introduce two different conditions, first breaking *Diff* to *SDiff*, and then breaking *SDiff*.

[3]The Euclidean signature is not essential at this stage and the results could be equally deduced in the Lorentzian case.

This strategy has been discussed and worked out in a different way in [122], see also [103] and [127] for a general discussion of partial gauge fixing. In the first step we choose a gauge-fixing functional $\mathcal{F}(g)$ and insert the standard Faddeev-Popov identity given by

$$1 = \Delta_{\mathcal{F}}(g) \int \mathcal{D}\phi \; \delta(\mathcal{F}(g^{\phi})) \,, \tag{6.19}$$

where $\Delta_{\mathcal{F}}(g)$ denotes the Faddeev-Popov determinant. The notation g^{ϕ} denotes the transformation of the metric generated by the longitudinal vectorfield $\nabla_\mu \phi$:

$$\delta_\phi g_{\mu\nu} = 2\nabla_\mu \nabla_\nu \phi \,. \tag{6.20}$$

We can now plug (6.19) in (6.17) leading to

$$\mathcal{Z}_{\text{Diff}} = \int \frac{\mathcal{D}h_{\mu\nu}}{V_{\text{Diff}}} \left(\Delta_{\mathcal{F}}(g) \int \mathcal{D}\phi \; \delta(\mathcal{F}(g^{\phi})) \right) \mathrm{e}^{-S_{\text{Diff}}[g(\bar{g};h)]} \,. \tag{6.21}$$

Following the standard steps we now use the gauge invariance of the measure, of the Faddeev-Popov determinant and the action and redefine the integration variable, to get

$$\mathcal{Z}_{\text{Diff}} = \int \frac{\mathcal{D}\phi \mathcal{D}h_{\mu\nu}}{V_{\text{Diff}}} \Delta_{\mathcal{F}}(g) \delta(\mathcal{F}(g)) \, \mathrm{e}^{-S_{\text{Diff}}[g(\bar{g};h)]} \,. \tag{6.22}$$

In [106, 107] it was shown that

$$V_{\text{Diff}} = \text{Det}\left(-\nabla^2\right) \times V_{\text{SDiff}} \times \int \mathcal{D}\phi \,, \tag{6.23}$$

where V_{SDiff} denotes the volume of the *SDiff* group. Hence,

$$\mathcal{Z}_{\text{Diff}} = \int \frac{\mathcal{D}h_{\mu\nu}}{V_{\text{SDiff}}} \frac{1}{\text{Det}\left(-\nabla^2\right)} \Delta_{\mathcal{F}}(g) \delta(\mathcal{F}(g)) \mathrm{e}^{-S_{\text{Diff}}[g(\bar{g};h)]} \,. \tag{6.24}$$

An explicit example of this first stage of gauge fixing is the unimodular gauge defined by

$$\mathcal{F}(g) = \det g_{\mu\nu} - \omega^2(x) \,, \tag{6.25}$$

$\omega(x)$ being a fixed scalar density. The delta function in (6.24) enforces that the full dynamical metric is unimodular. The corresponding Fadeev-Popov determinant is

$$\Delta_{\mathcal{F}}(g) = \text{Det}\left(\omega^2(x)(-\nabla^2)\right) \,. \tag{6.26}$$

The contribution due to $\omega^2(x)$ in (6.26) can be absorbed in a normalisation factor of the path integral and thereby it is harmless. Finally, by plugging (6.26) into (6.24), yields

$$\mathcal{Z}_{\text{Diff}} = \int \frac{\mathcal{D}h_{\mu\nu}}{V_{\text{SDiff}}} \delta(\det g_{\mu\nu} - \omega^2(x)) \mathrm{e}^{-S_{\text{Diff}}[g(\bar{g};h)]} \,. \tag{6.27}$$

Due to the presence of the delta functional in (6.27), the action in the Boltzmann factor collapses to its unimodular counterpart, i.e., $S_{\text{Diff}}[g(\bar{g};h)] \to S_{\text{SDiff}}[g(\bar{g};h)]$ where factors of $\sqrt{g}$

are replaced by $\omega(x)$ and when expanded in $h_{\mu\nu}$, the constraint $\mathcal{F}(g) = 0$ must be imposed. Eq.(6.27) is the path integral of UG with the unimodular measure $(\mathcal{D}h_{\mu\nu})_{\rm UG}$ defined by

$$(\mathcal{D}h_{\mu\nu})_{\rm UG} \equiv \mathcal{D}h_{\mu\nu}\, \delta(\det g_{\mu\nu} - \omega^2(x))\,, \tag{6.28}$$

i.e,

$$\mathcal{Z}_{\rm Diff} = \int \frac{(\mathcal{D}h_{\mu\nu})_{\rm UG}}{V_{\rm SDiff}} {\rm e}^{-S_{\rm SDiff}[g(\bar{g};h)]} \equiv \mathcal{Z}_{\rm SDiff}\,. \tag{6.29}$$

One particular parameterisation which is well-suited for the implementation of the unimodularity condition is the exponential split

$$g_{\mu\nu} = \bar{g}_{\mu\kappa}({\rm e}^{h})^{\kappa}{}_{\nu}\,. \tag{6.30}$$

Unimodularity of $g_{\mu\nu}$ is achieved by requiring the background to be unimodular ($\det\bar{g} = \omega^2(x)$) and that the flucuations $h_{\mu\nu}$ are traceless[4].

In order to complete the gauge-fixing procedure, one applies again the Faddeev-Popov method for a gauge condition which fixes the *SDiff* invariance. This is achieved, e.g., by taking the standard linear covariant gauges in quantum gravity and applying the transverse projector to it. We refer to [102, 106, 107, 116, 117, 123, 124, 129] for more details.

We remark that eq.(6.29) does not rely on the specific form of the gravitational action. Morever, if matter interactions were included (also of arbitrary form), the equivalence would still hold. In this case, the matter action $S_{\rm M}^{\rm Diff}(\varphi,\psi,A)$ is mapped to $S_{\rm M}^{\rm SDiff}(\varphi,\psi,A)$ with the replacement $\sqrt{g}\to\omega$ and fluctuations satisfying the constraint defined by the delta functional in (6.28). Thus, we expect that gravity-matter systems in a full diffeomorphism-invariant setting are equivalent, quantum-mechanically, to gravity-matter systems in the unimodular framework.

6.3 Non-minimal comparisons in Scalar-tensor theories

As an explicit check, we shall consider gravity non-minimally coupled to a scalar field and show that the one loop UV divergences are the same for GR and UG. This disagrees with [115], who claimed that a particular dimensionless combination of couplings, called Δ, has different beta functions in the two settings. In our calculation, the beta functions turn out to be the same. What is perhaps more important, we find that the beta functions of Δ are gauge-dependent, which may at least in part explain the discrepancy. Furthermore, the implementation of the unimodularity condition adopted in [115] is different from the one we use in this thesis. We therefore think that the question whether different formulations of quantum UG can lead to different physical predictions than GR remains still open. Here we perform an explicit calculation of one-loop divergences in gravity-matter systems, illustrating the quantum equivalence between *Diff*− and *SDiff*− invariant theories. In particular, we focus on scalar-tensor theories including non-minimal couplings between gravity and the scalar field.

[4]Another efficient method is the "densitized" parameterisation, see, e.g., [119, 128]. If one opts for less efficient implementations, the unimodularity condition becomes difficult to implement in practical calculations. Nevertheless, for the partial gauge-fixing associated with the gauge freedom (6.20), there seem to be no generation of quartic ghost terms [127] due to the fact that one just introduces a ghost-antighost pair.

6.3.1 Action

The beta functions of GR coupled to a scalar have been derived previously in, e.g., [130] for the general class of actions

$$S[\phi, g] = \int \mathrm{d}^d x \sqrt{g} \left(V(\phi) - F(\phi) R + \frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi \right) . \tag{6.31}$$

This includes an arbitrary potential V and arbitrary non-minimal couplings parametrised by the function F. If one expands V and F in Taylor series in ϕ, with the additional assumption of invariance under $\phi \to -\phi$,

$$V(\phi) = \mathcal{V} + \frac{1}{2} m^2 \phi^2 + \lambda \phi^4 \dots \, , \quad \mathcal{V} = \frac{\Lambda}{8\pi G_N} \tag{6.32}$$

$$F(\phi) = Z_N + \frac{1}{2} \xi \phi^2 + \dots \, , \quad Z_N = \frac{1}{16\pi G_N} \tag{6.33}$$

We are especially interested in the dimensionless couplings ξ and λ, whose leading one-loop beta functions are universal, and in dimensionless ratios of the dimensionful couplings, such as $G_N \Lambda$, $G_N m^2$, Λ / m^2, since their beta functions are also known to be less gauge- and parameterisation-dependent. In [130], the beta functions were computed by the use of the functional renormalisation group (FRG) equation which is based on a cutoff-like regularisation. Thus, power-law divergences are also taken into account. In [115], on the other hand, the authors employed dimensional regularisation which is blind to the power-law divergences. For a direct comparison, we would have to extract from the FRG the "universal" contributions, i.e., those related to logarithmic running. This is discussed in Appendix C.5. In the next section we directly extract the beta functions from the logarithmic divergences, calculated with heat kernel methods.

6.3.2 Dynamical gravitons: UG or GR in exponential parametrisation

We start from GR in the exponential parametrisation (6.30) and follow the procedure of [131]. We decompose the metric fluctuation in its irreducible spin 2, 1 and 0 components:

$$h_{\mu\nu} = h^{\mathrm{TT}}_{\mu\nu} + \bar{\nabla}_\mu \xi_\nu + \bar{\nabla}_\nu \xi_\mu + \bar{\nabla}_\mu \bar{\nabla}_\mu \sigma - \frac{1}{4} \bar{g}_{\mu\nu} \bar{\nabla}^2 \sigma + \frac{1}{4} \bar{g}_{\mu\nu} h \, , \tag{6.34}$$

with $\bar{\nabla}^\mu h^{\mathrm{TT}}_{\mu\nu} = 0$, $\bar{\nabla}^\mu \xi_\mu = 0$ and $\bar{g}^{\mu\nu} h_{\mu\nu} = h$. A redefinition of the fields σ and ξ_μ is performed in order to cancel the Jacobian generated by the York decomposition (6.34),

$$\xi'_\mu = \sqrt{-\bar{\nabla}^2 - \frac{\bar{R}}{4}} \xi_\mu \, , \quad \text{and} \quad \sigma' = \sqrt{-\bar{\nabla}^2} \sqrt{-\bar{\nabla}^2 - \frac{\bar{R}}{3}} \sigma \, . \tag{6.35}$$

We take the background metric $\bar{g}_{\mu\nu}$ to be a four-dimensional Euclidean maximally symmetric space. Then we choose the "unimodular physical gauge", which consists of setting to zero the spin one field ξ'_μ and the spin-0 field h. With these choices, the gauge fixed Hessian is

$$\begin{aligned} \tilde{S}^{(2)}_{\mathrm{grav}} = \int \mathrm{d}^4 x \sqrt{\bar{g}} \Bigg[& \frac{1}{4} F(\bar{\phi}) {h^{TT}}_{\mu\nu} \left(-\bar{\nabla}^2 + \frac{\bar{R}}{6} \right) h^{TT\mu\nu} - \frac{3}{32} F(\bar{\phi}) \sigma' (-\bar{\nabla}^2) \sigma' \\ & - \frac{3}{4} F'(\bar{\phi}) \delta\phi \sqrt{(-\bar{\nabla}^2) \left(-\bar{\nabla}^2 - \frac{\bar{R}}{3} \right)} \sigma' + \frac{1}{2} \delta\phi \left(-\bar{\nabla}^2 + V''(\bar{\phi}) - F''(\bar{\phi}) \bar{R} \right) \delta\phi \Bigg] . \end{aligned} \tag{6.36}$$

As a further simplification, we note that defining [5]

$$\sigma'' = \sigma' + 4\frac{F'(\bar{\phi})}{F(\bar{\phi})}\sqrt{\frac{-\bar{\nabla}^2 - \frac{\bar{R}}{3}}{-\bar{\nabla}^2}}\delta\phi\,, \tag{6.37}$$

the gauge fixed Hessian becomes diagonal,

$$\begin{aligned} S^{(2)}_{\rm grav} = \int \mathrm{d}^4x\sqrt{\bar{g}} \Bigg[&\frac{1}{4}F(\bar{\phi})h^{TT}{}_{\mu\nu}\left(-\bar{\nabla}^2 + \frac{\bar{R}}{6}\right)h^{TT\mu\nu} - \frac{3}{32}F(\bar{\phi})\sigma''(-\bar{\nabla}^2)\sigma'' \\ &+ \frac{1}{2}\delta\phi\left(-\bar{\nabla}^2 + V''(\bar{\phi}) - F''(\bar{\phi})\bar{R} + 3\frac{F'(\bar{\phi})^2}{F(\bar{\phi})}\left(-\bar{\nabla}^2 - \frac{\bar{R}}{d-1}\right)\right)\delta\phi\Bigg]\,. \end{aligned} \tag{6.38}$$

The unimodular physical gauge produces Faddeev-Popov ghost determinants

$$\Delta_{\rm FP=}\sqrt{\det{}_0(-\bar{\nabla}^2)}\sqrt{\det{}_1\left(-\bar{\nabla}^2 - \frac{\bar{R}}{4}\right)}\,, \tag{6.39}$$

with the subscripts 0 and 1 denoting the spin of the fields that the corresponding operators act on. Thus the one-loop partition function reads

$$Z = \mathrm{e}^{-S_{\rm grav}[\bar{\phi},\bar{g}]}\frac{\sqrt{\det\Delta_1}}{\sqrt{\det\Delta_2}\sqrt{\det\Delta_S}}\,, \tag{6.40}$$

where $\Delta_1 = -\bar{\nabla}^2 - \frac{\bar{R}}{4}$, $\Delta_0 = -\bar{\nabla}^2$ and

$$\Delta_S = -\bar{\nabla}^2 + E_S\;, \quad E_S = \frac{FV'' - (F'^2 + FF'')\bar{R}}{F + 3F'^2}\,. \tag{6.41}$$

This agrees with the standard result for GR with a cosmological constant, except for the appearance of the additional scalar determinant.

Consider now the same calculation in UG. The trace fluctuation h is absent from the degrees of freedom from the start and it is therefore not necessary to fix the corresponding gauge. The *SDiff* gauge can be fixed again by setting $\xi' = 0$. Altogether this produces the Faddeev-Popov determinant

$$\Delta^{\rm UG}_{\rm FP} = \sqrt{\det{}_1\left(-\bar{\nabla}^2 - \frac{\bar{R}}{4}\right)}\,. \tag{6.42}$$

On the other hand, as discussed in [106, 107], the factorization of the volume of *SDiff* produces and additional determinant $\sqrt{\det(-\bar{\nabla}^2)}$ which cancels the determinant coming from the integration over σ', so that the final result is again exactly (6.40). Notably, such an equivalence holds irrespective of the choice of $F(\phi)$.

In a standard perturbative approach, the beta functions can be read off from the logarithmic divergences. The one-loop effective action is

$$\Gamma = S + \frac{1}{2}\mathrm{Tr}\log\Delta_2 - \frac{1}{2}\mathrm{Tr}\log\Delta_1 + \frac{1}{2}\mathrm{Tr}\log\Delta_S\,, \tag{6.43}$$

[5] This change of variables has a trivial Jacobian.

and its divergent parts can be obtained from

$$\Gamma_{\rm div} = -\frac{1}{2}\frac{1}{16\pi^2}\log\left(\frac{\Lambda^2}{\mu^2}\right)\int \mathrm{d}^4x\sqrt{\bar{g}}\left[b_4(\Delta_2) - b_4(\Delta_1) + b_4(\Delta_S)\right] , \tag{6.44}$$

with Λ standing for an ultraviolet cutoff and μ being a reference scale. The first two contributions in (6.44) only give terms of order R^2 and are not relevant for the beta functions of interest. For Δ_S we have

$$\begin{aligned} -\frac{1}{2}\frac{1}{16\pi^2}b_4(\Delta_S) &= -\frac{1}{2}\frac{1}{16\pi^2}\left(\frac{1}{2}E_S^2 - \frac{1}{6}\bar{R}\,E_S + O(\bar{R}^2)\right) \\ &= -\frac{1}{64\pi^2}\frac{V''^2}{\left(1+3\frac{F'^2}{F}\right)} + \frac{1}{192\pi^2}\frac{1+6F''+9\frac{F'^2}{F}}{\left(1+3\frac{F'^2}{F}\right)}V''\bar{R} + O(\bar{R}^2) \;. \end{aligned} \tag{6.45}$$

Inserting (6.32) and (6.33) and expanding in powers of ϕ we obtain the beta functions

$$\begin{aligned} \beta_{\mathcal{V}} &= \frac{m^4}{32\pi^2} \\ \beta_{m^2} &= \frac{3\lambda}{2\pi^2}m^2 - \frac{6Gm^4\xi^2}{\pi} \\ \beta_{\lambda} &= \frac{9\lambda^2}{2\pi^2} - \frac{72Gm^2\lambda\xi^2}{\pi} \\ \beta_G &= -\frac{G^2m^2(1+6\xi)}{6\pi} \\ \beta_\xi &= \frac{\lambda(1+6\xi)}{4\pi^2} + \frac{Gm^2\xi^2(1-12\xi)}{\pi} \end{aligned} \tag{6.46}$$

We note that the leading terms are the same as for the pure scalar theory, discussed in Appendix C.5. Here we only kept correction terms linear in G. The explicit one-loop computation reported above leads to the same results in GR and UG, since (6.44) is the same in both cases. Moreover, we have kept only the contributions generated by the universal Q-functionals. The remaining terms are associated with power divergences and are not universal. The conclusion of this explicit calculation agrees with our statement in Sec. 6.2. In particular, the non-minimal scalar-gravity coupling does not change this conclusion.

As is well-known, quantum-gravity contributions to matter beta functions can be gauge dependent. The "unimodular physical gauge" can be obtained from the standard two-parameter linear covariant gauge condition for Diff-invariant theories, namely

$$\bar{\nabla}^\nu h_{\nu\mu} - \frac{1+\beta}{4}\bar{\nabla}_\mu h = \alpha b_\mu \,, \tag{6.47}$$

with b_μ being a fixed function, by taking the limits $\alpha \to 0$ and $\beta \to -\infty$. For generic α, β, the beta functions $\beta_{\mathcal{V}}$ and β_G are left unchanged, while the others become,

$$\begin{aligned} \beta_{m^2} &= \frac{3m^2\lambda}{2\pi^2} + \frac{2Gm^4(4\alpha - 3(2+(3-\beta)\xi)^2)}{(3-\beta)^2\pi} \,, \\ \beta_\lambda &= \frac{9\lambda^2}{2\pi^2} - \frac{8Gm^2\lambda(12-4\alpha+24(3-\beta)\xi+9(3-\beta)^2\xi^2)}{(3-\beta)^2\pi} \,, \\ \beta_\xi &= \frac{\lambda(1+6\xi)}{4\pi^2} - \frac{Gm^2}{12\pi}F(\alpha,\beta,\xi) \;. \end{aligned} \tag{6.48}$$

The contribution $F(\alpha, \beta, \xi)$ is lengthy and collected in the Appendix C.6. In the limit $\beta \to -\infty$, the beta functions turn out to be α-independent. It is also worth mentioning that the first two of these beta functions are also independent of another parameter that can be introduced in the definition of the measure, namely the use of a densitised metric as a quantum field, see, e.g., [128].

As a side comment, at one-loop order, the universal gravitational correction to the quartic coupling λ at vanishing non-minimal coupling is negative, irrespective of the values of the gauge parameter β, provided that[6] $\alpha < 3$,

$$\beta_\lambda\Big|_{\text{grav}} = -\lambda \frac{32Gm^2}{\pi} \frac{3-\alpha}{(3-\beta)^2} \,. \tag{6.49}$$

At $\alpha = 3$ or $\beta \to \pm\infty$, the contribution vanishes at one loop. In particular, in the unimodular physical gauge, the gravitational contribution vanishes at vanishing ξ. However, in such a gauge, if the non-minimal coupling is included, the contribution is always negative at leading order in G. Hence, such a contribution can balance the non-gravitational contribution to the one-loop running of λ - which is positive and leads to the well-known triviality problem. In order to circumvent the issues due to the gauge dependence, and of the non-universal power-law terms, one will have to to compute a gauge invariant physical observable possibly along the lines of [132].

6.3.3 A universal beta function?

In [115], it was argued that the dimensionless combination of couplings

$$\Delta = \frac{(Gm^2)^2}{\lambda} \,, \tag{6.50}$$

has a universal beta function and carries a physical meaning. By quantizing UG in the presence of non-minimally coupled scalar fields, the authors claim that the results differ in GR and UG. More precisely, taking into account the differences in notation, their result for UG is

$$\beta_\Delta^{\text{UG}} = \Delta \frac{-9\lambda + 2\pi Gm^2(-4 - 24\xi + 180\xi^2)}{6\pi^2} \tag{6.51}$$

while their result for GR is

$$\beta_\Delta^{\text{GR}} = \Delta \frac{-9\lambda + 2\pi Gm^2(-4 + 156\xi + 180\xi^2)}{6\pi^2} \,. \tag{6.52}$$

Hence, Δ would be a physical quantity able to distinguish GR and UG if the scalar field is non-minimally coupled to the gravitational field.

Using our previous calculations, we cannot distinguish UG and GR non-minimally coupled to scalars at one-loop simply because the path integrals are the same. In particular, in the unimodular physical gauge, we obtain

$$\beta_\Delta = \Delta \frac{-9\lambda + 2Gm^2\pi(-1 - 6\xi + 180\xi^2)}{6\pi^2} \,, \tag{6.53}$$

[6] In an Euclidean setting, α has to be non-negative.

which differs from either of the results above. These discrepancies may be ascribed to the fact that we are using a different parameterizarion of the metric and a different implementation of the unimodularity condition.

What is perhaps more important is that, even if we stick to our computation scheme, the quantity Δ is gauge dependent. In fact, in the linear covariant gauge (6.47), the result is

$$\beta_\Delta = \frac{\Delta(-9(3-\beta)^2\lambda - 2Gm^2\pi(48\alpha + \beta^2 A_1(\xi) + 6\beta A_2(\xi) - 27A_3(\xi)))}{6\pi^2(3-\beta)^2}\,, \tag{6.54}$$

with

$$A_1(\xi) = 1 + 6\xi - 180\xi^2\,, \quad A_2(\xi) = -1 + 66\xi + 180\xi^2\,, \quad A_3(\xi) = 5 + 46\xi + 60\xi^2\,. \tag{6.55}$$

Thus, even in the absence of ξ, the beta function of Δ is gauge dependent and comparing results for GR and UG would be problematic. We also remark that, in the limit $\beta \to \pm\infty$, eq.(6.54) reduces to (6.53) irrespective of α.

6.3.4 Dynamical gravitons: GR in linear parametrisation

So far, the explicit one-loop computations were performed using the exponential parameterisation of the metric. In this parameterisation, the unimodularity condition simply amounts to removing the trace mode of the gravitational field fluctuation $h_{\mu\nu}$. While field redefinitons, properly done, should not affect the result of physical quantities, there are several subtleties when changing from one parameterisation to another in quantum gravity. In this section, we present the one-loop results for the scalar-gravity system with a non-minimal coupling in the so-called linear parameterisation, i.e.,

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}\,, \tag{6.56}$$

in the linear covariant gauges (6.47). This system was studied, e.g., in [130], but the beta functions were computed with the functional renormalisation group and contained also non-universal terms. Here, we select just the universal contributions, that are related to logarithmic divergences. In a general gauge (α, β), the result is completely equivalent to (6.48) apart from the beta function of the non-minimal coupling β_ξ which reads

$$\beta_\xi = \frac{\lambda(1+6\xi)}{4\pi^2} - \frac{Gm^2}{12\pi} G(\alpha, \beta, \xi)\,, \tag{6.57}$$

where the explicit expression for $G(\alpha, \beta, \xi)$ is reported in Appendix C.6. In particular, if we take $\alpha \to 0$ and $\beta \to \pm\infty$, we obtain

$$G(0, \pm\infty, \xi) = 6(-13 + 10\xi^2 + 24\xi^3)\,, \tag{6.58}$$

which differs from $F(0, \pm\infty, \xi)$. The beta function of Δ in a general linear covariant gauge and in linear parameterisation is the same as (6.54). Hence, although gauge-dependent, Δ seems to display some kind of universality as far as different choices of parameterisation is concerned. This fact is not very surprising given that, the only beta function in the linear parameterizarion that differs from the exponential parameterisation at one-loop is β_ξ and it does not enter the definition of β_Δ.

In [133], Kamenshchick and Steinwachs (see, also, [134]) investigated the one-loop divergences of a more general theory than the one considered in this work. In particular, they have considered a scalar-gravity action $S_{\mathrm{KS}}[g,\Phi]$ expressed as[7]

$$S_{\mathrm{KS}}[g,\Phi] = \int \mathrm{d}^4x\sqrt{g}\left(V(\tilde{\Phi}) - F(\tilde{\Phi})R + \frac{1}{2}g^{\mu\nu}G(\tilde{\Phi})\nabla_\mu\Phi^a\nabla_\nu\Phi_a\right), \tag{6.59}$$

where $a = 1,\ldots,N$ and N is a positive integer. The functions V and F depend on $\tilde{\Phi} = \sqrt{\delta_{ab}\Phi^a\Phi^b}$. The gauge condition used in [133] is

$$F_\mu = \sqrt{F(\tilde{\Phi})}\left(\bar{\nabla}^\alpha h_{\alpha\mu} - \frac{1}{2}\bar{\nabla}_\mu h - \frac{F'(\tilde{\Phi})}{F(\tilde{\Phi})}n^a\bar{\nabla}_\mu\varphi_a\right), \tag{6.60}$$

with φ_a the scalar field fluctuations and $n^a = \Phi^a/\tilde{\Phi}$. Unfortunately, our gauge condition (6.47) is not deformable to this, and therefore we cannot directly compare our results with theirs. However, we can extract from their work the beta function of Δ in the gauge (6.60). The authors employed the linear parameterisation of the metric and the reduction to the a single-scalar non-minimally coupled to gravity is achieved by taking $N \to 1$, $\Phi^a = n^a = 1$ and $\tilde{\Phi} \to \phi$. Moreover, in order to have the same scalar-tensor action we discussed in this work, one has to take $G(\tilde{\Phi}) \to 1$ and their quantity s has to be identified as

$$s = -\frac{F}{F + 3F'^2} = \frac{-1 - 8\pi\xi G\phi^2}{1 + 8\pi\xi G\phi^2(1+6\xi)}\,. \tag{6.61}$$

The beta functionals of V and F are $\beta_V = 2\alpha_1$, $\beta_F = 2\alpha_2$, where α_1 and α_2 are given in their equations (48) and (49). From there we read off

$$\begin{aligned}
\beta_{m^2} &= \frac{3\lambda}{2\pi^2}m^2 - \frac{2Gm^4(2+4\xi+3\xi^2)}{\pi}\,,\\
\beta_\lambda &= \frac{9\lambda^2}{2\pi^2} - \frac{8Gm^2\lambda(2+8\xi+9\xi^2)}{\pi}\,,\\
\beta_G &= -\frac{G^2m^2(1+6\xi)}{6\pi}\,,\\
\beta_\xi &= \frac{\lambda(1+6\xi)}{4\pi^2} - \frac{Gm^2(13-16\xi-39\xi^2-36\xi^3)}{3\pi}\,,
\end{aligned} \tag{6.62}$$

which gives

$$\beta_\Delta = \Delta\frac{-9\lambda + 10\pi Gm^2(5+30\xi+36\xi^2)}{6\pi^2}\,. \tag{6.63}$$

This confirms once more that the first and last term in the fraction are universal, but not the other ones.

6.4 Conclusions

Disregarding the single global spacetime volume degree of freedom, we have shown at a formal path integral level that the classical equivalence between general (*Diff*–invariant) and unimod-

[7] The functions U, G, V of [133] correspond to $-F$, -1, $-V$ in our notation.

ular (*SDiff*–invariant) versions of gravity theories, can be maintained at the quantum level[8]. This is true independently of the choice of the action and also in the presence of matter.

We have then given an explicit one-loop example of this, by computing the universal parts of the beta functions of scalars coupled to gravity. In spite of significant differences in the two cases, the beta functions turn out to be the same. We have then compared these results to those of [115], who also made the same comparison. Our beta function for the dimensionless combination Δ differs from theirs both for GR and UG. The differences can probably to be ascribed at least in part to the different way they implement unimodularity. A more detailed analysis has shown that the beta function of Δ is actually gauge-dependent, so that it is not a sufficiently good test. There are two terms in the beta function of Δ that are the same in all gauges and are the same across all calculations we could find in the literature, whereas other terms have strong gauge dependence. For the future, it will be important to identify a genuinely universal combination of couplings, or another observable that can act as a benchmark.

We conclude with some comments on the cosmological constant. In UG, a "cosmological term" $\frac{\Lambda}{8\pi G}\int \mathrm{d}^4x\,\omega$ in the Lagrangian is just an additive, field-independent term that does not affect the equations of motion and can be absorbed in the overall normalisation of the functional integral. Thus, it has no physical effect. GR is only (classically) equivalent to UG if we impose that the total volume of spacetime is fixed. In this restricted theory the cosmological term in (6.2) is a Lagrange multiplier, whose value is ultimately related to the volume through the equations of motion.

Computations of the beta functions performed in the so-called unimodular gauge [131] show that the cosmological constant decouples from the system of beta functions. This resembles simpler calculations involving the functional renormalisation group, where a field-independent contribution is generated by the flow and can be cancelled by a proper normalisation of the vacuum energy, see [135]. This suggests that its quartic running is unphysical, This is in line with other hints coming from different directions [136–139]. This and related issues deserve to be investigated further.

[8]The equivalence of GR and UG in the presence of an independent connection, deserves a separate investigation, which is ongoing.

Chapter 7

Renormalisation of Poincaré gauge theories

After gaining some understanding of quantum structure of metric theories of gravity, we move on to consider some models that are subclasses of MAG. We will apply the off-diagonal heat kernel technique, explained in the section 1.5 to compute the one-loop RG flow.

In the case of antisymmetric MAG,one can view the frame field (or vierbein or tetrad) as a gauge field for translations, which can be combined with the Lorentz connection to form a Poincaré connection. The natural Lagrangian for such a theory is quadratic in the curvature of the Poincaré connection, i.e. quadratic in the curvature of the Lorentz connection and quadratic in torsion. Theories of this type are called Poincaré gauge theories (PGT).

We will show that PGTs are not renormalisable in strict sense: one loop effects generate terms that are not quadratic in curvature or torsion. This is a priori highly likely, since there exist dimension-four terms that cannot be written as squares of curvature. An enumeration and partial listing of such terms has been given in [86]. They are of the form $(\nabla T)^2$, $T^2\nabla T$ and T^4, where ∇ is the Levi-Civita (LC) covariant derivative. The only thing to check is that such terms are actually needed for the renormalisation of the theory, i.e. that there are divergences proportional to these new terms. It will be enough to look at terms of the form $(\nabla T)^2$.

The calculation of quantum effects in these theories is very cumbersome, for two reasons: the fields have many components, and the Lagrangian contains many terms. One can reduce the number of fields by working in coordinate rather than orthonormal frames, but this is a relatively small advantage: the 16 components of the tetrad are reduced to the 10 components of the metric, but the number of independent components of the connection remains 24. [1] We shall see that for the calculation we want to perform here, it is actually convenient to work with orthonormal frames. Regarding the Lagrangian, if we wanted to prove renormalizability, we would have to work with the most general one, but we want to prove the opposite, and for this purpose, it will be enough to consider a simple Lagrangian containing just two terms. We will find by an explicit one-loop calculation, that one needs to introduce counterterms proportional to other contractions of two curvatures, but also counterterms proportional to the square of covariant derivatives of torsion, which do not appear in the Lagrangian of PGT. This means that PGT is not renormalisable, or, in the more modern language of effective field theory (EFT), that at a given order of the low energy expansion one has to consider a larger

[1] The choice of frames is a choice of gauge. The theory in coordinate frames can be recovered by imposing that the tetrad is symmetric.

set of invariants in the Lagrangian.

The standard action for PGT is quadratic in curvature and torsion:

$$S(g,A) = -\frac{1}{2}\int d^dx\ \sqrt{|g|}\,\Big[- a_0 F + T^{\mu\rho\nu}\big(a_1 T_{\mu\rho\nu} + a_2 T_{\mu\nu\rho}\big) + a_3 T^\mu T_\mu$$
$$+ F^{\mu\nu\rho\sigma}\big(c_1 F_{\mu\nu\rho\sigma} + c_3 F_{\rho\sigma\mu\nu} + c_4 F_{\mu\rho\nu\sigma}\big) + F^{\mu\nu}\big(c_7 F_{\mu\nu} + c_8 F_{\nu\mu}\big) + c_{16} F^2\Big]\ , \tag{7.1}$$

where $T_\mu = T_\lambda{}^\lambda{}_\mu$. The non-consecutive numbering of coefficients is for compatibility with the action of more general MAGs in section 3.5.2. In writing this action, we have made two choices. We have chosen to work with coordinate bases, so as to have the lowest number of field components compatible with locality. One can write the action in orthonormal bases, simply changing the middle index of torsion and the last two indices of curvature from greek to latin. This is a mere gauge choice and is completely inconsequential to the physical content of the theory. We have also chosen to think of the action as a functional of the metric and of the independent gauge field A. This is the choice of variables that makes MAG more similar to a YM theory, and in [86] we called it the Cartan view of the theory. One can choose to present the theory in what we called the Einstein view, where the action is regarded as a functional of the metric and of the tortion (or equivalently of the contorsion). This change of dynamical variables is performed by using (1.32). In the Einstein view, the action consists of the action of higher derivative gravity (with the Hilbert term and three terms quadratic in R) plus terms of the form $R\nabla T$ and $(\nabla T)^2$. [2]

In the following we shall consider the case where the only nonzero couplings are c_1 and a_1. The Lagrangian in orthonormal frames can then be written

$$\mathcal{L} = -\tfrac{1}{2}\sqrt{|g|}g^{\mu\rho}g^{\nu\sigma}\eta_{ab}\left(c_1 F_{\mu\nu}{}^a{}_c F_{\rho\sigma}{}^b{}_d \eta^{cd} + a_1 T_\mu{}^a{}_\nu T_\rho{}^b{}_\sigma\right) \tag{7.2}$$

This writing clearly exposes the variables that have to be varied. Here the metric is to be regarded as a composite of the tetrads, as in (1.2) and F depends on A but not on the tetrad.

It is a general fact that calculations tend to be easier when one works in the Einstein formulation. In the following we will use mostly the Einstein formalism, in the sense that we will write covariant derivatives in terms of the Levi-Civita connection. For tensorial quantities such as curvatures F and R, they can coexist in formulas. In the end, to read off the beta functions in the Cartan basis, we obviously have to convert everything back to Cartan form.

7.1 Perturbative Expansion

The basic variables are the tetrad and connection. Their variations will be called

$$\delta\theta^a{}_\mu = X^a{}_\mu\ ,\quad \delta A_\mu{}^a{}_b = Z_\mu{}^a{}_b\ . \tag{7.3}$$

[2] Note that whereas in Cartan view torsion is a derived quantity, being constructed from the connection (in coordinate frames) or from the connection and tetrad (in orthonormal frames), in the Einstein view it has to be regarded as an independent field.

Then we have

$$
\begin{aligned}
\delta\sqrt{g} &= \sqrt{g}X^\rho{}_\rho \\
\delta^2\sqrt{g} &= \sqrt{g}\,(X^\rho{}_\rho X^\sigma{}_\sigma - X^{\rho\sigma}X_{\sigma\rho}) \\
\delta g^{\mu\nu} &= -X^{\mu\nu} - X^{\nu\mu} \\
\delta^2 g^{\mu\nu} &= 2(X^{\mu\rho}X^\nu{}_\rho + X^{\mu\rho}X_\rho{}^\nu + X^{\nu\rho}X_\rho{}^\mu) \\
\delta F_{\mu\nu}{}^a{}_c &= D_\mu Z_\nu{}^a{}_b - D_\nu Z_\mu{}^a{}_b + T_\mu{}^\rho{}_\nu Z_\rho{}^a{}_b \\
\delta^2 F_{\mu\nu}{}^a{}_c &= 2[Z_\mu, Z_\nu]^a{}_b \\
\delta T_\mu{}^a{}_\nu &= D_\mu X^a{}_\nu - D_\nu X^a + Z_\mu{}^a{}_\nu - Z_\nu{}^a{}_\mu + T_\mu{}^\rho{}_\nu X^a{}_\rho \\
\delta^2 T_\mu{}^a{}_\nu &= 2(Z_\mu{}^a{}_b X^b{}_\nu - Z_\nu{}^a{}_b X^b{}_\mu)\ .
\end{aligned}
\tag{7.4}
$$

Varying the action and using these relations one arrives at the Hessian, which is a quadratic form in X and Z. It is convenient to rewrite all D derivatives as ∇ derivatives plus terms linear in torsion. The terms with two derivatives are

$$
a_1 X_{\mu\nu}(-\bar\nabla^2 g^{\nu\sigma} + \bar\nabla^\sigma\bar\nabla^\nu)X^\mu{}_\sigma + c_1 Z_\mu{}^{\rho\sigma}(-\bar\nabla^2 g^{\mu\nu} + \bar\nabla^\nu\bar\nabla^\mu)Z_{\nu\rho\sigma}\ , \tag{7.5}
$$

where the bar over the covariant derivatives indicates that they are computed with the background metric. The occurrence of the nonminimal terms would greatly complicate the calculation, but can be avoided by choosing a suitable gauge. We observe that this is only possible using the vierbein formalism. Working in coordinate bases one only has the diffeomorphism invariance (4 parameters) and one can fix this gauge in such a way as to remove the nonminimal terms in the X-X sector. In order to remove the nonminimal terms in the Z-Z sector one needs in addition a Lorentz gauge fixing. We shall see this in detail in Section 7.3.

