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Delocalized Equivariant Elliptic Hochschild Homology

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Abstract: In this thesis we introduce the notion of *Elliptic Hochschild Homology* of derived stacks in characteristic zero. This notion is studied and some fundamental properties are shown, and it is computed in simple cases. We then introduce its *periodic cyclic* version and prove it recovers Grojnowski's equivariant elliptic cohomology of the analytification for quotient stacks.

In the second part of the thesis, we provide a notion of k-rationalized equivariant elliptic cohomology for $\mathbb{Q}$-algebras k, via adelic descent. We study the adelic decomposition of equivariant cohomology and K-theory and prove comparison theorems with periodic cyclic homology variants of the theories.

Finally, we collect partial results and ideas that will be explored in future work.

Self-plagiarism: Chapter 2 of this thesis is based on a paper written by the author in conjunction with his Ph.D. supervisor, Nicolò Sibilla. Some of the ideas appearing in Chapter 4 are inspired by or came up in discussions with Nicolò Sibilla and will be explored in future collaborations.

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Contents

CHAPTER 1

Introduction

1.1. An introduction to the thesis, but mostly rambling

Elliptic cohomology was constructed for the first time in the context of Chromatic Homotopy Theory. Previously, it appeared in hidden form in work of Ochanine [**53**] on *elliptic genera*, i.e. ring maps

$$\Omega^{\mathrm{or}} \to R$$

from the oriented cobordism ring to a ring R vanishing on manifolds isomorphic to the projectivization of an even-dimensional complex vector bundle over an oriented real closed manifold. In particular, the logarithm of the genus could always be written as an elliptic integral, which motivated the name.

In slightly more modern terms, elliptic cohomology arises in the context of Chromatic Homotopy Theory as the cohomology theory corresponding to a formal group obtained by completing an elliptic curve at its origin. In particular, given a base ring R and an elliptic curve E over R, we can define elliptic cohomology with coefficients in R as the following data:

Definition 1.1.1. *Elliptic cohomology with coefficients in a ring R is the data of:*

- *an elliptic curve over R;*
- *a complex oriented even (weakly) periodic cohomology theory A;*

subject to the relation that

- $A^0(\mathrm{pt}) = R$*;*
- $\mathrm{Spf} A^0(\mathbb{CP}^\infty) = E_{\hat{1}}$

where 1 *denotes the identity of E.*

In the definition above, not all elliptic curves are acceptable, only the ones that satisfy the criterion of *Landweber exactness* (see for example [**44**] for a brief explanation), that specifies which kinds of formal groups can give rise to complex oriented even periodic multiplicative cohomology theories. In the case of elliptic curves, Lurie explains in his survey of elliptic cohomology [**44**] that is it sufficient that the curve is represented by an étale map from $\mathrm{Spec}\, R$ to the (classical) moduli stack of elliptic curves $\mathcal{M}_{ell}$.

1.1.1. Witten's work. Interest in elliptic cohomology drastically increased after Witten realized a universal elliptic genus via Quantum Field Theory in his series of two papers [**81**] and [**82**]. His constructions link various objects together: on one hand, the partition function of a kind of string theory (as a 2D CFT), on the other hand the S^1-equivariant index of the Dirac-Ramond operator in the same context, and a universal elliptic genus, now known as *Witten genus*, which is a formal power series in a variable q that should be interpreted as the modulus of the elliptic curve E. His work takes a powerful — but in some sense incomplete, as pointed out by Lurie [**44**] — point of view: quantum mechanics on the loop space of a manifold X has a lot of information on the string theory on X itself. This viewpoint can be informally understood as forgetting the complex structure in the worldsheet E of the string theory sigma model, magically retaining important information that eventually leads to the Witten genus. In this context, the Dirac-Ramond operator can be interpreted as the S^1-equivariant Dirac operator on the loop space $\mathcal{L}X$, and its S^1-equivariant index as a character of the circle group can be analytically extended to the punctured disc $\mathbb{D}^*$; the new variable q can then be interpreted as the modulus of an elliptic curve. Giving the loop space $\mathcal{L}X$ the structure of a spin manifold is equivalent to attaching a *string* structure to the manifold X. In this setting, Witten proceeds to show that the Witten genus becomes a modular form (of mixed weight) up to a Dedekind eta function factor. The same result was later obtained by Zagier, without reference to physical arguments [**83**]. At the same time, Taubes [**72**], Brylinski [**15**] and Landweber [**36**] followed the point of view initiated by Witten in [**82**] and worked on developing the formalism of Dirac operators on loop spaces, in hopes to obtain the Witten genus via some generalization of the Atiyah–Singer index theorem — finally making rigorous all Witten's contructions.

It is Segal who makes this strong connection between elliptic cohomology and field theories the center of a conjecture. In particular, he proposes that geometric models for the cocycles in elliptic cohomology should come from two-dimensional conformal field theories [**63**], of which he develops an axiomatic definition (see [**65**]) in the same spirit as the notion of TQFT described by Atiyah [**5**]. This idea was then taken and refined by Stolz and Teichner. Their conjecture replaces conformal field theories with supersymmetric Euclidean field theories [**69**] [**70**]. This line of research has been extensively studied by Berwick-Evans and Berwick-Evans–Tripaty [**10**].

1.1.2. Equivariant elliptic cohomology following Grojnowski. It is folklore that, given a cohomology theory A, this will have a (genuine rationalized) S^1-equivariant counterpart only if the associated formal group is the completion of an algebraic group. Indeed, singular cohomology and topological K–theory have equivariant versions, as they are associated respectively to the formal affine line $\mathbb{A}^1_{\hat{0}}$ and the formal multiplicative group $\mathbb{G}_{m,\hat{1}}$. The reason behind this folklore statement is that equivariance is related to *decompleting* over this formal group, and this can only be done in the presence of an actual algebraic group. This decompletion procedure yields a quasicoherent sheaf on the algebraic group, encoding the equivariant cohomology theory. It is mostly an accident that for singular cohomology and K-theory the associated groups are affine, hence the equivariant theories can also be embodied by modules over rings.

This philosophy is strongly indebted to the work of Atiyah and Segal, and in particular to the celebrated *Atiyah–Segal completion theorem*[**3**], relating the completion of the G-equivariant K-theory of a topological space X at its augmentation ideal with the K-theory of the Borel quotient $X//G$. The augmentation ideal in $K_G(X)$, i.e. the ideal generated by virtual G-equivariant vector bundles of virtual rank 0, can also be thought of, for $G = S^1$, as the ideal associated to the closed immersion of the closed point 1 inside $\mathbb{G}_m$. This is the core of the decompletion leading to the equivariant theory: completing reduces the equivariant theory to its non-equivariant version. The combination of this phenomenon with *localization* principles allows one to obtain Atiyah–Segal completion statements over points different from the identity, by computing the Borel-equivariant K-theory of fixed loci in X associated to the point over which we are completing. The multiplication law of the group links all these completions together. A similar picture exists also for singular cohomology and was described by Roşu [**61**].

It was Grojnowski who constructed a rationalized S^1-equivariant elliptic cohomology theory in 1994 [**30**]. In the same year, Ginzburg, Kapranov and Vasserot produced a list of axioms that would determine S^1-equivariant elliptic cohomology uniquely [**25**]. Following the folklore ideas, Grojnowski constructs a quasi-coherent sheaf on E for any topological space X with an S^1-action. His construction is analytic in nature, and yields a quasi-coherent algebraic sheaf via the GAGA theorems of Serre. The original motivation for Grojnowski is representation theory and the Langlands program. In particular, constructing elliptic counterparts to affine algebras appearing in his previous work on Hilbert schemes in Algebraic Geometry [**29**]. His construction ended up having a strong resonating impact on homotopy theory, as it does indeed recover the correct rationalization of equivariant elliptic cohomology, whose general construction is still in development ([**24**],[**42**]).

Grojnowski's construction implements the Atiyah–Segal completion theorem and the localization principle in an analytic context, by assigning to a small analytic neighbourhood of a point e in E the ring

$$H_T^*(X^e) \otimes_{H_T(\mathrm{pt})} \mathcal{O}_E^{hol}(U - e)$$

The key idea behind this assignment is that localization allows to interchange the full space X with a sublocus of fixed points X^e canonically associated to e around said point. At the same time, singular cohomology has to be twisted by holomorphic functions on the elliptic curve around the identity. Nevertheless, locally, rationalized equivariant elliptic cohomology is controlled by Borel-equivariant singular cohomology: this is a deep reflection of the Atiyah–Segal completion theorem.

1.1.3. Level l equivariant elliptic cohomology. In his seminal paper [**30**] Grojnowski also proposes a notion of *level l* equivariant elliptic cohomology. For a choice of compact Lie group G and an adjoint-invariant non-degenerate symmetric bilinear form l on $\mathfrak{g} = \mathrm{Lie}(G)$, it is possible to construct a line bundle $\mathcal{L}_l$ on the G-equivariant elliptic cohomology of the point having l as its first Chern class. This construction is due to Looijenga. For G simple and simply connected, the space of choices for l becomes $\mathbb{Z}$, and the line bundle $\mathcal{L}_l$ becomes the lth tensor power of a base line bundle $\mathcal{L}$. In this setting, *level l G*-equivariant elliptic cohomology is defined to be the global sections

$$\mathcal{E}ll^l_G(X) = \Gamma(E_G, \mathcal{E}ll_G(X) \otimes \mathcal{L}_l)$$

where E_G is the G-equivariant elliptic cohomology of the point.

Level l equivariant elliptic cohomology of the point has a very deep meaning. Indeed, the global sections of $\mathcal{L}_l$ are the *non abelian theta functions* at level l, characters of special representations of a canonical central extension of the loop group $\mathcal{L}G$ of G (a good survey is in section 5 of [**44**], the original ideas date back to Ando's work [**1**]). Those representations are called *level l positive energy*, and are characterized by a finiteness condition which is akin to the condition of *rationality* of a vertex operator algebra. Indeed, non abelian theta functions appear also in the context of 2-dimensional Conformal Field Theory, as the *conformal blocks* of some models known as *Wess–Zumino–Witten models*, sigma models associated to the propagation of a string on a Lie group. Such models are canonically associated to a compact, simple, connected and simply connected Lie group and to an integer number l, also called the level. The symmetry algebra of the WZW model is an affine Lie algebra obtained from the Lie algebra of G and the level l. Indeed, the positive energy level l representations of the loop group of G are related to the representation theory of this affine Lie algebra and of some quantum groups cooked up from $\mathfrak{g}$, at specific roots of unity related to the level l. Similarly, there is a 3 dimensional Topological Quantum Field Theory, called *Chern–Simons theory*, which contains information on non-abelian theta functions. This model is also associated to such a Lie group G and an integer level l, and concerns the study of fields which are connections on principal G-bundles on a space. In particular, choosing the space to be an elliptic curve E times the real line, the model will have as a phase space the moduli space of flat principal G-connections on E, which also coincides with the G-equivariant elliptic cohomology of the point. Axelrod–Della Pietra–Witten observe in [**6**] that the Looijenga line bundles arise by geometric quantization in this context, and identify the non abelian theta functions with the space of states of G-Chern–Simons theory on $E \times \mathbb{R}$. This is an instance of *holography*: a 3 dimensional TQFT contains information about a 2 dimensional CFT sitting "at its boundary", ad allows to compute interesting quantities such as the conformal blocks, which in this specific example arise as the space of states.

This is the type of information that level l equivariant elliptic cohomology of the point contains. More recently, Distler–Sharpe proposed a physical model that relates to level l equivariant elliptic cohomology of a space X: *fibered* Wess–Zumino–Witten models [**20**]. Those are sigma models associated to the propagation of a string on the total space of a principal G-bundle on X.

1.1.4. The trichotomy of cubics. The three formal groups associated to singular cohomology, K-theory and elliptic cohomology — respectively the formal additive and multiplicative groups and the completion of an elliptic curve — all admit a decompletion, and indeed all three of these theories admit an equivariant version. This trichotomy is strongly related to a phenomenon in the theory of Lie groups. Fundamentally, one observes that for a compact Lie group G, G-equivariant cohomology of the point satisfies

$$\operatorname{Spec} H_G^{\oplus,0}(*) \simeq \mathfrak{g}_{\mathbb{C}}//G_{\mathbb{C}}$$

In the above expression, $H_G^{\oplus,0}$ denotes the degree zero part of the *sum*-$\mathbb{Z}_2$-periodization of equivariant singular cohomology, and $\mathfrak{g}_{\mathbb{C}} = \mathrm{Lie}(G) \otimes_{\mathbb{R}} \mathbb{C}$ is the complexified Lie algebra of G, i.e. the Lie algebra of the complexification $G_{\mathbb{C}}$ of G. The quotient is with respect to the adjoint action and taken in the GIT sense. Similarly, for K-theory, we have

$$\operatorname{Spec} K_G^0(*) \simeq G_{\mathbb{C}}//G_{\mathbb{C}}$$

In particular, if $G = S^1$,

$$\operatorname{Spec} H_{S^1}^{\oplus,0}(*) \simeq \mathbb{G}_a$$
$$\operatorname{Spec} K_{S^1}^0(*) \simeq \mathbb{G}_m$$

and we recover the decompletions of the formal groups associated to singular cohomology and K-theory. Elliptic cohomology is associated to the completion of an elliptic curve $\widehat{E}$ at its identity. We should think of E itself, following the examples of singular cohomology and K-theory, as being associated to S^1. Following Ben-Zvi, we call E the *elliptic group* associated to S^1. Similarly, for any compact Lie group G, we can construct the associated elliptic group

$$E_G = E_T/W = \mathcal{E}ll_G(*) = \mathcal{M}_G$$

i.e. the coarse moduli space of semi-stable principal G-bundles on E. Analytically, there are exponential maps

$$\mathfrak{g}_{\mathbb{C}} \to G_{\mathbb{C}} \to E_G$$

which can be thought of as geometric incarnations of equivariant Chern characters.

It is indeed possible to have a presentation of Lie algebras and Lie groups in terms of semi-stable bundles on a complex curve, indeed a cubic. In particular, we could define a notion of *cuspidal* group and *nodal* group associated to a compact Lie group G as the coarse moduli spaces of semi-stable degree zero principal G-bundles on the cusp and the nodal curve respectively. Calling C the cusp and N the nodal curve, we reserve the notation C_G and N_G for the cuspidal and nodal groups associated to G. It is a classical fact in algebraic geometry that there are isomorphisms

$$C_G \simeq \mathfrak{g}_{\mathbb{C}}//G_{\mathbb{C}}$$

$$N_G \simeq G_{\mathbb{C}}//G_{\mathbb{C}}$$

The notions of cuspidal and nodal group are redundant: they simply correspond to the notions of Lie algebra and Lie group itself, up to complexification and conjugation. The trichotomy of cubic curves is mirrored in Lie theory by the trichotomy Lie algebras–Lie groups–elliptic groups, and in homotopy theory by the trichotomy equivariant cohomology–equivariant K-theory–equivariant elliptic cohomology.

This trichotomy of groups can be interpreted in terms of three dimensional G-Chern–Simons theory: it amounts to extending it from the moduli stack of elliptic curves to the moduli stack of all cubics. There, its phase space over a curve C would correspond with the associated group C_G, giving rise to the Lie theoretic trichotomy from the trichotomy of cubics.

1.1.5. Hochschild homologies. Derived algebraic geometry offers yet another trichotomy. Given a quotient stack $[X/G]$ of a scheme X by the action of a smooth affine reductive algebraic group G, we can apply three different constructions:

(1) the *shifted tangent stack* $T_{[X/G]}[-1]$;
(2) the *derived loop space* $\mathcal{L}[X/G]$;
(3) the *derived stack of quasi-constant maps* $\underline{\mathrm{Map}}^0(E, [X/G])$ from an elliptic curve E.

The first two objects are quite classical, while the last one is introduced in Chapter 2 of this thesis. These three objects produce *Hochschild* variants of the three equivariant cohomology theories — they are their "partial" algebraic analogues.

The most classical of these correspondences is linked to the loop space. For a scheme X, its derived loop space

$$\mathcal{L}X = \underline{\mathrm{Map}}\left(S^1, X\right)$$

encodes the classical notion of Hochschild homology of X: the global sections of the structure sheaf

$$\mathcal{O}(\mathcal{L}X) \simeq \mathrm{HH}_*(X)$$

compute the *Hochschild homology* of X (or more precisely the Hochschild chains). On the other hand, the shifted tangent stack of X gives the *de Rham complex* of X:

$$\mathcal{O}(T_X[-1]) \simeq \mathrm{DR}^{-*}(X)$$

Ben-Zvi and Nadler, in their paper [**8**], obtain an equivalence

$$T_X[-1] \simeq \mathcal{L}X$$

for schemes X over a field of characteristic zero, based on a *Zariski codescent* result. Their equivalence recovers, after taking global sections, the HKR isomorphism. Under this isomorphism, as already explained by Toën and Vezzosi [**79**], the S^1-action on $\mathrm{HH}_*(X)$ induced by the loop rotation action on $\mathcal{L}X$ becomes a mixed structure on the de Rham complex, giving rise to the de Rham differential. In particular, for smooth schemes over $\mathbb{C}$, the Tate construction with respect to this action produces (up to $\mathbb{Z}_2$-periodization) the Betti cohomology of the analytification of X.

Replacing the target scheme with a quotient stack produces an unexpected phenomenon. Indeed, if we consider the loop space of the classifying stack of a reductive smooth affine group G,

$$\mathcal{L}BG \simeq [G//G]$$

we obtain the adjoint quotient of G rather then of its Lie algebra. This behaviour is expected from K-theory rather than from equivariant cohomology. On the other hand,

$$T_{BG}[-1] \simeq [\mathfrak{g}//G]$$

The Ben-Zvi–Nadler equivalence is indeed a reflection of the homotopy-theoretical phenomenon of the existence of logarithms in characteristic zero, identifying all formal group laws — and hence all complex oriented even periodic cohomology theories — after a base change to characteristic zero. At the same time, the equivariant theories are distinguishable even in characteristic zero, as their global behaviour is linked to the trichotomy of Lie algebras–Lie groups–elliptic groups. In particular, the simple observations above allow to infer a link between shifted tangent stacks of quotients and equivariant cohomology, and loop spaces of quotients and equivariant K-theory. Such ideas have been explored in the literature, mostly by Pantev–Toën–Vaquié–Vezzosi [**54**] and Calaque–Pantev–Toën–Vaquié–Vezzosi [**16**] in the shifted tangent setting and by Chen [**17**] and Halpern-Leistner–Pomerleano [**31**] in the loop space case. The result of their investigation is a characterization of the equivariant cohomology and equivariant K-theory in terms of Tate fixed points of the global sections of structure sheaves of the shifted tangent and the derived loop space respectively. Calaque–Pantev–Toën–Vaquié–Vezzosi can even prove their result for general Artin stacks.

In Chapter 3 we prove very similar statements, directly at the level of sheaves, using *adelic descent*. The goal of Chapter 2 is to prove an analogue of these theorems for equivariant elliptic cohomology, via the stack of *quasi-constant maps*

$$\underline{\mathrm{Map}}^0\,(E, [X/G])$$

One important feature of the algebraic constructions described above is that they provide *Hochschild homology* models to the topological theories. Traditionally, Hochschild homology plays a very relevant role as the recipient of the *Dennis trace* map from algebraic K-theory, and has been used to access the information retained by algebraic K-theory, which is notoriously hard to compute. Additionally, Hochschild homology is naturally non-commutative, hence can arise as an invariant of rings as much as an invariant of *categories*. In this sense, Hochschild homology is the categorification of the notion of *dimension* of a vector space.

There is an interesting approach to the definition of the dimension of a vector space via the notion of *one-dimensional topological quantum field theory*. Given a finite dimensional, i.e. *dualizable*, vector space V, the cobordism hypothesis allows us to conclude that there is a unique (essentially, at least) 1D (framed) TQFT, Z_V, whose value on the point is V itself. Then, the *partition function* of this theory — its evaluation on the circle S^1 — is exactly the dimension of V. There is no reason to consider the category of vector spaces as a target for our TQFT: in particular, we can consider a category of categories as target. The associated notion of dimension will reproduce Hochschild homology. In particular, for an associative algebra A, the Hochschild homology of A is given by

$$\mathrm{HH}_*(A) = \dim \mathrm{LMod}_A$$

the dimension of the category of left A-modules.

At the beginning of this section we introduced the trichotomy of shifted tangent stack–loop space–quasi-constant maps from an elliptic curve, and contemplated how they gave rise to Hochschild homology counterparts of the respective topological theory, obtained in rational coefficients via a Tate construction. We could ponder the possibility of those Hochschild homology theories to extend to non-commutative settings. The field-theoretic perspective is illuminating in this context.

We remarked that Hochschild homology is related to one-dimensional TQFT. Morally, when looking at the Hochschild homology induced by the shifted tangent bundle — that we will call *linearized* Hochschild homology — we should reduce by one the dimension of the field theory, to account for the lower chromatic level of the associated topological theory. This would produce a zero-dimensional TQFT, thus a trivial invariant. The correct object to look at is indeed $0|1$-dimensional TQFT, where we introduce a dimension in the *odd* direction, i.e. we consider bordisms which are supermanifolds of dimension $0|1$ rather then manifolds of dimension zero[1]. Such

[1]I would like to thank Joost Nuiten for suggesting this, as I was stuck with zero-dimensional theories trying to dig gold out of a coal mine.

a Hochschild homology should be very interesting to study, but let us turn to *elliptic* Hochschild homology. In this case, we need to increase the dimension by one, rather than decrease it. Moreover, elliptic curves over $\mathbb{C}$ are topologically all homeomorphic to $S^1 \times S^1$, the only structure distinguishing them is their complex, i.e *conformal*, structure: the correct object to look at is a *two-dimensional conformal field theory.*

The cobordism hypothesis links n-dimensional TQFTs with E_n-algebras; a similar link should exist between 2-dimensional CFTs and *vertex algebras*. These objects, very much like E_n-algebras in TQFTs, play the role of the "algebras of observables", the set of the relevant operators in the physical theory and the structure associated to the most fundamental manifold in the relevant bordism category — the point in (fully extended) TQFTs, the circle in 2d CFTs — in the mathematical formulation.

We honestly hope to be able to study these phenomena in the future. A mathematical theorem linking CFT and a formulation of elliptic Hochschild homology would be a starting point to understand the very deep relationship between elliptic cohomology itself and those physical theories, completing the picture initiated by Witten in the 80's.

1.2. Structure of the thesis, straight to the point, no bubbling

The thesis is structured in the following way. Chapter 1 is an introduction to the main topics, of informal and speculative nature. Some of the aspects discussed in this chapter will not be studied in the main chapters of the thesis, Chapter 2 and 3, but some partial results and discussion will be collected in Chapter 4.

Chapter 2 is the main chapter of the thesis. It is a 1:1 copy of the paper [**67**]. This paper has been written in conjunction with, and is the result of joint work with, my supervisor Nicolò Sibilla. In this paper we describe a notion of *elliptic Hochschild homology* for derived stacks in characteristic 0 (Definition 2.2.32). This notion is tailored to quotient stacks, our main application, but a more general definition appears in Chapter 4 as Definition 4.1.1. We then study this notion in Section 2.3, obtaining results of codescent with respect to the Zariski topology on target varieties, Corollary 2.3.10, and an analogous codescent result with respect to the equivariant Zariski topology on normal varieties, Theorem 2.3.14. In Section 2.4 we provide some computations of elliptic Hochschild homology of simple quotient stacks. These allow, together with the results in Section 2.3, to compute the elliptic Hochshild homology of toric varieties modulo their standard toric action, Theorem 2.4.5. Section 2.5 we provide some relevant properties of elliptic Hochschild homology that will be used in the last section of the paper. Most relevantly, a localization formula for the stack of quasi-constant maps, Theorem 2.5.2, and a formula for the completion of elliptic Hochschild homology at closed points of the base scheme E_T, Theorem 2.5.14. In the last section, Section 2.6, we define a Tate construction that allows to recover complexified equivariant elliptic cohomology from our notion of elliptic Hochschild homology. The relevant action is the natural E-action on

$$\underline{\mathrm{Map}}^0(E, [X/T])$$

for an elliptic curve E and a smooth variety X acted on by a torus T. The final theorem of the section and of the paper is Theorem 2.6.10, in which we show that the Tate construction indeed recovers Grojnowski's equivariant elliptic cohomology.

Chapter 3 is a 1:1 copy of the paper [**78**]. This paper is divided in two parts. The first part, Section 3.3, provides a construction of *k-rationalized equivariant elliptic cohomology* in terms of adelic descent, for a $\mathbb{Q}$-algebra k (Definition 3.3.8). This definition recovers Grojnowski's sheaf when $k = \mathbb{C}$ (Corollary 3.3.16). The second part of the paper is devoted to a discussion of equivariant cohomology and equivariant K-theory via Hochschild counterparts. Most of the results in this part of the paper were previously known in some form, due to Pantev–Toën–Vaquié–Vezzosi [**54**] and Calaque–Pantev–Toën–Vaquié–Vezzosi [**16**] for cohomology and to Halpern-Leistner–Pomerleano [**31**] for equivariant K-theory. The results of Calaque–Pantev–Toën–Vaquié–Vezzosi are much more general than what we can get with our methods, while our Theorem 3.5.2 generalizes Halpern-Leistner and Pomerleano's work. Similar results regarding a description of equivariant K-theory of manifolds in terms of periodic cyclc homology of their ring of C^∞ functions were known since the 90's. Relevant references include [**14**] [**13**]. In the second part of this paper, we describe equivariant cohomology (Theorem 3.5.14) and equivariant K-theory (Theorem 3.5.2) via Tate constructions respectively on the shifted tangent stack and derived loop space of quotients $[X/T]$ of smooth varieties by tori.

The last Chapter is devoted to partial results. This chapter will be updated until a few weeks before the defence of this thesis, most likely.

CHAPTER 2

Equivariant Elliptic Cohomology and Mapping Stacks I

Written by Nicolò Sibilla and Paolo Tomasini

2.1. Introduction

In this paper we give a new construction of equivariant elliptic cohomology via derived algebraic geometry. The use of techniques from derived algebraic geometry has become pervasive in the study of elliptic cohomology, especially after the groundbreaking work of Lurie. However our aims in this paper are more limited and somewhat different in spirit from the developments originating from Lurie's work. For starters, we are only interested in rational phenomena, and therefore we will work over a fixed ground field of characteristic zero. Our goal is providing a geometric interpretation of the equivariant elliptic cohomology of complex algebraic varieties equipped with the action of an algebraic group G. We will show that the G-equivariant elliptic cohomology of a variety X can be described in terms of functions over a certain substack of the mapping stack

$$\underline{\mathrm{Map}}\,(E, [X/G])$$

where E is an elliptic curve over k, and $[X/G]$ is the stacky quotient of X by G. Our approach is closely related to earlier works by other authors including Gorbounov–Malikov–Schechtman–Vaintrob [**26**] [**49**] [**47**] [**48**], Berwick-Evans [**9**] and Berwick-Evans–Tripathy [**10**], Costello [**18**] [**19**] and others.

This is the first article in a series. In this article we shall focus on the case when $G = T$ is an algebraic torus. We will introduce the substack of $\underline{\mathrm{Map}}\,(E, [X/T])$ parametrizing *almost constant maps* to $[X/T]$. We will show that functions over the stack of almost constant maps define a cohomology theory of stacks, which we call *elliptic Hochschild homology.* Under this assumption we will study the formal properties of elliptic Hochschild homology, and compute it in important classes of examples. In the final section of the paper we will prove that elliptic Hochschild homology recovers Grojnowski's rationalized equivariant elliptic cohomology. The comparison requires an additional step, namely passing to the Tate fixed points under a natural action. This is familiar from the theory of classical Hochschild homology, which recovers periodized de Rham cohomology only after passing to the Tate fixed points for the natural S^1-action. The relationship between elliptic Hochschild homology and elliptic cohomology is entirely parallel to this.

An in depth treatment of elliptic Hochschild homology in the case of the action of a general reductive group G is deferred to the follow-up paper [**66**].

Equivariant elliptic cohomology. Elliptic genera were first introduced by Ochanine in the 80's. Subsequently Witten introduced what is now called the Witten genus, which is a kind of universal elliptic genus, as the index of a Dirac operator on the loop space [**82**]. The work of Witten showed that elliptic genera had deep ties to quantum field theory [**81**], and spurred a great deal of research in the area. Elliptic cohomology was introduced in the late 80-s to provide a conceptual framework for the study of elliptic genera. Elliptic cohomologies are even periodic cohomology theories whose associated formal group law is isomorphic to the completion of an elliptic curve at the identity. They have been the focus of great interest within homotopy theory for the last thirty years. Giving a satisfactory construction of elliptic cohomology, and its universal variant Tmf, is highly non-trivial. The state-of-the art is provided by ongoing work of Lurie [**39**], [**40**], [**41**], which depends in a crucial way on the comprehensive foundations for ∞-categories and spectral geometry which he has been developing in a series of books [**46**], [**45**], [**43**], and by independent work of Gepner and Meier [**24**] in the equivariant case.

It was understood early on by Ginzburg–Kapranov–Vasserot and Grojnowski that rationalized equivariant elliptic cohomology should give rise to coherent sheaves over the elliptic curve itself [**25**] [**30**]. This fits into a well established paradigm, first evinced in Atiyah-Segal's work on equivariant K-theory, that turning on equivariance is closely related to decompleting. In particular, as the formal group law of elliptic cohomology theories is the completion of an elliptic curve E at the identity, the equivariant elliptic cohomology of a space X with an S^1-action should take values in coherent sheaves over E. Further, the stalks of this coherent sheaf can also be understood geometrically: they compute the Borel equivariant cohomology of various fixed points loci of X. Equivariance with respect to general Lie groups can be also understood in similar terms, and gives rise to coherent sheaves over the moduli space of G-bundles over E. In the influential article [**30**], Grojnowski gives a beautiful construction of rationalized equivariant elliptic cohomology which implements this picture. Our work in this paper provides in particular a geometric explanation of Grojnowski's construction in terms of the defomation theory of (almost constant) maps out of elliptic curves.

Elliptic cocycles and secondary Hochschild homology. One of the main challenges in elliptic cohomology is providing a geometric description of elliptic cocycles. Several influential proposals have been put forward starting from Segal's famous 1988 lecture at the Bourbaki seminar [**64**], and subsequent work of Stolz–Teichner [**69**], [**70**]. An implementation of this circle of ideas based on the concept of conformal nets has been pursued by Douglas, Bartels and Henriques in a series of works: we refer the reader to [**21**] for an overview of this important perspective.

A different, but related, point of view is that elliptic cohomology should be in some sense a categorification of K-theory. That fits with the heuristics that raising the chromatic level should be related to categorification. This suggests in particular that elliptic cocycles could be represented by appropriate categorified bundles. These ideas have been explored in [**7**]. Within algebraic geometry, this perspective has been taken up by Toën–Vezzosi, who introduced *secondary Hochschild homology* as a model of elliptic cohomology [**80**]. We refer the reader to the introduction of [**77**] for a beautiful discussion of these ideas. In this paper we do not attempt to provide a geometric interpretation of elliptic cocycles, although this is one of the broader goals of our project. However the definition of secondary Hochschild homology was an important motivation for our work, and thus it is useful to review it here and compare it with our construction.

Recall that the Hochschild homology of a scheme X is given by the global sections of the structure sheaf of the derived loop space of X,

$$\mathrm{HH}_*(X) \simeq \mathcal{O}(\mathcal{L}X)$$

Here the derived loop space $\mathcal{L}X = \mathrm{Map}(S^1, X)$ is the stack of maps from S^1 to X. Secondary Hochschild homology is defined as the global sections of the structure sheaf of the *double loop space* of X

$$\mathrm{HH}_*^{(2)}(X) = \mathcal{O}(\mathcal{L}\mathcal{L}X) \tag{1}$$

As a model of elliptic cohomology, $\mathrm{HH}_*^{(2)}(X)$ has several desirable features. First, the double loop space

$$\mathcal{L}\mathcal{L}X = \mathrm{Map}(S^1 \times S^1, X)$$

is the moduli space of maps out of a topological torus, which captures the underlying topology of an elliptic curve. Additionally categorified bundles yield cocycles in $\mathrm{HH}_*^{(2)}(X)$, as the heuristics on elliptic cohomology would dictate. We refer the reader to [**33, 34**] for additional information on secondary Hochschild homology and its properties.

On the other hand secondary Hochschild homology is insensitive to the complex moduli of elliptic curves, and therefore cannot be hoped to fully capture elliptic cohomology. Our construction can be described as a variant of (1), where we promote the topological torus $S^1 \times S^1$ to an elliptic curve E over a field k. When the ground field k is the field of complex numbers $\mathbb{C}$, and X is a complex scheme with an action of an algebraic group G, the resulting theory is closely related to the complexified (equivariant) elliptic cohomology of the analytification of X; and recovers it after passing to Tate fixed points for an appropriate action. One key difference with (1) is that the full mapping stack

$$\underline{\mathrm{Map}}\,(E, X)$$

is too large, in general, as there might be topologically non-trivial maps between E and X. From the perspective of elliptic cohomology, the only maps that contribute are the *almost constant* ones. As we explain next, this concept can be easily formalized.

Almost constant maps. We fix a ground field k of characteristic zero, and an elliptic curve E over k. Let T be an algebraic torus and let X be a variety equipped with a T-action. We denote by

$$\underline{\mathrm{Map}}^0\,(E, [X/T]) \subset \underline{\mathrm{Map}}\,(E, [X/T]) \tag{2}$$

the smallest clopen substack containing the trivial maps, i.e. the maps factoring as

$$E \to \mathrm{Spec}\,(k) \to [X/T]$$

We call $\underline{\mathrm{Map}}^0\,(E, [X/T])$ the stack of *almost constant maps*. In fact, the correct notion of quasi-constant maps is slightly more involved, and we refer the reader to Section 2.2.5 for a complete exposition of this point.

Working with maps that are close to being constant is familiar from many geometric contexts. The product structure on Chen–Ruan orbifold cohomology, for instance, is governed by almost constant maps out of marked rational curves. For an example which is closely related to our story recall that, in defining the Witten genus via Dirac operator on the loop space, Witten and Taubes actually work with *small loops*: i.e. with the normal bundle to constant loops, rather than with the full loop space. We regard almost constant maps out of E as an analogue, in our setting, of Witten and Taubes' small loops.

A key property of the stack of quasi-constant maps is that it satisfies a form of Zariski codescent on the target.

THEOREM A (Theorem 2.3.14). *Let $U_i \to X$ be a T-equivariant Zariski open cover. Then the natural map*

$$\varinjlim_i \underline{\mathrm{Map}}^0\,(E, [U_i/T]) \to \underline{\mathrm{Map}}^0\,(E, [X/T])$$

is an equivalence.

Theorem A plays a crucial role in our construction. Codescent fails in general for the full mapping stack, as topologically non-trivial maps will not factor through any equivariant Zariski open cover of X. The fact that codescent holds for almost constant maps should be viewed as a counterpart of the Mayer–Vietoris principle in elliptic cohomology. It is an interesting question to what extent codescent is a general feature of stacks of almost constant maps from an arbitrary source; we refer the reader to Remark 2.3.1.1 in the main text for additional comments on this point.

Elliptic Hochschild homology. The stack of almost constant maps carries a structure morphism

$$\underline{\mathrm{Map}}^0\,(E,[X/T]) \to \underline{\mathrm{Map}}^0\,(E,[pt/T]) \simeq \underline{\mathrm{Pic}}^0(E) \otimes_{\mathbb{Z}} \check{T}$$

where $\check{T}$ is the cocharacter lattice of T. The stack $\underline{\mathrm{Pic}}^0(E)$ is the connected component of the Picard stack of E which parametrizes degree 0 line bundles. We can rewrite this as

$$\underline{\mathrm{Pic}}^0(E) \otimes_{\mathbb{Z}} \check{T} \simeq \left(\mathrm{Pic}^0(E) \otimes_{\mathbb{Z}} \check{T}\right) \times [pt/T] \simeq \left(E \otimes_{\mathbb{Z}} \check{T}\right) \times [pt/T] \simeq E^n \times [pt/T]$$

where $\mathrm{Pic}^0(E)$ is the Picard scheme of E and n is the rank of T. In particular, we have a natural map

$$p : \underline{\mathrm{Map}}^0\,(E,[X/T]) \longrightarrow \mathrm{Pic}^0(E) \otimes_{\mathbb{Z}} \check{T}$$

We set $E_T := \mathrm{Pic}^0(E) \otimes_{\mathbb{Z}} \check{T}$.

The following is the most important definition of this article.

Definition B. *The T-equivariant* elliptic Hochschild homology *of X is*

$$\mathcal{HH}_E([X/T]) := p_*(\mathcal{O}_{\underline{\mathrm{Map}}^0(E,[X/T])}) \in \mathrm{QCoh}(E_T)$$

We denote by $\mathrm{HH}_E([X/T])$ *the global sections of* $\mathcal{HH}_E([X/T])$*.*

Note that, as a consequence of Theorem A, the sheaf $\mathcal{HH}_E([X/T])$ satisfies T-equivariant Zariski descent on X. That is, if $\mathcal{U} = \{U_i\}$ is a T-equivariant Zariski open cover of X

$$\mathcal{HH}_E([X/T]) \simeq \varprojlim_i \mathcal{HH}_E([U_i/T])$$

Remark 2.1.1. *The terminology* elliptic Hochschild homology *was already used by Moulinos-Robalo-Toën in the beautiful recent paper* [**51**] *to refer to a seemingly different construction. There are differences between our setting and theirs. They work over a p-adic ring of integers R; additionally they do not consider the equivariant setting, which is of primary importance for us.*

However the two notions are intimately related, and in fact equivalent when they overlap. In this article we place ourselves over a field k of characteristic 0 *because we are interested in establishing properties of* $\mathcal{HH}_E([X/T])$ *which only hold in that setting; and ultimately we want to set* $k = \mathbb{C}$ *and compare our theory with complexified equivariant elliptic cohomology of the analytification of* X*. Note however that our Definition B does not depend on the choice of ground ring, and therefore makes sense also over a ring of p-adic integers. We claim that if* X *is a derived scheme over* R *as considered in* [**51**]*, and* T *is the trivial group, then Definition B is equivalent to* *the elliptic Hochschild homology of X as defined in* [**51**]*. This comparison result will appear in forthcoming work. This justifies our usage of the term* elliptic Hochschild homology*, as it is compatible with its earlier definition in* [**51**].

