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L^1-optimal transport and functional inequalities in spaces with curvature dimension condition

Candidate:
Sara Farinelli

Advisor:
Prof. Fabio Cavalletti

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Content of the thesis

This thesis treats two main topics. The first one, contained in Chapter 2 and Chapter 3 and taken from [35] and [48], is the study of functional inequalities linked to optimal transport in spaces satisfying synthetic Ricci curvature lower bounds. The focus is on an indeterminacy estimate linking the Wasserstein distance and the perimeter measure and on optimal transport estimates related to eigenfunctions of the Laplacian. The results contain new contributions also in the classical setting. Two additional inequalities involving optimal transport and heat flow are presented in Appendix A.
The second topic, contained in Chapter 4 and taken from [34], concerns a proof of the existence of a solution to the Monge problem for the distance cost between absolutely continuous measures, with bounded densities, in an infinite product of $\mathsf{CD}(K, N)$ spaces with finite N provided that some additional hypotheses are satisfied.

Contents

Introduction

I Spaces with synthetic Ricci curvature lower bounds

The theory of metric measure spaces satisfying synthetic Ricci curvature lower bounds, born around twenty years ago, has faced in the last decades a very rapid growth.

The works of Cordero-Erasquin, McCann and Schmuckenshläger [47] and Sturm and Von Renesse [99] proved that for a smooth Riemannian manifold, having Ricci curvature bounded from below by K is equivalent to displacement convexity properties of an entropy functional along 2-Wasserstein geodesics. This second formulation, relying only on the metric of the manifold and on the volume measure, led Lott and Villani [73] and Sturm [93], [94] to the definition of a synthetic notion of lower Ricci curvature bounds for metric measure spaces (i.e. metric spaces endowed with a non negative Borel measure finite on bounded sets). This notion is called Curvature Dimension Condition $\mathsf{CD}(K,\infty)$ or $\mathsf{CD}(K,N)$, the second one encoding in a synthetic sense also an upper bound on the dimension of the space. The $\mathsf{CD}(K,N)$ condition is compatible with the smooth case in the sense that a smooth Riemannian manifold satisfies the $\mathsf{CD}(K,N)$ condition if and only if its dimension is less or equal than N and its Ricci curvature is greater or equal than K. $\mathsf{CD}(K,N)$ spaces are also closed under *measured Gromov Hausdorff convergence*, and thus contain limits of Riemannian manifolds.

Since the class of CD spaces includes also Finsler structures, the more restrictive Riemannian curvature dimension condition, $\mathsf{RCD}(K,N)$, was later introduced in [10] ([6] for the σ-finite setting) in the infinite dimensional case and in [58] in the finite dimensional case. This is defined by coupling the CD condition with the *Infinitesimal Hilbertianity condition*, which requires the Sobolev space $W^{1,2}$ to be an Hilbert space. We also recall the weaker $\mathsf{RCD}^*(K,N)$ condition (in correspondence with the weaker $\mathsf{CD}^*(K,N)$ condition introduced in [16]) which is now known to be equivalent to $\mathsf{RCD}(K,N)$ for finite measure after [37]. Moreover an equivalent distributional formulation of $\mathsf{RCD}(K,N)$ spaces using the so called Bakry-Émery curvature dimension condition is also available ([11, 55, 13]).

We finally mention a weaker variant of the $\mathsf{CD}(K,N)$ condition, namely the *Measure Contraction Property* defined through a convexity condition only along the 2-Wasserstein geodesics ending in a Dirac delta (see [78] and [94]). This weaker condition includes spaces, like the Heisenberg group which has been proved in [66] to not satisfy the $\mathsf{CD}(K,N)$ condition for any K and N but to satisfy the $\mathsf{MCP}(K,N)$ condition for suitable K and N. It is out of the scope of this note to give a deeper introduction to the history and developments of the theory and to give a complete list of references. During the treatment we will describe more in detail the aspects we are interested in.

In the following sections we introduce and present in detail the content of this thesis. In particular, Section II is an introduction to the works in Chapter 2 and 3. Section III is an introduction to Chapter 4. Finally Section IV contains description of some additional results contained in two appendices, respectively Appendix A and Appendix B.

II Indeterminacy estimates, eigenfunctions of the Laplacian and optimal transport

This is a short introduction to the works presented in Chapter 2 and Chapter 3 which treat similar topics, but use different approaches and tools. We give a detailed review of the content of them respectively in Section II.I and Section II.II.

In a smooth closed Riemannian manifold (M, g), an *eigenfunction of the Laplacian* operator Δ (the Laplace Beltrami operator of the manifold) of eigenvalue $\lambda > 0$ is a function f_λ which satisfies the equation

$$-\Delta f_\lambda = \lambda f_\lambda \quad \text{in } M.$$

The *nodal set* of an eigenfunction, namely its zero level set, has been object of an extensive investigation. An aspect of particular interest has been the study of its volume. Around 1980 Yau conjectured, in [100], that in a smooth closed N dimensional Riemannian manifold, eigenfunctions of the Laplacian of eigenvalue λ satisfy

$$C\sqrt{\lambda} \geq \mathcal{H}^{N-1}\{f_\lambda = 0\} \geq \frac{\sqrt{\lambda}}{C}, \tag{1}$$

where C depends only on the manifold and $\mathcal{H}^{N-1}$ is the $(N-1)$-dimensional Hausdorff measure. Among the many contributions we name Brüning [25] who solved the conjecture for $N = 2$, Donnelly and Fefferman [53] who settled the case of real analytic metrics, Colding and Minicozzi [46], Sogge and Zelditch [85, 86] who independently obtained the lower bound $\mathcal{H}^{N-1}(\{f_\lambda = 0\}) \geq c\lambda^{\frac{3-N}{4}}$, proved later also by Steinerberger [88]. A breakthrough on the problem has been obtained by Logunov in 2018, proving, for any $N \in \mathbb{N}$, a polynomial upper bound [70] and the complete lower bound [71]. For an overview and more references we refer to [72].

One could ask if analogous estimates hold for sum of eigenfunctions. In this direction, up to very recently, the only results were the Sturm Oscillation Theorem in dimension one (see [92], [19]), and an upper bound by Donnelly [52] in higher dimension.

In the first part of this thesis we focus on a new approach relating optimal transport and eigenfunctions introduced by Steinerberger (see [89]) to study the zero set of sum of them. This method turs out to be useful to prove that in a N-dimensional closed Riemannian manifold a function f which is a linear combination of eigenfunctions of eigenvalue bigger or equal than λ, satisfies

$$\mathcal{H}^{N-1}\{f = 0\} \geq \sqrt{\frac{\lambda}{\log(\lambda)^N}}\, C(\|f\|_{L^1}, \|f\|_{L^2}, \|f\|_{L^\infty}). \tag{2}$$

In [89] it is considered the case of 2-dimensional Riemannian manifolds, while in [82] the Euclidean case of any dimension (see also [90]). An analogous estimate has been proved in [30], without the logarithmic factor, for Riemannian manifolds of any dimension.

The idea to get (2) is to combine two inequalities: an upper bound of the type

$$W_1(f^+, f^-) \leq \sqrt{\frac{\log(\lambda)}{\lambda}}\, C(\|f\|_{L^1}, \|f\|_{L^2}, \|f\|_{L^\infty}), \tag{3}$$

for f linear combination of eigenfunctions of eigenvalue bigger or equal than λ with an indeterminacy estimate of the type

$$W_1(f^+, f^-)\mathcal{H}^{N-1}(\{f = 0\}) \geq C(\|f\|_{L^1}, \|f\|_{L^\infty}),$$

valid in a smooth closed N-dimensional Riemannian manifold for any continuous function f with zero integral (for a more detailed presentation on this type of inequality, see the next subsection). With $W_1(f^+, f^-)$ we mean $W_1(f^+\mathrm{Vol}, f^-\mathrm{Vol})$ which makes sense if f has zero integral, as it is the case of eigenfunctions.

We mention also that in relation to (3), Steinerberger in [91] conjectured that in a closed Riemannian manifold there exist constants C and c such that

$$\frac{C}{\sqrt{\lambda}}\|f_\lambda\|_{L^1}^{\frac{1}{p}} \geq W_p(f_\lambda^+, f_\lambda^-) \geq \frac{c}{\sqrt{\lambda}}\|f_\lambda\|_{L^1}^{\frac{1}{p}}. \tag{4}$$

His motivation behind this guess is the intuition that eigenfunctions oscillate at scale $\frac{1}{\sqrt{\lambda}}$ and so one would hope to transport their positive part into the negative one by moving everything by $\frac{1}{\sqrt{\lambda}}$ (see [91]). We mention that the upper bound for $p = 1$ was proved in [30].

This approach of using the tool of optimal transport to get a lower bound for the measure of the zero set of (linear combinations of) eigenfunctions is particularly suitable to the context of metric measure spaces satisfying synthetic lower Ricci curvature bounds where the smooth tools developed in the classical literature are not at our disposal. The links between indeterminacy estimates, eigenfunctions of the Laplacian and bounds on the Wasserstein distance between the positive part and the negative part of an eigenfunctions are the focus of Chapter 2 and Chapter 3. We give a detailed presentation of them in the two next subsections.

II.I Indeterminacy estimate via localization and lower bound for the nodal set of eigenfunctions

In this Section we review the results obtained in [35] that are presented in detail in Chapter 2.

Indeterminacy estimate I

Recently there has been an emerging interest in indeterminacy estimates of the following type. Let $f : [0, 1]^N \to \mathbb{R}$ be a continuous function with $\int_{[0,1]^N} f(x)\,\mathrm{d}x = 0$. Then

$$W_1(f^+\mathrm{d}x, f^-\mathrm{d}x)\mathcal{H}^{N-1}\{f = 0\} \geq C\left(\frac{\|f\|_{L^1}}{\|f\|_{L^\infty}}\right)^\alpha \|f\|_{L^1},$$

for some $\alpha \geq 1$.

First Steinerberger in [89] proved the inequality in dimension $N = 2$ with exponent $\alpha = 1$. Then Sagiv and Steinerberger proved the result in any dimension N with a dimensional exponent $4-\frac{1}{N}$ in [82]. In that work they also conjectured that the estimate should hold with exponent $\alpha = 1$. Carrol, Massaneda and Ortega-Cerdá extended the result to the setting of smooth closed Riemannian manifolds of any dimension N and improved the exponent to $2 - \frac{1}{N}$ in [30]. They also gave a proof of the fact that the estimate cannot hold with exponent smaller than $\alpha = 1$. A particular case of this inequality appeared recently also in [29].

The first part of Chapter 2, more precisely, Section 2.1 and Section 2.2, are devoted to the proof of an indeterminacy estimate with sharp exponent $\alpha = 1$, which holds in the general setting of metric measure spaces satisfying synthetic Ricci curvature lower bounds. In particular we prove the following result which is presented in its full generality in Theorem 2.2.2.

Theorem 1. *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an essentially non-branching m.m.s. satisfying the* $\mathsf{CD}(K, N)$ *condition with* $1 < N < +\infty$. *Let* $f : X \to \mathbb{R}$ *a continuous function with* $\int_X f\, \mathfrak{m} = 0$. *Assume also the existence of* $x_0 \in X$ *such that* $\int_X |f(x)|\, \mathsf{d}(x, x_0)\, \mathfrak{m}(dx) < \infty$. *Then*

$$W_1(f^+\mathfrak{m}, f^-\mathfrak{m}) \cdot \mathsf{Per}\left(\{x \in X\colon f(x) > 0\}\right) \geq C(K, D) \frac{\|f\|_{L^1}^2}{\|f\|_{L^\infty}}, \tag{5}$$

where $D = \mathrm{diam}\,(X)$.

The constant $C(K, D)$ is explicit and does not depend on D if $K \geq 0$, in which case the result is non trivial also in the unbounded case (see Theorem 2.2.2). The perimeter measure, which can be defined also in metric measure spaces, plays the role of the codimension-one Hausdorff measure. The class of spaces considered in the previous theorem includes smooth Riemannian manifolds and in particular the Euclidean space.

We get the sharp non dimensional exponent, $\alpha = 1$, thanks to the 1-dimensional localization technique which particularly fits to L^1-optimal transport problems (see Section III for an introduction and Section 1.6 for a more detailed treatment and references). In particular, localization allows to reduce the problem to one dimensional problems lying on weighted intervals with $\mathsf{CD}(K, \infty)$ densities if starting from a $\mathsf{CD}(K, N)$ space or with $\mathsf{MCP}(K, N)$ densities if starting from a $\mathsf{MCP}(K, N)$ space. To prove the estimate in weighted intervals we proceed in two steps. In Section 2.1.1 we first prove a sharp, in the exponent, indeterminacy estimate for the flat real interval, slightly different from the one already present in [91]. Notice that when dealing with the problem in one dimension, it is easier to get the sharp exponent and we remark that in dimension $N = 1$ and $N = 2$, the results already present were sharp. For completeness we add that recently the sharp and rigid version of the one dimensional indeterminacy estimate has been proved in [54]. The second step is to use part of the proof of estimate of Section 2.1.1, to prove an indeterminacy estimate still sharp in the exponent in the weighted interval, provided that the weighted measure satisfies some curvature dimension condition. In Section 2.1.2 we tackle the case of $\mathsf{CD}(K, \infty)$ densities while in Section 2.1.3 we go through the case of $\mathsf{MCP}(K, N)$ densities. The key idea is to exploit the log-concavity properties of the densities to get an indeterminacy estimate involving the corresponding perimeter measure.

Finally in Section 2.2, we "reintegrate" the estimate by using the localization theorem for $\mathsf{CD}(K, N)$ spaces or $\mathsf{MCP}(K, N)$ spaces. In passing from "one dimensional" geodesics to the whole space the same exponent is kept. Moreover in the $\mathsf{CD}(K, N)$ case, no dependence on the N appears.

Estimates for the measure of nodal sets of eigenfunctions

In the second part of Chapter 2, (Section 2.3) we apply the indeterminacy estimate proved before to get lower bounds for the measure of the nodal set of eigenfunctions of the Laplacian (and sums of them) in the non-smooth setting. Thanks to the weak notion of Laplacian which is available in the metric setting, (see Section 1.4.1) we can consider its eigenfunctions. We mention that even if the space satisfies some strong curvature assumption like the $\mathsf{RCD}(K,N)$ condition, where eigenfunctions of the Laplacian are continuous (see Remark 1.4.13 for more details and references) and the zero set is then a well defined closed set, in singular spaces no regularity results are at our disposal. As opposed to smooth manifolds, where the zero set is a codimension-one submanifold up to a codimension-two set (see [63], [28]), in our case it is not excluded yet that it has non-empty interior. This is because the weak unique continuation property for eigenfunctions is not known to hold. We mention for completeness in this regard the recent [51] which disproved the strong unique continuation property for harmonic functions in $\mathsf{RCD}(K,N)$ spaces, $N \geq 4$. So on one hand the zero set lacks of regularity, on the other side many of the tools developed to build the smooth theory are not available, as a consequence of the absence of a unique continuation result. For this reason the optimal transport approach which requires lower regularity is suitable for our setting. Notice, in this regard, that the estimate we proved in Theorem 3 involved, in place of the codimension-one measure of the zero set, the term $\mathsf{Per}(\{f>0\})$, which can be thought as the measure of the interface between the positive and negative part of f. From now on we refer to $\mathsf{Per}(\{f>0\})$ also as measure of the nodal set.

As said at the beginning of this section, the idea to get lower bound on $\mathsf{Per}(\{f_\lambda>0\})$, for f_λ eigenfunction of eigenvalue λ, is to link the indeterminacy estimate, with an upper bound on $W_1(f_\lambda^+, f_\lambda^-)$. In Subsection 2.3.1 we derive estimates for $W_1(f_\lambda^+, f_\lambda^-)$ in the cases of MCP and CD spaces. We obtain the following upper bound (see Lemma 2.3.1):

$$W_1(f_\lambda^+\mathfrak{m}, f_\lambda^-\mathfrak{m}) \leq \frac{\sqrt{\mathfrak{m}(X)}}{\sqrt{\lambda}}\|f_\lambda\|_{L^2}.$$

This combined with the indeterminacy estimate in (6) gives the lower bound in the case of a $\mathsf{CD}(K,N)$ space:

$$\mathsf{Per}(\{x\in X : f_\lambda(x)>0\}) \geq C_{K,D}\frac{\sqrt{\lambda}}{\sqrt{\mathfrak{m}(X)}}\cdot\frac{\|f_\lambda\|_{L^1}^2}{\|f_\lambda\|_{L^2}\|f_\lambda\|_{L^\infty}},$$

for every eigenfunction f_λ of eigenvalue $\lambda>0$, where $D=\operatorname{diam}(X)$ (see Theorem 2.3.2 for the precise statement). An analogous estimate holds in the MCP case (see Theorem 2.3.3). In Section 2.3.2 we tackle the problem in spaces satisfying the $\mathsf{RCD}(K,N)$ condition. The presence of a linear heat flow allows to get a better estimate of $W_1(f_\lambda^+, f_\lambda^-)$. The idea is to combine the fact that given f_λ eigenfunction of eigenvalue $\lambda>0$, $H_t(f_\lambda)=e^{-\lambda t}f_\lambda$ decays very fast to zero if λ is large, with a W_2-convergence estimate to the initial datum (see Theorem 2.3.4). In particular we show (see Proposition 2.3.6)

$$W_1(f_\lambda^+\mathfrak{m}, f_\lambda^-\mathfrak{m}) \leq C_{K,N,D}\sqrt{\frac{\log\lambda}{\lambda}}\|f_\lambda\|_{L^1}.$$

Again combining this with the indeterminacy estimate (6) and also a L^1-L^∞ estimate for eigenfunctions from [12] gives the following result.

Theorem 2. *Let $K,N\in\mathbb{R}$ with $N>1$. Let $(X,\mathsf{d},\mathfrak{m})$ be a m.m.s. verifying $\mathsf{RCD}(K,N)$, with $\operatorname{diam}(X)=D<\infty$ and $\mathfrak{m}(X)=1$. Let f_λ be an eigenfunction of the Laplacian of eigenvalue $\lambda>\max\{2,D^{-2}\}$. Then:*

$$\mathsf{Per}\left(\{x\in X\colon f_\lambda(x)>0\}\right) \geq \frac{1}{\bar{C}_{K,D,N}}\frac{1}{\sqrt{\log\lambda}}\lambda^{\frac{1-N}{2}}.$$

See Theorem 2.3.8 for a more detailed statement. It is worth to mention that in the smaller class of non-collapsed $\mathsf{RCD}(K,N)$ spaces, the perimeter coincides, as in the smooth setting, with the $(N-1)$-Hausdorff measure of the reduced boundary (see [3] and [24]). Finally the linearity of the Laplacian in the Infinitesimally Hilbertian case allows to prove similar lower bounds for linear combinations of eigenfunctions. This is treated in Section 2.4.

In conclusion, thanks to the optimal transport approach we are able to get results on the nodal sets of eigenfunctions even under very low regularity assumptions. We underline that our estimate is meaningful also in the case of an eigenfunction having a zero set with non-empty interior. Finally, by improving on the $L^1 - L^\infty$ estimate (to the one present in the smooth setting in [86]) and by getting the conjectured upper bound for W_1 in (4) one would be able to recover the estimate by Colding and Minicozzi.

II.II Indeterminacy estimate and lower bound for the Wassertein distances of eigenfunctions via heat flow

In this Section we review the results obtained in [48], in collaboration with Nicolò De Ponti, that are presented in detail in Chapter 3.

Indeterminacy estimate II

We dedicate Section 3.2 to a proof of the indeterminacy estimate in a different setting than in the previous chapter. As we already observed, estimate in Theorem 1 does not depend on the parameter N in the $\mathsf{CD}(K,N)$ condition. This suggests that an analogous estimate should hold in a space without upper bounds on the dimension: this is what we prove here. We underline that apart from the different setting, the interest in this estimate lies also on its proof which uses very different techniques from the previous presented. To state our first result we first need to recall the definition of Cheeger constant of a finite measure metric measure space $(X,\mathsf{d},\mathfrak{m})$:

$$h(X) := \inf\left\{\frac{\mathsf{Per}(A)}{\mathfrak{m}(A)} \,:\, A\subset X \text{ Borel with } 0<\mathfrak{m}(A)\le \frac{\mathfrak{m}(X)}{2}\right\}.$$

Theorem 3. *Let $(X,\mathsf{d},\mathfrak{m})$ be a space of finite measure satisfying the $\mathsf{RCD}(K,\infty)$ condition for some $K\in\mathbb{R}$. Let $f\in L^\infty(X)$ be such that $\int_X f\,\mathfrak{m}=0$ and $\int_X \mathsf{d}(\bar{x},x)\,|f(x)|\,\mathfrak{m}(\mathrm{d}x)<+\infty$ for some $\bar{x}\in X$. Then one has*

$$W_1(f^+\mathfrak{m}, f^-\mathfrak{m})\mathsf{Per}(\{f>0\}) \ge C(K,h(X))\left(\frac{\|f\|_{L^1}}{\|f\|_{L^\infty}}\right)\|f\|_{L^1}, \tag{6}$$

where $C(K,h(X))$ is explicit. If $K\ge 0$, $C(K,h(X))$ is a numeric constant.

It is important to remark that $C(K,h(X))$ is positive whenever $h(X)$ is positive (see Theorem 3.2.1). Observe in addition that we get again the sharp exponent discussed in the previous part.

The crucial ingredients of our proof are an inequality proved by Luise and Savaré in [74, Theorem 5.2], that we report in Proposition 3.1.1, linking the Wasserstein distance between two finite measures with the same total mass to the Hellinger distance (see Definition 1.3.1) between their evolution via the heat flow. As one can observe we get the result only for a subclass of $\mathsf{CD}(K,\infty)$ spaces. This is because our proof relied on linearity properties of the heat flow that are guaranteed only adding the infinitesimal Hilbertianity assumption (see Section 1.4).

Note that, differently from Theorem 1, we are not requiring the space to have finite diameter for $K < 0$. In particular, estimate (6) is meaningful also for spaces with finite measure, infinite diameter and positive Cheeger constant. One example of space satisfying these three assumptions can be found in [50]. We remark that in addition the assumption of having finite diameter implies the positivity of the Cheeger constant.

We then show that from the previous indeterminacy estimate it can directly be derived the analogous, sharp (in the exponent) indeterminacy estimate valid for W_p, for any $p \geq 1$. Namely, under the same hypotheses of Theorem 3 it holds:

$$W_p(f^+\mathfrak{m}, f^-\mathfrak{m})\mathsf{Per}(\{f>0\}) \geq C_p(h(X),K)\left(\frac{\|f\|_{L^1}}{\|f\|_{L^\infty}}\right)\|f\|_{L^1}^{\frac{1}{p}}.$$

We conclude Section 3.2 by showing an analogous indeterminacy estimate for the more general Hellinger-Kantorovich distance (see 1.10 for the definition). We do not report here the precise statement (see Theorem 3.2.7), but we notice that it is more refined if compared with Theorem 3, even if more implicit, and valid also for functions with non zero integral.