7.2 Gauge algebra

Due to the structure of the gauge group, the gauge fixing conditions for gravity in tetrad formulation (whether the connection is independent or not) is more complicated than imposing separate gauge conditions for diffeomorphisms and local Lorentz transformations. This kind of complication already occurs in the case of Yang-Mills fields coupled to gravity [140, 141]. In the case of Einstein-Cartan theory it has been discussed in [142–146]. We will broadly follow these references, but with some significant differences.

The fields of antisymmetric MAG are defined on OM, the bundle of orthonormal frames of the base manifold M, and its associated bundles and the action is invariant under the automorphisms of this bundle. One can parametrise locally this group by diffeomorphisms of M and local Lorentz transformations, acting in the standard way on Latin and Greek indices, respectively:

$$
\begin{aligned}
\delta^L_\omega \theta^a{}_\mu &= -\omega^a{}_b\theta^b{}_\mu\ , \\
\delta^L_\omega A_\mu{}^a{}_b &= D_\mu\omega^b{}_c\ , \\
\delta^D_v \theta^a{}_\mu &= \mathcal{L}_v\theta^a{}_\mu = v^\rho\partial_\rho\theta^a{}_\mu + \theta^a{}_\rho\partial_\mu v^\rho\ , \\
\delta^D_v A_\mu{}^a{}_b &= \mathcal{L}_v A_\mu{}^a{}_b\ ,
\end{aligned}
\tag{7.6}
$$

where $\omega_{ab} = -\omega_{ba}$ is an infinitesimal Lorentz gauge parameter and v^μ is an infinitesimal diffeomorphism (a vectorfield on M). Note that the latin indices are inert under this definition of diffeomorphism. The algebra of these transformations is

$$\begin{aligned}\left[\delta^L_{\omega_1},\delta^L_{\omega_2}\right] &= \delta^L_{[\omega_1,\omega_2]}\\ \left[\delta^D_{v_1},\delta^D_{v_2}\right] &= -\delta^D_{[v_1,v_2]}\\ \left[\delta^D_{v},\delta^L_{\omega}\right] &= \delta^L_{\mathcal{L}_v\omega}\end{aligned} \tag{7.7}$$

This shows that the local Lorentz transformations are a normal subgroup of the full gauge group, and the diffeomorphisms are the quotient of the full group by this subgroup.

Now we see that whereas the general fluctuation $X^a{}_\mu$ transforms properly under local Lorentz transformations, the gauge fluctuation $\delta_v\theta^a{}_\mu$ does not. This would become a serious obstacle in the following, because $\delta_v\theta^a{}_\mu$ is used in the construction of the ghost operator, and this definition would lead to a non-covariant ghost operator. [3] The solution consists in defining a modified action of diffeomorphisms on the fields, which consists of the original action defined above, plus an infinitesimal Lorentz transformation with a parameter $\epsilon^a{}_b = -v^\mu A_\mu{}^a{}_b \equiv -(v\cdot A)^a{}_b$. [4]

$$\tilde\delta^D_v = \delta^D_v - \delta^L_{v\cdot A}\ . \tag{7.8}$$

The action of these modified diffeomorphisms on the fields is

$$\begin{aligned}\tilde\delta^D_v\theta^a{}_\mu &= \theta^a{}_\rho\nabla_\mu v^\rho + v^\rho K_\rho{}^a{}_\mu\ ,\\ \tilde\delta^D_v A_\mu{}^a{}_b &= v^\rho F_{\rho\mu}{}^a{}_b\ ,\end{aligned} \tag{7.9}$$

where K is defined in (1.35). Their algebra is

$$\begin{aligned}\left[\tilde\delta^D_{v_1},\tilde\delta^D_{v_2}\right] &= -\tilde\delta^D_{[v_1,v_2]} - \delta^L_{F(v_1,v_2)}\\ \left[\tilde\delta^D_{v},\delta^L_{\omega}\right] &= 0\ .\end{aligned} \tag{7.10}$$

where $F(v_1,v_2)^a{}_b = v_1^\mu v_2^\nu F_{\mu\nu}{}^a{}_b$. This is just a different way of parametrizing the full gauge group of the theory, where the normal subgroup has remained untouched.

In background field calculations we have to define how to split the transformation of a field into transformations of its background and fluctuation parts. In the so called "quantum" transformations δ^Q the backgrounds are invariant and the whole transformation of the field is attributed to the fluctuation:

$$\begin{aligned}\delta^{QL}_v\bar\theta^a{}_\mu &= 0\ ,\\ \delta^{QL}_v\bar A_\mu{}^a{}_b &= 0\ ,\\ \delta^{QL}_v X^a{}_\mu &= -\omega^a{}_b\,\theta^b{}_\mu\ ,\\ \delta^{QL}_v Z_\mu{}^a{}_b &= D_\mu\omega^a{}_b\ ,\end{aligned} \tag{7.11}$$

$$\begin{aligned}\tilde\delta^{QD}_v\bar\theta^a{}_\mu &= 0\ ,\\ \tilde\delta^{QD}_v\bar A_\mu{}^a{}_b &= 0\ ,\\ \tilde\delta^{QD}_v X^a{}_\mu &= \theta^a{}_\rho\bar\nabla_\mu v^\rho + v^\rho K_\rho{}^a{}_\mu\ ,\\ \tilde\delta^{QD}_v Z_\mu{}^a{}_b &= v^\rho F_{\rho\mu}{}^a{}_b\ ,\end{aligned} \tag{7.12}$$

[3] One can try to covariantize $\mathcal{L}_v\theta^a{}_\mu$ by adding and subtracting $v^\rho\Gamma_\rho{}^\nu{}_\mu\theta^a{}_\mu$. However, the resulting covariant derivatives are only covariant under diffeomorphisms: the derivative acting on $\theta^a{}_\mu$ is not Lorentz-covariant, with the result that $\mathcal{L}_v\theta^a{}_\mu$ is not a Lorentz vector.

[4] Here we follow [142, 144, 145]. Alternatively one could also use $\epsilon^a{}_b = -v^\mu\Gamma_\mu{}^a{}_b \equiv -(v\cdot A)^a{}_b$, where $\Gamma_\mu{}^a{}_b$ are the components of the Levi-Civita connection in the orthonormal frame. This would lead to a simpler transformation for θ (the second term would be absent) but a more complicated one for A.

The "background" transformations δ^B are defined in such a way that the backgrounds transform as the original field (in particular, $\bar{A}$ transforms as a connection). In detail, the background Lorentz transformations are

$$\begin{aligned}
\delta^{BL}_\omega \bar{\theta}^a{}_\mu &= -\omega^a{}_b \bar{\theta}^b{}_\mu \ , \\
\delta^{BL}_\omega \bar{A}_\mu{}^a{}_b &= \bar{D}_\mu \omega^a{}_b \ , \\
\delta^{BL}_\omega X^a{}_\mu &= -\omega^a{}_c X^c{}_\rho \ , \\
\delta^{BL}_\omega Z_\mu{}^a{}_b &= -\omega^a{}_c Z_\mu{}^c{}_b + Z_\mu{}^a{}_c \omega^c{}_b \ ,
\end{aligned} \tag{7.13}$$

and the background diffeomorphisms are given by the Lie derivative on all fields. The background diffeomorphisms can be covariantized as above, in particular

$$\begin{aligned}
\bar{\delta}^{BD}_v \bar{\theta}^a{}_\mu &= \bar{\theta}^a{}_\rho \bar{\nabla}_\mu v^\rho + v^\rho \bar{K}_\rho{}^a{}_\mu \ , \\
\bar{\delta}^{BD}_v \bar{A}_\mu{}^a{}_b &= v^\rho \bar{F}_{\rho\mu}{}^a{}_b \ ,
\end{aligned} \tag{7.14}$$

7.3 Gauge fixed Hessian

We gauge fix by choosing the Lorentz-like gauge conditions

$$\begin{aligned}
\chi^\mu_D &= \bar{\nabla}^\nu X^\mu{}_\nu \\
\chi_L{}^a{}_b &= \bar{\nabla}^\nu Z_\nu{}^a{}_b \ .
\end{aligned} \tag{7.15}$$

In the latter expression it is understood that the covariant derivative is defined in terms of the background Levi-Civita connection for both types of indices. Then, the gauge fixing action is

$$S_{GF} = \int d^4x \sqrt{|g|} \left[\frac{a_1}{\alpha_D} \bar{g}_{\rho\sigma} \chi^\rho_D \chi^\sigma_D + \frac{c_1}{\alpha_L} \eta_{ac} \eta^{bd} \chi_L{}^a{}_b \chi_L{}^c{}_d \right] \ . \tag{7.16}$$

This breaks invariance under the "quantum" transformations while preserving invariance under the "background" transformations. Since the total background covariant derivative of the background tetrad is zero, in the second term we can harmlessly transform all the latin indices to greek ones. Then we only have background diffeomorphism invariance, and we do not need to worry about the covariantization that was discussed in the previous section. That discussion will only play a role in the definition of the ghost action.

Setting the parameters $\alpha_D = \alpha_L = 1$ (Feynman gauge), integrating by parts and commuting derivatives one gets

$$\begin{aligned}
S_{GF} = \int d^4x \sqrt{|g|} \Big[& - a_1 X_{\mu\nu} \bar{\nabla}^\sigma \bar{\nabla}^\nu X^\mu{}_\sigma - c_1 Z_\mu{}^{\rho\sigma} \bar{\nabla}^\nu \bar{\nabla}^\mu Z_{\nu\rho\sigma} \\
& + a_1 X_{\mu\nu} \bar{R}^{\nu\sigma} X^\mu{}_\sigma - a_1 X_{\mu\nu} \bar{R}^{\mu\rho\nu\sigma} X_{\rho\sigma} + c_1 Z_{\mu\alpha\beta} \bar{R}^{\mu\sigma} Z_\sigma{}^{\alpha\beta} - 2c_1 Z_{\mu\alpha\beta} \bar{R}^{\mu\rho\alpha\sigma} Z_{\rho\sigma}{}^{\beta]} \Big] \ .
\end{aligned} \tag{7.17}$$

We see that the first line exactly cancels the unwanted nonminimal terms in the Hessian.

At this point, rescaling

$$X^a{}_\mu \to \frac{1}{\sqrt{a_1}} X^a{}_\mu \ , \quad Z_\mu{}^a{}_b \to \frac{1}{\sqrt{c_1}} Z_\mu{}^a{}_b \tag{7.18}$$

and performing some integrations by parts, one can write the gauge-fixed Hessian in the form

$$H = \frac{1}{2}\int d^4x\sqrt{g}\begin{pmatrix}X & Z\end{pmatrix}\begin{pmatrix}-\nabla^2 + V^\mu_{XX}\nabla_\mu + W_{XX} & V^\mu_{XZ}\nabla_\mu + W_{XZ} \\ V^\mu_{ZX}\nabla_\mu + W_{ZX} & -\nabla^2 + V^\mu_{ZZ}\nabla_\mu + W_{ZZ}\end{pmatrix}\begin{pmatrix}X \\ Z\end{pmatrix} \tag{7.19}$$

where the V's and W's are matrices in the space of the fields, with the appropriate free indices. More compactly

$$H = \frac{1}{2}(\Psi, \mathcal{O}\Psi). \tag{7.20}$$

where $\Psi = \begin{pmatrix}X \\ Z\end{pmatrix}$ and

$$\mathcal{O} = -\nabla^2\mathbb{I} + \mathbb{V}^\sigma\nabla_\sigma + \mathbb{W}\ , \tag{7.21}$$

where

$$\mathbb{V} = \begin{pmatrix}V^\mu_{XX} & V^\mu_{XZ} \\ V^\mu_{ZX} & V^\mu_{ZZ}\end{pmatrix} \qquad \mathbb{V} = \begin{pmatrix}W_{XX} & V^\mu_{XZ} \\ V^\mu_{ZX} & V^\mu_{ZZ}\end{pmatrix} \tag{7.22}$$

The operator $\mathcal{O}$ must be self-adjoint, which implies the conditions

$$\begin{aligned}
V_{XX}^{[\mu\nu]\lambda[\alpha\beta]} &= -V_{XX}^{[\alpha\beta]\lambda[\mu\nu]} \\
V_{XZ}^{[\mu\nu]\lambda[\alpha\beta\gamma]} &= -V_{ZX}^{[\alpha\beta\gamma]\lambda[\mu\nu]} \\
V_{ZZ}^{[\mu\nu\rho]\lambda[\alpha\beta\gamma]} &= -V_{XX}^{[\alpha\beta\gamma]\lambda[\mu\nu\rho]} \\
W_{XX}^{[\mu\nu][\alpha\beta]} &= W_{XX}^{[\alpha\beta][\mu\nu]} - \nabla_\lambda V_{XX}^{[\mu\nu]\lambda[\alpha\beta]} \\
W_{XZ}^{[\mu\nu][\alpha\beta\gamma]} &= W_{ZX}^{[\alpha\beta\gamma][\mu\nu]} - \nabla_\lambda V_{ZX}^{[\alpha\beta\gamma]\lambda[\mu\nu]} \\
W_{ZZ}^{[\mu\nu\rho][\alpha\beta\gamma]} &= W_{ZZ}^{[\alpha\beta\gamma][\mu\nu\rho]} - \nabla_\lambda V_{ZZ}^{[\alpha\beta\gamma]\lambda[\mu\nu\rho]}
\end{aligned} \tag{7.23}$$

With the rescaling (7.18) the quantum fields X and Z have canonical dimension one, V has dimension 1 and W has dimension 2. We do not give the components of these tensors but just indicate the general structures that they contain:

$$\begin{aligned}
V_{XX} &\sim T \\
V_{XZ} \sim V_{ZX} &\sim \left(\sqrt{\frac{a_1}{c_1}}\ ,\ \sqrt{\frac{c_1}{a_1}}F\right) \\
V_{ZZ} &\sim T \\
W_{XX} &\sim \left(T^2\ ,\ \nabla T\ ,\ \frac{c_1}{a_1}F^2\right) \\
W_{XZ} \sim W_{ZX} &\sim \left(\sqrt{\frac{a_1}{c_1}}T\ ,\ \sqrt{\frac{c_1}{a_1}}TF\right) \\
W_{ZZ} &\sim \left(\frac{a_1}{c_1}\ ,\ F\ ,\ T^2\ ,\ \nabla T\right)\ .
\end{aligned} \tag{7.24}$$

Here terms without tensors have to be understood as combinations of the metric.

7.4 Ghost action

The gauge fixing has to be supplemented by the ghost action. We define the ghost operators

$$\delta^{QL}_\Sigma\chi_L = \Delta_{LL}\Sigma\ ,\quad \tilde{\delta}^{QD}_C\chi_L = \Delta_{LD}C\ ,\quad \delta^{QL}_\Sigma\chi_D = \Delta_{DL}\Sigma\ ,\quad \tilde{\delta}^{QD}_C\chi_D = \Delta_{DD}C\ . \tag{7.25}$$

Here we have the infinitesimal "quantum" transformations applied to the gauge fixing conditions, with the transformation parameters $\omega^a{}_b$ and v^μ replaced by the ghost fields $\Sigma^a{}_b$ and C^μ. Then the ghost action is given by

$$\begin{aligned} S_{gh} &= \int d^4x\sqrt{\bar{g}}\left[\bar{\Sigma}(\delta^{QL}_{\Sigma}\chi_L + \tilde{\delta}^{QD}_{C}\chi_L) + \bar{C}(\delta^{QL}_{\Sigma}\chi_D + \tilde{\delta}^{QD}_{C}\chi_D)\right] \\ &= \int d^4x\sqrt{\bar{g}}\,(\bar{\Sigma}\quad \bar{C})\begin{pmatrix}\Delta_{LL} & \Delta_{LD}\\ \Delta_{DL} & \Delta_{DD}\end{pmatrix}\begin{pmatrix}\Sigma\\ C\end{pmatrix} , \end{aligned} \tag{7.26}$$

where $\bar{\Sigma}_a{}^b$ and $\bar{C}_\mu$ are the antighost fields. All indices here have been suppressed for notational clarity. When one evaluates explicitly the infinitesimal transformations in (7.25), one obtains some operators of the form $\bar{\nabla}\nabla$, i.e. containing both the background and the full connection. However, we are ultimately only interested in the effective action at zero fluctuation fields, so we can identify the full and background fields. This means that the ghost operators are constructed entirely with background fields. Since the total covariant derivative of the background vierbein with the background LC connection is zero, we can write all formulas using only coordinate (greek) indices, without producing new terms. The ghost operators are then:

$$\begin{aligned} (\Delta_{LL}\Sigma)^\alpha{}_\beta &= \bar{\nabla}^2\Sigma^\alpha{}_\beta + \bar{\nabla}^\nu(\bar{K}_\nu{}^\alpha{}_\gamma\Sigma^\gamma{}_\beta - \Sigma^\alpha{}_\gamma\bar{K}_\nu{}^\gamma{}_\beta) \\ (\Delta_{LD}C)^\alpha{}_\beta &= \bar{\nabla}^\nu(\bar{F}_{\rho\nu}{}^\alpha{}_\beta C^\rho) \\ (\Delta_{DL}\Sigma)^\alpha &= \bar{\nabla}^\nu\Sigma^\alpha{}_\nu \\ (\Delta_{DD}C)^\alpha &= \bar{\nabla}^2 C^\alpha + \bar{\nabla}^\nu(K_\rho{}^\alpha{}_\nu C^\rho) \end{aligned} \tag{7.27}$$

7.5 One loop divergences and beta functions

The one-loop effective action is given by the classical action plus a quantum contribution

$$\Gamma = S + \Delta\Gamma^{(1)} . \tag{7.28}$$

The effective action could contain non-local terms, but these are related to infrared effects. We are interested here in the UV behaviour of the theory and in particular in the logarithmically divergent part, which is local. Thus we can write

$$\Gamma = \sum_i g_i \int d^4x\sqrt{g}\,\mathcal{L}_i$$

where $\mathcal{L}_i$ are dimension-four operators constructed with the fields and their covariant derivatives, and g_i are the corresponding (renormalized) dimensionless couplings. There is a similar expansion for the classical action S, whose the coefficients g_{Bi} are the bare couplings. In the presence of a momentum cutoff Λ, the logarithmically divergent part of Γ can be written

$$\Delta\Gamma^{(1)}_{log} = -B\log\left(\frac{\Lambda}{\mu}\right) , \tag{7.29}$$

where μ is a reference scale that has to be introduced for dimensional reasons and can be thought of as the renormalisation scale, and

$$B = \sum_i \int d^4x\sqrt{g}\,\beta_i\mathcal{L}_i , \tag{7.30}$$

where the coefficients β_i are defined by this equation. Thus

$$g_i(\mu) = g_{Bi}(\Lambda) - \beta_i \log\left(\frac{\Lambda}{\mu}\right) \ . \tag{7.31}$$

Here we assume that the bare couplings depend on the UV cutoff in such a way that the renormalized couplings are finite and as a consequence the renormalized couplings must depend on μ. Then we find that

$$\mu\frac{\partial g_i}{\partial\mu} = \beta_i \ , \tag{7.32}$$

so, as is well-known, we find that the coefficients of the logarithmic divergences are just the beta functions. Note that we can also think of B as

$$B = -\Lambda\frac{\partial\Delta\Gamma^{(1)}_{log}}{\partial\Lambda} \ . \tag{7.33}$$

For our theory,

$$\Delta\Gamma^{(1)} = \frac{1}{2}\mathrm{Tr}\log\mathcal{O} - \mathrm{Tr}\log\Delta_{gh} \ . \tag{7.34}$$

where both $\mathcal{O}$ and Δ_{gh} are operators of the form $-\bar{\nabla}^2 + V^\mu\bar{\nabla}_\mu + W$. For such an operator, the logarithmically divergent part of the kinetic operator is

$$\frac{1}{2}\mathrm{Tr}\log\Delta = -\frac{1}{2}\frac{1}{(4\pi)^2}log\left(\frac{\Lambda^2}{\mu^2}\right)\int d^4x\sqrt{g}b\,(\Delta) \ , \tag{7.35}$$

where

$$\begin{aligned} b\,(\Delta) = \tfrac{1}{180}&\left(\bar{R}_{\mu\nu\rho\sigma}\bar{R}^{\mu\nu\rho\sigma} - \bar{R}_{\mu\nu}\bar{R}^{\mu\nu} + \tfrac{5}{2}\bar{R}^2\right)\mathrm{tr}\mathbb{I} \\ &+ \tfrac{1}{2}\mathrm{tr}W^2 - \tfrac{1}{2}\mathrm{tr}W\bar{\nabla}_\mu V^\mu + \tfrac{1}{4}\mathrm{tr}\,WV_\mu V^\mu \\ &- \tfrac{1}{6}\bar{R}\,\mathrm{tr}W + \tfrac{1}{12}\bar{R}\,\mathrm{tr}\bar{\nabla}_\mu V^\mu - \tfrac{1}{24}\bar{R}\,\mathrm{tr}V_\mu V^\mu \\ &+ \tfrac{1}{12}\mathrm{tr}\Omega_{\mu\nu}\Omega^{\mu\nu} - \tfrac{1}{6}\mathrm{tr}\,\Omega_{\mu\nu}\bar{\nabla}^\mu V^\nu + \tfrac{1}{24}\mathrm{tr}\,\Omega_{\mu\nu}[V^\mu,V^\nu] \\ &+ \tfrac{1}{8}\mathrm{tr}\bar{\nabla}_\mu V^\mu\bar{\nabla}_\rho V^\rho - \tfrac{1}{8}\mathrm{tr}\bar{\nabla}_\mu V^\mu V_\rho V^\rho + \tfrac{1}{32}\mathrm{tr}V_\mu V^\mu V_\rho V^\rho \\ &+ \tfrac{1}{24}\mathrm{tr}(\bar{\nabla}_\mu V_\nu - \bar{\nabla}_\nu V_\mu)\bar{\nabla}^\mu V^\nu - \tfrac{1}{24}\mathrm{tr}\,\bar{\nabla}_\mu V_\nu[V^\mu,V^\nu] + \tfrac{1}{192}\mathrm{tr}\,[V_\mu,V_\nu][V^\mu,V^\nu] \ . \end{aligned} \tag{7.36}$$

The application of this formula was performed by computer algebra software. The obtained result for kinetic operator in X, Z sector (7.21) is

$$\begin{aligned} b\,(\Delta) = &\frac{67}{12}L_1^{FF} - \frac{71}{72}L_3^{FF} - \frac{43}{24}L_4^{FF} + \frac{2}{3}L_7^{FF} + \frac{179}{72}L_8^{FF} - \frac{55}{36}L_{16}^{FF} - \frac{25}{2}L_1^{FT} + \frac{5}{2}L_8^{FT} \\ &- \frac{29}{6}L_9^{FT} - \frac{8}{3}L_{13}^{FT} + \frac{25}{48}L_1^{TT} - \frac{21}{8}L_2^{TT} + \frac{3}{2}L_3^{TT} + \frac{29}{6}L_5^{TT} + \frac{3a_1}{4c_1}M_1^{TT} - \frac{7a_1}{4c_1}M_2^{TT} + \frac{2a_1}{c_1}M_3^{TT} \ , \end{aligned} \tag{7.37}$$

and for and for the ghost operator (7.27) we get

$$\begin{aligned} b\,(\Delta_{gh.}) = &\frac{31}{120}L_1^{FF} - \frac{997}{720}L_3^{FF} + \frac{473}{240}L_4^{FF} - \frac{39}{20}L_7^{FF} + \frac{1303}{720}L_8^{FF} + \frac{1}{9}L_{16}^{FF} - \frac{17}{10}L_1^{FT} - \frac{11}{12}L_8^{FT} \\ &+ \frac{67}{30}L_9^{FT} + \frac{77}{30}L_{13}^{FT} - \frac{9}{160}L_1^{TT} - \frac{97}{80}L_2^{TT} + \frac{13}{15}L_3^{TT} + \frac{27}{20}L_5^{TT} \ , \end{aligned} \tag{7.38}$$

Adding up these contributions as indicated in (7.34) we finally obtain

$$\begin{aligned}\Delta\Gamma^{(1)}_{log.div.} = & -\frac{1}{2}\frac{1}{16\pi^2} log\left(\frac{\Lambda^2}{\mu^2}\right)\int d^4x\sqrt{g}\left[\frac{76}{15}L_1^{FF} + \frac{107}{60}L_3^{FF} - \frac{86}{15}L_4^{FF} + \frac{137}{30}L_7^{FF}\right. \\ & - \frac{17}{15}L_8^{FF} - \frac{7}{4}L_{16}^{FF} - \frac{91}{10}L_1^{FT} + \frac{13}{3}L_8^{FT} - \frac{93}{10}L_9^{FT} - \frac{39}{5}L_{13}^{FT} + \frac{19}{30}L_1^{TT} \\ & \left. -\frac{1}{5}L_2^{TT} - \frac{7}{30}L_3^{TT} + \frac{32}{15}L_5^{TT} + \frac{3a_1}{4c_1}M_1^{TT} - \frac{7a_1}{4c_1}M_2^{TT} + \frac{2a_1}{c_1}M_3^{TT}\right] .\end{aligned} \tag{7.39}$$

In these formulae, the dimension four contributions have been disregarded. As expected, every term of the basis 3.28 is generated by quantum fluctuations.

Chapter 8

Field redefinitions

As we have already mentioned in sections 3.3.4 and 3.5.4, there exists a possibility to perform field redefinitions that may not just remove technically challenging nonminimal terms from the original action, but also remove local quantum corrections from the effective action. In this chapter, we will address this issue more thoroughly. First, we will study a combined effort of linear field redefinitions and a particular gauge choice, designed to cancel nonminimal terms. After, we will consider also nonlinear redefinitions and their connection with on-shell renormalisation.

8.1 Gauge-fixing and linear field redefinitions

In this section, we address the question of whether it is possible to cancel all the nonminimal terms that appear in MAG. Although the complete analysis is yet to be done, we show that the answer to this question is most likely negative.

8.1.1 Antisymmetric MAG

We note that the first three terms of type TT give contributions to the kinetic operator proportional to the Laplacian, whereas the others give non–minimal terms. Since these are undesirable, we will look for ways to get rid of them. First let us consider whether the nonminimal terms could be eliminated by a gauge choice.

For the $GL(4)$ gauge fixing we have

$$\mathcal{L}_{GL(4)} = \left(\zeta_0 \bar{g}_{\lambda\mu}\bar{g}^{\rho\nu} + \zeta_1 \bar{\delta}^{\nu}_{\lambda}\bar{\delta}^{\rho}_{\mu} + \zeta_2 \bar{\delta}^{\rho}_{\lambda}\bar{\delta}^{\nu}_{\mu}\right)\chi^{\lambda}{}_{\rho}\chi^{\mu}{}_{\nu} \tag{8.1}$$

where

$$\chi^{\mu}{}_{\nu} = \xi_1 \nabla^{\sigma} T_{\sigma}{}^{\mu}{}_{\nu} + \xi_2 \nabla^{\sigma} T_{\sigma\nu}{}^{\mu} + \xi_3 \nabla_{\sigma} T^{\mu\sigma}{}_{\nu} + \xi_4 \nabla^{\mu} T^{\sigma}{}_{\sigma\nu} + \xi_5 \nabla_{\nu} T_{\sigma}{}^{\sigma\mu} \tag{8.2}$$

Thus altogether there are eight gauge fixing parameters. The gauge fixing Lagrangian can be written in the form

$$\mathcal{L}_{GL(4)} = \sum_{i=1}^{9} \Delta b_i^{TT} H_i^{TT} \tag{8.3}$$

where

$$
\begin{aligned}
\Delta b_1^{TT} &= 0 \ , \\
\Delta b_2^{TT} &= 0 \ , \\
\Delta b_3^{TT} &= \zeta_0(\xi_4^2+\xi_5^2) + 2\zeta_1\xi_4\xi_5 \ , \\
\Delta b_4^{TT} &= \zeta_0(\xi_1^2+\xi_2^2) + 2\zeta_1\xi_1\xi_2 \ , \\
\Delta b_5^{TT} &= 2\zeta_0\xi_1\xi_2 + \zeta_1(\xi_1^2+\xi_2^2) \ , \\
\Delta b_6^{TT} &= (\zeta_0-\zeta_1)\xi_3^2 \ , \\
\Delta b_7^{TT} &= (\zeta_0-\zeta_1)(\xi_1-\xi_2)\xi_3 \ , \\
\Delta b_8^{TT} &= 2\zeta_0((\xi_1+\xi_3)\xi_4+(\xi_2-\xi_3)\xi_5) + 2\zeta_1((\xi_2-\xi_3)\xi_4+(\xi_1+\xi_3)\xi_5) \ , \\
\Delta b_9^{TT} &= 2\zeta_0\xi_4\xi_5 + \zeta_1(\xi_4^2+\xi_5^2) + \zeta_2(\xi_1+\xi_2-\xi_4-\xi_5)^2 \ .
\end{aligned}
\tag{8.4}
$$

One could now ask whether it is possible to choose the gauge parameters in such a way that

$$
\Delta b_i^{TT} = -b_i^{TT} \quad \text{for} \quad i = 4\ldots 9 \tag{8.5}
$$

so as to eliminate non–minimal terms from the kinetic operator.

A priori this may seem possible, because we have to fix 6 parameters in the full (gauge–fixed) Lagrangian, and we have at our disposal 8 gauge fixing parameters. However, the equations above are cubic in the variables and it is difficult to solve them. In fact, even though there are 8 gauge–fixing parameters, it is not even granted that the submanifold defined by the preceding equations has dimension 8. To find the dimension of this manifold we fix generic values of the gauge-fixing parameters and vary them

$$
\zeta_i \to \zeta_i + \delta\zeta_i \ , \qquad \xi_i \to \xi_i + \delta\xi_i \ .
$$

This leads to a change in the parameters Δb_i^{TT}. At linear order these changes are given by

$$
\begin{aligned}
\delta b_1^{TT} =& 0 \ , \\
\delta b_2^{TT} =& 0 \ , \\
\delta b_3^{TT} =& \delta\zeta_0(\xi_4^2+\xi_5^2) + 2\delta\zeta_1\xi_4\xi_5 + 2\delta\xi_4(\zeta_0\xi_4+\zeta_1\xi_5) + 2\delta\xi_5(\zeta_1\xi_4+\zeta_0\xi_5) \ , \\
\delta b_4^{TT} =& \delta\zeta_0(\xi_1^2+\xi_2^2) + 2\delta\zeta_1\xi_1\xi_2 + 2\delta\xi_1(\zeta_0\xi_1+\zeta_1\xi_2) + 2\delta\xi_2(\zeta_1\xi_1+\zeta_0\xi_2) \ , \\
\delta b_5^{TT} =& \delta\zeta_1(\xi_1^2+\xi_2^2) + 2\delta\zeta_0\xi_1\xi_2 + 2\delta\xi_1(\zeta_1\xi_1+\zeta_0\xi_2) + 2\delta\xi_2(\zeta_0\xi_1+\zeta_1\xi_2) \ , \\
\delta b_6^{TT} =& (\delta\zeta_0-\delta\zeta_1)\xi_3^2 + 2\delta\xi_3(\zeta_0-\zeta_1)\xi_3 \ , \\
\delta b_7^{TT} =& 2\delta\zeta_0(\xi_1-\xi_2)\xi_3 - 2\delta\zeta_1(\xi_1-\xi_2)\xi_3 \\
&+ 2\delta\xi_1(\zeta_0-\zeta_1)\xi_3 - 2\delta\xi_2(\zeta_0-\zeta_1)\xi_3 + 2\delta\xi_3(\zeta_0-\zeta_1)(\xi_1-\xi_2) \ , \\
\delta b_8^{TT} =& 2\delta\zeta_0((\xi_1+\xi_3)\xi_4+(\xi_2-\xi_3)\xi_5) + 2\delta\zeta_1((\xi_2-\xi_3)\xi_4+(\xi_1+\xi_3)\xi_5) \\
&+ 2\delta\xi_1(\zeta_0\xi_4+\zeta_1\xi_5) + 2\delta\xi_2(\zeta_1\xi_4+\zeta_0\xi_5) + 2\delta\xi_3(\zeta_0-\zeta_1)(\xi_4-\xi_5) \\
&+ 2\delta\xi_4(\zeta_1(\xi_2-\xi_3)+\zeta_0(\xi_1+\xi_3)) + 2\delta\xi_5(\zeta_0(\xi_2-\xi_3)+\zeta_1(\xi_1+\xi_3)) \ , \\
\delta b_9^{TT} =& 2\delta\zeta_0\xi_4\xi_5 + \delta\zeta_1(\xi_4^2+\xi_5^2) + \delta\zeta_2(\xi_1+\xi_2-\xi_4-\xi_5)^2 \\
&+ 2(\delta\xi_1+\delta\xi_2)\zeta_2(\xi_1+\xi_2-\xi_4-\xi_5) \\
&+ 2\delta\xi_4(\zeta_1\xi_4+\zeta_0\xi_5-\zeta_2(\xi_1+\xi_2-\xi_4-\xi_5)) + 2\delta\xi_5(\zeta_0\xi_4+\zeta_1\xi_5-\zeta_2(\xi_1+\xi_2-\xi_4-\xi_5)) \ .
\end{aligned}
\tag{8.6}
$$

The matrix of coefficients of this linear system has rank six. The null directions are the two obvious ones (δb_1^{TT} and δb_2^{TT}) and

$$-\frac{\xi_3}{\xi_1-\xi_2}\delta b_4^{TT}+\frac{\xi_3}{\xi_1-\xi_2}\delta b_5^{TT}-\frac{\xi_1-\xi_2}{\xi_3}\delta b_6^{TT}+\delta b_7^{TT}\ .$$

We can construct a non-degenerate submatrix by choosing the rows and columns labelled by

$$b_3^{TT},\ b_5^{TT},\ b_6^{TT},\ b_7^{TT},\ b_8^{TT},\ b_9^{TT}$$

and

$$\zeta_2,\xi_1,\xi_2,\xi_3,\xi_4,\xi_5\ .$$

It has determinant

$$32(\zeta_0-\zeta_1)^3(\zeta_0+\zeta_1)^2(\xi_1+\xi_2)\ \xi_3^2(\xi_1+\xi_2-\xi_4-\xi_5)^2(-\xi_2\xi_4+\ \xi_1\xi_5+\xi_3(\xi_4+\xi_5))\ .$$

For a generic point in this parameters space the determinant will be nonzero and then locally the manifold is six-dimensional. Now it seems that we have enough freedom to get rid of non–minimal terms. However we note that in order to find a non–degenerate minor it was necessary to include the direction b_3^{TT}, that we are not interested in. Thus we have one less parameter in the remaining subspace, and we conclude that it is not possible to satisfy (8.5) just by using the gauge fixing parameters.

Next we consider the effect of linear field redefinitions:

$$\delta T_{abc}=\alpha_1 T_{abc}+\alpha_2 T_{acb}+\alpha_3 g_{ab}T^d{}_{dc}\ . \tag{8.7}$$

At linear order this leads to the following changes in the Lagrangian coefficients:

$$\begin{aligned}
\delta b_1^{TT}&=2b_1^{TT}\alpha_1+2b_2^{TT}\alpha_2\ ,\\
\delta b_2^{TT}&=2b_2^{TT}\alpha_1+2b_1^{TT}\alpha_2\ ,\\
\delta b_3^{TT}&=2b_3^{TT}\alpha_1+(2b_1^{TT}+8b_3^{TT}-b_8^{TT})\alpha_3\ ,\\
\delta b_4^{TT}&=2b_4^{TT}\alpha_1+2b_5^{TT}\alpha_2\ ,\\
\delta b_5^{TT}&=2b_5^{TT}\alpha_1+(2b_4^{TT}+b_7^{TT})\alpha_2,\\
\delta b_6^{TT}&=2b_6^{TT}\alpha_1\ ,\\
\delta b_7^{TT}&=2b_7^{TT}\alpha_1+(2b_6^{TT}-b_7^{TT})\alpha_2\ ,\\
\delta b_8^{TT}&=2b_8^{TT}\alpha_1+2(b_4^{TT}+b_6^{TT}+b_7^{TT}+2b_8^{TT})\alpha_3\ ,\\
\delta b_9^{TT}&=2b_9^{TT}(\alpha_1+\alpha_2-\alpha_3)\ .
\end{aligned} \tag{8.8}$$

One can try to fix some of the b_i^{TT} in this way. When viewed as a system of equations for the α_i there is an apparent solution $\alpha_1=-1/2$, $\alpha_2=\alpha_3=0$. For such value, the Lagrangian becomes identically zero. In fact, this is not an infinitesimal transformation and has to be discarded. However, since the system has rank 3, one can solve three of the preceding equations and fix α_i as functions of the corresponding δb_i^{TT}. In this way one can remove three of the δb_i^{TT}.

One can then consider the simultaneous effect of changing the gauge parameters and redefining the field. To this end one has to consider the system of 9 linear equations where each of the δ_i^{TT} is equated to the sum of variations due to the change in the gauge parameters and to the field redefinition. It is just the system where the r.h.s. is the sum of the r.h.s. of the two systems considered above. The unknowns are the 11 parameters $\delta\zeta_i$, $\delta\xi_i$ and α_i.

Unlike the case when we only considered changes in the gauge condition, now it is possible to find a nondegenerate subsystem that does not involve b_3^{TT}. In fact, since we are not interested in what happens to b_i^{TT}, $i = 1, 2, 3$, we consider the subsystem of the equations with $i = 4, \ldots, 9$. The matrix of coefficients of this system is a 6×11 matrix and it still has rank 6. Thus we can remove any infinitesimal change of the coefficients b_i^{TT}, $i = 4, \ldots, 9$, by fixing six of the parameters $\delta\zeta_i$, $\delta\xi_i$ and α_i. It turns out that a suitable choice is $\delta\zeta_0$, $\delta\zeta_1$, $\delta\zeta_2$, α_1, α_2, α_3.