In this article we establish several fundamental formal properties of elliptic Hochschild homology. Some of our main results are a *localization theorem* for elliptic Hochschild homology, and a calculation of its *analytic stalks.* These results, which we will explain in the next section of this introduction, ought to be considered as direct analogues of the local structure of rationalized equivariant elliptic cohomology that was first described by Grojnowski in [**30**]. A more recent reference, which is more closely related to our work from a methodological standpoint, is the description of the local structure of Hochschild homology of global quotient stacks in [**17**]. Our results give a complete description of the local behaviour of elliptic Hochschild homology of quotient stacks, and are key to establish the comparison with Grojnowski's rationalized equivariant elliptic cohomology.

Before presenting these results however, let us explain two classes of examples for which $\mathcal{HH}_E(X)$ can be explicitly computed. The first observation is that, as expected, in the absence of a group action elliptic Hochschild homology coincides with ordinary Hochschild homology. This is an analogue of the fact that, in the non-equivariant regime, rationalization collapses all cohomology theories to singular cohomology.

Theorem C (Corollary 2.3.10). *Let T be the trivial group. Then there is an equivalence*

$$\mathrm{HH}_E(X) \simeq \mathrm{HH}_*(X)$$

Next, let us consider the case when X is a smooth toric variety equipped with the action of the maximal torus. Toric actions on toric varieties are treated at length in Section 2.4, for several reasons. First, calculations on affine spaces and projective spaces equipped with a torus actions are the cornerstone of our general structure results, as varieties equipped with a T-action admit an equivariant local embedding in affine space. Second, toric varieties provide fully computable examples of our theory owing to the codescent Theorem A. In particular, when X is a smooth toric variety equipped with the action of a maximal torus T we have the following result.

Assume that $k = \mathbb{C}$ and let X be a smooth toric variety equipped with the action of the maximal torus T. We denote by $\mathcal{E}ll^0_{T^{an}}(X^{an})$ the degree zero complexified T-equivariant elliptic cohomology of the analytification of X, viewed as a coherent sheaf over E_T

$$\mathcal{E}ll^0_{T^{an}}(X^{an}) \in \mathrm{Coh}(E_T)$$

Theorem D (Theorem 2.4.5). *There is an equivalence in* $\mathrm{Coh}(E_T)$

$$\mathcal{HH}_E([X/T]) \simeq \mathcal{E}ll^0_{T^{an}}(X^{an})$$

Theorem D follows from two ingredients. The first is the calculation of the elliptic Hochschild homology of the affine space $\mathbb{A}^N$ under an arbitrary torus action, which plays a key role in the proof of Theorems E and F. The second is codescent for almost constant maps, Theorem A.

Main theorems. Our main results are contained in the last two Sections of the article, Section 2.5 and 2.6. In Section 2.5 we establish two structure theorems that describe the local behaviour of elliptic Hochschild homology. They are exactly parallel to features of ordinary equivariant elliptic cohomology. As explained by Grojnowski, the local structure of elliptic cohomology is governed by information coming from the cohomology of fixed point loci. Further, on sufficiently small neighbourhoods of points of E_T, elliptic cohomology is equivalent to ordinary Borel equivariant cohomology of fixed point loci. In our setting, these two claims translate into the statements of Theorem E and Theorem F respectively. Analogous statements for ordinary Hochschild homology were proved by Chen in [**17**], which was an important inspiration for our work.

As pioneered by Grojnowski and explained by Roşu, we can associate to each closed point x of E_T a subgroup $T(x)$ of T. When T is of rank one, $T(x)$ is equal to T if x is non-torsion and is equal to μ_n if x is torsion of (exact) order n. We denote $X^{T(x)}$ the derived fixed locus of X under the induced $T(x)$-action. The classical fixed locus is given by the truncation $t_0(X^{T(x)})$.

Theorem E (Theorem 2.5.2). *Let X be a smooth variety over k. Then for any closed point $x \in E_T$ there exists a Zariski open neighborhood U of x such that the natural map*

$$\underline{\mathrm{Map}}^0\left(E, [t_0 X^{T(x)}/T]\right) \times_{E_T} U \to \underline{\mathrm{Map}}^0\left(E, [X/T]\right) \times_{E_T} U \tag{3}$$

induced by the inclusion $t_0 X^{T(x)} \to X$, is an equivalence.

Recall that $T \cong \operatorname{Spec} \mathrm{HH}_*([*/T])$. If Y is a stack with a T-action, the Hochschild homology $\mathrm{HH}_*([Y/T])$ carries an action of $\mathrm{HH}_*([*/T])$.

Theorem F (Theorem 2.5.14). *The étale stalk of $\mathcal{HH}_E([X/T])$ at a point x of E_T is equivalent to the completion of* $\mathrm{HH}_*([t_0 X^{T(x)}/T])$ *at* $1 \in T \cong \operatorname{Spec} \mathrm{HH}_*([*/T])$

Theorem E and F are key ingredients in the proof of the comparison between $\mathcal{HH}_E([X/T])$ and equivariant elliptic cohomology. As we already discussed, the first step is to introduce a *periodic cyclic* variant of $\mathcal{HH}_E([X/T])$, which we call *elliptic periodic cyclic homology* and denote $\mathcal{HP}_E([X/T])$. Elliptic Hochschild homology, just as classical Hochschild homology, is not homotopy invariant. Thus it cannot be hoped to coincide with elliptic cohomology on the nose. The fact that this discrepancy can be obviated by keeping track of an extra piece of data, in the form of a differential or of a cyclic action, is familiar from the classical story. However what exactly the cyclic action might be in the elliptic setting is a somewhat subtle issue.

In the ordinary setting periodic cyclic homology is calculated by passing to the Tate fixed points under the natural action of S^1 on Hochschild homology. Via the identification

$$\mathrm{HH}_*(X) \simeq \mathcal{O}(\mathcal{L}X)$$

this action comes from the action of S^1 on $\mathcal{L}X$ via loop rotation. However elliptic Hochschild homology $\mathcal{HH}_E([X/T])$ does not carry in a natural way an action of S^1. It is defined instead in terms of functions on a stack parametrizing maps out of the elliptic curve E

$$\mathcal{HH}_E([X/T]) := p_*(\mathcal{O}_{\underline{\mathrm{Map}}^0(E,[X/T])}) \in \mathrm{QCoh}(E_T).$$

Thus it is endowed with a natural E-action. It is not difficult to see however that this action is non-trivial only along the fibers of the structure map

$$\underline{\mathrm{Map}}^0\,(E,[X/T]) \to E_T$$

This simple observation allows us to reinterpret the E-action as a hidden S^1-action, *relative* to the base scheme E_T. The possibility to interpolate between a (cohomological) E-action and an action of the circle depends in a crucial way on the characteristic 0 assumption. Indeed it leverages the formality of the coherent cohomology of E, which is thus equivalent to the coalgebra of cochains on S^1 (since the latter is also formal). We stress that both of these formality statements fail away from characteristic zero. These identifications allow us to make sense of the Tate fixed points of elliptic Hochschild homology and we set

$$\mathcal{HP}_E([X/T]) := \mathcal{HH}_E([X/T])^{\mathrm{Tate}}$$

If X is a variety with a T-action, its elliptic periodic cyclic homology becomes an object in the Tate fixed points of the trivial S^1-action on $\mathrm{Perf}(E_T)$. The latter coincides with the $\mathbb{Z}_2$-*folding* of $\mathrm{Perf}(E_T)$, i.e. the category obtained by collapsing the natural $\mathbb{Z}$-grading on $\mathrm{Perf}(E_T)$ to a $\mathbb{Z}_2$-grading

$$\mathcal{HP}_E([X/T]) \in \mathrm{Perf}(E_T)^{\mathrm{Tate}} \simeq \mathrm{Perf}(E_T) \otimes_k k[u, u^{-1}]$$

where u is in degree 2.

Theorem G (Theorem 2.6.10). *Let $k = \mathbb{C}$. Let T be an algebraic torus of rank n acting on a smooth variety X. We have an isomorphism of $\mathbb{Z}_2$-periodic perfect complexes on E*

$$\mathcal{HP}_E([X/T]) \simeq \mathcal{E}ll_{T^{\mathrm{an}}}(X^{\mathrm{an}})$$

where $\mathcal{E}ll_{T^{\mathrm{an}}}(X^{\mathrm{an}})$ is the complexified T^{an}-equivariant elliptic cohomology of the analytification of X.

An analogue of Theorem G for classical Hochschild homology is due to Halpern-Leistner–Pomerleano. Their Theorem 2.17 in [**31**] shows that the periodic cyclic homology of suitable quotient stacks $[X/G]$ over the complex numbers is equivalent to the G^{an}-equivariant K-theory of the analytification X^{an} of X.

Let us comment briefly on the proof of Theorem G. It follows from Theorem F that the completions of $\mathcal{HP}_E([X/T])$ and $\mathcal{E}ll_{T^{\mathrm{an}}}(X^{\mathrm{an}})$ at points of E_T match. The formal completions at points of a smooth variety X can be thought of as algebraic analogues of small complex balls covering X. This might suggest that this information is sufficient to establish a global equivalence between $\mathcal{HP}_E([X/T])$ and $\mathcal{E}ll_{T^{\mathrm{an}}}(X^{\mathrm{an}})$. This is roughly correct, but the notion of completion has to be substantially enhanced. Completions at points are by themselves insufficient. The correct notion is provided by the *adeles* of a coherent sheaf, which are kinds of completed stalks labelled by flags of subvarieties. The adeles come with natural "restriction maps" which relate them, and give the adeles of a sheaf the structure of a cosimplicial complex. We review the theory of adelic descent in Section 3.2.2. The theory of adelic descent goes back to classic results of Weil for curves, while its generalization to higher dimensions is due to Parshin and Beilinson. We will mostly follow the modern formulation of Groechening [**28**], which is particularly convenient for our purposes.

Adelic methods are not quite sufficient to prove Theorem G. The reason is that, in general, the adeles of $\mathcal{HP}_E([X/T])$ and $\mathcal{E}ll_{T^{\mathrm{an}}}(X^{\mathrm{an}})$ are hard to compute owing to the fact that E_T is not affine. We work around this issue, by combining adelic descent with an induction on the rank of the torus T acting on X. This uses in a crucial way our second structure result on the local behaviour of elliptic Hochschild homology, i.e. Theorem E.

It is a natural question, see Roşu [**61**], whether equivariant cohomology theories can be defined via adelic methods. Grojnowski's original approach involves a careful choice of analytic open cover, and this limits its applicability to the complex setting. A fully adelic treatment would have several benefits, and in particular would work over an arbitrary characteristic zero base. This is the subject of work in progress of the second author [**78**].

Future work. As we explained this paper is the first in a series. In the forthcoming follow-up [**66**] we will complete the picture initiated in this article to account for equivariance under the action of an arbitrary reductive group G. In fact the techniques developed in this article are sufficient to tackle the general case, as it is possible to reduce the question of G-equivariance to T-equivariance for a maximal torus $T \subset G$, on condition of keeping track of the action of the Weyl group. The details will be spelled out in [**66**]. The project initiated in this article is part of a broader goal to obtain geometric descriptions of elliptic cohomology and Tmf, and their cocycles. One of the next objectives in this program will be giving a geometric description of rationalized Tmf, by working with stacks of almost constant maps out of the universal cubic curve. Along the way, we will obtain in particular new geometric descriptions of equivariant singular cohomology and equivariant K-theory in terms of almost constant maps out of cuspidal, and respectively nodal, cubic curves. All these questions will be pursued in future work.

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2.2. Preliminaries

Throughout the paper, k is a fixed field of characteristic zero. We use the term *variety* to mean a k-scheme which is integral, separated and of finite type. Unless we explicitly state otherwise, all geometric objects in the following are implicitly assumed to be defined over k.

In this paper we use the language of ∞-categories and derived algebraic geometry as developed by Jacob Lurie in [**46**] and [**43**]. We will mostly work over a field of characteristic zero, where derived rings can be modelled equivalently by simplicial commutative algebras or by commutative differential-graded algebras. In the setting of commutative cdga-s, foundations for derived algebraic geometry were developed by Toën and Vezzosi in [**75**] and [**76**].

2.2.1. Derived stacks. Let CAlg be the ∞-category of simplicial commutative rings, and $\mathrm{dAff} = \mathrm{CAlg}^{\mathrm{op}}$ be the ∞-category of derived affine schemes. We also use the name *derived rings* for simplicial commutative rings. Constant simplicial commutative rings embed fully faithfully in simplicial commutative rings, and the embedding has a left adjoint corresponding to the connected components functor, which is denoted π_0. We will refer to constant simplicial commutative rings as *underived, classical*, or *discrete* rings interchangeably.

The ∞-category of derived prestacks is the ∞-category of functors

$$\mathcal{P}(\mathrm{dAff}) := \mathrm{Fun}(\mathrm{dAff}^{\mathrm{op}}, \mathcal{S})$$

from simplicial commutative rings to the ∞-category of spaces. The ∞-category of derived stacks is the ∞-category of (hypercomplete) sheaves on dAff with respect to the étale topology, dSt. The category of derived stacks is naturally an ∞-topos. We denote by Spec the Yoneda embedding

$$\mathrm{Spec} : \mathrm{CAlg}^{\mathrm{op}} \to \mathcal{P}(\mathrm{CAlg}^{\mathrm{op}}).$$

Analogously, let $\mathcal{P}(\mathrm{Aff}) := \mathrm{Fun}(\mathrm{CRing}, \mathcal{S})$ be ∞-category of presheaves over classical commutative rings. The ∞-category of hypercomplete sheaves on CRing with respect to the étale topology is the category of *higher stacks*, St. These embed fully faithfully in derived stacks; the embedding has a right adjoint called the *truncation* functor and denoted by t_0. We refer to derived stacks which are equivalent to their truncation as *underived, classical* or *discrete.*

In a relative setting, given a simplicial commutative ring R, we define CAlg_R to be the ∞-category of commutative simplicial R-algebras. We denote the ∞-category of derived prestacks over R, $\mathcal{P}(\mathrm{CAlg}_R^{\mathrm{op}})$, and the ∞-category of stacks over R, dSt_R. If $R = k$ is a field of characteristic zero, the ∞-category CAlg_k is equivalent to the ∞-category $\mathrm{dg}-\mathrm{cAlg}_k^{\leqslant 0}$ of connective commutative dg algebras, i.e. concentrated in nonpositive degrees in cohomological indexing convention. In this paper we mostly work in this setting.

2.2.2. Effective epimorphisms, geometricity, connected components. *Effective epimorphisms* are the natural notion of surjective maps in an ∞-topos. Effective epimorphisms can be characterized as follows

DEFINITION 2.2.1 ([**46**], Corollary 6.2.3.5). *A morphism $f : X \to Y$ in an ∞-topos is an* effective epimorphism *if one of the following two equivalent conditions is satisfied:*

(1) f is a (-1)-truncated object in the ∞-topos $\mathrm{dSt}_{/Y}$ of derived stacks over Y
(2) The Čech nerve $\check{C}(f)$ is a simplicial resolution of Y.

This definition, in the special case of the ∞-topos of derived stacks, becomes equivalent to the following property:

PROPERTY 1. *A map of derived stacks $f : X \to Y$ is an effective epimorphism if for any representable* $\mathrm{Spec}\, S$ *and any map* $\mathrm{Spec}\, S \to Y$*, there exists an etale cover of S, $\{S_i\}$, such that for all i the composition* $\mathrm{Spec}\, S_i \to \mathrm{Spec}\, S \to Y$ *admits a lift* $\mathrm{Spec}\, S_i \to X$*.*

In other words, an effective epimorphism of derived stacks is an étale locally surjective map, just as in the classical theory of sheaves a surjective map of sheaves is a locally surjective map.

There is an important class of derived stacks called *geometric* (derived) stacks. This notion is a generalization to the setting of derived algebraic geometry of the more classical notion of Artin stack. Geometric stacks are characterized by an integer number called the *geometric level*. The definition is stated inductively on the level. We will state the definition in the context of smooth maps, for a discussion in full generality see for example [**57**].

Definition 2.2.2. *Let X be a derived stack.*

- *X is (-1)-geometric if it is an affine derived scheme. A map $f : X \to Y$ of derived stacks is (-1)-representable if for every map*
$$\operatorname{Spec} A \to Y$$
from a (-1)-geometric stack $\operatorname{Spec} A$, the fiber product $\operatorname{Spec} A \times_Y X$ is (-1)-geometric. A map is (-1)-smooth if it is (-1)-representable, and the induced map
$$\operatorname{Spec} A \times_Y X \to \operatorname{Spec} A$$
is a smooth map of affine derived schemes.
- *X is n-geometric if the diagonal map $X \to X \times X$ is $(n-1)$-representable and there exists an effective epimorphism $\coprod \operatorname{Spec} A_i \to X$, called an n-atlas for X, such that each map $\operatorname{Spec} A_i \to X$ is $(n-1)$-smooth. We say that X is geometric if it is n-geometric for some n.*
- *A map of derived stacks $X \to Y$ is n-representable if for every (-1)-geometric $\operatorname{Spec} A$ and any map $\operatorname{Spec} A \to Y$, the fiber product $\operatorname{Spec} A \times_Y X$ is n-geometric.*
- *A map $X \to Y$ is n-smooth if it is n-representable, and for any $\operatorname{Spec} A \to Y$ there exists a n-atlas $\coprod \operatorname{Spec} B_i$ of $\operatorname{Spec} A \times_Y X$ such that for all i the composition*
$$\operatorname{Spec} B_i \to \operatorname{Spec} A \times_Y X \to \operatorname{Spec} A$$
is smooth.

There is a notion of open and closed immersion of geometric stacks.

Definition 2.2.3. *Let $f : X \to Y$ be a morphism of derived geometric stacks. We say that f is an* open *(resp.* closed*)* immersion *if for any map $g : \operatorname{Spec} S \to Y$ the fiber product $\operatorname{Spec} S \times_Y X$ is a derived scheme, and the induced map $\operatorname{Spec} S \times_Y X \to \operatorname{Spec} S$ is an open (resp. closed) immersion of derived schemes.*

In the context of derived schemes, the above definition is equivalent to the definitions below.

Definition 2.2.4 ([**55**], Definition 4.2). *A map of derived schemes $f : X \to \operatorname{Spec} S$ with affine target is an* open immersion *if there exist affine derived schemes $\operatorname{Spec} A_i \to X$ over X such that the composite map $\coprod_i \operatorname{Spec} A_i \to X \to \operatorname{Spec} S$ is an effective epimorphism, and each composite $\operatorname{Spec} A_i \to X \to \operatorname{Spec} S$ exhibits A_i as a localization of S.*

Let $f : X \to Y$ be a morphism of derived schemes. f is an open immersion *if for any affine derived scheme $\operatorname{Spec} S$, the induced map $f_S : \operatorname{Spec} S \times_Y X \to \operatorname{Spec} S$ is an open immersion of derived schemes with affine target.*

Definition 2.2.5. *Let $f : X \to Y$ be a morphism of derived schemes. f is a* closed immersion *if the map on the underlying classical schemes $t_0 f : t_0 X \to t_0 Y$ is a closed immersion of classical schemes.*

We will also need the notion of connected component of a point in a derived geometric stack X. Let K be a field, and let $x : \operatorname{Spec} K \to X$ be a K-point. Consider the full subcategory of $\mathrm{dSt}_{/X}$ of open and closed maps to X whose image contains the K-point x:

$$\mathrm{Clopen}_x X = \{a : G \to X \text{ clopen map in dSt such that } x \text{ factors through } a\}$$

Definition 2.2.6. *The connected component of the point x in X, $X^{(x)}$, is an initial object in the ∞-category $\mathrm{Clopen}_x X$ (which always exists).*

It will be important to consider quasi-coherent sheaves on derived (pre)stacks.

Definition 2.2.7. *Let X be a prestack. The* ∞-category of quasi-coherent sheaves *on X, $\mathrm{QCoh}(X)$, is defined as the limit*

$$\mathrm{QCoh}(X) := \lim_{\{\operatorname{Spec} A \to X\}} \mathrm{Mod}_A$$

over the ∞-category of derived affine schemes with a map to X. Here Mod_A denotes the ∞-category of A-modules.

2.2.2.1. *$\mathbb{Z}_2$-folding of quasi-coherent sheaves.* We will be interested in dealing with a $\mathbb{Z}_2$-periodic version of the ∞-category of quasi-coherent sheaves. We review this object following Preygel [**59**]. There, he introduces a Tate construction on ∞-categories with an action of S^1 and this formalism recovers in particular $\mathbb{Z}_2$-folding, which is what we are interested in.

Definition 2.2.8 (Definition 1.2.3 in [**59**]). *Let $\mathcal{C}$ be a small stable k-linear idempotent complete ∞-category with an action of S^1, where k is a field of characteristic* *zero. Then the Tate construction of* $\mathcal{C}$ *with respect to this* S^1*-action is the tensor product of small stable* ∞*-categories*

$$\mathcal{C}^{\mathrm{Tate}} := \mathcal{C}^{S^1} \otimes_{\mathrm{Perf}(k[[u]])} \mathrm{Perf}(k((u)))$$

where u *is a variable of homological degree* -2*.*

In the following, we adopt Preygel's notation and omit Perf:

$$\mathcal{C}^{\mathrm{Tate}} = \mathcal{C}^{S^1} \otimes_{k[[u]]} k((u))$$

If $\mathcal{C}$ is not small but is equipped with a coherent t-structure (see Definition 4.2.7 in [**59**]), Preygel defines its Tate fixed points via a *regularization* procedure. The regularization of $\mathcal{C}$, $\mathcal{R}(\mathcal{C})$, is defined as the ind-completion of the full subcategory of *coherent* objects, i.e. bounded above objects C whose r-truncation, $\tau_{\leqslant r}C$, is compact for all r (see Definition 4.2.2 in [**59**]). Then Preygel defines (see Definition 1.3.4 in [**59**])

$$\mathcal{C}^{t\mathrm{Tate}} := \mathcal{R}(\mathcal{C}^{S^1}) \otimes_{k[[u]]} k((u))$$

Following Preygel, we refer to $k((u))$-linear ∞-categories as $\mathbb{Z}_2$-periodic.

If X is a Noetherian geometric stack, the standard t-structure on $\mathrm{QCoh}(X)$ is coherent (see Proposition 4.4.1 in [**59**]). In particular, we can consider the Tate construction with respect to a trivial S^1-action on $\mathrm{QCoh}(X)$.

Definition 2.2.9. *The* $\mathbb{Z}_2$-folding *of* $\mathrm{Perf}(X)$ *is the category*

$$\mathrm{Perf}(X)_{\mathbb{Z}_2} := \mathrm{Perf}(X)^{\mathrm{Tate}}$$

The $\mathbb{Z}_2$-folding *of* $\mathrm{QCoh}(X)$ *is the category*

$$\mathrm{QCoh}(X)_{\mathbb{Z}_2} := \mathrm{QCoh}(X)^{t\mathrm{Tate}}$$

2.2.3. Betti stacks and affinization. We define an important class of derived stacks, called *Betti stacks*, which correspond to spaces.

Definition 2.2.10. *Let* $X \in \mathcal{S}$ *be a space. The* Betti stack *associated to the space* X *is the sheafification of the constant presheaf* $\underline{X} : \mathrm{dAff}^{\mathrm{op}} \to \mathcal{S}$ *sending any derived affine* $\mathrm{Spec}\, A$ *to* X*. We abuse notation by denoting the Betti stack associated to a space* X *again by* X*.*

A particularly important example is the derived Betti stack S^1 associated to the circle. This stack plays a key role in this paper and in derived algebraic geometry more broadly. For instance, it will appear in a construction that we will describe in Section 2.2.4. This stack will also play a role through its *affinization*, which we now describe as a general construction in derived algebraic geometry.

The *affinization* of a derived stack is a fundamental notion, and is strictly related to the concept of *affine stack* which we review below. The construction has been introduced in the nonderived setting by Toën in [**73**], and studied in the derived context by Ben-Zvi and Nadler. A review is in [**51**]. We recall the construction here for the reader's convenience.

Let k be a discrete commutative ring. We denote by coCAlg_k the ∞-category of cosimplicial commutative algebras over k, and $\mathrm{St}_k \subset \mathrm{dSt}_k$ the ∞-category of classical higher stacks.

DEFINITION 2.2.11. *Denote by*

$$\mathrm{Spec}^{\Delta} : \mathrm{coCAlg} \to \mathrm{St}_k$$

the functor that sends a coconnective commutative k-algebra A to the functor it corepresents, i.e. the functor sending a simplicial commutative ring B to the space of maps $\mathrm{Map}_{\mathrm{coCAlg}}(A, B)$*. Stacks of the form* $\mathrm{Spec}^{\Delta} A$ *for some cosimplicial commutative k-algebra A will be called* affine stacks.[1]

The functor Spec^{Δ} has a left adjoint $\mathcal{O} : \mathrm{St}_k \to \mathrm{coCAlg}$. The composition

$$\mathrm{Spec}^{\Delta}\mathcal{O} : \mathrm{St}_k \to \mathrm{St}_k$$

is called the *affinization* functor.

We will consider the affinization of an elliptic curve E over a field k of characteristic zero, i.e. the affine stack $\mathrm{Spec}^{\Delta}\mathcal{O}(E)$. Over k the cdga $\mathcal{O}(E)$ is formal, and thus isomorphic to its cohomology, which is given by

$$H^i(E; \mathcal{O}_E) = \begin{cases} k \text{ if } i = 0 \text{ or } i = 1 \\ 0 \text{ else} \end{cases}$$

The cdga $\mathcal{O}(S^1) = C^*(S^1; k)$ is also formal, which implies

$$\mathrm{Aff}(E) \simeq \mathrm{Aff}(S^1)$$

Since the algebra of derived global functions on S^1 is isomorphic to the commutative dg-algebra $k[\epsilon]$ with the variable ϵ in (cohomological) degree 1, the affine stack $\mathrm{Aff}(S^1)$ is sometimes denoted also by $\mathbb{A}^1[1]$ and referred to as the *shifted affine line*.

REMARK 2.2.12. *Working over a field k of characteristic zero we can model simplicial commutative algebras over k via connective commutative dg-algebras. In this setting, Ben-Zvi and Nadler develop in* [**8**] *a similar construction. They consider the functor*

$$\mathrm{Spec} : \mathrm{dg} - \mathrm{cAlg}_k \to \mathrm{dSt}_k$$

sending A to the functor mapping $B \in \mathrm{dg} - \mathrm{cAlg}_k^{\leqslant 0}$ *to* $\mathrm{Map}_{\mathrm{dg}-\mathrm{cAlg}_k}(A, B)$*. This functor is right adjoint to $\mathcal{O}$, and the affinization functor is* $\mathrm{Spec}\,\mathcal{O} : \mathrm{dSt}_k \to \mathrm{dSt}_k$.

[1]The same notion is referred to as *coaffine stacks* by Lurie in [**38**].

2.2.4. Tangents and loops. The ∞-category of derived stacks dSt is Cartesian closed, hence it admits internal mapping objects, also known as *mapping stacks*.

Remark 2.2.13. *The notation we use for mapping spaces in ∞-categories is* Map, *in contrast with the notation for the mapping stack introduced below.*

Given two derived stacks X and Y, we denote their mapping stack as $\underline{\mathrm{Map}}\,(X, Y)$. As a presheaf on $\mathrm{CAlg}^{\mathrm{op}}$, the mapping stack is characterized by

$$\underline{\mathrm{Map}}\,(X, Y)\,(S) = \mathrm{Map}(X \times \operatorname{Spec} S, Y)$$

The mapping stack can also be defined relative to some base derived stack B: for X and Y over B, the mapping stack $\underline{\mathrm{Map}}\,(X, Y)_B$ relative to B is given by

$$\underline{\mathrm{Map}}\,(X, Y)_B\,(S) = \mathrm{Map}_{/B}(X \times_B \operatorname{Spec} S, Y)$$

Given a derived geometric stack X, we can build new derived stacks via universal constructions. In this paper we will consider the shifted tangent $T_X[-1]$, the unipotent loop space $\mathcal{L}^u X$, the loop space $\mathcal{L}X$ and the derived stack of quasi-constant maps $\underline{\mathrm{Map}}^0\,(E, X)$ from an elliptic curve E. The first three constructions are well documented in the literature, and we will recall them in this section, while the latter is new and the main object of study of this paper.

If X is a derived geometric stack its *cotangent complex* is the quasi-coherent sheaf

$$\mathbb{L}_X \in \mathrm{Qcoh}(X)$$

corepresenting the functor of derivations.

Definition 2.2.14. *The* shifted tangent bundle *of X, $T_X[-1]$, is the relative spectrum over X*

$$T_X[-1] := \operatorname{Spec}_{\mathcal{O}_X} \operatorname{Sym} \mathbb{L}_X[1]$$

The unipotent loop space *of X, $\mathcal{L}^u X$, is the mapping stack*

$$\mathcal{L}^u X := \underline{\mathrm{Map}}\,\big(\mathrm{Aff}(S^1), X\big)$$

The loop space *of X, $\mathcal{L}X$, is the mapping stack*

$$\mathcal{L}X := \underline{\mathrm{Map}}\,\big(S^1, X\big)$$

When X is a derived scheme, the shifted tangent bundle, the loop space and the unipotent loop space are all equivalent. This follows from the fact that these three objects are cosheaves over the Zariski site of a derived scheme, as explained in [**8**], and they coincide when the target X is affine. As an example, we recall the relevant codescent statement for the loop space:

LEMMA 2.2.15 (Lemma 4.2, [**8**]). *Let X be a derived scheme. The functor*

$$\begin{aligned} |X|_{\mathrm{Zar}} &\to \mathrm{dSt}_k \\ U &\mapsto \mathcal{L}U \end{aligned}$$

is a cosheaf.

All these constructions have formal counterparts, where one formally completes the stacks at a trivial locus corresponding to X. We start by recalling the notion of formal completion in derived algebraic geometry. A summary of this and related notions can be found for example in [**17**].

DEFINITION 2.2.16. *Let X and Y be derived stacks, and let $f : X \to Y$ be a map. We define the* formal completion *of Y at X, $\widehat{Y}_X$, as the derived stack whose functor of points is the following: for every ring S, the space $\widehat{Y}_X(S)$ is the space of commutative diagrams*

$$\begin{tikzcd} \operatorname{Spec} \pi_0(S)^{\mathrm{red}} \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} S \arrow[r] & Y \end{tikzcd}$$

where $\pi_0(S)^{\mathrm{red}}$ is the reduction of the discrete ring $\pi_0(S)$ considered as a constant simplicial commutative ring.

We can describe formal completions alternatively using the *de Rham* stack of a derived stack Y. Let $\mathrm{CRing}^{\mathrm{red}}$ be the category of reduced classical commutative rings, which embeds in the ∞-category of simplicial commutative rings CAlg as constant simplicial rings. In particular, we get a restriction functor

$$i^* : \mathcal{P}(\mathrm{dAff}) \to \mathcal{P}(\mathrm{CRing}^{\mathrm{red}^{\mathrm{op}}})$$

from derived prestacks to presheaves on $\mathrm{CRing}^{\mathrm{red}^{\mathrm{op}}}$. We can construct a right adjoint to this functor by sending a presheaf on $\mathrm{CRing}^{\mathrm{red}^{\mathrm{op}}}$ to its right Kan extension along i. We call this functor i_*.

DEFINITION 2.2.17. *Let Y be a derived prestack. Its* de Rham *prestack is the derived prestack $Y_{dR} = i_* i^* Y$.*

By definition, given a derived ring S, an S-point of Y_{dR} is a $\pi_0(S)^{\mathrm{red}}$-point of Y. Observe that the unit of the adjunction $\mathrm{Id} \to i_* i^*$ gives us a map $Y \to Y_{dR}$. Via the de Rham stack we can describe the formal completion of Y at X as the pullback

$$\begin{tikzcd} \widehat{Y}_X \arrow[r] \arrow[d] & X_{dR} \arrow[d] \\ Y \arrow[r] & Y_{dR} \end{tikzcd}$$

Now we can define the formal completions at X of the objects we described earlier.

Definition 2.2.18. *The* formal shifted tangent bundle *of X is the formal completion of the shifted tangent bundle of X at the zero section, and is denoted by $\widehat{T}_X[-1]$. The* formal loop space *of X is the formal completion of the loop space of X at the constant loops, and is denoted by $\widehat{\mathcal{L}}X$.*

2.2.5. Quasi-constant maps. We can now introduce one of the main objects of the paper, the *derived stack of quasi-constant maps* $\underline{\mathrm{Map}}^0(Y, X)$ between two derived stacks Y and X.

Let A be a finite abelian group isomorphic to a product of groups of roots of unity

$$A \simeq \prod_{i=1,\dots,r} \mu_{n_i} \tag{4}$$

For each map

$$\alpha : Y \to [\operatorname{Spec} k/A] \to X$$

let $(\underline{\mathrm{Map}}(Y, X))^{(\alpha)}$ be the connected component of $\underline{\mathrm{Map}}(Y, X)$ containing α.

Definition 2.2.19. *The* derived stack of quasi-constant maps *is the union*

$$\underline{\mathrm{Map}}^0(Y, X) := \bigcup_{\alpha} \underline{\mathrm{Map}}(Y, X)^{(\alpha)}$$

Remark 2.2.20. *The structure map $Y \to \operatorname{Spec} k$ gives a closed embedding*

$$X \simeq \underline{\mathrm{Map}}(\operatorname{Spec} k, X) \to \underline{\mathrm{Map}}(Y, X)$$

which factors through $\underline{\mathrm{Map}}^0(Y, X)$.

Remark 2.2.21. *Definition 2.2.19 is designed to work in the setting where the target is the quotient $[X/T]$ of a variety by the action of a torus, which is the framework we place ourselves in in this paper. Though adequate for what we plan to do in this article, Definition 2.2.19 is somewhat ad hoc. A more conceptual definition can be obtained by considering maps which are in a precise sense* of degree zero. *Consider the map*

$$\underline{\mathrm{Map}}(Y_{dR}, X) \to \underline{\mathrm{Map}}(Y, X) \tag{5}$$

induced by the unit $Y \to Y_{dR}$. We believe that in general quasi-constant maps should be definable as the image of this map, i.e. the smallest clopen subset of $\underline{\mathrm{Map}}(Y, X)$ such that the map (5) factors through it. As a reality check, consider the case when X is a classifying stack $[\operatorname{Spec} k/G]$, where G is a reductive algebraic group. The image of (5) is the stack classifying underlying G-bundles to flat G-bundles. These *are precisely degree 0 G-bundles, which is what we intend to model via quasi-constant maps.*

We will mostly use Definition 2.2.19 in the situation where the source Y is an elliptic curve E over a field k of characteristic zero.

REMARK 2.2.22. *The bundles classified by $\underline{\mathrm{Bun}}_T^0(E)$ are exactly the degree zero T–bundles on E, as those are the ones in the connected components of the T–bundles whose structure group can be reduced to A, for A as in* (4).

REMARK 2.2.23. *The variant of Definition 2.2.19 involving only connected components of constants maps is in general insufficient for our purposes. The issue arises from quotients $[X/T]$ that have points with finite non-trivial stabilizers. Consider $X = \mathbb{G}_m$ with an action of $T = \mathbb{G}_m$ with weight $w \neq 1$. The quotient $[X/T]$ is isomorphic to $[\operatorname{Spec} k/\mu_{|w|}]$. In this case, a simple equivariant elliptic cohomology computation dictates that there should be an isomorphism*

$$\underline{\mathrm{Map}}^0\left(E, [\mathbb{G}_m/\mathbb{G}_m]\right) \cong E[|w|] \times [\operatorname{Spec} k/\mu_{|w|}]$$

where $E[|w|]$ denotes the $|w|$-torsion points in E. In particular, this stack has $|w|$-many connected components. Definition 2.2.19 is designed precisely so as to reproduce this expected behaviour.

In the following Propositions we give sufficient conditions under which it is enough to look at connected components of constant maps.

PROPOSITION 2.2.24. *Let Y and X be derived schemes. Then the stack of quasi-constant maps $\underline{\mathrm{Map}}^0(Y, X)$ coincides with the union of the connected components of the constant maps. In particular, if X is connected, $\underline{\mathrm{Map}}^0(Y, X)$ is connected.*

PROOF. This is a direct consequence of the full embedding of derived schemes in derived stacks. □

PROPOSITION 2.2.25. *Let X be a variety with an action of an algebraic torus T, and let E be an elliptic curve over k. Assume the T-action on X is such that the T-orbits in X have connected stabilizers. Then the stack of quasi-constant maps $\underline{\mathrm{Map}}^0(E, [X/T])$ coincides with the union of the connected components of the constant maps. In particular, if X is connected, $\underline{\mathrm{Map}}^0(E, [X/T])$ is connected.*

PROOF. We will assume for simplicity that T is rank one. The general argument is a simple extension of the rank one case. This proof requires the codescent property, which will be proved in Section 2.3 as Theorem 2.3.14. It follows from that codescent result that points in $\underline{\mathrm{Map}}^0(E, [X/T])$ correspond to maps whose image is contained in a single T-orbit in X, which under our assumptions is either free or a fixed point.

Let $x \in X$ be a fixed point for the T-action. We need to consider maps from E to $[x/T]$ factoring through $[\operatorname{Spec} k/\mu_n]$ for all positive integers n. These maps classify T-bundles on E admitting a reduction of their structure group to μ_n. But these all lie in the connected component of $\underline{\operatorname{Pic}}(E)$ classifying degree zero bundles.

Now consider a free orbit O. Since T acts freely on O, we have that $[O/T] = \operatorname{Spec} k$. So any map to $[O/T]$ factors necessarily through the point. This concludes the proof. $\square$

An important class of examples satisfying the assumptions of Proposition 2.2.25 is given by toric varieties with the standard torus action.

COROLLARY 2.2.26. *Let X be a smooth toric variety equipped with standard action by the torus T and let E be an elliptic curve. Then the stack of quasi-constant maps $\underline{\operatorname{Map}}^0(E, [X/T])$ coincides with the connected component of the constant maps.*

Just as in the case of the shifted tangent bundle and of the loop space, we can consider a formal completion of the derived stack of quasi-constant maps.