Lower bound for the Wasserstein distances of eigenfunctions

In Section 3.3 we prove the lower bound in the conjecture of Steinerberger (4) for any $p \geq 1$: we state it in the setting of $\mathsf{RCD}(K,\infty)$ spaces, recalling that smooth closed Riemannian manifolds are included in the class of $\mathsf{RCD}(K,\infty)$ spaces provided that their Ricci curvature is bounded from below by K.

Theorem 4. *Let $M > 0$, $K \in \mathbb{R}$ and $(X,\mathsf{d},\mathfrak{m})$ be an $\mathsf{RCD}(K,\infty)$ space of finite measure. Then for any non-constant eigenfunction f_λ of the Laplacian, of eigenvalue $\lambda \geq M$ and satisfying $\int_X \mathsf{d}(\bar{x},x)|f_\lambda(x)|\,d\mathfrak{m}(x) < +\infty$ for some $\bar{x} \in X$, it holds*

$$W_p(f_\lambda^+\mathfrak{m}, f_\lambda^-\mathfrak{m}) \geq C(K,M,p)\frac{1}{\sqrt{\lambda}}\|f_\lambda\|_{L^1}^{\frac{1}{p}}.$$

The proof, for the case $p = 1$, is again based on the Wasserstein-Hellinger contraction estimate in Proposition 3.1.1. The case $p > 1$ directly follows as for the indeterminacy estimate. We point out that contemporarily to our result appeared in [77] a proof of the same lower bound, in the case of closed Riemannian manifolds. The argument there is completely different from our and uses fine properties of eigenfunctions and elliptic PDE's techniques.

III L^1-optimal transport

In this Section we describe briefly what is contained in Chapter 4. We first give a short general introduction and then we describe in detail our results.

III.I An introduction to L^1-optimal transport and localization

In a metric space (X,d), the L^1-optimal transport problem between two probability measures μ and ν, is the one of finding a minimizer for

$$T \mapsto \int_{X\times X} \mathsf{d}(x,T(x))\mu(\mathrm{d}x),$$

among all the Borel maps $T : X \to X$ which satisfy the compatibility condition $T_\sharp\mu = \nu$ (see (1.4) for the definition). A first attempt of solution for $X = \mathbb{R}^n$ and μ, $\nu \ll \mathcal{L}^n$, is given by Sudakov. His proof is based on reducing the problem to subregions of lower dimension of $\mathbb{R}^n$ on which the Kantorovich potential is linear. Even if his original argument in [95] contains a gap, his idea turns out to be effective for the solution of the problem. More in detail, his strategy is to fix a Kantorovich potential φ and partition almost all the space into maximal oriented segments (x, y), called *transport rays*, such that $\varphi(x) - \varphi(y) = |x - y|$. Via disintegration theory, one can reduce the problem to one dimensional transport problems, lying on the rays, between the conditional measures of μ and ν. If the marginals of μ are non atomic, then one can consider the monotone rearrangements along the rays and glue them to get an optimal map in the whole space. The gap in Sudakov's argument is in the justification that the marginals are non atomic and his approach is made rigorous later in [1]. Different proofs of the existence of L^1-optimal maps are present in [56], [96], [27], [15] (see also [57] for the setting of Riemannian manifolds).

In [21] the L^1-optimal transport problem is studied by Bianchini and Cavalletti in the setting of *non-branching* geodesic metric spaces. A metric space is called non-branching if two different geodesics cannot coincide for a positive amount of time (see Definition 1.1.7). Their approach is a generalization of the Sudakov decomposition where transport rays are geodesics of the space. The main difficulty is again in proving that the conditional measures given by the disintegration are non atomic: in particular some additional hypotheses on the space are needed. Everything is based on the good behaviour of the measure μ with respect to a suitable evolution of sets along transport rays. Informally said, for a set A, they build an evolution $(0, 1) \ni t \mapsto A_t$ which roughly "translates" the set along the rays and assume conditions of the type

$$\mu(A_t) \geq C\mu(A). \tag{7}$$

It turns out that curvature bounds on the space guarantee that this kind of assumptions are satisfied. For example, in [21], it is also shown that if $(X, \mathsf{d}, \mathfrak{m})$ is non-branching, satisfies the $\mathsf{MCP}(K, N)$ condition and $\mu \ll \mathfrak{m}$, then a version of assumption (7) is verified. The theory has also been extended to m.m.s. spaces satisfying a weaker assumption than non-branching, called *essentially non-branching* (see Definition 1.4.11). In particular in [32] it is proved the existence of L^1-optimal transport maps for spaces satisfying the $\mathsf{RCD}^*(K, N)$ condition, $N < +\infty$ which satisfy this assumption as shown in [80], the argument applies also to essentially non-branching $\mathsf{CD}(K, N)$ spaces, $N < +\infty$, after the work in [38]. See also [33] for an overview. Finally we mention that the results in [21] are applied in [31] to the possibly infinite dimensional case of Wiener spaces.

In Chapter 4 we approach the problem of optimal transport map in a different "infinite dimensional" case. The motivation for us to study this problem is the link of L^1-optimal transport with the theory of localization, which we now explain.

1-dimensional localization

Starting from a metric measure space, as we have mentioned above, curvature properties of the space guarantee that the partition into transport rays, starting from a 1-Lipschitz function, and the disintegration with respect to this partition, have better regularity. This is what is described more in detail by the *localization* theorem. Having its roots in convex geometry, the localization theory is developed in the setting of Riemannian manifolds by Klartag [68]. He proves that when disintegrating the volume measure of the manifold with

respect to a geodesic foliation, then the needles and the conditional measures keep the Ricci curvature lower bound. If $(X, \mathsf{d}, \mathfrak{m})$ is a metric measure space, it turns out analogously that disintegrating the reference measure $\mathfrak{m}$, with respect to a partition into transport rays, constructed starting from a one Lipschitz function, each ray endowed with the relative conditional measure is itself a one-dimensional metric measure space satisfying the same curvature bounds of the ambient space. This was proved in [39] in the case of essentially non-branching $\mathsf{CD}(K,N)$ spaces and extended in [42] to the MCP case (a first version in non-branching MCP setting was proved in [21] in a different formulation). For the precise statements, see Section 1.6. This localization property of the synthetic Ricci curvature lower bounds turned out to be a crucial tool in proving the *globalization theorem* for the CD condition (see [37]) and in proving sharp inequalities (also in quantitative form) in the non-smooth setting. We mention for example the Levy-Gromov isoperimetric inequality ([39] and [36]), the p-spectral gap and Sobolev inequalities (see [40]), the quantitative Obata's theorem (see [43]).
On the technical side, one of the main difficulties to obtain the localization of curvature bounds, is to prove the absolute continuity of the conditional measures, which is a step further the non-atomicity needed for the existence of the optimal maps and it is available so far only in the finite dimensional setting. Therefore our solution to the Monge problem in the $\mathsf{CD}(K,\infty)$ setting, besides its own relevance as one of the most natural question in the theory, is also a first step towards a complete understanding of the localization still missing at this level of generality.

III.II Existence of optimal maps in $\mathsf{CD}(K,\infty)$ product-spaces

We present the main results of Chapter 4 obtained in [34]. In Section 4.1 we gather the definitions and the main properties of the space we work in. Briefly, we consider spaces which are infinite product of finite dimensional non-branching $\mathsf{CD}(K,N)$ spaces. Namely take a sequence $\{(X_i, \mathsf{d}_i, \mathfrak{m}_i)\}_{i\in\mathbb{N}}$ of metric measure spaces, with $(X_i, \mathsf{d}_i, \mathfrak{m}_i)$ non-branching, $\mathfrak{m}_i(X_i) = 1$, and satisfying the $\mathsf{CD}(K, N_i)$ condition for $1 < N_i < +\infty$, with $\sum_{i\in\mathbb{N}} \operatorname{diam}(X_i)^2 < +\infty$. We assume in addition that for any $n \in \mathbb{N}$ the product of $\{(X_i, \mathsf{d}_i, \mathfrak{m}_i)\}_{1\le i\le n}$ is a $\mathsf{CD}(K, N(n)))$ space with $N(n) < +\infty$. Then we define

$$(X, \mathsf{d}, \mathfrak{m}) := \left(\prod_{i\in\mathbb{N}} X_i,\ \sqrt{\sum_{i\in\mathbb{N}} \mathsf{d}_i^2},\ \otimes_{i\in\mathbb{N}} \mathfrak{m}_i \right),$$

which is a $\mathsf{CD}(K,\infty)$ non-branching space. Our main result is the following.

Theorem 5. *Let μ and $\nu \in \mathcal{P}(X)$, with $\mu, \nu \ll \mathfrak{m}$ with bounded densities and $\mu \perp \nu$, then there exists an optimal map for the L^1-optimal transport problem between μ and ν.*

We now describe the strategy of the proof. In Section 4.2, we show a standard procedure to construct an optimal transport map, by gluing optimal maps along rays, provided that the partition into transport rays and the disintegration of the measure $\mathfrak{m}$ with respect to this partition satisfy some regularity properties. We then need to prove that the partition into transport rays and the disintegration with respect to this partition, in our setting, satisfy these properties. This is done in Section 4.3. The main goal is as said above, to build a positive evolution as in (7). We follow, in Section 4.3.1 an approximation scheme similar to the one used in [31] for Wiener spaces that consists in approximating the problem with finite dimensional problems. Informally for any $n \in \mathbb{N}$ we consider the optimal transport problem between

$$\mu_n := (P_n)_\sharp \mu, \quad \nu_n := (P_n)_\sharp \nu,$$

where $P_n : X \to X$ is the projection on the first n components of the space X (defined in (4.5)). For any n we take a Kantorovich potential between μ_n and ν_n and we build a positive evolution along its transport rays, $A \mapsto A_{t,n}$, for $t \in (0,1)$ with

$$\mathfrak{m}_n(A_{t,n}) \geq C\mu_n(A_{t,n}), \tag{8}$$

with $\mathfrak{m}_n = (P_n)_\sharp \mathfrak{m}$. The evolution is built by using the optimal plan $\pi_n \in \mathrm{Opt}_1(\mu_n, \nu_n)$ which is induced by gluing the monotone rearrangements along the transport rays. The main difficulty with respect to [31] is that in their case the finite dimensional approximations live in $\mathbb{R}^n$ while here we have to deal with more general metric measure spaces. What allows us to get this estimate here is that in any $P_n(X) \simeq \prod_{i=1}^n X_i$, which is a $\mathsf{CD}(K, N(n))$ space for a finite $N(n)$, we can use the Localization Theorem. Therefore we can perform the computations along the rays, where the plan is induced by the classical monotone rearrangement. The key point is that the constant C in (8) does not depend on the upper bound on the dimension $N(n)$ of the space $P_n(X)$, which goes to $+\infty$ for $n \to +\infty$. This independence of N, is obtained by using the $\mathsf{CD}(K,\infty)$ condition satisfied by the marginal measures along the rays. Finally by sending $n \to +\infty$ we get a positive evolution along rays of a Kantorovich potential between μ and ν, by using a $\pi \in \mathrm{Opt}_1(\mu,\nu)$ obtained as weak limit of π_n for $n \to +\infty$. This is done in Section 4.3.1. Using this result, in Section 4.3.2 we can prove that the partition and the disintegration are good enough. More precisely we first show that the transport rays do not have too many common extrema and then that the conditional measures are non atomic. We can finally apply the result on the glued monotone rearrangements proved before, to get the optimal map, in Section 4.4.

IV Additional results

IV.I Wasserstein-Hellinger inequality: the case $p > 2$

In Appendix A we present two inequalities partially inspired by the investigation of the topics in [48], obtained in collaboration with Nicolò De Ponti and Luca Tamanini, that will appear in a joint future work.

In particular the goal of the appendix is to prove an analogous of Proposition 3.1.1 (taken from [74] and valid for $p \leq 2$) in the case $p > 2$.

Proposition 1. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathsf{RCD}(K,\infty)$ metric measure space, $K \in \mathbb{R}$ and $\mathfrak{m}(X) < +\infty$. For $p > 2$ and $\mu_0, \mu_1 \in \mathcal{P}_p(X)$ it holds*

$$\frac{p}{\sqrt{p-1}}(R_K(t))^{\frac{1}{2}}\mathsf{He}_p(H_t^*\mu_0, H_t^*\mu_1) \leq W_p(\mu_0,\mu_1) \quad \forall\, t > 0,$$

where $R_K(t) := \begin{cases} \frac{e^{2Kt}-1}{K} & \text{if } K \neq 0, \\ 2t & \text{if } K = 0. \end{cases}$

We remark that this type of inequalities appeared, in the context of $\mathsf{RCD}(K,\infty)$ spaces, for the first time in [10, Corollary 6.9] for the case $p = 1$ as a consequence of the L^∞-to-Lipschitz regularization property of the heat flow. In [74], Luise and Savaré proved the inequality for the case $1 < p \leq 2$. Their proof is based on linking $W_p(\mu_0, \mu_1)$ to $\mathsf{He}_p(H_t^*\mu_0, H_t^*\mu_1)$ by using their dual dynamical formulations and the reverse Poincaré inequality for the heat semigroup. Namely if $(X, \mathsf{d}, \mathfrak{m})$ is a m.m.s. satisfying the $\mathsf{RCD}(K, \infty)$ condition, then any $f \in L^\infty$ satisfies for $t > 0$

$$R_K(t)|DH_t(f)|^2 \leq H_t(f^2) - H_t(f)^2, \tag{9}$$

from which by neglecting the negative term, taking the $\frac{q}{2}$ power and applying Jensen inequality follows

$$R_K(t)^{\frac{q}{2}}|DH_t(f)|^q \leq H_t(|f|^q), \tag{10}$$

for any $q \geq 2$. For the case $q < 2$, which corresponds by duality to the case $p > 2$ in Proposition 1, the lack of convexity does not allow to pass from (9) to (10). Therefore we prove, with a refined approach, an analogous refined estimate, of its own interest, valid for $q < 2$.

Proposition 2. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathsf{RCD}(K, \infty)$ metric measure space with $\mathfrak{m}(X) < +\infty$, $K \in \mathbb{R}$. Then for every $1 < q < 2$ and $f \in L^\infty(X)$ it holds*

$$(q-1)R_K(t)|DH_t(f)|_w^2 \leq (H_t(|f|^q))^{\frac{2}{q}} - (H_t(f))^2, \quad \mathfrak{m}\text{-a.e. in } X, \text{ for any } t > 0. \tag{11}$$

IV.II More regularity for the conditional measures

In Appendix B we show a partial result strictly linked to Chapter 4. We have explained in Section III.I, that starting from a m.m.s. $(X, \mathsf{d}, \mathfrak{m})$ and taking a 1-Lipschitz function, then one can consider the transport rays associated to this function. If the space has good properties, one can expect that when disintegrating the measure with respect to the partition into transport rays, which are geodesics, almost any marginal distribution is absolutely continuous with respect to the 1-Hausdorff measure of the corresponding ray. In this appendix, in the setting of Chapter 4, we start from μ and ν probability measures, absolutely continuous with respect to $\mathfrak{m}$, with bounded densities respectively ρ_μ and ρ_ν. We consider in $(X, \mathsf{d}, \mathfrak{m})$ as in Section 4.1 the evolution that we have constructed by taking a limit π of good approximating plans $\pi_n \in \mathrm{Opt}_1(\mu_n, \nu_n)$ (induced by the gluing of monotone maps). We show that if this evolution is d^2-monotone along the rays (as it it the case for any finite dimensional evolution constructed via π_n) then the conditional measures are absolutely continuous with respect to $\mathcal{H}^1$ in the sets where ρ_μ and ρ_ν are positive. See Theorem B.5 for the complete statement.

Chapter 1

Preliminaries

1.1 Basics about metric spaces

To fix notations, the space of Lipschitz functions on (X, d) will be denoted by $\mathrm{Lip}(X) = \mathrm{Lip}(X, \mathsf{d})$ while $\mathrm{Lip}_c(X) = \mathrm{Lip}_c(X, \mathsf{d})$ will be the subspace of compactly supported Lipschitz functions, $\mathrm{Lip}_b(X)$ is the set of Lipschitz and bounded functions and $\mathrm{Lip}_{bs}(X)$ is the set of Lipschitz functions with bounded support. If the function is locally Lipschitz in an open set A, i.e. for every $x \in A$, the function is Lipschitz in a neighbourhood of x, then we use the notation $\mathrm{Lip}_{loc}(A)$. We denote by $\mathrm{Lip}(f)$ the Lipschitz constant of a function f. We write $C_b(X)$ to denote the space of real valued, bounded and continuous functions on X.

By $\mathcal{M}(X)_+$ we denote the space of non negative Radon measures on X, by $\mathcal{M}(X)$ the space of finite, non-negative, Borel measures on X and $\mathcal{P}(X)$ the space of probability measures on X. We write $\mu \in \mathcal{M}_p(X)$ if $\mu \in \mathcal{M}(X)$ and there exists $\bar{x} \in X$ such that

$$\int_X \mathsf{d}(\bar{x}, x)^p \, \mu(\mathrm{d}x) < +\infty,$$

while $\mathcal{P}_p(X) \subset \mathcal{M}_p(X)$ denotes the subset of probability measures with finite p-moment. When X is endowed with a Borel measure $\mathfrak{m}$, we denote by $L^p(X, \mathfrak{m})$ the Lebesgue space of p-integrable ($\mathfrak{m}$-equivalence class of) functions, $p \in [1, \infty]$. For simplicity, we often write $L^p(X)$ (or L^p) in place of $L^p(X, \mathfrak{m})$. Finally, $\mathrm{B}_b(X)$ is the set of bounded Borel functions on X.

Now we fix the notion of weak convergence of probability measures that we will use and we recall a useful property of the space of probability measures when endowed with the weak convergence. We say that a sequence of probability measures $\{\mu_n\}_{n \in \mathbb{N}} \subset \mathcal{P}(X)$ *weakly converges* to $\mu \in \mathcal{P}(X)$ if for any $\varphi \in C_b(X)$

$$\lim_{n \to +\infty} \int_X \varphi \mu_n = \int_X \varphi \mu. \tag{1.1}$$

Remark 1.1.1. Let (X, d) be a metric space and let τ_w be the topology induced by the weak convergence. The space $(\mathcal{P}(X), \tau_w)$ is metrizable and compact if X is compact (see e.g. [61, Section 1.1]).

Given a subset A of a metric space (X, d) and a number $\varepsilon > 0$ we define the ε-enlargement of A as $(A)^\varepsilon := \{x \in X \ : \ \mathsf{d}(x, A) < \varepsilon\}$.
In a metric space one has a distance between sets, that is the following.

Definition 1.1.2 (Hausdorff distance and convergence)**.** Let (X, d) be a metric space and $A, B \subset X$. We define the *Hausdorff distance* between A and B as

$$\mathsf{d}_H(A, B) \coloneqq \inf\{\delta > 0 \ : \ A \subset (B)^\delta,\ B \subset (A)^\delta\}.$$

Moreover we say that a sequence $\{A_n\}_{n\in\mathbb{N}}$ Hausdorff converges to a set $B \subset X$ if $\lim_{n\to+\infty} \mathsf{d}_H(A_n, B) = 0$.

Note that, without further assumptions on the sets, the Hausdorff distance is only a pseudometric. We recall the following classical result (see e.g. [26, Theorem 7.3.7]).

Theorem 1.1.3. *Let (X, d) be a compact metric space and $\mathcal{C}$ be the set of non-empty closed subsets of X. Then $(\mathcal{C}, \mathsf{d}_H)$ is a compact metric space.*

The following lemma combines the lower semicontinuity of the weak convergence on closed sets with the Hausdorff convergence of sets.

Lemma 1.1.4. *Let (X, d) be a proper metric space. Let $\{\mu_n\}_{n\in\mathbb{N}}$ be a sequence of probability measures weakly converging to $\mu \in \mathcal{P}(X)$. Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of Borel sets converging Hausdorff to a bounded closed set A. Then*

$$\mu(A) \geq \limsup_{n\to+\infty} \mu_n(A_n).$$

Proof. We use the fact that for any A closed set

$$\mu(A) = \inf\left\{\int f\mu \colon f \in C_b(X),\ f \geq 0 \text{ and } f \geq 1 \text{ on a neighbourhood of } A\right\}.$$

So fix $\varepsilon > 0$ and take $f \in C_b(X)$, $f \geq 0$ such that $\int f\mu \leq \mu(A) + \varepsilon$. If n is sufficiently big, by definition of Hausdorff convergence, $f = 1$ on A_n. Therefore $\int f\mu_n \geq \mu_n(A_n)$ and by passing to the limits

$$\int f\mu = \lim_{n\to+\infty} \int f\mu_n \geq \limsup_{n\to+\infty} \mu_n(A_n).$$

□

We finally recall the notion of slope (or local Lipschitz constant) of a function.

Definition 1.1.5 (Slope)**.** Let (X, d) be a metric space and $u : X \to \mathbb{R}$ be a real valued function. We define the *slope* of f at the point $x \in X$ as

$$|Du|(x) := \begin{cases} \limsup_{y\to x} \frac{|u(x)-u(y)|}{d(x,y)} & \text{if } x \text{ is not isolated} \\ 0 & \text{otherwise.} \end{cases}$$

Curves in metric spaces

In a metric space (X, d) we denote by $C([0,1], X)$ the set of curves $\gamma : [0,1] \to X$ which are continuous. A curve $\gamma : [0,1] \to X$ is *absolutely continuous* if there exists $f \in L^1(0,1)$ such that for any s, $t \in [0,1]$, $\mathsf{d}(\gamma(s), \gamma(t)) \leq \int_s^t f(r)\mathrm{d}r$. For any absolutely continuous curve γ there exists for almost every $t \in [0,1]$ the limit

$$|\dot{\gamma}_t| := \lim_{h\to 0} \frac{\mathsf{d}(\gamma(t+h), \gamma(t))}{|h|},$$

that is called metric speed of γ. For any absolutely continuous curve γ it holds

$$\mathsf{d}(\gamma(0),\gamma(1)) \le \int_{(0,1)} |\dot{\gamma_t}| \, \mathrm{d}r. \tag{1.2}$$

A *geodesic* between the points $\gamma_0,\, \gamma_1 \in X$ is a minimizing constant speed curve $\gamma : [0,1] \to X$, $\gamma(0) = \gamma_0$, $\gamma(1) = \gamma_1$ which satisfies for any $s,\, t \in [0,1]$ the inequality

$$\mathsf{d}(\gamma(s),\gamma(t)) \le |s-t|\, \mathsf{d}(\gamma(0),\gamma(1)).$$

In particular actually equality holds for any $s,\, t \in [0,1]$. Equivalently a curve $\gamma \in C([0,1],X)$ is a geodesic if and only if γ is absolutely continuous and it satisfies

$$\int_{(0,1)} |\dot{\gamma_r}|^2 \, \mathrm{d}r = \mathsf{d}(\gamma(0),\gamma(1))^2 \tag{1.3}$$

and in particular by (1.2) also $|\dot{\gamma_r}| = \mathsf{d}(\gamma(0),\gamma(1))$ for any $r \in [0,1]$.