In summary, we have shown the following. For a given initial Lagrangian, suppose we are able to choose the gauge condition in such a way that conditions (8.5) are satisfied. Then the operator acting on the field ϕ is minimal. Then, assume that we vary infinitesimally the parameters b_i^{TT} in the initial Lagrangian. Generically, this will produce infinitesimal non–minimal terms in the operator. By means of a field redefinition and an infinitesimal change in the gauge parameters we can return to the condition of having a minimal operator. Note that we have not taken into account the $R\nabla T$ terms.

8.1.2 Symmetric MAG

We note that the first five terms of type QQ give contributions to the kinetic operator proportional to the Laplacian, whereas the others give non–minimal terms.

For the $GL(4)$ gauge fixing we have

$$\mathcal{L}_{GL(4)} = \left(\zeta_0 \bar{g}_{\lambda\mu}\bar{g}^{\rho\nu} + \zeta_1 \bar{\delta}^\nu_\lambda \bar{\delta}^\rho_\mu + \zeta_2 \bar{\delta}^\rho_\lambda \bar{\delta}^\nu_\mu\right) \chi^\lambda{}_\rho \chi^\mu{}_\nu \tag{8.9}$$

where

$$\begin{aligned}\chi_{\mu\nu} = {} & \xi_1 \nabla_\sigma Q^\sigma{}_{\mu\nu} + \xi_2 \nabla_\sigma Q_\mu{}^\sigma{}_\nu + \xi_3 \nabla_\sigma Q_\nu{}^\sigma{}_\mu + \xi_4 \nabla_\sigma Q_{\mu\nu}{}^\sigma + \xi_5 \nabla_\sigma Q_{\nu\mu}{}^\sigma \\ & + \xi_6 \nabla_\mu Q^\sigma{}_{\sigma\nu} + \xi_7 \nabla_\mu Q_{\nu\sigma}{}^\sigma + \xi_8 \nabla_\nu Q^\sigma{}_{\sigma\mu} + \xi_9 \nabla_\nu Q_{\mu\sigma}{}^\sigma\ .\end{aligned} \tag{8.10}$$

Thus altogether there are 12 gauge fixing parameters. The gauge fixing Lagrangian can be written in the form

$$\mathcal{L}_{GL(4)} = \sum_{i=1}^{9} \Delta b_i^{QQ} H_i^{QQ} \tag{8.11}$$

where the quantities Δb_i^{QQ} are presented in D.3. As in the antisymmetric case, this cubic equation is too complicated to solve, so we limit ourselves to an analysis of its linearisation.

The coefficients of this linear system form a 16×12 matrix of rank 8. Removing the rows that correspond to b_i^{TT}, $i = 1, 2, 3, 4, 5$ does not reduce the rank. Thus one can keep fixed 8 of the 11 parameters that produce nonminimal terms by readjusting the gauge fixing condition.

We can then consider the effect of field redefinitions. These are given by

$$\begin{aligned}\delta Q_{abc} = {} & \alpha_1 Q_{abc} + \alpha_2 Q_{bca} + \alpha_3 Q_{cab} + \alpha_4 g_{ab} Q^d{}_{dc} + \alpha_5 g_{ab} Q_{cd}{}^d \\ & + \alpha_6 g_{ac} Q^d{}_{db} + \alpha_7 g_{ac} Q_{bd}{}^d + \alpha_8 g_{bc} Q^d{}_{da} + \alpha_9 g_{bc} Q_{ad}{}^d\end{aligned} \tag{8.12}$$

and produce a variation of the Lagrangian parameters

$$\begin{aligned}
\delta b_1^{QQ} &= 2b_1^{QQ}\alpha_1 + 2b_2^{QQ}\alpha_2 \ , \\
\delta b_2^{QQ} &= 2b_2^{QQ}\alpha_1 + 2b_1^{QQ}\alpha_2 + 2(b_1^{QQ} + b_2^{QQ})\alpha_3 \ , \\
\delta b_3^{QQ} &= 2b_3^{QQ}\alpha_1 + (2b_3^{QQ} + b_5^{QQ})\alpha_2 + b_5^{QQ}\alpha_3 + (2b_1^{QQ} + b_{10}^{QQ} + 2b_2^{QQ} + 8b_3^{QQ} + b_5^{QQ})\alpha_4 \\
&\quad + (2b_1^{QQ} + 2b_3^{QQ} + b_5^{QQ})\alpha_6 + (b_{12}^{QQ} + 2(b_2^{QQ} + b_3^{QQ} + 2b_5^{QQ}))\alpha_8 \ , \\
\delta b_4^{QQ} &= 2b_4^{QQ}\alpha_1 + b_5^{QQ}\alpha_3 + (b_{11}^{QQ} + 2b_4^{QQ} + 4b_5^{QQ})\alpha_5 + (2b_2^{QQ} + 2b_4^{QQ} + b_5^{QQ})\alpha_7 \\
&\quad + (2b_1^{QQ} + b_{13}^{QQ} + 8b_4^{QQ} + b_5^{QQ})\alpha_9 \ ,
\end{aligned}$$

$$\begin{aligned}
\delta b_5^{QQ} &= 2b_5^{QQ}\alpha_1 + (2b_4^{QQ} + b_5^{QQ})\alpha_2 + 2(b_3^{QQ} + b_4^{QQ})\alpha_3 + (b_{11}^{QQ} + 2b_4^{QQ} + 4b_5^{QQ})\alpha_4 \\
&\quad + (2b_1^{QQ} + b_{10}^{QQ} + 2b_2^{QQ} + 8b_3^{QQ} + b_5^{QQ})\alpha_5 + (2b_2^{QQ} + 2b_4^{QQ} + b_5^{QQ})\alpha_6 \\
&\quad + (2b_1^{QQ} + 2b_3^{QQ} + b_5^{QQ})\alpha_7 + (2b_1^{QQ} + b_{13}^{QQ} + 8b_4^{QQ} + b_5^{QQ})\alpha_8 + (b_{12}^{QQ} + 2(b_2^{QQ} + b_3^{QQ} + 2b_5^{QQ}))\alpha_9 \ , \\
\delta b_6^{QQ} &= 2b_6^{QQ}\alpha_1 + b_9^{QQ}\alpha_2 \ , \\
\delta b_7^{QQ} &= 2b_7^{QQ}\alpha_1 + b_9^{QQ}\alpha_2 + 2b_8^{QQ}\alpha_3 \ , \\
\delta b_8^{QQ} &= 2b_8^{QQ}\alpha_1 + (2b_7^{QQ} + b_9^{QQ})\alpha_3 \ , \\
\delta b_9^{QQ} &= 2b_9^{QQ}\alpha_1 + 2(b_6^{QQ} + b_7^{QQ} + b_8^{QQ})\alpha_2 + (2b_6^{QQ} + b_9^{QQ})\alpha_3 \ , \\
\delta b_{10}^{QQ} &= 2b_{10}^{QQ}\alpha_1 + (2b_{10}^{QQ} + b_{11}^{QQ} + b_{12}^{QQ})\alpha_2 + (b_{11}^{QQ} + b_{12}^{QQ})\alpha_3 \\
&\quad + (4b_{10}^{QQ} + b_{11}^{QQ} + 2(b_6^{QQ} + b_7^{QQ} + b_9^{QQ}))\alpha_4 + (b_{10}^{QQ} + b_{11}^{QQ} + 2b_6^{QQ})\alpha_6 \\
&\quad + (b_{10}^{QQ} + 4b_{11}^{QQ} + 2b_8^{QQ} + b_9^{QQ})\alpha_8 \ ,
\end{aligned} \tag{8.13}$$

$$\begin{aligned}
\delta b_{11}^{QQ} &= 2b_{11}^{QQ}\alpha_1 + (b_{11}^{QQ} + b_{13}^{QQ})\alpha_2 + (b_{10}^{QQ} + b_{13}^{QQ})\alpha_3 \\
&\quad + (4b_{10}^{QQ} + b_{11}^{QQ} + 2(b_6^{QQ} + b_7^{QQ} + b_9^{QQ}))\alpha_5 + (b_{10}^{QQ} + b_{11}^{QQ} + 2b_6^{QQ})\alpha_7 \\
&\quad + (b_{10}^{QQ} + 4b_{11}^{QQ} + 2b_8^{QQ} + b_9^{QQ})\alpha_9 \ , \\
\delta b_{12}^{QQ} &= 2b_{12}^{QQ}\alpha_1 + (b_{12}^{QQ} + b_{13}^{QQ})\alpha_2 + (b_{10}^{QQ} + b_{13}^{QQ})\alpha_3 + (4b_{12}^{QQ} + b_{13}^{QQ} + 2b_8^{QQ})\alpha_4 \\
&\quad + (b_{12}^{QQ} + b_{13}^{QQ} + b_9^{QQ})\alpha_6 + (b_{12}^{QQ} + 4b_{13}^{QQ} + 2b_7^{QQ})\alpha_8 \ , \\
\delta b_{13}^{QQ} &= 2b_{13}^{QQ}\alpha_1 + (b_{11}^{QQ} + b_{12}^{QQ})\alpha_3 + (4b_{12}^{QQ} + b_{13}^{QQ} + 2b_8^{QQ})\alpha_5 + (b_{12}^{QQ} + b_{13}^{QQ} + b_9^{QQ})\alpha_7 \\
&\quad + (b_{12}^{QQ} + 4b_{13}^{QQ} + 2b_7^{QQ})\alpha_9 \ , \\
\delta b_{14}^{QQ} &= 2b_{14}^{QQ}\alpha_1 + b_{16}^{QQ}\alpha_2 + (b_{13}^{QQ} + 2b_{14}^{QQ} + b_{16}^{QQ})\alpha_5 \\
&\quad + (b_{11}^{QQ} + b_{13}^{QQ} + 2b_{14}^{QQ} + 4b_{16}^{QQ} + b_9^{QQ})\alpha_7 + (b_{11}^{QQ} + 8b_{14}^{QQ} + b_{16}^{QQ} + 2b_6^{QQ})\alpha_9) \\
\delta b_{15}^{QQ} &= 2b_{15}^{QQ}\alpha_1 + b_{16}^{QQ}\alpha_2 + (2b_{15}^{QQ} + b_{16}^{QQ})\alpha_3 + (b_{12}^{QQ} + 2b_{15}^{QQ} + b_{16}^{QQ})\alpha_4 \\
&\quad + (b_{10}^{QQ} + b_{12}^{QQ} + 8b_{15}^{QQ} + b_{16}^{QQ} + 2b_7^{QQ} + 2b_8^{QQ})\alpha_6 + (b_{10}^{QQ} + 2b_{15}^{QQ} + 4b_{16}^{QQ} + b_9^{QQ})\alpha_8 \\
\delta b_{16}^{QQ} &= 2b_{16}^{QQ}\alpha_1 + 2(b_{14}^{QQ} + b_{15}^{QQ})\alpha_2 + (2b_{14}^{QQ} + b_{16}^{QQ})\alpha_3 + (b_{13}^{QQ} + 2b_{14}^{QQ} + b_{16}^{QQ})\alpha_4 \\
&\quad + (b_{12}^{QQ} + 2b_{15}^{QQ} + b_{16}^{QQ})\alpha_5 + (b_{11}^{QQ} + b_{13}^{QQ} + 2b_{14}^{QQ} + 4b_{16}^{QQ} + b_9^{QQ})\alpha_6 \\
&\quad + (b_{10}^{QQ} + b_{12}^{QQ} + 8b_{15}^{QQ} + b_{16}^{QQ} + 2b_7^{QQ} + 2b_8^{QQ})\alpha_7 \\
&\quad + (b_{11}^{QQ} + 8b_{14}^{QQ} + b_{16}^{QQ} + 2b_6^{QQ})\alpha_8 + (b_{10}^{QQ} + 2b_{15}^{QQ} + 4b_{16}^{QQ} + b_9^{QQ})\alpha_9 \ .
\end{aligned}$$

This system has rank 9, so each parameter α_i can be used to modify one Lagrangian coefficient. This however falls short of the 11 parameters that contribute nonminimal terms.

When we put together the variations produced by the field redefinitions and the changes in the gauge parameters, we find a linear system of size 16×21 of rank 16. It is therefore possible to readjust all the coefficients in such a way as to maintain the minimality condition.

8.2 Nonlinear field redefinitions and on-shell reduction of the effective action

Let us denote all the dynamical fields of a theory as φ and consider the following infinitesimal redefinitions thereof:

$$\varphi \to \varphi + \Psi[\varphi], \tag{8.14}$$

where $\varphi \gg \Psi[\varphi]$. The corresponding change of the effective action is

$$\Gamma[\varphi] \to \Gamma[\varphi] + \frac{\delta \Gamma}{\delta \varphi} \Psi[\varphi]. \tag{8.15}$$

Therefore, terms proportional to the equations of motion can be (infinitesimally) eliminated from the effective action. We assume that the perturbation theory is applicable, which means that the quantum effective action can be expanded in powers of $\hbar$:

$$\Gamma = S + \sum_{k=1}^{\infty} \hbar^k \Gamma^{(k)} \tag{8.16}$$

where $\Gamma^{(k)}$ is colloquially referred to as "k-loop" effective action and

$$S \gg \hbar \Gamma^{(1)}, \qquad \Gamma^{(k)} \gg \hbar \Gamma^{(k+1)} \quad \forall k > 0. \tag{8.17}$$

Then, at one-loop level, we can separate terms of the effective action that are proportional to the equations of motion:

$$\Gamma^{(1)} \approx \Gamma^{(1)}_{on-shell} + \frac{\delta S}{\delta \varphi} \Psi[\varphi]. \tag{8.18}$$

Here the quantum action is replaced with the classical one because their difference would give contributions of the second order in $\hbar$. We conclude from (8.15) that the second terms can be removed. We call inessential those terms whose quantum corrections can be removed by field redefinitions and which therefore do not enter the observables. The other, essential terms, comprise what we will refer to as the on-shell effective action. Generalizing the same logic to higher orders we obtain:

$$\Gamma^{(k)} \approx \Gamma^{(k)}_{on-shell} + \frac{\delta}{\delta \varphi} \left(S + \sum_{l=1}^{k-1} \Gamma^{(l)} \right) \Psi[\varphi]. \tag{8.19}$$

This means that higher-order terms proportional to the equations of motion obtained from the lower-order terms can be eliminated from the effective action by appropriate field redefinitions. Note that by higher order we mean those related to the expansion in $\hbar$. This may or may not correspond to expansion in mass dimension. We stress here that the assumption about the perturbative character of the theory is important for our considerations.

There exists, of course, some ambiguity when it comes to choosing a basis of on-shell independent terms. Since in this note we are interested in the behaviour of the propagator, we will focus on reducing the number of independent contributions to it. In the following, we will write down equations neglecting any contributions to the potential and terms of mass dimension higher than 4. We will denote the corresponding "equality at the level of the flat space propagator" as $\simeq$.

In the following, we apply this recipe to Antisymmetric MAG. The general Lagrangian, including the terms of order up to four that contribute to the flat propagator, is given by (3.37), or (4.3) with $Q = 0$. We consider all possible contractions of the equations of motion with arbitrarily chosen functions of the fields, according to a recipe discussed above. In the formulae below integration over space-time is omitted and in their derivation integration by parts was used.

$$2\frac{\delta S}{\delta g_{\mu\nu}}g_{\mu\nu} \simeq m_1^{TT}M_{TT}^1 + m_2^{TT}M_{TT}^2 + m_3^{TT}M_{TT}^3 - m_0^2 R \tag{8.20}$$

$$2\frac{\delta S}{\delta g_{\mu\nu}}R_{\mu\nu} \simeq m_0^2 H_{RR}^2 - \frac{1}{2}m_0^2 H_{RR}^3 \tag{8.21}$$

$$2\frac{\delta S}{\delta g_{\mu\nu}}g_{\mu\nu}R \simeq -m_0^2 H_{RR}^3 \tag{8.22}$$

These three equations show that one-loop divergences of the types $m_0^2 R$, R^2 and R^2 can be reabsorbed into appropriate field redefinitions. This is well known in the case of metric gravity. However, in the case of gravity with torsion one can choose the function Ψ in (8.18) and get two additional equations

$$\begin{aligned} &2\frac{\delta S}{\delta g_{\mu\nu}}\nabla_\gamma T_{\mu\nu}{}^{\gamma} \simeq m_0^2 H_{RT}^5, \\ &2\frac{\delta S}{\delta g_{\mu\nu}}g_{\mu\nu}\nabla_\alpha T^\alpha \simeq -m_0^2 H_{RT}^3 + \frac{1}{2}m_0^2 H_{RT}^5, \end{aligned} \tag{8.23}$$

which show that the divergences in the mixing sector be reabsorbed in a similar manner.

$$\begin{aligned} &2\frac{\delta S}{\delta T_\mu}\nabla_\mu R \simeq -\left(2m_1^{TT} + m_2^{TT} + 3m_3^{TT}\right) H_{RT}^2 \\ &2\frac{\delta S}{\delta T_{\alpha\beta\gamma}}\nabla_\gamma R_{\beta\alpha} \simeq \left(2m_1^{TT} + m_2^{TT}\right) H_{RT}^3 + \frac{1}{2}m_3^{TT}H_{RT}^5 \end{aligned} \tag{8.24}$$

$$\begin{aligned} \frac{\delta S}{\delta T_{\alpha\beta\gamma}}T_{\alpha\beta\gamma} \simeq{}& \frac{1}{2}b_3^{RT}H_{RT}^3 + \frac{1}{2}b_5^{RT}H_{RT}^5 + b_1^{TT}H_{TT}^1 + b_2^{TT}H_{TT}^2 + b_3^{TT}H_{TT}^3 + b_4^{TT}H_{TT}^4 + b_5^{TT}H_{TT}^5 \\ &+ b_6^{TT}H_{TT}^6 + b_7^{TT}H_{TT}^7 + b_8^{TT}H_{TT}^8 + b_9^{TT}H_{TT}^9 + m_1^{TT}M_{TT}^1 + m_2^{TT}M_{TT}^2 + m_3^{TT}M_{TT}^3 \end{aligned} \tag{8.25}$$

$$\begin{aligned} \frac{\delta S}{\delta T_{\alpha\gamma\beta}}T_{\alpha\beta\gamma} \simeq{}& \frac{1}{2}b_3^{RT}H_{RT}^3 + \frac{1}{2}b_5^{RT}H_{RT}^5 + b_2^{TT}H_{TT}^1 + (2b_1^{TT} - b_2^{TT})H_{TT}^2 + b_3^{TT}H_{TT}^3 + (b_5^{TT} + \frac{1}{2}b_7^{TT})H_{TT}^4 \\ &+ (b_4^{TT} - \frac{1}{2}b_7^{TT})H_{TT}^5 + \frac{1}{2}b_7^{TT}H_{TT}^6 + (b_4^{TT} - b_5^{TT} + 2b_6^{TT} - \frac{1}{2}b_7^{TT})H_{TT}^7 + b_8^{TT}H_{TT}^8 \\ &+ b_9^{TT}H_{TT}^9 + m_2^{TT}M_{TT}^1 + (2m_1^{TT} - m_2^{TT})M_{TT}^2 + m_3^{TT}M_{TT}^3 \end{aligned} \tag{8.26}$$

$$\frac{\delta S}{\delta T_\alpha}T_\alpha \simeq \frac{1}{4}(b_3^{RT}+6b_5^{RT})H_{RT}^5+(2b_1^{TT}+b_2^{TT}+3b_3^{TT}+\frac{1}{2}b_8^{TT})H_{TT}^3+(b_4^{TT}+2b_6^{TT}+b_7^{TT}+\frac{3}{2}b_8^{TT})H_{TT}^8$$
$$+(b_4^{TT}+b_5^{TT}-\frac{1}{2}b_8^{TT}+3b_9^{TT})H_{TT}^9+(2m_1^{TT}+m_2^{TT}+3m_3^{TT})M_{TT}^3 \tag{8.27}$$

After considering all such combinations we conclude that all terms of the dimension up to four that contribute to the flat propagator can be rewritten in such a way that their small quantum corrections can be reabsorbed into field redefinitions. This means that at order four in mass dimension our one-loop EA is

$$\Gamma^{(1)} \approx \frac{\delta S}{\delta \varphi}\Psi[\varphi]+\textit{interaction terms}. \tag{8.28}$$

In our terminology, that means that all operators that contribute to the flat propagator are inessential at one-loop level.

Other projects

Whilst pursuing PhD degree, I have been involved in two other research projects, one of which was published and another is in progress. They have not been included in this thesis for their topic is somewhat different from the main line thereof. The main objectives and results will be briefly explained here.

Primordial Horndeski Cosmology

Horndeski theory is defined as the most general scalar-tensor theory with second-order field equations and incorporates several other well-studied theories. It is of especial interest for it admits a stable violation of the Null Energy Condition (NEC), and at that time there existed a hope that it could allow one to describe non-standard gravitational solutions such as Lorenzian wormholes and semiclosed worlds [147]. Another class of solutions of Horndeski theory concerns the cosmological domain. The stable violation of the NEC opens a plethora of possibilities for considering cosmological solutions such as bouncing Universe and genesis [148, 149]. In the latter scenario, Universe starts from an asymptotically flat space with an asymptotically small effective Planck mass, and then expands transiting into a hot epoch later on. As pointed out in [150] the effective Planck mass then goes to zero in the asymptotic past, which may signalise a potential strong coupling problem. It was of interest to see whether one can present a model that would describe the evolution of the Universe from the earliest times that is governed completely by the laws of classical field theory. We proved that in a wide range of parameters, the inverse time scale of classical evolution is much smaller than the strong coupling energy scale, thus showing the classical analysis of time evolution to be legitimate [151, 152]. The analysis was done at tree level and the strongest bound comes from the scalar sector of perturbations. It was recently proven that, surprisingly, the same bound from the scalar is the strongest one when also compared to all loop orders.

Essential Renormalisation Group

The Functional Renormalisation Group (FRG) is a non-perturbative approach to renormalisation that deals with a scale-dependent effective action called Effective Average Action (EAA). The Essential Renormalisation Group is a modern approach within FRG in which only essential couplings — the ones that cannot be changed by field redefinitions — are allowed to flow, whereas the inessential ones are kept fixed. An important notion is that of a theory space, which is defined as a space of all couplings of EAA. A physical theory can be represented as an RG trajectory in that space. By performing invertible field redefinitions one can alter a given trajectory, however any observables must obviously remain unchanged. All trajectories then fall into equivalence classes, with each member of a class describing the same physics. This approach was initially developed in [153] and applied to Quantum Einstein Gravity in [87], where the usefulness of nonlinear field redefinitions was advocated. The independent terms of the EAA in the fourth order of derivative expansion are the cosmological constant, the Hilbert–Einstein term, the Gauß–Bonnet (GB) invariant and the two terms quadratic in curvature. We identify the Einstein equivalence class which describes classical GR at large distances within the vicinity of the the Gaußian fixed point. Quantum Einstein Gravity is based on the existence of the ultraviolet (UV) Reuter fixed point which provides the UV completion of Einstein's theory. Within this class the only essential couplings are the Newton coupling and the one in front of the GB term, which however does not contribute to the dynamics. All the other couplings are inessential, including the cosmological constant, meaning that they can remain fixed along the flow. In an ongoing project, we study the gauge dependence of the critical exponent. Due to the approximations made, such as the derivative expansion, some degree of gauge dependence may appear, and the smaller it is, the better the applied approximations are. The main technical difficulty in studying gauge dependence in gravity are, again, nonminimal kinetic operators, which are usually cancelled by a gauge choice (de Donder gauge). Using the same code for tackling nonminimal operators mentioned in the previous section I computed the nonperturbative beta functions at order 4 in derivative expansion and used them to find the position of the Reuter fixed point. The gauge dependence of the critical exponent turned out to be reasonably small in a certain parameter range. At the moment we are investigating whether it is possible to further eliminate the dependence with another type of cutoff.

Outlook

Toward more general computation of divergences

As we have understood so far, in order to compute loop divergences in MAG one needs to develop a computational algorithm that can deal with nonminimal operators of the general form. Here we discuss a project in progress, in which such an algorithm was realised by means of computer algebra techniques and applied to the Antisymmetric MAG. The starting point in it is equation (1.54). The first task in it is to find the curved space propagator, which is the solution to the equation

$$F_\lambda G_\lambda = 1 \ . \tag{8.29}$$

This can be done in the following way. First, we find the solution of the corresponding equation in flat space:

$$F_0 \ G_0 = 1, \qquad G_0 = G_\lambda|_{p_\mu \to \nabla_\mu, \ R \to 0, \ T \to 0} \tag{8.30}$$

where the operator F_0 is obtained from F_λ by replacing the covariant derivatives with vectors (or performing a Fourier transform) and putting background curvatures and torsion tensors to zero. One can use an ansatz for G_0 with arbitrary coefficients and then find a solution for them. Then we replace momenta vectors back with covariant derivatives in G_0, ordering can be arbitrary. After plugging it back to (8.35) instead of the full G we get

$$F \ G_0|_{\nabla_\mu \to p_\mu} = 1 + M(\bar{\nabla}, \bar{R}, \bar{T}) \tag{8.31}$$

Now the full G can be expressed as a geometric series in $-M$:

$$G = G_0 \frac{1}{1+M} = G_0 \left[1 - M + M^2 - M^3 + M^4 - \dots\right] \tag{8.32}$$

And important notion at this stage is the one of the background dimensionality of a differential operator, which is defined as the mass dimension of the background quantities that enter in it. If derivatives act on the background quantity, they add 1 to the background dimensionality. If the operator contains several terms, the background dimensionality is the least one of them. As long as (8.30) is satisfied, M will contain terms of background dimensionality of at least one. We are looking for the one-loop contributions to the EA that are of the same form as the terms in (7.1), they will correspond to logarithmic divergences. Therefore, calculation can be seen as an expansion in terms of the background dimensionality, and we only need to keep the terms up to order 4. The geometric series in (8.32) is needed to be considered only up to order 4.

Let us look at the case when mass parameters are present in F. If the aforementioned procedure is applied directly, the inversion coefficients will come out momenta-dependant.

However, this complication is unnecessary due to the following [1]. Let us assign background dimensionality for masses. This by itself is harmless, since background dimensionality solely controls the expansion and tells us terms of what type we are looking for. Then F_0 will be the principal part of the operator and a solution for the inversion coefficients as rational functions of dimensionless couplings can be found relatively easily. Geometric series (8.32) will now be infinite, because when mass parameters are not small (in our case they can be of the order of Planck mass) and terms that are higher order in M will also be large. However, the same argument as before applies and such terms will contribute only to the convergent part of the EA [2].

Let us split the kinetic operator as

$$F(\lambda) = X(\lambda) + Y, \tag{8.33}$$

where X contains all terms with background dimension zero and lambda multiplies nonminimal terms and

$$X(\lambda) = F_{min.} + \lambda N. \tag{8.34}$$

We assume that F and X are invertable and positive definite, the latter means that all their eigenvalues are positive. Then we can define the "propagator" as the full (including curvature corrections) inverse of X

$$X(\lambda)G(\lambda) = \mathbf{1}. \tag{8.35}$$

and the unique positive definite square root of X. Introducing identity operators on the left and on the right we have

$$\begin{aligned}
&\frac{1}{2}Tr\ log\ [X+Y] \\
=&\frac{1}{2}Tr\ log\ \left[\sqrt{X}\frac{1}{\sqrt{X}}(X+Y)\frac{1}{\sqrt{X}}\sqrt{X}\right] \\
=&\frac{1}{2}Tr\ log\ X + \frac{1}{2}Tr\ log\ \left[1+\frac{1}{\sqrt{X}}Y\frac{1}{\sqrt{X}}\right] \\
=&\frac{1}{2}Tr\ log\ F_{min.} + \frac{1}{2}\int_0^1 d\lambda\ Tr\ [NG(\lambda)] + \frac{1}{2}Tr\ \left[YG - \frac{1}{2}YGYG\right]
\end{aligned} \tag{8.36}$$

In the third row we used the identity

$$tr\ log(AB) = tr\ logA + tr\ logB, \tag{8.37}$$

which is valid for any (non-commuting) positive definite operators, while in the last row we used (1.54) and the trace cyclic identity. Since Y contains terms proportional to at least one power of torsion or curvature we can replace G with its flat-space version G_0 in the last term:

$$Tr\ \left[YG - \frac{1}{2}YGYG\right] = Tr\ \left[YG_0(1-M) - \frac{1}{2}YG_0YG_0\right] \tag{8.38}$$

[1] I owe this trick to Christian Steinwachs.

[2] The only situation when things could go wrong is if operators G_0 or N contained inverse dimensions of mass, that could happen if we chose the field space metric to be dimensionfull. Then inside the traces of the second term in (1.54) ratios of dimensionfull couplings could appear. This explains the advantages of the simple form of the metric (A.4).

Then, using the definition of G (8.35) and the fact that the trace of unity does not contribute to logarithmic divergence we have:

$$Tr\left[G(\lambda)X(\lambda)\right]_{log.div.} = Tr\left[G|_{\lambda=0}\, F_{min.}\right]_{log.div.} = Tr1|_{log.div.} = 0 \tag{8.39}$$

Which allows us to rewrite the term under the integral as

$$\begin{aligned} &Tr\left[N\ G(\lambda)\right] = \frac{1}{\lambda}Tr\left[\left(X(\lambda) - F_{min.}\right)G(\lambda)\right] = \frac{1}{\lambda}Tr\left[1 - F_{min.}G(\lambda)\right] \\ &= \frac{1}{\lambda}Tr\left[G(\lambda)F_{min.}\right] = \frac{1}{\lambda}Tr\left[\left(G(\lambda) - G|_{\lambda=0}\right)F_{min.}\right]. \end{aligned} \tag{8.40}$$

Notice that these identities hold even when integrated over λ, because we first take the traces and then integrate. Therefore, within our approximations we obtain for the logarithmically divergent part of the effective action:

$$\begin{aligned} &\Gamma^{1-loop}_{log.div.} = \frac{1}{2}Tr\ log\ F_{min.} + \int_0^1 \frac{d\lambda}{\lambda}\ Tr\left[\frac{1}{2}\left(G(\lambda) - G|_{\lambda=0}\right)F_{min.}\right] \\ &+ Tr\ \left[\frac{1}{2}YG_0\left(1 - M - \frac{1}{2}YG_0\right)\right]_{\lambda=1} - Tr\ log\ \Delta_{gh.} - \frac{1}{2}Tr\ log\ \left(Y_{gh.}\right) \end{aligned} \tag{8.41}$$

By means of computer algebra, in principle, one can compute one-loop divergences in any theory of fields defined on curved space-time, that may dynamically mix with gravity. At this moment, such calculation for the general Poincaré gauge theory is in progress.

Appendix A

Background field method and kinetic operators

When the background field method is applied dynamical variables are split into their classical background value and quantum perturbations. Consider linear perturbations of the metric and torsion field about a generic background:

$$\begin{aligned} g_{\mu\nu} &= \bar{g}_{\mu\nu} + h^{\mu\nu} \\ T_{\rho}{}^{\lambda}{}_{\eta} &= \bar{T}_{\rho}{}^{\lambda}{}_{\eta} + t_{\rho}{}^{\lambda}{}_{\eta}, \end{aligned} \tag{A.1}$$

where the bar stands for background values. Using the condensed DeWitt notations the latter can be represented as a column φ^a, where index a carries both all the internal structure and the space-time dependence. We will call the linear space of all φ^a the configuration or field space V. Therefore, in the case at consideration we have:

$$\varphi^a = \begin{pmatrix} h_{\mu\nu} \\ t_{\rho}{}^{\lambda}{}_{\eta} \end{pmatrix} (x). \tag{A.2}$$

On the configuration space one can define metric, torsion and nonmetricity, which can be arbitrary in general. Since we have chosen to work with the Levi-Civita derivative the latter two vanish. The kinetic operator is defined as a linear differential operator acting on the space of linear perturbations of fields, $F : V \to V$. For operators of this type one can define determinant, trace and logarithm which are basis-independent, meaning give the same result if a linear transformation of fields φ^a with a unite Jacobian is applied. If the Jacobian is not unity the value of traces may differ, but such a change will be compensated by the corresponding change of the functional measure. The Hessian on the other hand is a bilinear form, acting from a Cartesian product of field spaces to real numbers, $H : V \times V \to \mathbb{R}$. One cannot define a trace of a Hessian. We must instead transform it into a proper operator first, by multiplying it with an inverse field space metric:

$$F^a{}_b = \mathcal{G}^{ac} H_{cb}. \tag{A.3}$$

The metric $\mathcal{G}$ is a bilinear form itself which in general can be quite arbitrary. The results for divergences will contain a nontrivial dependence on the choice of this metric. Let us focus on the simplest case of ultralocal metric, meaning not containing derivatives:

$$\mathcal{G}_{ab} = \begin{pmatrix} 1_s{}^{\alpha\beta,\ \mu\nu} - A\ g^{\alpha\beta} g^{\mu\nu} & 0 \\ 0 & 1_{as}{}^{\gamma}{}_{\delta}{}^{\zeta,\ \rho}{}_{\lambda}{}^{\eta} - B\ g^{\gamma\rho} g_{\delta\lambda} g^{\zeta\eta} - C\ \delta^{\zeta}_{\delta} \delta^{\eta}_{\lambda} g^{\gamma\rho} \end{pmatrix} \delta(x - x'). \tag{A.4}$$

It is nondegenerate if $A \neq 1/4$ and $b \neq 1$ and $B \neq -2$ and $C \neq -(2+B)/3$(in 4 dimensions). Here 1_s and 1_{as} are identities in the spaces of symmetric 2-rank and antisymmetric 3-rank tensors correspondingly. For $A = 1/2$ the metric part of it represents the DeWitt metric which is usually used in metric theories of gravity. In principle, there could be two additional overall factors in (A.4) which will however drop out from the final result.

Appendix B

Multiplication of pseudo-differential operators

The main sources of difficulty in such computations are the large number of terms in intermediate computations, the large number of derivatives in each term and the complicated tensor structure of each term. Operators under consideration contain background structures (such as curvatures, torsion tensors and derivatives acting on them) and several derivatives acting on perturbation fields.[1] When trying to calculate the product of two such operators, it is easy to get overwhelmed by the complexity. This is mainly the reason why there has not been much progress in the field since the late '90s (see however [78, 154–160]). In what follows we try to shed some light on how to manipulate such structures, aiming to accelerate the work in this direction.

From the technical point of view, the challenge is about writing an efficient computer code that can perform three following operations. The first one is multiplication of pseudo-differential operators, containing derivatives and rational functions of $\Box$'s. The second one is sorting the covariant derivatives, in such a way that all the contracted derivatives stand on the very right (or very left). The third one is the replacement of terms of mass dimension 4 with contracted terms from the Lagrangian (explained below). After such replacement, some previously uncontracted derivatives will become contracted, and one can try and sort the derivatives again. It is technically advantageous to perform the second and the third operations after each multiplication several times in order to reduce the number of derivatives in the expressions as much as possible before going to the next step.

It is more efficient to sort the derivatives so that they contract each other rather than trying to bring different terms to the same form.

Sometimes further simplification is possible due to the symmetries of the tensors involved. For example, a term of the following from:

$$R^{\mu\nu\rho\lambda}\nabla \dots \nabla_\mu \dots \nabla_\nu \dots \hat{h}_{\alpha\beta} \tag{B.1}$$

can be replaced with several terms each of them having two derivatives less than the original one however proportional to two Riemann tensors, after which the third operation can be

[1] In the case of dimension 4 truncation of Antisymmetric MAG we deal with 8-index background structures. Therefore, an operator acting in the space of perturbations of the torsion can have up to $8 + 3 \times 2 = 14$ uncontracted derivatives. The matrix M, which is what we expand in, cannot have more than 3 derivatives, however, it has around 1000 terms.

applied again, etc.

Contributions from commutators

For an arbitrary operator X and function f, using the Laplace transform, one can derive the following formula [79]:

$$[X, f(\Box)] = \sum_{n=1}^{\infty} \frac{1}{n!}(-1)^{n-1} \left[X \, , \Box \right]_n f^{(n)}(\Box) \tag{B.2}$$

We define the special product $\otimes$ of two pseudo-differential operators as the product computed in the usual way except that all (functions of) boxes are assumed to commute with derivatives:

$$O_1(\nabla) f_1(\Box) \otimes O_2(\nabla) f_2(\Box) \equiv O_1(\nabla) \otimes O_2(\nabla) f_1(\Box) f_2(\Box). \tag{B.3}$$

Technically this means that while computing it with a computer program boxes can be treated as mere constants. Then it is easy to see using (B.2) that for the usual product $\times$ of arbitrary operators L and R we have

$$L \times R = L \otimes R + L' \otimes [\Box, R] + \frac{1}{2} L'' \otimes [\Box, R]_2 + \frac{1}{3!} L''' \otimes [\Box, R]_3 + \ldots, \tag{B.4}$$

where primes stand for derivatives with respect to $\Box$ and the dots denote terms with higher background dimensionality. This way instead of computing all the corrections due to the commutators separately, one can put everything into one expression. For example, we have

$$\begin{aligned} M^2 &= M \otimes M + M' \otimes [\Box, M] + \frac{1}{2} M'' \otimes [\Box, M]_2 + \ldots, \\ M^3 &= M^2 \otimes M + (M^2)' \otimes [\Box, M] + \ldots, \\ M^4 &= M^3 \otimes M + \ldots, \end{aligned} \tag{B.5}$$

and for the curved space propagator we get from (8.32):

$$\begin{aligned} G =& G_0 \otimes \left(1 - M + M^2 - M^3 + M^4\right) + G_0' \otimes \left[\Box, -M + M^2 - M^3\right] \\ &+ \frac{1}{2} G_0'' \otimes \left[\Box, -M + M^2\right]_2 + \frac{1}{3!} G_0''' \otimes [\Box, -M]_3 + \ldots \end{aligned} \tag{B.6}$$

Replacement of dimension four terms inside traces

The third operation was discussed in [78], section 6, and concerns only the terms of the maximal background dimesionality that we allow in our truncation of the EA - in our case, four. Eventually, all we want to compute are traces, and such terms can be traces with metric only. This means we can replace them with scalar structures multiplied by tensors $\mathcal{T}(g)$ which are constructed with metric tensors only. Therefore, at any intermediate stage of the calculation we are allowed to make the following replacements:[2]

$$
\begin{aligned}
&\bar{R}_{\alpha\beta\gamma\delta}\bar{R}_{\mu\nu\rho\lambda} \to \mathcal{T}^{RR1}_{\alpha\beta\gamma\delta\mu\nu\rho\lambda}H^{RR}_1 + \mathcal{T}^{RR2}_{\alpha\beta\gamma\delta\mu\nu\rho\lambda}H^{RR}_2 + \mathcal{T}^{RR3}_{\alpha\beta\gamma\delta\mu\nu\rho\lambda}H^{RR}_3,\\
&\bar{\nabla}_\alpha\bar{T}_{\beta\gamma\delta}\bar{\nabla}_\mu\bar{T}_{\nu\rho\lambda} \to \sum_{i=1}^{9}\mathcal{T}^{TTi}_{\alpha\beta\gamma\delta\mu\nu\rho\lambda}H^{TT}_i,\\
&\bar{T}_{\beta\gamma\delta}\bar{\nabla}_\alpha\bar{\nabla}_\mu\bar{T}_{\nu\rho\lambda} \to -\sum_{i=1}^{9}\mathcal{T}^{TTi}_{\alpha\beta\gamma\delta\mu\nu\rho\lambda}H^{TT}_i,\\
&\bar{R}_{\mu\nu\rho\lambda}\bar{\nabla}_\mu\bar{T}_{\nu\rho\lambda} \to \mathcal{T}^{RT3}_{\alpha\beta\gamma\delta\mu\nu\rho\lambda}H^{RT}_3 + \mathcal{T}^{RT5}_{\alpha\beta\gamma\delta\mu\nu\rho\lambda}H^{RT}_5,\\
&\bar{T}_{\nu\rho\lambda}\bar{\nabla}_\mu\bar{R}_{\mu\nu\rho\lambda} \to -\mathcal{T}^{RT3}_{\alpha\beta\gamma\delta\mu\nu\rho\lambda}H^{RT}_3 - \mathcal{T}^{RT5}_{\alpha\beta\gamma\delta\mu\nu\rho\lambda}H^{RT}_5,
\end{aligned}
\tag{B.7}
$$

The structures $\mathcal{T}$ are complicated, but we only need to compute them once.