DEFINITION 2.2.27. *The derived stack of* formal maps*, denoted by $\widehat{\underline{\operatorname{Map}}}^0(Y, X)$, is the formal completion of $\underline{\operatorname{Map}}^0(Y, X)$ at the constant maps*

$$X \to \underline{\operatorname{Map}}^0(Y, X)$$

We conclude this subsection by studying the geometricity of the stack of quasi–constant maps.

PROPOSITION 2.2.28. *Let E be an elliptic curve over k. Then, if the target stack $\mathcal{X}$ is a finitely presented variety X or a quotient stack $[X/T]$ with X a variety, the stack of quasi-constant maps $\underline{\operatorname{Map}}^0(E, \mathcal{X})$ is 1-geometric.*

PROOF. Note that X is in particular finitely presented over k, thus this is a direct application of Theorem 5.1.1 in [**32**]. $\square$

2.2.6. Equivariant elliptic Hochschild homology.

In this section we define equivariant elliptic Hochschild homology, which is our main object of study in this article.

DEFINITION 2.2.29. *Let G be a smooth reductive algebraic group and Y be a scheme.*

- *The* derived stack of principal G-bundles *on Y is the mapping stack*

$$\underline{\operatorname{Bun}}_G(Y) := \underline{\operatorname{Map}}(Y, [\operatorname{Spec} k/G])$$

The derived stack of principal G-bundles of degree zero *on Y is*

$$\underline{\operatorname{Bun}}^0_G(Y) := \underline{\operatorname{Map}}^0(Y, [\operatorname{Spec} k/G])$$

- *Let $G = T$ be an algebraic torus of rank n. We set*

$$\underline{\mathrm{Pic}}(Y)_T := \underline{\mathrm{Bun}}_T(Y)$$

$$\underline{\mathrm{Pic}}^0(Y)_T := \underline{\mathrm{Bun}}^0_T(Y)$$

Remark 2.2.30. *When Y is connected, $\underline{\mathrm{Pic}}^0(Y)_T$ is the connected component of the trivial rank n bundle over Y.*

Remark 2.2.31. *When T is of rank 1, we write $\underline{\mathrm{Pic}}(Y)$ in place of $\underline{\mathrm{Pic}}_T(Y)$ and similarly for $\underline{\mathrm{Pic}}^0$.*

We will apply these definitions in the case when Y is an elliptic curve E over a field k of characteristic zero. Let $\check{T}$ be the cocharacter lattice of T. We have decompositions

$$\underline{\mathrm{Pic}}(E)_T \simeq (\mathrm{Pic}(E) \otimes_{\mathbb{Z}} \check{T}) \times [\mathrm{Spec}\, k/T], \quad \underline{\mathrm{Pic}}(E)^0_T \simeq E \otimes_{\mathbb{Z}} \check{T} \times [\mathrm{Spec}\, k/T]$$

where $\mathrm{Pic}(E)$ denotes the Picard scheme of E. In particular, these stacks are underived. In the rank 1 case,

$$\underline{\mathrm{Pic}}(E) \simeq \mathrm{Pic}(E) \times [\mathrm{Spec}\, k/\mathbb{G}_m]$$

$$\underline{\mathrm{Pic}}^0(E) \simeq \mathrm{Pic}^0(E) \times [\mathrm{Spec}\, k/\mathbb{G}_m] \simeq E \times [\mathrm{Spec}\, k/\mathbb{G}_m]$$

Although we are interested in torus actions, a few steps of our argument will depend on considering more general actions where T is the product of a torus T' and of a finite abelian group A isomorphic to a product of groups of roots of unity

$$A \simeq \prod_{i=1,\ldots,r} \mu_{n_i}$$

Note that there is an equivalence

$$\underline{\mathrm{Bun}}_A(E) \simeq \underline{\mathrm{Bun}}^0_A(E) \simeq \prod_{i=1,\ldots,r} \big(E[n_i] \times [\mathrm{Spec}\, k/\mu_{n_i}]\big)$$

where $E[n_i]$ denotes the n_i-torsion points in E. This induces

$$\underline{\mathrm{Bun}}^0_T(E) \simeq \big(E \otimes_{\mathbb{Z}} \check{T}' \times [\mathrm{Spec}\, k/T']\big) \times \Big(\prod_{i=1,\ldots,r} E[n_i] \times [\mathrm{Spec}\, k/\mu_{n_i}]\Big)$$

This stack carries a map towards its coarse moduli space

$$\underline{\mathrm{Bun}}^0_T(E) \to E \otimes_{\mathbb{Z}} \check{T}' \times \Big(\prod_{i=1,\ldots,r} E[n_i]\Big)$$

which we denote E_T. When T is a torus, fixing an isomorphism $T \simeq (\mathbb{G}_m)^n$ identifies this scheme with a product of $rk(T)$ copies of E.

Consider a variety X over k with the action of an abelian group T which decomposes as the product of a torus T' and a finite abelian group A as above. The structure map $X \to \operatorname{Spec} k$ induces

$$p' : \underline{\mathrm{Map}}^0 (E, [X/T]) \to \underline{\mathrm{Map}}^0 (E, [\operatorname{Spec} k/T])$$

We denote p the composition of p' with the projection $\underline{\mathrm{Bun}}^0_T(E) \to E_T$.

Definition 2.2.32. *The T-equivariant* elliptic Hochschild homology *of X is*

$$\mathcal{HH}_E([X/T]) := p_* \mathcal{O}_{\underline{\mathrm{Map}}^0(E,[X/T])} \in \mathrm{Qcoh}(E_T)$$

We denote by $\mathrm{HH}_E([X/T])$ *the global sections of the sheaf* $\mathcal{HH}_E([X/T])$*.*

We refer to $\mathcal{HH}_E([X/T])$ also as the *elliptic Hochschild homology* of $[X/T]$. We remark that $\mathcal{HH}_E([X/T])$ defines a cohomology theory for quotient stacks, at least in the weak sense that a T-equivariant map $Y \to X$ induces algebra maps

$$\mathcal{HH}_E([X/T]) \to \mathcal{HH}_E([Y/T])$$

and

$$\mathrm{HH}_E([X/T]) \to \mathrm{HH}_E([Y/T])$$

Additionally $\mathcal{HH}_E(-)$ satisfies a form of Mayer–Vietoris, Theorem 2.3.14. We remark that Definition 2.2.32 makes sense for general reductive algebraic groups.

2.2.7. Complexified Equivariant Elliptic Cohomology. Here we present a short review of rationalized equivariant elliptic cohomology. This object was axiomatically defined by Ginzburg–Kapranov–Vasserot in [**25**] and constructed by Grojnowski in [**30**]. We review Grojnowski's construction following mostly the more recent exposition found in [**23**] and [**62**]. Other reviews closer in style to the original can be found in [**2**], [**61**] and [**27**]. We remark that Grojnowski's paper only sketches the construction, and that the details were carried out by Roşu in [**60**].

Let X be a finite T-CW-complex, where T is a torus of rank n. We construct complex T-equivariant elliptic cohomology of X as an object in the $\mathbb{Z}_2$-periodic ∞-category of *complex analytic coherent* sheaves over the complex analytic variety $E_T := E \otimes_{\mathbb{Z}} \check{T}$, which is then viewed as an algebraic coherent complex via standard GAGA arguments, yielding

$$\mathcal{E}ll_T(X) \in \mathrm{Perf}(E_T)_{\mathbb{Z}_2}$$

Remark 2.2.33. *As E_T is a smooth Noetherian underived scheme,* $\mathrm{Perf}(E_T)_{\mathbb{Z}_2} \simeq \mathrm{Coh}(E_T)_{\mathbb{Z}_2}$. $\mathrm{Coh}(E_T)$ *is the* ∞-category of coherent sheaves*, i.e. the full subcategory of* $\mathrm{QCoh}(E_T)$ *spanned by bounded complexes having coherent homotopy sheaves.*

First, we set up some notation. Let $C_T^*(X)$ be the T-equivariant singular cochains on X, i.e. the singular cochains of the Borel construction $C^*(X//T)$. The *sum*-$\mathbb{Z}_2$-periodization of the T-equivariant singular cochains, denoted by $C_T^{\oplus,*}(X)$, is defined as

$$C_T^{\oplus,*}(X) := \bigoplus_{i\in\mathbb{Z}} C_T^{*+2i}(X;\mathbb{C})$$

Analogously, we introduce the *product*-$\mathbb{Z}_2$-periodization as

$$C_T^{\prod,*}(X) := \prod_{i\in\mathbb{Z}} C_T^{*+2i}(X;\mathbb{C})$$

Grojnowski's insight is that complexified equivariant elliptic cohomology is locally controlled by the singular equivariant cohomology of loci in X fixed by subgroups of T.

Definition 2.2.34. *Let e be a closed point of E_T. Let $S(e)$ be the set of subtori $T' \subset T$ such that e belongs to $E_{T'} \subset E_T$. Then set*

$$T(e) := \bigcap_{T'\in S(e)} T'$$

Remark 2.2.35. *In Section 2.6 we will need an extension of this notion to points of E_T which are not necessarily closed. Let $x \in E_T$ be any point. The subgroup $T(x)$ of T associated to x is the smallest subgroup of T such that $E_{T(x)}$ contains the closure of x, $\overline{\{x\}}$, i.e.*

$$T(x) = \bigcap_{K\subset T \mid \overline{\{x\}}\subset E_K} K$$

Moreover, we also set

$$T'(x) = T/T(x)$$

Remark 2.2.36. *If x is a non-closed point,* $\mathrm{rk}(T'(x)) \leqslant \mathrm{rk}(T) - 1$. *This will be relevant in our inductive proof of the comparison theorem 2.6.10.*

Remark 2.2.37. *Fixing an isomorphism $T \cong \mathbb{G}_m^n$ induces an isomorphism $E_T \cong E^n$. Under this identification, we can characterize the subgroup $T(e)$ for a closed point e as follows. Let $e = (e_1, \ldots, e_n) \in E^n$ and assume that*

- *$e_{i_1}, \ldots, e_{i_l} \in E$ are torsion with order $|e_{i_j}| = n_j$*
- *For all $k \notin \{i_1, \ldots, i_l\}$, $e_k \in E$ is not torsion*

Then, up to shuffling the factors,

$$T(e) = \Big(\prod_{i=i_1}^{i_l} \mu_{n_i}\Big) \times (\mathbb{G}_m)^{n-l} \subset T$$

We are now ready to construct complexified equivariant elliptic cohomology. First of all, recall that $C_T^{\oplus,*}(X)$ is a module over $C_T^{\oplus,0}(*)$. This is a formal commutative dg-algebra concentrated in degree zero, and in particular we have an equivalence

$$C_T^{\oplus,0}(*) \simeq H_T^{\oplus,0}(*) = \mathbb{C}[u_1, \ldots, u_{\mathrm{rk}(T)}]$$

Equivalently, we can regard the module $C_T^{\oplus,*}(X)$ as an object in $\mathrm{Perf}(E_T)_{\mathbb{Z}_2}$ (as X is a finite T-CW-complex). Let us call $\mathcal{H}_T(X)$ this object. By definition $\mathcal{H}_T(X)$ is a sheaf of algebras over $\operatorname{Spec} C_T^{\oplus,*}(*) \simeq \mathfrak{t}_{\mathbb{C}}$, which is the complexified Lie algebra of the torus T. Denote by $\mathcal{H}_T^{\mathrm{an}}(X)$ its analytification, i.e. the coherent analytic sheaf

$$\mathcal{H}_T^{\mathrm{an}}(X) := \mathcal{H}_T(X) \otimes_{\mathcal{O}_{\mathfrak{t}_{\mathbb{C}}}} \mathcal{O}_{\mathfrak{t}_{\mathbb{C}}}^{\mathrm{an}}$$

There is a quotient map

$$\exp^2 : \mathbb{A}^n_{\mathbb{C}} \to E_T$$

which is an isomorphism if restricted to sufficiently small analytic disks U in E_T. Let us call $\log^2$ its local inverse. Moreover, the group structure on E_T induces translation maps

$$\begin{aligned} \tau_e : & E_T \to E_T \\ & f \mapsto fe \end{aligned}$$

for all closed points e in E_T (we use multiplicative notation for the group operation on E_T). Then, for a closed point $e \in E_T$ and a sufficiently small analytic neighbourhood U_e of e, so that $U_1 \subset \tau_{e^{-1}}(U_e)$, we set

$$\mathcal{E}ll_T^{\mathrm{an}}(X)|_{U_e} := (\tau_e \circ \exp^2)_* \mathcal{H}_T^{\mathrm{an}}(X^{T(e)})|_{\log^2(e^{-1}U_e)}$$

As summarized in [**23**], these open sets cover E_T and transition isomorphisms between $\mathcal{E}ll_T^{\mathrm{an}}(X)|_{U_e}$ and $\mathcal{E}ll_T^{\mathrm{an}}(X)|_{U'_e}$ can be defined for all closed points e and e' in terms of the localization theorem in equivariant cohomology. These isomorphisms satisfy the cocycle identities and thus give rise to a complex-analytic sheaf denoted by $\mathcal{E}ll_T^{\mathrm{an}}(X)$.

We reserve the name $\mathcal{E}ll_T(X)$ for the algebraic sheaf obtained from $\mathcal{E}ll_T^{\mathrm{an}}(X)$ via GAGA.

REMARK 2.2.38. *Grojnowski's original construction involves singular cohomology rather than singular cochains. His construction can be obtained from ours by taking the cohomology sheaves of $\mathcal{E}ll_T(X)$.*

The completions of the periodic version of Grojnowski's sheaf over closed points x of E_T are given by a *product*-$\mathbb{Z}_2$-periodization of T-equivariant singular cohomology:

$$\mathcal{E}ll_T(X)_{\hat{x}} \simeq C_T^{\prod,*}(X^{T(x)}) \simeq C_T^{\oplus,*}(X^{T(x)}) \otimes_{C_T^{\oplus,0}(*)} \mathcal{O}_{E_T,\hat{x}}$$

where $\mathcal{O}_{E_T,\hat{x}}$ is a module over $C_T^{\oplus,0}(*) \simeq \mathcal{O}(\mathfrak{t}_{\mathbb{C}})$ via the completed multiplication map

$$\hat{\mu}_x : E_{T,\hat{1}} \simeq E_{T,\hat{x}}$$

and the identification $E_{T,\hat{1}} \simeq \mathfrak{t}_{\mathbb{C},\hat{0}}$.

2.2.8. Adelic descent. In the last section of this paper we make extensive use of adelic descent theory for n-dimensional schemes. This theory was first introduced by Parshin [**56**] and Beilinson [**11**]. A review of the fundamentals of this theory can be found in [**35**] and [**50**]. Modern developments of the theory include Groechenig's [**28**], which is also the main reference for the short reminder that follows.

Let X be a Noetherian scheme. For two points x and y we say $x \geqslant y$ if $y \in \overline{\{x\}}$. We let $|X|_k$ denote the set of k-*chains* on X, i.e. sequences of $k+1$ ordered points $(x_0 \geqslant \cdots \geqslant x_k)$ in X. If $k = 0$, we equivalently write $|X| = |X|_0$. Finally, given a subset $T \subset |X|_k$, we denote by

$${}_xT := \{\Delta \in |X|_{k-1} | (x \geqslant \Delta) \in T\}$$

This notation allows us to define sheaves of adèles on X for a subset $T \subset |X|_k$. The adèles are the unique family of exact functors

$$\mathsf{A}_X(T, -) : \mathrm{QCoh}(X) \to \mathrm{Mod}_{\mathcal{O}_X}$$

satisfying the following properties:

- $\mathsf{A}_X(T, -)$ commutes with directed colimits;
- if $\mathcal{F}$ is coherent and $k = 0$, $\mathsf{A}_X(T, \mathcal{F}) = \prod_{x \in T} \lim_{r \geqslant 0} \tilde{j}_{rx}\mathcal{F}$;
- if $\mathcal{F}$ is coherent and $k > 0$, $\mathsf{A}_X(T, \mathcal{F}) = \prod_{x \in |X|} \lim_{r \geqslant 0} \mathsf{A}_X({}_xT, \tilde{j}_{rx}\mathcal{F})$.

In the above, $\tilde{j}_{rx}$ denotes the functor $j_{rx\,*}j_{rx}^*$, where

$$j_{rx} : \operatorname{Spec} \mathcal{O}_{X,x}/\mathfrak{m}_x^r \to X$$

is the canonical immersion of an r-thickening of the point x. Here $\mathcal{O}_{X,x}$ is the local ring at x and $\mathfrak{m}_x$ is its maximal ideal.

The global sections $\Gamma(X, \mathsf{A}_X(T, \mathcal{F}))$ are denoted by $\mathbb{A}_X(T, \mathcal{F})$ and are the groups of adèles.

The sets $|X|_k$ can be assembled into a simplicial set: face and degeneracy maps are defined, respectively, by removing or repeating a point in a chain. We denote this simplicial set by $|X|_\bullet$. In particular, the sheaves of adèles assemble into a cosimplicial sheaf of $\mathcal{O}_X$–modules $\mathsf{A}_X(T_\bullet, \mathcal{F})$, for some $T_\bullet \subset |X|_\bullet$. If $T_\bullet = |X|_\bullet$, we denote this cosimplicial sheaf by $\mathsf{A}_X^\bullet(\mathcal{F})$ and its global sections by $\mathbb{A}_X^\bullet(\mathcal{F})$. If $\mathcal{F} = \mathcal{O}_X$, we denote the cosimplicial sheaves and groups of adèles by $\mathsf{A}_X^\bullet$ and $\mathbb{A}_X^\bullet$ respectively.

Similarly, there is a cosimplicial sheaf given by products of "local" adèles

$$[n] \mapsto \prod_{\Delta \in |X|_n} \mathsf{A}_X(\Delta, \mathcal{F})$$

Theorem 2.4.1 in [**35**] tells us that the natural inclusion of the full adèles into the product of local adèles respects the cosimplicial structures.

We conclude this section with two theorems that allow to reconstruct sheaves from their adelic decomposition.

THEOREM 2.2.39 (Theorem 3.1 in [**28**]). *Let X be a Noetherian scheme. Then there is an equivalence of symmetric monoidal ∞-categories*

$$\mathrm{Perf}^{\otimes}(X) \simeq \mathrm{TotPerf}^{\otimes}(\mathbb{A}_X)$$

The following theorem due to Beilinson appears as Theorem 1.16 in [**28**].

THEOREM 2.2.40 (Beilinson [**11**]). *Let $\mathcal{F}$ be a quasi-coherent sheaf on X. The augmentation $\mathcal{F} \to \mathsf{A}^{\bullet}_X(\mathcal{F})$ is a resolution of $\mathcal{F}$ by flasque $\mathcal{O}_X$-modules. In particular, the totalization of the adèles* $\mathrm{Tot}\mathbb{A}^{\bullet}_X(\mathcal{F})$ *computes the cohomology of $\mathcal{F}$.*

The objects we will consider in Section 2.6 belong rather to the $\mathbb{Z}_2$-*periodic* categories of perfect complexes. The arguments made by Groechenig in [**28**] also hold in this context, leading to completely parallel statements involving the $\mathbb{Z}_2$-periodic categories.

REMARK 2.2.41. *Let us remark that in Section 2.6 we will use a variant of Beilinson's theorem for perfect complexes, i.e. that the adelic descent data of a perfect complex, computed as in Beilinson's definition where we interpret the operations in the derived sense, recovers the original perfect complex after totalization. This follows from Theorem 3.1 in* [**28**].

2.3. Codescent for quasi-constant maps

In this section we prove that the stack of quasi-constant maps is Zariski local on the target. This behaviour is in sharp contrast with the full mapping stack, where locality on the target is essentially never satisfied. We will give a proof of this statement in the case when the source is an elliptic curve, which is the case that is most relevant for our applications, but we will also comment on extensions of our results to more general settings (see Section 2.3.1.1). One of the ingredients in our argument is a simple criterion that allows us to detect when an open immersions of geometric stacks is an equivalence, Proposition 2.3.3 below.

We start by proving a few simple general properties of the stack of quasi-constant maps.

Let S be a derived stack. Consider the functor

$$\underline{\mathrm{Map}}^0(S, -) : \mathrm{dSt}_k \to \mathrm{dSt}_k$$

If $f : X \to Y$ is a map in dSt_k, we denote by r_f the induced map

$$r_f : \underline{\mathrm{Map}}^0(S, X) \to \underline{\mathrm{Map}}^0(S, Y)$$

LEMMA 2.3.1. *The functor*

$$\underline{\mathrm{Map}}^0(S,-):\mathrm{dSt}_k\to\mathrm{dSt}_k$$

preserves limits. In particular, if $f:X\to Y$ is a map in dSt_k, there is an equivalence of simplicial objects in dSt_k

$$\underline{\mathrm{Map}}^0\left(S,\check{\mathrm{C}}(f)\right)\simeq\check{\mathrm{C}}(r_f)$$

PROOF. The functor from pointed stacks to stacks sending a pair (F,x) to the connected component of x preserves limits. Thus the statement follows from the analogous statement for $\underline{\mathrm{Map}}(S,-)$, which is obvious. The second part of the claim is a formal consequence of the first one. $\square$

LEMMA 2.3.2. *The functor*

$$\underline{\mathrm{Map}}^0(S,-):\mathrm{dSt}_k\to\mathrm{dSt}_k$$

preserves both open and closed immersions of derived stacks. That is, if $i:Y\to X$ is an open (resp. closed) immersion of derived stacks, then

$$r_i:\underline{\mathrm{Map}}^0(S,Y)\to\underline{\mathrm{Map}}^0(S,X)$$

is an open (resp. closed) immersion of derived stacks.

PROOF. We prove the statement for open immersions, as the proof for closed immersions is the same. We need to show that for every affine scheme $\mathrm{Spec}\,A$ and for every map

$$\mathrm{Spec}\,A\to\underline{\mathrm{Map}}^0(S,X)$$

the pullback map $\underline{\mathrm{Map}}^0(S,Y)\times_{\underline{\mathrm{Map}}^0(S,X)}\mathrm{Spec}\,A\to\mathrm{Spec}\,A$ is an open immersion of derived schemes.

Set $Z:=\underline{\mathrm{Map}}^0(S,Y)\times_{\underline{\mathrm{Map}}^0(S,X)}\mathrm{Spec}\,A$. Then by the universal property of the mapping stack, we obtain a pullback diagram

$$\begin{tikzcd} Z\times S \arrow[r] \arrow[d] & Y \arrow[d] \\ \mathrm{Spec}\,A\times S \arrow[r] & X \end{tikzcd}$$

Since the map $Y\to X$ is an open immersion, the map $Z\times S\to\mathrm{Spec}\,A\times S$ must be an open immersion. This map is the identity on S, so we conclude that $Z\to\mathrm{Spec}\,A$ is an open immersion; in particular Z is necessarily a derived scheme. $\square$

PROPOSITION 2.3.3 (Point-wise criterion). *Let X be an n-geometric derived stack and let*

$$\phi:\coprod_{\alpha\in I}U_\alpha\to X$$

be a coproduct of open immersions of derived stacks. Assume that for every field extension K of k and any map $f : \operatorname{Spec} K \to X$ there exists a lift

$$\begin{tikzcd} & \coprod_{\alpha\in I} U_\alpha \arrow[d, "\phi"] \\ \operatorname{Spec} K \arrow[r, "f"] \arrow[ur, dashed] & X \end{tikzcd}$$

Then the map ϕ is an effective epimorphism.

Proof. Since X is n-geometric, the map f factors through one of the affine schemes

$$\operatorname{Spec} A \to X$$

which compose the chosen atlas of X (up to trading K for a field extension). Open immersions of derived stacks are stable under base change, and thus the base change of ϕ along $\operatorname{Spec} A \to X$

$$\Big(\coprod_{\alpha\in I} U_\alpha\Big) \times_X \operatorname{Spec} A \simeq \coprod_{\alpha\in I}\Big(U_\alpha \times_X \operatorname{Spec} A\Big) \to \operatorname{Spec} A$$

is also a coproduct of open immersions. The equivalence above is a consequence of the universality of colimits in ∞-topoi. Each summand $V_\alpha := U_\alpha \times_X \operatorname{Spec} A$ is an open substack of $\operatorname{Spec} A$, and is therefore a derived scheme. Thus we can reduce to proving the claim when $X = \operatorname{Spec} A$ is an affine derived scheme and

$$\coprod_{\alpha\in I} V_\alpha \to X = \operatorname{Spec} A$$

is a disjoint union of open subschemes. Up to refining the cover $\{V_\alpha\}_{\alpha\in I}$, by taking affine open covers of each scheme V_α, we can also assume that the V_α-s are affine. Set $V_\alpha = \operatorname{Spec} A_\alpha$.

The existence of lifts in the affine situation is equivalent to the statement that the collection of maps of simplicial k-algebras

$$\{A \to A_\alpha\}_{\alpha\in I}$$

is a Zariski cover of the simplicial k-algebra A, i.e.

- all the k-algebras A_α are localizations of the algebra A at some elements $a_\alpha \in \pi_0 A$;
- the collection of the elements $\{a_\alpha\}_{\alpha\in I}$ generates $\pi_0 A$.

But this implies that the map $\coprod_{\alpha\in I} \operatorname{Spec} A_\alpha \to \operatorname{Spec} A$ is an effective epimorphism. □

2.3.1. Quasi-constant maps to varieties. Let X be a variety and E an elliptic curve over k. In this section we prove the following statement:

PROPOSITION 2.3.4. *The morphism*

$$\underline{\mathrm{Map}}^0\left(\mathrm{Aff}(E), X\right) \to \underline{\mathrm{Map}}^0\left(E, X\right)$$

induced by the unit $E \to \mathrm{Aff}(E)$ of the affinization is an equivalence of derived stacks.

The codescent property for $\underline{\mathrm{Map}}^0\left(E, X\right)$ will follow immediately from Proposition 2.3.4.

LEMMA 2.3.5. *Let $S \subset E$ be an affine open subset. Let U be an affine variety, and fix a locally closed embedding $U \subset \mathbb{P}^n$. Denote by $\overline{U}$ the closure of U. Then there are natural monomorphisms of stacks*

$$\underline{\mathrm{Map}}\left(S, U\right) \xrightarrow{\alpha} \underline{\mathrm{Map}}\left(E, \overline{U}\right) \xrightarrow{\beta} \underline{\mathrm{Map}}\left(E, \mathbb{P}^n\right)$$

that is, for every affine scheme Y the induced maps of sets

$$\pi_0\mathrm{Map}(S \times Y, U) \overset{\pi_0\alpha}{\to} \pi_0\mathrm{Map}(E \times Y, \overline{U}) \overset{\pi_0\beta}{\to} \pi_0\mathrm{Map}(E \times Y, \mathbb{P}^n)$$

are injective. Restricting to quasi-constant maps yields monomorphisms

$$\underline{\mathrm{Map}}^0\left(S, U\right) \xrightarrow{\alpha} \underline{\mathrm{Map}}^0\left(E, \overline{U}\right) \xrightarrow{\beta} \underline{\mathrm{Map}}^0\left(E, \mathbb{P}^n\right)$$

REMARK 2.3.6. *The notion of monomorphism we refer to in Lemma 2.3.5 is the notion of monomorphism in an ∞-category appearing in* [**46**, p. 575].

PROOF OF LEMMA 2.3.5. The inclusion $\overline{U} \subset \mathbb{P}^n$ determines a map $\underline{\mathrm{Map}}\left(E, \overline{U}\right) \xrightarrow{\beta} \underline{\mathrm{Map}}\left(E, \mathbb{P}^n\right)$ which has the desired properties. Let us define the map α. Let T be a proper and separated derived scheme. As E is a discrete one-dimensional scheme, the valuative criterion of properness[2] implies that there is an equivalence $\mathrm{Map}(E, T) \overset{\simeq}{\to} \mathrm{Map}(S, T)$. Defining α requires defining maps

$$\alpha_Y : \mathrm{Map}(S \times Y, U) \to \mathrm{Map}(E \times Y, \overline{U})$$

for every $Y \in \mathrm{dAff}$, that are natural in Y. We define α_Y as the composition

$$\mathrm{Map}(S\times Y, U) \overset{(a)}{\to} \mathrm{Map}_{\mathrm{dSt}/Y}(S\times Y, U\times Y) \overset{(b)}{\to} \mathrm{Map}_{\mathrm{dSt}/Y}(E\times Y, \overline{U}\times Y) \overset{(c)}{\to} \mathrm{Map}(E\times Y, \overline{U})$$

where

- $\mathrm{Map}_{\mathrm{dSt}/Y}(-,-)$ denotes the mapping space in the over-category dSt/Y
- on connected components, the map (a) is the assignment

$$(S \times Y \overset{f}{\to} U) \mapsto (S \times Y \overset{f\times \mathrm{pr}_Y}{\to} U \times Y)$$

[2]For a reference on the valuative criterion of properness in the derived setting, see for instance `https://www.preschema.com/lecture-notes/kdescent/lect6.pdf`.

- the map (b) is the morphism on mapping spaces given by the valuative criterion of properness, relative to the base scheme Y, as $E \times Y \to Y$ is a discrete curve over Y, and $\overline{U} \times Y \to Y$ is proper
- the map (c) is induced by the projection $\overline{U} \times Y \to \overline{U}$

The fact that α is natural in Y is clear. Also, it is easy to see that α induces sectionwise injections on connected components. As for the last statement, it follows from the fact that α and β preserve constant maps. Indeed, as the image of a connected stack under any map is connected, the maps α and β restrict to maps between the connected components of the constant maps, and by Proposition 2.2.24 the stack of quasi-constant maps with target a variety is the connected component of the constants. $\square$

Remark 2.3.7. *Let us make some more comments on the map α defined in Lemma 2.3.5. The map α can be factored as*

$$\begin{tikzcd} \underline{\mathrm{Map}}\left(S, \overline{U}\right) \arrow[dr, "\simeq"] & \\ \underline{\mathrm{Map}}\left(S, U\right) \arrow[u] \arrow[r, "\alpha"] & \underline{\mathrm{Map}}\left(E, \overline{U}\right) \end{tikzcd} \qquad \begin{tikzcd} \underline{\mathrm{Map}}^0\left(S, \overline{U}\right) \arrow[dr, "\simeq"] & \\ \underline{\mathrm{Map}}^0\left(S, U\right) \arrow[u] \arrow[r, "\alpha"] & \underline{\mathrm{Map}}^0\left(E, \overline{U}\right) \end{tikzcd}$$

Let us focus on then diagram on the left, as the one on the right is just obtained by restricting to quasi-constant maps. The vertical arrow is induced by the inclusion $U \to \overline{U}$, and thus is an open embedding as explained in Lemma 2.3.2. The diagonal arrow is an equivalence. It is the inverse of the natural map $\underline{\mathrm{Map}}\left(E, \overline{U}\right) \to \underline{\mathrm{Map}}\left(S, \overline{U}\right)$ given by restriction to S. As explained in the proof of Lemma 2.3.5, the fact that this map is an equivalence follows from the valuative criterion for properness.

Lemma 2.3.8. *Let $\mathrm{Aff}(E)$ be the affinization of E. Then the map*

$$\underline{\mathrm{Map}}^0\left(\mathrm{Aff}(E), X\right) \to \underline{\mathrm{Map}}^0\left(E, X\right)$$

is an open embedding.

Proof. Recall from Section 3.2, that there is an equivalence

$$\mathrm{Aff}(E) \simeq \mathrm{Aff}(S^1)$$

This implies that $\underline{\mathrm{Map}}\left(\mathrm{Aff}(E), X\right)$ is equivalent to the derived loop space $\mathcal{L}X$; in particular, the stack $\underline{\mathrm{Map}}\left(\mathrm{Aff}(E), X\right)$ is connected, and thus there is an identification

$$\underline{\mathrm{Map}}\left(\mathrm{Aff}(E), X\right) = \underline{\mathrm{Map}}^0\left(\mathrm{Aff}(E), X\right)$$

Now this also implies that $\underline{\mathrm{Map}}^0\left(\mathrm{Aff}(E), X\right)$ satisfies Zariski codescent on X. Indeed, this is easily proved for the loop space $\mathcal{L}X$; a reference is, for instance, Lemma 4.2 of [**8**].

Consider an affine open cover $\{U_i\}_{i \in I}$ of X. By Lemma 2.3.2, for every $i \in I$ the vertical arrows in the commutative diagram below are open inclusions

$$\begin{tikzcd} \underline{\mathrm{Map}}^0(\mathrm{Aff}(E), X) \arrow[r] & \underline{\mathrm{Map}}^0(E, X) \\ \underline{\mathrm{Map}}^0(\mathrm{Aff}(E), U_i) \arrow[u] \arrow[r, "\simeq"] \arrow[ur, "\subset"] & \underline{\mathrm{Map}}^0(E, U_i) \arrow[u] \end{tikzcd}$$

Further, the bottom arrow is an equivalence, by the universal property of the affinization. It follows that we have an open embedding $\underline{\mathrm{Map}}^0(\mathrm{Aff}(E), U_i) \to \underline{\mathrm{Map}}^0(E, X)$. Thus the map from the realization of the Čech nerve of the open substacks $\underline{\mathrm{Map}}^0(\mathrm{Aff}(E), U_i)$

$$\underline{\mathrm{Map}}^0(\mathrm{Aff}(E), X) \simeq |\underline{\mathrm{Map}}^0(\mathrm{Aff}(E), U_i)| \longrightarrow \underline{\mathrm{Map}}^0(E, X)$$

is also an open embedding. $\square$

Next we show Proposition 2.3.4 in the case when X is the projective space, as a stepping stone to the proof in the general case.

Lemma 2.3.9. *The morphism*

$$\underline{\mathrm{Map}}^0(\mathrm{Aff}(E), \mathbb{P}^n) \to \underline{\mathrm{Map}}^0(E, \mathbb{P}^n)$$

induced by the unit map $E \to \mathrm{Aff}(E)$, is an equivalence of derived stacks.

Proof. Let $\phi_{\mathcal{O}(1)} : \mathbb{P}^n \to [\mathrm{Spec}\, k/\mathbb{G}_m]$ be the classifying map of the bundle $\mathcal{O}(1)$ over $\mathbb{P}^n$. By evaluating the composition

$$\underline{\mathrm{Map}}^0(E, \mathbb{P}^n) \times \underline{\mathrm{Map}}(\mathbb{P}^n, [\mathrm{Spec}\, k/\mathbb{G}_m]) \longrightarrow \underline{\mathrm{Map}}(E, [\mathrm{Spec}\, k/\mathbb{G}_m]) = \underline{\mathrm{Pic}}(E)$$

at the point $\phi_{\mathcal{O}(1)}$ in the second factor, we obtain a map of stacks

$$\underline{\mathrm{Map}}^0(E, \mathbb{P}^n) \to \underline{\mathrm{Pic}}(E)$$

As connected stacks map to connected stacks, this map must factor through the inclusion $\underline{\mathrm{Pic}}^0(E) \subset \underline{\mathrm{Pic}}(E)$. This implies that for every map $f : E_K \to \mathbb{P}^n$ parametrized by a point of $\underline{\mathrm{Map}}^0(E, \mathbb{P}^n)$, the bundle $f^*\mathcal{O}(1)$ has degree 0. Further $f^*\mathcal{O}(1)$ must have non-trivial global sections (as it is the pull-back of a very ample bundle). These two properties imply that $f^*\mathcal{O}(1) \simeq \mathcal{O}_E$.

By Lemma 2.3.8, the statement we need to prove can be checked via Proposition 2.3.3. That is, we need to show that every map $f : E_K \to \mathbb{P}^n$ factors through some affine open subset $U \subset \mathbb{P}^n$. To show this, it is enough to check the set-theoretic condition that f induces a constant map between the set of geometric points of E and the set of geometric points of $\mathbb{P}^n$. Then it will be enough to choose as U an affine open neighbourhood of $f(p)$, where $p \in E$ is any geometric point. The fact that f is constant on geometric points follows immediately from the fact that $f^*\mathcal{O}(1) \simeq \mathcal{O}_E$. Indeed, the sections of $\mathcal{O}_E$ are constants, and therefore do not distinguish points. $\square$

Proof of Proposition 2.3.4. As in the case of $\mathbb{P}^n$, we start by observing that by Lemma 2.3.8 the natural map

$$\underline{\mathrm{Map}}^0(\mathrm{Aff}(E), X) \to \underline{\mathrm{Map}}^0(E, X)$$

is an open embedding. Thus to prove that it is an equivalence we can use the point-wise criterion. Consider a map $f : E_K \to X$ corresponding to a closed point $\mathrm{Spec}\, K \to \underline{\mathrm{Map}}^0(E, X)$. We need to show that f factors as

$$\begin{tikzcd} \mathrm{Aff}(E)_K \arrow[dr, "g"] & \\ E_K \arrow[u] \arrow[r, "f"] & X \end{tikzcd}$$

for some $g : \mathrm{Aff}(E)_K \to X$. Let $U \subset X$ be an affine open subset such that $S = E_K \times_U X$ is non-empty. Note $S \subset E_K$ is an affine open subset. Now fix a locally closed embedding $U \subset \mathbb{P}^n$, and let $\overline{U}$ be the closure of U. As $\mathbb{P}^n$ is proper there is a unique map $h : E_K \to \mathbb{P}^n$ which makes the following diagram commute

$$\begin{tikzcd} U \arrow[r] & \mathbb{P}^n \\ S \arrow[u, "f|_S"] \arrow[r] & E_K \arrow[u, "h"'] \end{tikzcd} \tag{6}$$

In fact more is true, namely S is the truncation of the fiber product: $S \cong t_0(U \times_{\mathbb{P}^n} E_K)$. By Lemma 2.3.5, the map h is parametrized by a closed point in $\underline{\mathrm{Map}}^0(E, \mathbb{P}^n)$. Thus, by Lemma 2.3.9, h factors through the affinization of E_K. We can complete diagram (6) to the following commutative diagram

$$\begin{tikzcd} \overline{U} \arrow[dr] & & \\ U \arrow[u] \arrow[r] & \mathbb{P}^n & \\ S \arrow[u, "f|_S"] \arrow[r] & E_K \arrow[u, "h"] \arrow[r] & \mathrm{Aff}(E)_K \arrow[ul] \arrow[uull, bend right, "h'"'] \end{tikzcd}$$

The commutativity of the external triangle with edge h' follows from the fact that the image of h must be contained in the closure of the image of f. In order to conclude we need to show that h' factors through U

$$\begin{tikzcd} \overline{U} & \\ U \arrow[u] & \mathrm{Aff}(E)_K \arrow[l, "g"'] \arrow[ul, "h'"'] \end{tikzcd}$$

This is easy to check. Indeed as U and $\overline{U}$ are schemes, we can replace $\mathrm{Aff}(E)_K$ with S^1_K and think in terms of loop spaces. It is a consequence of Zariski codescent for the loop space that if $\gamma : S^1_K \to \overline{U}$ is a loop, and $U \subset \overline{U}$ is an affine open subset, then the following two facts are equivalent

(1) The fiber product $S^1_K \times_{\overline{U}} U$ is non-empty
(2) The loop γ factors through U

By construction, the first condition is satisfied in our case. Thus h' factors through U, and this implies that h also factors through U. As a consequence f factors also through U, and this concludes the proof. □

Proposition 2.3.4 has the following useful consequence.