Remark 1.1.6. Given two geodesics $\gamma^1 : [0,1] \to X$, $\gamma^2 : [0,1] \to X$ such that $\gamma^1(1) = \gamma^2(0)$. The curve $\gamma : [0,1] \to X$ defined by concatenating the curve γ_2 after γ_1 and re-parametrized with constant speed is a geodesic if and only if $\mathsf{d}(\gamma^1(0),\gamma^1(1)) + \mathsf{d}(\gamma^2(0),\gamma^2(1)) = \mathsf{d}(\gamma^1(0),\gamma^2(1))$.

We call $\mathrm{Geo}(X) \subset C([0,1],X)$ the set of geodesics on X endowed with the sup-distance. A space (X,d) is a *geodesic space* if for any two points $x,\, y \in X$ there exists $\gamma \in \mathrm{Geo}(X)$ such that $\gamma(0) = x$, $\gamma(1) = y$.

Definition 1.1.7 (non-branching)**.** A subset $G \subset \mathrm{Geo}(X,\mathsf{d})$ of geodesics is called non-branching if for any $\gamma^1, \gamma^2 \in G$ the following holds:

$$\exists\, t \in (0,1) \colon \gamma^1(s) = \gamma^2(s) \quad \forall\ s \in [0,t] \implies \gamma^1(s) = \gamma^2(s) \quad \forall\ s \in [0,1].$$

(X,d) is called non-branching if $\mathrm{Geo}(X,\mathsf{d})$ is non-branching.

We define in addition the evaluation map at time $t \in [0,1]$, the map $\mathrm{e}_t : C([0,1],X) \to X$, $\mathrm{e}_t(\gamma) := \gamma(t)$.

1.2 Introduction to optimal transport

We recall here some classical fact about optimal transportation: for a more detailed treatment see e.g [5, Chapter 1], [7, Chapter 6], [98], [97].
Given two complete and separable metric spaces (X_1,d_1), (X_2,d_2) a Borel measure on X_1, Borel map $T : X_1 \to X_2$, the *pushforward of μ via T* is the Borel measure $T_\sharp\mu$ on X_2 defined as

$$T_\sharp\mu(A) = \mu(T^{-1}(A)) \tag{1.4}$$

for any $A \in B(X_2)$. A useful characterization of the pushforward measure is the following: $T_\sharp\mu = \nu$, with ν Borel measure on X_2, if and only if for any $\varphi : X_2 \to [0,+\infty]$ Borel

$$\int_{X_2} \varphi\, \nu = \int_{X_1} \varphi \circ T\, \mu. \tag{1.5}$$

Let $c : X_1 \times X_2 \to [0,+\infty)$ be a *cost function* and $\mu \in P(X_1)$, $\nu \in P(X_2)$. The *optimal transportation problem* admits two classical formulations.

Monge formulation

$$\inf_{T_\sharp\mu=\nu}\int_{X_1\times X_2} c(x,T(X))\mu(\mathrm{d}x) \qquad \text{(MP)}$$

Such a T is called *transport map* between μ and ν. A transport map is optimal if it realizes the infimum in (MP).
The set of *transport plans* between μ and $\nu\in\mathcal{P}(X)$ is

$$\mathrm{Adm}(\mu,\nu):=\{\pi\in\mathcal{P}(X\times X)\colon\ (P_1)_\sharp\pi=\mu,(P_2)_\sharp\pi=\nu\}\,.$$

Kantorovich formulation

$$\min_{\pi\in\mathrm{Adm}(\mu,\nu)}\int_{X_1\times X_2} c(x,y)\pi(\mathrm{d}x,\mathrm{d}y) \qquad \text{(KP)}$$

A transport plan is optimal if it realizes the infimum in (KP). Any transport map T between μ and ν induces a transport plan: $(Id,T)_\sharp\mu\in\mathrm{Adm}(\mu,\nu)$. While the existence of optimal transport maps is not guaranteed a priori by general conditions on the cost, if the cost c is lower semicontinuous and bounded from below then there exists a $\pi\in\mathrm{Adm}(\mu,\nu)$ which minimizes (KP). Moreover if the cost is continuous and μ is non atomic, then $\inf\text{(MP)}=\min\text{(KP)}$ (see [1]).

Conditions for optimality

In this part we recall how optimality for a plan is a property only of its support, introducing the notion of cyclical monotonicity and Kantorovich potential. Let $c:X_1\times X_2\to[0,+\infty)$ be a cost function. A set $\Gamma\subseteq X_1\times X_2$ is *c-cyclically monotone* if for any set $\{(x_i,y_i)\,i=1,\dots,n,\ n\in\mathbb{N}\}$ and for any σ permutation of $\{1,\dots,n\}$, it holds

$$\sum_{i=1}^n c(x_i,y_i)\le\sum_{i=1}^n c(x_i,y_{\sigma(i)}).$$

Given a function $\varphi:X_1\to\mathbb{R}\cup\{\pm\infty\}$, its *c-transform* is $\varphi^c:X_2\to\mathbb{R}\cup\{\pm\infty\}$ defined as

$$\varphi^c(y):=\inf_{x\in X_1} c(x,y)-\psi(x).$$

The definition is the symmetric one for the c-transform of $\varphi:X_2\to\mathbb{R}\cup\{\pm\infty\}$. A function $\varphi:X_1\to\mathbb{R}\cup\{-\infty\}$ is *c-concave* if there exists $\psi:X_2\to\mathbb{R}\cup\{\pm\infty\}$ such that $\varphi=\psi^c$. The *c-superdifferential* of a c-concave function $\varphi:X_1\to\mathbb{R}\cup\{-\infty\}$ is the set $\partial^c\varphi:=\{(x,y)\in X_1\times X_2\colon\varphi(x)+\varphi^c(y)=c(x,y)\}$. The c-superdifferential of a c-concave function is a c-cyclically monotone set.

Remark 1.2.1. If (X,d) is a metric space and $c(x,y)=\mathsf{d}(x,y)$, a function φ is c-cyclically monotone if and only if it is 1-Lipschitz. Moreover $\varphi^c=-\varphi$ and

$$\partial^c\varphi:=\{(x,y)\in X\times X\colon\varphi(x)-\varphi(y)=\mathsf{d}(x,y)\}.$$

Theorem 1.2.2 (Fundamental Theorem of optimal transport)**.** *Let $c:X_1\times X_2\to[0,+\infty)$ be a* cost function. *Assume that there exists $a\in L^1(X_1,\mu)$, $b\in L^1(X_2,\mu)$ such that $c(x,y)\le a(x)+b(y)$ for any $x\in X_1$, $y\in X_2$. For $\pi\in\mathrm{Adm}(\mu,\nu)$ the following are equivalent:*

1. π *is optimal,*

2. the set $\operatorname{supp}(\pi)$ *is c-cyclically monotone;*

3. there exists a c-concave function φ *such that* $\max\{\varphi, 0\} \in L^1(\mu)$ *and* $\operatorname{supp}(\pi) \subset \partial^c\varphi$.

The optimal transport problem admits a dual formulation that we recall.

Dual formulation

$$\sup_{\varphi(x)+\psi(y)\le c(x,y) \forall x\in X_1,\, y\in X_2} \int_{X_1} \varphi\,\mu + \int_{X_2} \psi\,\nu \qquad \text{(DP)}$$

where the supremum is among all $\varphi \in L^1(X_1, \mu)$, $\psi \in L^1(X_1, \mu)$. Under the same assumptions of the fundamental Theorem of Optimal Transport

$$\min(\mathrm{KP}) = \sup(\mathrm{DP}),$$

and the supremum in (DP) is attained among couples of the type (φ, φ^c) with φ c-concave. We call a φ such that (φ, φ^c) maximizes (DP) *Kantorovich Potential* for the problem.

Remark 1.2.3. The existence of a Kantorovich potential is guaranteed by point (3) of Theorem 1.2.2.

Moreover for any maximizing pair (φ, φ^c) for (DP) and for any π minimizer of (KP) it holds

$$\varphi(x) + \varphi^c(y) = c(x, y) \quad \pi \text{ a.e. } (x, y) \in X_1 \times X_2. \tag{1.6}$$

Some results in dimension one

In dimension one, under mild assumptions, the monotone rearrangement map is an explicit optimal transport map for the cost d^p for any $p \ge 1$. The following Theorem follows from e.g. [7, Theorem 6.0.2, Theorem 6.2.7] (see also [97]).

Theorem 1.2.4. *Let* μ *and* $\nu \in \mathcal{P}(\mathbb{R})$, μ *without atoms, compactly supported and let*

$$G(x) := \mu((-\infty, x)), \quad F(y) := \nu((-\infty, y))$$

be respectively the cumulative distribution functions of μ, ν*. Then the non decreasing map* $T_{mon} : \mathbb{R} \to \bar{\mathbb{R}}$ *defined as*

$$T_{mon}(x) := \sup\{y \in \mathbb{R} \colon F(y) \le G(x)\}$$

maps μ *into* ν *in the sense that* $(T_{mon})_\sharp\mu = \nu$*. In addition it is an optimal map for the optimal transport problem with cost* $d(x, y)^p$ *for any* $p \ge 1$*. If* $p > 1$ *it is the unique optimal transport map. Moreover, if* μ *and* ν *are absolutely continuous with respect to the Lebesgue measure, then* $T'_{mon} > 0$ μ *a.e..*

Remark 1.2.5. In the above theorem, if both μ and ν are absolutely continuous with respect to the Lebesgue measure, then the map T is μ-essentially invertible, meaning that T is injective outside a μ-negligible set.

The following measure theoretical results will be useful in computations involving optimal maps and densities in dimension one. For more detailed results, stated also in general dimension see [7, Section 5.5].

Theorem 1.2.6. *Let $\rho \in L^1(\mathbb{R})$ be a nonnegative function. Let $\mu = \rho\mathcal{L}^1$. Let $f : \mathbb{R} \to \mathbb{R}$ be a function and $E \subseteq \mathbb{R}$ Borel such that f is injective and differentiable on E and $\mu(\mathbb{R}\backslash E) = 0$. Then $f_\sharp\mu \ll \mathcal{L}^1$ if and only if $|f'| > 0$ μ a.e. in E and*

$$f_\sharp(\rho\mathcal{L}^1) = \frac{\rho}{|f'|} \circ f^{-1}_{|f(E)}\mathcal{L}^1.$$

Theorem 1.2.7. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Let $E \subseteq \mathbb{R}$ Borel such that f is differentiable and injective on E. Then*

$$\int_E \psi(x)\,|f'|\,(x)\,\mathrm{d}x = \int_{f(E)} \psi(f^{-1}(y))\,\mathrm{d}y$$

for any $\psi : \mathbb{R} \to [0, +\infty]$ Borel function.

1.3 Distances between measures

Wasserstein distance

Let (X, d) be a metric space. The Wasserstein distance between μ and ν with $\mu, \nu \in \mathcal{P}_p(X)$ is for $p \geq 1$,

$$W_p(\mu,\nu)^p := \inf\left\{\int_{X\times X} \mathsf{d}(x,y)^p\pi(\mathrm{d}x,\mathrm{d}y)\colon \pi \in \mathrm{Adm}(\mu,\nu)\right\},$$

where $\mathcal{P}_p(X)$ is defined at the beginning of Section 1.1. Denote the set of optimal transport plans for the cost $d^p : X \times X \to [0, +\infty)$ as

$$\mathrm{Opt}_p(\mu,\nu) := \left\{\pi \in \mathrm{Adm}(\mu,\nu)\colon \int_{X\times X} \mathsf{d}(x,y)^p\pi(\mathrm{d}x,\mathrm{d}y) = W_p(\mu,\nu)^p\right\}.$$

Notice that $W_p(\mu_1, \mu_2) = +\infty$ whenever $\mu_1(X) \neq \mu_2(X)$, but $W_p(\mu_1, \mu_2)$ is finite if μ_1, $\mu_2 \in \mathcal{M}_p(X)$ and have the same total mass. Moreover W_p is a distance on $\mathcal{P}_p(X)$ which metrizes the weak convergence of measures plus convergence of the p-moment (see e.g. [98]). In addition $(\mathcal{P}_p(X), W_p)$ is complete and separable if (X, d) is complete and separable.

It is also useful to recall that the space $(\mathcal{P}_2(X), W_2)$ is geodesic if and only if (X, d) is geodesic. A curve $[0, 1] \ni t \mapsto \mu_t \in \mathcal{P}_2(X)$ is a W_2-geodesic if and only if there exists $\nu \in \mathcal{P}(\mathrm{Geo}(X))$, such that $(e_t)_\sharp\nu = \mu_t$ for any $t \in [0, 1]$ and $(e_0, e_1)_\sharp\nu \in \mathrm{Opt}(\mu_0, \mu_1)$. The set of *optimal geodesic plans*, $\mathrm{OptGeo}(\mu_0, \mu_1)$ is the set of $\nu \in \mathcal{P}(\mathrm{Geo}(X))$ such that $(e_0, e_1)_\sharp\nu \in \mathrm{Opt}(\mu_0, \mu_1)$.

When (X, d) is a length metric space (see e.g. [26, Definition 2.1.6. 7.3.7] for the definition) one can prove a dynamic formulation of the Wasserstein distance (see for instance [74, Prooposition 2.10]):

$$\frac{1}{p}W_p^p(\mu_0,\mu_1) = \sup\Big\{\int_X \zeta_1\,\mu_1 - \int_X \zeta_0\,\mu_0,\ \zeta \in C^1([0,1],\mathrm{Lip}_b(X)),\ \partial_t\zeta_t + \frac{1}{q}|D\zeta_t|^q \leq 0\Big\}, \quad (1.7)$$

where we are using the notation $|Df|\,(x)$ for the slope of a Lipschitz function defined in Definition 1.1.5.

Throughout the thesis with a little abuse of notation, we will tacitly assume the Wasserstein distance to be defined on any couple of non-negative Borel measures (not necessarily of probability measures) having the same finite mass.

Hellinger and Hellinger-Kantorovich distance

Definition 1.3.1. Given μ_0, $\mu_1 \in \mathcal{M}(X)$ and $p \in [1, +\infty)$, the p-Hellinger distance He_p (also called Matusita distance) [65, 75] between μ_0 and μ_1 is defined as

$$\mathsf{He}_p^p(\mu_0, \mu_1) := \int_X \left|\rho_0^{1/p} - \rho_1^{1/p}\right|^p \mathrm{d}\lambda,$$

where λ is any dominating measure of μ_0, μ_1 and ρ_i are the relative densities: $\mu_i \ll \lambda$ and $\mu_i = \rho_i \lambda$ for $i = 0, 1$.

The case $p = 1$ corresponds to the classical total variation, the case $p = 2$ is the original distance studied by Hellinger.

An immediate consequence of the elementary inequality $|t - s| \geq \left|t^{1/2} - s^{1/2}\right|^2$ is that

$$\mathsf{He}_1(\mu_0, \mu_1) \geq \mathsf{He}_2^2(\mu_0, \mu_1) \qquad \text{for every } \mu_0, \mu_1 \in \mathcal{M}(X). \tag{1.8}$$

It is not difficult to show that all the p-Hellinger distances induce the same strong convergence of the total variation, and are complete distances on $\mathcal{M}(X)$.

It is useful to recall here that also the p-Hellinger distances admit a dynamic formulation for $p > 1$ [74, Proposition 2.8], specifically:

$$\mathsf{He}_p^p(\mu_0, \mu_1) = \sup\Big\{ \int_X \zeta_1\, \mu_1 - \int_X \zeta_0\, \mu_0,\ \zeta \in C^1([0,1], \mathrm{B}_b(X)),\ \partial_t \zeta_t + (p-1)\zeta_t^{\frac{p}{p-1}} \leq 0 \Big\}. \tag{1.9}$$

We also introduce the weighted Hellinger-Kantorovich distance HK_α, $\alpha > 0$, following the theory developed in [69]. For convenience we decided to introduce only its dynamical formulation: on a length metric space (X, d) it reads as follow

$$\mathsf{HK}_\alpha^2(\mu_0, \mu_1) := \sup\Big\{ \int_X \zeta_1\, \mu_1 - \int_X \zeta_0\, \mu_0,\ \ \zeta \in C^1([0,1], \mathrm{Lip}_b(X)),\ \partial_t \zeta_t + \frac{\alpha}{4}|D\zeta_t|^2 + \zeta_t^2 \leq 0 \Big\}. \tag{1.10}$$

Notice that $\mathsf{HK}_\alpha(\mu_0, \mu_1)$ is finite even if $\mu_0(X) \neq \mu_1(X)$ and one can prove that HK_α is indeed a distance on $\mathcal{M}(X)$.

For every two measures $\mu_0, \mu_1 \in \mathcal{M}(X)$ one can prove (see [69, Chapter 7]) the following relations between He_2, W_2 and HK

$$\mathsf{HK}_\alpha(\mu_0, \mu_1) \leq \mathsf{He}_2(\mu_0, \mu_1) \qquad \text{and} \qquad \lim_{\alpha \downarrow 0} \mathsf{HK}_\alpha(\mu_0, \mu_1) = \mathsf{He}_2(\mu_0, \mu_1), \tag{1.11}$$

$$\sqrt{\alpha}\, \mathsf{HK}_\alpha(\mu_0, \mu_1) \leq W_2(\mu_0, \mu_1) \qquad \text{and} \qquad \lim_{\alpha \uparrow +\infty} \sqrt{\alpha}\, \mathsf{HK}_\alpha(\mu_0, \mu_1) = W_2(\mu_0, \mu_1). \tag{1.12}$$

1.4 Metric measure spaces

With metric measure space, m.m.s. for short we will denote a triple $(X, \mathsf{d}, \mathfrak{m})$ where:

- (X, d) is a complete and separable metric space,
- $\mathfrak{m}$ is a non negative Borel measure finite on bounded sets.

1.4.1 Calculus on metric measure spaces

We recall the definition of Sobolev space on a metric measure space $(X, \mathsf{d}, \mathfrak{m})$, introduced in [45] and [84] (see also [59], [64], [22], [9], [8] for a more details and references). We first define the *Cheeger energy*, $\mathsf{Ch} : L^2(X, \mathfrak{m}) \to [0, +\infty]$, given by

$$\mathsf{Ch}(f) := \inf \left\{ \liminf_{n\to\infty} \frac{1}{2} \int |Df_n|^2 \, d\mathfrak{m} \colon f_n \in \mathrm{Lip}(X) \cap L^2(X, \mathfrak{m}), \, \|f_n - f\|_{L^2} \to 0 \right\}.$$

The domain of the Cheeger energy $D(\mathsf{Ch}) \subset L^2(X, \mathfrak{m})$ is defined as $D(\mathsf{Ch}) := \{f \in L^2(X, \mathfrak{m}) \colon \mathsf{Ch}(f) < +\infty\}$. We put

$$W^{1,2}(X, \mathsf{d}, \mathfrak{m}) := D(\mathsf{Ch}),$$

endowed with the norm $\|f\|^2_{W^{1,2}} := \|f\|^2_{L^2} + 2\mathsf{Ch}(f)$. Since Ch is a convex and lower semicontinuous functional over $L^2(X, \mathfrak{m})$, $W^{1,2}(X, \mathsf{d}, \mathfrak{m})$ is a Banach space. For simplicity, we will often drop the dependence of the metric measure structure and write $W^{1,2}(X)$ (or $W^{1,2}$) in place of $W^{1,2}(X, \mathsf{d}, \mathfrak{m})$ (the same for $L^2(X)$).
For any $f \in W^{1,2}(X)$, the Cheeger energy admits an integral representation

$$\mathsf{Ch}(f) = \frac{1}{2} \int_X |Df|^2_w \, d\mathfrak{m},$$

where $|Df|_w \in L^2(X)$ is called minimal weak upper gradient. For $f \in W^{1,2}(X) \cap \mathrm{Lip}(X)$

$$|Df|_w \leq \mathrm{Lip}(f), \quad \mathfrak{m}\text{-a.e.}. \tag{1.13}$$

Finally for any $f, u \in W^{1,2}(X)$ (see [58]), define the functions $D^{\pm} f(\nabla u) \in L^1(X)$ by

$$D^+ f(\nabla u) := \inf_{\varepsilon>0} \frac{|D(u + \varepsilon f)|^2_w - |Du|^2_w}{2\varepsilon},$$

while $D^- f(\nabla u)$ is obtained replacing $\inf_{\varepsilon>0}$ with $\sup_{\varepsilon<0}$. It holds that $\mathfrak{m}$-a.e.

$$\left| D^+ f(\nabla u) \right| \leq |Df|_w \, |Du|_w, \tag{1.14}$$

$$D^+ f(\nabla f) = |Df|^2_w = D^- f(\nabla f). \tag{1.15}$$

Metric Laplacian

We recall the definition of subdifferential for the functional Ch. Given $f \in W^{1,2}(X)$, we say that $g \in \partial^- \mathsf{Ch}(f)$, namely g is in the subdifferential of Ch at f, if

$$\int_X g(\psi - f) \, d\mathfrak{m} \leq \mathsf{Ch}(\psi) - \mathsf{Ch}(f) \qquad \forall \psi \in L^2(X).$$

Definition 1.4.1 (L^2-Laplacian, [9] (see also [58]))**.** The Laplacian $-\Delta f \in L^2(X, \mathfrak{m})$ of a function $f \in W^{1,2}(X, \mathsf{d}, \mathfrak{m})$ is the element of minimal $L^2(X, \mathfrak{m})$-norm in the sub-differential $\partial^- \mathsf{Ch}(f)$, provided the latter is non-empty. Accordingly a function $f \in W^{1,2}(X, \mathsf{d}, \mathfrak{m})$ is an eigenfunction of eigenvalue $\lambda > 0$ provided $-\Delta f = \lambda f$.

It will be clear from the proof of our results that the minimality requirement in the previous definition does not play any role: our main results will be valid for any element of the sub-differential.