Computing terms of Γ separately

In principle, one can compute all contributions to the EA up to a certain order in background dimensionality at once. In this way one has to keep the background arbitrary in order to distinguish the counterterms from each other. However, in practice it is easier to perform several separate computations, each of them aiming to a specific term (group of terms), while considering the backgrounds that are only necessary to distinguish such term(s) from the others. This way we compute the same EA, but piece by piece. Indeed, let us say we want to compute the counterterms proportional to the kinetic terms for torsion, $(\nabla T)^2$. This means that one can assume that the background metric is flat at any intermediate stage of computation. Furthermore,

$$[G_0, \Box] \overset{(\nabla T)^2}{\simeq} 0. \tag{B.8}$$

However,

$$[M, \Box] \overset{(\nabla T)^2}{\neq} 0. \tag{B.9}$$

The relevant contributions to the propagator can be then computed as

$$G \overset{(\nabla T)^2}{\simeq} G_0\left[1 + M + M\otimes M - M'\otimes[M,\Box] + \frac{1}{2}M''\otimes[M,\Box]_2\right], \tag{B.10}$$

where

$$M \overset{(\nabla T)^2}{\simeq} F\otimes G - 1. \tag{B.11}$$

When computing the running of m_0, α, β we set $\bar{T} = 0$. In the last computation we only keep the five contributions of the type $R^{\cdots}\nabla_{.}T_{\dots}$.

[2]In [78] it was correctly stated that after performing the Ricci decomposition (expressing the Riemann tensors in terms of Weyl tensor, traceless Ricci tensor and Ricci scalar) the number of invariants will be the lowest. However, the commutation of covariant derivatives produces Riemann tensors, and in this case one has to constantly jump from one basis to another.

Further remarks

To perform the calculation computer algebra packages *xTensor* [161], *Invar* [162, 163], *Sym-Manipulator* [164], and *xTras* [165] were used. It was found to be necessary to introduce some small modifications of certain functions of *xTras* to make them efficient for such involved computations. In short, one has to make sure that the *Expand* function is not applied whenever it is not needed. It remains to investigate whether the usage of other computer algebra systems such as *Cadabra* [166, 167] can give a significant gain in computational efficiency.

Appendix C

Trace technology

C.1 Universal Functional Traces

Here we discuss the procedure of calculating the traces of type

$$\mathrm{Tr}[\nabla_{\mu_1} \dots \nabla_{\mu_N} f(\Delta)] \tag{C.1}$$

with f being a smooth function, $\Delta = -\nabla^\mu \nabla_\mu + E$ is a minimal Laplace-type operator, ∇ is the Levi-Civita derivative and E contains no derivatives. They are commonly referred to as the Universal Functional Traces. The first step is to rewrite the expression in terms of symmetrised derivatives. Then for the function f we introduce its Laplace transform

$$f(\Delta) = \int_0^\infty ds \quad e^{-s\Delta} \tilde{f}(s)\,, \tag{C.2}$$

which allows us to write them in terms of the generalised heat kernel traces:

$$\begin{aligned} \mathrm{tr}[\nabla_{(\mu_1} \dots \nabla_{\mu_N)} f(\Delta)] &= \int d^d x \sqrt{g} \int_0^\infty ds \;\; \langle x | \, \nabla_{(\mu_1} \dots \nabla_{\mu_N)} e^{-s\Delta} \, | x \rangle \, \tilde{f}(s) = \\ &= Tr \int d^d x \sqrt{g} \, \int_0^\infty ds \; H_{(\mu_1 \dots \mu_N)}(x,s) \tilde{f}(s). \end{aligned} \tag{C.3}$$

Expressions of H via off-diagonal heat kernel can be found in [79]. Dependence on s can be factorised as

$$H_{(\mu_1 \dots \mu_N)}(x,s) = \frac{1}{(4\pi s)^{d/2}} \sum_{n \geq 0} s^{n - \lfloor N/2 \rfloor} K^{(n)}_{(\mu_1 \dots \mu_N)}(x)\,, \tag{C.4}$$

with $\lfloor x \rfloor$ being the floor function and

$$\begin{aligned} K^{(n)}(x) =& \overline{a_n} \\ K^{(n)}_\mu(x) =& \overline{\nabla_\mu a_n} \\ K^{(n)}_{(\mu\nu)}(x) =& -\frac{1}{2} g_{\mu\nu} \overline{a_n} + \overline{\nabla_{(\mu} \nabla_{\nu)} a_{n-1}} \\ K^{(n)}_{(\mu\nu\rho)}(x) =& -\frac{3}{2} g_{(\rho\nu} \overline{\nabla_{\mu)} a_n} + \overline{\nabla_{(\rho} \nabla_\nu \nabla_{\mu)} a_{n-1}} \\ K^{(n)}_{(\mu\nu\rho\lambda)}(s) =& \frac{3}{4} g_{(\lambda\rho} g_{\nu\mu)} \overline{a_n} - 3 g_{(\lambda\rho} \overline{\nabla_\nu \nabla_{\mu)} a_{n-1}} + \overline{\nabla_{(\lambda} \nabla_\rho \nabla_\nu \nabla_{\mu)} a_{n-2}} \end{aligned} \tag{C.5}$$

$$K^{(n)}_{(\mu\nu\rho\lambda\alpha)}(x) = \frac{15}{4} g_{(\alpha\lambda} g_{\rho\nu} \overline{\nabla_{\mu)} a_n} - 5 g_{(\alpha\lambda} \overline{\nabla_\rho \nabla_\nu \nabla_{\mu)} a_{n-1}} + \overline{\nabla_{(\alpha} \nabla_\lambda \nabla_\rho \nabla_\nu \nabla_{\mu)} a_{n-2}}$$

$$K^{(n)}_{(\mu\nu\rho\lambda\alpha\beta)}(x) = -\frac{15}{8} g_{(\beta\alpha} g_{\lambda\rho} g_{\nu\mu)} \overline{a_n} + \frac{45}{4} g_{(\beta\alpha} g_{\lambda\rho} \overline{\nabla_\nu \nabla_{\mu)} a_{n-1}} - \frac{15}{2} g_{(\beta\alpha} \overline{\nabla_\lambda \nabla_\rho \nabla_\nu \nabla_{\mu)} a_{n-2}} + \overline{\nabla_{(\beta} \nabla_\alpha \nabla_\lambda \nabla_\rho \nabla_\nu \nabla_{\mu)} a_{n-3}}$$

The bar stands for the coincidence limit of the off-diagonal heat kernel coefficients [76, 79]:

$$\begin{aligned}
\overline{a_0} &= 1\,, \\
\overline{\nabla_\mu a_0} &= 0\,, \\
\overline{\nabla_{(\nu} \nabla_{\mu)} a_0} &= \frac{1}{6} R_{\nu\mu}\,, \\
\overline{\nabla_{(\alpha} \nabla_\nu \nabla_{\mu)} a_0} &= \frac{1}{4} R_{(\nu\mu;\alpha)}\,, \\
\overline{\nabla_{(\beta} \nabla_\alpha \nabla_\nu \nabla_{\mu)} a_0} &= \frac{3}{10} R_{(\nu\mu;\alpha\beta)} + \frac{1}{12} R_{(\beta\alpha} R_{\nu\mu)} + \frac{1}{15} R_{\gamma(\beta|\delta|\alpha} R^{\gamma}{}_{\nu}{}^{\delta}{}_{\mu)}\,, \\
\overline{a_1} &= -E + \frac{1}{6} R\,, \\
\overline{\nabla_\mu a_1} &= -\frac{1}{2} E_{;\mu} - \frac{1}{6} \Omega_{\nu\mu;}{}^{\nu} + \frac{1}{12} R_{;\mu}\,, \\
\overline{\nabla_{(\nu} \nabla_{\mu)} a_1} &= -\frac{1}{3} E_{;(\mu\nu)} - \frac{1}{6} R_{\mu\nu} E - \frac{1}{6} \Omega_{\alpha(\mu;}{}^{\alpha}{}_{\nu)} + \frac{1}{6} \Omega_{\alpha(\nu} \Omega^{\alpha}{}_{\mu)} + \frac{1}{20} R_{;(\mu\nu)} - \frac{1}{60} \Delta R_{\nu\mu} \\
&\quad + \frac{1}{36} R R_{\nu\mu} - \frac{1}{45} R_{\nu\alpha} R^{\alpha}{}_{\mu} + \frac{1}{90} R_{\alpha\beta} R^{\alpha}{}_{\nu}{}^{\beta}{}_{\mu} + \frac{1}{90} R^{\alpha\beta\gamma}{}_{\nu} R_{\alpha\beta\gamma\mu}\,, \\
\overline{a_2} &= \frac{1}{6} \Delta E + \frac{1}{2} E^2 - \frac{1}{6} R E + \frac{1}{12} \Omega_{\mu\nu} \Omega^{\mu\nu} - \frac{1}{30} \Delta R + \frac{1}{72} R^2 - \frac{1}{180} R^2_{\mu\nu} + \frac{1}{180} R^2_{\mu\nu\alpha\beta}\,,
\end{aligned} \tag{C.6}$$

where Ω is defined in (1.49). Then after substituting (C.4) in (C.3) and introducing Q-functionals as

$$Q_m[f] := \int_0^\infty ds\ s^{-m} \tilde{f}(s)\,, \tag{C.7}$$

we obtain a master formula:

$$\mathrm{Tr}\left[\nabla_{(\mu_1} \dots \nabla_{\mu_N)} f(\Delta)\right] = \frac{1}{(4\pi)^{d/2}} \sum_{n\geq 0} Q_{-n+\frac{d}{2}+\lfloor N/2 \rfloor}[f] \cdot tr \int d^d x\ \sqrt{g} K^{(n)}_{(\mu_1 \dots \mu_N)}(x). \tag{C.8}$$

The Q-functionals here can be reexpressed as integrals over momenta, see [27]. For m positive integer we have

$$Q_r[f] = \frac{1}{\Gamma(r)} \int_0^\infty d\Delta\ \Delta^{r-1} f(\Delta), \tag{C.9}$$

whereas for non-positive integer we can choose k such that $m + k > 0$ and then

$$Q_r[f] = \frac{(-1)^k}{\Gamma(r+k)} \int_0^\infty d\Delta\ \Delta^{r+k-1} f^{(k)}(\Delta), \tag{C.10}$$

For r nonnegative integer:

$$Q_{-r}[f] = (-1)^r f^r(0), \tag{C.11}$$

and for $r = 0$

$$Q_0[f] = f(0). \tag{C.12}$$

This way all integrals over s are transformed into the integrals over the momenta.

C.2 Extracting logarithmic divergences

In order to regularise the momenta integrals let us introduce a momenta cutoff Λ. In the traces (C.8) the functions of our interest are:

$$f(\Delta) = \frac{1}{\Delta^m} \tag{C.13}$$

with positive integer m. Let us consider first the case when $r > 0$, $m \neq r$. Then from (C.9) we have

$$Q_r\left[\frac{1}{\Delta^m}\right] = \frac{1}{\Gamma(r)}\int_{\mu^2}^{\Lambda^2} d\Delta \cdot \Delta^{r-1-m} = \frac{1}{\Gamma(r)}\frac{\Delta^{r-m}}{r-m}\bigg|_{\mu^2}^{\Lambda^2}. \tag{C.14}$$

This expression is UV convergent for $r < m$ and power-law divergent for $r > m$. For $r = m > 0$ we get the logarithmically divergent contribution:

$$Q_m\left[\frac{1}{\Delta^m}\right] = \frac{1}{\Gamma(m)}\log\Delta|_{\mu^2}^{\Lambda^2} = \frac{1}{\Gamma(m)}log\left(\frac{\Lambda^2}{\mu^2}\right) \tag{C.15}$$

We will compute the traces with up to four uncontracted derivatives for further convenience and then give the general expression for any number of derivatives. First, we consider the case without uncontracted derivatives. The formula (C.8) gives

$$\mathrm{Tr}\left[\frac{1}{\Delta^m}\right] = \frac{1}{(4\pi)^{d/2}}\sum_{n\geq 0} Q_{\frac{d}{2}-n}\left[\frac{1}{\Delta^m}\right]\cdot tr\int d^dx\sqrt{g}K^{(n)}(x). \tag{C.16}$$

This expression is logarithmically divergent if $n = d/2 - m$, $m > 0$.

From here on we set $d = 4$. Using the first line of (C.5) we get:

$$\mathrm{Tr}\left[\frac{1}{\Delta^m}\right]_{log.div.} = \frac{1}{(4\pi)^2}\frac{1}{\Gamma(m)}log\left(\frac{\Lambda^2}{\mu^2}\right)\cdot tr\int d^4x\sqrt{g}\;\overline{a_{2-m}} \tag{C.17}$$

This formula is inapplicable when $m = 0$ and the expression on the lhs is in fact finite, see below. It is convergent for $m \geq 3$.

For the case of a single uncontracted derivative, in four space-time dimensions,

$$Tr\left[\nabla_\mu\frac{1}{\Delta^m}\right] = \frac{1}{16\pi^2}\sum_{n\geq 0} Q_{2-n}\left[\frac{1}{\Delta^m}\right]\cdot tr\int d^4x\sqrt{g}K_\mu^{(n)}(x) \tag{C.18}$$

which gives logarithmic divergence if $n = 2 - m$. Then the two nontrivial contributions that we obtain from (C.5) are

$$\begin{aligned}
Tr\left[\nabla_\mu\frac{1}{\Delta}\right] &= \frac{1}{16\pi^2}log\left(\frac{\Lambda^2}{\mu^2}\right)\cdot tr\int d^4x\sqrt{g}\;\overline{\nabla_\mu a_1},\\
Tr\left[\nabla_\mu\frac{1}{\Delta^2}\right] &= \frac{1}{16\pi^2}log\left(\frac{\Lambda^2}{\mu^2}\right)\cdot tr\int d^4x\sqrt{g}\;\overline{\nabla_\mu a_0},\\
Tr\left[\nabla_\mu\frac{1}{\Delta^m}\right] &= 0 \quad \text{for} \quad m \geq 3.
\end{aligned} \tag{C.19}$$

For traces with two uncontracted derivatives, in four space-time dimensions,

$$Tr\left[\nabla_{(\mu}\nabla_{\nu)}\frac{1}{\Delta^m}\right] = \frac{1}{16\pi^2}\sum_{n\geq 0} Q_{3-n}\left[\frac{1}{\Delta^m}\right]\cdot tr\int d^4x\sqrt{g}K^{(n)}_{(\mu\nu)}(x) \tag{C.20}$$

gives logarithmic divergence if $n = 3 - m$. Therefore,

$$\begin{aligned}
Tr\left[\nabla_{(\mu}\nabla_{\nu)}\frac{1}{\Delta}\right] &= \frac{1}{16\pi^2}log\left(\frac{\Lambda^2}{\mu^2}\right)\cdot tr\int d^4x\sqrt{g}\left(-\frac{1}{2}g_{\mu\nu}\overline{a_2} + \overline{\nabla_{(\mu}\nabla_{\nu)}a_1}\right), \\
Tr\left[\nabla_{(\mu}\nabla_{\nu)}\frac{1}{\Delta^2}\right] &= \frac{1}{16\pi^2}log\left(\frac{\Lambda^2}{\mu^2}\right)\cdot tr\int d^4x\sqrt{g}\left(-\frac{1}{2}g_{\mu\nu}\overline{a_1} + \overline{\nabla_{(\mu}\nabla_{\nu)}a_0}\right) \\
Tr\left[\nabla_{(\mu}\nabla_{\nu)}\frac{1}{\Delta^3}\right] &= \frac{1}{16\pi^2}\frac{1}{2}\ log\left(\frac{\Lambda^2}{\mu^2}\right)\cdot tr\int d^4x\sqrt{g}\left(-\frac{1}{2}g_{\mu\nu}\overline{a_0}\right) \\
Tr\left[\nabla_{(\mu}\nabla_{\nu)}\frac{1}{\Delta^m}\right] &= 0 \quad \text{for} \quad m \geq 4
\end{aligned} \tag{C.21}$$

For traces with three uncontracted derivatives, again in four space-time dimensions,

$$Tr\left[\nabla_{(\mu}\nabla_{\nu}\nabla_{\rho)}\frac{1}{\Delta^m}\right] = \frac{1}{16\pi^2}\sum_{n\geq 0} Q_{3-n}\left[\frac{1}{\Delta^m}\right]\cdot tr\int d^4x\sqrt{g}K^{(n)}_{(\mu\nu\rho)}(x) \tag{C.22}$$

again gives logarithmic divergence if $n = 3 - m$. We have

$$\begin{aligned}
Tr\left[\nabla_{(\mu}\nabla_{\nu}\nabla_{\rho)}\frac{1}{\Delta}\right] &= \frac{1}{16\pi^2}log\left(\frac{\Lambda^2}{\mu^2}\right)\cdot tr\int d^4x\sqrt{g}\left(-\frac{3}{2}g_{(\rho\nu}\overline{\nabla_{\mu)}a_2} + \overline{\nabla_{(\rho}\nabla_{\nu}\nabla_{\mu)}a_1}\right), \\
Tr\left[\nabla_{(\mu}\nabla_{\nu}\nabla_{\rho)}\frac{1}{\Delta^2}\right] &= \frac{1}{16\pi^2}log\left(\frac{\Lambda^2}{\mu^2}\right)\cdot tr\int d^4x\sqrt{g}\left(-\frac{3}{2}g_{(\rho\nu}\overline{\nabla_{\mu)}a_1} + \overline{\nabla_{(\rho}\nabla_{\nu}\nabla_{\mu)}a_0}\right) \\
Tr\left[\nabla_{(\mu}\nabla_{\nu}\nabla_{\rho)}\frac{1}{\Delta^3}\right] &= \frac{1}{16\pi^2}\frac{1}{2}\ log\left(\frac{\Lambda^2}{\mu^2}\right)\cdot tr\int d^4x\sqrt{g}\left(-\frac{3}{2}g_{(\rho\nu}\overline{\nabla_{\mu)}a_0}\right) \\
Tr\left[\nabla_{(\mu}\nabla_{\nu}\nabla_{\rho)}\frac{1}{\Delta^m}\right] &= 0 \quad \text{for} \quad m \geq 4
\end{aligned} \tag{C.23}$$

And finally, traces with four uncontracted derivatives are

$$Tr\left[\nabla_{(\mu}\nabla_{\nu}\nabla_{\rho}\nabla_{\lambda)}\frac{1}{\Delta^m}\right] = \frac{1}{16\pi^2}\sum_{n\geq 0} Q_{4-n}\left[\frac{1}{\Delta^m}\right]\cdot tr\int d^4x\sqrt{g}K^{(n)}_{(\mu\nu\rho\lambda)}(x) \tag{C.24}$$

again give logarithmic divergence if $n = 4 - m$. We have

$$\begin{aligned}
Tr\left[\nabla_{(\mu}\nabla_{\nu}\nabla_{\rho}\nabla_{\lambda)}\frac{1}{\Delta}\right] &= \frac{1}{16\pi^2}log\left(\frac{\Lambda^2}{\mu^2}\right) tr\int d^4x\sqrt{g}\left(\frac{3}{4}g_{(\lambda\rho}g_{\nu\mu)}\overline{a_3} - 3g_{(\lambda\rho}\overline{\nabla_{\nu}\nabla_{\mu)}a_2} + \overline{\nabla_{(\lambda}\nabla_{\rho}\nabla_{\nu}\nabla_{\mu)}a_1}\right), \\
Tr\left[\nabla_{(\mu}\nabla_{\nu}\nabla_{\rho}\nabla_{\lambda)}\frac{1}{\Delta^2}\right] &= \frac{1}{16\pi^2}log\left(\frac{\Lambda^2}{\mu^2}\right) tr\int d^4x\sqrt{g}\left(\frac{3}{4}g_{(\lambda\rho}g_{\nu\mu)}\overline{a_2} - 3g_{(\lambda\rho}\overline{\nabla_{\nu}\nabla_{\mu)}a_1} + \overline{\nabla_{(\lambda}\nabla_{\rho}\nabla_{\nu}\nabla_{\mu)}a_0}\right) \\
Tr\left[\nabla_{(\mu}\nabla_{\nu}\nabla_{\rho}\nabla_{\lambda)}\frac{1}{\Delta^3}\right] &= \frac{1}{16\pi^2}\frac{1}{2}\ log\left(\frac{\Lambda^2}{\mu^2}\right) tr\int d^4x\sqrt{g}\left(\frac{3}{4}g_{(\lambda\rho}g_{\nu\mu)}\overline{a_1} - 3g_{(\lambda\rho}\overline{\nabla_{\nu}\nabla_{\mu)}a_0}\right) \\
Tr\left[\nabla_{(\mu}\nabla_{\nu}\nabla_{\rho}\nabla_{\lambda)}\frac{1}{\Delta^4}\right] &= \frac{1}{16\pi^2}\frac{1}{6}\ log\left(\frac{\Lambda^2}{\mu^2}\right) tr\int d^4x\sqrt{g}\ \frac{3}{4}g_{(\lambda\rho}g_{\nu\mu)}\overline{a_0} \\
Tr\left[\nabla_{(\mu}\nabla_{\nu}\nabla_{\rho}\nabla_{\lambda)}\frac{1}{\Delta^m}\right] &= 0 \quad \text{for} \quad m \geq 5
\end{aligned} \tag{C.25}$$

In general, for traces of the type (C.8) with (C.13), the logarithmic divergence comes from the n-th term of the expansion for which the following equation is satisfied:

$$-n + \frac{d}{2} + \lfloor N/2 \rfloor = m \tag{C.26}$$

or

$$n = \frac{d}{2} + \lfloor N/2 \rfloor - m. \tag{C.27}$$

Therefore, for any positive integer m:

$$\mathrm{Tr}\left[\nabla_{(\mu_1} \dots \nabla_{\mu_N)} \frac{1}{\Delta^m}\right]_{log.div.} = \frac{1}{(4\pi)^{d/2}} \frac{1}{\Gamma(m)} log\left(\frac{\Lambda^2}{\mu^2}\right) \cdot tr \int d^d x \ \sqrt{g} K^{\left(\frac{d}{2}+\lfloor N/2 \rfloor - m\right)}_{(\mu_1 \dots \mu_N)}(x) \tag{C.28}$$

The case of $m = 0$ should be considered separately. For $r > 0$ gives power divergences

$$Q_r[1] = \frac{1}{\Gamma(r)} \int_{\mu^2}^{\Lambda^2} d\Delta \cdot \Delta^{r-1} = \frac{1}{\Gamma(r)} \frac{\Delta^r}{r}\bigg|_{\mu^2}^{\Lambda^2}, \tag{C.29}$$

whilst for $r = m = 0$ it is convergent

$$Q_0[1] = 1. \tag{C.30}$$

Therefore we conclude that

$$\mathrm{Tr}\left[\nabla_{(\mu_1} \dots \nabla_{\mu_N)}\right]_{log.div.} = 0. \tag{C.31}$$

C.3 Second order minimal operator

Let us now consider a minimal Laplace-type operator of the form

$$\Delta + E \ , \tag{C.32}$$

acting on fields that may carry either spacetime or internal indices, where

$$\Delta = \nabla^2$$

and E is an endomorphism (a linear map acting on the fields, without derivatives). The covariant derivative ∇ is constructed with the Levi-Civita connection on spacetime indices, plus any other connection may be present that acts on the internal indices.

First, we perform a simple consistency check and derive a formula for the logarithmically divergent contribution of this operator using (C.17). Since we are looking at high momenta divergences we can assume that $\Delta \gg E$ when E contains no or one derivative. Then, using the relation

$$tr \ log(AB) = tr \ logA + tr \ logB, \tag{C.33}$$

which is valid for any positive definite operators A and B, and expanding the logarithm we have:

$$\begin{aligned} &\frac{1}{2} Tr \ log\left(\Delta + E\right) = \frac{1}{2} Tr \ log\left[\left(1 + E\frac{1}{\Delta}\right)\Delta\right] = \\ &= \frac{1}{2} Tr \ log\Delta + \frac{1}{2} Tr\left[E\frac{1}{\Delta} - \frac{1}{2} E\frac{1}{\Delta}E\frac{1}{\Delta} + \frac{1}{3} E\frac{1}{\Delta}E\frac{1}{\Delta}E\frac{1}{\Delta} - \dots\right]. \end{aligned} \tag{C.34}$$

For the divergent part, the expansion is finite because E possesses a positive background dimensionality. Then in four space-time dimensions we have

$$\frac{1}{2}\mathrm{Tr}\log\Delta|_{log.div.} = -\frac{1}{32\pi^2}\int d^4x\sqrt{g}\ \log\left(\frac{\Lambda^2}{\mu^2}\right)\overline{a_2\left(\Delta\right)} \tag{C.35}$$

Focusing on the case when E does not contain derivatives and commutes with Δ we get from (C.17):

$$\mathrm{Tr}\left[E\frac{1}{\Delta}\right]_{log.div.} = \frac{1}{16\pi^2}log\left(\frac{\Lambda^2}{\mu^2}\right)\cdot tr\int d^4x\sqrt{g}\ \overline{a_1\left(\Delta\right)}E, \tag{C.36}$$

$$\mathrm{Tr}\left[E^2\frac{1}{\Delta^2}\right]_{log.div.} = \frac{1}{16\pi^2}log\left(\frac{\Lambda^2}{\mu^2}\right)\cdot tr\int d^4x\sqrt{g}\ \overline{a_0\left(\Delta\right)}E^2, \tag{C.37}$$

$$\mathrm{Tr}\left[E^m\frac{1}{\Delta^m}\right]_{log.div.} = 0 \quad for\ m\geq 3. \tag{C.38}$$

When $\Delta = -g^{\mu\nu}\nabla_\mu\nabla_\nu$ we have $a_0 = 1$ and $a_1 = R/6$, and we get

$$\begin{aligned}&\frac{1}{2}Tr\left[log\Delta + E\frac{1}{\Delta} - \frac{1}{2}E\frac{1}{\Delta}E\frac{1}{\Delta} + \dots\right]_{log.div.} = \\ &-\frac{1}{32\pi^2}\int d^4x\sqrt{g}\ log\left(\frac{\Lambda^2}{\mu^2}\right)\left[\overline{a_2\left(\Delta\right)} - \overline{a_1\left(\Delta\right)}E + \frac{1}{2}\overline{a_0\left(\Delta\right)}E^2 - \dots\right]\end{aligned} \tag{C.39}$$

using (1.48) for $E = 0$ we obtain

$$\frac{1}{2}Tr\ log\left(\Delta + E\right) = -\frac{1}{32\pi^2}\int d^4x\sqrt{g}\ log\left(\frac{\Lambda^2}{\mu^2}\right)\overline{a_2\left(\Delta + E\right)}, \tag{C.40}$$

which is exactly what was expected.

Now let us consider an operator of the form

$$\Delta + V + W\ , \tag{C.41}$$

where $V = V^\mu\nabla_\mu$ and W is an endomorphism.

Now we find

$$\begin{aligned}\frac{1}{2}\mathrm{Tr}\log\left(\Delta + V + W\right) =&\frac{1}{2}\mathrm{Tr}\log\Delta + \frac{1}{2}\mathrm{Tr}\log\left(1 + V\frac{1}{\Delta} + W\frac{1}{\Delta}\right) \\ =&\frac{1}{2}\mathrm{Tr}\log\Delta + \frac{1}{2}\mathrm{Tr}\Big[V\frac{1}{\Delta} + W\frac{1}{\Delta} - \frac{1}{2}V\frac{1}{\Delta}V\frac{1}{\Delta} - V\frac{1}{\Delta}W\frac{1}{\Delta} - \frac{1}{2}W\frac{1}{\Delta}W\frac{1}{\Delta} + \dots\Big].\end{aligned} \tag{C.42}$$

We stop the expansion at this order since we are not interested in terms with three or more powers of V or W. The leading term is

$$\frac{1}{2}\mathrm{Tr}\log\Delta = -\frac{1}{32\pi^2}\log\left(\frac{\Lambda^2}{\mu^2}\right)\int d^4x\sqrt{g}\ \mathrm{tr}\left[\frac{1}{180}\left(R_{\mu\nu\rho\lambda}R^{\mu\nu\rho\lambda} - R_{\mu\nu}R^{\mu\nu} + \frac{5}{2}R^2 + 6\Box R\right) + \frac{1}{12}\Omega_{\mu\nu}^2\right] \tag{C.43}$$

The term linear in W is

$$\frac{1}{2}\mathrm{Tr}\log\left[W\frac{1}{\Delta}\right] = \frac{1}{2}\frac{1}{(4\pi)^2}\log\left(\frac{\Lambda^2}{\mu^2}\right)\int d^4x\sqrt{g}\,\frac{1}{6}R\,\mathrm{tr}W\ . \tag{C.44}$$

and the term linear in V is

$$\frac{1}{2}\mathrm{Tr}\log\left[V^\mu\nabla_\mu\frac{1}{\Delta}\right]=\frac{1}{2}\frac{1}{(4\pi)^2}\log\left(\frac{\Lambda^2}{\mu^2}\right)\int d^4x\sqrt{g}\left(-\frac{1}{12}\right)R\,\mathrm{tr}\nabla_\mu V^\mu\ . \tag{C.45}$$

In order to evaluate the remaining terms with the universal functional traces computed in the preceding section, we have to bring all inverse powers of Δ together in the extreme right. This involves commutators of V and W with the inverse of Δ. Such commutators can be computed from (B.2), leading to the expansion

$$\left[X,\frac{1}{\Delta}\right]=-\left[X,\Delta\right]\frac{1}{\Delta^2}-\left[X,\Delta\right]_2\frac{1}{\Delta^2}+\dots \tag{C.46}$$

for an arbitrary operator X. The remaining multiple commutators in the r.h.s. have to be computed separately in each case.

We start from the term with two W's, that is easier. We find that

$$[W,\Delta]=(\nabla^2W)+2(\nabla^\mu W)\nabla_\mu\ . \tag{C.47}$$

Since this multiplies $1/\Delta^3$, the first trace is finite. Similarly, the double commutator $[W,\Delta]_2$ contains at most two ∇ on the right, and since this multiplies $1/\Delta^4$, the second trace is also finite. This means that

$$\frac{1}{2}\left(-\frac{1}{2}\right)\mathrm{Tr}\left[W\frac{1}{\Delta}W\frac{1}{\Delta}\right]\approx-\frac{1}{4}\mathrm{Tr}\left[W^2\frac{1}{\Delta^2}\right]=-\frac{1}{4}\frac{1}{(4\pi)^2}\log\left(\frac{\Lambda^2}{\mu^2}\right)\int d^4x\sqrt{g}\,\mathrm{tr}\,W^2\ . \tag{C.48}$$

It is a bit more complicated to evaluate the mixed V-W term. First of all, we can write

$$\begin{aligned}\frac{1}{2}(-1)\mathrm{Tr}\left[V^\mu\nabla_\mu\frac{1}{\Delta}W\frac{1}{\Delta}\right]&=-\frac{1}{2}\mathrm{Tr}\left[V^\mu\nabla_\mu\left(W\frac{1}{\Delta}-\left[W,\frac{1}{\Delta}\right]\right)\frac{1}{\Delta}\right]\\&=-\frac{1}{2}\mathrm{Tr}\left[V^\mu(\nabla_\mu W)\frac{1}{\Delta^2}\right]-\frac{1}{2}\mathrm{Tr}\left[V^\mu W\nabla_\mu\frac{1}{\Delta^2}\right]+\frac{1}{2}\mathrm{Tr}\left[V^\mu\nabla_\mu\left[W,\frac{1}{\Delta}\right]\frac{1}{\Delta}\right]\end{aligned} \tag{C.49}$$

The first term is equal to

$$-\frac{1}{2}\frac{1}{(4\pi)^2}\log\left(\frac{\Lambda^2}{\mu^2}\right)\int d^4x\sqrt{g}\,\mathrm{tr}\,V^\mu\nabla_\mu W\ .$$

The second vanishes because it contains $\overline{\nabla_\mu a_0}$. For the third we use (C.47) to write it as

$$-\frac{1}{2}\mathrm{Tr}\left[2(V^\rho\nabla^\mu W)\nabla_\rho\nabla_\mu\frac{1}{\Delta^3}\right]=\frac{1}{4}\frac{1}{(4\pi)^2}\log\left(\frac{\Lambda^2}{\mu^2}\right)\int d^4x\sqrt{g}\,\mathrm{tr}\,V^\mu\nabla_\mu W\ ,$$

plus terms that contain fewer free ∇'s and are therefore convergent. Adding these two contributions we find

$$-\frac{1}{2}\mathrm{Tr}\left[V^\mu\nabla_\mu\frac{1}{\Delta}W\frac{1}{\Delta}\right]=\frac{1}{4}\frac{1}{(4\pi)^2}\log\left(\frac{\Lambda^2}{\mu^2}\right)\int d^4x\sqrt{g}\,\mathrm{tr}\,W\nabla_\mu V^\mu\ . \tag{C.50}$$

Finally, we come to the term with two V's, which are the most complicated ones. Using the commutation rules,

$$
\begin{aligned}
-\frac{1}{4}\mathrm{Tr}\left[V^\mu\nabla_\mu\frac{1}{\Delta}V^\nu\nabla_\nu\frac{1}{\Delta}\right] &= -\frac{1}{4}\mathrm{Tr}\left[V^\mu\nabla_\mu V^\nu\nabla_\nu\frac{1}{\Delta^2}+V^\mu\nabla_\mu\left[V^\nu\nabla_\nu,\frac{1}{\Delta}\right]\frac{1}{\Delta}\right]\\
&= -\frac{1}{4}\mathrm{Tr}\left[(V^\mu\nabla_\mu V^\nu)\nabla_\nu\frac{1}{\Delta^2}+V^\mu V^\nu\nabla_{(\mu}\nabla_{\nu)}\frac{1}{\Delta^2}+\frac{1}{2}V^\mu V^\nu[\nabla_\mu,\nabla_\nu]\frac{1}{\Delta^2}\right.\\
&\qquad\left.+V^\mu\nabla_\mu[V^\nu\nabla_\nu,\Delta]\frac{1}{\Delta^3}+V^\mu\nabla_\mu[V^\nu\nabla_\nu,\Delta]_2\frac{1}{\Delta^4}+\dots\right]
\end{aligned}
\tag{C.51}
$$

The first term vanishes because it contains $\overline{\nabla_\mu a_0}$. The second evaluates to

$$
-\frac{1}{4}\frac{1}{(4\pi)^2}\log\left(\frac{\Lambda^2}{\mu^2}\right)\int d^4x\sqrt{g}\left(-\frac{1}{12}R\,\mathrm{tr}\,V_\mu V^\mu+\frac{1}{6}R_{\mu\nu}\mathrm{tr}\,V^\mu V^\nu\right)\ . \tag{C.52}
$$

The third term is equal to

$$
-\frac{1}{8}\mathrm{Tr}\left[V^\mu V^\nu\Omega_{\mu\nu}\frac{1}{\Delta^2}\right]=-\frac{1}{16}\frac{1}{(4\pi)^2}\log\left(\frac{\Lambda^2}{\mu^2}\right)\int d^4x\sqrt{g}\,\mathrm{tr}\,\Omega_{\mu\nu}[V^\mu,V^\nu]\ . \tag{C.53}
$$

In the fourth term, we need the single commutator

$$
[V^\mu\nabla_\mu,\Delta]=V^\mu\left(\nabla^\rho\Omega_{\rho\mu}\right)+V^\mu\left(R_{\mu\rho}-2\Omega_{\mu\rho}\right)\nabla^\rho+\left(\Box V^\mu\right)\nabla_\mu+2\left(\nabla^\rho V^\mu\right)\nabla_\rho\nabla_\mu\ . \tag{C.54}
$$

There is an additional ∇ on the left that needs to be brought to the right. When this is done, there are terms with zero, one, two or three free ∇'s. Because of the presence of $1/\Delta^3$, only the ones with two or three ∇'s are logarithmically divergent. The term with three ∇'s gives zero because it contains $\overline{\nabla_\mu a_0}$. The remaining terms with two ∇'s give finally

$$
\begin{aligned}
&-\frac{1}{4}\mathrm{Tr}\left[V^\mu\nabla_\mu\left[V^\nu\nabla_\nu,\Delta\right]\frac{1}{\Delta^3}\right]\Big|_{log.div.}\\
&=\frac{1}{16}\frac{1}{(4\pi)^2}\log\left(\frac{\Lambda^2}{\mu^2}\right)\int d^4x\sqrt{g}\left(\mathrm{tr}\,V_\rho\nabla^2V^\rho+2\mathrm{tr}\,V^\mu\nabla_\mu\nabla_\rho V^\rho+R_{\mu\nu}\mathrm{tr}V^\mu V^\nu+\mathrm{tr}\,\Omega_{\mu\nu}[V^\mu,V^\nu]\right)\ .
\end{aligned}
\tag{C.55}
$$

For the last term, one needs to evaluate the double commutator. It gives

$$
[V^\mu\nabla_\mu,\Delta]_2=-4(\nabla^\rho\nabla^\mu V^\nu)\nabla_\rho\nabla_\mu\nabla_\nu+\dots
$$

where the ellipses stand for terms with fewer free ∇'s. When we further act with $V^\lambda\nabla_\lambda$ from the left and bring ∇_λ to the right, because of the presence of $1/\Delta^4$, the only term that gives a log divergence is

$$
V^\lambda\nabla_\lambda[V^\mu\nabla_\mu,\Delta]_2\approx-4V^\lambda(\nabla^\rho\nabla^\mu V^\nu)\nabla_\lambda\nabla_\rho\nabla_\mu\nabla_\nu\ .
$$

We can assume that the covariant derivatives are totally symmetrised, because all the commutators will decrease the number of ∇'s and produce convergent terms of higher dimension. The result for the fifth term is then

$$
\begin{aligned}
&-\frac{1}{4}\mathrm{Tr}\left[V^\mu\nabla_\mu\left[V^\nu\nabla_\nu,\Delta\right]_2\frac{1}{\Delta^4}\right]\Big|_{log.div.}\\
&=-\frac{1}{24}\frac{1}{(4\pi)^2}\log\left(\frac{\Lambda^2}{\mu^2}\right)\int d^4x\sqrt{g}\left(\mathrm{tr}\,V_\rho\nabla^2V^\rho+\mathrm{tr}\,V^\mu\nabla_\mu\nabla_\rho V^\rho+\mathrm{tr}\,V^\mu\nabla_\nu\nabla_\mu V^\nu\right)\ .
\end{aligned}
\tag{C.56}
$$

We can now sum up all the contributions. There is some freedom in the presentation of the result, and we choose the following way. The terms of the form $V\nabla\nabla V$ sum up to

$$\frac{1}{48}\mathrm{tr}\left(V_\rho\nabla^2 V^\rho + 4V^\rho\nabla_\rho\nabla_\sigma V^\sigma - 2V^\rho\nabla_\sigma\nabla_\rho V^\sigma\right) \ .$$

Half of the third term is left alone, and in the other half, we commute the covariant derivatives, using

$$[\nabla_\mu,\nabla_\nu]V^\rho = R_{\mu\nu}{}^\rho{}_\sigma V^\sigma + [\Omega_{\mu\nu}, V^\rho]$$

In this way, we generate a term cancelling the Ricci term, while the other adds to the ΩVV term. All the remaining terms of the form $V\nabla\nabla V$ are then integrated by parts. In this way, we arrive at our final result

$$\begin{aligned}
&\frac{1}{2}\mathrm{Tr}\log\left(\Delta + V^\mu\nabla_\mu + W\right)|_{log.div.} = \\
&-\frac{1}{2}\frac{1}{(4\pi)^2}\log\left(\frac{\Lambda^2}{\mu^2}\right)\int d^4x\sqrt{g}\,\mathrm{tr}\left[\frac{1}{180}\left(R_{\mu\nu\rho\lambda}R^{\mu\nu\rho\lambda} - R_{\mu\nu}R^{\mu\nu} + \frac{5}{2}R^2\right)\right. \\
&+\frac{1}{2}W^2 - \frac{1}{6}R\,W - \frac{1}{2}W\nabla_\mu V^\mu + \frac{1}{12}R\,\nabla_\mu V^\mu - \frac{1}{24}R\,V^\mu V_\mu \\
&+\frac{1}{12}\,\Omega_{\mu\nu}\Omega^{\mu\nu} - \frac{1}{6}\,\Omega_{\mu\nu}\nabla^\mu V^\nu + \frac{1}{24}\,\Omega_{\mu\nu}[V^\mu, V^\nu] \\
&\left.+\,\frac{1}{8}\,\nabla_\mu V^\mu\nabla_\nu V^\nu + \frac{1}{24}\,\left(\nabla_\mu V_\nu\nabla^\mu V^\nu - \nabla_\mu V_\nu\nabla^\nu V^\mu\right)\right]
\end{aligned} \tag{C.57}$$

We emphasise that there are further log divergences containing higher powers of V and W, that we are not interested in here.