Corollary 2.3.10. *Let X be a variety, and let $|X|_{\mathrm{Zar}}$ denote its small Zariski site.*

(1) The assignment mapping an open subset $U \in |X|_{\mathrm{Zar}}$ to the stack of quasi-constant maps $\underline{\mathrm{Map}}^0(E, U) \in \mathrm{dSt}_k$ defines a cosheaf on $|X|_{\mathrm{Zar}}$

$$\underline{\mathrm{Map}}^0(E, -) : |X|_{\mathrm{Zar}} \to \mathrm{dSt}_k$$

(2) The natural map $\mathcal{L}U \to \underline{\mathrm{Map}}^0(E, U)$ defines an equivalence of dSt_k-valued cosheaves on $|X|_{\mathrm{Zar}}$

$$\mathcal{L}(-) \simeq \underline{\mathrm{Map}}^0(E, -) : |X|_{\mathrm{Zar}} \to \mathrm{dSt}_k$$

Proof. This an immediate consequence of Proposition 2.3.4. Indeed by Proposition 2.3.4 for every $U \in |X|_{\mathrm{Zar}}$ there is an equivalence

$$\mathcal{L}(U) \simeq \underline{\mathrm{Map}}^0(E, U)$$

which is natural for maps in $|X|_{\mathrm{Zar}}$. This implies that $\underline{\mathrm{Map}}^0(E, -)$ is equivalent to $\mathcal{L}(-)$ as dSt_k-valued precosheaves on $|X|_{\mathrm{Zar}}$. As $\mathcal{L}(-)$ is a cosheaf, it follows that $\underline{\mathrm{Map}}^0(E, -)$ is also a cosheaf. □

2.3.1.1. *Some generalizations of Proposition 2.3.4.* We formulated Proposition 2.3.4 for an elliptic curve E, as this is the case we will be interested in the remainder of the paper. However the statement holds more generally for any smooth and proper curve C over k, namely the unit map $C \to \mathrm{Aff}(C)$ induces an equivalence

$$\underline{\mathrm{Map}}^0\,(\mathrm{Aff}(C), X) \to \underline{\mathrm{Map}}^0\,(C, X)$$

The proof we have given for the case $C = E$ extends without variations to this more general setting.

Corollary 2.3.10 also generalizes to the case of a general smooth and proper curve C. In characteristic 0, there is an equivalence between $\mathrm{Aff}(C)$ and the affinization of the Betti stack of a wedge of g circles S_g, where g is the genus of C

$$\mathrm{Aff}(C) \simeq \mathrm{Aff}(S_g) \simeq k[\varepsilon_1, \dots, \varepsilon_g]$$

where

- $k[\varepsilon_1, \dots, \varepsilon_g]$ denotes the square-zero extension of k by g generators with $\deg(\varepsilon_i) = 1$
- the last equivalence follows from the fact that, in characteristic zero, the cdga-s
$$\mathrm{Hom}(\mathcal{O}_C, \mathcal{O}_C) \quad \text{and} \quad C^*_{sing}(S_g)$$
are both formal and therefore quasi-equivalent to their cohomology.

This implies in particular that there are equivalences

$$\underline{\mathrm{Map}}^0\,(C, X) \simeq \underline{\mathrm{Map}}^0\,(S_g, X) \simeq \mathcal{L}X \times_X \dots \times_X \mathcal{L}X$$

where the last one follows from the presentation of S_g as an iterated push-out

$$S_g \simeq S^1 \coprod_{pt} S^1 \coprod_{pt} \cdots \coprod_{pt} S^1$$

A simple observation which we have used repeatedly is that $\mathcal{L}X$ is local with respect to the Zariski topology on X. The argument in Lemma 4.2 of [**8**] immediately extends to show that $\underline{\mathrm{Map}}^0\,(S_g, X)$ also defines a cosheaf on $|X|_{\mathrm{Zar}}$. It follows that the conclusions of Corollary 2.3.10 apply to the case of a general smooth and proper curve, and in particular $\underline{\mathrm{Map}}^0\,(C, X)$ defines a cosheaf on $|X|_{\mathrm{Zar}}$.

It is natural to ask whether Proposition 2.3.4 and Corollary 2.3.10 are in fact general features of quasi-constant maps, beyond the curve case. As this will not play any role in the sequel, we leave these as open questions without attempting to answer them in this article.

QUESTION 2.3.11. *Let X be a scheme.*

(1) Under what assumptions on T does $\underline{\mathrm{Map}}^0\,(T, X)$ define a cosheaf of stacks on $|X|_{Zar}$?

(2) Under what assumptions on T is there an equivalence of stacks

$$\underline{\mathrm{Map}}^0\,(\mathrm{Aff}(T), X) \simeq \underline{\mathrm{Map}}^0\,(T, X)\ ?$$

REMARK 2.3.12. *It should be possible to extend Proposition 2.3.4 to the setting where the target X is an algebraic space satisfying suitable properties. The key observation should be that, although in general loops are not local for the étale topology, they are when the target is a scheme or an algebraic space. As this extra generality is not essential for our intended applications, we will not pursue this any further.*

2.3.2. Quasi-constant maps to global quotient stacks. Let X be a variety over k, and assume that X carries an action of an algebraic torus T.

DEFINITION 2.3.13. *The small T-*equivariant Zariski site *of X, which we denote by $|X|^T_{\mathrm{Zar}}$, is the site having*

(1) as objects, schemes U equipped with an action of T, and a T-equivariant open immersion $U \to X$;
(2) as morphisms, T-equivariant open immersions $V \to U$ over X;
(3) as covering families, jointly surjective families of T-equivariant immersions.

The main result of this Section is Theorem 2.3.14 below, which generalizes the first part of Corollary 2.3.10 to the setting of stacks that are global quotients of varieties by a T-action. We remark that the second part of Corollary 2.3.10 fails in the presence of a T-action, and this is a key feature differentiating elliptic Hochschild homology from ordinary Hochschild homology. Another important difference with Corollary 2.3.10 is that in the statement of Theorem 2.3.14 we require X to be normal. The reason is that the argument we will give relies on Sumihiro's Theorem [**71**], which does not hold in general without the normality assumption.

THEOREM 2.3.14. *Assume that X is normal. Then the assignment mapping $U \in |X|^T_{\mathrm{Zar}}$ to the stack of quasi-constant maps*

$$\underline{\mathrm{Map}}^0\,(E, [U/T]) \in \mathrm{dSt}_k$$

defines a cosheaf on $|X|^T_{\mathrm{Zar}}$,

$$\underline{\mathrm{Map}}^0\,(E, [-/T]) : |X|^T_{\mathrm{Zar}} \to \mathrm{dSt}_k$$

If X satisfies the conclusion of Theorem 2.3.14, we say that $\underline{\mathrm{Map}}^0\,(E, [X/T])$ satisfies T-equivariant Zariski codescent on $[X/T]$. In its main lines the proof of Theorem 2.3.14 closely parallels the argument given in the Section 2.3.1 in non-equivariant case. However there are a few minor subtleties that arise when taking into account the T-action.

REMARK 2.3.15. *Let us stress that Theorem 2.3.14 fails if we consider finer topologies such as the étale or smooth topology on* $[X/T]$. *Consider for instance the case when* $T = \mathbb{G}_m$ *and* $X = \operatorname{Spec} k$. *Then* $\operatorname{Spec} k \to [\operatorname{Spec} k/\mathbb{G}_m]$ *is a smooth cover, but*

$$\underline{\mathrm{Map}}^0(E, \operatorname{Spec} k) \simeq \operatorname{Spec} k \to \underline{\mathrm{Map}}^0(E, [\operatorname{Spec} k/\mathbb{G}_m]) \simeq \underline{\mathrm{Pic}}^0(E)$$

clearly is not.

We start by proving Theorem 2.3.14 in two important special cases, namely when X is isomorphic to an affine space or to a projective space. Up to fixing an isomorphism

$$T \cong (\mathbb{G}_m)^n$$

and applying a change of basis to $\mathbb{A}^N$, we can diagonalize the action of T, which can then be written in standard form as follows. For every $\lambda = (\lambda_1, \dots, \lambda_n) \in (\mathbb{G}_m)^n \cong T$, and $(z_1 \dots z_N) \in \mathbb{A}^N$

$$(\lambda_1, \dots, \lambda_n) \cdot (z_1 \dots z_N) = (\prod_{i=1}^{n} \lambda_i^{w_i^1} z_1, \dots, \prod_{i=1}^{n} \lambda_i^{w_i^N} z_N)$$

for an appropriate collection of integers $\{w_i^j\}$, called *weights*.

LEMMA 2.3.16. *The stack of quasi-constant maps* $\underline{\mathrm{Map}}^0\left(E, [\mathbb{A}^N/T]\right)$ *satisfies* T-*equivariant Zariski codescent on* $[\mathbb{A}^N/T]$.

PROOF. We fix an isomorphism $T \cong (\mathbb{G}_m)^n$ and a diagonalization of the T-action. Maps to $[\mathbb{A}^N/(\mathbb{G}_m)^n]$ classify the datum of

(1) n line bundles $\mathcal{L}_i$, $i \in \{1, \dots, n\}$
(2) and N sections $s_j \in H^0(\otimes_{i=1}^{i=n} \mathcal{L}_i^{w_i^j})$, $j \in \{1, \dots, N\}$

There is a natural map

$$\underline{\mathrm{Map}}^0\left(E, [\mathbb{A}^N/T]\right) \to \underline{\mathrm{Map}}^0(E, [\operatorname{Spec} k/T]) \simeq \underline{\mathrm{Pic}}^0(E)^N$$

which forgets the information on sections.

By Proposition 2.3.3, it is sufficient to show that given a T-equivariant open cover $\{U_i\}_{i \in I}$ of $\mathbb{A}^N$ and a map

$$f : E \times_{\operatorname{Spec} k} \operatorname{Spec} K = E_K \to [\mathbb{A}^N/T]$$

corresponding to a closed point of $\underline{\mathrm{Map}}^0\left(E, [\mathbb{A}^N/T]\right)$, there is an open subset U_i such that f factors as

$$\begin{tikzcd} & {[U_i/T]} \arrow[d] \\ E_K \arrow[ur, dashed] \arrow[r, "f"] & {[\mathbb{A}^N/T]} \end{tikzcd}$$

We will prove this by showing the stronger claim that f has to factor through the image of a T-orbit in $[\mathbb{A}^N/T]$. As we discussed, giving the map f is the same as giving n line bundles $\mathcal{L}_i$ on E_K and N sections s_j of appropriate tensor powers of the $\mathcal{L}_i$-s. As f is parametrized by a point in $\underline{\mathrm{Map}}^0\left(E, [\mathbb{A}^N/T]\right)$ all the $\mathcal{L}_i$-s have degree zero. Thus the sections s_j are all constant. This immediately implies that the image of f is a single orbit of the T-action. $\square$

Next, let us consider a T-action on the projective space $\mathbb{P}^N$. Up to fixing an isomorphism $T \cong (\mathbb{G}_m)^n$ and applying an automorphism of $\mathbb{P}^N$, we can put the T-action in the following standard form. For every $\lambda = (\lambda_1, \ldots, \lambda_n) \in (\mathbb{G}_m)^n \cong T$, and $[z_0, \ldots, z_N] \in \mathbb{P}^N$

$$\lambda \cdot [z_0, z_1 \ldots z_N] = [z_0, \prod_{i=1}^n \lambda_i^{w_i^1} z_1, \ldots, \prod_{i=1}^n \lambda_i^{w_i^N} z_N] \in \mathbb{P}^N \tag{7}$$

for an appropriate collection of integers $\{w_i^j\}$. In particular, we can assume that the standard toric affine open cover of $\mathbb{P}^N$ is T-equivariant.

LEMMA 2.3.17. *The stack of quasi-constant maps $\underline{\mathrm{Map}}^0\left(E, [\mathbb{P}^N/T]\right)$ satisfies T-equivariant Zariski codescent on $[\mathbb{P}^N/T]$.*

PROOF. The proof strategy is the same as for Lemma 2.3.16. Namely, consider a map

$$f : E_K \to [\mathbb{P}^N/T]$$

classified by a point of $\underline{\mathrm{Map}}^0\left(E, [\mathbb{P}^N/T]\right)$. We need to show that f factors through the image of a T-orbit in $[\mathbb{P}^N/T]$. Now we make the following observations:

(1) all line bundles on $\mathbb{P}^N$ admit a T-equivariant structure;
(2) there exists a $r > 0$ such that there exists a non-vanishing T-equivariant section σ of $\mathcal{O}_{\mathbb{P}^N}(r)$;
(3) we can further assume that, in point (2), $r = 1$ on condition of replacing, if needed, $\mathbb{P}^N$ with a larger projective space $\mathbb{P}^M$ equipped with a T-action and such that there is a T-equivariant Veronese embedding

$$\mathbb{P}^N \to \mathbb{P}^M$$

Although these are all standard facts, let us sketch a proof. We start with (1). We identify $\mathbb{P}^N$ with the projectivization $\mathbb{P}(V)$ of the vector space V which is dual to $H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1))$. Note that we can lift the T-action on $\mathbb{P}^N$ to a linear action on V. This turns V into a T-representation. Taking the dual representation gives a T-action on $H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1))$. This induces a T-equivariant structure on the line bundle $\mathcal{O}_{\mathbb{P}^N}(1)$. Taking tensor powers and duals generates T-equivariant structures on all the bundles $\mathcal{O}_{\mathbb{P}^N}(m)$, for all integers m.

Let us consider (2) next. The existence of such a section σ for some r is equivalent to fact that the GIT quotient $\mathbb{P}^N//T$ is non-empty, i.e. it is equivalent to the existence of a semistable point of $\mathbb{P}^N$ with respect to the T-action. A semistable point in $\mathbb{P}^N$ with respect to a linear action is by definition a semistable point of the lift of the action to the vector space $V \cong k^{N+1}$ such that $\mathbb{P}^N \cong \mathbb{P}(V)$, i.e. a point in V such that the closure of its orbit under the T-action does not contain 0. The existence of a semistable point is clear after we write the action in standard form (see equation (7) above). Indeed, the point $(1, 0, 0, \ldots, 0)$ in k^{N+1} is fixed by the lift of the action, and hence it is in particular semistable, as the closure of its orbit does not contain the point 0. This implies that the point $[1 : 0 : 0 : \cdots : 0]$ in $\mathbb{P}^N$ is also semistable, hence the GIT quotient $\mathbb{P}^N//T$ is nonempty.

As for point (3), i.e. the reduction to the case $r = 1$, it is sufficient to linearize the action of T on $\mathbb{P}^N$ by choosing an equivariant structure on $\mathcal{O}_{\mathbb{P}^N}(r)$ which makes σ into an equivariant section. As familiar from GIT, the choice of linearization yields a T-equivariant embedding

$$\mathbb{P}^N \to \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^N}(r))^\vee) \cong \mathbb{P}^M$$

In particular, we obtain an isomorphism of T-modules

$$H^0(\mathbb{P}^M, \mathcal{O}_{\mathbb{P}^M}(1)) \cong H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(r))$$

which restricts to the spaces of equivariant sections

$$H^0(\mathbb{P}^M, \mathcal{O}_{\mathbb{P}^M}(1))^T \cong H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(r))^T$$

hence a bijective correspondence between the T-equivariant sections of the two bundles. Note that f factors through the image of a T-orbit if and only the composite map

$$E_K \to [\mathbb{P}^N/T] \to [\mathbb{P}^M/T]$$

factors through the image of a T-orbit. Thus, from the perspective of the argument we are carrying out, we can harmlessly replace $\mathbb{P}^N$ with $\mathbb{P}^M$. We do this implicitly in the sequel, and in particular assume that $r = 1$ and σ is linear, but we will not rename either $\mathbb{P}^N$ or σ.

Consider the map ϕ classifying the line bundle $\mathcal{O}_{[\mathbb{P}^N/T]}(r)$, $\phi : [\mathbb{P}^N/T] \to [\operatorname{Spec} k/\mathbb{G}_m]$. If the map f is classified by a geometric point of $\underline{\operatorname{Map}}^0\left(E, [\mathbb{P}^N/T]\right)$, the pull-back line bundle $f^*(\mathcal{O}_{[\mathbb{P}^N/T]}(r))$ is classified by the composition

$$\operatorname{Spec}(K) \xrightarrow{f} \underline{\operatorname{Map}}^0\left(E, [\mathbb{P}^N/T]\right) \xrightarrow{\phi} \underline{\operatorname{Map}}^0\left(E, [\operatorname{Spec} k/\mathbb{G}_m]\right) \cong \underline{\operatorname{Pic}}^0(E)$$

In particular, $f^*(\mathcal{O}_{[\mathbb{P}^N/T]}(r))$ is a degree-zero line bundle on E.

Let $H \subset \mathbb{P}^N$ be the zero locus of σ, and consider its complement U. As σ is linear H is a hyperplane and U is a T-equivariant open subset of $\mathbb{P}^N$ isomorphic to $\mathbb{A}^N$. We claim that the image of f is entirely contained either in H or in U. Indeed, assume to the contrary that the image of f intersects both H and U. The pull-back section $f^*\sigma \in H^0(E, f^*\mathcal{O}_{[\mathbb{P}^N/T]}(r))$ is not constant, as the image of f intersects both the zero-locus of σ and its complement. However the line bundle $f^*\mathcal{O}_{[\mathbb{P}^N/T]}(r)$ must be of degree zero, and therefore its sections are necessarily constant. Thus as we claimed f factors either through

$$[\mathbb{A}^N/T] \cong [U/T] \subset [\mathbb{P}^N/T] \quad \text{or through} \quad [H/T] \subset [\mathbb{P}^N/T]$$

In the first case, we are done by Lemma 2.3.16. In the second case, f factors through the lower dimensional space $[H/T] \cong [\mathbb{P}^{N-1}/T]$, and we can conclude by induction on the dimension N: note indeed that the base case of the statement $N = 1$ is clear, as the complement of a T-invariant open subset $\mathbb{A}^1 \cong U \subset \mathbb{P}^1$ is a fixed point, which is a T-orbit. $\square$

We are now ready to prove Theorem 2.3.14.

Proof of Theorem 2.3.14. This proof is very similar to that of Proposition 2.3.4. By Sumihiro's Theorem [**71**] there exist a T-equivariant open cover $\{U_i\}_{i\in I}$ of X such that each U_i is affine. Further any T-equivariant open cover of X can be refined to such a affine T-equivariant open cover. Thus we can restrict, without loss of generality, to covers in $|X|^T_{\mathrm{Zar}}$ given by disjoint unions of affines equipped with a T-action.

Let $\{U_i \to X\}_{i\in I}$ be such a cover. By the pointwise criterion, it is enough to show that for every quasi-constant map $f : E_K \to [X/T]$ there exists an $i \in I$ such that there is a factorization

$$\begin{tikzcd} & {[U_i/T]} \arrow[d] \\ E_K \arrow[r, "f"] \arrow[ur, dashed] & {[X/T]} \end{tikzcd}$$

Let U_i be such that the fiber product

$$S := [U_i/T] \times_{[X/T]} E_K$$

is non-empty, and fix a T-equivariant embedding

$$U_i \to \mathbb{A}^N \subset \mathbb{P}^N$$

where $\mathbb{A}^N$ are $\mathbb{P}^N$ equipped with a suitable T-action; the existence of such an embedding is an application of Lemma 5.2 in [**22**]. Note that $[\mathbb{P}^N/T]$ satisfies the valuative criterion for properness: as a consequence, there always exists a (non unique) arrow

h making the diagram

$$\begin{tikzcd}
{[U_i/T]} \arrow[r] & {[\mathbb{P}^n/T]} \\
S \arrow[u, "f|_S"] \arrow[r] & E_K \arrow[u, "h"'] \arrow[ul, dashed]
\end{tikzcd}$$

commute. Thus, by Lemma 2.3.17, h admits a lift represented by the dashed arrow in the diagram. ☐

REMARK 2.3.18. *$[\mathbb{P}^N/T]$ satisfies the valuative criterion of properness as T-torsors are locally trivial in the Zariski topology, and thus every T-torsor on the fraction field of a DVR has a section. Indeed, let R be a DVR and consider a map $m : \operatorname{Spec}\operatorname{Frac}R \to [\mathbb{P}^N/T]$. Let us show that it admits a lift*

$$\bar{m} : \operatorname{Spec}(R) \to [\mathbb{P}^N/T]$$

The map m classifies a diagram of the form

$$\begin{tikzcd}
P \arrow[r] \arrow[d] & \mathbb{P}^N \arrow[d] \\
\operatorname{Spec}(\operatorname{Frac}(R)) \arrow[u, dashed, bend left] \arrow[ur, "n"] \arrow[r, "m"] & {[\mathbb{P}^N/T]}
\end{tikzcd}$$

where P is a T-torsor; as P is trivial we can pick a section (represented by a dashed arrow) and this induces a lift n. Applying the ordinary valuative criterion for properness to $\mathbb{P}^n$, we obtain a map $\bar{n}$ which makes the following diagram commute

$$\begin{tikzcd}
\operatorname{Spec}(R) \arrow[r, "\bar{n}"] & \mathbb{P}^N \arrow[d] \\
\operatorname{Spec}(\operatorname{Frac}(R)) \arrow[u] \arrow[r, "n"] & \mathbb{P}^N
\end{tikzcd}$$

Then the composite map

$$\bar{m} : \operatorname{Spec}(R) \xrightarrow{\bar{n}} \mathbb{P}^N \to [\mathbb{P}^N/T]$$

is the desired lift of m.

2.4. The local model, and toric varieties.

In this section we compute the elliptic Hochschild homology of quotient stacks of the form

$$[\mathbb{A}^l \times \mathbb{G}_m^k/T] \tag{8}$$

where T is an algebraic torus. This calculation is relatively straightforward, but important. It is obtained by combining the results of propositions 2.4.1, 2.4.3 and

2.4.4. It will also play a role in the next section as, via Luna slice theorem, we will often be able to reduce our arguments to this local case. Then, using codescent, we compute the elliptic Hochschild homology of smooth toric varieties with their standard torus action. This will give us a broad supply of geometrically interesting examples for which elliptic Hochschild homology can be explicitly described. Finally, in Theorem 2.4.5 we show that when $k = \mathbb{C}$ this coincides with the complexified equivariant cohomology of the analytification of X.

We begin this section with the following simple observation, which is an analogue of the Künneth isomorphism for mapping stacks.

PROPOSITION 2.4.1 (Künneth formula). *Let G be an algebraic group and let X and Y be G-varieties. Then*

$$\underline{\mathrm{Map}}^0(E, [X \times Y/G]) \simeq \underline{\mathrm{Map}}^0(E, [X/G]) \times_{\underline{\mathrm{Map}}^0(E,[\mathrm{Spec}\, k/G])} \underline{\mathrm{Map}}^0(E, [Y/G])$$

where G acts on $X \times Y$ diagonally. Similarly, if we equip the product $X \times Y$ with the product action, we have that

$$\underline{\mathrm{Map}}^0(E, [X \times Y/G \times G]) \simeq \underline{\mathrm{Map}}^0(E, [X/G]) \times \underline{\mathrm{Map}}^0(E, [Y/G])$$

PROOF. Note that if X and Y are two G-varieties, then

$$[X \times Y/G] \simeq [X/G] \times_{[\mathrm{Spec}\, k/G]} [Y/G]$$

for the diagonal G-action, and

$$[X \times Y/G \times G] \simeq [X/G] \times [Y/G]$$

for the product G-action. The formulas follow from the fact that $\underline{\mathrm{Map}}^0(E, -)$ preserves limits. $\square$

2.4.1. The local model. In this section we consider quotient stacks of the form $[\mathbb{A}^l \times \mathbb{G}_m^k/T]$. Iterated applications of the Künneth formula allow us to break down the computations for product $\mathbb{A}^l \times \mathbb{G}_m^k$ to the cases of $\mathbb{A}^1$ and $\mathbb{G}_m$, and thus we will limit ourselves to describe these explicitly. This will be achieved in a sequence of Propositions. Let us consider trivial actions first.

PROPOSITION 2.4.2. *Let X be a variety over k and let T be an algebraic torus acting trivially on X. Then*

$$\mathcal{HH}_E([X/T]) = \mathrm{HH}_*(X) \otimes_k \mathcal{O}_{E_T}$$

PROOF. Since the action is trivial, we have that $[X/T] \simeq X \times BT$. The proposition follows because $\underline{\mathrm{Map}}^0(E, -)$ preserves products. $\square$

PROPOSITION 2.4.3. *Let T be an algebraic torus of rank r acting on $\mathbb{A}^1$. Assume that the T-action is non-trivial. Then there is an equivalence*

$$\mathcal{HH}_E([\mathbb{A}^1/T]) \simeq \mathcal{O}_{E_T}$$

PROOF. Choosing an isomorphism $T \cong \mathbb{G}_m^r$, the action in coordinates becomes

$$\lambda \cdot z = \prod_{i=1}^{r} \lambda_i^{w_i} z$$

Recall that the stack $\underline{\mathrm{Map}}^0(E, [\mathbb{A}^1/\mathbb{G}_m^r])$ classifies r-tuples $\{\mathcal{L}_i\}_{i=1}^r$ of degree zero line bundles on E together with a section $s \in H^0(\otimes_{i=1}^r \mathcal{L}_i^{w_i})$. In particular, we obtain the following description of $\underline{\mathrm{Map}}^0(E, [\mathbb{A}^1/T])$, when the action is non-trivial. Let Z be the closed subscheme of E_T cut out by the equation

$$\prod_{i=1}^{r} e_i^{w_i} = 1 \tag{9}$$

Then we have a push-out diagram

$$\begin{tikzcd}
{[Z \times \operatorname{Spec} k/T]} \arrow[r] \arrow[d, "0"] & \underline{\mathrm{Pic}}^0(E)_T \arrow[d] \\
{[Z \times \mathbb{A}^1/T]} \arrow[r] & \underline{\mathrm{Map}}^0(E, [\mathbb{A}^1/T])
\end{tikzcd}$$

where the left vertical map is the zero section of the projection $[Z \times \mathbb{A}^1/T] \to [Z/T]$, and T acts trivially on E_T and in particular on Z. Note that if T is rank 1, Z is a finite subset of the torsion points in E_T. In particular, the coarse moduli space of the stack $\underline{\mathrm{Map}}^0(E, [\mathbb{A}^1/T])$ is given by E_T, which implies that

$$p_* \mathcal{O}_{\underline{\mathrm{Map}}^0(E,[\mathbb{A}^1/T])} \simeq \mathcal{O}_{E_T}$$

which ends the proof. $\square$

PROPOSITION 2.4.4. *Let T be an algebraic torus of rank r acting on $\mathbb{G}_m$. Let*

$$p : \underline{\mathrm{Map}}^0(E, [\mathbb{G}_m/T]) \to E_T$$

be the structure map. Let $Z \subset E_T$ be the closed susbscheme of E_T cut out by equation (9)*. Then, as soon as the action is nontrivial, we have an equivalence*

$$\mathcal{H}\mathcal{H}_E([\mathbb{G}_m/T]) \simeq \mathcal{O}_Z$$

as quasi-coherent sheaves on E_T.

PROOF. The proof of this proposition is the same as the proof of proposition 2.4.3, and in fact the geometry is simpler. The stack $\underline{\mathrm{Map}}^0(E, [\mathbb{G}_m/T])$ classifies r-tuples $\{\mathcal{L}_i\}_{i=1}^r$ of degree zero line bundles on E, and *trivializations* of the tensor product

$$\bigotimes_{i=1}^{r} \mathcal{L}_i^{w_i} \simeq \mathcal{O}_E \tag{10}$$

where the w_i-s are the weights of the action. Thus the stack $\underline{\mathrm{Map}}^0(E, [\mathbb{G}_m/T])$ sits over the locus $Z \subset E_T$ of those bundles satisfying (10). $\square$

2.4.2. The case of maximal tori. In this subsection we compute the equivariant elliptic Hochschild homology of smooth toric varieties equipped with the toric action of their maximal torus. In particular, we prove the following theorem:

THEOREM 2.4.5. *Let X be a smooth (normal) toric variety over $\mathbb{C}$, and let the maximal torus T act on X either with the standard toric action or with weights $\{w_1, \ldots, w_n\}$ all non-zero. Then there is an isomorphism of coherent sheaves on E_T*

$$\mathcal{HH}_E([X/T]) \simeq \mathcal{E}ll_T^0(X^{\mathrm{an}})$$

where X^{an} denotes the analytification of X, and $\mathcal{E}ll_T^0(X^{\mathrm{an}}) \simeq \pi_0\mathcal{E}ll_T(X^{\mathrm{an}})$ is 0-th homotopy sheaf of the complexified T-equivariant elliptic cohomology of X^{an}.

PROOF OF THEOREM 2.4.5. As X is a toric variety, we can find a T-equivariant open cover $\mathcal{U}$ by products $\mathbb{A}^n \times \mathbb{G}_m^k$. Let us assume for simplicity that $k = 0$, as the case $k \neq 0$ is basically the same due to the Künneth formula. By the codescent property we know that the natural map

$$\coprod_{U_i \in \mathcal{U}} \underline{\mathrm{Map}}^0(E, [U_i/T]) \to \underline{\mathrm{Map}}^0(E, [X/T])$$

induced by the cover $\mathcal{U}$ is an effective epimorphism, and that the functor

$$\underline{\mathrm{Map}}^0(E, [-/T]) : |X|^T_{\mathrm{Zar}} \to \mathrm{dSt}_k$$

is a cosheaf, as explained in theorem 2.3.14. This implies that the functor

$$\mathcal{HH}_E([-/T]) : |X|^T_{\mathrm{Zar}} \to \mathrm{Qcoh}(E_T)$$

is a sheaf, and in particular $\mathcal{HH}_E([X/T])$ is obtained as the totalization of the cosimplicial object $\mathcal{HH}_E([\mathcal{U}_\bullet/T])$, where $\mathcal{U}_\bullet$ is the Cech nerve of the cover $\mathcal{U}$. Similarly, T-equivariant elliptic cohomology satisfies Mayer-Vietoris which implies that we can compute the quasi-coherent sheaf $\mathcal{E}ll_T^0(X)$ as the totalization of the cosimplicial object $\mathcal{E}ll_T^0(\mathcal{U}_\bullet)$. Propositions 2.4.3 and 2.4.4 imply that this two cosimplicial objects coincide, hence they have the same totalization. $\square$

2.5. Equivariant Elliptic Hochschild Homology

In this section we study the local behaviour of equivariant elliptic Hochschild homology $\mathcal{HH}_E([X/T])$ when X is a smooth variety over k equipped with an action of T. We relate this to Hochschild homology and equivariant elliptic cohomology. In particular, this involves completing the quasi-constant maps at the constant maps and comparing this with the completion of the loop space at the constant loops. A localization phenomenon, combined with the group structure on the elliptic curve

E, allows for the computation of the completions over all closed points of E_T. Our main results in this section are direct analogues of theorems established by Chen in [**17**] in the setting of ordinary Hochschild homology.

First, we need the following definition.

Definition 2.5.1. *Let G be a derived group scheme acting on a derived stack X. The* derived fixed locus *of the G-action is the following fiber product of derived stacks*

$$\begin{tikzcd} X^G \arrow[r] \arrow[d] & X \times G \arrow[d, "(\pi,a)"] \\ X \arrow[r, "\Delta"] & X \times X \end{tikzcd}$$

where π is the projection and a is the action map.

From now on all fixed loci are assumed to be derived.

2.5.1. The localization formula for quasi-constant maps. In this section we establish a localization theorem for quasi-constant maps. This is an analogue of Theorem 3.1.12 in [**17**].

Theorem 2.5.2 (Localization formula). *Let X be a smooth variety over k equipped with an action of an algebraic torus T. Then for any closed point $e \in E_T$ there exists a Zariski open set $U \subset E_T$ containing e such that the natural map*

$$\underline{\mathrm{Map}}^0\left(E, [t_0 X^{T(e)}/T]\right) \times_{E_T} U \to \underline{\mathrm{Map}}^0\left(E, [X/T]\right) \times_{E_T} U \tag{11}$$

induced by the inclusion of the classical fixed locus $t_0 X^{T(e)} \to X$, is an equivalence.

In the above, $T(e)$ is the subgroup of T as in Definition 3.2.1. To prove this theorem we need to establish some preliminary results first.

Lemma 2.5.3 (Localization for the affine space). *Consider the n-dimensional affine space $\mathbb{A}^n$, equipped with an action of a torus T of rank k. Fix a closed point e in E_T. Then there exists a Zariski open $U(e)$ in E_T such that the natural map*

$$\phi : \underline{\mathrm{Map}}^0\left(E, [t_0(\mathbb{A}^n)^{T(e)}/T]\right) \times_{E_T} U(e) \to \underline{\mathrm{Map}}^0\left(E, [\mathbb{A}^n/T]\right) \times_{E_T} U(e)$$

induced by the inclusion of the classical fixed locus $t_0(\mathbb{A}^n)^{T(e)} \to \mathbb{A}^n$ is an equivalence.

Proof. In the case of $[\mathbb{A}^n/T]$ we can describe explicitly the open $U(e)$. First of all, observe that, since the action of T on $\mathbb{A}^n$ is linear, the fixed loci will be a linear subspace, in particular there exists a natural number p such that $t_0(\mathbb{A}^n)^{T(e)} \cong \mathbb{A}^p$. We distinguish two situations:

(1) $e = 1$ in E_T. In this case, the statement is true for any $U(e)$;

(2) $e \neq 1$. In this case, the open set $U(e)$ can be described in terms of the weights $\{w_i^j\}$ of the action.

Let us focus on the second case. Recall that the stack $[\operatorname{Spec} k/T]$ classifies principal T-bundles or equivalently k-tuples of line bundles $\{\mathcal{L}_i\}_{i=1}^k$, while $[\mathbb{A}^n/T]$ classifies such k-tuples $\{\mathcal{L}_i\}_{i=1}^k$ together with an n-tuple of sections $s_j \in H^0(\otimes_{i=1}^k \mathcal{L}_i^{w_i^j})$ for all values of j in $\{1, \dots, n\}$. Since degree zero bundles on elliptic curves have sections if and only if they are trivial, the stacks $\underline{\mathrm{Map}}^0(E, [\mathbb{A}^n/T])$ and $\underline{\mathrm{Map}}^0(E, [\mathbb{A}^p/T])$ will differ only over the locus of those points $f = (f_1, \dots, f_n)$ in E_T such that $\Sigma_{i=1}^k f_i^{w_i^j} = 0$ for more than p values of j in $\{1, \dots, n\}$. This is because, for points f of this kind, there will be more then p line bundles of the form $\otimes_{i=1}^k \mathcal{L}_i^{w_i^j}$ (indexed by j) that admit non-vanishing sections, hence n-tuples of sections may differ from p-tuples of sections. Then, the open $U(e)$ is defined by removing from E_T the locus of the points f having this property. □

REMARK 2.5.4. *The lemma above can be viewed as a statement about deformation theory of bundles with sections on elliptic curves. In particular, it is possible to compute the relative cotangent complex of the map*

$$\underline{\mathrm{Map}}^0\left(E, [t_0(\mathbb{A}^n)^{T(e)}/T]\right) \to \underline{\mathrm{Map}}^0(E, [\mathbb{A}^n/T])$$

Its vanishing on closed points lying over the Zariski open $U(e)$ depends on the fact that nontrivial degree zero line bundles on E have no non-zero sections.

2.5.1.1. *The localization theorem on closed points.* Before proceeding with the proof of Theorem 2.5.2, we will explain why the statement is true on closed points. This will clarify the geometry underlying Theorem 2.5.2. Further the partial results we will obtain in this section will actually be needed in the course of the proof of Theorem 2.5.2, which we will present in the next section.

In section 2.3 we have shown that the derived stack of quasi-constant maps from E satisfies a codescent property with respect to equivariant Zariski open covers. We proved this by showing that the images of the total spaces of principal T-bundles are always contained inside T-orbits in the target space X. This property of quasi-constant maps allows us to show that the map (12) is a homotopy equivalence on geometric points, as the type of T-orbit selects which bundles are allowed to map to it.

PROPOSITION 2.5.5 (Localization formula on geometric points). *Let X be a smooth variety over k equipped with an action of an algebraic torus T. Then for any closed point $e \in E_T$ there exists a Zariski open set $U \subset E_T$ containing e such that the natural map*

$$\underline{\mathrm{Map}}^0\left(E, [t_0 X^{T(e)}/T]\right) \times_{E_T} U \to \underline{\mathrm{Map}}^0(E, [X/T]) \times_{E_T} U \tag{12}$$

is a homotopy equivalence on geometric points.