The following is a version of the integration by parts formula for the Laplacian, which is a consequence of [58, Proposition 4.9].

Proposition 1.4.2. *For every $f \in W^{1,2}(X)$ and every $h \in \partial^- \mathsf{Ch}(f)$ it holds that*

$$\int_X D^- u(\nabla f)\, \mathfrak{m} \le - \int_X hu\, \mathfrak{m} \le \int_X D^+ u(\nabla f)\, \mathfrak{m}, \tag{1.16}$$

for each $u \in W^{1,2}(X)$.

Remark 1.4.3. It is straightforward to check that any eigenfunction has zero mean, provided $\mathfrak{m}(X) < \infty$. Here we only sketch the argument when X is proper. Consider any sequence (χ_n) of 1-Lipschitz functions with bounded support and values in $[0,1]$ such that $\chi_n \equiv 1$ in $B_n(\bar{x})$, for some fixed $\bar{x} \in X$. Since we are assuming X to be proper, $\chi_n \in \mathrm{Lip}_c(X) \subseteq W^{1,2}(X)$ and therefore

$$\int D^- \chi_n(\nabla f)\, \mathfrak{m} \le \lambda \int \chi_n f\, \mathfrak{m} \le \int D^+ \chi_n(\nabla f)\, \mathfrak{m};$$

for both quantities,

$$\left| \int D^{\pm} \chi_n(\nabla f)\, \mathfrak{m} \right| \le \int_{X \setminus B_n(\bar{x})} |\nabla f|_w\, \mathfrak{m}$$

that are both converging to zero, provided $\mathfrak{m}(X) < \infty$, giving $\int f\, \mathfrak{m} = 0$ by dominated convergence theorem.

Infinitesimal Hilbertianity and heat flow

We give an equivalent definition of the *Infinitesimal Hilbertianity* condition introduced in [58] (see Definition 4.19 and Proposition 4.22 for equivalent definitions).

Definition 1.4.4. A m.m.s. $(X, \mathsf{d}, \mathfrak{m})$ is *Infinitesimally Hilbertian* if the Cheeger energy Ch is a quadratic form on $W^{1,2}(X, \mathsf{d}, \mathfrak{m})$, i.e. for every f and $g \in W^{1,2}(X, \mathsf{d}, \mathfrak{m})$ the following equality is satisfied

$$\mathsf{Ch}(f+g) + \mathsf{Ch}(f-g) = 2\mathsf{Ch}(f) + 2\mathsf{Ch}(g).$$

In an Infinitesimally Hilbertian space, the subdifferential of Ch where non empty is single valued. Moreover the Laplacian is a linear operator. A property that is useful for us is that for any f and $g \in W^{1,2}(X, \mathsf{d}, \mathfrak{m})$

$$D^+ f(\nabla g) = D^- f(\nabla g) \quad \mathfrak{m} - \text{a.e.}.$$

We call, following [58],

$$\nabla f \cdot \nabla g := D^+ f(\nabla g) = D^- f(\nabla g).$$

It holds that $W^{1,2}(X, \mathsf{d}, \mathfrak{m}) \ni f \mapsto \nabla f \cdot \nabla g$ is linear and $\nabla f \cdot \nabla g = \nabla g \cdot \nabla f$ for any $f,\, g \in W^{1,2}(X, \mathsf{d}, \mathfrak{m})$.

It follows from Proposition 1.4.2, that given $f \in W^{1,2}(X, \mathsf{d}, \mathfrak{m})$ and its Laplacian $-\Delta f$, if it exists, satisfy for any $g \in W^{1,2}(X, \mathsf{d}, \mathfrak{m})$

$$\int_X -\Delta f\, g\, \mathfrak{m} = \int_X \nabla f \cdot \nabla g\, \mathfrak{m}. \tag{1.17}$$

From the convexity and lower semicontinuity of Ch and from the fact that $W^{1,2}(X)$ is dense in $L^2(X)$, it follows, using the theory of gradient flows in Hilbert spaces, that for any

$f \in L^2(X)$ there exists a unique locally absolutely continuous curve $t \mapsto H_t(f)$, $t \in (0, +\infty)$, with values in $L^2(X)$, which satisfies

$$\begin{cases} \frac{d}{dt} H_t f = -\partial^- \mathsf{Ch}(H_t f) & \text{a.e. } t > 0, \\ \lim_{t\to 0} H_t f = f & \text{in } L^2(X). \end{cases}$$

$\{H_t\}_{t\geq 0}$ is called the heat semigroup and for any $t > 0$, $f \mapsto H_t f$ is a linear contraction in $L^2(X)$. By the density of $L^2(X) \cap L^p(X)$ in $L^p(X)$, it can be extended to a semigroup of linear contractions in any $L^p(X)$, $p \geq 1$. It can also be extended to $L^\infty(X)$ and it is known that $H_t f$, for $f \in L^\infty(X)$, admits an integral representation via the heat kernel.

We remark that in our setting, the heat semigroup satisfies the maximum principle:

$$H_t f \leq C \quad \text{ if } f \leq C \quad \mathfrak{m}\text{-a.e.}\,, \tag{1.18}$$

from which follows that it is sign preserving. Moreover, H_t is also measure preserving

$$\int_X H_t f \, d\mathfrak{m} = \int_X f \, d\mathfrak{m}, \quad \forall f \in L^1(X), \quad \forall\, t > 0.$$

If one considers the evolution at time t via the heat flow of an eigenfunction f_λ of eigenvalue λ, then

$$H_t f_\lambda = e^{-\lambda t} f_\lambda. \tag{1.19}$$

Perimeter and sets of finite perimeter

Given a metric measure space one can introduce a notion of perimeter which extends the classical one in $\mathbb{R}^n$. We recall the notion of sets of finite perimeter in a m.m.s. taken from [76] (see also the more recent [4]).

Definition 1.4.5 (Perimeter)**.** Let $E \in \mathcal{B}(X)$, where $\mathcal{B}(X)$ denotes the class of Borel sets of (X, d), and let $A \subset X$ be open. We define the perimeter of E relative to A as:

$$\mathsf{Per}(E; A) := \inf \left\{ \liminf_{n\to\infty} \int_A |Du_n| \,\mathfrak{m} \colon\ u_n \in \mathrm{Lip}_{loc}(A),\ u_n \to \chi_E \text{ in } L^1_{loc}(A, \mathfrak{m}) \right\},$$

where $|Du|\,(x)$ is the slope of u at the point x (see Definition 1.1.5). If $\mathsf{Per}(E; X) < \infty$, we say that E is a *set of finite perimeter*. We denote $\mathsf{Per}(E; X)$ with $\mathsf{Per}(E)$.

When E is a fixed set of finite perimeter, the map $A \mapsto \mathsf{Per}(E; A)$ is the restriction to open sets of a finite Borel measure on X, defined as

$$\mathsf{Per}(E; B) := \inf \{\mathsf{Per}(E; A) \colon A \text{ open}, A \supset B\}\,.$$

1.4.2 Synthetic notions of Ricci curvature bounds

In this section we recall the main definitions and properties that we will need about spaces satisfying synthetic lower Ricci curvature bounds. For more background and references on the topic we refer to ([2, 98, 73, 93, 94, 9, 11, 10, 55, 14, 13]).

Distorsion coefficients

We first need to recall the definitions of distortion coefficients. Given $K \in \mathbb{R}$ and $N \in (0, \infty]$, define:

$$D_{K,N} := \begin{cases} \frac{\pi}{\sqrt{K/N}} & K > 0\ ,\ N < \infty \\ +\infty & \text{otherwise} \end{cases}. \tag{1.20}$$

In addition, given $t \in [0, 1]$ and $0 < \theta < D_{K,N}$, define:

$$\sigma_{K,N}^{(t)}(\theta) := \frac{\sin(t\theta\sqrt{\frac{K}{N}})}{\sin(\theta\sqrt{\frac{K}{N}})} = \begin{cases} \frac{\sin(t\theta\sqrt{\frac{K}{N}})}{\sin(\theta\sqrt{\frac{K}{N}})} & K > 0\ ,\ N < \infty \\ t & K = 0 \text{ or } N = \infty\ , \\ \frac{\sinh(t\theta\sqrt{\frac{-K}{N}})}{\sinh(\theta\sqrt{\frac{-K}{N}})} & K < 0\ ,\ N < \infty \end{cases}$$

and set $\sigma_{K,N}^{(t)}(0) = t$ and $\sigma_{K,N}^{(t)}(\theta) = +\infty$ for $\theta \geq D_{K,N}$. Given $K \in \mathbb{R}$ and $N \in (1, \infty]$, the *distortion coefficients* are defined as:

$$\tau_{K,N}^{(t)}(\theta) := t^{\frac{1}{N}} \sigma_{K,N-1}^{(t)}(\theta)^{1-\frac{1}{N}}.$$

When $N = 1$, set $\tau_{K,1}^{(t)}(\theta) = t$ if $K \leq 0$ and $\tau_{K,1}^{(t)}(\theta) = +\infty$ if $K > 0$. Now we are ready to define the first synthetic notion of Ricci curvature lower bound.

CD condition

Given a metric measure space $(X, \mathsf{d}, \mathfrak{m})$, we define the *Boltzman entropy* $\mathrm{Ent} : \mathcal{P}_2(X) \to [-\infty, +\infty]$, as

$$\mathrm{Ent}(\mu) := \begin{cases} \int_X \rho \log(\rho)\, \mathfrak{m}, & \mu = \rho\mathfrak{m}, \\ +\infty & \text{otherwise.} \end{cases}$$

We recall that it is well defined if for some a, $b \geq 0$ and for some $\bar{x} \in X$ $\mathfrak{m}(B_r(\bar{x})) \leq ae^{br^2}$ for all $r > 0$, which is also a natural assumption in this setting (see [94]). Note that we will mainly work with spaces with finite measure which trivially satisfy the latter condition.

We are ready to introduce the first notion of synthetic Ricci curvature lower bound for a metric measure space ([93, 94, 73]).

Definition 1.4.6 ($\mathsf{CD}(K, \infty)$ condition)**.** $(X, \mathsf{d}, \mathfrak{m})$ verifies the $\mathsf{CD}(K, \infty)$ condition for some $K \in \mathbb{R}$, if for any pair of probability measures $\mu_0, \mu_1 \in \mathcal{P}_2(X)$ $\mu_0, \mu_1 \ll \mathfrak{m}$ and $\mathrm{Ent}(\mu_i) < \infty$, $i = 0, 1$, there exists $(\nu_s)_{s\in[0,1]} \in \mathrm{Geo}(\mathcal{P}_2(X))$, with $\nu_0 = \mu_0$, $\nu_1 = \mu_1$ such that for any $t \in [0, 1]$

$$\mathrm{Ent}(\nu_t) \leq (1 - t)\mathrm{Ent}(\nu_0) + t\mathrm{Ent}(\nu_1) - \frac{K}{2} t(1 - t)W_2(\mu_0, \mu_1)^2).$$

It will be useful for us to recall that the $\mathsf{CD}(K, \infty)$ condition is stable under the notion of convergence for metric measure spaces induced by the $\mathbb{D}$-distance introduced in [94] (see also [60]).

Definition 1.4.7. Given $(X, \mathsf{d}, \mathfrak{m})$ and $(X', \mathsf{d}', \mathfrak{m}')$ normalized m.m.s.

$$\mathbb{D}((X, \mathsf{d}, \mathfrak{m}), (X', \mathsf{d}', \mathfrak{m}')) = \inf \left(\int_{X\times X'} \hat{\mathsf{d}}^2(\psi(x), \psi'(x'))\, \hat{q}(\mathrm{d}x, \mathrm{d}x') \right)^{\frac{1}{2}}$$

where the infimum is taken among all $(\hat{M}, \hat{\mathsf{d}})$ metric spaces, $\psi : X \to \hat{M}$, $\psi' : X' \to \hat{M}$ isometric embeddings, $\hat{q} \in \mathcal{P}(X \times X')$ coupling of $\mathfrak{m}$ and $\mathfrak{m}'$.

The $\mathsf{CD}(K,\infty)$ condition can be refined by considering an entropy which takes into account also a parameter N playing the role of an upper bound on the dimension. The *N-Rényi entropy* with respect to the base measure $\mathfrak{m}$, $\mathcal{E}_N : \mathcal{P}_2(X) \to [-\infty, 0]$ is defined as

$$\mathcal{E}_N(\mu) := -\int_X \rho^{1-\frac{1}{N}}\, \mathfrak{m},$$

where $\mu = \rho\mathfrak{m} + \mu^\perp$ is the Lebesgue decomposition of μ with respect to $\mathfrak{m}$ with $\mu^\perp$ singular with respect to $\mathfrak{m}$.

Definition 1.4.8 ($\mathsf{CD}(K,N)$ condition)**.** $(X, \mathsf{d}, \mathfrak{m})$ verifies the $\mathsf{CD}(K,N)$ condition for some $K \in \mathbb{R}$, $N \in (1,\infty)$ if for any pair of probability measures $\mu_0, \mu_1 \in \mathcal{P}_2(X)$ with bounded support and with $\mu_0, \mu_1 \ll \mathfrak{m}$, there exists $\nu \in \mathrm{OptGeo}(\mu_0,\mu_1)$ such that $\mu_t := (\mathrm{e}_t)_\sharp \nu \ll \mathfrak{m}$ and

$$\mathcal{E}_{N'}(\mu_t) \leq \int \left\{ \tau_{K,N'}^{(1-t)}(\mathsf{d}(\gamma(0),\gamma(1)))\rho_0^{-\frac{1}{N'}}(\gamma(0)) + \tau_{K,N'}^{(t)}(\mathsf{d}(\gamma(0),\gamma(1)))\rho_1^{-\frac{1}{N'}}(\gamma(1)) \right\} \nu(\mathrm{d}\gamma)$$

for any $N' \geq N$, $t \in [0,1]$.

For our purposes it is crucial to recall the following definitions (see e.g. [37, Appendix A]).

Definition 1.4.9 (CD densities)**.** Let I be a real interval and h a non-negative function defined on I, then

- h is a $\mathsf{CD}(K,N)$ density, for $K, N \in \mathbb{R}$ and $N \in (1,\infty)$, if for all $x_0, x_1 \in I$ and $t \in [0,1]$:
$$h(tx_1 + (1-t)x_0)^{\frac{1}{N-1}} \geq \sigma_{K,N-1}^{(t)}(|x_1 - x_0|)h(x_1)^{\frac{1}{N-1}} + \sigma_{K,N-1}^{(1-t)}(|x_1 - x_0|)h(x_0)^{\frac{1}{N-1}};$$
- h is a $\mathsf{CD}(K,\infty)$ density if for all $x_0, x_1 \in I$ and $t \in [0,1]$:
$$\log h(tx_1 + (1-t)x_0) \geq t \log h(x_1) + (1-t)\log h(x_0) + \frac{K}{2}t(1-t)(x_1 - x_0)^2;$$
- h is a $\mathsf{CD}(K,1)$ density on I iff $K \leq 0$ and h is constant on the interior of I.

For the proof of the following result see [37, Theorem A.2].

Lemma 1.4.10. *If h is a $\mathsf{CD}(K,N)$ density on an interval $I \subset \mathbb{R}$ then the m.m.s. $(\bar{I}, |\cdot|, h(t)\mathcal{L}^1)$ verifies the $\mathsf{CD}(K,N)$ condition. Conversely, if the m.m.s. $(\mathbb{R}, |\cdot|, \mu)$ verifies $\mathsf{CD}(K,N)$ and $\bar{I} = \mathrm{supp}(\mu)$ is not a point, then $\mu \ll \mathcal{L}^1$ and there exists a version of the density $h = \frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{L}^1}$ which is a $\mathsf{CD}(K,N)$ density on I.*

We will restrict ourselves to spaces which satisfy the CD condition and additionally satisfy the following generalized notion of non-branching introduced in Definition 1.1.7.

Definition 1.4.11. $(X, \mathsf{d}, \mathfrak{m})$ is called essentially non-branching if for any $\mu_0, \mu_1 \ll \mathfrak{m}$ with $\mu_0, \mu_1 \in \mathcal{P}_2(X)$ any $\nu \in \mathrm{OptGeo}(\mu_0, \mu_1)$ is concentrated on a Borel non-branching subset $G \subset \mathrm{Geo}(X, \mathsf{d})$.

This definition was introduced in [80] by Rajala and Sturm. The restriction to essentially non-branching spaces is natural and facilitates avoiding pathological cases. One example is the failure of the local-to-global property for a general $\mathsf{CD}(K,N)$ in [79], property that has been recently verified in [37] under the assumption of essentially non-branching (and finite $\mathfrak{m}$).

RCD **condition**

The CD condition can be strengthened to get a "more Riemannian" structure. This was done in the infinite dimensional setting giving rise to the definition of *Riemannian curvature dimension condition* $\mathsf{RCD}(K,\infty)$ in [10] (see also [6] for the case of σ-finite measure). The finite dimensional counterpart was introduced in [58].

Definition 1.4.12. $(X,\mathsf{d},\mathfrak{m})$ m.m.s satifies the $\mathsf{RCD}(K,N)$ condition for $K\in\mathbb{R}$ and $N\in[1,+\infty]$ if it satisfies the $\mathsf{CD}(K,N)$ condition and it is Infinitesimally Hilbertian.

If $(X,\mathsf{d},\mathfrak{m})$ satisfies the $\mathsf{RCD}(K,\infty)$ condition then for any $f\in L^\infty(X,\mathfrak{m})$ we have that H_tf belongs to the space $\mathrm{Lip}_b(X)$, with the bound [49, Proposition 3.1]

$$\begin{aligned} \|\,|DH_tf|_w\,\|_{L^\infty} &\le \sqrt{\frac{2K}{\pi(e^{2Kt}-1)}}\,\|f\|_{L^\infty} \quad \text{if } K\neq 0,\\ \|\,|DH_tf|_w\,\|_{L^\infty} &\le \sqrt{\frac{1}{\pi t}}\,\|f\|_{L^\infty} \quad \text{if } K=0, \end{aligned} \tag{1.21}$$

(which is sharp in the case $K>0$).

From properties (1.18) and (1.21), H_t maps $\mathcal{C}_b(X)$ into itself, so it is defined its adjoint operator $H_t^*:\mathcal{P}(X)\to\mathcal{P}(X)$ that satisfies

$$H_t^*(\rho\mathfrak{m}) = H_t(\rho)\mathfrak{m} \tag{1.22}$$

for any probability density $\rho\in L^1_+(X,\mathfrak{m})$ (see [11, Proposition 3.2] for details).

Finally if one considers the metric gradient flow of the Boltzmann entropy Ent in $\mathcal{P}_2(X,\mathsf{d})$. it has been proven in [9] (the result is valid in $\mathsf{CD}(K,\infty)$ spaces) that for any $\mu\in D(\mathrm{Ent})$ there exists a unique gradient flow of Ent starting from μ (for details we refer to [9]). This gives rise to a semigroup $(\mathfrak{H}_t)_{t\ge0}$ on $P_2(X,\mathsf{d})$ defined by $\mathfrak{H}_t\mu=\mu_t$ where μ_t is the unique gradient flow of Ent starting from μ.

In [9] it is proven that the identification of the two gradient flows holds: if $(X,\mathsf{d},\mathfrak{m})$ is a $\mathsf{CD}(K,\infty)$ space and $f\in L^2(X,\mathfrak{m})$ such that $f\mathfrak{m}=\mu\in\mathcal{P}_2(X,\mathsf{d})$, then

$$\mathfrak{H}_t\mu = (H_tf)\,\mathfrak{m}, \qquad \forall\, t\ge 0. \tag{1.23}$$

Remark 1.4.13. We mention that in a $\mathsf{RCD}(K,\infty)$ space of finite measure, the condition $\mathrm{diam}\,(X)<\infty$, or $K>0$ implies that the embedding of $W^{1,2}(X)$ into $L^2(X)$ is compact (see [60, Proposition 6.7] and [50, Theorem 2.17]). This implies the existence of a basis of $L^2(X)$ formed by eigenfunctions corresponding to a diverging sequence of eigenvalues.

Moreover if in addition the space satisfies the $\mathsf{RCD}(K,N)$ condition for a finite N, then eigenfunctions are Lipschitz continuous (see [12, Proposition 7.1]).

MCP **condition**

For our purposes we also need to introduce a weaker variant of CD called Measure Contraction Property, $\mathsf{MCP}(K,N)$ in short, introduced separately by Ohta [78] and Sturm [94] with two definitions that slightly differ in general metric spaces, but that coincide on essentially non-branching spaces.

Definition 1.4.14 ($\mathsf{MCP}(K,N)$)**.** A m.m.s. $(X,\mathsf{d},\mathfrak{m})$ is said to satisfy $\mathsf{MCP}(K,N)$ if for any $o\in\mathrm{supp}(\mathfrak{m})$ and $\mu_0\in\mathcal{P}_2(X,\mathsf{d},\mathfrak{m})$ of the form $\mu_0=\frac{1}{\mathfrak{m}(A)}\mathfrak{m}_{|A}$ for some Borel set

$A \subset X$ with $0 < \mathfrak{m}(A) < \infty$ (and with $A \subset B(o, \pi\sqrt{(N-1)/K})$ if $K > 0$), there exists $\nu \in \mathrm{Opt}(\mu_0, \delta_o)$ such that:

$$\frac{1}{\mathfrak{m}(A)}\mathfrak{m} \geq (\mathrm{e}_t)_\sharp(\tau_{K,N}^{(1-t)}(\mathsf{d}(\gamma_0,\gamma_1))^N \nu(d\gamma)) \quad \forall t \in [0,1]. \tag{1.24}$$

If $(X, \mathsf{d}, \mathfrak{m})$ is a m.m.s. verifying $\mathsf{MCP}(K,N)$, then $(\mathrm{supp}(\mathfrak{m}), \mathsf{d})$ is Polish, proper and it is a geodesic space. With no loss in generality for our purposes we will assume that $X = \mathrm{supp}(\mathfrak{m})$.

It has been proven (see e.g. [94, 44]) that the MCP condition is implied by the CD condition and there are examples of spaces satisfying the first one and not satisfying the second one (see e.g. [66]).

In analogy to the case of CD densities, we recall the following definition of MCP density followed by a result describing the link between MCP density and MCP condition.