C.4 Other trace formulae

We have fields with values in a vectorbundle V with connection and covariant derivative D_μ. The "gravitational" part of the connection is assumed to be Levi-Civita. We denote $\Delta = -D^2$. Tr stands for functional trace and tr for the finite dimensional trace in the vectorbundle.

Let $F(\Delta)$ be some function of the Laplacian. From [79] we can derive formulae for traces of a function of Δ with insertions of covariant derivatives.

We define the Q-functionals

$$\begin{aligned}
Q_m(F) &= \int_0^\infty ds s^{-m}\tilde{F}(s) \\
&= \frac{1}{\Gamma[m]}\int_0^\infty dz z^{m-1}F(z).
\end{aligned} \tag{C.58}$$

$\tilde{F}$ being the Laplace transform of F. In this section all Q-functionals will always be evaluated on the same function F, therefore we shall often omit the argument for notational compactness:

$$Q_m = Q_M[F] \ .$$

The trace of $F(\Delta)$ can be computed with the usual formula

$$\mathrm{Tr}F = \frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\left\{Q_2[F]b_0 + Q_1[F]b_2 + Q_0[F]b_4 + \ldots\right\} \ . \tag{C.59}$$

where b_n are the usual heat kernel coefficients of the operator Δ. Since the operator will always be the same, we shall not need to write $b_n(\Delta)$.

The untraced heat kernel coefficients are bilocal: $\mathbf{b}_n(x, x')$. The overbar denotes evaluation at coincident points: $\mathbf{b}_n(x) \equiv \mathbf{b}_n(x, x)$ is an endomorphisms in the fibre of the vectorbundle over x. Its trace is $b_n(x) \equiv \mathrm{tr}\mathbf{b}_n(x)$. Below we will need the covariant derivative of the coefficients. The quantity $D_{\mu_1} \dots D_{\mu_m}\mathbf{b}_n$ is the coincidence limit of the covariant derivative of $\mathbf{b}_n$ and is not to be confused with $D_{\mu_1} \dots D_{\mu_m}\mathbf{b}_n$, the covariant derivative of the coincidence limit of $\mathbf{b}_n$. Our notation is related to that of [79] by

$$\mathbf{b}_{2n} = A_n \ .$$

Now let $\mathbf{E}$ be an endomorphism in the vectorbundle. We have

$$\mathrm{Tr}\mathbf{E}F = \frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\left\{Q_2[F]\mathrm{tr}\mathbf{E}\mathbf{b}_0 + Q_1[F]\mathrm{tr}\mathbf{E}\mathbf{b}_2 + Q_0[F]\mathrm{tr}\mathbf{E}\mathbf{b}_4 + \dots\right\} \ . \tag{C.60}$$

Now let $\mathbf{V} = \mathbf{V}^\mu D_\mu$ where $\mathbf{V}^\mu$ are endomorphisms of the vectorbundle.

$$\begin{aligned}
\mathrm{Tr}\mathbf{V}F &= \int ds\tilde{F}(s)\mathrm{Tr}\left(\mathbf{V}^\mu D_\mu e^{-s\Delta}\right) \\
&= \int ds\tilde{F}(s)\frac{1}{(4\pi s)^2}\int d^4x\sqrt{g}\,\mathrm{tr}\left\{\mathbf{V}^\mu D_\mu\mathbf{b}_0 + s\mathbf{V}^\mu D_\mu\mathbf{b}_2 + s^2\mathbf{V}^\mu D_\mu\mathbf{b}_4 + \dots\right\} \\
&= \frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\left\{Q_2\mathrm{tr}\mathbf{V}^\mu D_\mu\mathbf{b}_0 + Q_1\mathrm{tr}\mathbf{V}^\mu D_\mu\mathbf{b}_2 + Q_0\mathrm{tr}\mathbf{V}^\mu D_\mu\mathbf{b}_4 + \dots\right\} \ .
\end{aligned} \tag{C.61}$$

Now let $\mathbf{W} = \mathbf{W}^{\mu\nu}D_\mu D_\nu$ where $\mathbf{W}^{\mu\nu}$ are endomorphisms of the vectorbundle, symmetric in μ, ν.

$$\begin{aligned}
\mathrm{Tr}\left(\mathbf{W}F\right) &= \int ds\tilde{F}(s)\mathrm{Tr}\left(\mathbf{W}^{\mu\nu}D_\mu D_\nu e^{-s\Delta}\right) \\
&= \int ds\tilde{F}(s)\frac{1}{(4\pi s)^2}\int d^4x\sqrt{g}\,\mathrm{tr}\Bigg\{\frac{1}{s}\mathbf{W}^{\mu\nu}\left(-\frac{1}{2}g_{\mu\nu}\mathbf{b}_0\right) \\
&+ \mathbf{W}^{\mu\nu}\left(-\frac{1}{2}g_{\mu\nu}\mathbf{b}_2 + D_{(\mu}D_{\nu)}\mathbf{b}_0\right) + s\mathbf{W}^{\mu\nu}\left(-\frac{1}{2}g_{\mu\nu}\mathbf{b}_4 + D_{(\mu}D_{\nu)}\mathbf{b}_2\right) + \dots\Bigg\} \\
&= \frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\Bigg\{-\frac{1}{2}Q_3\mathrm{tr}\mathbf{W}^\mu{}_\mu\mathbf{b}_0 + Q_2\left(-\frac{1}{2}\mathrm{tr}\mathbf{W}^\mu{}_\mu\mathbf{b}_2 + \mathrm{tr}\mathbf{W}^{\mu\nu}D_{(\mu}D_{\nu)}\mathbf{b}_0\right) \\
&+ Q_1\left(-\frac{1}{2}\mathrm{tr}\mathbf{W}^\mu{}_\mu\mathbf{b}_4 + \mathrm{tr}\mathbf{W}^{\mu\nu}D_{(\mu}D_{\nu)}\mathbf{b}_2\right) + \dots\Bigg\} \ .
\end{aligned} \tag{C.62}$$

Now let $\mathbf{Y} = \mathbf{Y}^{\mu\nu\rho}D_\mu D_\nu D_\rho$ where $\mathbf{Y}^{\mu\nu\rho}$ are endomorphisms of the vectorbundle, totally symmetric in μ, ν, ρ.

$$\begin{aligned}
\mathrm{Tr}\,(\mathbf{Y}F) &= \int ds\tilde{F}(s)\mathrm{Tr}\left(\mathbf{Y}^{\mu\nu\rho}D_\mu D_\nu D_\rho e^{-s\Delta}\right)\\
&= \int ds\tilde{F}(s)\frac{1}{(4\pi s)^2}\int d^4x\sqrt{g}\,\mathrm{tr}\left\{\frac{1}{s}\mathbf{Y}^{\mu\nu\rho}\left(-\frac{3}{2}g_{(\mu\nu}D_{\rho)}\mathbf{b}_0\right)\right.\\
&\left.+\mathbf{Y}^{\mu\nu\rho}\left(-\frac{3}{2}g_{(\mu\nu}D_{\rho)}\mathbf{b}_2+D_{(\mu}D_\nu D_{\rho)}\mathbf{b}_0\right)+\dots\right\} \qquad \text{(C.63)}\\
&= \frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\left\{-\frac{3}{2}Q_3\mathrm{tr}\mathbf{Y}_\mu{}^{\mu\rho}D_\rho\mathbf{b}_0\right.\\
&\left.+Q_2\left(-\frac{3}{2}\mathrm{tr}\mathbf{Y}_\mu{}^{\mu\rho}D_\rho\mathbf{b}_2+\mathrm{tr}\mathbf{Y}^{\mu\nu\rho}D_{(\mu}D_\nu D_{\rho)}\mathbf{b}_0\right)+\dots\right\}.
\end{aligned}$$

Finally, let $\mathbf{X}=\mathbf{X}^{\mu\nu\rho\sigma}D_\mu D_\nu D_\rho D_\sigma$ where $\mathbf{X}^{\mu\nu\rho\sigma}$ are endomorphisms of the vectorbundle, totally symmetric in μ, ν, ρ, σ.

$$\begin{aligned}
\mathrm{Tr}\,(\mathbf{X}F) &= \int ds\tilde{F}(s)\mathrm{Tr}\left(\mathbf{X}^{\mu\nu\rho\sigma}D_\mu D_\nu D_\rho D_\sigma e^{-s\Delta}\right)\\
&= \int ds\tilde{F}(s)\frac{1}{(4\pi s)^2}\int d^4x\sqrt{g}\,\mathrm{tr}\left\{\frac{1}{s^2}\mathbf{X}^{\mu\nu\rho\sigma}\left(-\frac{3}{4}g_{(\mu\nu}g_{\rho\sigma)}\mathbf{b}_0\right)+\dots\right\} \qquad \text{(C.64)}\\
&= \frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\left\{\frac{3}{4}Q_4\mathrm{tr}\mathbf{X}_{\mu\nu}{}^{\mu\nu}\mathbf{b}_0+\dots\right\}.
\end{aligned}$$

C.5 Derivation of beta functions from the Functional Renormalisation Group

In the literature, the beta functions of gravity, with or without matter, have been often calculated in the Functional Renormalisation Group (FRG) framework, see, e.g., [27, 168–170] for reviews on the subject). Since the FRG is based on a momentum cutoff, the beta functions contain terms proportional to powers of the cutoff, that are not seen with other techniques. In this appendix, we discuss the way in which one can recover from the FRG the standard one-loop beta functions that one would see, e.g. in dimensional regularisation. For a more detailed discussion of the relation between the FRG and dimensional regularisation we refer to [171].

In the FRG, a cutoff function R_k is introduced by hand in the quadratic part of the action, in order to suppress the contribution to the functional integral of modes with (Euclidean) momenta smaller than a cutoff scale k. This leads to a coarse-grained effective action Γ_k which coincides with the full effective action at $k=0$. The flowing action Γ_k obeys the flow equation

$$k\frac{\mathrm{d}}{\mathrm{d}k}\Gamma_k \equiv \partial_t\Gamma_k = \frac{1}{2}\mathrm{Tr}\left[(\Gamma_k^{(2)}+R_k)^{-1}\partial_t R_k\right], \qquad \text{(C.65)}$$

where $\Gamma_k^{(2)}$ is the Hessian constructed from Γ_k. For our present purposes, it will be enough to consider a simple case of scalar fields in a background metric, with a Hessian of the form

$\Gamma_k^{(2)} = \Delta$ where Δ is a Laplace-type operator:

$$\Delta = -\nabla^2 + E \; ; \quad E = m^2 + 12\lambda\phi^2 - \xi R \; . \tag{C.66}$$

This derives from a scalar action containing a potential and a non-minimal coupling to gravity. Then the r.h.s. of the flow equation is a function $W(\Delta)$ that, for constant ϕ, can be evaluated as

$$\mathrm{Tr}W(\Delta) = \frac{1}{(4\pi)^{d/2}}\Big[Q_{d/2}(W)B_0(\Delta) + Q_{d/2-1}(W)B_2(\Delta) + \ldots + Q_0(W)B_d(\Delta) + \ldots\Big] \tag{C.67}$$

with the Q-functionals defined as

$$Q_n(W) = \frac{(-1)^k}{\Gamma(n+k)}\int_0^\infty \mathrm{d}z\, z^{n+k-1} W^{(k)}(z) \, . \tag{C.68}$$

In eq.(C.68), $n \in \mathbf{R}$, $W^{(k)}(z)$ stands for the k-th derivative of W with respect to z. If $n > 0$, then $k = 0$. Otherwise, k is a positive integer such that $n + k > 0$. The heat kernel coefficients are $B_n(\Delta) = \int \mathrm{d}^d x\sqrt{g}\ \mathrm{Tr}\ b_n(\Delta)$, where

$$\begin{aligned} b_0 &= 1\, , \qquad b_2 = \frac{R}{6} - E\, , \\ b_4 &= \frac{1}{180}\left(R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta} - R^{\mu\nu}R_{\mu\nu} + \frac{5}{2}R^2\right) - \frac{1}{6}RE + \frac{1}{2}E^2\, . \end{aligned} \tag{C.69}$$

In the flow equation, we are interested in computing Q-functionals of the form

$$W(z) = \frac{\partial_t R_k(z)}{(P_k(z))^m}\, , \tag{C.70}$$

where $P_k(z) = z + R_k(z)$. If, $m = n + 1$, then one can show that

$$Q_n\left(\frac{\partial_t R_k}{P_k^{n+1}}\right) = \frac{2}{\Gamma(n+1)}\, , \tag{C.71}$$

is "universal", i.e. independent of the shape of R_k. For certain cutoff schemes the denominator in the function W is $P_k + E$, and

$$Q_n\,(W) = Q_n\left(\frac{\partial_t R_k}{(P_k + E)^m}\right)\, , \tag{C.72}$$

are, in general, non-universal quantities. Nevertheless, one can extract universal parts of each Q-functional defined in eq.(C.72) by expanding in E:

$$Q_n\left(\frac{\partial_t R_k}{(P_k + E)^m}\right) = Q_n\left(\frac{\partial_t R_k}{P_k^m}\left(1 - m\frac{E}{P_k} + \frac{m(m+1)}{2}\frac{E^2}{P_k^2} - \frac{m(m+1)(m+2)}{3!}\frac{E^3}{P_k^3} + \ldots\right)\right)\, , \tag{C.73}$$

and exploiting the linearity of the Q-functionals to pick up the contribution which satisfies $n = m + 1$.

Consider first a "type III" cutoff, see, e.g., [27, 172]. The beta functional is

$$\dot{\Gamma}_k = \frac{1}{32\pi^2}\left[Q_2\left(\frac{\dot{R}_k}{P_k}\right)B_0(\Delta) + Q_1\left(\frac{\dot{R}_k}{P_k}\right)B_2(\Delta) + Q_0\left(\frac{\dot{R}_k}{P_k}\right)B_4(\Delta) + \dots\right]. \tag{C.74}$$

Only the last term is universal. Thus

$$\dot{\Gamma}_k\Big|_{\text{univ}} = \frac{2}{32\pi^2}\int \mathrm{d}^4x\sqrt{g}\, b_4(\Delta)\,. \tag{C.75}$$

The relevant terms (up to linear order in R which are not total derivatives) are

$$\begin{aligned} b_4 &\sim \frac{1}{2}E^2 - \frac{1}{6}R\,E \\ &\sim \frac{1}{2}m^4 + 72\lambda^2\phi^4 + 12\lambda m^2\phi^2 + \left(\xi + \frac{1}{6}\right)m^2R + 2\lambda(6\xi+1)\phi^2R\,. \end{aligned}$$

From here one reads off the beta functions

$$\begin{aligned} \beta_{\mathcal{V}} &= \frac{m^4}{32\pi^2}\,, \\ \beta_{m^2} &= \frac{3\lambda m^2}{2\pi^2}\,, \\ \beta_{\lambda} &= \frac{9\lambda^2}{2\pi^2}\,, \\ \beta_{Z_N} &= \frac{1+6\xi}{96\pi^2}m^2\,, \\ \beta_{\xi} &= \frac{\lambda(1+6\xi)}{4\pi^2}\,. \end{aligned} \tag{C.76}$$

The same result can be obtained in a more laborious way using a "type I" cutoff. In this case

$$\dot{\Gamma}_k = \frac{1}{32\pi^2}\left[Q_2\left(\frac{\dot{R}_k}{P_k+E}\right)B_0(-\nabla^2) + Q_1\left(\frac{\dot{R}_k}{P_k+E}\right)B_2(-\nabla^2) + Q_0\left(\frac{\dot{R}_k}{P_k+E}\right)B_4(-\nabla^2) + \dots\right]. \tag{C.77}$$

The universal terms come from all three pieces in this expression when one expands in E: the third term in the expansion for Q_2, the second for Q_1 and the leading term for Q_0. In the latter term, $B_4(-\nabla^2)$ is of order R^2 and does not concern us. The rest is

$$\begin{aligned} \dot{\Gamma}_k &\sim \frac{1}{32\pi^2}\left[Q_2\left(\frac{\dot{R}_k}{P_k^3}\right)E^2B_0(-\nabla^2) + Q_1\left(\frac{\dot{R}_k}{P_k^2}\right)(-E)B_2(-\nabla^2) + \dots\right], \\ &= \frac{1}{32\pi^2}\int \mathrm{d}^4x\sqrt{g}\left[E^2 - 2E\frac{1}{6}R + \dots\right], \end{aligned}$$

which is clearly the same as before, and therefore leads to the same beta functions. Note that the universal parts of the beta functions are those that come from the dimensionless Q-functionals and therefore are independent of k, see eq.C.71.

C.6 Some General Expressions

In this appendix, we collect some long expressions that were omitted in the main text. In particular, the beta function of the non-minimal coupling ξ depends on the choice of metric parameterisation. Hence, in the exponential parameterisation in a general linear covariant gauge (6.47), the factor F in eq.(6.48) is

$$F(\alpha,\beta,\xi) = -4\frac{F_1(\alpha,\beta,\xi) - 3\left(F_2(\beta,\xi) + F_3(\beta,\xi) + F_4(\beta,\xi) + F_5(\beta,\xi) + F_6(\beta,\xi)\right)}{(3-\beta)^4}\,, \quad \text{(C.78)}$$

with

$$\begin{aligned}
F_1(\alpha,\beta,\xi) &= 24\alpha^2 + 2\alpha(\beta^2(24\xi+1) - 18\beta(4\xi+1) - 27)\,,\\
F_2(\beta,\xi) &= \beta^4\xi^2(12\xi-1)\,,\\
F_3(\beta,\xi) &= -4\beta^3\xi\left(36\xi^2+9\xi-1\right)\,,\\
F_4(\beta,\xi) &= \beta^2\left(648\xi^3+342\xi^2+36\xi-2\right)\,,\\
F_5(\beta,\xi) &= -12\beta\left(108\xi^3+81\xi^2+15\xi+1\right)\,,\\
F_6(\beta,\xi) &= 9\left(108\xi^3+99\xi^2+12\xi-2\right)\,.
\end{aligned}$$

As for the linear parameterisation, the expression for $G(\alpha,\beta,\xi)$ in (6.57) is

$$G(\alpha,\beta,\xi) = 2\frac{G_1(\alpha,\beta) + G_2(\alpha,\beta,\xi) + 3G_4(\beta,\xi)}{(3-\beta)^4} \quad \text{(C.79)}$$

with

$$\begin{aligned}
G_1(\alpha,\beta) &= -3\alpha^2(3(\beta-6)\beta((\beta-6)\beta+18)+259)\,,\\
G_2(\alpha,\beta,\xi) &= \alpha(2\beta(\beta(3(\beta-12)\beta+24\xi+194)-396)-432\xi+630)\,,\\
G_3(\beta,\xi) &= -13\beta^4+112\beta^3-458\beta^2+24(\beta-3)^4\xi^3\,,\\
G_4(\beta,\xi) &= \left(2(\beta(5\beta-42)+117)(\beta-3)^2\xi^2 - 8(\beta(5\beta-12)+27)(\beta-3)\xi + 888\beta - 657\right)\,.
\end{aligned} \quad \text{(C.80)}$$

Appendix D

Other useful MAG formulae

D.1 Lagrangian bases

D.1.1 Antisymmetric MAG in Cartan form: the leftover terms

At the end of Section 3.3.2 we give two bases for the dimension-four terms of the type. Here we give the formulas for the remaining invariants as linear combinations of the basis elements. Using the first basis (3.27)

$$\begin{aligned}
L_3^{FF} &= L_1^{FF} - 3/2L_1^{TT} - L_2^{TT} + 3L_4^{TT} + L_5^{TT} - 1/2L_6^{TT} + 2L_7^{TT} \ , \\
L_4^{FF} &= 1/2(L_1^{FF} - L_1^{TT}) + L_4^{TT} \ , \\
L_8^{FF} &= L_7^{FF} - L_3^{TT} - 1/2L_6^{TT} + 2L_8^{TT} + L_9^{TT} \ , \\
L_1^{FT} &= L_{13}^{FT} + 1/2L_1^{TT} - L_4^{TT} - L_7^{TT} + L_8^{TT} \ , \\
L_3^{FT} &= 2L_{13}^{FT} + 1/2L_1^{TT} + L_2^{TT} - L_4^{TT} - L_5^{TT} - 1/2L_6^{TT} - 2L_7^{FF} + 2L_8^{FF} \ , \\
L_4^{FT} &= L_{13}^{FT} - 1/2L_6^{TT} + L_8^{TT} \ , \\
L_5^{FT} &= 2L_{13}^{FT} - 2L_7^{TT} + 2L_8^{TT} \ , \\
L_8^{FT} &= -1/2L_{21}^{FT} - L_3^{TT} + L_8^{TT} + L_9^{TT} \ , \\
L_9^{FT} &= -1/2L_{21}^{FT} \ , \\
L_{14}^{FT} &= L_{13}^{FT} - L_7^{TT} + L_8^{TT} \ , \\
L_{17}^{FT} &= 1/2L_6^{TT} - L_8^{TT} \ .
\end{aligned} \tag{D.1}$$

Using the second basis (3.28)

$$
\begin{aligned}
L_3^{FT} &= -L_3^{FF} + 2L_4^{FF} - 2L_7^{FF} + 2L_8^{FF} - 2L_8^{FT} + 2L_9^{FT} + 2L_{13}^{FT} \ , \\
L_4^{FT} &= -L_7^{FF} + L_8^{FF} - L_8^{FT} + L_9^{FT} + L_{13}^{FT} \ , \\
L_5^{FT} &= -L_1^{FF} + 2L_4^{FF} + 2L_1^{FT} \ , \\
L_{14}^{FT} &= 1/2(-L_1^{FF} + 2L_4^{FF} + 2L_1^{FT}) \ , \\
L_{17}^{FT} &= L_7^{FF} - L_8^{FF} + L_8^{FT} - L_9^{FT} \ , \\
L_{21}^{FT} &= -2L_9^{FT} \ , \\
L_4^{TT} &= 1/2(-L_1^{FF} + 2L_4^{FF} + L_1^{TT}) \ , \\
L_6^{TT} &= -L_1^{FF} + 2L_3^{FF} - 2L_4^{FF} + 4L_7^{FF} - 4L_8^{FF} + 4L_1^{FT} + 4L_8^{FT} - 4L_9^{FT} - 4L_{13}^{FT} + 2L_2^{TT} - 2L_5^{TT} \ , \\
L_7^{TT} &= L_3^{FF} - 2L_4^{FF} + L_7^{FF} - L_8^{FF} + L_1^{FT} + L_8^{FT} - L_9^{FT} - L_{13}^{FT} + L_2^{TT} - L_5^{TT} \ , \\
L_8^{TT} &= -1/2L_1^{FF} + L_3^{FF} - L_4^{FF} + L_7^{FF} - L_8^{FF} + 2L_1^{FT} + L_8^{FT} - L_9^{FT} - 2L_{13}^{FT} + L_2^{TT} - L_5^{TT} \ , \\
L_9^{TT} &= 1/2L_1^{FF} - L_3^{FF} + L_4^{FF} - L_7^{FF} + L_8^{FF} - 2L_1^{FT} + 2L_{13}^{FT} - L_2^{TT} + L_3^{TT} + L_5^{TT} \ .
\end{aligned}
\tag{D.2}
$$

D.1.2 Symmetric MAG in Cartan form: the leftover terms

At the end of Section 3.4.2 we give two bases for the dimension-four terms of the type. Here we give the formulas for the remaining invariants as linear combinations of the basis elements.

Using the first basis (3.54)

$$
\begin{aligned}
L_2^{FF} &= -L_1^{FF} + L_1^{QQ} - L_6^{QQ} \ , \\
L_3^{FF} &= 1/2(2L_1^{FF} - 3L_1^{QQ} + 2L_2^{QQ} + 3L_6^{QQ} + 2L_7^{QQ} - 4L_9^{QQ}) \ , \\
L_8^{FF} &= 1/4(4L_7^{FF} - L_4^{QQ} + L_{14}^{QQ}) \ , \\
L_9^{FF} &= L_7^{FF} - 2L_{18}^{FQ} + L_{24}^{FQ} + L_3^{QQ} - L_5^{QQ} + L_7^{QQ} - 2L_{10}^{QQ} + L_{12}^{QQ} - L_{15}^{QQ} + L_{16}^{QQ} \ , \\
L_{10}^{FF} &= 1/4(4L_7^{FF} - 8L_{18}^{FQ} + 4L_{24}^{FQ} - L_4^{QQ} + 4L_8^{QQ} - 4L_{11}^{QQ} - 4L_{12}^{QQ} + 4L_{13}^{QQ} + L_{14}^{QQ}) \ , \\
L_{11}^{FF} &= 1/2(-2L_7^{FF} + 2L_{18}^{FQ} - L_{24}^{FQ} + L_5^{QQ} - L_{12}^{QQ} + L_{15}^{QQ} - L_{16}^{QQ}) \ , \\
L_{12}^{FF} &= 1/4(-4L_7^{FF} + 4L_{18}^{FQ} - 2L_{24}^{FQ} + L_4^{QQ} + 2L_{11}^{QQ} - 2L_{12}^{QQ} - 2L_{13}^{QQ} - L_{14}^{QQ} + 2L_{15}^{QQ}) \ , \\
L_1^{FQ} &= 1/2(L_1^{QQ} - L_6^{QQ}) \ , \\
L_2^{FQ} &= -L_{16}^{FQ} + L_{18}^{FQ} + 1/2L_1^{QQ} - L_2^{QQ} - 1/2L_6^{QQ} - L_7^{QQ} + 2L_9^{QQ} \ , \\
L_3^{FQ} &= 1/2(2L_{16}^{FQ} - 2L_{18}^{FQ} - L_1^{QQ} + L_6^{QQ} + 2L_7^{QQ} - 2L_9^{QQ}) \ , \\
L_4^{FQ} &= L_{16}^{FQ} - L_{18}^{FQ} \ , \\
L_5^{FQ} &= -L_{16}^{FQ} + L_{18}^{FQ} - L_7^{QQ} + L_9^{QQ} \ , \\
L_6^{FQ} &= 1/2(L_{24}^{FQ} - L_5^{QQ} + L_{12}^{QQ} - L_{15}^{QQ} + L_{16}^{QQ}) \ , \\
L_7^{FQ} &= 1/2(L_{24}^{FQ} + L_{12}^{QQ} - L_{15}^{QQ}) \ , \\
L_8^{FQ} &= 1/2(L_{23}^{FQ} - L_4^{QQ} + L_{13}^{QQ} + L_{14}^{QQ} - L_{16}^{QQ}) \ , \\
L_9^{FQ} &= 1/2(L_{23}^{FQ} + L_{13}^{QQ} - L_{16}^{QQ}) \ , \\
L_{10}^{FQ} &= 1/2(-L_{24}^{FQ} - 2L_3^{QQ} + L_5^{QQ} + 2L_{10}^{QQ} - L_{12}^{QQ} + L_{15}^{QQ} - L_{16}^{QQ}) \ ,
\end{aligned}
\tag{D.3}
$$

$$
\begin{aligned}
L^{FQ}_{11} &= 1/2(-L^{FQ}_{24} + L^{QQ}_{12} - L^{QQ}_{15}) \ , \\
L^{FQ}_{12} &= 1/2(-L^{FQ}_{23} + L^{QQ}_{4} - 2L^{QQ}_{5} + 2L^{QQ}_{11} - L^{QQ}_{13} - L^{QQ}_{14} + L^{QQ}_{16}) \ , \\
L^{FQ}_{13} &= 1/2(-L^{FQ}_{23} + L^{QQ}_{13} - L^{QQ}_{16}) \ , \\
L^{FQ}_{14} &= 1/2(L^{QQ}_{5} - L^{QQ}_{16}) \ , \\
L^{FQ}_{15} &= 1/2(L^{QQ}_{4} - L^{QQ}_{14}) \ , \\
L^{FQ}_{17} &= -L^{FQ}_{16} + L^{QQ}_{9} - L^{QQ}_{10} \ , \\
L^{FQ}_{19} &= 1/2(2L^{FQ}_{18} + L^{QQ}_{11} - L^{QQ}_{13}) \ , \\
L^{FQ}_{20} &= -L^{FQ}_{18} + L^{QQ}_{7} - L^{QQ}_{10} \ , \\
L^{FQ}_{21} &= 1/2(-2L^{FQ}_{18} + 2L^{QQ}_{8} - L^{QQ}_{11} - 2L^{QQ}_{12} + L^{QQ}_{13}) \ , \\
L^{FQ}_{22} &= 1/2(L^{QQ}_{11} - L^{QQ}_{13}) \ .
\end{aligned}
\tag{D.4}
$$

Using the second basis (3.55)

$$
\begin{aligned}
L^{FQ}_{1} &= 1/2(L^{FF}_{1} + L^{FF}_{2}) \ , \\
L^{FQ}_{2} &= -L^{FF}_{2} - L^{FF}_{3} - L^{FQ}_{16} + L^{FQ}_{18} \ , \\
L^{FQ}_{3} &= 1/2(-L^{FF}_{1} - L^{FF}_{2} + 2L^{FF}_{9} + 2L^{FF}_{11} + 2L^{FQ}_{10} - 2L^{FQ}_{17}) \ , \\
L^{FQ}_{4} &= L^{FQ}_{16} - L^{FQ}_{18} \ , \\
L^{FQ}_{5} &= -L^{FF}_{9} - L^{FF}_{11} - L^{FQ}_{10} + L^{FQ}_{17} \ , \\
L^{FQ}_{6} &= -L^{FF}_{7} - L^{FF}_{11} + L^{FQ}_{18} \ , \\
L^{FQ}_{7} &= -L^{FF}_{7} - L^{FF}_{11} + L^{FQ}_{14} + L^{FQ}_{18} \ , \\
L^{FQ}_{8} &= -2L^{FF}_{7} + 2L^{FF}_{8} + 2L^{FF}_{11} - 2L^{FF}_{12} + L^{FQ}_{12} + L^{FQ}_{23} \ , \\
L^{FQ}_{9} &= 2L^{FF}_{11} - 2L^{FF}_{12} + L^{FQ}_{12} + L^{FQ}_{23} \ , \\
L^{FQ}_{13} &= 2L^{FF}_{11} - 2L^{FF}_{12} + L^{FQ}_{12} \ , \\
L^{FQ}_{15} &= 2(L^{FF}_{7} - L^{FF}_{8}) \ , \\
L^{FQ}_{19} &= -L^{FF}_{7} + L^{FF}_{8} - L^{FF}_{11} + L^{FF}_{12} + L^{FQ}_{14} + L^{FQ}_{18} \ , \\
L^{FQ}_{20} &= L^{FF}_{9} + L^{FF}_{11} + L^{FQ}_{10} \ , \\
L^{FQ}_{21} &= L^{FF}_{10} + L^{FF}_{12} + L^{FQ}_{11} \ , \\
L^{FQ}_{22} &= -L^{FF}_{7} + L^{FF}_{8} - L^{FF}_{11} + L^{FF}_{12} + L^{FQ}_{14} \ , \\
L^{FQ}_{24} &= -L^{FF}_{7} - L^{FF}_{11} - L^{FQ}_{11} + L^{FQ}_{14} + L^{FQ}_{18} \ ,
\end{aligned}
\tag{D.5}
$$

$$
\begin{aligned}
L_2^{QQ} &= 1/2L_1^{FF} + 3/2L_2^{FF} + L_3^{FF} - L_9^{FF} - L_{11}^{FF} - L_{10}^{FQ} + 2L_{16}^{FQ} + 2L_{17}^{FQ} - L_{18}^{FQ} + L_{10}^{QQ} \ , \\
L_3^{QQ} &= L_7^{FF} + L_{11}^{FF} - L_{10}^{FQ} - L_{18}^{FQ} + L_{10}^{QQ} \ , \\
L_4^{QQ} &= 4L_7^{FF} - 4L_8^{FF} + L_{14}^{QQ} \ , \\
L_5^{QQ} &= 2L_7^{FF} - 2L_8^{FF} - 2L_{11}^{FF} + 2L_{12}^{FF} - 2L_{12}^{FQ} - L_{23}^{FQ} + L_{11}^{QQ} \ , \\
L_6^{QQ} &= -L_1^{FF} - L_2^{FF} + L_1^{QQ} \ , \\
L_7^{QQ} &= L_9^{FF} + L_{11}^{FF} + L_{10}^{FQ} + L_{18}^{FQ} + L_{10}^{QQ} \ , \\
L_8^{QQ} &= -L_7^{FF} + L_8^{FF} + L_{10}^{FF} - L_{11}^{FF} + 2L_{12}^{FF} + L_{11}^{FQ} + L_{14}^{FQ} + L_{18}^{FQ} + L_{12}^{QQ} \ , \\
L_9^{QQ} &= L_{16}^{FQ} + L_{17}^{FQ} + L_{10}^{QQ} \ , \\
L_{13}^{QQ} &= 2L_7^{FF} - 2L_8^{FF} + 2L_{11}^{FF} - 2L_{12}^{FF} - 2L_{14}^{FQ} + L_{11}^{QQ} \ , \\
L_{15}^{QQ} &= L_7^{FF} + L_{11}^{FF} - L_{11}^{FQ} - L_{14}^{FQ} - L_{18}^{FQ} + L_{12}^{QQ} \ , \\
L_{16}^{QQ} &= 2L_7^{FF} - 2L_8^{FF} - 2L_{11}^{FF} + 2L_{12}^{FF} - 2L_{12}^{FQ} - 2L_{14}^{FQ} - L_{23}^{FQ} + L_{11}^{QQ} \ .
\end{aligned} \tag{D.6}
$$