PROOF. Let us choose a T-orbit in X, O, generated by a closed point x with stabilizer T_x. Then considering the mapping stack to $O \simeq T/T_x$ we obtain

$$\underline{\mathrm{Map}}^0\left(E, [O/T]\right) \simeq \underline{\mathrm{Map}}^0\left(E, [\operatorname{Spec} k/T_x]\right)$$

which is the classifying stack of degree zero T_x-bundles on E. Hence, we conclude that only the principal T-bundles that admit a reduction of the structure group from T to T_x are allowed to map to the orbit O. In particular, as the subscheme $t_0X^{T(e)}$ is a union of orbits of the form T/S, where S is a subgroup of T containing $T(e)$, the bundles that admit a reduction of the structure group to $T(e)$ are allowed to map to the complement of $t_0(X^{T(e)})$. In order to have that the map (12) is a homotopy equivalence on K-points for an algebraically closed field K, we need to remove maps from those bundles. To do so, it is sufficient to remove the bundles that admit a reduction of the structure group to a subgroup of $T(e)$, and this is implemented by restricting the mapping stack to a Zariski open U of E_T. □

2.5.1.2. *The proof.* We are now ready to prove Theorem 2.5.2. The key ingredient in the proof is Luna's slice theorem [**22**], which allows us to reduce to the case of a linear action on affine space, which was treated in Lemma 2.5.3.

PROOF OF THEOREM 2.5.2. By codescent we can assume X is affine. Our goal is to prove that the map

$$\phi : \underline{\mathrm{Map}}^0\left(E, [t_0X^{T(e)}/T]\right) \times_{E_T} U \to \underline{\mathrm{Map}}^0\left(E, [X/T]\right) \times_{E_T} U \tag{13}$$

is an equivalence, for a Zariski open subset U of E_T. We choose a U that makes Proposition 2.5.5 hold.

First we prove that this map is étale. Choose a point x of $t_0(X^{T(e)})$ such that the orbit Tx is closed. The Luna slice theorem applied to x gives us a locally closed smooth subvariety V of X closed under the action of the stabilizer T_x of x such that the natural T-equivariant map $\psi : T \times^{T_x} V \to X$ is étale and has image given by a Zariski open Z of X. We have an induced commutative diagram

$$\begin{tikzcd}
\underline{\mathrm{Map}}^0\left(E, [t_0(T \times^{T_x} V)^{T(e)}/T]\right) \times_{E_T} U \arrow[r, "\phi_V"] \arrow[d, "i_e"] & \underline{\mathrm{Map}}^0\left(E, [T \times^{T_x} V/T]\right) \times_{E_T} U \arrow[d, "i"] \\
\underline{\mathrm{Map}}^0\left(E, [t_0(X)^{T(e)}/T]\right) \times_{E_T} U \arrow[r, "\phi"] & \underline{\mathrm{Map}}^0\left(E, [X/T]\right) \times_{E_T} U
\end{tikzcd}$$

for the mapping stacks. First note that the map (13) is locally finitely presented (see Remark 2.5.6), hence we only need to show it is formally étale, i.e. its relative cotangent complex vanishes. To do so, we first observe that the vertical maps have vanishing relative cotangent complex. The argument is the same for the left and the right one. As for the right vertical composition, choose an S-point

$$x : \operatorname{Spec} S \to \underline{\mathrm{Map}}^0\left(E, [T \times^{T_x} V/T]\right)$$

We need to show that the relative cotangent vanishes at any such S-point. We apply Halpern-Leistner and Preygel's Proposition 5.1.10 in [**32**]:

$$\mathbb{L}_{\underline{\mathrm{Map}}(X,Y),f} \simeq \pi_+ f^* \mathbb{L}_Y$$

where the S-point $f : \operatorname{Spec} S \to \underline{\mathrm{Map}}(X, Y)$ is viewed as a map $f : \operatorname{Spec} S \times X \to Y$ and π_+ is a left adjoint to the pullback along the projection $\pi : \operatorname{Spec} S \times X \to \operatorname{Spec} S$.

The pullback $x^*\mathbb{L}_i$ is

$$x^*\mathbb{L}_i \simeq \pi_+ x^* \mathbb{L}_\psi \simeq 0$$

as Luna's slice theorem guarantees that the map ψ is étale. In particular, we obtain that the map i (and similarly i_e) is formally étale. One consequence of this fact is that we have an equivalence

$$i_e^*\mathbb{L}_\phi \xrightarrow{\simeq} \mathbb{L}_{\phi_V} \tag{14}$$

To see this, recall that for a commutative triangle

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "h"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

of derived stacks, there is an induced cofiber (and fiber) sequence of the relative cotangent complexes:

$$f^*\mathbb{L}_g \to \mathbb{L}_h \to \mathbb{L}_f$$

(see for example Corollary 1.44 in [**37**]).

We get two cofiber sequences:

$$\phi_V^*\mathbb{L}_i \to \mathbb{L}_{i\circ\phi_V} \to \mathbb{L}_{\phi_V}$$

$$i_e^*\mathbb{L}_\phi \to \mathbb{L}_{\phi\circ i_e} \to \mathbb{L}_{i_e}$$

Since we know that the two relative contangent complexes $\mathbb{L}_i$ and $\mathbb{L}_{i_e}$ vanish, we get equivalences

$$\mathbb{L}_{i\circ\phi_V} \xrightarrow{\simeq} \mathbb{L}_{\phi_V}$$

$$i_e^*\mathbb{L}_\phi \xrightarrow{\simeq} \mathbb{L}_{\phi\circ i_e}$$

Since there is an equivalence $\phi \circ i_e \simeq i \circ \phi_V$, we conclude that the equivalence (14) holds.

We now show ϕ_V is formally étale. Recall that, for smooth closed points $x \in X$, the Luna étale slice theorem gives us an additional map $V \to T_{V,x}$ from V to its tangent space at x which is T_x-equivariant and étale onto its image, which is an open subscheme of $T_{V,x}$. We have a further commutative diagram

$$\begin{tikzcd} \underline{\mathrm{Map}}^0\left(E,[t_0(T\times^{T_x}V)^{T(e)}/T]\right)\times_{E_T}U \arrow[r, "\phi_V"] \arrow[d, "j_e"] & \underline{\mathrm{Map}}^0\left(E,[T\times^{T_x}V/T]\right)\times_{E_T}U \arrow[d, "j"] \\ \underline{\mathrm{Map}}^0\left(E,[t_0(T\times^{T_x}T_{V,x})^{T(e)}/T]\right)\times_{E_T}U \arrow[r, "\phi_x"] & \underline{\mathrm{Map}}^0\left(E,[T\times^{T_x}T_{V,x}/T]\right)\times_{E_T}U \end{tikzcd}$$

and, reasoning as in the previous case, we obtain an equivalence

$$j_e^*\mathbb{L}_{\phi_x}\xrightarrow{\simeq}\mathbb{L}_{\phi_V}$$

But the map

$$\underline{\mathrm{Map}}^0\left(E,[t_0(T\times^{T_x}T_{V,x})^{T(e)}/T]\right)\times_{E_T}U\to\underline{\mathrm{Map}}^0\left(E,[T\times^{T_x}T_{V,x}/T]\right)\times_{E_T}U$$

is an equivalence by an application of lemma 2.5.3. In particular, we conclude that the map ϕ_V is formally étale and deduce that $i_e^*\mathbb{L}_\phi\simeq 0$ from (14).

Let us observe that, in the case of algebraic actions of tori on affine varieties, there is a sufficient supply of points with closed orbit, i.e. there is a collection of closed points in X such that the orbit they span is closed, and the images of the Luna slice maps ψ at each of these points form an open cover of X. Indeed, recall that in an affine variety with an action of an algebraic group G, for every orbit O there is a unique closed orbit in the complement $\overline{O}-O$, where $\overline{O}$ is the closure of O. Moreover, the Zariski open sets Z given by images of the étale slice maps $\psi: T\times^{T_x}V\to X$ are saturated, that means that given a point $z\in Z$ and any other point $x\in X$, if the intersection $\overline{Tz}\bigcap\overline{Tx}$ of the closure of the orbits is non-empty, then $x\in Z$. In particular, given an orbit O in X, there always exists a point $x\in X$ whose orbit is closed, and such that the image Z of the Luna slice at the point x contain the orbit O. As a consequence, it is always possible to cover X with images of Luna slices.

Now we can conclude: for each induced map i_e relative to each of these points we know that $i_e^*\mathbb{L}_\phi$ vanishes, and the coproduct of all the maps i_e is an étale effective epimorphism by equivariant Zariski codescent. This is enough to prove that $\mathbb{L}_\phi=0$, and since ϕ is locally finitely presented we obtain that it is étale, as we desired to show.

We now argue that (13) is an equivalence. Since it is étale and a closed immersion, it is also an open immersion; in particular it exhibits $\underline{\mathrm{Map}}^0\left(E,[t_0X^{T(e)}/T]\right)$ as a union of connected components of $\underline{\mathrm{Map}}^0(E,[X/T])$. Since $\underline{\mathrm{Map}}^0(E,[X/T])$ is a union of connected components of $\underline{\mathrm{Map}}(E,[X/T])$ by definition, checking that the map (13) is an equivalence amounts only to understanding if its image contains all such connected components, which can be checked on geometric points. But

Proposition 2.5.5 tells us that such map is a homotopy equivalence on the spaces of closed points, and this concludes the proof. $\square$

REMARK 2.5.6. *We make the following observation. Let $f : X \to Y$ is a locally finitely presented map of derived stacks. The induced map $\underline{\mathrm{Map}}^0(E, X) \to \underline{\mathrm{Map}}^0(E, Y)$ is locally finitely presented (the proof goes like that of Lemma 2.3.2). In the sitiuation of the proof of Theorem 2.5.2, the map $[t_0 X^{T(e)}/T] \to [X/T]$ is locally finitely presented, as the map $t_0 X^{T(e)} \to X$ is a locally finitely presented map of (classical) schemes. Recall indeed that, as $t_0 X^{T(e)}$ and X are varieties over a characteristic zero field k, the map $t_0 X^{T(e)} \to X$ is locally finitely presented (see for example Lemma 29.21.11 in* [**68**, Tag 01TO]*).*

A consequence of the localization formula is the following description of the fibers of the structure map $p' : \underline{\mathrm{Map}}^0(E, [X/T]) \to \underline{\mathrm{Pic}}^0(E)_T$.

COROLLARY 2.5.7. *Let X be a smooth variety over k. Then given a closed point*

$$\bar{e} : \operatorname{Spec} K \to \underline{\mathrm{Pic}}^0(E)_T$$

the fiber of the structure map $p' : \underline{\mathrm{Map}}^0(E, [X/T]) \to \underline{\mathrm{Pic}}^0(E)_T$ over the point $\bar{e}$ is given by the derived fixed locus $X^{T(e)}$, where e is the closed point in E_T corresponding to the composition $\operatorname{Spec} K \xrightarrow{\bar{e}} \underline{\mathrm{Pic}}^0(E) \to E_T$.

PROOF. Since the closed immersion $\bar{e} : \operatorname{Spec} K \to \underline{\mathrm{Pic}}^0(E)_T$ factors through $\underline{U_e} = U_e \times BT$, where U_e is the Zariski open determined by Proposition 2.5.2 for the point e, we can apply the same proposition to reduce the computation to the fiber

$$\left(\underline{\mathrm{Map}}^0(E, [X/T]) \times_{\underline{\mathrm{Pic}}^0(E)_T} \underline{U_e}\right) \times_{\underline{U_e}} \bar{e} \simeq \left(\underline{\mathrm{Map}}^0\left(E, [t_0 X^{T(e)}/T]\right) \times_{\underline{\mathrm{Pic}}^0(E)_T} \underline{U_e}\right) \times_{\underline{U_e}} \bar{e}$$

which in turn is the fiber of the structure map $p' : \underline{\mathrm{Map}}^0\left(E, [t_0 X^{T(e)}/T]\right) \to \underline{\mathrm{Pic}}^0(E)_T$ over the point $\bar{e}$.

Let $T(e)$ be the subgroup associated to the point e in E_T. The map $\bar{e} : \operatorname{Spec} K \to \underline{\mathrm{Pic}}^0(E)_T$ will factor through the stack $\underline{\mathrm{Bun}}^0_{T(e)}(E)$, hence we can compute the fiber using the following pasting of pullback diagrams:

$$\begin{tikzcd}
F_e \arrow[r] \arrow[d] & \underline{\mathrm{Map}}^0\left(E, [t_0 X^{T(e)}/T(e)]\right) \arrow[r] \arrow[d] & \underline{\mathrm{Map}}^0\left(E, [t_0 X^{T(e)}/T]\right) \arrow[d] \\
\operatorname{Spec} K \arrow[r, "\bar{e}"] & \underline{\mathrm{Bun}}^0_{T(e)}(E) \arrow[r] & \underline{\mathrm{Pic}}^0(E)_T
\end{tikzcd}$$

where we called F_e the fiber we are interested in computing. The square on the right is a pullback since $\underline{\mathrm{Map}}^0(E,-)$ commutes with limits, and the diagram

$$\begin{tikzcd} {[t_0X^{T(e)}/T(e)]} \arrow[r] \arrow[d, "p_e"] & {[t_0X^{T(e)}/T]} \arrow[d] \\ {[\operatorname{Spec} k/T(e)]} \arrow[r] & {[\operatorname{Spec} k/T]} \end{tikzcd}$$

is a pullback. The calculation of F_e follows easily from the observation that, by definition, $T(e)$ acts trivially on $t_0X^{T(e)}$, and in particular we have that

$$\underline{\mathrm{Map}}^0\left(E,[t_0X^{T(e)}/T(e)]\right) \simeq \underline{\mathrm{Map}}^0\left(E,t_0X^{T(e)} \times BT(e)\right) \simeq \underline{\mathrm{Map}}^0\left(E,t_0X^{T(e)}\right) \times \underline{\mathrm{Bun}}^0_{T(e)}(E)$$

hence the map p_e necessarily has fiber over $\bar{e}$ given by $\underline{\mathrm{Map}}^0\left(E,t_0X^{T(e)}\right)$, as p_e is isomorphic to the projection to $\underline{\mathrm{Bun}}^0_{T(e)}(E)$. Then, by Corollary 1.0.1 in [**17**]

$$\underline{\mathrm{Map}}^0\left(E,t_0X^{T(e)}\right) \simeq \mathcal{L}\, t_0X^{T(e)} \simeq X^{T(e)}$$

□

REMARK 2.5.8. *We established Theorem 2.5.2 for closed points $e \in E_T$, but it holds also for non-closed points $x \in E_T$ with the notion of subgroup $T(x)$ associated to one such point introduced in Remark 2.2.35. Indeed, if x is any point in E_T, its closure $\overline{\{x\}}$ contains at least one closed point e such that the two subgroups $T(e)$ and $T(x)$ coincide. We can then declare the Zariski open subset U_x of E_T realizing localization for the point x to be the open subset U_e associated to the closed point e, as $x \in U_e$. If we do so, the statement of Theorem 2.5.2 extends to closed points.*

2.5.2. The local structure of the quasi-constant maps. We now compute the completions of elliptic Hochschild homology at closed points of E_T.

Recall that for a derived stack $\mathcal{X}$, there is a natural map

$$\mathcal{X} \xrightarrow{\simeq} \underline{\mathrm{Map}}(\operatorname{Spec} k,\mathcal{X}) \to \underline{\mathrm{Map}}^0(E,\mathcal{X})$$

induced by the structure morphism $E \to \operatorname{Spec} k$. We call the completion of this map *the completion of* $\underline{\mathrm{Map}}^0(E,\mathcal{X})$ *at the constant maps* or *formal maps* from E to $\mathcal{X}$. There is an analogous map for the loop space $\mathcal{L}\mathcal{X}$, and its formal completion is usually called the *formal loop space.*

REMARK 2.5.9. *The formal completion of* $\underline{\mathrm{Map}}^0(E,\mathcal{X})$ *at the constant maps is the same as that of* $\underline{\mathrm{Map}}(E,\mathcal{X})$*, as* $\underline{\mathrm{Map}}^0(E,\mathcal{X})$ *is a collection of connected components of* $\underline{\mathrm{Map}}(E,\mathcal{X})$ *containing the constant maps.*

Proposition 2.5.10. *Let $\mathcal{X}$ be a derived stack with affine diagonal over a field k of characteristic zero, and E be an elliptic curve over k. There is a natural map between formal completions at the constant maps*

$$\psi : \hat{\mathcal{L}}\mathcal{X} \to \widehat{\underline{\mathrm{Map}}}^0 (E, \mathcal{X})$$

Further, ψ is an equivalence.

Proposition 2.5.10 is closely related to results that have already appeared in the literature in slightly different settings, and in particular to Theorem 6.9 of [**8**]. The point is that the deformation theory of quasi-constant maps out of E near the constant maps is controlled by the affinization of E. As the latter is equivalent to the affinization of S^1 the completion of $\underline{\mathrm{Map}}^0 (E, \mathcal{X})$ is equivalent to the completion of $\mathcal{L}X$.

Proof. We will show that the map

$$\psi : \hat{\mathcal{L}}\mathcal{X} \simeq \widehat{\underline{\mathrm{Map}}}^0 (\mathrm{Aff}(E), \mathcal{X}) \to \widehat{\underline{\mathrm{Map}}}^0 (E, \mathcal{X})$$

induces an equivalence of the pullback of the cotangent complexes to the constant maps.

Define the maps

$$\begin{aligned} u &: \underline{\mathrm{Map}} (\mathrm{Aff}(E), \mathcal{X}) \to \underline{\mathrm{Map}} (E, \mathcal{X}) \\ c' &: \mathcal{X} \simeq \underline{\mathrm{Map}} (\operatorname{Spec} k, \mathcal{X}) \to \underline{\mathrm{Map}} (\mathrm{Aff}(E), \mathcal{X}) \\ c &: \mathcal{X} \simeq \underline{\mathrm{Map}} (\operatorname{Spec} k, \mathcal{X}) \to \underline{\mathrm{Map}} (E, \mathcal{X}) = u \circ c' \end{aligned}$$

given by composition with the unit of the affinization $E \to \mathrm{Aff}(E)$ and with the structure maps $\mathrm{Aff}(E) \to \operatorname{Spec} k$ and $E \to \operatorname{Spec} k$ respectively. The map u induces a fiber-cofiber sequence

$$u^*\mathbb{L}_{\underline{\mathrm{Map}}(E,\mathcal{X})} \to \mathbb{L}_{\underline{\mathrm{Map}}(\mathrm{Aff}(E),\mathcal{X})} \to \mathbb{L}_u$$

of quasi-coherent sheaves on $\underline{\mathrm{Map}} (\mathrm{Aff}(E), \mathcal{X})$. Here, $\mathbb{L}_u$ denotes the relative cotangent complex of the map u. Pulling this back along the map c' we get a fiber-cofiber sequence

$$c^*\mathbb{L}_{\underline{\mathrm{Map}}(E,\mathcal{X})} \to c'^*\mathbb{L}_{\underline{\mathrm{Map}}(\mathrm{Aff}(E),\mathcal{X})} \to c'^*\mathbb{L}_u$$

of quasi–coherent sheaves on $\mathcal{X}$. Our goal is to show that $c'^*\mathbb{L}_u$ vanishes.

To do so, we show the stronger fact that for any constant map $x : \operatorname{Spec} S \to \mathcal{X}$ the map of the based loop spaces

$$\Omega_x(u) : \Omega_x\underline{\mathrm{Map}} (\mathrm{Aff}(E), \mathcal{X}) \to \Omega_x\underline{\mathrm{Map}} (E, \mathcal{X})$$

is an equivalence. Indeed, the cotangent complex of a based loop space is a shift of the original cotangent complex: let $F : \mathrm{dAff}^{\mathrm{op}} \to \mathcal{S}$ be a prestack, and let $x : \operatorname{Spec} S \to F$

be a S-point of F. The point x canonically induces a point $\delta_x : \operatorname{Spec} S \to \Omega_x F$, and the following relation holds:

$$\mathbb{L}_{F,x} \simeq \mathbb{L}_{\Omega_x F, \delta_x}[-1]$$

To show that $\Omega_x(u)$ is an equivalence, observe that the based loop spaces have a presentation as mapping stacks: in particular we have equivalences

$$\Omega_x \underline{\operatorname{Map}}(\operatorname{Aff}(E), \mathcal{X}) \simeq \underline{\operatorname{Map}}(\operatorname{Aff}(E), \Omega_x \mathcal{X})$$
$$\Omega_x \underline{\operatorname{Map}}(E, \mathcal{X}) \simeq \underline{\operatorname{Map}}(E, \Omega_x \mathcal{X})$$

obtained by applying the tensor-hom adjunction twice on different factors. Since $\mathcal{X}$ has affine diagonal the based loop space $\Omega_x \mathcal{X}$ is an affine scheme, hence we have an identification $\underline{\operatorname{Map}}(\operatorname{Aff}(E), \Omega_x \mathcal{X}) \simeq \underline{\operatorname{Map}}(E, \Omega_x \mathcal{X})$. This completes the proof. □

REMARK 2.5.11. *Similar arguments appear also in Ben-Zvi and Nadler's paper* [**8**].

REMARK 2.5.12. *The equivalence in Proposition 2.5.10 is clearly natural in the second variable. Note also that the fact that the map* $\underline{\operatorname{Map}}(\operatorname{Aff}(E), \mathcal{X}) \to \underline{\operatorname{Map}}(E, \mathcal{X})$ *has vanishing cotangent complex over the constant maps holds in considerable generality, as the only restriction is that the mapping stacks have to admit a cotangent complex over the loci of constant maps. For instance, it remains true if E replaced with any smooth variety.*

We will now apply Proposition 2.5.10 to compute the completion of $\mathcal{HH}_E([X/T])$ at the identity of E_T. Denote by

$$i : \{1_{E_T}\} \to E_T \quad j : \{1_T\} \to T$$

the inclusion of the identity elements. Let $\widehat{E_T}$ be the completion of E_T at i and denote by

$$\widehat{i} : \widehat{E_T} \to E_T$$

the natural map. Similarly, let $\widehat{T}$ be the completion of T at j and denote by

$$\widehat{j} : \widehat{T} \to T$$

the natural map.

Following Remark 2.4.9 in [**17**], we define the derived completion of a quasi-coherent sheaf $\mathcal{F}$ on E_T at the identity element as the pull-push

$$\widehat{i}_* \widehat{i}^* \mathcal{F}$$

COROLLARY 2.5.13. *The derived completion of* $\mathcal{HH}_E([X/T])$ *at the identity of* E_T

$$\widehat{i}_* \widehat{i}^* \mathcal{HH}_E([X/T])$$

is the push-forward along $\widehat{i}$ *of the completion of* $\mathrm{HH}_*([X/T])$ *at the prime ideal corresponding to the point* $1 \in T \cong \operatorname{Spec}\mathrm{HH}_*([\operatorname{Spec} k/T])$.

PROOF. Consider the following pullback diagram:

$$\begin{tikzcd} \widehat{\underline{\mathrm{Map}}}^0(E,[X/T]) \arrow[r, "\widehat{i}_X"] \arrow[d, "\widehat{p}"] & \underline{\mathrm{Map}}^0(E,[X/T]) \arrow[d, "p"] \\ \widehat{E_T} \arrow[r, "\widehat{i}"] & E_T \end{tikzcd}$$

Base-changing along this diagram, we can substitute $\widehat{i}^* p_*$ with $\widehat{p}_* \widehat{i}_X^*$ in $\widehat{i}_* \widehat{i}^* \mathcal{HH}_E([X/T])$.

There is an analogous pullback square for the loop space

$$\begin{tikzcd} \widehat{\mathcal{L}}[X/T] \arrow[r, "\widehat{j}_X"] \arrow[d, "\widehat{q}"] & \mathcal{L}[X/T] \arrow[d, "q"] \\ \widehat{T} \arrow[r, "\widehat{j}"] & T \end{tikzcd}$$

We may consider similar completions for the loop space, namely $\widehat{j}_* \widehat{j}^* q_* \mathcal{O}_{\mathcal{L}[X/T]}$. Base-changing along the pullback square for the loop space we may rewrite this completion as

$$\widehat{j}_* \widehat{j}^* q_* \mathcal{O}_{\mathcal{L}[X/T]} \simeq \widehat{j}_* \widehat{q}_* \widehat{j}_X^* \mathcal{O}_{\mathcal{L}[X/T]} \simeq \widehat{j}_* \widehat{q}_* \mathcal{O}_{\widehat{\mathcal{L}}[X/T]} \tag{15}$$

Now, $\widehat{j}_* \widehat{q}_* \mathcal{O}_{\widehat{\mathcal{L}}[X/T]}$ is a quasi-coherent sheaf on T, which is affine, hence it is completely determined by its global sections, which are given by the completion of the Hochschild homology module of $[X/T]$ at the maximal ideal corresponding to the identity element of the torus T.

Proposition 2.5.10 provides an identification of the maps $\widehat{p}$ and $\widehat{q}$. In particular, when evaluating the expression $\widehat{i}_* \widehat{i}^* \mathcal{HH}_E([X/T])$ we obtain

$$\widehat{i}_* \widehat{i}^* \mathcal{HH}_E([X/T]) \simeq \widehat{i}_* \widehat{p}_* \mathcal{O}_{\widehat{\underline{\mathrm{Map}}}^0(E,[X/T])} \simeq \widehat{i}_* \widehat{q}_* \mathcal{O}_{\widehat{\mathcal{L}}[X/T]}$$

□

The completions over other closed points e of E_T can be computed from the completion at the identity using the localization formula explained in 2.5.1 and the group structure on E_T.

For a closed point e of E_T, let $\widehat{i}_e : \widehat{E}_{T,e} \to E_T$ be the natural map from the derived formal completion of E_T at the closed point e to E_T. Moreover, call $\widehat{\mu}_e : \widehat{E}_T \to \widehat{E}_{T,e}$ the completion of the map of multiplication by e, which is an equivalence of formal derived schemes. In particular, the group structure on E_T gives canonical identifications between completions at different closed points.

THEOREM 2.5.14. *The (derived) completion of* $\mathcal{HH}_E([X/T])$ *at the closed point* e *of* E_T, $(\widehat{i}_e)_*\widehat{i}_e^*\mathcal{HH}_E([X/T])$, *is the completion of* $\mathrm{HH}_*([t_0X^{T(e)}/T])$ *at the prime ideal corresponding to the point* $1 \in T = \operatorname{Spec}\mathrm{HH}_*([\operatorname{Spec} k/T])$.

PROOF. Using a similar base change procedure as in the proof of Corollary 2.5.13 we can rewrite the derived completion of $\mathcal{HH}_E([X/T])$ at e as

$$(\widehat{i}_e)_*\widehat{i}_e^*\mathcal{HH}_E([X/T]) \simeq (\widehat{i}_e)_*(\widehat{p}_e)_*\mathcal{O}_{\underline{\mathrm{Map}}^0(E,[X/T])}$$

where $p_{\widehat{e}} : \widehat{\underline{\mathrm{Map}}}^0(E,[X/T])_e \to \widehat{E}_{T,e}$ is the completion at e of the structure map p. Proposition 2.5.2 gives us a Zariski open U of E_T containing e such that

$$\underline{\mathrm{Map}}^0\left(E,[t_0X^{T(e)}/T]\right) \times_{E_T} U \to \underline{\mathrm{Map}}^0(E,[X/T]) \times_{E_T} U$$

is an equivalence, which implies that the completion at e of $\mathcal{HH}_E([X/T])$ is equivalent to the completion at the same point of the sheaf given by $\mathcal{HH}_E([t_0X^{T(e)}/T])$.

Consider the following pullback diagram:

$$\begin{tikzcd}
\widehat{\underline{\mathrm{Map}}}^0\left(E,[t_0X^{T(e)}/T]\right) \arrow[r, "\widehat{\mu}_{e,X}"] \arrow[d, "\widehat{p}"] & \widehat{\underline{\mathrm{Map}}}^0\left(E,[t_0X^{T(e)}/T]\right)_e \arrow[d, "\widehat{p}_e"] \\
\widehat{E}_T \arrow[r, "\widehat{\mu}_e"] & \widehat{E}_{T,e}
\end{tikzcd}$$

Since $\widehat{\mu}_e$ is an equivalence, we have the following relation:

$$(\widehat{p}_e)_*\mathcal{O}_{\widehat{\underline{\mathrm{Map}}}^0\left(E,[t_0X^{T(e)}/T]\right)_e} \simeq (\widehat{\mu}_e)_*\widehat{p}_*\mathcal{O}_{\widehat{\underline{\mathrm{Map}}}^0\left(E,[t_0X^{T(e)}/T]\right)}$$

Corollary 2.5.13 implies that we can identify $\widehat{p}_*\mathcal{O}_{\widehat{\underline{\mathrm{Map}}}^0\left(E,[t_0X^{T(e)}/T]\right)}$ with $\widehat{q}_*\mathcal{O}_{\widehat{\mathcal{L}}[t_0X^{T(e)}/T]}$. By plugging in this equivalence in the previous expression we obtain

$$(\widehat{p}_e)_*\mathcal{O}_{\widehat{\underline{\mathrm{Map}}}^0\left(E,[t_0X^{T(e)}/T]\right)_e} \simeq (\widehat{\mu}_e)_*\widehat{q}_*\mathcal{O}_{\widehat{\mathcal{L}}[t_0X^{T(e)}/T]}$$

as quasi-coherent sheaves on the formal completion $\widehat{E}_{T,e}$. In order to obtain the desired quasi-coherent sheaf on E_T we need to take the pushforward along $\widehat{i}_e$:

$$(\widehat{i}_e)_*\widehat{i}_e^*\mathcal{HH}_E([X/T]) \simeq (\widehat{i}_e)_*(\widehat{\mu}_e)_*\widehat{q}_*\mathcal{O}_{\widehat{\mathcal{L}}[t_0X^{T(e)}/T]}$$

Since the map $\widehat{i} : \widehat{E}_T \to E_T$ factors as $\widehat{\mu}_e : \widehat{E}_T \to \widehat{E}_{T,e}$ followed by $\widehat{i}_e : \widehat{E}_{T,e} \to E_T$, we can rewrite the previous formula as

$$(\widehat{i}_e)_*\widehat{i}_e^*\mathcal{HH}_E([X/T]) \simeq (\widehat{i}_e)_*(\widehat{\mu}_e)_*\widehat{q}_*\mathcal{O}_{\widehat{\mathcal{L}}[t_0X^{T(e)}/T]} \simeq \widehat{i}_*\widehat{q}_*\mathcal{O}_{\widehat{\mathcal{L}}[t_0X^{T(e)}/T]}$$

and this completes the proof. □

Theorem 2.5.14 will play a major role in the proof of a general comparison theorem between a periodic cyclic version of elliptic Hochschild homology and equivariant elliptic cohomology in the sense of Grojnowski. This is the content of Section 2.6.

Remark 2.5.15. *Using equation* (15) *we can write*

$$(\widehat{i}_e)_*\widehat{i}_e^*\mathcal{HH}_E([X/T]) \simeq \widehat{i}_*\widehat{q}_*\mathcal{O}_{\widehat{\mathcal{L}}[t_0X^{T(e)}/T]} \simeq \widehat{i}_*\widehat{j}^*q_*\mathcal{O}_{\mathcal{L}[t_0X^{T(e)}/T]}$$

where $q : \mathcal{L}[X/T] \to T$ *is the natural map.*

We denote $q_*\mathcal{O}_{\mathcal{L}[t_0X^{T(e)}/T]}$ as $\mathcal{HH}([X/T])$. The global sections of this sheaf (over $T \cong \operatorname{Spec} \mathrm{HH}([\operatorname{Spec} k/T])$) are given by $\mathrm{HH}([X/T])$. Call $\widehat{k} : \widehat{\mathfrak{t}} \to \mathfrak{t}$ the map from the completion at 0 of the Lie algebra of T to the Lie algebra of T, and by $\mathcal{H}([X/T])$ the quasi-coherent sheaf on $\mathfrak{t} \cong \operatorname{Spec} \mathrm{H}_T(*)$ having as global sections the $\mathbb{Z}_2$-periodized T-equivariant cohomology of X^{an}, $\mathrm{H}_T^{\oplus,*}(X^{\mathrm{an}})$. Then we have

$$(\widehat{i}_e)_*\widehat{i}_e^*\mathcal{HH}_E([X/T]) \simeq \widehat{i}_*\widehat{q}_*\mathcal{O}_{\widehat{\mathcal{L}}[t_0X^{T(e)}/T]} \simeq \widehat{i}_*\widehat{j}^*\mathcal{HH}([t_0X^{T(e)}/T])$$
$$(\widehat{i}_e)_*\widehat{i}_e^*\mathcal{E}ll_T(X^{\mathrm{an}}) \simeq \widehat{i}_*\widehat{k}^*\mathcal{H}([t_0(X^{\mathrm{an}})^{T(e)}/T])$$

as the completions of equivariant elliptic cohomology over E_T compute Borel equivariant cohomology. If we replace Hochschild homology by periodic cyclic homology by taking Tate fixed points with respect to the canonical S^1-action, the two completions become equivalent by means of Chen's Theorem 4.3.2 in [**17**].

In the next section we explain how the natural E-action on $\underline{\mathrm{Map}}^0(E,[X/T])$ induces S^1-actions on the adelic descent data of elliptic Hochschild homology, and use this action to define the *elliptic periodic cyclic homology* of $[X/T]$. We will show that this object recovers Grojnowski's equivariant elliptic cohomology of the analytification.

2.6. The adelic Tate construction

2.6.1. The action of the elliptic curve E. Let $\mathcal{X}$ be a derived stack over k. The multiplication map $\mu : E \times E \to E$ induces a global E-action on $\underline{\mathrm{Map}}^0(E,\mathcal{X})$.

Remark 2.6.1. *Let X be a variety over a field k of characteristic zero. As explained in* [**8**] *the unit map*

$$E \to \mathrm{Aff}(E)$$

is a group homomorphism. This implies that the canonical equivalence

$$\mathcal{L}X \simeq \underline{\mathrm{Map}}^0(\mathrm{Aff}(E),X) \to \underline{\mathrm{Map}}^0(E,X)$$

intertwines the E-action on the mapping stack on the right with the $\mathrm{Aff}(E) \simeq \mathrm{Aff}(S^1) \simeq B\mathbb{G}_a$*-action on the mapping stack on the left, i.e. the HKR isomorphism is equivariant with respect to the relevant actions.*

In the following lemma we characterize the E-action on the stack of quasi-constant maps from E to BT.

LEMMA 2.6.2. *The E-action on $\underline{\mathrm{Map}}^0(E, BT)$ induces a trivial action on the coarse moduli space E_T.*

PROOF. Without loss of generality, let us restrict to the case when T has rank 1. The triviality of the action on the coarse moduli space is a consequence of the fact that degree zero line bundles on elliptic curves can be presented as maps of abelian groups from E to $B\mathbb{G}_m$. In particular, consider the following maps:

- the action map
$$E \times \underline{\mathrm{Pic}}^0(E) \to \underline{\mathrm{Pic}}^0(E)$$
that is adjoint to the map
$$\underline{\mathrm{Map}}^0(E, B\mathbb{G}_m) \to \underline{\mathrm{Map}}^0(E \times E, B\mathbb{G}_m)$$
induced by composition with the multiplication map $\mu : E \times E \to E$;
- the map that classifies the box product of line bundles
$$E \times \underline{\mathrm{Pic}}^0(E) \to \underline{\mathrm{Pic}}^0(E)$$

The latter is constructed by adjunction from the map
$$\underline{\mathrm{Map}}^0(E, B\mathbb{G}_m) \to \underline{\mathrm{Map}}^0(E \times E, B\mathbb{G}_m)$$
which is obtained from the following composition:
$$E \times \underline{\mathrm{Map}}^0(E, B\mathbb{G}_m) \times E \times \underline{\mathrm{Map}}^0(E, B\mathbb{G}_m) \xrightarrow{\mathrm{ev}\times\mathrm{ev}} B\mathbb{G}_m \times B\mathbb{G}_m \xrightarrow{m} B\mathbb{G}_m$$
In the above, ev is the evaluation map, and $m : B\mathbb{G}_m \times B\mathbb{G}_m \to B\mathbb{G}_m$ is the multiplication on $B\mathbb{G}_m$. Adjoining this map, we obtain
$$\underline{\mathrm{Map}}^0(E, B\mathbb{G}_m) \times \underline{\mathrm{Map}}^0(E, B\mathbb{G}_m) \to \underline{\mathrm{Map}}^0(E \times E, B\mathbb{G}_m)$$
We further compose the above map with the diagonal Δ:
$$\underline{\mathrm{Map}}^0(E, B\mathbb{G}_m) \xrightarrow{\Delta} \underline{\mathrm{Map}}^0(E, B\mathbb{G}_m) \times \underline{\mathrm{Map}}^0(E, B\mathbb{G}_m) \to \underline{\mathrm{Map}}^0(E \times E, B\mathbb{G}_m)$$
to obtain the map classifying the box product. This map is equivalent to the projection to $\underline{\mathrm{Pic}}^0(E)$
$$E \times \underline{\mathrm{Pic}}^0(E) \to \underline{\mathrm{Pic}}^0(E)$$
which corresponds to the trivial action map.

The condition that degree zero line bundles on elliptic curves correspond to maps of abelian groups from E to $B\mathbb{G}_m$ implies that the maps on the coarse moduli spaces induced by the action map and the map classifying the box product are isomorphic. $\square$

REMARK 2.6.3. *Since the E-action on $\underline{\mathrm{Map}}^0(E,\mathcal{X})$ is induced by the group structure on E, for any map of derived stacks*

$$f:\mathcal{X}\to\mathcal{Y}$$

the map induced by composition

$$\underline{\mathrm{Map}}^0(E,\mathcal{X})\to\underline{\mathrm{Map}}^0(E,\mathcal{Y})$$

is E-equivariant. In particular, the structure map

$$p:\underline{\mathrm{Map}}^0(E,[X/T])\to E_T$$

is E-equivariant. In the above, E acts trivially on E_T as this is the coarse moduli space of $\underline{\mathrm{Pic}}^0(E)_T$, on which the action is trivial by Lemma 2.6.2.

2.6.2. Adelic Tate construction for elliptic cohomology. We now describe how the global E-action allows us to perform a Tate construction on the sheaf $\mathcal{HH}_E([X/T])$.

Consider $\pi: E\times E_T\to E_T$ as a group scheme over E_T; this object acts on $\underline{\mathrm{Map}}^0(E,[X/T])$ in the category of derived stacks over E_T.