Definition 1.4.15 ($\mathsf{MCP}(K,N)$ density)**.** Given $K, N \in \mathbb{R}$ and $N \in (1, \infty)$, a non-negative function h defined on an interval $I \subset \mathbb{R}$ is called a $\mathsf{MCP}(K,N)$ density on I if for all $x_0, x_1 \in I$ and $t \in [0,1]$:

$$h(tx_1 + (1-t)x_0) \geq \sigma_{K,N-1}^{(1-t)}(|x_1 - x_0|)^{N-1} h(x_0). \tag{1.25}$$

Lemma 1.4.16. *If h is a $\mathsf{MCP}(K,N)$ density on an interval $I \subset \mathbb{R}$ then the m.m.s. $(I, |\cdot|, h(t)dt)$ verifies $\mathsf{MCP}(K,N)$. Conversely, if the m.m.s. $(\mathbb{R}, |\cdot|, \mu)$ verifies $\mathsf{MCP}(K,N)$ and $I = \mathrm{supp}(\mu)$ is not a point, then $\mu \ll L^1$ and there exists a version of the density $h = \frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{L}^1}$ which is a $\mathsf{MCP}(K,N)$ density on I.*

The inequality (1.25) implies several known properties that we collect in what follows. We define for $\kappa \in \mathbb{R}$ the function $s_\kappa : [0, +\infty) \to \mathbb{R}$ (on $[0, \pi/\sqrt{\kappa})$ if $\kappa > 0$)

$$s_\kappa(\theta) := \begin{cases} (1/\sqrt{\kappa}) \sin(\sqrt{\kappa}\theta) & \text{if } \kappa > 0, \\ \theta & \text{if } \kappa = 0, \\ (1/\sqrt{-\kappa}) \sinh(\sqrt{-\kappa}\theta) & \text{if } \kappa < 0. \end{cases} \tag{1.26}$$

We assume $I = (a,b)$ with $a, b \in \mathbb{R}$; hence (1.25) implies

$$\left(\frac{s_{K/(N-1)}(b-x_1)}{s_{K/(N-1)}(b-x_0)}\right)^{N-1} \leq \frac{h(x_1)}{h(x_0)} \leq \left(\frac{s_{K/(N-1)}(x_1-a)}{s_{K/(N-1)}(x_0-a)}\right)^{N-1}, \tag{1.27}$$

for $x_0 \leq x_1$. Hence denoting with $D = b - a$ the length of I, for any $\varepsilon > 0$ it follows that

$$\sup\left\{\frac{h(x_1)}{h(x_0)} : x_0, x_1 \in [a+\varepsilon, b-\varepsilon]\right\} \leq C_\varepsilon, \tag{1.28}$$

where C_ε only depends on K, N, provided $2\varepsilon \leq D \leq \frac{1}{\varepsilon}$. In particular, $\mathsf{MCP}(K,N)$ densities will be locally Lipschitz in the interior of their domain and continuous on its closure (see [42] for details).

1.5 L^1 optimal transport setting

What is contained in this section is taken from [21], [33], [42] (see also [39]). Since a large part of the thesis is based on L^1-optimal transport, we report all the statements that we need with the proofs, when possible.

1.5.1 Disintegration of measures

Given a measure space $(X, \mathscr{R})$, and a measure $\mathfrak{m}$ on $\mathscr{R}$ we say that a set $A \subseteq X$ is $\mathfrak{m}$-measurable if it belongs to the $\mathfrak{m}$-completion of the σ algebra $\mathscr{R}$ obtained by adding to $\mathscr{R}$ all the subsets of sets having zero $\mathfrak{m}$-measure. Analogously a function $f : X \to \mathbb{R}$ is measurable if counterimages of real Borel sets are $\mathfrak{m}$-measurable.

It is useful for what follows to recall that given a measure space $(X, \mathscr{R})$ and a function $\mathfrak{Q} : X \to Q$, with Q a set, we can endow Q with the *push forward σ-algebra* $\mathscr{Q}$ of $\mathscr{R}$:

$$C \in \mathscr{Q} \quad \Longleftrightarrow \quad \mathfrak{Q}^{-1}(C) \in \mathscr{X},$$

that is the biggest σ-algebra on Q such that $\mathfrak{Q}$ is measurable. Moreover, given a measure $\mathfrak{m}$ on $(X, \mathscr{R})$, we can define a measure $\mathfrak{q}$ on $(Q, \mathscr{Q})$ by pushing forward $\mathfrak{m}$ via $\mathfrak{Q}$, i.e. $\mathfrak{q} := \mathfrak{Q}_\sharp \, \mathfrak{m}$.

Definition 1.5.1 (Disintegation via a map)**.** Let $(X, \mathscr{R})$ be a measure space and $\mathfrak{m}$ a measure on it, Q a set and $\mathfrak{Q} : X \to Q$ a map. A *disintegration* of $\mathfrak{m}$ *consistent with* $\mathfrak{Q}$ is a map:

$$Q \ni \alpha \longmapsto \mathfrak{m}_\alpha \in \mathcal{P}(X, \mathscr{R})$$

such that:

1. for all $B \in \mathscr{R}$, the map $\alpha \mapsto \mathfrak{m}_\alpha(B)$ is $\mathfrak{q}$-measurable;
2. for all $B \in \mathscr{R}$ and $C \in \mathscr{Q}$, the following consistency condition holds:

$$\mathfrak{m}\left(B \cap \mathfrak{Q}^{-1}(C)\right) = \int_C \mathfrak{m}_\alpha(B) \, \mathfrak{q}(d\alpha).$$

A disintegration of $\mathfrak{m}$ is called *strongly consistent with respect to* $\mathfrak{Q}$ if in addition:

(3) for $\mathfrak{q}$-a.e. $\alpha \in Q$, $\mathfrak{m}_\alpha$ is concentrated on $\mathfrak{Q}^{-1}(\alpha)$.

The measures $\mathfrak{m}_\alpha$ are sometimes called marginal distributions.

We will focus in the disintegration in the following particular case.

Remark 1.5.2. Let $(X, \mathscr{R})$ be a measure space, Q a set of indexes and $\{X_\alpha\}_{\alpha \in Q}$ is a *partition* of X. One can take the following quotient map: $\mathfrak{Q} : X \to Q$, the map which associates to any element of X the index of the element of the partition to which it belongs (quotient map), i.e.

$$\mathfrak{Q}(x) = \alpha \iff x \in X_\alpha.$$

We endow Q with the quotient σ-algebra $\mathscr{Q}$ and the quotient measure $\mathfrak{q}$ as described above obtaining the quotient measure space $(Q, \mathscr{Q}, \mathfrak{q})$. When a disintegration $\alpha \mapsto \mathfrak{m}_\alpha$ of $\mathfrak{m}$ is (strongly) consistent with the quotient map $\mathfrak{Q}$, we will say that it is (strongly) consistent with the partition.

Let $\{X_\alpha\}_{\alpha \in Q}$ be a partition of X, a set $S \subset X$ is a section for the partition if for any $\alpha \in Q$, $S \cap X_\alpha$ is a singleton $\{x_\alpha\}$. By the axiom of choice, a section S always exists. A set of indexes Q and a section S can always be identified via the map $Q \ni \alpha \mapsto x_\alpha \in S$. A set $S_\mathfrak{m}$ is an $\mathfrak{m}$-section if there exists $Y \in \mathscr{R}$ with $\mathfrak{m}(X \setminus Y) = 0$ such that the partition $Y = \cup_{\alpha \in Q_\mathfrak{m}} (X_\alpha \cap Y)$ has section $S_\mathfrak{m}$, where $Q_\mathfrak{m} = \{\alpha \in Q; X_\alpha \cap Y \neq \emptyset\}$.

A σ-algebra $\mathcal{A}$ is *countably generated* if there exists a countable family of sets so that $\mathcal{A}$ coincides with the smallest σ-algebra containing them.

Now we recall a version of an existence and uniqueness result of the disintegration taken from [20, Theorem A.7, Proposition A.9].

Theorem 1.5.3. *Assume that* $(X, \mathscr{R}, \mathfrak{m})$ *is a countably generated probability space and that* $\{X_\alpha\}_{\alpha\in Q}$ *is a partition of* X*. Then there exists an essentially unique disintegration* $\alpha \mapsto \mathfrak{m}_\alpha$ *consistent with the partition.*

If in addition $\mathscr{R}$ *contains all singletons, then the disintegration is strongly consistent if and only if there exists a* $\mathfrak{m}$*-section* $S_\mathfrak{m} \in \mathscr{R}$ *of the partition such that the* σ*-algebra on* $S_\mathfrak{m}$ *induced by the quotient-map contains the trace* σ*-algebra* $\mathscr{R} \cap S_\mathfrak{m} := \{A \cap S_\mathfrak{m}; A \in \mathscr{R}\}$*.*

We will always apply the previous theorem in the following particular case.

Remark 1.5.4. Let (X, d) be a separable metric space. A sufficient condition for having a unique strongly consistent disintegration for a Borel measure $\mathfrak{m}$ on X with respect to a partition, is to have a section for the partition which is Borel and for which the quotient map is Borel. Indeed in this case the trace σ algebra is the sigma algebra induced by the Borel sets. In addition since the quotient map is Borel then the pre-image of any Borel set is Borel, therefore the push-forward σ-algebra contains the trace σ-algebra.

In the following lemma it is shown that under assumption of existence and uniqueness of the disintegration, if we disintegrate, with respect to the same map, two measures which are absolutely continuous one with respect to the other, then the absolutely continuity is preserved both by the pushforward measures and by the marginal distributions.

Lemma 1.5.5. *Let* $(X, \mathscr{R})$ *be a countably generated measure space and* $\mathfrak{Q} : X \to Q$ *a map and* $\mathscr{Q}$ *the pushforward sigma algebra on* Q*. Let* $\mathfrak{m}$ *and* μ *be measures on* $(X, \mathscr{R})$ *with* $\mu = \rho_\mu \mathfrak{m}$*. Assume that we have disintegrations of* $\mathfrak{m}$ *and* μ consistent with $\mathfrak{Q}$,

$$Q \ni q \longmapsto \mathfrak{m}_q \in \mathcal{P}(X, \mathscr{R}),$$
$$Q \ni q \longmapsto \mu_q \in \mathcal{P}(X, \mathscr{R}),$$

with $\mathfrak{Q}_\sharp \mathfrak{m} = \mathfrak{q}_\mathfrak{m}$*,* $\mathfrak{Q}_\sharp \mu = \mathfrak{q}_\mu$*. Then, setting* $l(q) := \int_X \rho_\mu \mathfrak{m}_q$*, we have*

$$\mathfrak{q}_\mu = l(q)\mathfrak{q}_\mathfrak{m} \tag{1.29}$$

and $\mathfrak{q}_\mu$ *a.e.* $q \in \mathscr{Q}$*,* $l(q) \neq 0$ *and*

$$\mu_q = \frac{\rho_\mu}{l(q)}\mathfrak{m}_q.$$

Proof. Take $A \in \mathscr{Q}$. Then

$$\mathfrak{q}_\mu(A) = \mu(\mathfrak{Q}^{-1}(A)) = \int_Q \int_{\mathfrak{Q}^{-1}(A)} \rho_\mu \mathfrak{m}_q \, \mathfrak{q}_\mathfrak{m}(\mathrm{d}q) = \int_A \int_X \rho_\mu \mathfrak{m}_q \, \mathfrak{q}_\mathfrak{m}(\mathrm{d}q) = \int_A l(q)\mathfrak{q}_\mathfrak{m}(\mathrm{d}q).$$

Since it holds for any $A \in \mathscr{Q}$, then the first claim in the statement follows. It also follows that

$$\mu = \int_Q \rho_\mu \mathfrak{m}_q \, \mathfrak{q}_\mathfrak{m}(\mathrm{d}q) = \int_Q \mu_q \, \mathfrak{q}_\mu(\mathrm{d}q) = \int_Q \mu_q l(q) \, \mathfrak{q}_\mathfrak{m}(\mathrm{d}q),$$

which by the uniqueness of the disintegration means that $\mathfrak{q}_\mathfrak{m}$ a.e. $q \in \mathcal{Q}$, $\mu_q l(q) = \rho_\mu \mathfrak{m}_q$. Moreover from (1.29) $l(q) \neq 0$ for $\mathfrak{q}_\mu$-a.e. $q \in \mathcal{Q}$ from which also the second part follows. □

1.5.2 Main definitions

For a more detailed reference about what follows see e.g. [1] and [15] for works about L^1-optimal transport in the euclidean space, [21] for the generalization to the general setting of geodesic metric spaces. Since large part of this thesis is based on L^1-optimal transport results we tried to keep the exposition self-contained when possible.

Let (X, d) be a geodesic metric space. Let $\varphi : X \to \mathbb{R}$ be a 1-Lipschitz function. Define the following d-monotone set associated to φ:

$$\Gamma_\varphi := \{(x, y) \in X \times X : \ \varphi(x) - \varphi(y) = \mathsf{d}(x, y)\}.$$

Define in addition:

- the *transport relation* associated to φ

$$R_\varphi := \Gamma_\varphi \cap \Gamma_\varphi^{-1};$$

  where $\Gamma_\varphi^{-1} := \{(x, y) \in X \times X : \ (y, x) \in \Gamma_\varphi\}$;

- for any $x \in X$, the section of Γ_φ through x, $\Gamma_\varphi(x) := \{y \in X : \ (x, y) \in \Gamma_\varphi\}$, and analogously $\Gamma_\varphi^{-1}(x)$. Set also $R_\varphi(x) := \Gamma_\varphi(x) \cup \Gamma_\varphi^{-1}(x)$;

- the *transport set with end points*,

$$\mathcal{T}_\varphi^e := P_1(R_\varphi \setminus \{(x, x) : \ x \in X\});$$

  equivalently $x \in \mathcal{T}_\varphi^e \iff$ there exists $y \neq x$ such that $(x, y) \in R_\varphi$,

- the set of *initial* and *final points*,

$$\begin{aligned}\mathfrak{a}_\varphi :=& \{x \in \mathcal{T}_\varphi^e : \nexists\, z \in X,\, x \neq z : \, z \in \Gamma_\varphi^{-1}(x)\},\\ \mathfrak{b}_\varphi :=& \{x \in \mathcal{T}_\varphi^e : \nexists\, z \in X,\, x \neq z : \, z \in \Gamma_\varphi(x)\};\end{aligned}$$

- the set of forward (and respectively backward) *branching points*:

$$\begin{aligned}A_\varphi^+ &:= \{x \in \mathcal{T}_\varphi^e : \exists z, w \in \Gamma_\varphi(x), (z, w) \notin R_\varphi\},\\ A_\varphi^- &:= \{x \in \mathcal{T}_\varphi^e : \exists z, w \in \Gamma_\varphi^{-1}(x), (z, w) \notin R_\varphi\},\end{aligned}$$

  more in general we define the set of *branching points* as $A_\varphi^+ \cup A_\varphi^-$;

- the *non-branching transport set*,

$$\mathcal{T}_\varphi^{nb} := \mathcal{T}_\varphi^e \setminus (A_\varphi^+ \cup A_\varphi^-). \tag{1.30}$$

Remark 1.5.6. We notice that if $x \in A_\varphi^+$ and $y \in \Gamma_\varphi^{-1}(x)$, then $y \in A_\varphi^+$. Analogously if $x \in A_\varphi^-$ and $y \in \Gamma_\varphi(x)$, then $y \in A_\varphi^-$.

Remark 1.5.7. If the space is non-branching, the set of branching points can still be non empty. However, in this case $A_\varphi^+ \subseteq \mathfrak{a}_\varphi$ and $A_\varphi^- \subseteq \mathfrak{b}_\varphi$. Indeed take $x \in A_\varphi^+$. Then there exists z, w such that $(x, z) \in \Gamma_\varphi$, $(x, w) \in \Gamma_\varphi$ and $(z, w) \notin R_\varphi$. Assume by contradiction that $x \notin \mathfrak{a}_\varphi$, then there exists $y \neq x$ such that $(y, x) \in \Gamma_\varphi$. Note that the points y, x, z, w, are all distinct. We can observe that

$$\mathsf{d}(y, x) + \mathsf{d}(x, z) = \varphi(y) - \varphi(x) + \varphi(x) - \varphi(z) = \varphi(y) - \varphi(z) \leq \mathsf{d}(y, z),$$

therefore $\mathsf{d}(y,x)+\mathsf{d}(x,z) = \mathsf{d}(y,z)$ and the same holds with y, x, w. Therefore from Remark 1.1.6 there exists two geodesic γ^1 and γ^2 satisfying $\gamma^1(0) = y$, $\gamma^1(1) = z$, $\gamma^2(0) = y$, $\gamma^2(1) = w$, both satisfying $\gamma^i(\bar{t}) = x$ for $i = 1, 2$ and $\bar{t} \in (0, 1)$ and coinciding on $t \in [0, \bar{t}]$. this contradicts the non-branching hypothesis. Analogously we can show that $A^-_\varphi \subseteq \mathfrak{b}_\varphi$.

We recall some notions of measure theory (see [87]) and collect here some measurability observations about the sets we constructed.

Definition 1.5.8. Let X be a Polish space. A set $B \subseteq X$ is called analytic if it is the projection of a Borel set of $X \times X$. A subset of X is universally measurable if it is $\mathfrak{m}$ measurable for any Borel measure $\mathfrak{m}$ on X.

Proposition 1.5.9. *Let X be a Polish space. Analytic sets are universally measurable i.e. if A is analytic then it is universally measurable (see [67, Theorem 21.10]).*

Remark 1.5.10. Projections of analytic sets are analytic (see [87, Proposition 4.1.2]).

Lemma 1.5.11. *Let (X, d) be a metric space and φ be a 1-Lipschitz function. Consider the sets constructed above. Then*

1. *R_φ is closed;*
2. *$\mathcal{T}^e_\varphi$ is universally measurable; moreover if the space is proper it is σ-compact;*
3. *A^+_φ and A^-_φ are universally measurable; moreover if the space is proper they are σ-compact;*
4. *$\mathcal{T}^{nb}_\varphi$ is universally measurable; moreover if the space is proper it is Borel;*
5. *$\mathfrak{a}_\varphi$ and $\mathfrak{b}_\varphi$ are universally measurable; moreover if the space is proper they are Borel.*

Proof. 1. it follows from the fact that φ continuous;

2. $\mathcal{T}^e_\varphi$ is the projection of the difference of two closed sets, so it is the projection of a Borel set and therefore analytic, hence by Proposition 1.5.9 it is universally measurable. For the second part call $D := \{(x, x) \colon x \in X\}$ and D^ε its open ε enlargement,

$$R_\varphi \setminus D = \cup_{n\in\mathbb{N}} \left(R_\varphi \cap (D^{\frac{1}{n}})^c \right).$$

Therefore $R_\varphi \setminus D$ is countable union of closed sets and so σ-compact if the space is proper. Hence $\mathcal{T}^e_\varphi$ is σ-compact.

3. It is immediate from the definition that

$$A^+_\varphi = P_1 \left((\mathcal{T}^e_\varphi \times R^c_\varphi) \cap (P^{-1}_{1,2}(\Gamma_\varphi) \cap (P^{-1}_{1,3}(\Gamma_\varphi) \right)$$

where $P_{1,2}$ (resp. $P_{1,3}$) is the projection $X \times X \times X \to X \times X$ onto the first and second (resp. first and third) component. Therefore A^+_φ is the projection of an analytic set, so analytic and universally measurable. If the space is proper $\mathcal{T}^e_\varphi$ is σ-compact as shown in point (2) and R_φ^c is an open set in a metric space and so countable union of closed sets and therefore σ-compact. Moreover $P_{1,2}^{-1}(\Gamma_\varphi), P_{1,2}^{-1}(\Gamma_\varphi)$ are closed and thus σ-compact since Γ_φ is closed and the projections are continuous. The proof for A_φ^- is analogous.

4. It follows from the fact that $\mathcal{T}_\varphi^{nb} = \mathcal{T}_\varphi^e \setminus (A_\varphi^+ \cup A_\varphi^-)$ and the previous points.

5. We need to observe that

$$\mathfrak{a}_\varphi^c \cap \mathcal{T}_\varphi^e = P_1(\mathcal{T}_\varphi^e \times X \cap \Gamma_\varphi^{-1}).$$

So arguing as above $\mathfrak{a}_\varphi^c \cap \mathcal{T}_\varphi^e$ is universally measurable. Therefore $\mathfrak{a}_\varphi = (\mathfrak{a}_\varphi^c \cap \mathcal{T}_\varphi^e)^c \cap \mathcal{T}_\varphi^e$ is universally measurable. The Borelianity in a proper space follows as in the previous cases. $\mathfrak{b}_\varphi$ can be treated in the same way.

□

Notice that if two points are in relation through R_φ then any couple of points in a geodesic connecting them is also in relation as shown in the following result.

Lemma 1.5.12. *Let γ be a geodesic with $(\gamma(0), \gamma(1)) \in \Gamma_\varphi$. Then $(\gamma(s), \gamma(t)) \in \Gamma_\varphi$ for any $s \le t \in [0, 1]$.*

Proof. Let $t \ge s$,

$$\begin{aligned}
&\mathsf{d}(\gamma(s), \gamma(t)) \ge \varphi(\gamma(s)) - \varphi(\gamma(t)) \\
&= \varphi(\gamma(s)) - \varphi(\gamma(0)) + \varphi(\gamma(0)) - \varphi(\gamma(1)) + \varphi(\gamma(1)) - \varphi(\gamma(t)) \\
&\ge \mathsf{d}(\gamma(0), \gamma(1)) - \mathsf{d}(\gamma(s), \gamma(0)) - \mathsf{d}(\gamma(1), \gamma(t)) = \mathsf{d}(\gamma(s), \gamma(t)).
\end{aligned}$$

□

R_φ is in general not an equivalence relation, in particular it is not necessarily transitive. In the following proposition we show that it is when restricted to the non-branching transport set.

Proposition 1.5.13. *The relation $R_\varphi \cap (\mathcal{T}_\varphi^{nb} \times \mathcal{T}_\varphi^{nb})$ is an equivalence relation. In particular R_φ defines a partition on $\mathcal{T}_\varphi^{nb}$ into equivalence classes.*

Proof. The relation is reflexive and symmetric by definition of Γ_φ and R_φ. We need only to check transitivity.
Let x, y, z be distinct elements of $\mathcal{T}_\varphi^{nb}$, such that $(x, y) \in R_\varphi$ and $(y, z) \in R_\varphi$, then we have the following cases. If $y \in \Gamma_\varphi(x)$ and $z \in \Gamma_\varphi(y)$ one has

$$\mathsf{d}(x, z) \ge \varphi(x) - \varphi(z) = \varphi(x) - \varphi(y) + \varphi(y) - \varphi(z) = \mathsf{d}(x, y) + \mathsf{d}(y, z) \ge \mathsf{d}(x, z),$$

so equality follows and $(z, x) \in R_\varphi$. If $y \in \Gamma_\varphi(x)$ and $z \in \Gamma_\varphi^{-1}(y)$ then $x, z \in \Gamma_\varphi^{-1}(y)$, and since $y \notin A_\varphi^-$, then $(x, z) \in R_\varphi$. If $y \in \Gamma_\varphi^{-1}(x)$ and $z \in \Gamma_\varphi(y)$, then $x, z \in \Gamma_\varphi(y)$, and since $y \notin A_\varphi^+$, then as before $(x, z) \in R_\varphi$. If $y \in \Gamma_\varphi^{-1}(x)$ and $z \in \Gamma_\varphi^{-1}(y)$ then $x \in \Gamma_\varphi(y)$ and $y \in \Gamma_\varphi(z)$ so as in the first case $(x, z) \in R_\varphi$. □

We call $[x]_\varphi$ the equivalence class of an element $x \in \mathcal{T}_\varphi^{nb}$ determined by the equivalence relation $R_\varphi \cap (\mathcal{T}_\varphi^{nb} \times \mathcal{T}_\varphi^{nb})$. Note that $[x]_\varphi = R_\varphi(x) \cap \mathcal{T}_\varphi^{nb}$.