D.1.3 General MAG in Cartan form: the leftover terms

We give the formulas mentioned at the end of Section 3.5.2. Using the first basis (3.80)

$$
\begin{aligned}
L_2^{FF} &= -L_1^{FF} + L_1^{QQ} - L_6^{QQ} \ , \\
L_3^{FF} &= L_1^{FF} - 3/2L_1^{QQ} + L_2^{QQ} + 3/2L_6^{QQ} + L_7^{QQ} - 2L_9^{QQ} - 3/2L_1^{TT} - L_2^{TT} + 3L_4^{TT} + L_5^{TT} \\
&\quad - 1/2L_6^{TT} + 2L_7^{TT} + 4L_1^{TQ} - 4L_4^{TQ} + 4L_7^{TQ} \ , \\
L_4^{FF} &= 1/2L_1^{FF} - L_1^{QQ} + L_2^{QQ} + L_6^{QQ} + L_7^{QQ} - 2L_9^{QQ} - 1/2L_1^{TT} + L_4^{TT} + 2L_1^{TQ} - 2L_4^{TQ} + 2L_7^{TQ} \ , \\
L_5^{FF} &= -1/2L_1^{FF} + 1/2L_1^{QQ} - 1/2L_6^{QQ} + 1/2L_1^{TT} - L_4^{TT} - L_1^{TQ} + L_4^{TQ} - L_7^{TQ} \ , \\
L_6^{FF} &= 1/2L_1^{FF} - 1/2L_1^{TT} + L_4^{TT} \ , \\
L_8^{FF} &= L_7^{FF} - 1/4L_4^{QQ} + 1/4L_{14}^{QQ} - L_3^{TT} - 1/2L_6^{TT} + 2L_8^{TT} + L_9^{TT} - L_3^{TQ} + L_9^{TQ} - L_{12}^{TQ} \ , \\
L_9^{FF} &= L_7^{FF} - 2L_{18}^{FQ} + L_{24}^{FQ} + L_3^{QQ} - L_5^{QQ} + L_7^{QQ} - 2L_{10}^{QQ} + L_{12}^{QQ} - L_{15}^{QQ} + L_{16}^{QQ} \\
&\quad - 2L_2^{TQ} + 2L_8^{TQ} - 2L_{13}^{TQ} \ ,
\end{aligned} \tag{D.7}
$$

$$
\begin{aligned}
L_{10}^{FF} &= L_7^{FF} - 2L_{18}^{FQ} + L_{24}^{FQ} - 1/4L_4^{QQ} + L_8^{QQ} - L_{11}^{QQ} - L_{12}^{QQ} + L_{13}^{QQ} + 1/4L_{14}^{QQ} \\
&\quad - L_3^{TQ} + 2L_5^{TQ} + L_9^{TQ} - 2L_{10}^{TQ} + 2L_{11}^{TQ} - L_{12}^{TQ} - L_3^{TT} - 1/2L_6^{TT} + 2L_8^{TT} + L_9^{TT} \ , \\
L_{11}^{FF} &= -L_7^{FF} + L_{18}^{FQ} - 1/2L_{24}^{FQ} + 1/2L_5^{QQ} - 1/2L_{12}^{QQ} + 1/2L_{15}^{QQ} - 1/2L_{16}^{QQ} + L_2^{TQ} - L_8^{TQ} + L_{13}^{TQ} \ , \\
L_{12}^{FF} &= -L_7^{FF} + L_{18}^{FQ} - 1/2L_{24}^{FQ} + 1/4L_4^{QQ} + 1/2L_{11}^{QQ} - 1/2L_{12}^{QQ} - 1/2L_{13}^{QQ} - 1/4L_{14}^{QQ} + 1/2L_{15}^{QQ} \\
&\quad + L_3^{TT} + 1/2L_6^{TT} - 2L_8^{TT} - L_9^{TT} + L_3^{TQ} - L_5^{TQ} - L_9^{TQ} + L_{10}^{TQ} - L_{11}^{TQ} + L_{12}^{TQ} \ , \\
L_{13}^{FF} &= 1/2(L_4^{QQ} - L_{14}^{QQ}) \ , \\
L_{14}^{FF} &= 1/4(-L_4^{QQ} + L_{14}^{QQ} - 2L_3^{TQ} + 2L_9^{TQ} - 2L_{12}^{TQ}) \ , \\
L_{15}^{FF} &= 1/4(L_4^{QQ} - 2L_5^{QQ} + 2L_{11}^{QQ} - 2L_{13}^{QQ} - L_{14}^{QQ} + 2L_{16}^{QQ} + 2L_3^{TQ} - 2L_9^{TQ} + 2L_{12}^{TQ}) \ ,
\end{aligned}
$$

(D.8)

$$
\begin{aligned}
L_1^{FT}=&L_{13}^{FT}+1/2L_1^{TT}-L_4^{TT}-L_7^{TT}+L_8^{TT}-L_7^{TQ}+1/2L_9^{TQ}\ ,\\
L_2^{FT}=&-L_{13}^{FT}-1/2L_1^{TT}+L_4^{TT}+L_7^{TT}-L_8^{TT}+L_1^{TQ}-L_4^{TQ}+L_7^{TQ}-1/2L_9^{TQ}\ ,\\
L_3^{FT}=&2L_{13}^{FT}+1/2L_1^{TT}+L_2^{TT}-L_4^{TT}-L_5^{TT}-1/2L_6^{TT}-2L_7^{TT}+2L_8^{TT}\\
&-L_1^{TQ}+L_4^{TQ}+L_5^{TQ}-L_6^{TQ}-L_7^{TQ}+L_9^{TQ}\ ,\\
L_4^{FT}=&L_{13}^{FT}-1/2L_6^{TT}+L_8^{TT}+1/2L_9^{TQ}\ ,\\
L_5^{FT}=&2L_{13}^{FT}-2L_7^{TT}+2L_8^{TT}+L_9^{TQ}\ ,\\
L_6^{FT}=&-L_{13}^{FT}+1/2L_6^{TT}-L_8^{TT}-L_5^{TQ}+L_6^{TQ}-1/2L_9^{TQ}\ ,\\
L_7^{FT}=&-2L_{13}^{FT}+2L_7^{TT}-2L_8^{TT}+2L_7^{TQ}-L_9^{TQ}\ ,\\
L_8^{FT}=&1/2(-L_{21}^{FT}-2L_3^{TT}+2L_8^{TT}+2L_9^{TT}-L_3^{TQ}+L_{11}^{TQ}-L_{12}^{TQ}+L_{13}^{TQ})\ ,\\
L_9^{FT}=&1/2(-L_{21}^{FT}+L_{11}^{TQ}+L_{13}^{TQ})\ ,\\
L_{10}^{FT}=&1/2L_{21}^{FT}+L_3^{TT}-L_8^{TT}-L_9^{TT}-L_2^{TQ}+1/2L_3^{TQ}+L_{10}^{TQ}-1/2L_{11}^{TQ}+1/2L_{12}^{TQ}-1/2L_{13}^{TQ}\ ,
\end{aligned}
\tag{D.9}
$$

$$
\begin{aligned}
L_{11}^{FT}=&1/2(L_{21}^{FT}+L_{11}^{TQ}+L_{13}^{TQ})\ ,\\
L_{12}^{FT}=&1/2(L_3^{TQ}+L_{12}^{TQ})\ ,\\
L_{14}^{FT}=&L_{13}^{FT}-L_7^{TT}+L_8^{TT}+1/2L_9^{TQ}\ ,\\
L_{15}^{FT}=&-L_{13}^{FT}+L_6^{TQ}-L_8^{TQ}\ ,\\
L_{16}^{FT}=&-L_{13}^{FT}+L_7^{TT}-L_8^{TT}+L_7^{TQ}-1/2L_9^{TQ}\ ,\\
L_{17}^{FT}=&1/2L_6^{TT}-L_8^{TT}-1/2L_9^{TQ}\ ,\\
L_{18}^{FT}=&-1/2L_6^{TT}+L_8^{TT}+L_5^{TQ}-L_8^{TQ}+1/2L_9^{TQ}\ ,\\
L_{19}^{FT}=&1/2L_9^{TQ}\ ,\\
L_{20}^{FT}=&L_9^{TQ}\ ,
\end{aligned}
\tag{D.10}
$$

$$
\begin{aligned}
L_1^{FQ}=&1/2(L_1^{QQ}-L_6^{QQ}) \ ,\\
L_2^{FQ}=&-L_{16}^{FQ}+L_{18}^{FQ}+1/2L_1^{QQ}-L_2^{QQ}-1/2L_6^{QQ}-L_7^{QQ}+2L_9^{QQ}-L_1^{TQ}+L_4^{TQ}-L_7^{TQ} \ ,\\
L_3^{FQ}=&L_{16}^{FQ}-L_{18}^{FQ}-1/2L_1^{QQ}+1/2L_6^{QQ}+L_7^{QQ}-L_9^{QQ}+L_1^{TQ}-L_4^{TQ}+L_7^{TQ} \ ,\\
L_4^{FQ}=&L_{16}^{FQ}-L_{18}^{FQ} \ ,\\
L_5^{FQ}=&-L_{16}^{FQ}+L_{18}^{FQ}-L_7^{QQ}+L_9^{QQ} \ ,\\
L_6^{FQ}=&1/2(L_{24}^{FQ}-L_5^{QQ}+L_{12}^{QQ}-L_{15}^{QQ}+L_{16}^{QQ}-2L_2^{TQ}+2L_8^{TQ}-2L_{13}^{TQ}) \ ,\\
L_7^{FQ}=&1/2(L_{24}^{FQ}+L_{12}^{QQ}-L_{15}^{QQ}) \ ,\\
L_8^{FQ}=&1/2(L_{23}^{FQ}-L_4^{QQ}+L_{13}^{QQ}+L_{14}^{QQ}-L_{16}^{QQ}-2L_3^{TQ}+2L_9^{TQ}-2L_{12}^{TQ}) \ ,\\
L_9^{FQ}=&1/2(L_{23}^{FQ}+L_{13}^{QQ}-L_{16}^{QQ}) \ ,\\
L_{10}^{FQ}=&-1/2L_{24}^{FQ}-L_3^{QQ}+1/2L_5^{QQ}+L_{10}^{QQ}-1/2L_{12}^{QQ}+1/2L_{15}^{QQ}-1/2L_{16}^{QQ}+L_2^{TQ}-L_8^{TQ}+L_{13}^{TQ} \ ,\\
L_{11}^{FQ}=&1/2(-L_{24}^{FQ}+L_{12}^{QQ}-L_{15}^{QQ}) \ ,\\
L_{12}^{FQ}=&-1/2L_{23}^{FQ}+1/2L_4^{QQ}-L_5^{QQ}+L_{11}^{QQ}-1/2L_{13}^{QQ}-1/2L_{14}^{QQ}+1/2L_{16}^{QQ}+L_3^{TQ}-L_9^{TQ}+L_{12}^{TQ} \ ,\\
L_{13}^{FQ}=&1/2(-L_{23}^{FQ}+L_{13}^{QQ}-L_{16}^{QQ}) \ ,\\
L_{14}^{FQ}=&1/2(L_5^{QQ}-L_{16}^{QQ}) \ ,
\end{aligned}
\tag{D.11}
$$

$$
\begin{aligned}
L_{15}^{FQ}=&1/2(L_4^{QQ}-L_{14}^{QQ}) \ ,\\
L_{17}^{FQ}=&-L_{16}^{FQ}+L_9^{QQ}-L_{10}^{QQ} \ ,\\
L_{19}^{FQ}=&L_{18}^{FQ}+1/2L_{11}^{QQ}-1/2L_{13}^{QQ}-L_5^{TQ}+L_{10}^{TQ}-L_{11}^{TQ} \ ,\\
L_{20}^{FQ}=&-L_{18}^{FQ}+L_7^{QQ}-L_{10}^{QQ} \ ,\\
L_{21}^{FQ}=&-L_{18}^{FQ}+L_8^{QQ}-1/2L_{11}^{QQ}-L_{12}^{QQ}+1/2L_{13}^{QQ}+L_5^{TQ}-L_{10}^{TQ}+L_{11}^{TQ} \ ,\\
L_{22}^{FQ}=&1/2(L_{11}^{QQ}-L_{13}^{QQ}) \ .
\end{aligned}
\tag{D.12}
$$

Using the second basis (3.81)

$$
\begin{aligned}
L_2^{FT}=&1/2L_2^{FF}-L_5^{FF}-L_{14}^{FT} \ ,\\
L_3^{FT}=&-L_3^{FF}+L_4^{FF}-L_5^{FF}-L_7^{FF}+L_8^{FF}-L_{14}^{FF}-L_8^{FT}+L_9^{FT}+L_{13}^{FT}-L_{15}^{FT}+L_{18}^{FT} \ ,\\
L_4^{FT}=&-L_7^{FF}+L_8^{FF}-L_{14}^{FF}-L_8^{FT}+L_9^{FT}+L_{13}^{FT} \ ,\\
L_5^{FT}=&2L_{14}^{FT} \ ,\\
L_6^{FT}=&L_{15}^{FT}-L_{18}^{FT} \ ,\\
L_7^{FT}=&L_1^{FF}-2L_6^{FF}-2L_1^{FT} \ ,\\
L_{10}^{FT}=&-L_{11}^{FF}+L_{12}^{FF}-L_{15}^{FF}+L_9^{FT}+L_{18}^{FT}+L_{21}^{FT} \ ,\\
L_{11}^{FT}=&L_9^{FT}+L_{21}^{FT} \ ,\\
L_{16}^{FT}=&1/2L_1^{FF}-L_6^{FF}-L_1^{FT} \ ,\\
L_{17}^{FT}=&L_7^{FF}-L_8^{FF}+L_{14}^{FF}+L_8^{FT}-L_9^{FT} \ ,\\
L_{19}^{FT}=&1/2L_{13}^{FF}+L_{14}^{FF}+L_{12}^{FT} \ ,\\
L_{20}^{FT}=&L_{13}^{FF}+2L_{14}^{FF}+2L_{12}^{FT} \ ,
\end{aligned}
\tag{D.13}
$$

$$
\begin{aligned}
L_1^{FQ}&=1/2\left(L_1^{FF}+L_2^{FF}\right)\ ,\\
L_2^{FQ}&=-L_4^{FF}-L_5^{FF}-L_{16}^{FQ}+L_{18}^{FQ}\ ,\\
L_3^{FQ}&=-L_5^{FF}-L_6^{FF}+L_9^{FF}+L_{11}^{FF}+L_{10}^{FQ}-L_{17}^{FQ}\ ,\\
L_4^{FQ}&=L_{16}^{FQ}-L_{18}^{FQ}\ ,\\
L_5^{FQ}&=-L_9^{FF}-L_{11}^{FF}-L_{10}^{FQ}+L_{17}^{FQ}\ ,\\
L_6^{FQ}&=-L_7^{FF}-L_{11}^{FF}+L_{18}^{FQ}\ ,\\
L_7^{FQ}&=-L_8^{FF}-L_{12}^{FF}+L_{19}^{FQ}\ ,\\
L_8^{FQ}&=2L_{14}^{FF}-2L_{15}^{FF}+L_{12}^{FQ}+L_{23}^{FQ}\ ,\\
L_9^{FQ}&=-2L_{15}^{FF}+L_{12}^{FQ}+L_{23}^{FQ}\ ,\\
L_{13}^{FQ}&=-2L_{15}^{FF}+L_{12}^{FQ}\ ,\\
L_{15}^{FQ}&=L_{13}^{FF}\ ,\\
L_{20}^{FQ}&=L_9^{FF}+L_{11}^{FF}+L_{10}^{FQ}\ ,\\
L_{21}^{FQ}&=L_{10}^{FF}+L_{12}^{FF}+L_{11}^{FQ}\ ,\\
L_{22}^{FQ}&=L_{14}^{FF}+L_{15}^{FF}+L_{14}^{FQ}\ ,\\
L_{24}^{FQ}&=-L_8^{FF}-L_{12}^{FF}-L_{11}^{FQ}+L_{19}^{FQ}\ ,
\end{aligned}
\tag{D.14}
$$

$$
\begin{aligned}
L_4^{TT} =& -1/2L_1^{FF} + L_6^{FF} + 1/2L_1^{TT} \ , \\
L_6^{TT} =& -L_2^{FF} + 2L_3^{FF} - 2L_4^{FF} + 4L_5^{FF} + 4L_7^{FF} - 4L_8^{FF} + 4L_{14}^{FF} + 4L_8^{FT} - 4L_9^{FT} - 4L_{13}^{FT} \\
& + 4L_{14}^{FT} + 2L_2^{TT} - 2L_5^{TT} \ , \\
L_7^{TT} =& -1/2L_2^{FF} + L_3^{FF} - L_4^{FF} + 2L_5^{FF} + L_7^{FF} - L_8^{FF} + L_{14}^{FF} + L_8^{FT} - L_9^{FT} - L_{13}^{FT} + L_{14}^{FT} \\
& + L_2^{TT} - L_5^{TT} \ , \\
L_8^{TT} =& -1/2L_2^{FF} + L_3^{FF} - L_4^{FF} + 2L_5^{FF} + L_7^{FF} - L_8^{FF} - 1/2L_{13}^{FF} + L_8^{FT} - L_9^{FT} - L_{12}^{FT} \\
& - 2L_{13}^{FT} + 2L_{14}^{FT} + L_2^{TT} - L_5^{TT} \ , \\
L_9^{TT} =& 1/2L_2^{FF} - L_3^{FF} + L_4^{FF} - 2L_5^{FF} - L_7^{FF} + L_8^{FF} + 1/2L_{13}^{FF} + 2L_{12}^{FT} + 2L_{13}^{FT} - 2L_{14}^{FT} \\
& - L_2^{TT} + L_3^{TT} + L_5^{TT} \ , \\
L_2^{QQ} =& L_4^{FF} + 2L_5^{FF} + L_6^{FF} - L_9^{FF} - L_{11}^{FF} - L_{10}^{FQ} + 2L_{16}^{FQ} + 2L_{17}^{FQ} - L_{18}^{FQ} + L_{10}^{QQ} \ , \\
L_3^{QQ} =& L_7^{FF} + L_{11}^{FF} - L_{10}^{FQ} - L_{18}^{FQ} + L_{10}^{QQ} \ , \\
L_4^{QQ} =& 2L_{13}^{FF} + L_{14}^{QQ} \ , \\
L_5^{QQ} =& -2L_{14}^{FF} + 2L_{15}^{FF} - 2L_{12}^{FQ} - L_{23}^{FQ} + L_{11}^{QQ} \ , \\
L_6^{QQ} =& -L_1^{FF} - L_2^{FF} + L_1^{QQ} \ , \\
L_7^{QQ} =& L_9^{FF} + L_{11}^{FF} + L_{10}^{FQ} + L_{18}^{FQ} + L_{10}^{QQ} \ , \\
L_8^{QQ} =& L_{10}^{FF} + L_{12}^{FF} + L_{11}^{FQ} + L_{19}^{FQ} + L_{12}^{QQ} \ , \\
L_9^{QQ} =& L_{16}^{FQ} + L_{17}^{FQ} + L_{10}^{QQ} \ , \\
L_{13}^{QQ} =& -2L_{14}^{FF} - 2L_{15}^{FF} - 2L_{14}^{FQ} + L_{11}^{QQ} \ , \\
L_{15}^{QQ} =& L_8^{FF} + L_{12}^{FF} - L_{11}^{FQ} - L_{19}^{FQ} + L_{12}^{QQ} \ , \\
L_{16}^{QQ} =& -2L_{14}^{FF} + 2L_{15}^{FF} - 2L_{12}^{FQ} - 2L_{14}^{FQ} - L_{23}^{FQ} + L_{11}^{QQ} \ , \\
L_2^{TQ} =& L_{11}^{FF} - L_{12}^{FF} + L_{15}^{FF} - L_8^{FT} - L_9^{FT} - L_{18}^{FT} - L_{21}^{FT} + L_{10}^{TQ} \ , \\
L_3^{TQ} =& 2L_{12}^{FT} - L_{12}^{TQ} \ , \\
L_4^{TQ} =& -1/2L_2^{FF} + L_5^{FF} - L_1^{FT} + L_{14}^{FT} + L_1^{TQ} \ , \\
L_5^{TQ} =& L_{14}^{FF} + L_{15}^{FF} + L_{14}^{FQ} + L_{18}^{FQ} - L_{19}^{FQ} + L_{10}^{TQ} - L_{11}^{TQ} \ , \\
L_6^{TQ} =& -L_7^{FF} + L_8^{FF} + L_{15}^{FF} + L_{14}^{FQ} + L_{18}^{FQ} - L_{19}^{FQ} - L_8^{FT} + L_9^{FT} + L_{13}^{FT} + L_{15}^{FT} - L_{18}^{FT} \\
& + L_{10}^{TQ} - L_{11}^{TQ} \ , \\
L_7^{TQ} =& 1/2L_1^{FF} - L_6^{FF} - L_1^{FT} + L_{14}^{FT} \ , \\
L_8^{TQ} =& -L_7^{FF} + L_8^{FF} + L_{15}^{FF} + L_{14}^{FQ} + L_{18}^{FQ} - L_{19}^{FQ} - L_8^{FT} + L_9^{FT} - L_{18}^{FT} + L_{10}^{TQ} - L_{11}^{TQ} \ , \\
L_9^{TQ} =& L_{13}^{FF} + 2L_{14}^{FF} + 2L_{12}^{FT} \ , \\
L_{13}^{TQ} =& 2L_9^{FT} + L_{21}^{FT} - L_{11}^{TQ} \ .
\end{aligned}
\tag{D.15}
$$

D.2 Maps

D.2.1 General MAG in Einstein form: $\phi \leftrightarrow TQ$ map

Concerning the dimension-two terms, the relation between the couplings in the Einstein form and those in the Cartan form has already been given in (3.10). Given the Lagrangian in the form (3.8), it can be rewritten in the form (3.11), where the couplings are related as follows:

$$\begin{aligned}
m_1^{TT} &= 1/4(3m_1^{\phi\phi} - 3m_2^{\phi\phi} + m_3^{\phi\phi} + m_4^{\phi\phi} - m_5^{\phi\phi})\ , \\
m_2^{TT} &= 1/2(m_1^{\phi\phi} - m_2^{\phi\phi} + m_3^{\phi\phi} + m_4^{\phi\phi} - m_5^{\phi\phi})\ , \\
m_3^{TT} &= m_6^{\phi\phi} + m_7^{\phi\phi} - m_9^{\phi\phi}\ , \\
m_1^{QQ} &= 1/4(3m_1^{\phi\phi} - m_2^{\phi\phi} + 3m_3^{\phi\phi} - m_4^{\phi\phi} - m_5^{\phi\phi})\ , \\
m_2^{QQ} &= 1/2(-m_1^{\phi\phi} + m_2^{\phi\phi} - m_3^{\phi\phi} + m_4^{\phi\phi} + m_5^{\phi\phi})\ , \\
m_3^{QQ} &= 1/4(m_6^{\phi\phi} + m_7^{\phi\phi} + m_8^{\phi\phi} - m_9^{\phi\phi} + m_{10}^{\phi\phi} - m_{11}^{\phi\phi})\ , \\
m_4^{QQ} &= m_7^{\phi\phi}\ , \\
m_5^{QQ} &= 1/2(-2m_7^{\phi\phi} + m_9^{\phi\phi} + m_{11}^{\phi\phi})\ , \\
m_1^{TQ} &= -2m_1^{\phi\phi} + 2m_2^{\phi\phi} - 2m_3^{\phi\phi} + m_5^{\phi\phi}\ , \\
m_2^{TQ} &= 1/2(2m_6^{\phi\phi} + 2m_7^{\phi\phi} - 2m_9^{\phi\phi} + m_{10}^{\phi\phi} - m_{11}^{\phi\phi})\ , \\
m_3^{TQ} &= -2m_7^{\phi\phi} + m_9^{\phi\phi}\ .
\end{aligned} \tag{D.16}$$

Similarly, the dimension-four terms are related as follows: [1]:

$$\begin{aligned}
b_3^{RT} &= b_8^{R\phi} - b_9^{R\phi}\ , & b_5^{RT} &= b_{11}^{R\phi} - b_{12}^{R\phi}\ , \\
b_4^{RQ} &= 1/2(b_7^{R\phi} - b_8^{R\phi} + b_9^{R\phi})\ , & b_5^{RQ} &= b_8^{R\phi}\ , \\
b_6^{RQ} &= 1/2(b_{10}^{R\phi} - b_{11}^{R\phi} + b_{12}^{R\phi})\ , & b_7^{RQ} &= b_{11}^{R\phi}\ ,
\end{aligned} \tag{D.17}$$

$$\begin{aligned}
b_1^{TT} &= 1/4(3b_1^{\phi\phi} - 3b_2^{\phi\phi} + b_3^{\phi\phi} + b_4^{\phi\phi} - b_5^{\phi\phi})\ , \\
b_2^{TT} &= 1/2(b_1^{\phi\phi} - b_2^{\phi\phi} + b_3^{\phi\phi} + b_4^{\phi\phi} - b_5^{\phi\phi})\ , \\
b_3^{TT} &= b_6^{\phi\phi} + b_7^{\phi\phi} - b_9^{\phi\phi}\ , \\
b_4^{TT} &= 1/2(b_{12}^{\phi\phi} - b_{13}^{\phi\phi} + b_{14}^{\phi\phi} + b_{15}^{\phi\phi} + b_{16}^{\phi\phi} + b_{17}^{\phi\phi} - b_{22}^{\phi\phi} - b_{23}^{\phi\phi})\ , \\
b_5^{TT} &= 1/2(-b_{12}^{\phi\phi} + b_{13}^{\phi\phi} + b_{14}^{\phi\phi} + b_{15}^{\phi\phi} + b_{16}^{\phi\phi} + b_{17}^{\phi\phi} - b_{22}^{\phi\phi} - b_{23}^{\phi\phi})\ , \\
b_6^{TT} &= 1/4(b_{12}^{\phi\phi} - b_{13}^{\phi\phi} + b_{14}^{\phi\phi} - b_{15}^{\phi\phi} + b_{16}^{\phi\phi} - b_{17}^{\phi\phi} + b_{18}^{\phi\phi} - b_{19}^{\phi\phi} - b_{20}^{\phi\phi} + b_{21}^{\phi\phi} - b_{22}^{\phi\phi} + b_{23}^{\phi\phi})\ , \\
b_7^{TT} &= 1/2(2b_{12}^{\phi\phi} - 2b_{13}^{\phi\phi} + b_{18}^{\phi\phi} - b_{19}^{\phi\phi} - b_{20}^{\phi\phi} + b_{21}^{\phi\phi})\ , \\
b_8^{TT} &= b_{24}^{\phi\phi} - b_{25}^{\phi\phi} - b_{27}^{\phi\phi} + b_{28}^{\phi\phi}\ , \\
b_9^{TT} &= b_{34}^{\phi\phi} + b_{35}^{\phi\phi} - b_{38}^{\phi\phi}\ ,
\end{aligned} \tag{D.18}$$

[1]We are choosing the basis (3.71) and basis (3.73)

$$
\begin{aligned}
&b_1^{QQ} = 1/4(3b_1^{\phi\phi} - b_2^{\phi\phi} + 3b_3^{\phi\phi} - b_4^{\phi\phi} - b_5^{\phi\phi}) \ , \\
&b_2^{QQ} = 1/2(-b_1^{\phi\phi} + b_2^{\phi\phi} - b_3^{\phi\phi} + b_4^{\phi\phi} + b_5^{\phi\phi}) \ , \qquad b_3^{QQ} = b_7^{\phi\phi} \ , \\
&b_4^{QQ} = 1/4(b_6^{\phi\phi} + b_7^{\phi\phi} + b_8^{\phi\phi} - b_9^{\phi\phi} + b_{10}^{\phi\phi} - b_{11}^{\phi\phi}) \ , \qquad b_5^{QQ} = 1/2(-2b_7^{\phi\phi} + b_9^{\phi\phi} + b_{11}^{\phi\phi}) \ , \\
&b_6^{QQ} = 1/4(b_{12}^{\phi\phi} + b_{13}^{\phi\phi} + b_{14}^{\phi\phi} + b_{15}^{\phi\phi} + b_{16}^{\phi\phi} + b_{17}^{\phi\phi} - b_{18}^{\phi\phi} - b_{19}^{\phi\phi} + b_{20}^{\phi\phi} + b_{21}^{\phi\phi} - b_{22}^{\phi\phi} - b_{23}^{\phi\phi}) \ , \\
&b_7^{QQ} = 1/2(b_{12}^{\phi\phi} - b_{13}^{\phi\phi} + b_{14}^{\phi\phi} + b_{15}^{\phi\phi} + b_{16}^{\phi\phi} - b_{17}^{\phi\phi} - b_{20}^{\phi\phi} + b_{21}^{\phi\phi}) \ , \\
&b_8^{QQ} = 1/2(-b_{12}^{\phi\phi} + b_{13}^{\phi\phi} + b_{14}^{\phi\phi} + b_{15}^{\phi\phi} - b_{16}^{\phi\phi} + b_{17}^{\phi\phi} + b_{20}^{\phi\phi} - b_{21}^{\phi\phi}) \ , \\
&b_9^{QQ} = 1/2(-2b_{14}^{\phi\phi} - 2b_{15}^{\phi\phi} + b_{18}^{\phi\phi} + b_{19}^{\phi\phi} + b_{22}^{\phi\phi} + b_{23}^{\phi\phi}) \ , \\
&b_{10}^{QQ} = b_{28}^{\phi\phi} \ , \qquad b_{11}^{QQ} = 1/2(b_{27}^{\phi\phi} - b_{28}^{\phi\phi} + b_{29}^{\phi\phi}) \ , \quad b_{12}^{QQ} = 1/2(b_{25}^{\phi\phi} - b_{28}^{\phi\phi} + b_{31}^{\phi\phi}) \ , \\
&b_{13}^{QQ} = 1/4(b_{24}^{\phi\phi} - b_{25}^{\phi\phi} + b_{26}^{\phi\phi} - b_{27}^{\phi\phi} + b_{28}^{\phi\phi} - b_{29}^{\phi\phi} + b_{30}^{\phi\phi} - b_{31}^{\phi\phi} + b_{32}^{\phi\phi}) \ , \\
&b_{14}^{QQ} = 1/4(b_{33}^{\phi\phi} + b_{34}^{\phi\phi} + b_{35}^{\phi\phi} - b_{36}^{\phi\phi} + b_{37}^{\phi\phi} - b_{38}^{\phi\phi}) \ , \\
&b_{15}^{QQ} = b_{34}^{\phi\phi} \ , \qquad b_{16}^{QQ} = 1/2(-2b_{34}^{\phi\phi} + b_{36}^{\phi\phi} + b_{38}^{\phi\phi}) \ ,
\end{aligned} \tag{D.19}
$$

$$
\begin{aligned}
&b_1^{TQ} = -2b_1^{\phi\phi} + 2b_2^{\phi\phi} - 2b_3^{\phi\phi} + b_5^{\phi\phi} \ , \qquad b_2^{TQ} = -2b_7^{\phi\phi} + b_9^{\phi\phi} \ , \\
&b_3^{TQ} = b_6^{\phi\phi} + b_7^{\phi\phi} - b_9^{\phi\phi} + (b_{10}^{\phi\phi} - b_{11}^{\phi\phi})/2 \ , \\
&b_4^{TQ} = 1/2(-2b_{14}^{\phi\phi} - 2b_{15}^{\phi\phi} - 2b_{16}^{\phi\phi} - 2b_{17}^{\phi\phi} + b_{18}^{\phi\phi} + b_{19}^{\phi\phi} - b_{20}^{\phi\phi} - b_{21}^{\phi\phi} + 2b_{22}^{\phi\phi} + 2b_{23}^{\phi\phi}) \ , \\
&b_5^{TQ} = 1/2(-2b_{12}^{\phi\phi} + 2b_{13}^{\phi\phi} - 2b_{16}^{\phi\phi} + 2b_{17}^{\phi\phi} - b_{18}^{\phi\phi} + b_{19}^{\phi\phi} + 2b_{20}^{\phi\phi} - 2b_{21}^{\phi\phi} + b_{22}^{\phi\phi} - b_{23}^{\phi\phi}) \ , \\
&b_6^{TQ} = 1/2(-2b_{12}^{\phi\phi} + 2b_{13}^{\phi\phi} + 2b_{14}^{\phi\phi} + 2b_{15}^{\phi\phi} + b_{20}^{\phi\phi} - b_{21}^{\phi\phi} - b_{22}^{\phi\phi} - b_{23}^{\phi\phi}) \ , \\
&b_7^{TQ} = 1/2(2b_{12}^{\phi\phi} - 2b_{13}^{\phi\phi} + 2b_{14}^{\phi\phi} + 2b_{15}^{\phi\phi} - b_{20}^{\phi\phi} + b_{21}^{\phi\phi} - b_{22}^{\phi\phi} - b_{23}^{\phi\phi}) \ , \\
&b_8^{TQ} = b_{25}^{\phi\phi} - b_{28}^{\phi\phi} \ , \qquad b_9^{TQ} = 1/2(b_{24}^{\phi\phi} - b_{25}^{\phi\phi} + b_{26}^{\phi\phi} - b_{27}^{\phi\phi} + b_{28}^{\phi\phi} - b_{29}^{\phi\phi}) \ , \\
&b_{10}^{TQ} = b_{27}^{\phi\phi} - b_{28}^{\phi\phi} \ , \qquad b_{11}^{TQ} = 1/2(b_{24}^{\phi\phi} - b_{25}^{\phi\phi} - b_{27}^{\phi\phi} + b_{28}^{\phi\phi} + b_{30}^{\phi\phi} - b_{31}^{\phi\phi}) \ , \\
&b_{12}^{TQ} = 1/2(-2b_{34}^{\phi\phi} - 2b_{35}^{\phi\phi} + b_{36}^{\phi\phi} - b_{37}^{\phi\phi} + 2b_{38}^{\phi\phi}) \ , \qquad b_{13}^{TQ} = 2b_{34}^{\phi\phi} - b_{38}^{\phi\phi} \ .
\end{aligned} \tag{D.20}
$$

D.2.2 Einstein ↔ Cartan map

Here we report the linear map between the coefficients of the general MAG Lagrangian in the Cartan form and in the Einstein form. In order not to rely on a particular basis, we give the general relation between the linearly dependent terms, namely the map from the 99 c-type coefficients to the 53 b-type coefficients. In order to derive the map between coefficients in fixed bases, one has to remove from the r.h.s. all the c-coefficients that are not part of the Cartan basis and from the l.h.s. all the b-coefficients that are not part of the Einstein basis (this happens only in the b^{RT} and b^{RQ} sectors).