REMARK 2.6.4. *The pushforward $\pi_*\mathcal{O}_{E\times E_T}$ is a sheaf of Hopf algebras on E_T. The action relative to E_T gives to $p_*\mathcal{O}_{\underline{\mathrm{Map}}^0(E,[X/T])}=\mathcal{HH}_E([X/T])$ the structure of a comodule over this sheaf of Hopf algebras in* $\mathrm{QCoh}(E_T)$*.*

Over a field k of characteristic zero, the equivalence $\mathrm{Aff}(E)\simeq\mathrm{Aff}(S^1)$ induces an equivalence

$$\pi_*\mathcal{O}_{E\times E_T}\simeq\pi_*\mathcal{O}_{S^1\times E_T}$$

where $\pi: S^1\times E_T\to E_T$ is viewed as a group scheme over E_T. In particular, $\mathcal{HH}_E([X/T])$ receives a comodule structure over $\pi_*\mathcal{O}_{S^1\times E_T}$.

The elliptic curve E acts trivially on E_T itself according to lemma 2.6.2. This action induces an S^1-action on the category $\mathrm{QCoh}(E_T)$, by letting S^1 act trivially on E_T. In particular, by the discussion above, the E-action on $\underline{\mathrm{Map}}^0(E,[X/T])$ induces a lift of $\mathcal{HH}_E([X/T])$ to the S^1-invariant category $\mathrm{QCoh}(E_T)^{S^1}$.

DEFINITION 2.6.5. *The* elliptic periodic cyclic homology *of* $[X/T]$

$$\mathcal{HP}_E([X/T])$$

is the image of the pair given by $\mathcal{HH}_E([X/T])$ and its S^1-action in the category $\mathrm{QCoh}(E_T)_{\mathbb{Z}_2}$.

REMARK 2.6.6. *The E-action on E_T restricts to each term of its adelic decoposition* $\mathrm{Spec}\,\mathbb{A}^\bullet_{E_T}$*, and moreover the adelic descent data $\mathbb{A}^\bullet_E(\mathcal{HH}_E([X/T]))$ obtain an action of S^1 by similar arguments to those above. Let $\mathbb{A}^\bullet_E(\mathcal{HH}_E([X/T]))^{tS^1}$ be the* *cosimplicial object obtained by applying level-wise the Tate construction with respect to this S^1-action. In particular, we have the following equivalence:*

$$\mathbb{A}^\bullet_E(\mathcal{HP}_E([X/T])) \simeq \mathbb{A}^\bullet_E(\mathcal{HH}_E([X/T]))^{tS^1}$$

This behaviour justifies the name adelic *Tate constuction.*

Remark 2.6.7. *Under the assumption that X is a smooth variety over a field k, the elliptic periodic cyclic homology of $[X/T]$ is a $\mathbb{Z}_2$-periodic perfect complex on E_T. Indeed, by the discussion in section 2.5, the coherence of this complex is controlled by finite generation of $H_{T^{\mathrm{an}}}(X^{\mathrm{an}})$ as a $H_{T^{\mathrm{an}}}(*)$-module. As explained in the discussion in the proof of Corollary 4.3.21 in* [**17**]*, X being a quasi-compact algebraic space is sufficient for this finite generation requirement, as the T^{an}-equivariant cohomology can be computed by a double complex whose E_1-page is given by $H(X) \otimes H_{T^{\mathrm{an}}}(*)$. Moreover, as soon as $k = \mathbb{C}$ the analytification of X has the same homotopy type of a finite CW-complex, ensuring that Grojnowski's equivariant elliptic cohomology is also an object in the $\mathbb{Z}_2$-periodic category* $\mathrm{Perf}(E_T)_{\mathbb{Z}_2}$.

Remark 2.6.8. *On the other hand, by the same arguments as in Remark 2.6.7, the coherence of the complex $\mathcal{HH}_E([X/T])$ is controlled by finite generation of the Hochschild homology of X. In particular, a sufficient condition for the coherence of $\mathcal{HH}_E([X/T])$ is X be proper.*

2.6.2.1. *The rank one case.* Since the proof of the comparison theorem is an induction on the rank of the torus, we start by proving it for tori of rank 1.

Proposition 2.6.9 (Comparison theorem, rank one case). *Let $k = \mathbb{C}$. Let T be an algebraic torus of rank* 1 *acting on a smooth variety X. We have an isomorphism of $\mathbb{Z}_2$-periodic coherent sheaves on E*

$$\mathcal{HP}_E([X/T]) \simeq \mathcal{E}ll_{T^{\mathrm{an}}}(X^{\mathrm{an}})$$

where $\mathcal{E}ll_{T^{\mathrm{an}}}(X^{\mathrm{an}})$ denotes complexified equivariant elliptic cohomology of the analytification of X. Moreover, this equivalence is natural with respect to X.

Proof. The proof of this theorem is based on adelic descent in dimension one.

Let us start with the adèles with respect to closed points e of E. In this case, the adèles are given by completion at such points. In particular, Theorem 2.5.14 gives us the desired equivalence. The adèle given by completion at the generic point corresponds to the generic fiber. The equivalence of such adèles comes from observing that the generic fiber can be computed via the localization theorem. Let $c : E \to \operatorname{Spec} k$ be the structure morphism. Then we have a canonical isomorphism

$$\tilde{j}_\eta \mathcal{HP}_E([X/T]) \simeq \tilde{j}_\eta c^* \mathrm{HP}(t_0 X^T)$$

for elliptic Hochschild homology, and similarly

$$\tilde{j}_\eta \mathcal{E}ll_{T^{\mathrm{an}}}(X^{\mathrm{an}}) \simeq \tilde{j}_\eta c^* C_{dR}^{\oplus,*}((t_0 X^T)^{\mathrm{an}})$$

for equivariant elliptic cohomology. Indeed, the localization theorem gives us a canonical equivalence

$$\tilde{j}_\eta \mathcal{HP}([X/T]) \simeq \tilde{j}_\eta \mathcal{HP}([t_0 X^{T(\eta)}/T])$$

and since $T(\eta) = T$, T acts trivially on $t_0 X^T$ and we have canonical equivalences

$$\tilde{j}_\eta \mathcal{HP}([t_0 X^{T(\eta)}/T]) \simeq \tilde{j}_\eta c^* \mathcal{HP}([t_0 X^{T(\eta)}/T'(\eta)]) \simeq \tilde{j}_\eta c^* \mathrm{HP}(t_0 X^T)$$

The HKR theorem, as in Proposition 4.4 of [**8**] induces an equivalence

$$c^* \mathrm{HP}(t_0 X^T) \simeq c^* C_{dR}^{\oplus,*}((t_0 X^T)^{\mathrm{an}})$$

which in turn gives an equivalence of the adèles with respect to the generic point.

Similarly, for a chain $\Delta = (\eta, x)$ where η is the generic point and x is a closed point, note that

$$\mathbb{A}_E(\Delta, \mathcal{F}) = \mathbb{A}_E(x, \tilde{j}_\eta \mathcal{F})$$

for a perfect complex $\mathcal{F}$. The same HKR theorem induces a canonical equivalence

$$\tilde{j}_\eta \mathcal{HP}([X/T]) \simeq \tilde{j}_\eta \mathcal{E}ll_{T^{\mathrm{an}}}(X^{\mathrm{an}})$$

and thus we have that

$$\mathbb{A}_E(\Delta, \mathcal{HP}_E([X/T])) \simeq \mathbb{A}_E\left(\Delta, \mathcal{E}ll_{T^{\mathrm{an}}}(X^{\mathrm{an}})\right)$$

Since this equivalence is induced by a canonical isomorphism of perfect complexes, it is compatible with the coface map corresponding to removing the point x from the chain Δ. Compatibility with the coface map induced by removing the point η has to be tested separately. In particular, we need to check that the following diagram commutes:

$$\begin{tikzcd}
\mathbb{A}_E\left((x), \mathcal{HP}_E([X/T])\right) \arrow[r] \arrow[d] & \mathbb{A}_E((x), \mathcal{E}ll_{T^{\mathrm{an}}}(X^{\mathrm{an}})) \arrow[d] \\
\mathbb{A}_E(\Delta, \mathcal{HP}_E([X/T])) \arrow[r] & \mathbb{A}_E(\Delta, \mathcal{E}ll_{T^{\mathrm{an}}}(X^{\mathrm{an}}))
\end{tikzcd}$$

We can rewrite the above diagram as

$$\begin{tikzcd}
\mathcal{HP}_E([X/T])_{\widehat{x}} \arrow[r] \arrow[d] & \mathcal{E}ll_{T^{\mathrm{an}}}(X^{\mathrm{an}})_{\widehat{x}} \arrow[d] \\
\mathrm{HP}(t_0(X^T)) \otimes_k \mathrm{Frac}\mathcal{O}_{E,\widehat{x}} \arrow[r] & C_{dR}^{\oplus,*}(t_0(X^T)^{\mathrm{an}}) \otimes_k \mathrm{Frac}\mathcal{O}_{E,\widehat{x}}
\end{tikzcd}$$

since the adèles with respect to the chain Δ correspond to

$$\left(\tilde{j}_\eta \mathcal{HP}_E([X/T])\right)_{\widehat{x}} \simeq \left(\mathrm{HP}(t_0(X^T)) \otimes_k k(E)\right) \otimes_{\mathcal{O}_{E,x}} \mathcal{O}_{E,\widehat{x}} \simeq \mathrm{HP}(t_0(X^T)) \otimes_k \mathrm{Frac}\mathcal{O}_{E,\widehat{x}}$$

Commutativity of this diagram then follows from the compatibility of the HKR isomorphisms as in Proposition 4.4 of [**8**] and Theorem 4.3.2 of [**17**] with pullbacks and with the base change to $k(E)$. Indeed, we can rewrite $\mathrm{HP}(t_0(X^T)) \otimes_k \mathrm{Frac}\mathcal{O}_{E,\widehat{x}}$ as

$$\mathrm{HP}(t_0(X^T)) \otimes_k \mathrm{Frac}\mathcal{O}_{E,\widehat{x}} \simeq \mathrm{HP}([t_0(X^T)/T])_{\widehat{1}} \otimes_{\mathcal{O}_{E,x}} k(E)$$

and $C^{\oplus,*}_{dR}(t_0(X^T)^{\mathrm{an}}) \otimes_k \mathrm{Frac}\mathcal{O}_{E,\widehat{x}}$ as

$$C^{\oplus,*}_{dR}(t_0(X^T)^{\mathrm{an}}) \otimes_k \mathrm{Frac}\mathcal{O}_{E,\widehat{x}} \simeq C^{\prod,*}_{dR}(t_0(X^T)^{\mathrm{an}}) \otimes_{\mathcal{O}_{E,x}} k(E)$$

We now show the naturality of the equivalence with respect to X. We need to show that if $f : Y \to X$ is a map, the following diagram commutes

$$\begin{tikzcd}
\mathcal{HP}_E([X/T]) \arrow[r] \arrow[d] & \mathcal{E}ll_{T^{\mathrm{an}}}(X^{\mathrm{an}}) \arrow[d] \\
\mathcal{HP}_E([Y/T]) \arrow[r] & \mathcal{E}ll_{T^{\mathrm{an}}}(Y^{\mathrm{an}})
\end{tikzcd}$$

The corresponding diagram of adèles commutes as the vertical maps are given by pullback in periodic cyclic homology (for the left arrow) and de Rham cohomology (for the right arrow); commutativity follows specifically from the compatibility of the HKR theorem with pullbacks, both in its form as Proposition 4.4 in [**8**] and as Theorem 4.3.2 in [**17**]. □

2.6.2.2. *The general case.* We proceed with the proof of the main result of this Section. As already anticipated, the general case follows inductively from the rank 1 case.

THEOREM 2.6.10 (Comparison theorem). *Let $k = \mathbb{C}$. Let T be an algebraic torus of rank n acting on a smooth variety X. There is an isomorphism of $\mathbb{Z}_2$-periodic coherent sheaves on E*

$$\mathcal{HP}_E([X/T]) \simeq \mathcal{E}ll_{T^{\mathrm{an}}}(X^{\mathrm{an}})$$

where $\mathcal{E}ll_{T^{\mathrm{an}}}(X^{\mathrm{an}})$ denotes the complexified equivariant elliptic cohomology of the analytification of X. Moreover, this equivalence is natural with respect to X.

PROOF. We reason by induction, as anticipated.

Our inductive hypothesis gives us an equivalence

$$\mathcal{HP}_E([X/K]) \simeq \mathcal{E}ll_{K^{\mathrm{an}}}(X^{\mathrm{an}})$$

for any torus K of rank strictly smaller than n. Further, this equivalence is natural with respect to X. The base case when $\mathrm{rk}(T) = 1$ was proved as Proposition 2.6.9. Now let T be an algebraic torus of tank n. We will produce an equivalence

$$\mathcal{HP}_E([X/T]) \simeq \mathcal{E}ll_{T^{\mathrm{an}}}(X^{\mathrm{an}})$$

natural with respect to X. As in the rank one case, we use adelic descent. In the case of closed points $e \in E_T$, the equivalence

$$\mathbb{A}_{E_T}\left((e), \mathcal{HP}_E([X/T])\right) \simeq \mathbb{A}_{E_T}\left((e), \mathcal{E}ll_{T^{\mathrm{an}}}(X^{\mathrm{an}})\right)$$

is Theorem 2.5.14 together with Theorem 4.3.2 in [**17**]. Now let $\Delta = (x > x_1 > \cdots > x_k) \in |E_T|_k$ be a chain of length $k > 1$ on E_T, and let $\Delta' = (x_1 > \cdots > x_k)$. By definition we have

$$\mathbb{A}_{E_T}\left(\Delta, \mathcal{HP}_E([X/T])\right) \simeq \lim_{r \geqslant 0} \mathbb{A}_{E_T}\left(\Delta', \tilde{j}_{rx}\mathcal{HP}_E([X/T])\right)$$

The localization theorem provides an equivalence

$$\tilde{j}_{rx}\mathcal{HP}_E([X/T]) \simeq \tilde{j}_{rx}c_x^*\mathcal{HP}_E([t_o X^{T(x)}/T'(x)])$$

By the inductive hypothesis there is an equivalence

$$\mathcal{HP}_E([t_o X^{T(x)}/T'(x)]) \simeq \mathcal{E}ll_{T'(x)^{\mathrm{an}}}(t_0(X^{T(x)})^{\mathrm{an}})$$

Indeed x is not a closed point, hence the rank of $T'(x)$ is necessarily smaller than n. Finally, this equivalence of quasi-coherent complexes induces an equivalence

$$\lim_{r \geqslant 0} \mathbb{A}_{E_T}\left(\Delta', \tilde{j}_{rx}c_x^*\mathcal{HP}_E([X/T])\right) \simeq \lim_{r \geqslant 0} \mathbb{A}_{E_T}\left(\Delta', \tilde{j}_{rx}c_x^*\mathcal{E}ll_{T'(x)^{\mathrm{an}}}(t_0(X^{T(x)})^{\mathrm{an}})\right)$$

which, by the computation carried out above, means that we obtain by composition a canonical isomorphism

$$\phi_\Delta : \mathbb{A}_{E_T}\left(\Delta, \mathcal{HP}_E([X/T])\right) \simeq \mathbb{A}_{E_T}\left(\Delta, \mathcal{E}ll_{T^{\mathrm{an}}}(X^{\mathrm{an}})\right)$$

All the equivalences between adelic groups produced by the above argument come from equivalences of objects in $\mathrm{Perf}(E_T)_{\mathbb{Z}_2}$, thus they are all compatible with the coface maps induced by the operation of removing a point x_i from the chain Δ, for $i \in \{1, \ldots, k\}$. The coface map induced by removing x has to be treated separately: in this case we cannot reduce to the sheaves $\tilde{j}_{rx}\mathcal{HP}_E([X/T])$ for one of the two adelic groups involved. Indeed, by definition, the adèle $\mathbb{A}_{E_T}\left(\Delta', \mathcal{HP}_E([X/T])\right)$ is a limit of adèles of the sheaves $\tilde{j}_{rx_1}\mathcal{HP}_E([X/T])$ rather than $\tilde{j}_{rx}\mathcal{HP}_E([X/T])$.

Let us consider now the case of the coface map induced by removing the point x from the chain Δ. For this specific case, we switch to the complexes of adèles as opposed to their global sections. We need to check the commutativity of the following diagram:

$$\begin{tikzcd}
\mathsf{A}_{E_T}(\Delta', \mathcal{HP}_E([X/T])) \arrow[r] \arrow[d] & \mathsf{A}_{E_T}(\Delta', \mathcal{E}ll_{T^{\mathrm{an}}}(X^{\mathrm{an}})) \arrow[d] \\
\mathsf{A}_{E_T}(\Delta, \mathcal{HP}_E([X/T])) \arrow[r] & \mathsf{A}_{E_T}(\Delta, \mathcal{E}ll_{T^{\mathrm{an}}}(X^{\mathrm{an}}))
\end{tikzcd}$$

where

- Δ' is obtained from Δ by removing x
- the horizontal arrows are the isomorphisms obtained above via the inductive hypothesis
- the vertical arrows are the coface maps

For the sheaves of adèles we have a decomposition

$$\mathsf{A}_X(\Delta, \mathcal{F}) \simeq \mathcal{F} \otimes_{\mathcal{O}_X} \mathsf{A}_X(\Delta, \mathcal{O}_X)$$

which allows us to write the above diagram as

$$\begin{tikzcd}
\mathsf{A}_{E_T}(\Delta', \mathcal{HP}_E([X/T])) \arrow[r, "\simeq"] \arrow[d, "\simeq"] & \mathsf{A}_{E_T}(\Delta', \mathcal{E}ll_{T^{\mathrm{an}}}(X^{\mathrm{an}})) \arrow[d, "\simeq"] \\
c_{x_1}^* \mathcal{HP}_E([t_0(X^{T(x_1)})/T'(x_1)]) \otimes_{\mathcal{O}_X} \mathsf{A}_X(\Delta', \mathcal{O}_X) \arrow[r, "\simeq"] \arrow[d] & c_{x_1}^* \mathcal{E}ll_{T'(x_1)^{\mathrm{an}}}(t_0(X^{T(x_1)})^{\mathrm{an}}) \otimes_{\mathcal{O}_X} \mathsf{A}_X(\Delta', \mathcal{O}_X) \arrow[d] \\
c_x^* \mathcal{HP}_E([t_0(X^{T(x)})/T'(x)]) \otimes_{\mathcal{O}_X} \mathsf{A}_X(\Delta, \mathcal{O}_X) \arrow[r, "\simeq"] \arrow[d] & c_x^* \mathcal{E}ll_{T'(x)^{\mathrm{an}}}(t_0(X^{T(x)})^{\mathrm{an}}) \otimes_{\mathcal{O}_X} \mathsf{A}_X(\Delta, \mathcal{O}_X) \arrow[d] \\
\mathsf{A}_{E_T}(\Delta, \mathcal{HP}_E([X/T])) \arrow[r, "\simeq"] & \mathsf{A}_{E_T}(\Delta, \mathcal{E}ll_{T^{\mathrm{an}}}(X^{\mathrm{an}}))
\end{tikzcd}$$

Since $x > x_1$, c_x factors as the composition

$$c_x : E_T \xrightarrow{c_{x_1}} E_{T'(x_1)} \xrightarrow{c_{x_1,x}} E_{T'(x)}$$

In particular, the two middle vertical maps in the diagram factor as the tensor product of a pullback map along the inclusion $t_0(X^{T(x)}) \to t_0(X^{T(x_1)})$ and the coface map for the adèles of the structure sheaf. If $k > 1$ or $k = 1$ and x_1 is not closed, the bottom and top squares commute. By inductive hypothesis on naturality, the middle square also commutes, and thus we obtain the desired equivalence.

If $k = 1$ and x_1 is closed, $T'(x_1)$ might have the same rank as T. In this situation, we apply Proposition 3.2.1 in [**35**]:

$$\mathbb{A}_{E_T}((x_0, x_1), \mathcal{HP}_E([X/T])) = C_{x_0} S_{x_0}^{-1} C_{x_1} S_{x_1}^{-1} \mathcal{HP}_E([X/T])$$

where S_p^{-1} is localization at p and C_p is the functor that sends a quasi-coherent complex $M = \mathrm{colim}_i N_i$, with N_i perfect complex, to $\mathrm{colim}_i N_{i,\widehat{p}}$. The coface map is the natural map

$$C_{x_1} S_{x_1}^{-1} \mathcal{HP}_E([X/T]) \to C_{x_0} S_{x_0}^{-1} C_{x_1} S_{x_1}^{-1} \mathcal{HP}_E([X/T])$$

and similarly for elliptic cohomology. The inductive hypothesis and localization identify the isomorphism $C_{x_1} S_{x_1}^{-1} \mathcal{HP}_E([X/T]) \simeq C_{x_1} S_{x_1}^{-1} \mathcal{E}ll_{T^{\mathrm{an}}}(X^{\mathrm{an}})$ with the HKR isomorphism (together with Theorem 2.5.14), which implies that the relevant diagram commutes.

To finish the argument, we need to prove naturality. This follows from the inductive hypothesis in the following way: choose $f : Y \to X$ and consider the diagram of adèles associated to naturality, for a chain $\Delta \in |E_T|_k$:

$$\begin{tikzcd} \mathbb{A}_{E_T}(\Delta, \mathcal{HP}_E([X/T])) \arrow[r] \arrow[d] & \mathbb{A}_{E_T}(\Delta, \mathcal{E}ll_{T^{\mathrm{an}}}(X^{\mathrm{an}})) \arrow[d] \\ \mathbb{A}_{E_T}(\Delta, \mathcal{HP}_E([Y/T])) \arrow[r] & \mathbb{A}_{E_T}(\Delta, \mathcal{E}ll_{T^{\mathrm{an}}}(Y^{\mathrm{an}})) \end{tikzcd}$$

By localization, the above diagram is obtained by applying the functor $\mathbb{A}_{E_T}(\Delta, -)$ to the diagram

$$\begin{tikzcd} c_x^*\mathcal{HP}_E([t_0X^{T(x)}/T'(x)]) \arrow[r] \arrow[d] & c_x^*\mathcal{E}ll_{T'(x)^{\mathrm{an}}}((t_0X^{T(x)})^{\mathrm{an}}) \arrow[d] \\ c_x^*\mathcal{HP}_E([t_0Y^{T(x)}/T'(x)]) \arrow[r] & c_x^*\mathcal{E}ll_{T'(x)^{\mathrm{an}}}((t_0Y^{T(x)})^{\mathrm{an}}) \end{tikzcd}$$

which commutes by the inductive hypothesis. □

From Theorem 2.6.10 we deduce the following corollary:

Corollary 2.6.11. *Let $k = \mathbb{C}$. Let T be an algebraic torus of rank n acting on a smooth variety X. We have an isomorphism of $\mathbb{Z}_2$-periodic coherent sheaves on E*

$$\pi_*\mathcal{HP}_E([X/T]) \simeq \mathcal{E}ll^*_{T^{\mathrm{an}}}(X^{\mathrm{an}})$$

where $\mathcal{E}ll^*_{T^{\mathrm{an}}}(X^{\mathrm{an}})$ denotes the collection of homotopy sheaves of complexified equivariant elliptic cohomology of the analytification of X, i.e. the classical version of equivariant elliptic cohomology due to Grojnowski. Moreover, this equivalence is natural with respect to X.

Remark 2.6.12. *We expect elliptic Hochschild homology to encode 2-categorical information on the stack $[X/T]$. This is the most exciting future direction of our work, as it could shed light on the much studied problem of constructing geometric representatives of elliptic cocycles. As a reality check we remark that, in contrast with ordinary Hochschild homology, $\mathcal{HH}_E([X/T])$ and $\mathcal{HP}_E([X/T])$ are* not *invariants of* $\mathrm{Perf}([X/T])$*. This follows immediately from Theorem 2.6.10 and the main result of* [**62**]*. The category* $\mathrm{Perf}([X/T])$ *can be viewed as the universal recipient of 1-categorical information on $[X/T]$. This confirms the expectation that elliptic Hochschild homology detects information which is not 1-categorical in nature.*

CHAPTER 3

Adelic decomposition for Equivariant Elliptic Cohomology

3.1. Introduction

A powerful method to study complexified equivariant cohomology theories is to regard them as *quasi-coherent* sheaves of algebras on the decompletion of the formal group associated to their non-equivariant incarnation. Based on this principle, Grojnowski proposed a first construction of complexified equivariant elliptic cohomology in [**30**]. At around the same time Ginzburg–Kapranov–Vasserot gave an axiomatic description of equivariant elliptic cohomology in [**25**]. The details of Grojnowski's construction were worked out by Roşu in [**60**]. Roşu applies similar methods in [**61**] to build complexified equivariant K-theory. The upshot is that elliptic cohomology and K-theory can be constructed complex-analytically from the singular cohomology of fixed point loci. In this same paper, Roşu states that such a description of complexified equivariant K-theory can also be obtained algebraically, via completions. Clearly, a byproduct of this construction would be a lift from the complexification to rationalization with respect to a general field of characteristic zero. Unfortunately, Roşu never completed this program. This is one of our main contributions in this paper.

Our goal in this paper is to obtain a purely algebraic description of equivariant K-theory and of Grojnowski's equivariant elliptic cohomology via adelic methods. In the last forty years elliptic cohomology has been intensely studied for its importance in homotopy theory — for example Greenlees' approach via his algebraic model for rational G-spectra [**27**] and Ganter's [**23**] — and its relevance in mathematical physics and the theory of representations of loop groups (see Witten's [**81, 82**] for the relationship with string theory and Dirac operators on loop spaces, or Ando's [**1**] that highlights the relationship with loop groups). Nevertheless, the original question by Roşu of a purely algebraic construction remains unanswered.

We will achieve this description via *adelic descent*. Adelic descent has its origins in algebraic number theory and algebraic geometry, as a tool to study curves. One of the earliest applications of adelic methods is a celebrated theorem of André Weil, that describes principal G-bundles on curves in terms of the adelic ring of the curve. This theory was generalized to n-dimensional Noetherian schemes by Parshin [**56**] and Beilinson [**11**]. Roughly, adelic descent yields a description of coherent sheaves on

Noetherian schemes in terms of their completion at chains of points on the scheme. A recent formulation is due to Groechenig [**28**], who proved an *adelic reconstruction theorem* in terms of an equivalence of ∞-categories of perfect complexes on a Noetherian scheme X and perfect complexes on the adelic decomposition of X.

Rationalized equivariant elliptic cohomology. In Section 3.3 we apply adelic descent to study G-equivariant elliptic cohomology for compact Lie groups G. Our approach is encoded in Definition 3.3.8 and Definition 3.3.17. The key case is when $G = T$ is a torus. Here the construction proceeds by induction on the rank of T. If $k = \mathbb{C}$ is the field of the complex numbers, our definition recovers Grojnowski's: this is the content of Corollary 3.3.16. Finally, we provide a chain-level presentation of Grojnowski's sheaf: this is the content of the remarks in Subsection 3.3.4. This presentation uses the same inductive construction of Definition 3.3.8 and rests on the formality of the algebra $C^*(BT)$ of cochains on the classifying space of T.

Adelic descent for equivariant cohomology and K-theory. In Section 3.4 we complete Roşu's program by giving a description of rationalized equivariant K-theory in terms of adelic descent data. This shows that rationalized equivariant K-theory can be constructed in a purely algebraic manner out of the singular cohomology of fixed loci, thus implementing Roşu's proposal. Our construction has several advantages, with respect to Roşu's original paper [**61**]. First, it works over all fields of characteristic zero, rather then just over the complex numbers, and it recovers equivariant K-theory directly, rather than its extension by the holomorphic functions over the complexification of the torus.

We perform similar computations also for equivariant cohomology. In particular, this shows that elliptic cohomology, K-theory and singular cohomology can ultimately be built out of some basic local data — always expressed in terms of Borel-equivariant singular cohomology of fixed loci — and an induction based on localization theorems. This implements a principle that first appears in celebrated work of Atiyah–Segal on equivariant K-theory.

Comparisons with periodic cyclic counterparts. In Section 3.5 we compare rationalized equivariant cohomology and K-theory with *periodic cyclic homology theories* constructed respectively from the *shifted tangent bundle* and the *derived loop space* of a quotient stack. The case of rationalized equivariant K-theory was considered by Halpern-Leistner–Pomerleano in [**31**]. There, they establish an equivalence between the periodic cyclic homology of well-behaved quotient stacks $[X/G]$ over $\mathbb{C}$ and the the G-equivariant topological K-theory of the analytification of X. Here we use adelic descent to prove the following theorem:

THEOREM A (Theorem 3.5.2). *Let X be a smooth variety over $\mathbb{C}$ acted on by an algebraic torus T. There is an equivalence of $\mathbb{Z}_2$-periodic coherent sheaves on T*

$$\pi_* \mathcal{HP}([X/T]) \simeq \mathcal{K}_{T^{\mathrm{an}}}(X^{\mathrm{an}})$$

which is natural in X with respect to T-equivariant maps.

Our approach differs in several respects from the one of [**31**]. First, we obtain a more general theorem, as we drop technical assumptions such as quasi-projectivity and the existence of a semi-complete KN stratification. It is true that in the present iteration we treat only torus actions, whereas actions by general algebraic groups are considered in [**31**]. Our Theorem A however can be extended to all reductive groups without difficulty, by keeping track of the action of the Weyl group, this extension will appear in a future version of this paper. Secondly, our methods are much more elementary, as they only depend on localization in topological K-theory. We hope that our result might be used to provide more examples in which the *lattice conjecture* holds, i.e. the equivalence between the complexification of Blanc's topological K-theory of a dg category and its periodic cyclic homology. We will return to this in future work.

Halpern-Leistner and Pomerleano remark that their theorem follows from an identification at the level of cochains. Theorem A has a similar lift, Theorem 3.5.7, which requires a cochain model for equivariant K-theory. Such a model can be constructed again using adelic descent with the same arguments made in the elliptic setting in Section 3.3.

We remark that similar comparison results are known in differential geometry since the 90's. In this context, the equivariant K-theory of a smooth manifold can be recovered from the periodic cyclic homology of the algebra of C^∞-functions on the manifold itself. Relevant references include [**14**] and [**13**].

A similar picture holds in the case of equivariant cohomology, where the *shifted tangent stack* takes on the role played by the loop space in the case of K-theory. This story is well-known, and follows immediately from work of Calaque–Pantev–Toën–Vaquié–Vezzosi [**16**]. We include it in the paper, as we give a different argument based on adelic methods. This allows to treat on the same footing equivariant elliptic cohomology, K-theory and singular cohomology.

THEOREM B (Theorem 3.5.14). *Let X be a smooth variety acted on by an algebraic torus T. There is an isomorphism of $\mathbb{Z}_2$-periodic perfect complexes on* $\mathfrak{t}$

$$\mathcal{HP}^{lin}([X/T]) \simeq \mathcal{H}_{T^{\mathrm{an}}}(X^{\mathrm{an}})$$

where the left-hand is linearized *periodic cyclic homology, and the right-hand are the equivariant singular cochains $C^{\oplus,*}_{T^{\mathrm{an}}}(X^{\mathrm{an}})$ viewed as a sheaf on* $\mathfrak{t} = \operatorname{Spec} H^{\oplus,0}_{T^{\mathrm{an}}}(*) \simeq \operatorname{Spec} C^{\oplus,0}_{T^{\mathrm{an}}}(*)$.

We call *linearized periodic cyclic homology* the Tate fixed points of the mixed structure on the de Rham complex of $[X/T]$. A necessary requirement for the proof via adelic descent is a *localization formula* for the shifted tangent stack, Proposition 3.5.13. To the best of the author's knowledge, this localization phenomenon is new and might be of independent interest. Localization for the loop space has been extensively studied in [**17**].

The case of equivariant elliptic cohomology has been investigated in [**67**], where the geometric object taking the role of the loop space and the shifted tangent stack is an appropriate derived stack of *quasi-constant* maps from an elliptic curve E over a $\mathbb{Q}$-algebra k. The appropriate notion of *elliptic periodic cyclic homology* is introduced, and proved to be equivalent to Grojnowski's equivariant elliptic cohomology when $k = \mathbb{C}$. Thus our results in this paper, together with [**67**], provide a unified treatement of elliptic cohomology, K-theory and singular cohomology and their comparisons with corresponding invariants defined via derived algebraic geometry.

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3.2. Preliminaries

For the preliminary section of this paper we refer to the Preliminaries section in the paper [**67**]. For the reader's convenience, we review some basic material.

3.2.1. Complexified Equivariant Elliptic Cohomology.

Complexified equivariant elliptic cohomology was axiomatically defined by Ginzburg–Kapranov–Vasserot in [**25**] and constructed by Grojnowski in [**30**]. We follow mostly the more recent exposition found in [**23**] and [**62**]. Other reviews closer in style to the original can be found in [**2**], [**60**] and [**27**]. We remark that Grojnowski's paper only sketches the construction, but the details were carried out by Roşu in [**60**].

Let X be a finite T-CW-complex, where T is real torus of rank n. Complex T-equivariant elliptic cohomology of X is defined by first constructing a coherent sheaf of $\mathbb{Z}_2$-graded algebras $\mathcal{E}ll_T^{\mathrm{an}}(X)$ over the complex manifold

$$E_T := E \otimes_{\mathbb{Z}} \check{T}$$

and then viewing it as an algebraic coherent sheaf via standard GAGA arguments, yielding

$$\mathcal{E}ll_T(X) \in \mathrm{Qcoh}(E_T)^{\mathbb{Z}_2}$$

The construction we review is as a $\mathbb{Z}_2$-periodic rather than $\mathbb{Z}_2$-graded coherent sheaf. The two constructions are completely equivalent.

Grojnowski's main insight is that, rationally, equivariant elliptic cohomology is locally given by equivariant cohomology of loci in X which are fixed by some subgroups of T indexed by points of E_T.

DEFINITION 3.2.1. *Let e be a point of E_T. Let $S(e)$ be the set of subtori $T' \subset T$ such that the closure of e, $\overline{\{e\}}$, belongs to $E_{T'} \subset E_T$. Then we define*

$$T(e) := \bigcap_{T' \in S(e)} T'$$

Let

$$H_T^{\oplus,*}(X) = \bigoplus_{i \in \mathbb{Z}} H_T^{*+2i}(X; \mathbb{C})$$

be the *sum*-$\mathbb{Z}_2$-periodization of T-equivariant singular cohomology of X, with complex coefficients. This is a module over the even T-equivariant cohomology of the point

$$H_T^{\oplus,0}(*) = \mathbb{C}[u_1, \dots, u_n]$$

or equivalently a quasi-coherent sheaf over $\operatorname{Spec} H_T^{\oplus,0}(*) \simeq \mathfrak{t}_{\mathbb{C}} \simeq \mathbb{A}^n_{\mathbb{C}}$, where $\mathfrak{t}_{\mathbb{C}}$ is the complexified Lie algebra of T. Let us call $\mathcal{H}_T(X)$ this quasi-coherent sheaf.

REMARK 3.2.2. *Under our assumptions, $\mathcal{H}_T(X)$ is a* coherent *sheaf on $\mathfrak{t}_{\mathbb{C}}$.*

We denote by $\mathcal{H}_T^{\mathrm{an}}(X)$ the analytification, i.e. the coherent sheaf

$$\mathcal{H}_T^{\mathrm{an}}(X) = \mathcal{H}_T(X) \otimes_{\mathcal{O}_{\mathfrak{t}_{\mathbb{C}}}} \mathcal{O}^{\mathrm{an}}_{\mathfrak{t}_{\mathbb{C}}}$$

There is a quotient map

$$\exp^2 : \mathfrak{t}_{\mathbb{C}} \to E_T$$

which is an isomorphism if restricted to sufficiently small analytic disks U in E_T. Let us call $\log^2$ the local inverse. Moreover, the group structure (we use multiplicative notation) on E_T induces translation maps

$$\begin{aligned} \tau_e : & E_T \to E_T \\ & f \mapsto fe \end{aligned}$$

for all closed points e in E_T. Then, for a closed point $e \in E_T$ and a sufficiently small analytic neighbourhood U_e of e, we set

$$\mathcal{E}ll_T^{\mathrm{an}}(X)|_{U_e} = (\tau_e \circ \exp^2)_* \mathcal{H}_T^{\mathrm{an}}(X^{T(e)})|_{\log^2(e^{-1}U_e)}$$

The algebraic coherent sheaf obtained from $\mathcal{E}ll_T^{\mathrm{an}}(X)$ via GAGA is denoted $\mathcal{E}ll_T(X)$.

The completions of Grojnowski's sheaf over closed points e of E_T can be expressed in terms of the *product*-$\mathbb{Z}_2$-periodization of T-equivariant singular cohomology:

$$\mathcal{E}ll_T(X)_{\hat{e}} = H_T^{\prod,*}(X^{T(e)}) \simeq H_T^{\oplus,*}(X^{T(e)}) \otimes_{\mathcal{O}(\mathfrak{t}_{\mathbb{C}})} \mathcal{O}_{E_T,\hat{e}}$$

where $\mathcal{O}_{E_T,\hat{e}}$ is a module over $\mathcal{O}(\mathfrak{t}_{\mathbb{C}})$ via the completed multiplication map $\hat{\mu}_e : E_{T,\hat{1}} \to E_{T,\hat{e}}$ and the identification $E_{T,\hat{1}} \simeq \mathfrak{t}_{\mathbb{C},\hat{0}}$.

3.2.2. Adelic descent. We now review adelic descent theory for n-dimensional schemes. This theory was first introduced by Parshin [**56**] and Beilinson [**11**]. A review of this theory can be found in [**35**] and [**50**]. Recently, Groechenig [**28**] made a very relevant contribution to the theory. His paper is the main reference for the short reminder that follows.