1.5.3 Structure of the transport set

We start with a remark.

Remark 1.5.14. For any $x \in \mathcal{T}_\varphi^{nb}$, $\varphi_{|R_\varphi(x)}$ is an isometry. Indeed if $[x]_\varphi$ is a point, we are done. Otherwise let y, z be two different points of $[x]_\varphi$. Then $(y, z) \in R_\varphi$ (by definition of equivalence class). We assume without loss of generality that $(y, z) \in \Gamma_\varphi$ and therefore $\varphi(y) - \varphi(z) = \mathsf{d}(y, z)$. $\varphi_{|[x]_\varphi}$ is an isometry.

Lemma 1.5.15. *For every $x \in \mathcal{T}_\varphi^{nb}$ the set $\varphi([x]_\varphi)$ is an interval $I \subseteq \mathbb{R}$ (open closed or none of them). In particular $[x]_\varphi$ is isometric to I.*

Proof. It is enough to show that $\varphi([x]_\varphi)$ is connected. If $\varphi([x]_\varphi)$ is a point we are done taking as I a degenerate interval. Otherwise let $a < b \in \mathbb{R}$ such that $\varphi(z) = a$, $\varphi(w) = b$ for $z, w \in [x]_\varphi$. Let $\gamma : [0, 1] \to X$ be a geodesic such that $\gamma(0) = w$, $\gamma(1) = z$. We show that $\gamma(s) \in [x]_\varphi$ for any $s \in [0, 1]$ from which follows by continuity of φ that $\varphi([x]_\varphi)$ is connected. Let $(w, z) \in R_\varphi$ by definition of equivalence class and since $\varphi(w) > \varphi(z)$, $(w, z) \in \Gamma_\varphi$. Therefore we can apply 1.5.12 to deduce that $(\gamma(s), \gamma(1)) \in \Gamma_\varphi$ and $(\gamma(0), \gamma(s)) \in \Gamma_\varphi$ for any $s \in [0, 1]$. It remains to show that $\gamma(s) \in \mathcal{T}_\varphi^{nb}$. We need to prove that $\gamma(s) \notin A_\varphi^+ \cup A_\varphi^-$. Assume that $\gamma(s) \in A_\varphi^+$, then this would imply that also $x = \gamma(0) \in A_\varphi^+$ (see Remark 1.5.6) which is false. Assume that $\gamma(s) \in A_\varphi^-$, then this would imply that also $y = \gamma(1) \in A_\varphi^-$ which is still false. So $\gamma(s) \in \mathcal{T}_\varphi^{nb}$ and $(\gamma(0), \gamma(s)) \in \Gamma(\varphi)$ and hence $\gamma(s) \in [x]_\varphi$. The second part follows from Remark 1.5.14. $\square$

Remark 1.5.16. For any $x \in \mathcal{T}_\varphi^{nb}$, it follows analogously that $\varphi_{|R_\varphi(x)}$ is an isometry and $\varphi(R_\varphi(x))$ is connected.

1-Lipschitz functions and disintegration

Given a 1-Lipschitz function φ we apply the Disintegration result of section 1.5.1 to the partition of the non-branching transport set constructed through the equivalence relation induced by φ. In order to apply the Disintegration Theorem to the latter partition, we need a set of indexes. We will choose as set of indexes a section for the partition. Since we want the disintegration to be strongly consistent, we need the section to satisfy some measurability properties. This is possible thanks to the following lemma.

Lemma 1.5.17. *Let (X, d) be a proper and geodesic metric space and φ be a 1-Lipschitz function. Let $\mathcal{T}_\varphi^{nb}$ be the set and R_φ the equivalence relation constructed above. Consider the measure space $(\mathcal{T}_\varphi^{nb}, \mathcal{B}(\mathcal{T}_\varphi^{nb}))$. There exists a section $\mathcal{Q} \in \mathcal{B}(\mathcal{T}_\varphi^{nb})$ for the partition into equivalence classes given by R_φ, for which the quotient map*

$$f_\varphi : (\mathcal{T}_\varphi^{nb}, \mathcal{B}(\mathcal{T}_\varphi^{nb})) \to (\mathcal{Q}, \sigma(\mathcal{Q}))$$

is Borel, where $\sigma(\mathcal{Q}) = \mathcal{B}(\mathcal{Q})$ is the trace sigma algebra on Q.

Proof. See [33] Proposition 3.4.8. $\square$

Let $\mathcal{Q}$ be the set of Lemma 1.5.17. We denote for any $q \in \mathcal{Q}$,

$$X_q^\varphi := \{x \in \mathcal{T}_\varphi^{nb} \colon f_\varphi(x) = q\} = [q]_\varphi, \tag{1.31}$$

the equivalence class of the element $x \in \mathcal{T}_\varphi^{nb}$ such that $f_\varphi(x) = q$. From now on when referring to the equivalence classes, we will use the notation $\{X_q^\varphi\}_{q \in \mathcal{Q}}$ dropping the previous one of $\{[q]_\varphi\}_{q\in\mathcal{Q}}$. Sometimes we will drop the dependence on φ using X_q in place of X_q^φ. We call each X_q a transport ray. The following proposition shows that Disintegration Theorem 1.5.3 holds in the measure space $(\mathcal{T}_\varphi^{nb}, \mathcal{B}(\mathcal{T}_\varphi^{nb}))$ with respect to the partition into equivalence classes $\{X_q^\varphi\}_{q\in Q}$.

Proposition 1.5.18. *Let (X, d) be a proper metric space. Let φ be a 1-Lipschitz function, $\mathcal{T}_\varphi^{nb}$ the set defined above. Let $\mathfrak{m}$ be a finite Borel measure on X. Consider the space $(\mathcal{T}_\varphi^{nb}, \mathcal{B}(\mathcal{T}_\varphi^{nb}))$ and $\mathfrak{m}_{|\mathcal{T}_\varphi^{nb}}$. Then there is an essentially unique strongly consistent disintegration of $\mathfrak{m}_{|\mathcal{T}_\varphi^{nb}}$ with respect to the partition into equivalence classed $\{X_q^\varphi\}_{q\in\mathcal{Q}}$, i.e. there exists a $\mathfrak{q} := (f_\varphi)_\sharp \mathfrak{m}_{|\mathcal{T}_\varphi^{nb}}$ essentially unique map $\mathcal{Q} \ni q \mapsto \mathfrak{m}_q \in \mathcal{P}(\mathcal{T}_\varphi^{nb})$ such that*

$$\mathfrak{m}_{|\mathcal{T}_\varphi^{nb}} = \int_{\mathcal{Q}} \mathfrak{m}_q \mathfrak{q}(dq) \tag{1.32}$$

and $\mathfrak{m}_q$ is concentrated on X_q for $\mathfrak{q}$-a.e. q, where $\mathcal{Q}$ and f_φ are any Borel section of the partition and any Borel quotient map.

Proof. Since our space $(X, d, \mathfrak{m})$ is separable, then $\mathcal{B}(X)$ is countably generated and so is $\mathcal{B}(\mathcal{T}_\varphi^{nb})$. By Lemma 1.5.17 the section $\mathcal{Q}$ is $\mathfrak{m}_{|\mathcal{T}_\varphi^{nb}}$-measurable since it is Borel. Moreover the sigma algebra on $\mathcal{Q}$ induced by the quotient map f_φ contains the trace sigma algebra of Borel sets since the map f_φ is Borel. This means that we can apply the complete version of the Disintegration Theorem 1.5.3. So there exists an essentially unique disintegration strongly consistent with the partition $\{X_q^\varphi\}_{q\in Q}$. □

Remark 1.5.19. The measure $\mathfrak{q}$ in Proposition 1.5.18 is a Borel measure on $\mathcal{Q} \subseteq X$. This follows from the fact that f_φ is Borel.

Remark 1.5.20. Up to renormalizing one can choose $\mathfrak{q} \in \mathcal{P}(\mathcal{Q})$ and consequently $\mathfrak{m}_q \in \mathcal{M}(X)$ not necessarily of total mass equal to one.

Parametrization of the non-branching transport set

We have shown in Lemma 1.5.15 that any equivalence class is isometric to a real interval. Informally, this means that we have constructed for our space a sort of system of coordinates lying on $\mathcal{Q} \times \mathbb{R}$. Now we show that this choice of coordinates can be done in a measurable way. Recall more precisely that we have proved in Lemma 1.5.15 that given a 1-Lipschitz function φ, fixed the section $\mathcal{Q}$ for the partition of $\mathcal{T}_\varphi^{nb}$, for any $q \in \mathcal{Q}$, X_q is isometric to a real interval I_q. In particular, up to reparametrization, we have isometries, for any q, $\gamma_q^\varphi : I_q \to X_q$ satisfying $\gamma_q^\varphi(0) = q$.

Definition 1.5.21 (Ray map)**.** Set $\mathrm{Dom}(g) := \cup_{q\in\mathcal{Q}}\{q\} \times I_q$

$$\begin{aligned} g : \mathrm{Dom}(g) \subset Q \times \mathbb{R} &\to \mathcal{T}_\varphi^{nb} \\ (q, t) &\mapsto \gamma_q^\varphi(t), \end{aligned}$$

where $\gamma_q^\varphi : I_q \to X$ is the isometry defined above. Notice that it holds: $\mathrm{graph}(g) = S$, with

$$\begin{aligned} S := &\Big\{(q, t, x) \in \mathcal{Q} \times [0, +\infty) \times \mathcal{T}_\varphi^{nb} : (q, x) \in \Gamma_\varphi,\ \mathsf{d}(q, x) = t\Big\} \\ &\cup \Big\{(q, t, x) \in \mathcal{Q} \times (-\infty, 0] \times \mathcal{T}_\varphi^{nb} : (x, q) \in \Gamma_\varphi,\ \mathsf{d}(q, x) = -t\Big\}. \end{aligned} \tag{1.33}$$

We are not calling $\{\gamma_q^\varphi\}_{q\in\mathcal{Q}}$ geodesics since they are not parametrized in $[0,1]$ and they can have infinite length.

Proposition 1.5.22. *Let (X,d) be a proper and geodesic metric measure space and φ be a 1-Lipschitz function. Let g be the map defined in Definition 1.5.21. Then the set $Dom(g)$ is analytic and the map $g : Dom(g) \to \mathcal{T}_\varphi^{nb}$ is Borel. In addition g is a bijection with $g^{-1} : \mathcal{T}_\varphi^{nb} \to Dom(g)$ also Borel.*

We recall here a measurability result that we use in the proof of the Proposition. It is taken from Theorem 4.5.2 in [87].

Theorem 1.5.23. *Let X and Y be two Polish spaces. Let $A \subseteq X$ be analytic and $f : A \to Y$. Then if* $\mathrm{graph}(f)$ *is analytic, f is Borel measurable.*

Proof of Proposition 1.5.22. We claim that the set S in (1.33) is Borel. We show that the first term of the union is Borel. The second is analogous.

$$\Big\{(q,t,x) \in \mathcal{Q} \times [0,+\infty) \times \mathcal{T}_\varphi^{nb} : (q,x) \in \Gamma_\varphi,\ \mathsf{d}(q,x) = t\Big\}$$
$$= (\mathcal{Q} \times [0,+\infty) \times \mathcal{T}_\varphi^{nb}) \cap \{(q,t,x) \colon (q,x) \in \Gamma_\varphi,\ t \in \mathbb{R}\} \cap \{(q,t,x) : \mathsf{d}(q,x) - t = 0\}$$

where $\mathcal{Q}\times[0,+\infty)\times\mathcal{T}_\varphi^{nb}$ is Borel since $\mathcal{Q}$ and $\mathcal{T}_\varphi^{nb}$ are Borel and $\{(q,t,x) \colon (q,x) \in \Gamma_\varphi,\ t \in \mathbb{R}\}$ and $\{(q,t,x) \colon (q,x) \in \Gamma_\varphi,\ t \in \mathbb{R}\}$ are closed. Therefore $P_{1,2}(S) = \mathrm{Dom}(g)$ is analytic. In addition thanks to Theorem 1.5.23 one has that the map g is Borel. The fact that g^{-1} is Borel follows from the fact that for $x \in \mathcal{T}_\varphi^{nb}$, $g^{-1}(x) = (f_\varphi(x), \varphi(x) - \varphi(f_\varphi(x)))$ where f_φ is the Borel quotient map defined above. □

1.6 One dimensional localization

In this section we will state some of the results in this context, that will be useful for us without pretension of being exhaustive.

Localization of the $\mathsf{CD}(K,N)$ and $\mathsf{MCP}(K,N)$ conditions

The construction made in the previous section turned out to inherit curvature properties of the space. Informally we have that if $(X,\mathsf{d},\mathfrak{m})$ is a m.m.s. and φ is a 1-Lipschitz function,we can consider the construction made in the previous Section, namely we can built the set $\mathcal{T}_\varphi^{nb}$ and partition it into equivalence classes $\{X\}_{q\in\mathcal{Q}}$ constructed through the equivalence relation Γ_φ. By applying the Disintegration Theorem we get a disintegration of the measure $\mathfrak{m}_{|\mathcal{T}_\varphi^{nb}}$, $q \mapsto \mathfrak{m}_q$ with almost any $\mathfrak{m}_q$ concentrated on X_q. If the space is essentially non-branching and satisfies the $\mathsf{CD}(K,N)$ condition or the $\mathsf{MCP}(K,N)$ condition almost all the spaces $(X_q, \mathsf{d}_{|X_q\times X_q}, \mathfrak{m}_q)$ satisfy the same synthetic Ricci curvature lower bound.

First we remark that if the space satisfies some curvature assumption like the MCP condition and is essentially non-branching, then the sets constructed through the 1-Lipschitz function enjoy some additional properties. The first one concerns the negligibility of the branching points. It was proved first in [32] for $\mathsf{RCD}(K,N)$ spaces relying on existence and uniqueness of L^2-optimal maps ([62]), in particular after [38] it extends to the MCP case.

Proposition 1.6.1. *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an essentially non-branching m.m.s. satisfying the* $\mathsf{MCP}(K,N)$ *condition. Let* φ *be a* 1*-Lipschitz function. Let* A^+_φ *and* A^-_φ *be the sets of forward and backwards branching points defined in Section 1.5.2. Then*

$$\mathfrak{m}(A^+_\varphi) = \mathfrak{m}(A^-_\varphi) = 0.$$

This in particular implies that one is able to get a disintegration of the measure not only on $\mathcal{T}^{nb}_\varphi$ but on the whole transport set $\mathcal{T}^e_\varphi$.

The second good property implied by the curvature assumptions, is the maximality of the equivalence classes. In particular under the same assumptions of the previous proposition it holds

$$R_\varphi(q) = \bar{X}_q \supseteq X_q \supset \overset{\circ}{R}_\varphi(q), \text{ for } \mathfrak{q} - \text{a.e. } q \in \mathcal{Q},$$

where $\mathcal{Q}$ is a Borel section for the partition, f_φ a Borel quotient map, $\mathfrak{q} := (f_\varphi)\sharp \mathfrak{m}_{|\mathcal{T}^{nb}_\varphi}$ and $\overset{\circ}{R}_\varphi(q)$ is $R_\varphi(q)$ without its endpoints (recall Remark 1.5.16). This was proved in [37] relying only on existence and uniqueness of L^2-optimal maps and Proposition 1.6.1.

Finally equivalence classes inherits curvature properties of the space as stated in the following theorems.

Theorem 1.6.2. *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an essentially non-branching m.m.s. with* $\mathfrak{m}(X) < +\infty$*, verifying the* $\mathsf{CD}(K,N)$ *(resp.* $\mathsf{MCP}(K,N)$*) condition for* $K \in \mathbb{R}$*,* $N \in [1, +\infty)$*. Let* φ *be a* 1*-Lipschitz function. Consider the partition of the set* $\mathcal{T}^{nb}_\varphi$ *into equivalence classes* $\{X^\varphi_q\}_{q \in \mathcal{Q}}$*,* $\mathcal{Q}$ *a Borel section for the partition and* f_φ *a Borel quotient map. Consider in addition the strongly consistent disintegration of* $\mathfrak{m}_{|\mathcal{T}^{nb}_\varphi}$

$$\mathfrak{m}_{|\mathcal{T}^{nb}_\varphi} = \int_{\mathcal{Q}} \mathfrak{m}_q \mathfrak{q}(dq),$$

with $\mathfrak{q} := (f_\varphi)\sharp \mathfrak{m}_{|\mathcal{T}^{nb}_\varphi}$*. Then for* $\mathfrak{q}$ *a.e.* $q \in \mathcal{Q}$*,* $(\bar{X}_q, \mathsf{d}, \mathfrak{m}_q)$ *satisfies the* $\mathsf{MCP}(K,N)$ *(resp.* $\mathsf{CD}(K,N)$*) condition.*

In the particular case where φ is a Kantorovich potential for an L^1-optimal transport problem the partition which inherits curvature properties of the space is also the one which allows to reduce the optimal transport problem to one dimensional transport problems.

Theorem 1.6.3. *Under the hypotheses and notations of the previous theorem, assume that* φ *is a Kantorovich potential for the* L^1*-optimal trasport problem between* $f^+\mathfrak{m}$ *and* $f^-\mathfrak{m}$ *where* $f : X \to \mathbb{R}$ *is* $\mathfrak{m}$*-integrable such that* $\int_X f\, \mathfrak{m} = 0$ *and* $\int_X |f(x)|\, \mathsf{d}(x, x_0)\, \mathfrak{m}(dx) < \infty$ *for some* $x_0 \in X$*.*

Then

- $\mathfrak{q}$*-almost every* $q \in \mathcal{Q}$*, it holds* $\int_{X_q} f\, \mathfrak{m}_q = 0$*,*
- $f = 0$ $\mathfrak{m}$*-a.e. in* $(\mathcal{T}^{nb}_\varphi)^c$*.*

Theorem 1.6.2 and 1.6.3 were proved in the $\mathsf{CD}(K,N)$ case in [39], while the MCP version is discussed in [42]. A localization result under the stronger assumption of non-branching, in the MCP case already appeared in [21].

We finally report for completeness the Localization in its classical statement which is presented in a more implicit version, including also the case of σ-finite measures proved in [42].

Theorem 1.6.4. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an essentially non-branching metric measure space with* $\mathrm{supp}(\mathfrak{m}) = X$*. Let $f : X \to \mathbb{R}$ be $\mathfrak{m}$-integrable such that $\int_X f \, \mathfrak{m} = 0$ and assume the existence of $x_0 \in X$ such that $\int_X |f(x)| \, \mathsf{d}(x, x_0) \, \mathfrak{m}(\mathrm{d}x) < \infty$.*

Assume also $(X, \mathsf{d}, \mathfrak{m})$ verifies $\mathsf{CD}(K, N)$ (resp. $\mathsf{MCP}(K, N)$) condition for some $K \in \mathbb{R}$ and $N \in [1, \infty)$.

Then the space X can be written as the disjoint union of two sets Z and $\mathcal{T}$ with $\mathcal{T}$ admitting a partition $\{X_q\}_{q \in Q}$ and a corresponding disintegration of $\mathfrak{m}_{|\mathcal{T}}$ such that:

$$\mathfrak{m}_{|\mathcal{T}} = \int_Q \mathfrak{m}_q \, \mathfrak{q}(\mathrm{d}q),$$

where $\mathfrak{q}$ is a Borel probability measure over $Q \subset X$ such that $\mathfrak{Q}_\sharp(\mathfrak{m}_{|\mathcal{T}}) \ll \mathfrak{q}$, with $\mathfrak{Q}$ the quotient map associated to the partition and the map $Q \ni q \mapsto \mathfrak{m}_q \in \mathcal{M}_+(X)$ satisfying the following properties:

- *for any $\mathfrak{m}$-measurable set B, the map $q \mapsto \mathfrak{m}_q(B)$ is $\mathfrak{q}$-measurable;*
- *for $\mathfrak{q}$-a.e. $q \in Q$, $\mathfrak{m}_q$ is concentrated on $\mathfrak{Q}^{-1}(q) = X_q$ (strong consistency);*
- *For $\mathfrak{q}$-almost every $q \in Q$, it holds $\int_{X_q} f \, \mathfrak{m}_q = 0$ and $f = 0$ $\mathfrak{m}$-a.e. in Z.*
- *For $\mathfrak{q}$-almost every $q \in Q$, the set X_q is a geodesic (even more a transport ray) and the one dimensional m.m.s. $(X_q, \mathsf{d}, \mathfrak{m}_q)$ verifies $\mathsf{CD}(K, N)$ (resp. $\mathsf{MCP}(K, N)$).*

Moreover, fixed any $\mathfrak{q}$ as above such that $\mathfrak{Q}_\sharp(\mathfrak{m}_{|\mathcal{T}}) \ll \mathfrak{q}$, the disintegration is $\mathfrak{q}$-essentially unique.

The following remark together with the localization theorem will allow us to work in the weighted interval.

Remark 1.6.5. In Theorem 1.6.2 and Theorem 1.6.4, for $\mathfrak{q}$-almost every $q \in Q$ the space $(X_q, d_{|X_q}, \mathfrak{m}_q)$ is isomorphic to $(I_q, |\cdot|, \tilde{\mathfrak{m}}_q)$ via the inverse of the ray map $g(q, \cdot)^{-1} : X_q \to I_q$ with I_q real interval. In particular Lemma 1.4.10 (resp. 1.4.16) implies that

$$g(q, \cdot)^{-1}_\sharp \mathfrak{m}_q = \tilde{\mathfrak{m}}_q = h_q \cdot \mathcal{L}^1$$

with h_q is a $\mathsf{CD}(K, N)$ (resp. $\mathsf{MCP}(K, N)$) density.