$$
\begin{aligned}
&b_1^{RR} = c_1^{FF} - c_2^{FF} + c_3^{FF} + (c_4^{FF} - c_5^{FF} + c_6^{FF})/2 \ , \\
&b_2^{RR} = c_7^{FF} + c_8^{FF} + c_9^{FF} + c_{10}^{FF} - c_{11}^{FF} - c_{12}^{FF} \ , \\
&b_3^{RR} = c_{16}^{FF} \ ,
\end{aligned} \tag{D.21}
$$

$$
\begin{aligned}
b_1^{RT} + 2b_2^{RT} + b_3^{RT} &= 8c_1^{FF} - 8c_2^{FF} + 8c_3^{FF} + 4c_4^{FF} - 4c_5^{FF} + 4c_6^{FF} + 2c_7^{FF} + 2c_8^{FF} + 2c_9^{FF} + 2c_{10}^{FF} \\
&\quad - 2c_{11}^{FF} - 2c_{12}^{FF} + c_1^{FT} - c_2^{FT} + 2c_3^{FT} + c_4^{FT} + 2c_5^{FT} \\
&\quad - c_6^{FT} - 2c_7^{FT} + c_{13}^{FT} + c_{14}^{FT} - c_{15}^{FT} - c_{16}^{FT} \, , \\
-1/2 b_4^{RT} + b_5^{RT} &= c_7^{FF} + c_8^{FF} + c_9^{FF} + c_{10}^{FF} - c_{11}^{FF} - c_{12}^{FF} + 4c_{16}^{FF} \\
&\quad - 1/2(c_8^{FT} + c_9^{FT}) + 1/2(c_{10}^{FT} + c_{11}^{FT}) + c_{21}^{FT} \, , \\
b_1^{RQ} + b_4^{RQ} &= -4c_1^{FF} + 4c_2^{FF} - 4c_3^{FF} - 2c_4^{FF} + 2c_5^{FF} - 2c_6^{FF} - c_7^{FF} - c_8^{FF} - c_9^{FF} - c_{10}^{FF} \\
&\quad + c_{11}^{FF} + c_{12}^{FF} - c_2^{FQ} + c_3^{FQ} + c_4^{FQ} - c_5^{FQ} + c_{16}^{FQ} - c_{17}^{FQ} \, , \\
-b_1^{RQ} + b_5^{RQ} &= 4c_1^{FF} - 4c_2^{FF} + 4c_3^{FF} + 2c_4^{FF} - 2c_5^{FF} + 2c_6^{FF} + 2c_7^{FF} + 2c_8^{FF} - c_{11}^{FF} - c_{12}^{FF} \\
&\quad + c_2^{FQ} - c_3^{FQ} - c_4^{FQ} + c_5^{FQ} + c_{18}^{FQ} + c_{19}^{FQ} - c_{20}^{FQ} - c_{21}^{FQ} \, , \\
b_3^{RQ} + 2b_6^{RQ} &= -c_7^{FF} - c_8^{FF} - c_9^{FF} - c_{10}^{FF} + c_{11}^{FF} + c_{12}^{FF} - 4c_{16}^{FF} \\
&\quad + c_8^{FQ} + c_9^{FQ} - c_{12}^{FQ} - c_{13}^{FQ} + 2c_{23}^{FQ} \, , \\
b_2^{RQ} + 2b_7^{RQ} &= 2c_9^{FF} + 2c_{10}^{FF} - c_{11}^{FF} - c_{12}^{FF} + 4c_{16}^{FF} + c_6^{FQ} + c_7^{FQ} - c_{10}^{FQ} - c_{11}^{FQ} + 2c_{24}^{FQ} \, ,
\end{aligned}
\tag{D.22}
$$

$$
\begin{aligned}
b_1^{TT} &= c_1^{TT} + (6c_1^{FF} - 6c_2^{FF} + c_4^{FF} - c_5^{FF} + c_6^{FF} + 2c_1^{FT} - 2c_2^{FT} + 2c_3^{FT})/4 \, , \\
b_2^{TT} &= c_2^{TT} + (2c_1^{FF} - 2c_2^{FF} + c_4^{FF} - c_5^{FF} + c_6^{FF} + 2c_3^{FT})/2 \, , \\
b_3^{TT} &= c_3^{TT} + c_7^{FF} + c_9^{FF} - c_{11}^{FF} - c_8^{FT} + c_{10}^{FT} \, , \\
b_4^{TT} &= c_4^{TT} + (-2c_1^{FF} + 2c_2^{FF} + 4c_3^{FF} + c_4^{FF} - c_5^{FF} + c_6^{FF} + c_7^{FF} + c_8^{FF} + c_9^{FF} + c_{10}^{FF} \\
&\quad - c_{11}^{FF} - c_{12}^{FF} - c_1^{FT} + c_2^{FT} + c_4^{FT} + 2c_5^{FT} - c_6^{FT} - 2c_7^{FT} + c_{13}^{FT} + c_{14}^{FT} - c_{15}^{FT} - c_{16}^{FT})/2 \, , \\
b_5^{TT} &= c_5^{TT} + (2c_1^{FF} - 2c_2^{FF} + 4c_3^{FF} + c_4^{FF} - c_5^{FF} + c_6^{FF} + c_7^{FF} + c_8^{FF} + c_9^{FF} + c_{10}^{FF} - c_{11}^{FF} \\
&\quad - c_{12}^{FF} + c_1^{FT} - c_2^{FT} + c_4^{FT} + 2c_5^{FT} - c_6^{FT} - 2c_7^{FT} + c_{13}^{FT} + c_{14}^{FT} - c_{15}^{FT} - c_{16}^{FT})/2 \, , \\
b_6^{TT} &= c_6^{TT} + (-2c_1^{FF} + 2c_2^{FF} - 4c_3^{FF} - c_4^{FF} + c_5^{FF} - c_6^{FF} + c_7^{FF} - c_8^{FF} + c_9^{FF} - c_{10}^{FF} \\
&\quad - c_{11}^{FF} + c_{12}^{FF} - 2c_3^{FT} - 2c_4^{FT} + 2c_6^{FT} + 2c_{17}^{FT} - 2c_{18}^{FT})/4 \, , \\
b_7^{TT} &= c_7^{TT} - 2c_1^{FF} + 2c_2^{FF} - c_4^{FF} + c_5^{FF} - c_6^{FF} \\
&\quad + (-c_1^{FT} + c_2^{FT} - 2c_3^{FT} + c_4^{FT} - 2c_5^{FT} - c_6^{FT} + 2c_7^{FT} + c_{13}^{FT} - c_{14}^{FT} - c_{15}^{FT} + c_{16}^{FT})/2 \, , \\
b_8^{TT} &= c_8^{TT} - 2c_7^{FF} - 2c_9^{FF} + 2c_{11}^{FF} + c_8^{FT} - c_{10}^{FT} - c_{13}^{FT} + c_{15}^{FT} - c_{17}^{FT} + c_{18}^{FT} \, , \\
b_9^{TT} &= c_9^{TT} + c_8^{FF} + c_{10}^{FF} - c_{12}^{FF} + 4c_{16}^{FF} - c_9^{FT} + c_{11}^{FT} + 2c_{21}^{FT} \, ,
\end{aligned}
\tag{D.23}
$$

$$
\begin{aligned}
b_1^{QQ} &= c_1^{QQ} + (6c_1^{FF} - 2c_2^{FF} - c_4^{FF} - c_5^{FF} + 3c_6^{FF})/4 + (c_1^{FQ} + c_2^{FQ} - c_3^{FQ})/2 \ , \\
b_2^{QQ} &= c_2^{QQ} + (-2c_1^{FF} + 2c_2^{FF} + c_4^{FF} + c_5^{FF} - c_6^{FF})/2 - c_2^{FQ} \ , \\
b_3^{QQ} &= c_3^{QQ} + c_9^{FF} - c_{10}^{FQ} \ , \\
b_4^{QQ} &= c_4^{QQ} + (c_7^{FF} + c_9^{FF} - c_{11}^{FF} + 2c_{13}^{FF} - c_{14}^{FF} + c_{15}^{FF})/4 + (-c_8^{FQ} + c_{12}^{FQ} + c_{15}^{FQ})/2 \ , \\
b_5^{QQ} &= c_5^{QQ} + (-2c_9^{FF} + c_{11}^{FF} - c_{15}^{FF} - c_6^{FQ} + c_{10}^{FQ} - 2c_{12}^{FQ} + c_{14}^{FQ})/2 \ , \\
b_6^{QQ} &= c_6^{QQ} + (-2c_1^{FF} - 2c_2^{FF} + 4c_3^{FF} + 3c_4^{FF} - c_5^{FF} - c_6^{FF} + c_7^{FF} + c_8^{FF} + c_9^{FF} + c_{10}^{FF} \\
&\quad - c_{11}^{FF} - c_{12}^{FF})/4 + (-c_1^{FQ} - c_4^{FQ} + c_5^{FQ} - c_{16}^{FQ} + c_{17}^{FQ})/2 \ , \\
b_7^{QQ} &= c_7^{QQ} + (-2c_1^{FF} + 2c_2^{FF} + c_4^{FF} + c_5^{FF} - c_6^{FF} + c_7^{FF} + c_8^{FF} + c_9^{FF} - c_{10}^{FF} \\
&\quad - c_2^{FQ} + c_3^{FQ} - c_4^{FQ} - c_5^{FQ} + c_{18}^{FQ} + c_{19}^{FQ} + c_{20}^{FQ} - c_{21}^{FQ})/2 \ , \\
b_8^{QQ} &= c_8^{QQ} + (2c_1^{FF} - 2c_2^{FF} + 2c_3^{FF} + c_4^{FF} - c_5^{FF} + c_6^{FF} + c_7^{FF} + c_8^{FF} - c_9^{FF} + c_{10}^{FF} \\
&\quad + c_2^{FQ} - c_3^{FQ} - c_4^{FQ} + c_5^{FQ} + c_{18}^{FQ} + c_{19}^{FQ} - c_{20}^{FQ} + c_{21}^{FQ})/2 \ , \\
b_9^{QQ} &= c_9^{QQ} - 2c_3^{FF} - 2c_4^{FF} - c_7^{FF} - c_8^{FF} + (c_{11}^{FF} + c_{12}^{FF})/2 \\
&\quad + (c_2^{FQ} + c_3^{FQ} + 3c_4^{FQ} - c_5^{FQ} + 2c_{16}^{FQ} - c_{18}^{FQ} - c_{19}^{FQ} + c_{20}^{FQ} + c_{21}^{FQ})/2 \ , \\
b_{10}^{QQ} &= c_{10}^{QQ} - 2c_9^{FF} + c_{10}^{FQ} - c_{17}^{FQ} - c_{20}^{FQ} \ , \\
b_{11}^{QQ} &= c_{11}^{QQ} + (2c_9^{FF} - c_{11}^{FF} + c_{15}^{FF} + 2c_{12}^{FQ} - c_{16}^{FQ} + c_{17}^{FQ} - c_{18}^{FQ} + c_{20}^{FQ} + c_{22}^{FQ})/2 \ , \\
b_{12}^{QQ} &= c_{12}^{QQ} + (2c_9^{FF} - 2c_{10}^{FF} - c_{11}^{FF} - c_{12}^{FF} + c_6^{FQ} + c_7^{FQ} - c_{10}^{FQ} + c_{11}^{FQ} - 2c_{21}^{FQ})/2 \ , \\
b_{13}^{QQ} &= c_{13}^{QQ} + (-c_7^{FF} - c_8^{FF} - c_9^{FF} + c_{10}^{FF} + c_{11}^{FF} - c_{15}^{FF} \\
&\quad + c_8^{FQ} + c_9^{FQ} - c_{12}^{FQ} + c_{13}^{FQ} - c_{19}^{FQ} + c_{21}^{FQ} - c_{22}^{FQ})/2 \ , \\
b_{14}^{QQ} &= c_{14}^{QQ} + (c_8^{FF} + c_{10}^{FF} - c_{12}^{FF} - 2c_{13}^{FF} + c_{14}^{FF} - c_{15}^{FF})/4 + c_{16}^{FF} \\
&\quad + (-c_9^{FQ} + c_{13}^{FQ} - c_{15}^{FQ} - 2c_{23}^{FQ})/2 \ , \\
b_{15}^{QQ} &= c_{15}^{QQ} + c_{10}^{FF} + c_{16}^{FF} - c_{11}^{FQ} + c_{24}^{FQ} \ , \\
b_{16}^{QQ} &= c_{16}^{QQ} + (-2c_{10}^{FF} + c_{12}^{FF} + c_{15}^{FF} - 4c_{16}^{FF} - c_7^{FQ} + c_{11}^{FQ} - 2c_{13}^{FQ} - c_{14}^{FQ} + 2c_{23}^{FQ} - 2c_{24}^{FQ})/2 \ ,
\end{aligned}
\tag{D.24}
$$

$$
\begin{aligned}
b_1^{TQ} &= c_1^{TQ} - 4c_1^{FF} + 4c_2^{FF} + c_5^{FF} - 2c_6^{FF} + c_2^{FT} - c_3^{FT} - c_2^{FQ} + c_3^{FQ} \ , \\
b_2^{TQ} &= c_2^{TQ} - 2c_9^{FF} + c_{11}^{FF} - c_{10}^{FT} - c_6^{FQ} + c_{10}^{FQ} \ , \\
b_3^{TQ} &= c_3^{TQ} + c_7^{FF} + c_9^{FF} - c_{11}^{FF} - c_{14}^{FF}/2 + c_{15}^{FF}/2 + (-c_8^{FT} + c_{10}^{FT} + c_{12}^{FT})/2 - c_8^{FQ} + c_{12}^{FQ} \ , \\
b_4^{TQ} &= c_4^{TQ} - 4c_3^{FF} - 2c_4^{FF} + c_5^{FF} - c_7^{FF} - c_8^{FF} - c_9^{FF} - c_{10}^{FF} + c_{11}^{FF} + c_{12}^{FF} \\
&\quad + (-c_1^{FT} - c_2^{FT} - c_4^{FT} - 2c_5^{FT} + c_6^{FT} + 2c_7^{FT} - c_{13}^{FT} - c_{14}^{FT} + c_{15}^{FT} + c_{16}^{FT})/2 \\
&\quad + c_4^{FQ} - c_5^{FQ} + c_{16}^{FQ} - c_{17}^{FQ} \ , \\
b_5^{TQ} &= c_5^{TQ} + 2c_1^{FF} - 2c_2^{FF} + 2c_3^{FF} + c_4^{FF} - c_5^{FF} + c_6^{FF} - c_9^{FF} + c_{10}^{FF} + (c_{11}^{FF} - c_{12}^{FF})/2 \\
&\quad + c_3^{FT} - c_6^{FT} + c_{18}^{FT} + (c_2^{FQ} - c_3^{FQ} - c_4^{FQ} + c_5^{FQ} + c_{18}^{FQ} - c_{19}^{FQ} - c_{20}^{FQ} + c_{21}^{FQ})/2 \ , \\
b_6^{TQ} &= c_6^{TQ} + 2c_1^{FF} - 2c_2^{FF} + 2c_3^{FF} + c_4^{FF} - c_5^{FF} + c_6^{FF} + c_7^{FF} + c_8^{FF} - (c_{11}^{FF} + c_{12}^{FF})/2 \\
&\quad + (c_1^{FT} - c_2^{FT} + c_4^{FT} + 2c_5^{FT} + c_6^{FT} - 2c_7^{FT} + c_{13}^{FT} + c_{14}^{FT} + c_{15}^{FT} - c_{16}^{FT} \\
&\quad + c_2^{FQ} - c_3^{FQ} - c_4^{FQ} + c_5^{FQ} + c_{18}^{FQ} + c_{19}^{FQ} - c_{20}^{FQ} - c_{21}^{FQ})/2 \ ,
\end{aligned}
\tag{D.25}
$$

$$
\begin{aligned}
b_7^{TQ} &= c_7^{TQ} - 2c_1^{FF} + 2c_2^{FF} + 2c_3^{FF} + c_4^{FF} - c_6^{FF} + c_7^{FF} + c_8^{FF} - (c_{11}^{FF} + c_{12}^{FF})/2 \\
&\quad + (-c_1^{FT} + c_2^{FT} + c_4^{FT} + 2c_5^{FT} - c_6^{FT} + 2c_7^{FT} + c_{13}^{FT} + c_{14}^{FT} - c_{15}^{FT} + c_{16}^{FT})/2 \\
&\quad + (-c_2^{FQ} + c_3^{FQ} - c_4^{FQ} + c_5^{FQ} + c_{18}^{FQ} + c_{19}^{FQ} - c_{20}^{FQ} - c_{21}^{FQ})/2 \ , \\
b_8^{TQ} &= c_8^{TQ} + 2c_9^{FF} - c_{11}^{FF} - c_{15}^{FT} - c_{18}^{FT} + c_6^{FQ} - c_{10}^{FQ} \ , \\
b_9^{TQ} &= c_9^{TQ} + (-2c_7^{FF} - 2c_9^{FF} + 2c_{11}^{FF} + c_{14}^{FF} - c_{15}^{FF} \\
&\qquad - c_{13}^{FT} + c_{15}^{FT} - c_{17}^{FT} + c_{18}^{FT} + c_{19}^{FT} + 2c_{20}^{FT} + 2c_8^{FQ} - 2c_{12}^{FQ})/2 \ , \\
b_{10}^{TQ} &= c_{10}^{TQ} + 2c_9^{FF} - c_{11}^{FF} + c_{10}^{FT} - c_{16}^{FQ} + c_{17}^{FQ} - c_{18}^{FQ} + c_{20}^{FQ} \ , \\
b_{11}^{TQ} &= c_{11}^{TQ} - c_7^{FF} - c_8^{FF} - c_9^{FF} + c_{10}^{FF} + c_{11}^{FF} + (c_8^{FT} + c_9^{FT} - c_{10}^{FT} + c_{11}^{FT})/2 - c_{19}^{FQ} + c_{21}^{FQ} \ , \\
b_{12}^{TQ} &= c_{12}^{TQ} + (-2c_8^{FF} - 2c_{10}^{FF} + 2c_{12}^{FF} - c_{14}^{FF} + c_{15}^{FF} - 8c_{16}^{FF} \\
&\qquad + c_9^{FT} - c_{11}^{FT} + c_{12}^{FT} - 2c_{21}^{FT} + 2c_9^{FQ} - 2c_{13}^{FQ} + 4c_{23}^{FQ})/2 \ , \\
b_{13}^{TQ} &= c_{13}^{TQ} + 2c_{10}^{FF} - c_{12}^{FF} + 4c_{16}^{FF} + c_{11}^{FT} + c_{21}^{FT} + c_7^{FQ} - c_{11}^{FQ} + 2c_{24}^{FQ} \ .
\end{aligned}
\tag{D.26}
$$

D.2.3 Map with decomposed variables

$$
\begin{aligned}
m_1 &= \frac{2}{3}m_1^{TT} + \frac{1}{3}m_2^{TT} + m_3^{TT}, \\
m_2 &= \left(-m_1^{TT} + m_2^{TT}\right)/6, \\
m_3 &= m_1^{TT} + \frac{1}{2}m_2^{TT}, \\
m_4 &= \frac{5}{18}m_1^{QQ} - \frac{1}{18}m_2^{QQ} + m_3^{QQ}, \\
m_5 &= \frac{4}{9}m_1^{QQ} + \frac{1}{9}m_2^{QQ} + m_4^{QQ}, \\
m_6 &= -\frac{2}{9}m_1^{QQ} + \frac{4}{9}m_2^{QQ} + m_5^{QQ} \\
m_7 &= m_1^{QQ} + m_2^{QQ}, \\
m_8 &= m_1^{QQ} - \frac{1}{2}m_2^{QQ}, \\
m_9 &= \frac{1}{3}m_1^{TQ} - m_2^{TQ}, \\
m_{10} &= -\frac{1}{3}m_1^{TQ} - m_3^{TQ}, \\
m_{11} &= m_1^{TQ} \ ,
\end{aligned}
\tag{D.27}
$$

$$\begin{aligned}
r_1 &= \frac{1}{6}b_3^{RT} + b_5^{RT}, \\
r_2 &= b_3^{RT}, \\
r_3 &= \frac{2}{9}b_4^{RQ} + \frac{1}{18}b_5^{RQ} + b_6^{RQ}, \\
r_4 &= \frac{1}{9}b_4^{RQ} + \frac{5}{18}b_5^{RQ} + b_7^{RQ}, \\
r_5 &= b_4^{RQ} + b_5^{RQ}, \\
r_6 &= b_4^{RQ} - \frac{1}{2}b_5^{RQ} \ ,
\end{aligned} \tag{D.28}$$

$$\begin{aligned}
d_1 &= -\frac{2}{3}b_1^{TT} - \frac{1}{3}b_2^{TT} - b_3^{TT} - \frac{1}{9}b_4^{TT} - \frac{2}{9}b_6^{TT} - \frac{1}{9}b_7^{TT} - \frac{1}{3}b_8^{TT}, \\
d_2 &= -\frac{2}{9}b_4^{TT} - \frac{1}{3}b_5^{TT} + \frac{2}{9}b_6^{TT} + \frac{1}{9}b_7^{TT} + \frac{1}{3}b_8^{TT} - b_9^{TT}, \\
d_3 &= \left(3b_1^{TT} - 3b_2^{TT} + b_4^{TT} - b_5^{TT} + b_6^{TT} - b_7^{TT}\right)/18, \\
d_4 &= \left(-b_4^{TT} + b_5^{TT} - b_6^{TT} + b_7^{TT}\right)/18, \\
d_5 &= -b_1^{TT} - \frac{1}{2}b_2^{TT},
\end{aligned} \tag{D.29}$$

$$\begin{aligned}
d_6 &= -b_4^{TT} - 2b_6^{TT} - b_7^{TT}, \\
d_7 &= -b_5^{TT} + 2b_6^{TT} + b_7^{TT}, \\
d_8 &= -\frac{2}{3}b_4^{TT} - \frac{4}{3}b_6^{TT} - \frac{2}{3}b_7^{TT} - b_8^{TT}, \\
d_9 &= \frac{1}{3}b_4^{TT} - \frac{1}{3}b_5^{TT} - \frac{2}{3}b_6^{TT} + \frac{1}{6}b_7^{TT}, \\
d_{10} &= -\frac{5}{18}b_1^{QQ} + \frac{1}{18}b_2^{QQ} - b_4^{QQ} - \frac{1}{162}b_6^{QQ} - \frac{13}{162}b_7^{QQ} + \frac{5}{162}b_8^{QQ} + \frac{1}{81}b_9^{QQ} + \frac{1}{18}b_{11}^{QQ} - \frac{5}{18}b_{13}^{QQ},
\end{aligned} \tag{D.30}$$

$$\begin{aligned}
d_{11} &= -\frac{41}{162}b_6^{QQ} + \frac{7}{162}b_7^{QQ} - \frac{11}{162}b_8^{QQ} + \frac{1}{162}b_9^{QQ} - \frac{2}{9}b_{11}^{QQ} + \frac{1}{9}b_{13}^{QQ} - b_{14}^{QQ}, \\
d_{12} &= -\frac{4}{9}b_1^{QQ} - \frac{1}{9}b_2^{QQ} - b_3^{QQ} - \frac{8}{81}b_6^{QQ} - \frac{5}{81}b_7^{QQ} + \frac{4}{81}b_8^{QQ} - \frac{2}{81}b_9^{QQ} - \frac{2}{9}b_{10}^{QQ} + \frac{1}{9}b_{12}^{QQ}, \\
d_{13} &= -\frac{4}{81}b_6^{QQ} - \frac{16}{81}b_7^{QQ} - \frac{25}{81}b_8^{QQ} - \frac{1}{81}b_9^{QQ} - \frac{1}{9}b_{10}^{QQ} - \frac{4}{9}b_{12}^{QQ} - b_{15}^{QQ}, \\
d_{14} &= \frac{2}{9}b_1^{QQ} - \frac{4}{9}b_2^{QQ} - b_5^{QQ} + \frac{4}{81}b_6^{QQ} + \frac{7}{81}b_7^{QQ} - \frac{11}{81}b_8^{QQ} - \frac{7}{162}b_9^{QQ} + \frac{1}{18}b_{10}^{QQ} - \frac{2}{9}b_{11}^{QQ} - \frac{5}{18}b_{12}^{QQ} + \frac{1}{9}b_{13}^{QQ}, \\
d_{15} &= \frac{2}{81}b_6^{QQ} - \frac{10}{81}b_7^{QQ} + \frac{8}{81}b_8^{QQ} - \frac{22}{81}b_9^{QQ} - \frac{2}{9}b_{10}^{QQ} - \frac{1}{9}b_{11}^{QQ} + \frac{1}{9}b_{12}^{QQ} - \frac{4}{9}b_{13}^{QQ} - b_{16}^{QQ},
\end{aligned} \tag{D.31}$$

$$
\begin{aligned}
d_{16} &= -b_1^{QQ} - b_2^{QQ},\\
d_{17} &= -b_6^{QQ} - b_7^{QQ} - b_8^{QQ} - b_9^{QQ},\\
d_{18} &= -b_1^{QQ} + \frac{1}{2}b_2^{QQ},\\
d_{19} &= -b_6^{QQ} - \frac{1}{2}b_8^{QQ} + b_9^{QQ},\\
d_{20} &= -b_7^{QQ} + b_8^{QQ},\\
d_{21} &= -2b_6^{QQ} + b_7^{QQ} + b_8^{QQ} - \frac{1}{2}b_9^{QQ},\\
d_{22} &= \frac{2}{9}b_6^{QQ} - \frac{4}{9}b_7^{QQ} - \frac{4}{9}b_8^{QQ} - \frac{1}{9}b_9^{QQ} - b_{11}^{QQ} - b_{13}^{QQ},\\
d_{23} &= -\frac{8}{9}b_6^{QQ} - \frac{2}{9}b_7^{QQ} - \frac{2}{9}b_8^{QQ} - \frac{5}{9}b_9^{QQ} - b_{10}^{QQ} - b_{12}^{QQ},\\
d_{24} &= -\frac{1}{9}b_6^{QQ} - \frac{11}{18}b_7^{QQ} + \frac{7}{18}b_8^{QQ} + \frac{5}{36}b_9^{QQ} + \frac{1}{2}b_{11}^{QQ} - b_{13}^{QQ},\\
d_{25} &= \frac{4}{9}b_6^{QQ} + \frac{4}{9}b_7^{QQ} - \frac{5}{9}b_8^{QQ} - \frac{1}{18}b_9^{QQ} + \frac{1}{2}b_{10}^{QQ} - b_{12}^{QQ},
\end{aligned}
\tag{D.32}
$$

$$
\begin{aligned}
d_{26} &= -\frac{1}{3}b_1^{TQ} + b_3^{TQ} - \frac{1}{54}b_4^{TQ} - \frac{1}{9}b_5^{TQ} - \frac{1}{54}b_6^{TQ} + \frac{5}{54}b_7^{TQ} + \frac{1}{3}b_9^{TQ} - \frac{1}{18}b_{10}^{TQ} + \frac{5}{18}b_{11}^{TQ},\\
d_{27} &= -\frac{7}{27}b_4^{TQ} + \frac{1}{9}b_5^{TQ} + \frac{2}{27}b_6^{TQ} - \frac{1}{27}b_7^{TQ} - \frac{1}{3}b_9^{TQ} + \frac{2}{9}b_{10}^{TQ} - \frac{1}{9}b_{11}^{TQ} - b_{12}^{TQ},\\
d_{28} &= \frac{1}{3}b_1^{TQ} + b_2^{TQ} + \frac{2}{27}b_4^{TQ} + \frac{1}{9}b_5^{TQ} + \frac{2}{27}b_6^{TQ} - \frac{1}{27}b_7^{TQ} + \frac{1}{3}b_8^{TQ} + \frac{2}{9}b_{10}^{TQ} - \frac{1}{9}b_{11}^{TQ},\\
d_{29} &= \frac{1}{27}b_4^{TQ} - \frac{1}{9}b_5^{TQ} - \frac{8}{27}b_6^{TQ} - \frac{5}{27}b_7^{TQ} - \frac{1}{3}b_8^{TQ} + \frac{1}{9}b_{10}^{TQ} + \frac{4}{9}b_{11}^{TQ} - b_{13}^{TQ},
\end{aligned}
\tag{D.33}
$$

$$
\begin{aligned}
d_{30} &= -\frac{1}{18}b_4^{TQ} - \frac{1}{3}b_5^{TQ} - \frac{1}{18}b_6^{TQ} + \frac{5}{18}b_7^{TQ} + b_9^{TQ},\\
d_{31} &= \frac{2}{9}b_4^{TQ} + \frac{1}{3}b_5^{TQ} + \frac{2}{9}b_6^{TQ} - \frac{1}{9}b_7^{TQ} + b_8^{TQ},\\
d_{32} &= \frac{1}{3}(b_4^{TQ} + b_6^{TQ} + b_7^{TQ}) + b_{10}^{TQ} + b_{11}^{TQ},\\
d_{33} &= -\frac{1}{6}b_4^{TQ} - \frac{1}{2}b_5^{TQ} - \frac{1}{6}b_6^{TQ} + \frac{1}{3}b_7^{TQ} - \frac{1}{2}b_{10}^{TQ} + b_{11}^{TQ},\\
d_{34} &= \left(-b_5^{TQ} + b_6^{TQ} - b_7^{TQ}\right)/6,\\
d_{35} &= -b_4^{TQ} - b_6^{TQ} - b_7^{TQ},\\
d_{36} &= -b_1^{TQ},\\
d_{37} &= -b_4^{TQ} + b_5^{TQ} + b_6^{TQ},\\
d_{38} &= -2b_5^{TQ} - b_6^{TQ} + b_7^{TQ},
\end{aligned}
\tag{D.34}
$$

D.3 Some variational expressions

$$
\begin{aligned}
\Delta b_1^{QQ} &= 0 \ , \\
\Delta b_2^{QQ} &= 0 \ , \\
\Delta b_3^{QQ} &= \zeta_0(\xi_6^2+\xi_8^2) + 2\zeta_1\xi_6\xi_8 \ , \\
\Delta b_4^{QQ} &= \zeta_0(\xi_7^2+\xi_9^2) + 2\zeta_1\xi_7\xi_9 \ , \\
\Delta b_5^{QQ} &= 2\zeta_0(\xi_6\xi_7+\xi_8\xi_9) + 2\zeta_1(\xi_6\xi_9+\xi_7\xi_8) \ , \\
\Delta b_6^{QQ} &= (\zeta_0+\zeta_1)\xi_1^2 \ , \\
\Delta b_7^{QQ} &= \zeta_0(\xi_2^3+\xi_3^2+\xi_4^2+\xi_5^2+2\xi_2\xi_4+2\xi_3\xi_5) + 2\zeta_1(\xi_2+\xi_4)(\xi_3+\xi_5) \ , \\
\Delta b_8^{QQ} &= 2\zeta_0(\xi_2+\xi_5)(\xi_3+\xi_4) + \zeta_1((\xi_2+\xi_4)^2+(\xi_3+\xi_5)^2) \ , \\
\Delta b_9^{QQ} &= 2(\zeta_0+\zeta_1)\xi_1(\xi_2+\xi_3+\xi_4+\xi_5) \ , \\
\Delta b_{10}^{QQ} &= 2\zeta_0((\xi_1+\xi_2+\xi_4)\xi_6+(\xi_1+\xi_3+\xi_5)\xi_8) \\
&\quad + 2\zeta_1((\xi_1+\xi_3+\xi_5)\xi_6+(\xi_1+\xi_2+\xi_4)\xi_8) \ , \\
\Delta b_{11}^{QQ} &= 2\zeta_0((\xi_1+\xi_2+\xi_4)\xi_7+(\xi_1+\xi_3+\xi_5)\xi_9) \\
&\quad + 2\zeta_1((\xi_1+\xi_3+\xi_5)\xi_7+(\xi_1+\xi_2+\xi_4)\xi_9) \ , \\
\Delta b_{12}^{QQ} &= 2\zeta_0((\xi_3+\xi_5)\xi_6+(\xi_2+\xi_4)\xi_8) + 2\zeta_1((\xi_2+\xi_4)\xi_6+(\xi_3+\xi_5)\xi_8) \\
\Delta b_{13}^{QQ} &= 2\zeta_0((\xi_3+\xi_5)\xi_7+(\xi_2+\xi_4)\xi_9) + 2\zeta_1((\xi_2+\xi_4)\xi_7+(\xi_3+\xi_5)\xi_9) \\
\Delta b_{14}^{QQ} &= 2\zeta_0\xi_7\xi_9 + 2\zeta_1(\xi_7^2+\xi_9^2) + 2\zeta_2(\xi_1+\xi_7+\xi_9)^2 \\
\Delta b_{15}^{QQ} &= 2\zeta_0\xi_6\xi_8 + 2\zeta_1(\xi_6^2+\xi_8^2) + 2\zeta_2(\xi_2+\xi_3+\xi_4+\xi_5+\xi_6+\xi_8)^2 \\
\Delta b_{16}^{QQ} &= 2\zeta_0(\xi_6\xi_9+\xi_7\xi_8) + 2\zeta_1(\xi_6\xi_7+\xi_8\xi_9) + 2\zeta_2(\xi_2+\xi_3+\xi_4+\xi_5+\xi_6+\xi_8)(\xi_1+\xi_7+\xi_9) \ .
\end{aligned}
\tag{D.35}
$$

And the linearised version of the same relations is

$$
\begin{aligned}
\delta b_1^{QQ} &= 0 \ , \\
\delta b_2^{QQ} &= 0 \ , \\
\delta b_3^{QQ} &= \delta\zeta_0(\xi_6^2+\xi_8^2) + 2\delta\zeta_1\xi_6\xi_8 + 2\delta\xi_6(\zeta_0\xi_6+\zeta_1\xi_8) + 2\delta\xi_8(\zeta_0\xi_8+\zeta_1\xi_6) \ , \\
\delta b_4^{QQ} &= \delta\zeta_0(\xi_7^2+\xi_9^2) + 2\delta\zeta_1\xi_7\xi_9 + 2\delta\xi_7(\zeta_0\xi_7+\zeta_1\xi_9) + 2\delta\xi_9(\zeta_0\xi_9+\zeta_1\xi_7) \ ,
\end{aligned}
\tag{D.36}
$$

$$
\begin{aligned}
\delta b_5^{QQ} &= 2\delta\zeta_0(\xi_6\xi_7+\xi_8\xi_9) + 2\delta\zeta_1(\xi_6\xi_9+\xi_7\xi_8) \\
&\quad + 2\delta\xi_6(\zeta_0\xi_7+\zeta_1\xi_9) + 2\delta\xi_7(\zeta_0\xi_6+\zeta_1\xi_8) + 2\delta\xi_8(\zeta_0\xi_9+\zeta_1\xi_7) + 2\delta\xi_9(\zeta_0\xi_8+\zeta_1\xi_6) \ , \\
\delta b_6^{QQ} &= (\delta\zeta_0+\delta\zeta_1)\xi_1^2 + 2\delta\xi_1(\zeta_0+\zeta_1)\xi_1 \ , \\
\delta b_7^{QQ} &= \delta\zeta_0((\xi_2+\xi_4)^2+(\xi_3+\xi_5)^2) + 2\delta\zeta_1(\xi_2+\xi_4)(\xi_3+\xi_5) \\
&\quad + 2\delta\xi_2(\zeta_0(\xi_2+\xi_4)+\zeta_1(\xi_3+\xi_5)) + 2\delta\xi_3(\zeta_1(\xi_2+\xi_4)+\zeta_0(\xi_3+\xi_5)) \\
&\quad + 2\delta\xi_4(\zeta_0(\xi_2+\xi_4)+\zeta_1(\xi_3+\xi_5)) + 2\delta\xi_5(\zeta_1(\xi_2+\xi_4)+\zeta_0(\xi_3+\xi_5)) \ ,
\end{aligned}
\tag{D.37}
$$

$$
\begin{aligned}
\delta b_8^{QQ} &= 2\delta\zeta_0(\xi_2+\xi_4)(\xi_3+\xi_5)+\delta\zeta_1((\xi_2+\xi_4)^2+(\xi_3+\xi_5)^2)\\
&+2\delta\xi_2(\zeta_1(\xi_2+\xi_4)+\zeta_0(\xi_3+\xi_5))+2\delta\xi_3(\zeta_0(\xi_2+\xi_4)+\zeta_1(\xi_3+\xi_5))\\
&+2\delta\xi_4(\zeta_1(\xi_2+\xi_4)+\zeta_0(\xi_3+\xi_5))+2\delta\xi_5(\zeta_0(\xi_2+\xi_4)+\zeta_1(\xi_3+\xi_5))\ ,\\
\delta b_9^{QQ} &= 2\delta\zeta_0\xi_1(\xi_2+\xi_3+\xi_4+\xi_5)-2\delta\zeta_1\xi_1(\xi_2+\xi_3+\xi_4+\xi_5)+2\delta\xi_1(\zeta_0+\zeta_1)(\xi_2+\xi_3+\xi_4+\xi_5)\\
&+2\delta\xi_2(\zeta_0+\zeta_1)\xi_1+2\delta\xi_3(\zeta_0+\zeta_1)\xi_1+2\delta\xi_4(\zeta_0+\zeta_1)\xi_1+2\delta\xi_5(\zeta_0+\zeta_1)\xi_1\ ,
\end{aligned}
\tag{D.38}
$$

$$
\begin{aligned}
\delta b_{10}^{QQ} &= 2\delta\zeta_0((\xi_1+\xi_2+\xi_4)\xi_6+(\xi_1+\xi_3+\xi_5)\xi_8)+2\delta\zeta_1((\xi_1+\xi_3+\xi_5)\xi_6+(\xi_1+\xi_2+\xi_4)\xi_8)\\
&+2\delta\xi_1(\zeta_0+\zeta_1)(\xi_6+\xi_8)+2\delta\xi_2(\zeta_0\xi_6+\zeta_1\xi_8)+2\delta\xi_3(\zeta_1\xi_6+\zeta_0\xi_8)\\
&+2\delta\xi_4(\zeta_0\xi_6+\zeta_1\xi_8)+2\delta\xi_5(\zeta_1\xi_6+\zeta_0\xi_8)\\
&+2\delta\xi_6(\zeta_0(\xi_1+\xi_2+\xi_4)+\zeta_1(\xi_1+\ \xi_3+\xi_5))+2\delta\xi_8(\zeta_1(\xi_1+\xi_2+\xi_4)+\zeta_0(\xi_1+\ \xi_3+\xi_5))\ ,\\
\delta b_{11}^{QQ} &= 2\delta\zeta_0((\xi_1+\xi_2+\xi_4)\xi_7+(\xi_1+\xi_3+\xi_5)\xi_9)+2\delta\zeta_1((\xi_1+\xi_3+\xi_5)\xi_7+(\xi_1+\xi_2+\xi_4)\xi_9)\\
&+2\delta\xi_1(\zeta_0+\zeta_1)(\xi_7+\xi_9)+2\delta\xi_2(\zeta_0\xi_7+\zeta_1\xi_9)+2\delta\xi_3(\zeta_1\xi_7+\zeta_0\xi_9)+2\delta\xi_4(\zeta_0\xi_7+\zeta_1\xi_9)\\
&+2\delta\xi_5(\zeta_1\xi_7+\zeta_0\xi_9)+2\delta\xi_7(\zeta_0(\xi_1+\xi_2+\xi_4)+\zeta_1(\xi_1+\ \xi_3+\xi_5))\\
&+2\delta\xi_9(\zeta_1(\xi_1+\xi_2+\xi_4)+\zeta_0(\xi_1+\ \xi_3+\xi_5))\ ,
\end{aligned}
\tag{D.39}
$$

$$
\begin{aligned}
\delta b_{12}^{QQ} &= 2\delta\zeta_0((\xi_3+\xi_5)\xi_6+(\xi_2+\xi_4)\xi_8)+2\delta\zeta_1((\xi_2+\xi_4)\xi_6+(\xi_3+\xi_5)\xi_8)\\
&+2\delta\xi_2(\zeta_1\xi_6+\zeta_0\xi_8)+2\delta\xi_3(\zeta_0\xi_6+\zeta_1\xi_8)+2\delta\xi_4(\zeta_1\xi_6+\zeta_0\xi_8)+2\delta\xi_5(\zeta_0\xi_6+\zeta_1\xi_8)\\
&+2\delta\xi_6(\zeta_1(\xi_2+\xi_4)+\zeta_0(\xi_3+\xi_5))+2\delta\xi_8(\zeta_0(\xi_2+\xi_4)+\zeta_1(\xi_3+\xi_5))\ ,\\
\delta b_{13}^{QQ} &= 2\delta\zeta_0((\xi_3+\xi_5)\xi_7+(\xi_2+\xi_4)\xi_9)+2\delta\zeta_1((\xi_2+\xi_4)\xi_7+(\xi_3+\xi_5)\xi_9)\\
&+2\delta\xi_2(\zeta_1\xi_7+\zeta_0\xi_9)+2\delta\xi_3(\zeta_0\xi_7+\zeta_1\xi_9)+2\delta\xi_4(\zeta_1\xi_7+\zeta_0\xi_9)+2\delta\xi_5(\zeta_0\xi_7+\zeta_1\xi_9)\\
&+2\delta\xi_7(\zeta_1(\xi_2+\xi_4)+\zeta_0(\xi_3+\xi_5))+2\delta\xi_9(\zeta_0(\xi_2+\xi_4)+\zeta_1(\xi_3+\xi_5))\ ,
\end{aligned}
\tag{D.40}
$$

$$
\begin{aligned}
\delta b_{14}^{QQ} &= 2\delta\zeta_0\xi_7\xi_9+\delta\zeta_1(\xi_7^2+\xi_9^2)+\delta\zeta_2(\xi_1+\xi_7+\xi_9)^2\\
&+2\delta\xi_1\zeta_2(\xi_1+\xi_7+\xi_9)\\
&+2\delta\xi_7(\zeta_1\xi_7+\zeta_0\xi_9+\zeta_2(\xi_1+\xi_7+\xi_9))\\
&+2\delta\xi_9(\zeta_0\xi_7+\zeta_1\xi_9+\zeta_2(\xi_1+\xi_7+\xi_9))\\
\delta b_{15}^{QQ} &= 2\delta\zeta_0\xi_6\xi_8+\delta\zeta_1(\xi_6^2+\xi_8^2)+\delta\zeta_2(\xi_2+\xi_3+\xi_4+\xi_5+\xi_6+\xi_8)^2\\
&+2(\delta\xi_2+\delta\xi_3+\delta\xi_4+\delta\xi_5)\zeta_2(\xi_2+\xi_3+\xi_4+\xi_5+\xi_6+\xi_8)\\
&+2\delta\xi_6(\zeta_1\xi_6+\zeta_0\xi_8+\zeta_2(\xi_2+\ \xi_3+\xi_4+\xi_5+\xi_6+\xi_8))\\
&+2\delta\xi_8(\zeta_0\xi_6+\zeta_1\xi_8+\zeta_2(\xi_2+\ \xi_3+\xi_4+\xi_5+\xi_6+\xi_8))\\
\delta b_{16}^{QQ} &= 2\delta\zeta_0(\xi_7\xi_8+\xi_6\xi_9)+2\delta\zeta_1(\xi_6\xi_7+\xi_8\xi_9)+2\delta\zeta_2(\xi_2+\xi_3+\xi_4+\xi_5+\xi_6+\xi_8)(\ \xi_1+\xi_7+\xi_9)\\
&+2\delta\xi_1\zeta_2(\xi_2+\xi_3+\xi_4+\xi_5+\xi_6+\xi_8)+2(\delta\xi_2+\delta\xi_3+\delta\xi_4+\delta\xi_5)\zeta_2(\xi_1+\xi_7+\xi_9)\\
&+2\delta\xi_6(\zeta_1\xi_7+\zeta_0\xi_9+\zeta_2(\xi_1+\ \xi_7+\xi_9))+2\delta\xi_7(\zeta_1\xi_6+\zeta_0\xi_8+\zeta_2(\xi_2+\ \xi_3+\xi_4+\xi_5+\xi_6+\xi_8))\\
&+2\delta\xi_8(\zeta_0\xi_7+\zeta_1\xi_9+\zeta_2(\xi_1+\ \xi_7+\xi_9))+2\delta\xi_9(\zeta_0\xi_6+\zeta_1\xi_8+\zeta_2(\xi_2+\ \xi_3+\xi_4+\xi_5+\xi_6+\xi_8))\ .
\end{aligned}
\tag{D.41}
$$

Bibliography

[1] A. Einstein, “Zur einheitlichen Feldtheorie”, Preussische Akademie der Wissenschaften, Phys.-math. Klasse, Sitzungsberichte **1928, 2–7**, 10.1002/3527608958.ch38 (1929).