Let X be a Noetherian scheme. For two points x and y we say $x \geqslant y$ if $y \in \overline{\{x\}}$. We let $|X|_k$ denote the set of k-*chains* on X, i.e. sequences of $k+1$ ordered points $(x_0 \geqslant \cdots \geqslant x_k)$ in X. If $k = 0$, we equivalently write $|X| = |X|_0$. Finally, for a subset $T \subset |X|_k$, we call

$${}_xT := \{\Delta \in |X|_{k-1} | (x \geqslant \Delta) \in T\}$$

We are now ready to define sheaves of adèles on X for a choice of $T \subset |X|_k$. The adèles are the unique family of exact functors

$$\mathsf{A}_X(T, -) : \mathrm{QCoh}(X) \to \mathrm{Mod}_{\mathcal{O}_X}$$

such that:

- $\mathsf{A}_X(T, -)$ commutes with directed colimits;
- if $\mathcal{F}$ is coherent and $k = 0$, $\mathsf{A}_X(T, \mathcal{F}) = \prod_{x \in T} \lim_{r \geqslant 0} \tilde{j}_{rx}\mathcal{F}$;
- if $\mathcal{F}$ is coherent and $k > 0$, $\mathsf{A}_X(T, \mathcal{F}) = \prod_{x \in |X|} \lim_{r \geqslant 0} \mathsf{A}_X({}_xT, \tilde{j}_{rx}\mathcal{F})$.

The notation we use is borrowed from [**50**]. $\tilde{j}_{rx}$ denotes the functor $j_{rx\,*}j_{rx}^*$, where

$$j_{rx} : \operatorname{Spec} \mathcal{O}_{X,x}/\mathfrak{m}_x^r \to X$$

is the canonical immersion of an r-thickening of the point x. $\mathcal{O}_{X,x}$ is the local ring at x and $\mathfrak{m}_x$ its maximal ideal.

We denote by $\mathbb{A}_X(T, \mathcal{F})$ the global sections $\Gamma(X, \mathsf{A}_X(T, \mathcal{F}))$, and call them *the groups of adèles.*

The sets $|X|_k$ admit the structure of a simplicial set by defining face and degeneracy maps by the operations of removing a point from a chain or doubling one up respectively. Let this simplicial set be denoted by $|X|_\bullet$. This implies that the sheaves of adèles assemble into a cosimplicial sheaf of $\mathcal{O}_X$-modules $\mathsf{A}_X(T_\bullet, \mathcal{F})$, for some $T_\bullet \subset |X|_\bullet$. In the case $T_\bullet = |X|_\bullet$, this cosimplicial sheaf is denoted by $\mathsf{A}_X^\bullet(\mathcal{F})$, and its global sections by $\mathbb{A}_X^\bullet(\mathcal{F})$. If $\mathcal{F} = \mathcal{O}_X$, the notation we reserve is $\mathsf{A}_X^\bullet$ and $\mathbb{A}_X^\bullet$ respectively.

It is possible to consider the cosimplicial sheaf of "products of local adèles"

$$[n] \mapsto \prod_{\Delta \in |X|_n} \mathsf{A}_X(\Delta, \mathcal{F})$$

i.e. we choose $T = \{\Delta\}$ and then take the product over all chains Δ. The content of Theorem 2.4.1 in [**35**] is that the natural inclusion of the adèles into this product respects the cosimplicial structures.

Adelic descent theory allows to reconstruct sheaves from their *adelic descent data*, i.e. the data of the sheaves of adèles or their global sections. We state two theorems on adelic descent for Noetherian n-dimensional schemes. The first one is due to Groechenig and holds for perfect complexes in the context of small ∞-categories.

Theorem 3.2.3 (Theorem 3.1 in [**28**]). *Let X be a Noetherian scheme. Then adelic reconstruction is an equivalence of symmetric monoidal ∞-categories*

$$\mathrm{Perf}^{\otimes}(X) \simeq \mathrm{TotPerf}^{\otimes}(\mathbb{A}_X)$$

The following theorem due to Beilinson appears as Theorem 1.16 in [**28**]. This is a classical theorem for the abelian category of quasi-coherent sheaves.

Theorem 3.2.4 (Beilinson [**11**]). *Let $\mathcal{F}$ be a quasi-coherent sheaf on X. The augmentation $\mathcal{F} \to \mathsf{A}^{\bullet}_X(\mathcal{F})$ is a resolution of $\mathcal{F}$ by flasque $\mathcal{O}_X$-modules. In particular, the totalization of the adèles $\mathrm{Tot}\mathbb{A}^{\bullet}_X(\mathcal{F})$ computes the cohomology of $\mathcal{F}$.*

Remark 3.2.5. *The arguments made by Groechenig in* [**28**] *hold without variations in the context of $\mathbb{Z}_2$-periodic sheaves. This is the context we are interested in, as we are dealing with $\mathbb{Z}_2$-periodic cohomology theories.*

3.2.3. Shifted tangent bundles and loop spaces. In this subsection we recall two fundamental objects in derived algebraic geometry that will be used in the second part of this paper. For general references on derived algebraic geometry in the context of E_∞-rings, see Lurie's work [**46**], [**45**] and [**43**]. In the context of simplicial commutative rings/cdgas the theory has been developed by Toën and Vezzosi in [**75**] and [**76**]. For a short review, see the preliminaries section of [**67**].

Let X be a derived stack, and $\mathbb{L}_X$ be its cotangent complex.

Definition 3.2.6. *The* shifted tangent stack *of X is the derived stack*

$$T_X[-1] := \mathrm{Spec}_{\mathcal{O}_X} \mathrm{Sym}\, \mathbb{L}_X[1]$$

Definition 3.2.7. *The* derived loop space *of X is the stack*

$$\mathcal{L}X := \underline{\mathrm{Map}}\left(S^1, X\right)$$

In the above, $\underline{\mathrm{Map}}(-,-)$ denotes the mapping stack as derived stacks.

Remark 3.2.8. *Since there is an equivalence $S^1 \simeq * \coprod_{* \coprod *} *$, we have that*

$$\mathcal{L}X \simeq X \times_{X \times X} X$$

The algebra of (derived) global sections of the structure sheaf of S^1 is formal over a field of characteristic zero k. In particular, over k this algebra is isomorphic to $k[\epsilon]$, where ϵ is a variable in cohomological degree one. In particular, the cosimplicial spectrum in the sense of [**73**], $\operatorname{Spec} k[\epsilon] = \operatorname{Spec} \mathcal{O}(S^1)$, is the *affinization* of the circle S^1 (see also [**51**] for more on affinization and affine stacks). This allows us to introduce a third object:

Definition 3.2.9. *The* unipotent loop space *of X is the stack*

$$\mathcal{L}^u X := \underline{\operatorname{Map}}\,(\operatorname{Spec} k[\epsilon], X)$$

If X is a scheme over k, Ben-Zvi and Nadler establish a Zariski codescent result for the loop space, Lemma 4.2 in [**8**], which has the consequence that for schemes the three objects described above coincide (Proposition 4.4 in [**8**]), as they do for affine schemes. In particular, taking global sections gives a form of the HKR theorem, as the global sections of the loop space compute Hochschild homology and the global sections of the shifted tangent bundle compute the de Rham complex of X.

We will be interested in the shifted tangent bundle and in the derived loop space of quotient stacks. In the following few lines, we review the case of classifying stacks.

Example 3.2.10. *Let G be a smooth affine reductive algebraic group over k. The shifted tangent bundle of BG, $T_{BG}[-1]$, is given by*

$$T_{BG}[-1] \simeq [\mathfrak{g}/G]$$

*where $\mathfrak{g}$ is the Lie algebra of G and G acts on $\mathfrak{g}$ via the adjoint representation. In particular, the affinization of the shifted tangent is given by the GIT quotient, $\mathfrak{g}//G$. This example can be found in [**8**]. It depends on the fact that the cotangent complex of BG, pulled back along $a : \operatorname{Spec} k \to BG$, is given by:*

$$a^* \mathbb{L}_{[\operatorname{Spec} k/G]} \simeq \mathfrak{g}^\vee[-1]$$

where $\mathfrak{g}^\vee$ is the dual to the Lie algebra $\mathfrak{g}$. The shift places $\mathfrak{g}^\vee$ in degree zero, so that in the end

$$\operatorname{Sym} \mathbb{L}_{BG}[1] \simeq \operatorname{Sym}(\mathfrak{g}^\vee)^G$$

Example 3.2.11. *Let G be a smooth affine reductive algebraic group over k. The derived loop space of BG, $\mathcal{L}BG$, is given by*

$$\mathcal{L}BG \simeq [G/G]$$

where G acts on itself via the adjoint action. In particular, the affinization of the derived loop space is given by the GIT quotient, $G//G$. This follows because the *derived loop space of BG classifies G-local systems on the circle, which are given exactly by $[G/G]$. For more details, see* [**8**].

Remark 3.2.12. *Let G be a compact Lie group. By work of Atiyah–Bott* [**4**] *we have:*

$$\operatorname{Spec} H_G^{\oplus,0}(*;\mathbb{C}) \simeq \mathfrak{g}_{\mathbb{C}}//G_{\mathbb{C}}$$

where $\mathfrak{g}_{\mathbb{C}}$ and $G_{\mathbb{C}}$ denote the complexifications of the Lie algebra of G and of G itself respectively. Similarly, we have

$$\operatorname{Spec} K_G^0(*) \otimes_{\mathbb{Z}} \mathbb{C} \simeq G_{\mathbb{C}}//G_{\mathbb{C}}$$

In particular, over the complex numbers,

$$\operatorname{Aff}(T_{BG}[-1]) \simeq \operatorname{Spec} H_{G^c}^{\oplus,0}(*)$$

$$\operatorname{Aff}(\mathcal{L}BG) \simeq \operatorname{Spec} K_{G^c}^0(*)$$

Here, G^c is the maximal compact subgroup of G (G is the complexification of G^c).

3.3. The Adelic Decomposition of Equivariant Elliptic Cohomology

In this section we define k-rationalized T-equivariant elliptic cohomology, where k is a $\mathbb{Q}$-algebra, for finite T-CW complexes. The main point in the construction is the localization theorem, which allows us to describe this object inductively via its adelic descent data. When the torus is S^1, the adelic descent data for k-rationalized equivariant elliptic cohomology can be described in terms of Borel equivariant singular cohomology with coefficients in k.

3.3.1. The rank one case. We begin with a definition of k-rationalized S^1-equivariant elliptic cohomology of finite T-CW-complexes.

Definition 3.3.1. *Let k be a $\mathbb{Q}$-algebra and E be an elliptic curve over k, and let X be a finite $T = S^1$-CW-complex. We define k-rationalized T-equivariant elliptic cohomology as the coherent sheaf $\mathcal{E}ll_T(X)$ on E such that:*

- *For a closed point $e \in E$, $\mathcal{E}ll_T(X;k)_{\hat{e}} = H_T^{\oplus,*}(X^{T(e)};k) \otimes_{\mathcal{O}(\mathfrak{t})} \mathcal{O}_{E,\hat{e}}$;*
- *$\mathcal{E}ll_T(X;k)_{\hat{\eta}} = H^{\oplus,*}(X^T;k) \otimes_k \mathcal{O}_{E,\eta}$;*
- *$\mathcal{E}ll_T(X;k)_{\widehat{(\eta>e)}} = H^{\oplus,*}(X^T;k) \otimes_k \operatorname{Frac}\mathcal{O}_{E,\hat{e}}$.*

The (reduced) cosimplicial structure is induced by the cosimplicial structure on the adèles for E and pullback and change of group maps in equivariant singular cohomology:

- *the map $\mathcal{E}ll_T(X;k)_{\hat{\eta}} \to \prod_{e\in|E|_{cl}} \mathcal{E}ll_T(X;k)_{\widehat{(\eta>e)}}$ is given by the identity tensored with the coface map $\mathcal{O}_{E,\eta} \to \operatorname{Frac}\mathcal{O}_{E,\hat{e}}$ of the adèles of $\mathcal{O}_E$;*

- *the map* $\prod_{e\in|E|_{cl}} \mathcal{E}ll_T(X;k)_{\hat{e}} \to \prod_{e\in|E|_{cl}} \mathcal{E}ll_T(X;k)_{\widehat{(\eta>e)}}$ *is given by the product of tensor products of pullback maps in cohomology along inclusions* $X^T \hookrightarrow X^{T(e)}$ *and coface maps* $\mathcal{O}_{E,\hat{e}} \to \mathrm{Frac}\mathcal{O}_{E,\hat{e}}$ *of the adèles of* $\mathcal{O}_E$.

In the above definition, $|E|_{cl}$ denotes the set of closed points of E.

REMARK 3.3.2. *In the above definition we only describe the* reduced *cosimplicial structure of the adèles, and not the full cosimplicial structure. This is enough. See for example* [**50**] *for a description of the reduced adèles in the one dimensional case, and of the relevant (reduced) cosimplicial structure.*

PROPOSITION 3.3.3. *Let* X *be a finite* $T = S^1$*-CW-complex. Let* $\mathcal{E}ll_T(X)$ *be Grojnowski's* T*-equivariant elliptic cohomology. The adelic descent data for* $\mathcal{E}ll_T(X)$ *is*

- *For a closed point* $e \in E$, $\mathcal{E}ll_T(X)_{\hat{e}} = H_T^{\oplus,*}(X^{T(e)};\mathbb{C}) \otimes_{\mathcal{O}(\mathfrak{t})} \mathcal{O}_{E,\hat{e}}$*;*
- $\mathcal{E}ll_T(X)_{\hat{\eta}} = H^{\oplus,*}(X^T;\mathbb{C}) \otimes_k \mathcal{O}_{E,\eta}$*;*
- $\mathcal{E}ll_T(X)_{\widehat{(\eta>e)}} = H^{\oplus,*}(X^T;\mathbb{C}) \otimes_k \mathrm{Frac}\mathcal{O}_{E,\hat{e}}$.

The reduced cosimplicial structure is given by the coface maps for the adèles of $\mathcal{O}_E$ *and pullback maps in singular cohomology.*

PROOF. The case of closed points is explained in [**23**]. For the generic point, the localization theorem yields

$$\mathcal{E}ll_T(X)_{\hat{\eta}} \simeq \left(c_\eta^* H^{\oplus,*}(X^{T(\eta)};\mathbb{C})\right)_{\hat{\eta}} \simeq H^{\oplus,*}(X^T;\mathbb{C}) \otimes_k \mathcal{O}_{E,\eta}$$

For the chain $(\eta > e)$, by definition we have

$$\mathcal{E}ll_T(X)_{\widehat{(\eta>e)}} = \mathbb{A}_E((e), \tilde{j}_\eta \mathcal{E}ll_T(X))$$

hence by localization

$$\mathbb{A}_E((e), \tilde{j}_\eta \mathcal{E}ll_T(X)) \simeq H^{\oplus,*}(X^T;\mathbb{C}) \otimes_k \mathrm{Frac}\mathcal{O}_{E,\hat{e}}$$

We now describe the cosimplicial structure. The map

$$\mathbb{A}_E((\eta), \mathcal{E}ll_{S^1}(X)) \to \mathbb{A}_E((\eta > e), \mathcal{E}ll_{S^1}(X))$$

is given by the identity on $H^{\oplus,*}(X^T)$ tensored with the coface map relative to the adèles for the structure sheaf of E. The map

$$\mathbb{A}_E((e), \mathcal{E}ll_{S^1}(X)) \to \mathbb{A}_E((\eta > e), \mathcal{E}ll_{S^1}(X))$$

is given by the pullback along the inclusion $X^{T(e)} \hookrightarrow X$ in singular cohomology tensored with the coface map as in the case above. □

The above proposition shows that, if $k = \mathbb{C}$, Definition 3.3.1 recovers S^1-equivariant elliptic cohomology in the sense of Grojnowski:

COROLLARY 3.3.4. *Let X be a finite S^1-CW-complex. Then*

$$\mathcal{E}ll_{S^1}(X;\mathbb{C}) \simeq \mathcal{E}ll_{S^1}(X)$$

We now construct pullback and change of group maps in S^1-equivariant elliptic cohomology with coefficients in k.

LEMMA 3.3.5. *A T-equivariant map $f : X \to Y$ induces a pullback map*

$$f^* : \mathcal{E}ll_{S^1}(Y;k) \to \mathcal{E}ll_{S^1}(X;k)$$

PROOF. We can give the adelic decomposition of f^* as follows. For a reduced chain Δ on E, we declare

$$\mathbb{A}_E(\Delta, f^*) : H_T^{\oplus,*}(Y^{T(\Delta)};k) \otimes_k \mathcal{O}_{E,\eta} \to H_T^{\oplus,*}(X^{T(\Delta)};k) \otimes_k \mathcal{O}_{E,\eta}$$

to be the tensor product of the pullback map $H_T^{\oplus,*}(Y^{T(\Delta)};k) \to H_T^{\oplus,*}(X^{T(\Delta)};k)$ in equivariant cohomology relative to the map $f^{T(\Delta)} : X^{T(\Delta)} \to Y^{T(\Delta)}$. Here we denote by $T(\Delta)$ the subgroup $T(x)$, where x is the maximal point in Δ.

The coface maps in Definition 3.3.1 are defined to be the tensor product of pullback maps in singular cohomology and the coface maps for the adèles for E. In particular, the maps $\mathbb{A}_E(\Delta, f^*)$ assemble into a cosimplicial map

$$\mathbb{A}_E^\bullet(f^*) : \mathbb{A}_E^\bullet(\mathcal{E}ll_{S^1}(Y;k)) \to \mathbb{A}_E^\bullet(\mathcal{E}ll_{S^1}(X;k))$$

as pullbacks commute with pullbacks. □

LEMMA 3.3.6. *Let X be a finite S^1-CW-complex, and $1 : * \to S^1$ be the identity of S^1. The map 1 induces a map of sheaves on E*

$$\mathcal{E}ll_{S^1}(X;k) \to (1_E)_* H^{\oplus,*}(X;k)$$

where $1_E : E_1 \simeq \operatorname{Spec} k \to E \simeq E_{S^1}$ is the identity section of E.

PROOF. By adjunction, we are allowed to describe the map

$$(1_E)^* \mathcal{E}ll_{S^1}(X;k) \to H^{\oplus,*}(X;k)$$

instead. The left hand side is the fiber of k-rationalized equivariant elliptic cohomology over the identity section of E, and is given exactly by $H^{\oplus,*}(X;k)$. Hence, the map is an isomorphism. □

LEMMA 3.3.7. *Let $S^1 \to S^1/S^1 \simeq *$ be the quotient map, and X be a finite CW-complex. This map induces a map of sheaves on E*

$$H^{\oplus,*}(X;k) \to c_* \mathcal{E}ll_{S^1}(X;k)$$

where $c : E \to \operatorname{Spec} k$ is the structure map of E. In the above, S^1 acts on X through the homomorphism $S^1 \to S^1/S^1$, i.e. trivially.

Proof. By adjunction, we can equivalently describe the map

$$c^* H^{\oplus,*}(X;k) \to \mathcal{E}ll_{S^1}(X;k)$$

This map is actually an isomorphism. Indeed, as S^1 acts on X via the homomorphism $S^1 \to S^1/S^1$, the equivariant elliptic cohomology $\mathcal{E}ll_{S^1}(X;k)$ is the tensor product

$$H^{\oplus,*}(X;k) \otimes_k \mathcal{O}_E$$

which is exactly the pullback of $H^{\oplus,*}(X;k)$ along the structure map $c : E \to \operatorname{Spec} k$. $\square$

3.3.2. The higher rank case. We define $\mathcal{E}ll_T(X;k)$ for higher rank tori via induction on the rank of the torus T. This allows us to avoid describing the adèles with respect to a chain $\Delta = (x, x_1, \dots, x_p)$ in terms of singular cohomology whenever x_p is not closed. If x_p is closed it is possible to give such a description.

Definition 3.3.8. *Let k be a $\mathbb{Q}$-algebra and E be an elliptic curve over k, T be a real torus of rank n and X be a finite T-CW-complex. The k-rationalized T-equivariant elliptic cohomology of X is the coherent sheaf $\mathcal{E}ll_T(X;k)$ on E_T inductively defined by the following adelic descent data:*

- *given a reduced chain $\Delta = (x, x_1, \dots, x_p)$,*
  $\mathsf{A}_{E_T}(\Delta, \mathcal{E}ll_T(X;k)) = c_x^* \mathcal{E}ll_{T'(x)}(X^{T(x)};k) \otimes_{\mathcal{O}_{E_T}} \mathcal{O}_{E_T,\hat{\Delta}}$;
- *given a reduced chain $\Delta = (x, x_1, \dots, x_p)$, if x_p is closed,*
  $\mathsf{A}_{E_T}(\Delta, \mathcal{E}ll_T(X;k)) = H_T^{\oplus,*}(X^{T(x)};k) \otimes_{\mathcal{O}(\mathfrak{t})} \mathcal{O}_{E_T,\hat{\Delta}}$.

Here $\mathcal{E}ll_{T'(x)}(X^{T(x)};k)$ is k-rationalized $T'(x)$-equivariant elliptic cohomology of $X^{T(x)}$. The cosimplicial structure is the following:

- *if the chain $\Delta = (x, x_1, \dots, x_p)$ is such that x_p is closed we use Proposition 3.2.2 in [**35**], so that the relevant coface maps for removing a point of the chain are given by tensor products of the corresponding coface maps for the adèles for the structure sheaf of E_T and pullback maps in equivariant cohomology;*
- *If x_p is not closed, the coface maps are tensor products of the corresponding maps of the adèles of $\mathcal{O}_{E_T}$ and pullback and change of group maps in equivariant elliptic cohomology with respect to tori of rank strictly smaller than* $\operatorname{rk}(T)$.

Remark 3.3.9. *As in Definition 3.3.1, we give only the reduced cosimplicial structure and deal with reduced chains. Moreover, as x is not a closed point, the torus $T'(x)$ is of rank strictly smaller than the rank of T.*

Remark 3.3.10. *If we remove a point x_i, $i \in \{1, \dots, p-1\}$, the pullback map in equivariant cohomology is the identity, since the point x is not removed. This map* *differs from the identity only when we remove x from Δ. Similarly, if we remove x_p from Δ, we need to use the presentation in terms of equivariant elliptic cohomology for smaller rank tori when describing the associated coface map, as x_{p-1} is not closed.*

To ensure that Definition 3.3.8 is well posed, we need to produce pullback and restriction maps inductively.

LEMMA 3.3.11. *Let $f : X \to Y$ be a T-equivariant map, where T is a torus of rank n. Assume that k-rationalized equivariant elliptic cohomology has pullbacks with respect to $\tilde{T}$-equivariant maps, where $\tilde{T}$ is any torus of rank strictly smaller than n. Moreover, assume that change of group maps with respect to maps of tori exist in k-rationalized $\tilde{T}$-equivariant elliptic cohomology. Then f induces a pullback map*

$$f^* : \mathcal{E}ll_T(Y; k) \to \mathcal{E}ll_T(X; k)$$

in k-rationalized T-equivariant elliptic cohomology.

PROOF. We construct these maps using the adelic decomposition described in Definition 3.3.8. In the case x_p is not closed, we use pullback maps for equivariant elliptic cohomology relative to smaller rank tori, which exist by the inductive hypothesis. Indeed, the maps we need to construct is

$$\mathsf{A}_{E_T}(\Delta, \mathcal{E}ll_T(Y; k)) \to \mathsf{A}_{E_T}(\Delta, \mathcal{E}ll_T(X; k))$$

for all reduced chains $\Delta = (x, x_1, \dots, x_p)$, respecting the reduced cosimplicial structure. The map above is constructed then as the tensor product

$$c_x^* \mathcal{E}ll_{T'(x)}(X^{T(x)}; k) \otimes_{\mathcal{O}_{E_T}} \mathcal{O}_{E_T, \hat{\Delta}} \to c_x^* \mathcal{E}ll_{T'(x)}(Y^{T(x)}; k) \otimes_{\mathcal{O}_{E_T}} \mathcal{O}_{E_T, \hat{\Delta}}$$

of the pullback along $f^{T(x)} : X^{T(x)} \to Y^{T(x)}$ with the identity of $\mathcal{O}_{E_T, \hat{\Delta}}$. If x_p is closed, we equivalently use the presentation of the adèles in terms of equivariant cohomology. In this case, the map between the adèles becomes the tensor product

$$H_T^{\oplus,*}(X^{T(x)}; k) \otimes_{\mathcal{O}(\mathfrak{t})} \mathcal{O}_{E_T, \hat{\Delta}} \to H_T^{\oplus,*}(Y^{T(x)}; k) \otimes_{\mathcal{O}(\mathfrak{t})} \mathcal{O}_{E_T, \hat{\Delta}}$$

of the pullback in equivariant cohomology along $f^{T(x)} : X^{T(x)} \to Y^{T(x)}$ with the identity of $\mathcal{O}_{E_T, \hat{\Delta}}$. The commutativity with the cosimplicial structure is clear, as pullbacks commute with pullbacks. $\square$

The proof of the next lemma works very similarly to that of the previous lemma.

LEMMA 3.3.12. *Let $T_1 \to T_2$ be a group homomorphism, and T_1 and T_2 be tori of rank smaller or equal to n. Assume that k-rationalized equivariant elliptic cohomology has pullbacks with respect to $\tilde{T}$-equivariant maps, where $\tilde{T}$ is any torus of rank strictly smaller than n. Moreover, assume that change of group maps with respect to maps of* *tori exist in k-rationalized $\tilde{T}$-equivariant elliptic cohomology. Then we have a change of group map*

$$\mathcal{E}ll_{T_2}(Y;k) \to c_{1,2*}\mathcal{E}ll_{T_1}(X;k)$$

In the above, $c_{1,2}$ is the projection map

$$E_{T_1} \to E_{T_2}$$

PROOF. As in the proof of Lemma 3.3.11, the relevant map between the adèles relative to the reeduced chain Δ is given by a tensor product of the identity of the adèle $\mathcal{O}_{E_T,\hat{\Delta}}$ with the change of group map in either k-rationalized equivariant elliptic cohomology with respect to tori of smaller rank if x_p is not closed or in singular cohomology for x_p closed. $\square$

As a consequence of the two lemmas 3.3.11 and 3.3.12, we obtain that k-rationalized T-equivariant elliptic cohomology has pullbacks with respect to T-equivariant maps and change of group maps with respect to homomorphisms of tori.

COROLLARY 3.3.13. *Let $f : X \to Y$ be a T-equivariant map. Then f induces a pullback map*

$$f^* : \mathcal{E}ll_T(Y;k) \to \mathcal{E}ll_T(X;k)$$

Let $T_1 \to T_2$ be a group homomorphism, and T_1 and T_2 be tori. We have an induced change of group map

$$\mathcal{E}ll_{T_2}(Y;k) \to c_{1,2*}\mathcal{E}ll_{T_1}(X;k)$$

where $c_{1,2}$ is the projection map

$$E_{T_1} \to E_{T_2}$$

We now observe that, if $k = \mathbb{C}$, our definition recovers Grojnowski's. To do so, we need to compute the adèles of Grojnowski's sheaf first.

LEMMA 3.3.14. *Given a chain $\Delta = (x, x_1, \ldots, x_p)$ we have that*

$$\mathsf{A}_{E_T}(\Delta, \mathcal{E}ll_T(X)) \simeq c_x^*\mathcal{E}ll_{T'(x)}(X^{T(x)}) \otimes_{\mathcal{O}_{E_T}} \mathcal{O}_{E_T,\hat{\Delta}}$$

PROOF. By definition,

$$\mathsf{A}_{E_T}(\Delta, \mathcal{E}ll_T(X)) = \lim_{r\geqslant 0} \mathsf{A}_{E_T}({}_x\Delta, \tilde{j}_{rx}\mathcal{E}ll_T(X))$$

The localization theorem dictates that

$$\tilde{j}_{rx}\mathcal{E}ll_T(X) \simeq \tilde{j}_{rx}\mathcal{E}ll_T(X^{T(x)}) \simeq \tilde{j}_{rx}c_x^*\mathcal{E}ll_{T'(x)}(X^{T(x)})$$

hence

$$\begin{aligned}\mathsf{A}_{E_T}(\Delta, \mathcal{E}ll_T(X)) &\simeq \lim_{r\geqslant 0} \mathsf{A}_{E_T}({}_x\Delta, \tilde{j}_{rx}c_x^*\mathcal{E}ll_{T'(x)}(X^{T(x)})) \\ &\simeq \mathsf{A}_{E_T}(\Delta, c_x^*\mathcal{E}ll_{T'(x)}(X^{T(x)}))\end{aligned}$$

$\square$

Whenever the chain $\Delta = (x, x_1, \dots, x_p)$ is such that x_p is closed, it is easy to obtain a description of the adèles in terms of singular cohomology:

LEMMA 3.3.15. *Given a chain $\Delta = (x, x_1, \dots, x_p)$ with x_p closed, we have that*

$$\mathsf{A}_{E_T}(\Delta, \mathcal{E}ll_T(X)) = H_T^{\oplus,*}(X^{T(x)}) \otimes_{\mathcal{O}(\mathfrak{t})} \mathcal{O}_{E_T,\hat{\Delta}}$$

PROOF. This is a simple application of Proposition 3.2.1 from [**35**], together with the description of the adèles at closed points. The C_{x_i} operations do not affect the singular cohomology modules as they are finitely presented over $H_T^{\oplus}$, since this ring is Noetherian and X is a finite T-CW-complex. We can replace the equivariant cohomology of the locus fixed by $T(x)$ with the equivariant cohomology of the locus fixed by $T(x_p)$ by applying the localization formula. □

The two Lemmas 3.3.14 and 3.3.15 allow us to deduce immediately the following Corollary:

COROLLARY 3.3.16. *Let X be a finite T-CW-complex. Then*

$$\mathcal{E}ll_T(X; \mathbb{C}) \simeq \mathcal{E}ll_T(X)$$

PROOF. Lemmas 3.3.14 and 3.3.15 give us a description of the adèles of Grojnowski's equivariant elliptic cohomology relative to reduced chains Δ. The cosimplicial structure corresponds exactly to the one described in Definition 3.3.8, hence we obtain an isomorphism of the "product of local adèles"

$$\prod_{\Delta \in |E_T|_\bullet} \mathsf{A}_{E_T}(\Delta, \mathcal{E}ll_T(X; \mathbb{C})) \simeq \prod_{\Delta \in |E_T|_\bullet} \mathsf{A}_{E_T}(\Delta, \mathcal{E}ll_T(X))$$

As the adèles embed in the product of local adèles as a cosimplicial subcomplex, we obtain the isomorphism

$$\mathcal{E}ll_T(X; \mathbb{C}) \simeq \mathcal{E}ll_T(X)$$

□

3.3.3. Extension to compact Lie groups. In this subsection we extend the construction from tori to compact Lie groups. There is a canonical way to do so, and is explained in [**23**]. Let G be a compact Lie group, and call T its maximal torus and W the Weyl group. Then call

$$E_G = E_T/W$$

DEFINITION 3.3.17. *The k-rationalized G-equivariant elliptic cohomology of a finite G-CW-complex X is the coherent sheaf of W-invariants on E_G:*

$$\mathcal{E}ll_G(X; k) := \mathcal{E}ll_T(X; k)^W$$

3.3.4. Cochain-level variant. The same construction we developed in this section can be adapted to a cochain-level variant. The key to this construction is the formality of the algebra of cochains on the Borel quotient $*//T$. Let

$$C_T^{\oplus,*}(X) := C^{\oplus,*}(X//T)$$

denote the sum-$\mathbb{Z}_2$-periodization of the cochains on the Borel construction $X//T$. The algebra $C_T^{\oplus,0}(*)$ is formal, in particular $C_T^{\oplus,0}(*) \simeq \mathcal{O}(\mathfrak{t})$. Hence we can redefine the adelic descent data given in Definition 3.3.8 by replacing the cohomology of $X^{T(x)}//T$ with its cochains. The same inductive procedure yields, by an application of Groechenig's Theorem 3.1 in [**28**], an object in $\mathrm{Perf}(E_T)$ (as X is a finite T-CW-complex), that we denote by

$$\mathcal{CE}ll_T(X)$$

If the base field k is taken to be the field of the complex numbers, this object can be equivalently constructed following Grojnowski's methodology. See the Preliminaries in [**67**] for the details.

3.4. The Adelic Decomposition for Equivariant Cohomology and K-theory

In this section we study the adelic description of equivariant singular cohomology and equivariant K-theory, summarized in the following two lemmas. These lemmas will play a role in section 3.5, where we construct a periodic cyclic homology model for these two cohomology theories.

3.4.1. Equivariant cohomology. We work over a field k of characteristic zero. Throughout this section, we denote by

$$\mathcal{H}_T(X)$$

the quasi-coherent sheaf on

$$\mathfrak{t}^{alg} = \mathbb{A}^1_k \otimes_{\mathbb{Z}} \check{T}$$

whose global sections are

$$\Gamma(\mathfrak{t}^{alg}, \mathcal{H}_T^{\oplus,*}(X)) = H_T(X;k)$$

The adelic decomposition of singular cohomology is obtained as a simple application of the localization formula and the observation stated in Groechenig's [**28**] after Remark 1.9: if X is an affine Noetherian scheme and $\mathcal{F}$ a quasi-coherent sheaf, $\mathbb{A}^n_X(\mathcal{F}) \simeq \mathcal{F}(X) \otimes_{\mathcal{O}_X(X)} \mathbb{A}^n_X$.

First we need to introduce the notion of subgroup of T associated to an element of the Lie algebra $\mathfrak{t}$.

DEFINITION 3.4.1. *Let $\xi \in \mathfrak{t}^{alg}$ be a point of $\mathfrak{t}^{alg}$. We define*

$$T(\xi) \leqslant T$$

to be the smallest subgroup of T such that the closure of the point ξ, $\overline{\{\xi\}}$, is contained in

$$\mathfrak{t}^{alg}_{T(\xi)} := \mathbb{A}^1_k \otimes_{\mathbb{Z}} \check{T}(\xi)$$

where $\check{T}(\xi)$ is the cocharacter lattice of $T(\xi)$.

LEMMA 3.4.2. *Let X be a finite T-CW-complex and let $\Delta = (\xi > \xi_1 > \cdots > \xi_k)$ be a chain of length k on $\mathfrak{t}_{\mathbb{C}}$. Then*

$$\mathbb{A}_{\mathfrak{t}^{alg}}(\Delta, \mathcal{H}_T(X)) \simeq H_T^{\oplus}(X^{T(\xi)}) \otimes_{\mathcal{O}(\mathfrak{t}^{alg})} \mathbb{A}_{\mathfrak{t}^{alg}}(\Delta, \mathcal{O}_{\mathfrak{t}^{alg}})$$

PROOF. Groechenig's observation yields

$$\mathbb{A}_{\mathfrak{t}^{alg}}(\Delta, \mathcal{H}_T(X)) \simeq H_T^{\oplus}(X) \otimes_{\mathcal{O}(\mathfrak{t}^{alg})} \mathbb{A}_{\mathfrak{t}^{alg}}(\Delta, \mathcal{O}_{\mathfrak{t}^{alg}})$$

and the localization theorem implies that the pullback along the inclusion $X^{T(\xi)} \hookrightarrow X$ becomes an isomorphism after tensoring with the adèles $\mathbb{A}_{\mathfrak{t}^{alg}}(\Delta, \mathcal{O}_{\mathfrak{t}^{alg}})$ over the base ring $\mathcal{O}(\mathfrak{t}^{alg})$. □

3.4.2. Equivariant K-theory. The adelic decomposition of equivariant K-theory can be proved as in the singular cohomology case, as long as the chain starts from a closed point. We work over a field k of characteristic zero. In particular, we denote by

$$\mathcal{K}_T(X)$$

the quasi-coherent sheaf on

$$T^{alg} := \mathbb{G}_{m,k} \otimes_{\mathbb{Z}} \check{T}$$

with global sections given by

$$\Gamma(T^{alg}, \mathcal{K}_T(X)) = K_T^*(X) \otimes_{\mathbb{Z}} k$$

Similarly to the equivariant cohomology case, we need to introduce the notion of subgroup of T associated to an element of T.

DEFINITION 3.4.3. *Let $x \in T^{alg}$ be a point of T^{alg}. We define*

$$T(x) \leqslant T$$

to be the smallest subgroup of T such that the closure $\overline{\{x\}}$ of the point x is contained in

$$T(x)^{alg} = \mathbb{G}_m \otimes_{\mathbb{Z}} \check{T}(x)$$

where $\check{T}(x)$ is the cocharacter lattice of $T(x)$.

LEMMA 3.4.4. *Let X be a finite T-CW-complex and let $\Delta = (x > x_1 > \cdots > x_k)$ be a chain of length k on $T_{\mathbb{C}}$, such that x_k is a closed point. Then*

$$\mathbb{A}_{T_{\mathbb{C}}}(\Delta, \mathcal{K}_T(X)) \simeq H_T^{\oplus}(X^{T(x)}) \otimes_{\mathcal{O}(\mathfrak{t})} \mathbb{A}_{T_{\mathbb{C}}}(\Delta, \mathcal{O}_{T_{\mathbb{C}}})$$

PROOF. We apply Proposition 3.2.1 in [**35**] to the formula computing the completion of equivariant K-theory at a closed point:

$$\mathcal{K}_T(X)_{\widehat{x_k}} \simeq H_T^{\prod,*}(X^{T(x_k)}) \simeq H_T^{\oplus,*}(X^{T(x_k)}) \otimes_{\mathcal{O}(\mathfrak{t})} \mathcal{O}_{T,\widehat{x_k}}$$

where we identified the completions $\mathcal{O}_{\mathfrak{t},\hat{0}} \simeq \mathcal{O}_{T,\hat{1}} \simeq \mathcal{O}_{T,\widehat{x_k}}$ (the former isomorphism is the exponential while the latter follows from the group structure on T). The cohomology groups $H_T(X^{T(x)})$ are finitely presented over $H(BT)$ since X is a finite T-CW-complex. The final statement follows from the localization formula. □

We observe at this point that Lemma 3.4.4 can be used as input data for an inductive construction akin to the one explained in Section 3.3 in the elliptic case. All the proofs of Lemmas 3.3.5, 3.3.6, 3.3.7, 3.3.11, 3.3.12 and Corollary 3.3.13 go through almost unchanged — the role of E is now played by $\mathbb{G}_m$ — thus the same exact construction produces k-rationalized equivariant K-theory out of singular cohomology and induction. Indeed, we could propose the following

DEFINITION 3.4.5. *Let k be a $\mathbb{Q}$-algebra and E be an elliptic curve over k, T be a real torus of rank n and X be a finite T-CW-complex. The* adelic k-rationalized T-equivariant K-theory of X

$$\mathcal{K}_T^{ad}(X;k)$$

is the coherent sheaf $\mathcal{K}_T(X;k)$ on T^{alg} inductively defined by the following adelic descent data:

- *given a reduced chain $\Delta = (x, x_1, \ldots, x_p)$,* $\mathsf{A}_{T^{alg}}(\Delta, \mathcal{K}_T^{ad}(X;k)) = c_x^* \mathcal{K}_{T'(x)}^{ad}(X^{T(x)};k) \otimes_{\mathcal{O}_{T^{alg}}} \mathcal{O}_{T^{alg},\hat{\Delta}}$*;*
- *given a reduced chain $\Delta = (x, x_1, \ldots, x_p)$, if x_p is closed,* $\mathsf{A}_{T^{alg}}(\Delta, \mathcal{K}_T^{ad}(X;k)) = H_T^{\oplus,*}(X^{T(x)};k) \otimes_{\mathcal{O}(\mathfrak{t})} \mathcal{O}_{T^{alg},\hat{\Delta}}$.