Localization of the $\mathsf{CD}(K, \infty)$ condition under $\mathsf{MCP}(\bar{K}, N)$ condition

Theorem 1.6.2 is not known for a general $\mathsf{CD}(K, \infty)$ spaces. We state here a version of the localization of this condition under the additional assumption that the space satisfies additionally the $\mathsf{MCP}(K', N')$ for some $K', N' \in \mathbb{R}$ (with possibly K' different from K). This assumption excludes all the technical issues and the proof of the following localization result just follows as the one of Theorem 1.6.4.

Theorem 1.6.6. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an essentially non-branching metric measure space with* $\mathrm{supp}(\mathfrak{m}) = X$*. Let $f : X \to \mathbb{R}$ be $\mathfrak{m}$-integrable such that $\int_X f \, \mathfrak{m} = 0$ and assume the existence of $x_0 \in X$ such that $\int_X |f(x)| \, \mathsf{d}(x, x_0) \, \mathfrak{m}(dx) < \infty$.*

Assume also $(X, \mathsf{d}, \mathfrak{m})$ verifies $\mathsf{CD}(K, \infty)$ and $\mathsf{MCP}(K', N')$ conditions for some $K, K' \in \mathbb{R}$ and $N' \in [1, \infty)$.

Then the space X can be written as the disjoint union of two sets Z and $\mathcal{T}$ with $\mathcal{T}$ admitting a partition $\{X_q\}_{q\in Q}$ and a corresponding disintegration of $\mathfrak{m}_{|\mathcal{T}}$ such that:

$$\mathfrak{m}_{|\mathcal{T}} = \int_Q \mathfrak{m}_q \, \mathfrak{q}(\mathrm{d}q),$$

where $\mathfrak{q}$ is a Borel probability measure over $Q \subset X$ such that $\mathfrak{Q}_\sharp(\mathfrak{m}_{|\mathcal{T}}) \ll \mathfrak{q}$ and the map $Q \ni q \mapsto \mathfrak{m}_q \in \mathcal{M}_+(X)$ satisfies the following properties:

- *for any $\mathfrak{m}$-measurable set B, the map $q \mapsto \mathfrak{m}_q(B)$ is $\mathfrak{q}$-measurable;*
- *for $\mathfrak{q}$-a.e. $q \in Q$, $\mathfrak{m}_q$ is concentrated on $\mathfrak{Q}^{-1}(q) = X_q$ (strong consistency);*
- *For $\mathfrak{q}$-almost every $q \in Q$, it holds $\int_{X_q} f \, \mathfrak{m}_q = 0$ and $f = 0$ $\mathfrak{m}$-a.e. in Z.*
- *For $\mathfrak{q}$-almost every $q \in Q$, the set X_q is a geodesic (even more a transport ray) and the one dimensional m.m.s. $(X_q, \mathsf{d}, \mathfrak{m}_q)$ verifies $\mathsf{CD}(K, \infty)$.*

Moreover, fixed any $\mathfrak{q}$ as above such that $\mathfrak{Q}_\sharp(\mathfrak{m}_{|\mathcal{T}}) \ll \mathfrak{q}$, the disintegration is $\mathfrak{q}$-essentially unique.

Chapter 2

Indeterminacy estimate via localization and lower bound for the nodal set of eigenfunctions

In this chapter we report and describe the results obtained in [35].

2.1 One dimensional indeterminacy estimates

As already described in the introduction, the first part of the chapter is devoted to the proof of an indeterminacy estimate in the setting of metric spaces with synthetic Ricci curvature lower bounds. The first step towards the proof consists in proving an indeterminacy estimate in the standard real interval, that is what we do in the following subsection. We then will use part of the proof of the Euclidean indeterminacy estimate in Section 2.1.2 and 2.1.3 to derive the corresponding version for intervals with weighted Lebesgue measure, respectively with weight being a CD density or MCP density.

2.1.1 Indeterminacy estimate in the standard real interval

We first need to fix some notations.

Definition 2.1.1. Given an interval $I = [x, y] \subset \mathbb{R}$ (with x, y possibly $-\infty$ or $+\infty$) and a set A open in I, we define the *counting boundary* $CB(A, I) \subset \mathrm{Int}(I)$ as follows. Let $C_1, \ldots, C_n$ be the connected components of A in I (which are intervals open in I), with n possibly $+\infty$. In particular $\cup_{k=1}^{n} \bar{C}_k = \cup_{k=1}^{m} [a_k, b_k]$, with $[a_k, b_k] \subset I$ disjoint, with m possibly $+\infty$, $a_k, b_k \in \mathbb{R} \cup \{\pm\infty\}$. Then we set

$$CB(A, I) := \cup_{k=1}^{m} \{a_k, b_k\} \setminus \{x, y\}.$$

Definition 2.1.2. Given a function $f : I \to \mathbb{R}$ with zero mean, with I real closed interval, possibly of infinite length, satisfying the hypotheses of Theorem 1.6.4 we say that the transport of f goes along a unique transport ray if applying Theorem 1.6.4 one has that the partition in $\{X_q\}_q$ is made of only one element.

Remark 2.1.3. Given $f \in L^1(I)$ with $\int_I f(x)\,\mathrm{d}x = 0$. If the transport of f goes along a unique transport ray, then $\varphi(x) = x$ (or $\varphi(x) = -x$) is a Kantorovich potential for the L^1-optimal transport problem between $f^+\mathcal{L}^1$ and $f^-\mathcal{L}^1$.

We can now state and prove our first estimate.

Proposition 2.1.4. *Let $f\colon [0,1] \to \mathbb{R}$ be a continuous function having zero mean w.r.t Lebesgue measure, i.e.*

$$\int_{(0,1)} f^+(x)\,\mathrm{d}x = \int_{(0,1)} f^-(x)\,\mathrm{d}x,$$

and assume that the transport of f goes along a unique transport ray, (see Definition 2.1.2). Then it holds:

$$W_1(f^+\mathcal{L}^1, f^-\mathcal{L}^1)\,\mathcal{H}^0\left(CB(\{x \mid f(x)>0\},[0,1])\right) \geq \frac{\|f^+\|^2_{L^1(0,1)}}{2\min\{\|f^+\|_{L^\infty(0,1)}, \|f^-\|_{L^\infty(0,1)}\}}. \tag{2.1}$$

Proof. **Step 1** We claim that given two non-negative functions $h, g \in L^\infty(0,1)$ such that $\int_{[0,1]} h(x)\,dx = \int_{[0,1]} g(x)\,dx$ and which satisfy the following condition on the supports: there exists $\bar{x} \in (0,1)$ such that

$$\operatorname{supp}\{h\} \subseteq [0,\bar{x}], \quad \operatorname{supp}\{g\} \subseteq [\bar{x},1], \tag{2.2}$$

then one has

$$W_1(h\mathcal{L}^1, g\mathcal{L}^1) \geq \frac{1}{2}\frac{\|h\|^2_{L^1}}{\min\{\|h\|_{L^\infty}, \|g\|_{L^\infty}\}}. \tag{2.3}$$

Indeed we can consider the two following rearrangement of the masses

$$r_h\mathcal{L}^1 := \|h\|_{L^\infty}\,\chi_{(\bar{x}-\tau_h,\bar{x})}\mathcal{L}^1, \quad r_g\mathcal{L}^1 := \|g\|_{L^\infty}\,\chi_{(\bar{x},\bar{x}+\tau_g)}\mathcal{L}^1,$$

with τ_h and τ_g chosen so that the total mass of $r_h\mathcal{L}^1$ is the same total mass of $h\mathcal{L}^1$, and the same for $r_g\mathcal{L}^1$ and $g\mathcal{L}^1$. We notice that by direct calculation it holds

$$W_1(r_h\mathcal{L}^1, r_g\mathcal{L}^1) = \frac{1}{2}\left(\frac{\|h\|^2_{L^1(0,1)}}{\|h\|_{L^\infty(0,1)}} + \frac{\|g\|^2_{L^1(0,1)}}{\|g\|_{L^\infty(0,1)}}\right), \tag{2.4}$$

and then we observe that

$$W_1(h\mathcal{L}^1, g\mathcal{L}^1) \geq W_1(r_h\mathcal{L}^1, r_g\mathcal{L}^1). \tag{2.5}$$

Indeed for any π optimal transport plan between $h\mathcal{L}^1$ and $g\mathcal{L}^1$, one has

$$\begin{aligned} W_1(h\mathcal{L}^1, g\mathcal{L}^1) &= \int |x-y|\pi(\mathrm{d}x\mathrm{d}y) = \int_{(\bar{x},1)} (y-\bar{x})g(y)\,\mathrm{d}y + \int_{(0,\bar{x})} (\bar{x}-x)h(x)\,\mathrm{d}x \\ &= \int_{(\bar{x},1)} yg(y)\,\mathrm{d}y + \int_{(0,\bar{x})} -xh(x)\,\mathrm{d}x \geq \int_{(\bar{x},1)} yr_g(y)\,\mathrm{d}y + \int_{(0,\bar{x})} -xr_h(x)\,\mathrm{d}x \end{aligned}$$

where the last inequality follow from the two following observations:

- $g - r_g \leq 0$ in $(\bar{x}, \bar{x}+\tau_g)$ and $g - r_g \geq 0$ in $(\bar{x}+\tau_g, 1)$,
- $h - r_h \leq 0$ in $(0, \bar{x}-\tau_h)$ and $h - r_h \geq 0$ in $(\bar{x}-\tau_h, \bar{x})$,

and the fact that for a function $\psi : (0,+\infty) \to \mathbb{R}$ with zero mean and such that $\psi \leq 0$ in $(0,a)$ and $\psi \geq 0$ in $(a,+\infty)$ it holds that $\int_{(0,+\infty)} x\psi(x)\,dx \geq 0$. So finally putting (2.4) and (2.5) together we obtain

$$W_1(h\mathcal{L}^1, g\mathcal{L}^1) \geq \frac{\|h\|^2_{L^1(0,1)}}{2\min\{\|h\|_{L^\infty(0,1)}, \|g\|_{L^\infty(0,1)}\}}.$$

Step 2 Let $f : [0,1] \to \mathbb{R}$ be as in the hypotheses.
We define the sets D_k as follows: let $C_1, \ldots, C_n$ be the connected components of $\{x \in [0,1] \mid f(x) > 0\}$, with n possibly $+\infty$, then the sets $\{D_k\}_k$ are the closed disjoint intervals such that

$$\cup_{k=1}^n \bar{C}_k = \cup_{k=1}^m D_k.$$

We observe that if $m = +\infty$ then $\mathcal{H}^0(CB(\{x \mid f(x) > 0\}, [0,1])) = +\infty$ and the statement is trivially true. So we assume that $m < +\infty$.

Let $T : [0,1] \to \mathbb{R}$ be an optimal transport map for the problem.
We prove the following **claim:**

$$W_1(f^+_{|D_k}\mathcal{L}^1, T_\sharp(f^+_{|D_k}\mathcal{L}^1)) \geq \frac{1}{2}\frac{\|f^+_{|D_k}\|^2_{L^1(0,1)}}{\min\{\|f^+\|_{L^\infty(0,1)}, \|f^-\|_{L^\infty(0,1)}\}}, \quad \forall\, 1 \leq k \leq m.$$

Proof of the claim.
We observe that $T_\#(f^+_{|D_k}\mathcal{L}^1) \leq f^-\mathcal{L}^1$ (actually equality holds), for any k, so in particular it is absolutely continuous with respect to the Lebesgue measure. We consider

$$h := f^+_{|D_k}, \quad g := \frac{\mathrm{d}T_\sharp(f^+_{|D_k}\mathcal{L}^1)}{\mathrm{d}\mathcal{L}^1}, \tag{2.6}$$

and we notice that they satisfy the hypotheses of the previous step. Indeed

$$\mathrm{supp}(h) \subseteq D_k, \quad \mathrm{supp}(g) \subseteq T(D_k) \text{ and } T(x) \geq x \quad \forall\, x \in D_k \text{ or } T(x) \leq x \quad \forall\, x \in D_k.$$

To see this observe that, since the transport of f goes along a unique transport ray, we have that either $u(x) := -x$ or $u(x) := x$ is a Kantorovich potential for the problem. Assuming without loss of generality $u(x) = -x$ and using the definition of Kantorovich potential we have that each couple $(x, T(x))$ with $x \in \mathrm{supp} f^+$ satisfies $u(x) - u(T(x)) = |x - T(x)|$ so in particular $T(x) = x + |x - T(x)|$ and $T(x) \geq x$. The claim follows by applying the result of the previous step to h and g and observing that $\|g\|_{L^\infty} \leq \|f^-\|_{L^\infty}$.

Once that we proved the claim, we notice that being the sets C_k disjoint, we can sum over k the inequalities (2.6), and we get

$$W_1(f^+\mathcal{L}^1, f^-\mathcal{L}^1) = \sum_{k=1}^m W_1(f^+_{|D_k}\mathcal{L}^1, T_\sharp(f^+_{|D_k}\mathcal{L}^1))) \geq \frac{1}{2}\sum_{k=1}^m \frac{\|f^+_{|D_k}\|^2_{L^1(0,1)}}{\min\{\|f^+\|_{L^\infty(0,1)}, \|f^-\|_{L^\infty(0,1)}\}}.$$

Applying Cauchy-Schwartz inequality the result follows:

$$\begin{aligned} W_1(f^+\mathcal{L}^1, f^-\mathcal{L}^1) &\geq \frac{1}{2\min\{\|f^+\|_{L^\infty(0,1)}, \|f^-\|_{L^\infty(0,1)}\}}\frac{1}{m}\left(\sum_{k=1}^m \|f^+_{|D_k}\|_{L^1(0,1)}\right)^2 \\ &= \frac{1}{2m}\frac{\|f^+\|^2_{L^1(0,1)}}{\min\{\|f^+\|_{L^\infty(0,1)}, \|f^-\|_{L^\infty(0,1)}\}}, \end{aligned}$$

and we can conclude by observing that $m \leq \mathcal{H}^0(CB(\{x \mid f(x) > 0\}, [0,1]))$. □

Remark 2.1.5. We observe that in the preceding proposition the fact that the interval in which we are working in is exactly $[0,1]$ plays no role, so it analogously holds for an interval $[a,b]$ or in general for intervals of infinite length provided that the function f is in L^1.

Remark 2.1.6. Proposition 2.1.4 holds also if the hypothesis that the transport of f goes along a unique transport ray is removed: the only difference being a worse constant in the inequality (4 in place of 2). The result can be obtained in this case with an analogous proof, by observing that each D_k, in the notation of the proof of Proposition 2.1.4, can be decomposed as $D_k := D_k^+ \cup D_k^-$ where $D_k^+ := \{x \in D_k : T(x) > x\}$ and $D_k^- := \{x \in D_k : T(x) < x\}$. By performing the same computations as in the previous proof, the inequality follows.

As already mentioned in the introduction, a slightly different version of the following Proposition 2.1.4 was already present in the literature [91, Theorem 4] while a sharp and rigid version can be found in [54].

2.1.2 One dimensional densities: the case of $\mathsf{CD}(K,\infty)$ densities

Before stating the main result of this section, we include the following fact about the perimeter in the weighted one dimensional case. For a proof we refer to [41, Proposition 3.1] where there is an analogous statement for $\mathsf{CD}(K,N)$ densities.

Lemma 2.1.7. *Let $\mathfrak{m} = h\mathcal{L}^1$ be a non-negative measure on $\mathbb{R}$, with h a $\mathsf{CD}(K,\infty)$ density on its support, which in particular is an interval. Let E be an open set in $\operatorname{supp}(\mathfrak{m})$. It holds*

$$\mathsf{Per}_h(E) = \sum_{x \in CB(E, \operatorname{supp}(\mathfrak{m}))} h(x),$$

where Per_h is the Perimeter functional in the space $(\operatorname{supp}(\mathfrak{m}), |\cdot|, h\mathcal{L}^1)$ and $CB(\cdot,\cdot)$ is defined in Definition 2.1.1.

We now obtain the one-dimensional estimate for a weighted Lebesgue measure, where the weight is a $\mathsf{CD}(K,\infty)$ density. As before, we will first consider the case of functions defined on a compact interval $[0,D]$ and then we will discuss the non-compact case in the following Remark 2.1.9.

Proposition 2.1.8. *Let $h\colon [0,D] \to [0,+\infty)$ be a $\mathsf{CD}(K,\infty)$-density (recall Definition 1.4.9). Let $f\colon [0,D] \to \mathbb{R}$ be a continuous function having zero mean w.r.t the measure $h\mathcal{L}^1$: $\int_{(0,D)} f(x)h(x)\,\mathrm{d}x = 0$. Assume also that the transport of fh goes along a unique transport ray. Then it holds*

$$W_1(f^+h\mathcal{L}^1, f^-h\mathcal{L}^1)\left(\sum_{x\in CB(\{f>0\},[0,D])} h(x)\right) \geq \frac{\|fh\|^2_{L^1(0,D)}}{8C_{K,D}\|f\|_{L^\infty(0,D)}}, \tag{2.7}$$

(see Definition 2.1.1 for the definition of $CB(.,.)$) where

$$C_{K,D} := \begin{cases} 1 & K \geq 0, \\ e^{-KD^2/2} & K < 0. \end{cases} \tag{2.8}$$

Proof. **Step 1** We make the following preliminary observation. From $\mathsf{CD}(K,\infty)$ assumption it follows that the map

$$[0,D] \ni x \mapsto \log h(x) + K\frac{(x-\bar{x})^2}{2},$$

is concave. In particular, for each $\bar{x} \in (0,D)$ either is increasing in $[0,\bar{x}]$ or is decreasing in $[\bar{x},D]$. Hence in the first case

$$\log h(x) + K\frac{(x-\bar{x})^2}{2} \leq \log h(\bar{x}), \qquad \forall\, x \in [0,\bar{x}];$$

while in the second case:

$$\log h(x) + K\frac{(x-\bar{x})^2}{2} \leq \log h(\bar{x}), \qquad \forall\, x \in [\bar{x},D];$$

The combination of the two previous inequalities yields

$$\min\{\|h\|_{L^\infty[0,\bar{x}]}, \|h\|_{L^\infty[\bar{x},D]}\} \leq h(\bar{x}) C_{K,D}, \tag{2.9}$$

where $C_{K,D}$ is defined in (2.8).
Similarly to Step 1 of the previous proof we make a base estimate that we will use in the next step: we take two non negative bounded functions $f, g : [0,D] \to \mathbb{R}$ such that $\int_{[0,D]} fh\,dx = \int_{[0,D]} gh\,dx$, satisfying

$$\operatorname{supp}\{f\} \subseteq [0,\bar{x}], \quad \operatorname{supp}\{g\} \subseteq [\bar{x},D]. \tag{2.10}$$

We can now apply (2.3) to fh, gh (recalling that h is positive) and

$$\begin{aligned} W_1(fh\mathcal{L}^1, gh\mathcal{L}^1) &\geq \frac{\|fh\|^2_{L^1(0,D)}}{2\min\{\|fh\|_{L^\infty(0,D)}, \|gh\|_{L^\infty(0,D)}\}} \\ &\geq \frac{\|fh\|^2_{L^1(0,D)}}{2C_{K,D}\max\{\|f\|_{L^\infty(0,D)}, \|g\|_{L^\infty(0,D)}\}h(\bar{x})}, \end{aligned} \tag{2.11}$$

where the second inequality follows from (2.9).

Step 2 Consider $C_1, \ldots, C_n$ the connected components of $\{f > 0\}$ with n possibly $+\infty$. As in Definition 2.1.1 we consider the set $\cup_{k=1}^n \bar{C}_k$. We observe that it is the union of disjoint closed intervals: $\cup_{k=1}^n \bar{C}_k = \cup_{k=1}^m [a_k, b_k]$, with m possibly $+\infty$. We will proceed as in the proof of Proposition 2.1.4: we consider an optimal transport map T and we obtain

$$W_1(f^+h\mathcal{L}^1, f^-h\mathcal{L}^1) = \sum_{k=1}^m W_1(f^+h\mathcal{L}^1_{|(a_k,b_k)}, T_\sharp(f^+h\mathcal{L}^1_{|(a_k,b_k)})).$$

Then we can apply (2.11) (as in the previous proof using the fact that the transport of f^+h

into f^-h goes along a unique transport ray) to obtain:

$$\begin{aligned} W_1(f^+h\mathcal{L}^1, f^-h\mathcal{L}^1) &= \sum_{k=1}^m W_1(f^+h\mathcal{L}^1_{|(a_k,b_k)}, T_\sharp(f^+h\mathcal{L}^1_{|(a_k,b_k)})) \\ &\geq \sum_{k=1}^m \frac{\|f^+h\|^2_{L^1(a_k,b_k)}}{2C_{K,D}\|f\|_{L^\infty(a_k,b_k)}(h(a_k)+h(b_k))} \\ &\geq \frac{1}{2C_{K,D}\|f\|_{L^\infty(0,D)}} \sum_{k=1}^m \frac{\|f^+h\|^2_{L^1(a_k,b_k)}}{(h(a_k)+h(b_k))} \\ &\geq \frac{\|fh\|^2_{L^1(0,D)}}{8C_{K,D}\|f\|_{L^\infty(0,D)}\sum_{k=1}^m(h(a_k)+h(b_k))}, \end{aligned}$$

with the convention that if $a_k = 0$ (resp. $b_k = D$) the term $h(a_k)$ (resp. $h(b_k)$) does not appear. From this we get

$$8C_{K,D}W_1(f^+h\mathcal{L}^1, f^-h\mathcal{L}^1)\left(\sum_{k=0}^m (h(a_k)+h(b_k))\right) \geq \frac{\|fh\|^2_{L^1(0,D)}}{\|f\|_{L^\infty(0,D)}},$$

with the same convention on $h(0)$, $h(D)$ as above, from which the conclusion follows. □

Remark 2.1.9. The case of non-compact intervals of definition holds without modifications. The only relevant case is $K \geq 0$ and $D = \infty$ indeed for $K < 0$ and $D = \infty$, the claim becomes empty. Notice that D plays a role only in (2.9) where, in the relevant cases, it becomes independent on D.

2.1.3 One dimensional densities: the case of $\mathsf{MCP}(K,N)$ densities

We now address the case of an $\mathsf{MCP}(K,N)$-density. As it is clear from the proof of Proposition 2.1.8, the only place where the $\mathsf{CD}(K,\infty)$ assumption has been used is to ensure $h > 0$ over $(0,D)$ and to derive (2.9). A similar estimate, with suitable variations, can be obtained also for $\mathsf{MCP}(K,N)$-densities.