[2] T. Sauer, “Field equations in teleparallel spacetime: Einstein’s fernparallelismus approach towards unified field theory”, Historia Math. **33**, 399 (2006), arXiv:physics/0405142.

[3] J. W. Maluf, “The teleparallel equivalent of general relativity”, Annalen Phys. **525**, 339 (2013), arXiv:1303.3897 [gr-qc].

[4] J. Beltrán Jiménez, L. Heisenberg, and T. S. Koivisto, “The Geometrical Trinity of Gravity”, Universe **5**, 173 (2019), arXiv:1903.06830 [hep-th].

[5] Y.-F. Cai, S. Capozziello, M. De Laurentis, and E. N. Saridakis, “f(T) teleparallel gravity and cosmology”, Rept. Prog. Phys. **79**, 106901 (2016), arXiv:1511.07586 [gr-qc].

[6] N. J. Pop$\lambda awski$, “Cosmology with torsion: An alternative to cosmic inflation”, Phys. Lett. B **694**, [Erratum: Phys.Lett.B 701, 672–672 (2011)], 181 (2010), arXiv:1007.0587 [astro-ph.CO].

[7] N. J. Poplawski, “Nonsingular, big-bounce cosmology from spinor-torsion coupling”, Phys. Rev. D **85**, 107502 (2012), arXiv:1111.4595 [gr-qc].

[8] D. Benisty, E. I. Guendelman, A. van de Venn, D. Vasak, J. Struckmeier, and H. Stoecker, “The dark side of the torsion: dark energy from propagating torsion”, Eur. Phys. J. C **82**, 264 (2022), arXiv:2109.01052 [astro-ph.CO].

[9] A. Jiménez-Cano, “Review of gravitational wave solutions in quadratic metric-affine gauge gravity”, Int. J. Geom. Meth. Mod. Phys. **19**, 2240004 (2022), arXiv:2203.03936 [gr-qc].

[10] S. Bahamonde, K. F. Dialektopoulos, C. Escamilla-Rivera, G. Farrugia, V. Gakis, M. Hendry, M. Hohmann, J. Levi Said, J. Mifsud, and E. Di Valentino, “Teleparallel gravity: from theory to cosmology”, Rept. Prog. Phys. **86**, 026901 (2023), arXiv:2106.13793 [gr-qc].

[11] M. Hohmann, “Scalar-torsion theories of gravity I: general formalism and conformal transformations”, Phys. Rev. D **98**, 064002 (2018), arXiv:1801.06528 [gr-qc].

[12] M. Hohmann and C. Pfeifer, “Scalar-torsion theories of gravity II: $L(T, X, Y, \phi)$ theory”, Phys. Rev. D **98**, 064003 (2018), arXiv:1801.06536 [gr-qc].

[13] M. Hohmann, “Scalar-torsion theories of gravity III: analogue of scalar-tensor gravity and conformal invariants”, Phys. Rev. D **98**, 064004 (2018), arXiv:1801.06531 [gr-qc].

[14] D. Iosifidis, “Metric-Affine Gravity and Cosmology/Aspects of Torsion and non-Metricity in Gravity Theories”, PhD thesis (2019), arXiv:1902.09643 [gr-qc].

[15] E. Barrientos, F. S. N. Lobo, S. Mendoza, G. J. Olmo, and D. Rubiera-Garcia, “Metric-affine f(R,T) theories of gravity and their applications”, Phys. Rev. D **97**, 104041 (2018), arXiv:1803.05525 [gr-qc].

[16] N. J. Poplawski, “Big bounce from spin and torsion”, Gen. Rel. Grav. **44**, 1007 (2012), arXiv:1105.6127 [astro-ph.CO].

[17] K. Krasnov and R. Percacci, “Gravity and Unification: A review”, Class. Quant. Grav. **35**, 143001 (2018), arXiv:1712.03061 [hep-th].

[18] M. Blagojević and F. W. Hehl, eds., *Gauge Theories of Gravitation: A Reader with Commentaries* (World Scientific, Singapore, 2013).

[19] G. 't Hooft and M. J. G. Veltman, “One loop divergencies in the theory of gravitation”, Ann. Inst. H. Poincare Phys. Theor. A **20**, 69 (1974).

[20] K. S. Stelle, “Renormalization of Higher Derivative Quantum Gravity”, Phys. Rev. D **16**, 953 (1977).

[21] R. Percacci, “Gravity from a Particle Physicists' perspective”, PoS **ISFTG**, edited by C. Pinheiro, A. S. de Arruda, H. Blas, and G. O. Pires, 011 (2009), arXiv:0910.5167 [hep-th].

[22] C. F. Steinwachs, “Towards a unitary, renormalizable and ultraviolet-complete quantum theory of gravity”, 10.3389/fphy.2020.00185 (2020), arXiv:2004.07842 [hep-th].

[23] C. F. Steinwachs, “Towards a unitary, renormalizable and ultraviolet-complete quantum theory of gravity”, 10.3389/fphy.2020.00185 (2020), arXiv:2004.07842 [hep-th].

[24] R. Martini, G. P. Vacca, and O. Zanusso, “Perturbative approaches to non-perturbative quantum gravity”, (2022), arXiv:2210.13910 [hep-th].

[25] S. Weinberg, “Critical Phenomena for Field Theorists”, in 14th International School of Subnuclear Physics: Understanding the Fundamental Constitutents of Matter (Aug. 1976).

[26] S. Weinberg, “ULTRAVIOLET DIVERGENCES IN QUANTUM THEORIES OF GRAVITATION”, in *General Relativity: An Einstein Centenary Survey* (1980), pp. 790–831.

[27] R. Percacci, *An Introduction to Covariant Quantum Gravity and Asymptotic Safety*, Vol. 3, 100 Years of General Relativity (World Scientific, 2017).

[28] C. Pagani and R. Percacci, “Quantum gravity with torsion and non-metricity”, Class. Quant. Grav. **32**, 195019 (2015), arXiv:1506.02882 [gr-qc].

[29] M. Reuter and G. M. Schollmeyer, “The metric on field space, functional renormalization, and metric–torsion quantum gravity”, Annals Phys. **367**, 125 (2016), arXiv:1509.05041 [hep-th].

[30] J. F. Donoghue, “Introduction to the effective field theory description of gravity”, in Advanced School on Effective Theories (June 1995), arXiv:gr-qc/9512024.

[31] P. D. Mannheim, “Solution to the ghost problem in fourth order derivative theories”, Found. Phys. **37**, edited by L. P. Horwitz, 532 (2007), arXiv:hep-th/0608154.

[32] A. Salvio and A. Strumia, “Agravity”, JHEP **06**, 080 (2014), arXiv:1403.4226 [hep-ph].

[33] D. Anselmi and M. Piva, “The Ultraviolet Behavior of Quantum Gravity”, JHEP **05**, 027 (2018), arXiv:1803.07777 [hep-th].

[34] D. Anselmi and M. Piva, “Quantum Gravity, Fakeons And Microcausality”, JHEP **11**, 021 (2018), arXiv:1806.03605 [hep-th].

[35] J. F. Donoghue and G. Menezes, “Unitarity, stability and loops of unstable ghosts”, Phys. Rev. D **100**, 105006 (2019), arXiv:1908.02416 [hep-th].

[36] B. Holdom, “Ultra-Planckian scattering from a QFT for gravity”, Phys. Rev. D **105**, 046008 (2022), arXiv:2107.01727 [hep-th].

[37] B. Holdom, “Photon-photon scattering from a UV-complete gravity QFT”, JHEP **04**, 133 (2022), arXiv:2110.02246 [hep-ph].

[38] C. Marzo, “Radiatively stable ghost and tachyon freedom in metric affine gravity”, Phys. Rev. D **106**, 024045 (2022), arXiv:2110.14788 [hep-th].

[39] E. Sezgin and P. van Nieuwenhuizen, “New Ghost Free Gravity Lagrangians with Propagating Torsion”, Phys. Rev. D **21**, 3269 (1980).

[40] E. Sezgin, “Class of Ghost Free Gravity Lagrangians With Massive or Massless Propagating Torsion”, Phys. Rev. D **24**, 1677 (1981).

[41] R. Percacci and E. Sezgin, “New class of ghost- and tachyon-free metric affine gravities”, Phys. Rev. D **101**, 084040 (2020), arXiv:1912.01023 [hep-th].

[42] Y.-C. Lin, M. P. Hobson, and A. N. Lasenby, “Ghost and tachyon free Poincaré gauge theories: A systematic approach”, Phys. Rev. D **99**, 064001 (2019), arXiv:1812.02675 [gr-qc].

[43] D. E. Neville, “Gravity Theories With Propagating Torsion”, Phys. Rev. D **21**, 867 (1980).

[44] S. M. Christensen, “SECOND AND FOURTH ORDER INVARIANTS ON CURVED MANIFOLDS WITH TORSION”, (1979).

[45] N. h. Barth and S. m. Christensen, “ARBITRARY SPIN FIELD EQUATIONS ON CURVED MANIFOLDS WITH TORSION”, J. Phys. A **16**, 543 (1983).

[46] C. A. Hernaski, A. A. Vargas-Paredes, and J. A. Helayel-Neto, “A Discussion on Massive Gravitons and Propagating Torsion in Arbitrary Dimensions”, Phys. Rev. D **80**, 124012 (2009), arXiv:0905.1068 [hep-th].

[47] L. Fabbri, “Re-normalizable Chern–Simons extension of propagating torsion theory”, Eur. Phys. J. Plus **135**, 700 (2020), arXiv:2004.14776 [physics.gen-ph].

[48] R. T. Hammond, “Propagating torsion as a massive vector field”, Class. Quant. Grav. **7**, 2107 (1990).

[49] A. Saa, “Propagating torsion from first principles”, Gen. Rel. Grav. **29**, 205 (1997), arXiv:gr-qc/9609011.

[50] P. Baikov, M. Hayashi, N. Nelipa, and S. Ostapchenko, “Ghost and tachyon free gauge invariant, Poincare, affine and projective Lagrangians”, Gen. Rel. Grav. **24**, 867 (1992).

[51] R. G. Deveras, "On unitarity of a linearized Yang-Mills formulation for massless and massive gravity with propagating torsion", Int. J. Mod. Phys. A **25**, 4911 (2010), arXiv:1009.0455 [gr-qc].

[52] A. A. Tseytlin, "On the Poincare and De Sitter Gauge Theories of Gravity With Propagating Torsion", Phys. Rev. D **26**, 3327 (1982).

[53] M. Reuter and F. Saueressig, *Quantum Gravity and the Functional Renormalization Group: The Road towards Asymptotic Safety* (Cambridge University Press, Jan. 2019).

[54] E. S. Fradkin and A. A. Tseytlin, "Renormalizable asymptotically free quantum theory of gravity", Nucl. Phys. B **201**, 469 (1982).

[55] I. G. Avramidi and A. O. Barvinsky, "ASYMPTOTIC FREEDOM IN HIGHER DERIVATIVE QUANTUM GRAVITY", Phys. Lett. B **159**, 269 (1985).

[56] J. F. Donoghue, "General relativity as an effective field theory: The leading quantum corrections", Phys. Rev. D **50**, 3874 (1994), arXiv:gr-qc/9405057.

[57] C. P. Burgess, "Quantum gravity in everyday life: General relativity as an effective field theory", Living Rev. Rel. **7**, 5 (2004), arXiv:gr-qc/0311082.

[58] J. F. Donoghue, "The effective field theory treatment of quantum gravity", AIP Conf. Proc. **1483**, edited by W. A. Rodrigues, R. Kerner, G. Pires, and C. Pinheiro, 73 (2012), arXiv:1209.3511 [gr-qc].

[59] N. Arkani-Hamed, H. Georgi, and M. D. Schwartz, "Effective field theory for massive gravitons and gravity in theory space", Annals Phys. **305**, 96 (2003), arXiv:hep-th/0210184.

[60] W. D. Goldberger and I. Z. Rothstein, "An Effective field theory of gravity for extended objects", Phys. Rev. D **73**, 104029 (2006), arXiv:hep-th/0409156.

[61] G. Gubitosi, F. Piazza, and F. Vernizzi, "The Effective Field Theory of Dark Energy", JCAP **02**, 032 (2013), arXiv:1210.0201 [hep-th].

[62] J. K. Bloomfield, É. É. Flanagan, M. Park, and S. Watson, "Dark energy or modified gravity? An effective field theory approach", JCAP **08**, 010 (2013), arXiv:1211.7054 [astro-ph.CO].

[63] K. S. Stelle, "Classical Gravity with Higher Derivatives", Gen. Rel. Grav. **9**, 353 (1978).

[64] F. W. Hehl, J. D. McCrea, E. W. Mielke, and Y. Ne'eman, "Metric affine gauge theory of gravity: Field equations, Noether identities, world spinors, and breaking of dilation invariance", Phys. Rept. **258**, 1 (1995), arXiv:gr-qc/9402012.

[65] A. Cucchieri, M. Porrati, and S. Deser, "Tree level unitarity constraints on the gravitational couplings of higher spin massive fields", Phys. Rev. D **51**, 4543 (1995), arXiv:hep-th/9408073.

[66] C. Marzo, "Ghost and tachyon free propagation up to spin 3 in Lorentz invariant field theories", Phys. Rev. D **105**, 065017 (2022), arXiv:2108.11982 [hep-ph].

[67] D. Anselmi, "Quantum field theories of arbitrary-spin massive multiplets and Palatini quantum gravity", JHEP **07**, 176 (2020), arXiv:2006.01163 [hep-th].

[68] M. Piva, "Massive higher-spin multiplets and asymptotic freedom in quantum gravity", Phys. Rev. D **105**, 045006 (2022), arXiv:2110.09649 [hep-th].

[69] M. A. Kurkov, F. Lizzi, and D. Vassilevich, "High energy bosons do not propagate", Phys. Lett. B **731**, 311 (2014), arXiv:1312.2235 [hep-th].

[70] M. Reuter, "Nonperturbative evolution equation for quantum gravity", Phys. Rev. D **57**, 971 (1998), arXiv:hep-th/9605030.

[71] O. Lauscher and M. Reuter, "Is quantum Einstein gravity nonperturbatively renormalizable?", Class. Quant. Grav. **19**, 483 (2002), arXiv:hep-th/0110021.

[72] M. Reuter and F. Saueressig, "Quantum Einstein Gravity", New J. Phys. **14**, 055022 (2012), arXiv:1202.2274 [hep-th].

[73] J. S. Schwinger, "On gauge invariance and vacuum polarization", Phys. Rev. **82**, edited by K. A. Milton, 664 (1951).

[74] B. S. DeWitt, "Dynamical theory of groups and fields", Conf. Proc. C **630701**, edited by C. DeWitt and B. DeWitt, 585 (1964).

[75] D. V. Vassilevich, "Heat kernel expansion: User's manual", Phys. Rept. **388**, 279 (2003), arXiv:hep-th/0306138.

[76] A. O. Barvinsky and G. A. Vilkovisky, "The Generalized Schwinger-Dewitt Technique in Gauge Theories and Quantum Gravity", Phys. Rept. **119**, 1 (1985).

[77] D. Benedetti, K. Groh, P. F. Machado, and F. Saueressig, "The Universal RG Machine", JHEP **06**, 079 (2011), arXiv:1012.3081 [hep-th].

[78] B. Knorr, "The derivative expansion in asymptotically safe quantum gravity: general setup and quartic order", SciPost Phys. Core **4**, 020 (2021), arXiv:2104.11336 [hep-th].

[79] K. Groh, F. Saueressig, and O. Zanusso, "Off-diagonal heat-kernel expansion and its application to fields with differential constraints", (2011), arXiv:1112.4856 [math-ph].

[80] L. Järv, M. Rünkla, M. Saal, and O. Vilson, "Nonmetricity formulation of general relativity and its scalar-tensor extension", Phys. Rev. D **97**, 124025 (2018), arXiv:1802.00492 [gr-qc].

[81] J. Beltrán Jiménez, L. Heisenberg, and T. S. Koivisto, "The Geometrical Trinity of Gravity", Universe **5**, 173 (2019), arXiv:1903.06830 [hep-th].

[82] V. Vitagliano, "The role of nonmetricity in metric-affine theories of gravity", Class. Quant. Grav. **31**, 045006 (2014), arXiv:1308.1642 [gr-qc].

[83] J. Beltrán Jiménez, L. Heisenberg, and T. Koivisto, "Coincident General Relativity", Phys. Rev. D **98**, 044048 (2018), arXiv:1710.03116 [gr-qc].

[84] A. Conroy and T. Koivisto, "The spectrum of symmetric teleparallel gravity", Eur. Phys. J. C **78**, 923 (2018), arXiv:1710.05708 [gr-qc].

[85] J. Beltrán Jiménez, L. Heisenberg, D. Iosifidis, A. Jiménez-Cano, and T. S. Koivisto, "General teleparallel quadratic gravity", Phys. Lett. B **805**, 135422 (2020), arXiv:1909.09045 [gr-qc].

[86] A. Baldazzi, O. Melichev, and R. Percacci, "Metric-Affine Gravity as an effective field theory", Annals Phys. **438**, 168757 (2022), arXiv:2112.10193 [gr-qc].

[87] A. Baldazzi and K. Falls, “Essential Quantum Einstein Gravity”, Universe **7**, 294 (2021), arXiv:2107.00671 [hep-th].

[88] R. Rivers, ““Lagrangian theory for neutral massive spin-2 fields””, Il Nuovo Cimento **34 386** (1964).

[89] K. J. Barnes, “Lagrangian Theory for the Second-Rank Tensor Field”, J. Math. Phys. **6, 788** (1965).

[90] A. Aurilia and H. Umezawa, “Theory of High-Spin Fields”, Phys. Rev. **182, 1682**, https://doi.org/10.1103/PhysRev.182.1682 (1969).

[91] R. Schimidt Bittencourt, “Unitariedade de modelos de spin-3”, master thesis, Sao Paulo State University.

[92] D. Iosifidis and T. Koivisto, “Scale transformations in metric-affine geometry”, Universe **5**, 82 (2019), arXiv:1810.12276 [gr-qc].

[93] D. Sauro, R. Martini, and O. Zanusso, “Projective transformations in metric-affine and Weylian geometries”, (2022), arXiv:2208.10872 [hep-th].

[94] C. Fronsdal, “Massless Fields with Integer Spin”, Phys. Rev. D **18**, 3624 (1978).

[95] P. Baekler, N. Boulanger, and F. W. Hehl, “Linear connections with propagating spin-3 field in gravity”, Phys. Rev. D **74**, 125009 (2006), arXiv:hep-th/0608122.

[96] J. Colin, R. Mohayaee, M. Rameez, and S. Sarkar, “Evidence for anisotropy of cosmic acceleration”, Astron. Astrophys. **631**, L13 (2019), arXiv:1808.04597 [astro-ph.CO].

[97] R. Mohayaee, M. Rameez, and S. Sarkar, “Do supernovae indicate an accelerating universe?”, Eur. Phys. J. ST **230**, 2067 (2021), arXiv:2106.03119 [astro-ph.CO].

[98] A. Salvio, “Inflating and reheating the Universe with an independent affine connection”, Phys. Rev. D **106**, 103510 (2022), arXiv:2207.08830 [hep-ph].

[99] S. W. Hawking, “Space-Time Foam”, Nucl. Phys. B **144**, 349 (1978).

[100] L. Smolin, “The Quantization of unimodular gravity and the cosmological constant problems”, Phys. Rev. D **80**, 084003 (2009), arXiv:0904.4841 [hep-th].

[101] I. D. Saltas, “UV structure of quantum unimodular gravity”, Phys. Rev. D **90**, 124052 (2014), arXiv:1410.6163 [hep-th].

[102] D. Benedetti, “Essential nature of Newton’s constant in unimodular gravity”, Gen. Rel. Grav. **48**, 68 (2016), arXiv:1511.06560 [hep-th].

[103] A. Padilla and I. D. Saltas, “A note on classical and quantum unimodular gravity”, Eur. Phys. J. C **75**, 561 (2015), arXiv:1409.3573 [gr-qc].

[104] D. J. Burger, G. F. R. Ellis, J. Murugan, and A. Weltman, “The KLT relations in unimodular gravity”, (2015), arXiv:1511.08517 [hep-th].

[105] E. Alvarez, S. Gonzalez-Martin, and C. P. Martin, “Unimodular Trees versus Einstein Trees”, Eur. Phys. J. C **76**, 554 (2016), arXiv:1605.02667 [hep-th].

[106] R. de León Ardón, N. Ohta, and R. Percacci, “Path integral of unimodular gravity”, Phys. Rev. D **97**, 026007 (2018), arXiv:1710.02457 [gr-qc].

[107] R. Percacci, “Unimodular quantum gravity and the cosmological constant”, Found. Phys. **48**, 1364 (2018), arXiv:1712.09903 [gr-qc].

[108] S. González-Martın and C. P. Martin, "Unimodular Gravity and General Relativity UV divergent contributions to the scattering of massive scalar particles", JCAP **01**, 028 (2018), arXiv:1711.08009 [hep-th].

[109] S. Gonzalez-Martin and C. P. Martin, "Scattering of fermions in the Yukawa theory coupled to Unimodular Gravity", Eur. Phys. J. C **78**, 236 (2018), arXiv:1802.03755 [hep-th].

[110] G. P. De Brito, A. Eichhorn, and A. D. Pereira, "A link that matters: Towards phenomenological tests of unimodular asymptotic safety", JHEP **09**, 100 (2019), arXiv:1907.11173 [hep-th].

[111] L. Smolin, "Unimodular loop quantum gravity and the problems of time", Phys. Rev. D **84**, 044047 (2011), arXiv:1008.1759 [hep-th].

[112] S. Yamashita, "Hamiltonian analysis of unimodular gravity and its quantization in the connection representation", Phys. Rev. D **101**, 086007 (2020), arXiv:2003.05083 [gr-qc].

[113] L. Baulieu, "Unimodular Gauge in Perturbative Gravity and Supergravity", Phys. Lett. B **808**, 135591 (2020), arXiv:2004.05950 [hep-th].

[114] L. Baulieu, "Unimodular gauge and ADM gravity path integral", Phys. Lett. B **831**, 137179 (2022), arXiv:2012.01116 [hep-th].

[115] M. Herrero-Valea and R. Santos-Garcia, "Non-minimal Tinges of Unimodular Gravity", JHEP **09**, 041 (2020), arXiv:2006.06698 [hep-th].

[116] G. P. de Brito and A. D. Pereira, "Unimodular quantum gravity: Steps beyond perturbation theory", JHEP **09**, 196 (2020), arXiv:2007.05589 [hep-th].

[117] G. P. de Brito, A. D. Pereira, and A. F. Vieira, "Exploring new corners of asymptotically safe unimodular quantum gravity", Phys. Rev. D **103**, 104023 (2021), arXiv:2012.08904 [hep-th].

[118] E. Álvarez, S. González-Martın, M. Herrero-Valea, and C. P. Martın, "Unimodular Gravity Redux", Phys. Rev. D **92**, 061502 (2015), arXiv:1505.00022 [hep-th].

[119] E. Álvarez, S. González-Martın, M. Herrero-Valea, and C. P. Martın, "Quantum Corrections to Unimodular Gravity", JHEP **08**, 078 (2015), arXiv:1505.01995 [hep-th].

[120] R. Bufalo, M. Oksanen, and A. Tureanu, "How unimodular gravity theories differ from general relativity at quantum level", Eur. Phys. J. C **75**, 477 (2015), arXiv:1505.04978 [hep-th].

[121] S. Upadhyay, M. Oksanen, and R. Bufalo, "BRST Quantization of Unimodular Gravity", Braz. J. Phys. **47**, 350 (2017), arXiv:1510.00188 [hep-th].

[122] B. Fiol and J. Garriga, "Semiclassical Unimodular Gravity", JCAP **08**, 015 (2010), arXiv:0809.1371 [hep-th].

[123] A. Eichhorn, "On unimodular quantum gravity", Class. Quant. Grav. **30**, 115016 (2013), arXiv:1301.0879 [gr-qc].

[124] A. Eichhorn, "The Renormalization Group flow of unimodular f(R) gravity", JHEP **04**, 096 (2015), arXiv:1501.05848 [gr-qc].

[125] N. Ohta, R. Percacci, and A. D. Pereira, “$f(R, R^2_{\mu\nu})$ at one loop”, Phys. Rev. D **97**, 104039 (2018), arXiv:1804.01608 [hep-th].

[126] S. Gonzalez-Martin and C. P. Martin, “Do the gravitational corrections to the beta functions of the quartic and Yukawa couplings have an intrinsic physical meaning?”, Phys. Lett. B **773**, 585 (2017), arXiv:1707.06667 [hep-th].

[127] F. Ferrari, “Partial Gauge Fixing and Equivariant Cohomology”, Phys. Rev. D **89**, 105018 (2014), arXiv:1308.6802 [hep-th].

[128] N. Ohta, R. Percacci, and A. D. Pereira, “Gauges and functional measures in quantum gravity I: Einstein theory”, JHEP **06**, 115 (2016), arXiv:1605.00454 [hep-th].

[129] E. Alvarez, A. F. Faedo, and J. J. Lopez-Villarejo, “Ultraviolet behavior of transverse gravity”, JHEP **10**, 023 (2008), arXiv:0807.1293 [hep-th].

[130] G. Narain and R. Percacci, “Renormalization Group Flow in Scalar-Tensor Theories. I”, Class. Quant. Grav. **27**, 075001 (2010), arXiv:0911.0386 [hep-th].

[131] R. Percacci and G. P. Vacca, “Search of scaling solutions in scalar-tensor gravity”, Eur. Phys. J. C **75**, 188 (2015), arXiv:1501.00888 [hep-th].

[132] M. B. Fröb, “Gauge-invariant quantum gravitational corrections to correlation functions”, Class. Quant. Grav. **35**, 055006 (2018), arXiv:1710.00839 [gr-qc].

[133] C. F. Steinwachs and A. Y. Kamenshchik, “One-loop divergences for gravity non-minimally coupled to a multiplet of scalar fields: calculation in the Jordan frame. I. The main results”, Phys. Rev. D **84**, 024026 (2011), arXiv:1101.5047 [gr-qc].

[134] I. L. Shapiro and H. Takata, “One loop renormalization of the four-dimensional theory for quantum dilaton gravity”, Phys. Rev. D **52**, 2162 (1995), arXiv:hep-th/9502111.

[135] S. Lippoldt, “Renormalized Functional Renormalization Group”, Phys. Lett. B **782**, 275 (2018), arXiv:1804.04409 [hep-th].

[136] C. Pagani and M. Reuter, “Background Independent Quantum Field Theory and Gravitating Vacuum Fluctuations”, Annals Phys. **411**, 167972 (2019), arXiv:1906.02507 [gr-qc].

[137] C. Pagani and M. Reuter, “Why the Cosmological Constant Seems to Hardly Care About Quantum Vacuum Fluctuations: Surprises From Background Independent Coarse Graining”, Front. in Phys. **8**, 214 (2020).

[138] M. Becker and M. Reuter, “Background Independent Field Quantization with Sequences of Gravity-Coupled Approximants”, Phys. Rev. D **102**, 125001 (2020), arXiv:2008.09430 [gr-qc].

[139] J. F. Donoghue, “Cosmological constant and the use of cutoffs”, Phys. Rev. D **104**, 045005 (2021), arXiv:2009.00728 [hep-th].

[140] R. Jackiw, “GAUGE COVARIANT CONFORMAL TRANSFORMATIONS”, Phys. Rev. Lett. **41**, 1635 (1978).

[141] J.-E. Daum, U. Harst, and M. Reuter, “Running Gauge Coupling in Asymptotically Safe Quantum Gravity”, JHEP **01**, 084 (2010), arXiv:0910.4938 [hep-th].

[142] J. .-. Daum and M. Reuter, “Renormalization Group Flow of the Holst Action”, Phys. Lett. B **710**, 215 (2012), arXiv:1012.4280 [hep-th].

[143] D. Benedetti and S. Speziale, "Perturbative quantum gravity with the Immirzi parameter", JHEP **06**, 107 (2011), arXiv:1104.4028 [hep-th].

[144] J.-E. Daum and M. Reuter, "Running Immirzi Parameter and Asymptotic Safety", PoS **CNCFG2010**, edited by K. N. Anagnostopoulos, D. Bahns, H. Grosse, N. Irges, and G. Zoupanos, 003 (2010), arXiv:1111.1000 [hep-th].

[145] J. E. Daum and M. Reuter, "Einstein-Cartan gravity, Asymptotic Safety, and the running Immirzi parameter", Annals Phys. **334**, 351 (2013), arXiv:1301.5135 [hep-th].

[146] U. Harst and M. Reuter, "A new functional flow equation for Einstein–Cartan quantum gravity", Annals Phys. **354**, 637 (2015), arXiv:1410.7003 [hep-th].

[147] V. A. Rubakov, "Can Galileons support Lorentzian wormholes?", Teor. Mat. Fiz. **187**, 338 (2016), arXiv:1509.08808 [hep-th].

[148] P. Creminelli, M. A. Luty, A. Nicolis, and L. Senatore, "Starting the Universe: Stable Violation of the Null Energy Condition and Non-standard Cosmologies", JHEP **12**, 080 (2006), arXiv:hep-th/0606090.

[149] P. Creminelli, A. Nicolis, and E. Trincherini, "Galilean Genesis: An Alternative to inflation", JCAP **11**, 021 (2010), arXiv:1007.0027 [hep-th].

[150] T. Kobayashi, "Generic instabilities of nonsingular cosmologies in Horndeski theory: A no-go theorem", Phys. Rev. D **94**, 043511 (2016), arXiv:1606.05831 [hep-th].

[151] Y. A. Ageeva, O. A. Evseev, O. I. Melichev, and V. A. Rubakov, "Horndeski Genesis: strong coupling and absence thereof", EPJ Web Conf. **191**, edited by V. E. Volkova, Y. V. Zhezher, D. G. Levkov, V. A. Rubakov, and V. A. Matveev, 07010 (2018), arXiv:1810.00465 [hep-th].

[152] Y. Ageeva, O. Evseev, O. Melichev, and V. Rubakov, "Toward evading the strong coupling problem in Horndeski genesis", Phys. Rev. D **102**, 023519 (2020), arXiv:2003.01202 [hep-th].

[153] A. Baldazzi, R. B. A. Zinati, and K. Falls, "Essential renormalisation group", SciPost Phys. **13**, 085 (2022), arXiv:2105.11482 [hep-th].

[154] M. S. Ruf and C. F. Steinwachs, "Renormalization of generalized vector field models in curved spacetime", Phys. Rev. D **98**, 025009 (2018), arXiv:1806.00485 [hep-th].

[155] M. S. Ruf and C. F. Steinwachs, "Quantum effective action for degenerate vector field theories", Phys. Rev. D **98**, 085014 (2018), arXiv:1809.04601 [hep-th].

[156] A. O. Barvinsky, M. Herrero-Valea, and S. M. Sibiryakov, "Towards the renormalization group flow of Horava gravity in $(3+1)$ dimensions", Phys. Rev. D **100**, 026012 (2019), arXiv:1905.03798 [hep-th].

[157] C. F. Steinwachs, "Non-perturbative quantum Galileon in the exact renormalization group", JCAP **04**, 038 (2021), arXiv:2101.07271 [hep-th].

[158] A. O. Barvinsky, A. V. Kurov, and S. M. Sibiryakov, "Beta functions of (3+1)-dimensional projectable Hořava gravity", Phys. Rev. D **105**, 044009 (2022), arXiv:2110.14688 [hep-th].

[159] A. O. Barvinsky and W. Wachowski, "Heat kernel expansion for higher order minimal and nonminimal operators", Phys. Rev. D **105**, 065013 (2022), arXiv:2112.03062 [hep-th].

[160] Y. Kluth and D. F. Litim, "Functional renormalization for f(R$\mu\nu\rho\sigma$) quantum gravity", Phys. Rev. D **106**, 106022 (2022), arXiv:2202.10436 [hep-th].

[161] J. Martın-Garcıa, "xTensor, A fast manipulator of tensor expressions", (2002–2007).

[162] J. M. Martin-Garcia, R. Portugal, and L. R. U. Manssur, "The Invar Tensor Package", Comput. Phys. Commun. **177**, 640 (2007), arXiv:0704.1756 [cs.SC].

[163] J. M. Martin-Garcia, D. Yllanes, and R. Portugal, "The Invar tensor package: Differential invariants of Riemann", Comput. Phys. Commun. **179**, 586 (2008), arXiv:0802.1274 [cs.SC].

[164] Thomas Bäckdahl, "SymManipulator", (2011–2021).

[165] T. Nutma, "xTras : A field-theory inspired xAct package for mathematica", Comput. Phys. Commun. **185**, 1719 (2014), arXiv:1308.3493 [cs.SC].

[166] K. Peeters, "Introducing Cadabra: A Symbolic computer algebra system for field theory problems", (2007), arXiv:hep-th/0701238.

[167] O. Castillo-Felisola, D. T. Price, and M. Scomparin, "Cadabra and Python algorithms in General Relativity and Cosmology I: Generalities", (2022), arXiv:2210.00005 [gr-qc].

[168] N. Dupuis, L. Canet, A. Eichhorn, W. Metzner, J. M. Pawlowski, M. Tissier, and N. Wschebor, "The nonperturbative functional renormalization group and its applications", Phys. Rept. **910**, 1 (2021), arXiv:2006.04853 [cond-mat.stat-mech].

[169] J. M. Pawlowski, "Aspects of the functional renormalisation group", Annals Phys. **322**, 2831 (2007), arXiv:hep-th/0512261.

[170] H. Gies, "Introduction to the functional RG and applications to gauge theories", Lect. Notes Phys. **852**, 287 (2012), arXiv:hep-ph/0611146.

[171] A. Baldazzi, R. Percacci, and L. Zambelli, "Functional renormalization and the $\overline{\text{MS}}$ scheme", Phys. Rev. D **103**, 076012 (2021), arXiv:2009.03255 [hep-th].

[172] A. Codello, R. Percacci, and C. Rahmede, "Investigating the Ultraviolet Properties of Gravity with a Wilsonian Renormalization Group Equation", Annals Phys. **324**, 414 (2009), arXiv:0805.2909 [hep-th].