The cosimplicial structure is the following:

- *if the chain $\Delta = (x, x_1, \ldots, x_p)$ is such that x_p is closed, the relevant coface maps for removing a point of the chain are given by tensor products of the corresponding coface maps for the adèles for the structure sheaf of T^{alg} and pullback maps in equivariant cohomology;*
- *If x_p is not closed, the coface maps are tensor products of the corresponding maps of the adèles of $\mathcal{O}_{T^{alg}}$ and pullback and change of group maps in adelic k-rationalized equivariant K-theory with respect to tori of rank strictly smaller than* $\mathrm{rk}(T)$.

This definition is well-posed, as the rank one case can be treated as in the case of elliptic cohomology and all the lemmas needed hold, as discussed above. Then Lemma 3.4.4 proves the following

THEOREM 3.4.6. *Let X be a finite T-CW-complex, for T a real torus. Then*

$$\mathcal{K}_T^{ad}(X;k) \simeq \mathcal{K}_T(X)$$

as $\mathbb{Z}_2$-periodic sheaves on T^{alg}.

PROOF. This proof goes the same as the proof of Corollary 3.3.16. Lemma 3.4.4 gives an equivalence

$$\prod_{\Delta\in|T^{alg}|_\bullet} \mathsf{A}_{T^{alg}}(\Delta, \mathcal{E}ll_T(X;\mathbb{C})) \simeq \prod_{\Delta\in|T^{alg}|_\bullet} \mathsf{A}_{T^{alg}}(\Delta, \mathcal{E}ll_T(X))$$

which gives the desired isomorphism. □

Theorem 3.4.6 is an algebraic incarnation of Roşu's theorem [**61**]. Indeed, Roşu proves that it is possible to construct the analytic extension of complexified equivariant K-theory only in terms of singular cohomology of fixed loci in X. Our Theorem 3.4.6 proves his result without the need to extend scalars by holomorphic functions, and moreover holds over all fields k of characteristic zero. In particular, it answers the question posed by Roşu himself of the existence of such an algebraic construction.

REMARK 3.4.7. *In particular, if G is a compact Lie group, we could define* adelic *G-equivariant K-theory, $\mathcal{K}_G^{ad}(X;k)$, as the Weyl invariants of the adelic T-equivariant K-theory, for a maximal torus T. Then it would follow immediately that $\mathcal{K}_G^{ad}(X;k) \simeq \mathcal{K}_G(X)$.*

3.5. Geometric presentations for Equivariant Cohomology and K-theory

In this section we provide a presentation of equivariant K-theory and equivariant singular cohomology of the analytification of a smooth algebraic variety via Hochschild homology counterparts, in terms of objects in derived algebraic geometry. This will be a byproduct of the adelic decomposition for equivariant cohomology and K-theory. A similar presentation appears in [**67**] in the context of rationalized equivariant elliptic cohomology. The K-theory case has been analysed by Halpern-Leistner–Pomerleano in [**31**]; their proof does not make use of adelic descent techniques but rather of Blanc's topological K-theory of a dg-category over $\mathbb{C}$ [**12**]. They prove the equivalence for smooth quasi-projective schemes acted on by a reductive group so that the quotient admits a *semi-complete* KN stratification, Definition 1.1 of [**31**]. Our proof is for tori, rather than general reductive groups, but we do not require quasi-projectivity and the existence of a semi-complete KN stratification.

Moreover, our proof is sufficiently compact, and uses more elementary mathematics compared to Halpern-Leistner and Pomerleano's.

To the best of the author's knowledge, the analogous statement for equivariant cohomology — Theorem 3.5.14 — does not appear in the literature in this form, but is well-known. In particular, a version of the theorem for general Artin stacks follows immediately from work of Calaque–Pantev–Toën–Vaquié–Vezzosi [**16**].

Remark 3.5.1. *The construction appearing in* [**67**] *for rationalized equivariant elliptic cohomology can be compared, over any field k of characteristic zero, with a* de Rham *variant of the adelic construction of equivariant elliptic cohomology in Section 3.3. In particular, the main result in* [**67**], *Theorem 6.10, would extend over a general field k of characteristic zero to an isomorphism with such a de Rham variant.*

Throughout this section we work over a fixed base field k of characteristic 0.

3.5.1. Equivariant K-theory. In this subsection we prove a small variation of Halpern-Leistner–Pomerleano's Theorem 2.17 in [**31**]. They prove that, if X is a smooth quasi-projective scheme acted on by an algebraic group G so that $[X/G]$ admits a semi-complete KN stratification (see Definition 1.1 in [**31**]) there is an equivalence

$$\mathrm{HP}(\mathrm{Perf}[X/G]) \simeq K^{top}([X/G]) \otimes \mathbb{C}$$

between the periodic cyclic homology of $[X/G]$ and Blanc's topological K-theory of the dg-category $\mathrm{Perf}([X/G])$.

The theorem we prove in this section is the following:

Theorem 3.5.2. *Let X be a smooth variety over $\mathbb{C}$ acted on by an algebraic torus T. There is an equivalence of $\mathbb{Z}_2$-periodic coherent sheaves on $\mathcal{T}$*

$$\pi_* \mathcal{HP}([X/T]) \simeq \mathcal{K}_{T^{\mathrm{an}}}(X^{\mathrm{an}})$$

which is natural in X with respect to T-equivariant maps.

In the statement above, $\mathcal{HP}([X/T])$ denotes the $\mathbb{Z}_2$-periodic quasi-coherent complex on

$$\mathcal{T} = \operatorname{Spec} \mathrm{HP}^0([\operatorname{Spec} \mathbb{C}/T])$$

associated to periodic cyclic homology, viewed as a module over $\mathrm{HP}^0([\operatorname{Spec}\mathbb{C}/T])$. Similarly, $\mathcal{K}_{T^{\mathrm{an}}}(X^{\mathrm{an}})$ is the $\mathbb{Z}_2$-periodic quasi-coherent sheaf on

$$\mathcal{T} = \operatorname{Spec} K^0_{T^{\mathrm{an}}}(*) \otimes_{\mathbb{Z}} \mathbb{C}$$

associated to the $\mathbb{C}$-rationalized topological K-theory $K_{T^{\mathrm{an}}}(X^{\mathrm{an}})$ viewed as a module over $K^0_{T^{\mathrm{an}}}(*) \otimes_{\mathbb{Z}} \mathbb{C}$.

Remark 3.5.3. *Theorem 3.5.2 does not require any assumption on quasi-projectivity and the existence of a semi-complete KN stratification. Moreover, the proof is streamlined and elementary compared to that of Theorem 2.17 in* [**31**]. *One consequence of Theorem 3.5.2 is that it provides more examples in which the lattice conjecture holds. To extend to Theorem 3.5.2 to reductive groups, we need to keep track of the Weyl group action. This will appear in a future version of this paper.*

Remark 3.5.4. *The assumption that X is a variety is enough to ensure that $\mathcal{HP}([X/T])$ belongs to $\mathrm{Coh}(T)$. Indeed, it is enough that X is a quasi-compact algebraic space (see the discussion in* [**17**] *in the proof of Corollary 4.3.21).*

Remark 3.5.5. *Let X be a variety acted on by an algebraic torus T. Let*

$$q : \mathcal{L}[X/T] \to T$$

be the structure map. We define:

$$\mathcal{HH}([X/T] := q_*\mathcal{O}_{\mathcal{L}[X/T]}$$

The S^1-action on the loop space equips this quasi-coherent sheaf with a lift from $\mathrm{QCoh}(T)$ to $\mathrm{QCoh}(T)^{S^1}$ (where S^1 acts trivially on T). $\mathcal{HP}([X/T])$ is equivalently the image of $\mathcal{HH}([X/T])$ with its S^1-action in the $\mathbb{Z}_2$-periodic category of quasi-coherent sheaves

$$\mathrm{QCoh}(T)^{\mathbb{Z}_2}$$

defined as the Tate construction of $\mathrm{QCoh}(T)$ with respect to a trivial action of S^1 on T. Such procedure has been defined by Preygel in [**59**]. *For small k-linear categories with an action of S^1, it amounts to a base change of the S^1-fixed locus from $k[[u]]$ to $k((u))$. For presentable categories there is an additional* regularization *step involving t-structures, as explained in* [**59**].

The arguments we produce to prove Theorem 3.5.2 are parallel to the ones appearing in the proof of Proposition 6.8 and Theorem 6.9 in [**67**].

Lemma 3.5.6. *Let X be a smooth variety over $\mathbb{C}$ acted on by an algebraic torus T of rank 1. There is an equivalence of $\mathbb{Z}_2$-periodic coherent sheaves*

$$\pi_*\mathcal{HP}([X/T]) \simeq \mathcal{K}_{T^{\mathrm{an}}}(X^{\mathrm{an}})$$

which is natural in X with respect to T-equivariant maps.

Proof. The adelic decomposition for the sheaf of rationalized equivariant K-theory has been computed in Section 3.4. In the case of periodic cyclic homology, the adelic decomposition follows from Chen's Theorem 4.3.2 in [**17**]:

$$\pi_*\mathcal{HP}([X/T])_{\hat{z}} \simeq H^{\oplus,*}_{T^{\mathrm{an}}}(t_0 X^{T(z),\mathrm{an}}) \otimes_{\mathcal{O}(\mathfrak{t})} \mathcal{O}_{T,\hat{z}}$$

and Chen's localization theorem for the loop space, Theorem 3.2.12 in [**17**]:

$$\pi_* \mathcal{HP}([X/T])_{\widehat{\eta}} \simeq H^{\oplus,*}(t_0 X^{T,\mathrm{an}}) \otimes_{\mathcal{O}(\mathfrak{t})} \mathcal{O}_{T,\eta}$$

The isomorphisms satisfy the compatibility conditions necessary to promote them to isomorphisms of cosimplicial groups. Indeed, the diagram

$$\begin{tikzcd}
\pi_*\mathrm{HP}([X/T])_{\widehat{z}} \arrow[r] \arrow[d] & H^{\oplus,*}_{dR,T}(t_0(X^{T(z)})^{\mathrm{an}}) \otimes_{\mathcal{O}(\mathfrak{t})} \mathcal{O}_{T,\widehat{z}} \arrow[d] \\
\pi_*\mathrm{HP}(t_0(X^T)) \otimes_k \mathrm{Frac}\mathcal{O}_{T,\widehat{z}} \arrow[r] & H^{\oplus,*}_{dR}(t_0(X^T)^{\mathrm{an}}) \otimes_k \mathrm{Frac}\mathcal{O}_{T,\widehat{z}}
\end{tikzcd}$$

commutes as a consequence of the compatibility of Chen and Ben-Zvi–Nadler's HKR theorems 4.3.2 in[**17**] and Proposition 4.4 in [**8**] with pullbacks, and the diagram

$$\begin{tikzcd}
\pi_*\mathrm{HP}([X/T])_{\eta} \arrow[r] \arrow[d] & H^{\oplus,*}_{dR}(t_0(X^T)^{\mathrm{an}}) \otimes_{\mathcal{O}(\mathfrak{t})} \mathcal{O}_{T,\eta} \arrow[d] \\
\pi_*\mathrm{HP}(t_0(X^T)) \otimes_k \mathrm{Frac}\mathcal{O}_{T,\widehat{z}} \arrow[r] & H^{\oplus,*}_{dR}(t_0(X^T)^{\mathrm{an}}) \otimes_k \mathrm{Frac}\mathcal{O}_{T,\widehat{z}}
\end{tikzcd}$$

commutes as the bottom isomorphism is the tensor product of the top one with $\mathcal{O}_{T,\widehat{z}}$.

Naturality of the isomorphism is a consequence of the compatibility with T-equivariant maps of the HKR isomorphisms provided by Theorem 4.3.2 in [**17**] and Proposition 4.4 in [**8**]. □

Theorem 3.5.2 follows from Lemma 3.5.6 applying induction on the rank of T.

PROOF OF THEOREM 3.5.2. We first focus on the analysis of chains $\Delta = (x > x_1 > \cdots > x_p)$ where x_p is a closed point of T. In this case, we have an explicit description of the adelic groups in terms of singular cohomology for both $\mathcal{HP}$ and $\mathcal{K}$. An application of Huber's Proposition 3.2.1 in [**35**] to Theorem 4.3.2 from Chen's [**17**] yields

$$\mathbb{A}_T(\Delta, \mathcal{HP}([X/T])) \simeq H^{\oplus,*}_{T^{\mathrm{an}}}(t_0(X^{T(x)})^{\mathrm{an}}) \otimes_{\mathcal{O}(\mathfrak{t})} \mathcal{O}_{T,\Delta}$$

In the above, the subgroup $T(x)$ is determined according to Definition 3.5.9. We also applied the localization theorem (Theorem 3.2.12 in [**17**]) to further restrict from $t_0 X^{T(x_p)}$ to $t_0 X^{T(x)}$. Since X is a variety over $\mathbb{C}$, the equivariant cohomology of the fixed loci is always finitely generated as a module over the equivariant cohomology of the point.

For K-theory, Lemma 3.4.4 gives the same formula:

$$\mathbb{A}_T(\Delta, \mathcal{K}_{T^{\mathrm{an}}}(X^{\mathrm{an}})) \simeq H^{\oplus,*}_{T^{\mathrm{an}}}(t_0(X^{T(x)})^{\mathrm{an}}) \otimes_{\mathcal{O}(\mathfrak{t})} \mathcal{O}_{T,\Delta}$$

The cosimplicial structure is induced by pullback maps in equivariant cohomology and the coface/codegeneracy maps for the cosimplicial adelic group of $\mathcal{O}_T$, in both cases.

We now assume x_p is not a closed point. In this situation, we apply induction on the rank of the torus T. Since T is affine, we have decompositions

$$\mathbb{A}_T(\Delta, \mathcal{HP}([X/T])) \simeq \mathrm{HP}([t_0 X^{T(x)}/T]) \otimes_{\mathcal{O}(T)} \mathcal{O}_{T,\Delta}$$

$$\mathbb{A}_T(\Delta, \mathcal{K}_{T^{\mathrm{an}}}(X^{\mathrm{an}})) \simeq K_{T^{\mathrm{an}}}(t_0(X^{T(x)})^{\mathrm{an}})) \otimes_{\mathcal{O}(T)} \mathcal{O}_{T,\Delta}$$

where we are allowed to restrict to the locus fixed by $T(x)$ by the localization theorem respectively for the loop space (Theorem 3.1.12 in [**17**]) and in equivariant K-theory.

Observing that $T(x)$ acts trivially on $t_0 X^{T(x)}$, we can reduce the equivariance group to $T'(x)$:

$$\mathrm{HP}([t_0 X^{T(x)}/T]) \simeq \mathrm{HP}([t_0 X^{T(x)}/T'(x)]) \otimes_{\mathcal{O}(T'(x))} \mathcal{O}(T)$$

$$K_{T^{\mathrm{an}}}(t_0(X^{T(x)})^{\mathrm{an}})) \simeq K_{T'(x)^{\mathrm{an}}}(t_0(X^{T(x)})^{\mathrm{an}})) \otimes_{\mathcal{O}(T'(x))} \mathcal{O}(T)$$

Then the inductive hypothesis produces an equivalence of the adelic groups relative to the chain Δ. The cosimplicial structure is controlled by the coface/codegeneracy maps of the adelic group relative to $\mathcal{O}_T$, hence the isomorphism produced above is compatible with the cosimplicial maps induced by removing points x_i, $i \in \{1, \ldots, p\}$, from the chain Δ. The only coface map that needs to be analysed separately is the one relative to removing the point x from the chain Δ. In this case, naturality with respect to X of the equivalence (which is assumed inductively) allows us to conclude the compatibility. $\square$

In their paper [**31**], Halpern-Leistner and Pomerleano remark that their theorem 2.17 follows from a chain level identification. The same is true in our case. Indeed, we can produce a cochain-level version of rationalized equivariant K-theory via adelic descent, using the same techniques that we use to define rationalized equivariant elliptic cohomology in Section 3.3. Let us call this object $\mathcal{CK}_T(X)$. We impose its adèles at a chain $\Delta = (x, x_1, \ldots, x_p)$ with x_p closed to be given by

$$\mathbb{A}_T(\Delta, \mathcal{CK}_T(X)) = C_T^{\oplus,*}(X^{T(x)}) \otimes_{\mathcal{O}(\mathfrak{t})} \mathbb{A}_T(\Delta, \mathcal{O}_T)$$

where $C_T^{\oplus,*}(X^{T(x)})$ are the $\mathbb{Z}_2$-periodized T-equivariant cochains on $X^{T(x)}$ with coefficients in k, i.e. the $\mathbb{Z}_2$-periodization of the cochains on the Borel construction $X^{T(x)}//T$. The cochains $C_T^{\oplus,*}(X^{T(x)})$ are a module over $C_T^{\oplus,0}(*)$, which is formal, hence $C_T^{\oplus,0}(*) \simeq \mathcal{O}(\mathfrak{t})$. The perfect complex $\mathcal{CK}_T(X)$ is then constructed following the same inductive techniques in Section 3.3.

The same proof of Theorem 3.5.2 then proves a cochain-level variant:

THEOREM 3.5.7. *Let X be a smooth variety over $\mathbb{C}$ acted on by an algebraic torus T. There is an equivalence of objects in* $\mathrm{Perf}(T)$

$$\mathcal{HP}([X/T]) \simeq \mathcal{CK}_{T^{\mathrm{an}}}(X^{\mathrm{an}})$$

which is natural in X with respect to T-equivariant maps.

REMARK 3.5.8. *The hypothesis that X is a variety is sufficient to have that $\mathcal{CK}_T(X^{\mathrm{an}})$ is in* $\mathrm{Perf}(T)$*. Indeed, it belongs to* $\mathrm{Coh}(T)$ *as the analytification of a variety has the same homotopy type of a finite CW-complex, and T is smooth and Noetherian.*

3.5.2. Equivariant cohomology. In this subsection we prove an analogue of Theorem 3.5.2 for equivariant singular cohomology, using the same technique as in the K-theory case. The equivalence at the level of global sections is a consequence of work of Calaque–Pantev–Toën–Vaquié–Vezzosi [**16**]. Their work indeed proves the statement in far grater generality, for any Artin stack in characteristic zero. Our proof is tailored to quotient stacks, and for the statement at the level of sheaves. Nevertheless, since the base space $\mathfrak{g}//G$ is an affine scheme, we do not get a more general theorem with our techniques. We include these results here for completeness. In particular, this allows us to observe that our techniques — based on adelic descent and induction — adapt to treat equivariant cohomology as well as equivariant K-theory and equivariant elliptic cohomology, that is discussed in the paper [**67**].

First we define the relevant notion of Hochschild homology in this context.

DEFINITION 3.5.9. *Let $\mathcal{X}$ be a derived stack. We define the* linearized Hochschild homology *of $\mathcal{X}$ as the global sections*

$$\mathrm{HH}^{lin}(\mathcal{X}) := \mathcal{O}(T_{\mathcal{X}}[-1])$$

as an object of Mod_k*. Let X be a scheme acted on by a reductive algebraic group G. Let*

$$r : T_{[X/G]}[-1] \to \mathfrak{g}//G$$

be the structure map. We define:

$$\mathcal{HH}^{lin}([X/G]) := r_* \mathcal{O}_{T_{[X/G]}[-1]}$$

REMARK 3.5.10. *The* de Rham complex *of $\mathcal{X}$,* $\mathrm{DR}(\mathcal{X})$*, is a canonical structure of graded mixed complex on the linearized Hochschild homology of $\mathcal{X}$. This is explained in* [**54**].

The next step is defining a periodic cyclic version of linearized Hochschild homology. The first ingredient we need is a global $B\widehat{\mathbb{G}}_a$-action on the shifted tangent stack

$$T_{[X/G]}[-1]$$

This action will be used to perform a Tate construction similar to the procedure used in [**67**] in the context of equivariant elliptic cohomology, where the relevant action is that of the elliptic curve E. In our setting, the $B\widehat{\mathbb{G}}_a$-action comes from the identification

$$T_{\mathcal{X}}[-1] \simeq \underline{\mathrm{Map}}\left(B\widehat{\mathbb{G}}_a, \mathcal{X}\right)$$

for any derived stack $\mathcal{X}$ over k. This equivalence is explained in [**52**].

Lemma 3.5.11. *Let G be a smooth reductive algebraic group over k. The $B\widehat{\mathbb{G}}_a$-action on*

$$T_{BG}[-1]$$

induces a trivial action of $B\widehat{\mathbb{G}}_a$ on the affinization $\mathfrak{g}//G$.

Proof. This is a consequence of the fact that any differential p-form on BG over a field of characteristic zero is canonically closed, which implies that the de Rham differential acts trivially. This fact is explained in Section 5 of [**74**]. The $B\widehat{\mathbb{G}}_a$-action on $T_{BG}[-1]$ induces a mixed structure on the global sections

$$\mathcal{O}(T_{BG}[-1])$$

as there is an identification

$$\mathrm{QCoh}(BB\widehat{\mathbb{G}}_a) \simeq \mathrm{QCoh}(BB\mathbb{G}_a) \simeq \mathrm{QCoh}(BS^1)$$

which can be found in [**58**]. In particular, the triviality of the de Rham differential implies that the mixed structure on $\mathcal{O}(T_{BG}[-1])$ is trivial, which in turn induces a trivial $B\widehat{\mathbb{G}}_a$-action on

$$\mathrm{Spec}\, \mathcal{O}(T_{BG}[-1]) \simeq \mathfrak{g}//G$$

□

We follow the same steps as in [**67**]: Lemma 3.5.11 allows us to promote the action of $B\widehat{\mathbb{G}}_a$ on the shifted tangent stack

$$T_{[X/G]}[-1]$$

to an action relative to the base $\mathrm{Aff}(T_{BG}[-1]) \simeq \mathfrak{g}//G$ of the group stack $B\widehat{\mathbb{G}}_a \times \mathfrak{g}//G$ as an object of the comma ∞-category

$$(\mathrm{dSt}_k)_{/\mathfrak{g}//G}$$

This action induces on the quasi-coherent complex

$$\mathcal{HH}^{lin}([X/G])$$

the structure of a comodule over the Hopf algebra object

$$\pi_* \mathcal{O}_{B\widehat{\mathbb{G}}_a \times \mathfrak{g}//G}$$

internal to the ∞-category $\mathrm{QCoh}(\mathfrak{g}//G)$, for $\pi : B\widehat{\mathbb{G}}_a \times \mathfrak{g}//G \to \mathfrak{g}//G$ the canonical projection. This corresponds to a lift

$$\mathcal{HH}^{lin}([X/G]) \in \mathrm{QCoh}(\mathfrak{g}//G)^{S^1}$$

DEFINITION 3.5.12. *Let X be a scheme acted on by a reductive smooth algebraic group G. The* linearized periodic cyclic homology *of $[X/G]$*

$$\mathcal{HP}^{lin}([X/G])$$

is the image of $\mathcal{HH}^{lin}([X/G])$, as an object of $\mathrm{QCoh}(\mathfrak{g}//G)^{S^1}$, inside the $\mathbb{Z}_2$-periodic ∞-category $\mathrm{QCoh}(\mathfrak{g}//G)^{\mathbb{Z}_2}$.

The $\mathbb{Z}_2$-periodic ∞-category $\mathrm{QCoh}(\mathfrak{g}//G)^{\mathbb{Z}_2}$ is obtained following Preygel's definition of *Tate construction* for ∞-categories described in [**59**], equipping $\mathfrak{g}//G$ with a trivial S^1-action.

As a preliminary to the theorem, we need a localization formula for the shifted tangent stack of a quotient. We give an analytic-topology version of the localization theorem.

PROPOSITION 3.5.13. *Let X be a smooth variety over $\mathbb{C}$ acted on by an algebraic torus T. For any closed point ξ of $\mathfrak{t}$ there exists a analytic open neighbourhood U_ξ of ξ in $\mathfrak{t}$ such that the map induced by the inclusion $t_0X^{T(\xi)} \hookrightarrow X$*

$$T_{[t_0X^{T(\xi)}/T]}[-1]^{\mathrm{an}} \times_{\mathfrak{t}} U_\xi \to T_{[X/T]}[-1]^{\mathrm{an}} \times_{\mathfrak{t}} U_\xi$$

is an equivalence of derived analytic stacks.

PROOF. This fact is a simple consequence of the localization formula for the loop space, Theorem 3.1.12 of [**17**], and the analytic non-equivariant Chern character. Localization for the loop space $\mathcal{L}[X/T]$ implies that, for a closed point $z \in T$, there is an analytic neighbourhood U_z of z such that the map

$$\mathcal{L}[t_0X^{T(z)}/T]^{\mathrm{an}} \times_{T^{\mathrm{an}}} U_z \to \mathcal{L}[X/T]^{\mathrm{an}} \times_{T^{\mathrm{an}}} U_z$$

induced by the inclusion $t_0X^{T(z)} \hookrightarrow X$ is an equivalence of analytic derived stacks. We choose a closed point z of T such that $T(z) = T(\xi)$. Such a point always exists, and in particular is contained in a small analytic neighbourhood of $\exp(\xi)$. By localization for the loop space, there is an analytic neighbourhood U_z of z such that

$$\mathcal{L}[t_0X^{T(z)}/T]^{\mathrm{an}} \times_{T^{\mathrm{an}}} U_z \to \mathcal{L}[X/T]^{\mathrm{an}} \times_{T^{\mathrm{an}}} U_z$$

is an equivalence. We can assume U_z small enough that the preimage via exp is a disjoint union of analytic disks in $\mathfrak{t}$. Call U_ξ the only such disk containing ξ. Then, the exponential map restricted to U_ξ

$$T_{[t_0X^{T(\xi)}/T]}[-1]^{\mathrm{an}} \times_{\mathfrak{t}} U_\xi \to T_{[X/T]}[-1]^{\mathrm{an}} \times_{\mathfrak{t}} U_\xi$$

is the equivalence we were seeking. $\square$

We can finally prove the following theorem:

THEOREM 3.5.14. *Let X be a smooth variety over $\mathbb{C}$ acted on by an algebraic torus T. There is an isomorphism of $\mathbb{Z}_2$-periodic perfect complexes on* $\mathfrak{t}$

$$\mathcal{HP}^{lin}([X/T]) \simeq \mathcal{H}_{T^{\mathrm{an}}}(X^{\mathrm{an}})$$

where on the left-hand side we have the homotopy sheaves of linear periodic cyclic homology, and on the right-hand side the perfect complex on $\mathfrak{t} = \operatorname{Spec} H^{\oplus,0}_{T^{\mathrm{an}}}(*) \simeq \operatorname{Spec} C^{\oplus,0}_{T^{\mathrm{an}}}(*)$ *whose global sections are the equivariant singular cochains* $C^{\oplus,*}_{T^{\mathrm{an}}}(X^{\mathrm{an}})$.

PROOF. The proof of this theorem is the same as for Theorem 3.5.2, so we give a very short treatment. It is based on the localization formula for the shifted tangent of a quotient, Proposition 3.5.13, and on the same inductive argument of the proof of Theorem 3.5.2. In this situation, the localization theorem for the shifted tangent stack of $[X/T]$ gives an identification of the completion

$$\widehat{T}_{[X/T]}[-1]$$

with the completion

$$\widehat{T}_{[t_0 X^{T(\xi)}/T]}[-1]$$

which is induced by the inclusion $t_0 X^{T(\xi)} \hookrightarrow X$. In particular, we obtain

$$\mathbb{A}_{\mathfrak{t}}(\Delta, \mathcal{HP}^{lin}([X/T])) = \mathrm{HP}([t_0 X^{T(\xi)}/T]) \otimes_{\mathcal{O}(\mathfrak{t})} \mathcal{O}_{\mathfrak{t},\hat{\Delta}}$$

for a chain $\Delta = (\xi > \xi_1 > \cdots > \xi_0)$ on $\mathfrak{t}$, where ξ_0 is a closed point. In this context, the group $T(\xi)$ is the one appearing in Definition 3.4.1. On the other hand, for the complex $\mathcal{H}_{T^{\mathrm{an}}}(X^{\mathrm{an}})$ we have

$$\mathbb{A}_{\mathfrak{t}}(\Delta, \mathcal{H}_{T^{\mathrm{an}}}(X^{\mathrm{an}})) = C^{\oplus,*}_{T^{\mathrm{an}}}((t_0 X^{T(\xi)})^{\mathrm{an}}) \otimes_{\mathcal{O}(\mathfrak{t})} \mathcal{O}_{\mathfrak{t},\hat{\Delta}}$$

If ξ_0 is not a closed point, and to deal with the cosimplicial structure, we use the exact same inductive argument as in the proof of Theorem 3.5.2. Indeed, if $\mathrm{rk}(T) = 1$ the conclusion is immediate. $\square$

CHAPTER 4

Partial results and future perspectives

4.1. Partial results

In this section we collect some facts we proved which did not appear yet outside of this thesis. First of all, we give a new definition of quasi-constant maps that makes sense for all target stacks. Recall that any map in an ∞-topos can be factored essentially uniquely as an effective epimorphism followed by a monomorphism.

DEFINITION 4.1.1. *Let X and Y be derived stacks. The* derived stack quasi-constant maps *from Y to X, $\underline{\mathrm{Map}}^0(Y,X)$, is the smallest clopen stack containing the image of the map induced by the unit of the de Rham stack of Y, $\underline{\mathrm{Map}}(Y_{dR},X) \to \underline{\mathrm{Map}}(Y,X)$:*

$$\begin{tikzcd} \underline{\mathrm{Map}}(Y_{dR},X) \arrow[r] \arrow[d] & \underline{\mathrm{Map}}(Y,X) \\ \underline{\mathrm{Map}}^0(Y,X) \arrow[ur] & \end{tikzcd}$$

CONJECTURE 4.1.2. *Let E be an elliptic curve over a field k of characteristic zero and X be either a variety over k or a quotient stack $[Y/T]$ of a variety Y by an algebraic torus T over k. Then Definition 4.1.1 is equivalent to Definition 2.2.19.*

4.2. Future perspectives

In this section we discuss potential future developments of the material contained in the thesis. The main two aspects to be discussed are an extension of the constructions carried out in the thesis to positive characteristic — and eventually to the sphere spectrum — and non-commutative variants. Other small aspects are discussed, such as the presence of 1-shifted symplectic structures on $\underline{\mathrm{Map}}^0(E,BG)$ depending on forms on $\mathfrak{g}$.

4.2.1. Extension to characteristic p. Let us consider the shifted tangent stack of a quotient stack together with its canonical map

$$r : T_{[X/G]}[-1] \to \mathfrak{g}/\!/G$$

Over the field $\mathbb{C}$ of the complex numbers, the Tate construction on the global sections

$$\mathcal{O}(T_{[X/G]})^{tS^1} \simeq H_G^{\oplus,*}(X^{\mathrm{an}})$$

computes the Betti cohomology of the analytification of X. Similarly, for a field k of characteristic zero, the same object computes the G-equivariant algebraic de Rham cohomology of X.

One might ask what happens when k is of positive characteristics. I was suggested by Bertrand Toën that the correct object would be "equivariant crystalline cohomology". The Tate construction might be much harder to perform, as there would be no obvious reason for the action on the base to be trivial. Nevertheless, one might assume that it is still possible to do a Tate construction, and this would define three equivariant variations of crystalline cohomology: a "plain" version over $\mathfrak{g}//G$, a K-theoretic version over $G//G$, and an elliptic version over E_G. This theory would deviate from the topological counterpart, and in particular we should not expect there to be a Grojnowski picture for equivariant elliptic cohomology in the positive characteristic setup.

4.2.2. Non-commutative analogue. Let us focus on classical Hochschild homology first. In this case, the non-commutative formulation amounts to extending the notion of Hochschild homology from commutative algebras to non-commutative algebras, and from schemes to *non-commutative schemes*, i.e. stable ∞-categories. This can be done by observing that the definition of Hochschild homology of a commutative algebra $A \in \mathrm{CAlg}$

$$\mathrm{HH}_*(A) = A \otimes_{A\otimes A} A$$

can be extended easily to non-commutative ones:

$$\mathrm{HH}_*(A) = A \otimes_{A\otimes A^{\mathrm{op}}} A$$

More interesting is the approach via *topological quantum field theory*. The Hochschild homology of A, $\mathrm{HH}^*(A)$, can indeed be constructed as the partition function of (framed) a one-dimensional TQFT Z_A canonically attached to the commutative algebra A. By the Cobordism Hypothesis, we are allowed to construct such a TQFT by assigning its *algebra of observables*. In the case of Hochschild homology of a commutative algebra A, this is the ∞-category of its modules, Mod_A. The partition function of Z_A, i.e. evaluation on the circle cobordism, gives

$$Z_A(S^1) = \mathrm{HH}(A) = \dim \mathrm{Mod}_A$$

the Hochschild homology of A, i.e. the dimension of its category of modules. This picture generalizes obviously to give a notion of Hochschild homology of a fully dualizable object in $\mathrm{Pr}^{L,st}$, its dimension.

In the case of *elliptic Hochschild homology*, we expect that the role of the one-dimensional TQFT should be taken by a *two-dimensional conformal field theory*. The correct non-commutative objects admitting such an invariant would be vertex algebras as opposed to associative algebras — or E_2-algebras, in the case of two-dimensional topological theories. Similarly to the case of classical Hochschild homology, this would extend to modular tensor categories (over the complex numbers), as modules over a vertex algebra form such categorical structures.

Algebraically, such construction will mimic the topological case. The replacement for the cobordism hypothesis would be a conjectured equivalence between mathematical theories of two-dimensional CFTs:

- vertex algebras and chiral algebras;
- conformal nets;
- Segal chiral CFTs.

We are mostly interested in the first and last items in the list above. A Segal chiral CFT is a functorial axiomatization of a CFT, hence that would play the role of the TQFT. The vertex algebra plays the role of the associative algebra. The procedure is as follows:

(1) select a vertex algebra V, or a modular tensor category $\mathcal{C}$;
(2) to this data, there is an associated Segal 2d CFT Z_V or $Z_{\mathcal{C}}$;
(3) we evaluate this CFT on a chosen elliptic curve E to obtain an object $\mathrm{HH}_E(V)$ (or $\mathrm{HH}_E(\mathcal{C})$) in $\mathrm{Mod}_{\mathbb{C}}$.

We remark that, in order to get an object in $\mathrm{Mod}_{\mathbb{C}}$, we need to suitably categorify the theory (which is trickier for Segal CFTs than for TQFTs: in particular we need to deloop the vertex algebra twice, i.e. consider $\mathrm{Mod}_{\mathrm{Mod}_V}$, as value on the circle).

4.2.2.1. *CFT in positive characteristic.* One might hope that the relationship between elliptic cohomology and CFT could be pushed beyond the complexified setting. This is probably not possible, unless we accept that the cohomology theory arising in this context is some version of *algebraic* elliptic cohomology. The first step is an extention of the notion of CFT to positive characteristics. There are many potential approaches to such extension:

- one first (and most likely incomplete) formulation would be a variant of Segal CFTs for rigid analytic geometry;
- one second approach would be to develop the correct notion of vertex algebras in characteristic p — a notion akin to *partition* Lie algebras.

4.2.3. Shifted symplectic structures. Let l be a symmetric, bilinear, non-degenerate, adjoint-invariant form on $\mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of a compact, connected, simply connected, simple Lie group G. Such a form corresponds to an element $l \in H^4(BG;\mathbb{Z})$. Classically, this is the data necessary to construct the corresponding Looijenga line bundle, that has relevance in the theory of representations of loop groups — its global sections are *nonabelian theta functions*.

This data is also equivalent to a *2-shifted symplectic structure* on the classifying stack BG. In particular, this induces a 1-shifted symplectic structure on $\underline{\mathrm{Map}}^0(E, BG)$. One could consider quantizations of this structure. Pavel Safronov suggested that its deformation quantization is related to the theory of *elliptic quantum groups*. We could also consider its *geometric quantization*, if it exists. In that case, the resulting prequantum $\mathbb{G}_m$-gerbe would be defined by the same set of data as a Looijenga line bundle. This *Looijenga gerbe* might have interesting global sections, i.e. the category of positive energy representations of the loop group of G.

The main question is the existence of the Looijenga gerbe. This would be the main issue to solve.

4.2.4. Quantizations of quasi-coherent sheaves on E. Let $G = \mathbb{G}_m$ for simplicity. We want to make sense of the following slogan:

level l elliptic cohomology of the point is a quantization of the elliptic cohomology of the point.

This slogan is clear from the physical viewpoint. The G-equivariant elliptic cohomology of the point is (in degree zero) E itself. The choice of a level $l \in H^4(BG;\mathbb{Z})$ induces a symplectic structure on E whose prequantum line bundle is the Loojenga line bundle at level l. The global sections of this bundle, the theta functions, are the fiber over E in the moduli stack of elliptic curves of the bundle of conformal blocks of Wess–Zumino–Witten theory, and in particular they give the level l equivariant elliptic cohomology of the point.

What we expect is that the quantization could be carried out at the level of the category $\mathrm{QCoh}(E)$, whose monoidal unit is the equivariant elliptic cohomology of the point: the geometric quantization of E should correspond to some quantization of the stable ∞-category $\mathrm{QCoh}(E)$ itself. I do not know if such theory exists already. We would then observe that under this process the unit $\mathcal{O}_E = \mathcal{E}ll^0_G(*)$ is mapped exactly to this space of global sections, for some mysterious reason.

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