Lemma 2.1.10. *Let $h : [0,D] \to [0,\infty]$ be an $\mathsf{MCP}(K,N)$-density for some real parameters K,N with $N \geq 1$. Then for any $\bar{x} \in [0,D]$ the following estimates holds true:*

$$\min\{\|h\|_{L^\infty[0,\bar{x}]}, \|h\|_{L^\infty[\bar{x},D]}\} \leq h(\bar{x})C_{K,N,D}, \tag{2.12}$$

where

$$C_{K,N,D} := \begin{cases} 2^{N-1} & K \geq 0 \\ 2^{N-1}e^{\sqrt{-K(N-1)}\frac{D}{2}} & K < 0. \end{cases} \tag{2.13}$$

Proof. The claim will follow from simple manipulations of (1.27). For clarity we recall it: for all $0 \leq x_0 \leq x_1 \leq D$

$$\left(\frac{s_{K/(N-1)}(D-x_1)}{s_{K/(N-1)}(D-x_0)}\right)^{N-1} \leq \frac{h(x_1)}{h(x_0)} \leq \left(\frac{s_{K/(N-1)}(x_1)}{s_{K/(N-1)}(x_0)}\right)^{N-1};$$

indeed for $x \in [0,\bar{x}]$

$$h(x) \leq \left(\frac{s_{K/(N-1)}(D-x)}{s_{K/(N-1)}(D-\bar{x})}\right)^{N-1} h(\bar{x}) \leq \frac{h(\bar{x})}{s_{K/(N-1)}(D-\bar{x})^{N-1}} \sup_{0\leq x\leq\bar{x}} s_{K/(N-1)}(D-x)^{N-1}$$

and for $x \in [\bar{x}, D]$

$$h(x) \leq \left(\frac{s_{K/(N-1)}(x)}{s_{K/(N-1)}(\bar{x})}\right)^{N-1} h(\bar{x}) \leq \frac{h(\bar{x})}{s_{K/(N-1)}(\bar{x})^{N-1}} \sup_{\bar{x} \leq x \leq D} s_{K/(N-1)}(x)^{N-1}.$$

Then if $K \geq 0$, in particular h will be $\mathsf{MCP}(0, N)$ giving

$$\sup_{0 \leq x \leq \bar{x}} h(x) \leq h(\bar{x}) \left(\frac{D}{D - \bar{x}}\right)^{N-1}, \qquad \sup_{\bar{x} \leq x \leq D} h(x) \leq h(\bar{x}) \left(\frac{D}{\bar{x}}\right)^{N-1},$$

and therefore

$$\min\{\|h\|_{L^\infty[0,\bar{x}]}, \|h\|_{L^\infty[\bar{x},D]}\} \leq h(\bar{x}) D^{N-1} \min\{1/(D - \bar{x}), 1/\bar{x}\}^{N-1} \leq 2^{N-1} h(\bar{x}),$$

proving the inequality if $K \geq 0$. If $K < 0$, arguing analogously one gets

$$\min\{\|h\|_{L^\infty[0,\bar{x}]}, \|h\|_{L^\infty[\bar{x},D]}\} \leq h(\bar{x}) 2^{N-1} e^{\sqrt{-K(N-1)}\frac{D}{2}},$$

concluding the proof. □

Putting together the proof of Proposition 2.1.8 and Lemma 2.1.10 we straightforwardly obtain the next

Proposition 2.1.11. *Let $h\colon [0, D] \to [0, +\infty)$ be an $\mathsf{MCP}(K, N)$-density. Let $f\colon [0, D] \to \mathbb{R}$ be a continuous function having zero mean w.r.t the measure with density h: $\int_{(0,D)} f(x)h(x)\, dx = 0$. Assume also that the transport of fh goes along a unique transport ray: $\int_{(0,s)} f(x)h(x)\, dx \geq 0$ for all $s \in [0, D]$. Then it holds*

$$W_1(f^+ h\mathcal{L}^1, f^- h\mathcal{L}^1) \left(\sum_{\{x \in CB(\{f>0\},[0,D])\}} h(x) \right) \geq \frac{\|fh\|^2_{L^1(0,D)}}{8 C_{K,N,D} \|f\|_{L^\infty(0,D)}}, \tag{2.14}$$

where $C_{K,N,D}$ is given by (2.13)*.*

Remark 2.1.12. The case of non-compact intervals of definition holds again without modifications. The only relevant case here will be $K = 0$ and $D = \infty$; if $K > 0$, then MCP implies that $D < D_{K,N}$ (see (1.20)) while if $K < 0$ and $D = \infty$, the claim becomes empty. Notice that D plays a role only in (2.9) that is the content of Lemma 2.1.10.

2.2 Indeterminacy estimate via localization

We now use the one-dimensional estimates of the previous section to deduce the following sharp indeterminacy estimates. We first observe the following fact.

Lemma 2.2.1. *Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space, $E \subseteq X$ be a Borel set. Assume that we are given a strongly consistent disintegration of $\mathfrak{m}_{|\mathcal{T}}$ associated to the partition $\{X_q\}_{q \in \mathcal{Q}}$ induced by a 1-Lipschitz function φ, as given in Theorem 1.6.4:*

$$\mathfrak{m}_{|\mathcal{T}} = \int_Q \mathfrak{m}_q \, \mathfrak{q}(\mathrm{d}q),$$

where $\mathfrak{q}$ is a Borel probability measure over $\mathcal{Q} \subset X$ such that and $\mathfrak{m}_q \in \mathcal{M}_+(X)$. Then it holds

$$\mathsf{Per}(E) \geq \int_Q \mathsf{Per}_q(E_q) \, \mathfrak{q}(\mathrm{d}q),$$

where $E_q = E \cap X_q$ and Per_q is the perimeter functional in the space $(\bar{X}_q, \mathsf{d}, \mathfrak{m}_q)$.

Proof. Let $\{f_n\}_n \in \mathrm{Lip}_{\mathrm{loc}}(X)$ be a sequence of functions converging in $L^1(X,\mathfrak{m})$ to χ_E. Then, by disintegration

$$0 = \lim_{n\to+\infty} \int_X |f_n(x) - \chi_E(x)|\, \mathfrak{m}_{|\mathcal{T}}(\mathrm{d}x) = \lim_{n\to+\infty} \int_Q \int_{X_q} |f_n(x) - \chi_E(x)|\, \mathfrak{m}_q(\mathrm{d}x)\, \mathfrak{q}(\mathrm{d}q),$$

so up to extracting a subsequence, that we call again $\{f_n\}$, we have that for $\mathfrak{q}$-a.e. $q \in Q$

$$\lim_{n\to+\infty} \int_{X_q} |f_n(x) - \chi_E(x)|\, \mathfrak{m}_q(\mathrm{d}x) = 0.$$

Recalling that each $\mathfrak{m}_q$ is concentrated on X_q and denoting $E_q := E \cap X_q$, we have that $(f_n)_{|X_q}$ converges on $L^1(X_q, \mathfrak{m}_q)$ to χ_{E_q} for $\mathfrak{q}$-a.e $q \in Q$. We observe in addition that if f_n is Lipschitz then $(f_n)_{|X_q}$ is Lipschitz as well with a smaller local Lipschitz constant. Hence, taken $\{f_n\}_n \in \mathrm{Lip}_{\mathrm{loc}}(X)$ a sequence of functions attaining in the limit $\mathsf{Per}(E)$, we have that

$$\begin{aligned}
\mathsf{Per}(E) &= \liminf_{n\to\infty} \int_X |\nabla f_n|\, \mathfrak{m} \geq \liminf_{n\to\infty} \int_X |\nabla f_n|\, \mathfrak{m}_{|\mathcal{T}} = \liminf_{n\to\infty} \int_Q \int_{X_q} |\nabla f_n|\, \mathfrak{m}_q\, \mathfrak{q}(\mathrm{d}q) \\
&\geq \liminf_{n\to\infty} \int_Q \int_{X_q} |\nabla (f_n)_{|X_q}|\, \mathfrak{m}_q\, \mathfrak{q}(\mathrm{d}q) \geq \int_Q \liminf_{n\to\infty} \int_{X_q} |\nabla (f_n)_{|X_q}|\, \mathfrak{m}_q\, \mathfrak{q}(\mathrm{d}q) \\
&\geq \int_Q \mathsf{Per}_q(E_q)\, \mathfrak{m}_q\, \mathfrak{q}(\mathrm{d}q),
\end{aligned}$$

and the claim follows. □

Now we can state our main result concerning indeterminacy estimates.

Theorem 2.2.2. *Let $K, K', N \in \mathbb{R}$ with $N > 1$. Let $(X, \mathsf{d}, \mathfrak{m})$ be an essentially non-branching m.m.s. satisfying either $\mathsf{CD}(K,N)$ or $\mathsf{MCP}(K',N)$ and $\mathsf{CD}(K,\infty)$. Let $f \in L^1(X,\mathfrak{m})$ a continuous function or, alternatively, $f \in W^{1,2}(X,\mathsf{d},\mathfrak{m})$ be such that $\int_X f\,\mathfrak{m} = 0$. Assume also the existence of $x_0 \in X$ such that $\int_X |f(x)|\, \mathsf{d}(x,x_0)\, \mathfrak{m}(dx) < \infty$. Then the following indeterminacy estimate is valid:*

$$W_1(f^+\mathfrak{m}, f^-\mathfrak{m}) \cdot \mathsf{Per}\left(\{x \in X \colon f(x) > 0\}\right) \geq \frac{\|f\|^2_{L^1(X,\mathfrak{m})}}{8C_{K,D}\|f\|_{L^\infty(X,\mathfrak{m})}}, \tag{2.15}$$

where $D = \mathrm{diam}\,(X)$ and

$$C_{K,D} := \begin{cases} 1 & K \geq 0, \\ e^{-KD^2/2} & K < 0. \end{cases}$$

Remark 2.2.3. Notice that curvature assumptions $\mathsf{CD}(K,N)$ and $\mathsf{MCP}(K,N)$ imply $D < \infty$ only in the range $K > 0$ and $N \in (1,\infty)$. Hence under the second set of assumptions ($\mathsf{MCP}(K',N')$ and $\mathsf{CD}(K,\infty)$), the result (2.15) for $K \geq 0$ gives a non-trivial bound also in the non-compact case $D = \infty$.

Proof of Theorem 2.2.2. Given f as in the assumptions, we can use localization result (Theorem 1.6.4 and Theorem 1.6.6) yielding a decomposition of the space X as $X = Z \cup \mathcal{T}$, where f is zero $\mathfrak{m}$-a.e. in Z and $\mathcal{T}$ can be partitioned into $\{X_q\}_q$ with q in a Borel set $Q \subset X$, and a disintegration of $\mathfrak{m}$,

$$\mathfrak{m}_{|\mathcal{T}} = \int_Q \mathfrak{m}_q\, \mathfrak{q}(\mathrm{d}q),$$

with $\mathfrak{q}$ Borel probability measure with $\mathfrak{q}(Q) = 1$ and $Q \ni q \mapsto \mathfrak{m}_q \in \mathcal{M}_+(X)$ satisfying the properties of Theorem 1.6.4. In particular, $(\bar{X}_q, \mathsf{d}, \mathfrak{m}_q)$ is a $\mathsf{CD}(K,N)$ space (or $\mathsf{CD}(K,\infty)$ see Theorem 1.6.6), $\int_{X_q} f\, \mathfrak{m}_q = 0$ and every X_q is a transport ray associated to the L^1-optimal transport of $f^+\mathfrak{m}$ into $f^-\mathfrak{m}$.

As proven in [42, Proposition 4.4.] for the case of signed distance functions, $\mathfrak{q}$ can be identified with a test plan, see [9, Definition 5.1]; hence, if $f \in W^{1,2}(X, \mathsf{d}, \mathfrak{m})$, by the identification between different definitions of Sobolev spaces [9, Theorem 6.2], for $\mathfrak{q}$-a.e. q the function f restricted to the geodesic X_q is Sobolev and therefore continuous.

As said in Remark 1.6.5, we have an isomorphism between each space $(\bar{X}_q, \mathsf{d}, \mathfrak{m}_q)$ and spaces $(I_q, |\cdot|, h_q \cdot \mathcal{L}^1)$, with I_q a real interval (of possible infinite length) satisfying the same $\mathsf{CD}(K,N)$ (or $\mathsf{CD}(K,\infty)$) condition, $\int_{I_q} f_q(x) h_q(x)\, dx = 0$ being f_q the corresponding of $f_{|X_q}$ through the isomorphism and I_q transport ray for f_q. Whenever possible, for simplicity of notation, we will use $f = f_q$.

So now we can apply Proposition 2.1.8 and we have that $\mathfrak{q}$-a.e. $q \in Q$ it holds

$$W_1(f_q^+ h_q \mathcal{L}^1, f_q^- h_q \mathcal{L}^1) \left(\sum_{x \in CB(\{f_q>0\}, \bar{I}_q)} h_q(x) \right) \geq \frac{\|f\|^2_{L^1(X_q, \mathfrak{m}_q)}}{8 C_{K,D} \|f\|_{L^\infty(X_q,\mathfrak{m}_q)}}. \tag{2.16}$$

By Lemma 2.1.7 $\sum_{x \in CB(\{f_q>0\}, \bar{I}_q)} h_q(x) = \mathsf{Per}_{h_q}(\{x \in \bar{I}_q \colon f(x) > 0\})$, hence using the isomorphisms of metric measure spaces we have

$$W_1(f^+\mathfrak{m}_q, f^-\mathfrak{m}_q)\, \mathsf{Per}_q(\{x \in X_q \colon f_q(x) > 0\}) \geq \frac{\|f\|^2_{L^1(X_q, \mathfrak{m}_q)}}{8 C_{K,D} \|f\|_{L^\infty(X_q,\mathfrak{m}_q)}}.$$

where Per_q is the perimeter in $(\bar{X}_q, \mathsf{d}, \mathfrak{m}_q)$ and Per_{h_q} in $(\bar{I}_q, |\cdot|, h_q \cdot \mathcal{L}^1)$. In the previous factor we have tacitly used that $C_{K,D} \geq C_{K,D_q}$, where D_q is the length of X_q. Integrating the square root of the inequality with respect to the measure $\mathfrak{q}$ on Q and applying Holder inequality, we get

$$\begin{aligned}
&\left(\int_Q W_1(f^+\mathfrak{m}_q, f^-\mathfrak{m}_q)\, \mathfrak{q}(dq) \right)^{\frac{1}{2}} \left(\int_Q \mathsf{Per}_q(\{x \in X_q \colon f_q(x) > 0\})\, \mathfrak{q}(dq) \right)^{\frac{1}{2}} \\
&\geq \int_Q \Big(W_1(f^+\mathfrak{m}_q, f^-\mathfrak{m}_q) \cdot \mathsf{Per}_q(\{x \in X_q \colon f_q(x) > 0\}) \Big)^{\frac{1}{2}} \mathfrak{q}(dq) \\
&\geq \int_Q \frac{\|f\|_{L^1(X_q,\mathfrak{m}_q)}}{2\sqrt{2C_{K,D}} \|f\|^{\frac{1}{2}}_{L^\infty(X_q,\mathfrak{m}_q)}}\, \mathfrak{q}(\mathrm{d}q) \\
&\geq \frac{1}{2\sqrt{2C_{K,D}} \|f\|^{\frac{1}{2}}_{L^\infty(X,\mathfrak{m})}} \int_Q \int_{X_q} |f(x)| \mathfrak{m}_q(\mathrm{d}x)\, \mathfrak{q}(\mathrm{d}q) \\
&= \frac{\|f\|_{L^1(X,\mathfrak{m})}}{2\sqrt{2C_{K,D}} \|f\|^{\frac{1}{2}}_{L^\infty(X,\mathfrak{m})}}.
\end{aligned}$$

Clearly $\int_Q W_1(f^+\mathfrak{m}_q, f^-\mathfrak{m}_q)\, \mathfrak{q}(\mathrm{d}q) = W_1(f^+\mathfrak{m}, f^-\mathfrak{m})$; therefore

$$W_1(f^+\mathfrak{m}, f^-\mathfrak{m})^{\frac{1}{2}} \left(\int_Q \mathsf{Per}_q(\{x \in X_q \colon f(x) > 0\})\, \mathfrak{q}(dq) \right)^{\frac{1}{2}} \geq \frac{\|f\|_{L^1(X,\mathfrak{m})}}{2\sqrt{2C_{K,D}} \|f\|^{\frac{1}{2}}_{L^\infty(X,\mathfrak{m})}}.$$

The conclusion follows using Lemma 2.2.1. □

Repeating the same argument of the previous proof and using Proposition 2.1.11, we also obtain the analogous estimate for spaces verifying the weaker $\mathsf{MCP}(K,N)$; as expected, weaker curvature assumptions yields a dependence on the dimension of the estimate.

Theorem 2.2.4. *Let $K, N \in \mathbb{R}$ with $N > 1$. Let $(X, \mathsf{d}, \mathfrak{m})$ be an essentially non-branching m.m.s. verifying $\mathsf{MCP}(K,N)$.*

Let $f \in L^1(X, \mathfrak{m})$ a continuous function or, alternatively, $f \in W^{1,2}(X, \mathsf{d}, \mathfrak{m})$ be such that $\int_X f\, \mathfrak{m} = 0$. Assume also the existence of $x_0 \in X$ such that $\int_X |f(x)|\, \mathsf{d}(x, x_0)\, \mathfrak{m}(\mathrm{d}x) < \infty$. Then the following indeterminacy estimate is valid:

$$W_1(f^+\mathfrak{m}, f^-\mathfrak{m}) \cdot \mathsf{Per}\left(\{x \in X : f(x) > 0\}\right) \geq \frac{\|f\|_{L^1}^2}{8C_{K,N,D}\|f\|_{L^\infty}}, \tag{2.17}$$

where $\operatorname{diam}(X) = D$ *and*

$$C_{K,N,D} := \begin{cases} 2^{N-1} & K \geq 0, \\ 2^{N-1} e^{\sqrt{-K(N-1)}\frac{D}{2}} & K < 0. \end{cases}$$

Remark 2.2.5. With the same argument of Corollary 3.2.5 one can obtain the indeterminacy estimate involving W_p still sharp in the exponent.

2.3 Estimates for the measure of nodal sets of eigenfunctions

In this section we obtain lower bounds on the nodal set of eigenfunctions under curvature assumptions. Building on the previous Theorem 2.2.2 and Theorem 2.2.4, this will reduce to finding an upper bound on the W_1 distance between the positive and the negative part of the eigenfunction.

2.3.1 Measure of the nodal set under MCP and CD

The first version of the upper bound on W_1 combines integration by parts and the duality formulation of optimal transport.

Lemma 2.3.1. *Let $(X, \mathsf{d}, \mathfrak{m})$ be a bounded m.m.s. verifying $\mathsf{MCP}(K,N)$. Let f be an eigenfunction of the Laplacian with eigenvalue $\lambda \neq 0$ accordingly to Definition 1.4.1 and assume moreover the existence of $x_0 \in X$ such that $\int_X |f(x)|\, \mathsf{d}(x, x_0)\, \mathfrak{m}(dx) < \infty$.*

Then

$$W_1(f^+\mathfrak{m}, f^-\mathfrak{m}) \leq \frac{\sqrt{\mathfrak{m}(X)}}{\sqrt{\lambda}}\|f\|_{L^2}.$$

Proof. First from Remark 1.4.3, $\int f\, \mathfrak{m} = 0$ and, by definition, $f \in W^{1,2}(X, \mathsf{d}, \mathfrak{m})$. By assumption Kantorovich duality has a solution and therefore there exists a 1-Lipschitz Kantorovich potential $u : X \to \mathbb{R}$ such that

$$W_1(f^+\mathfrak{m}, f^-\mathfrak{m}) = \int_X (f^+(x) - f^-(x))u(x)\, \mathfrak{m}(\mathrm{d}x) = \int_X f(x)u(x)\, \mathfrak{m}(\mathrm{d}x). \tag{2.18}$$

Since f is a eigenfunction in the sense of Definition 1.4.1, then the following integration by-parts formula

$$\int_X D^-g(\nabla f)\, \mathfrak{m} \leq \lambda \int_X gf\, \mathfrak{m} \leq \int_X D^+g(\nabla f)\, \mathfrak{m},$$

is valid for any $g \in W^{1,2}(X, \mathsf{d}, \mathfrak{m})$ from Proposition 1.4.2.

From the boundedness it follows that $u \in W^{1,2}(X, \mathsf{d}, \mathfrak{m})$, hence together with (2.18) gives

$$W_1(f^+\mathfrak{m}, f^-\mathfrak{m}) \le \frac{1}{\lambda}\int_X D^+u(\nabla f)\,\mathfrak{m} \le \frac{1}{\lambda}\int_X |Du|_w\,|Df|_w\;\mathfrak{m} \le \frac{\mathrm{Lip}(u)}{\lambda}\int_X |Df|_w\;\mathfrak{m},$$

where we used the fact that $|D^{\pm}u(\nabla f)| \le |Du|_w\,|Df|_w$ and (1.13). Then by Holder inequality we have

$$\int_X |Df|_w\;\mathfrak{m} \le \mathfrak{m}(X)^{\frac{1}{2}}\left(\int_X |Df|_w^2\;\mathfrak{m}\right)^{\frac{1}{2}} = \mathfrak{m}(X)^{\frac{1}{2}}\left(\int_X D^-f(\nabla f)\,\mathfrak{m}\right)^{\frac{1}{2}} \le \mathfrak{m}(X)^{\frac{1}{2}}\sqrt{\lambda}\left(\int_X f^2\,\mathfrak{m}\right)^{\frac{1}{2}}$$

noticing that $Df^+(\nabla f) = |Df|_w^2$ (see (1.15)) and f itself as test-function. □

Putting together Lemma 2.3.1 and the previous result we obtain the next result.

Theorem 2.3.2. *Let $(X, \mathsf{d}, \mathfrak{m})$ be a bounded essentially non-branching m.m.s. verifying either $\mathsf{CD}(K, N)$ or $\mathsf{MCP}(K', N')$ and $\mathsf{CD}(K, \infty)$.*

Let f be an eigenfunction of the Laplacian of eigenvalue $\lambda > 0$ accordingly to Definition 1.4.1 and assume moreover the existence of $x_0 \in X$ such that $\int_X |f(x)|\,\mathsf{d}(x, x_0)\,\mathfrak{m}(\mathrm{d}x) < \infty$. Then the following estimate on the size of its nodal set holds true:

$$\mathsf{Per}(\{x \in X : f(x) > 0\}) \ge \frac{\sqrt{\lambda}}{8C_{K,D}\sqrt{\mathfrak{m}(X)}} \cdot \frac{\|f\|_{L^1}^2}{\|f\|_{L^2}\|f\|_{L^\infty}},$$

where $D = \mathrm{diam}\,(X)$ and

$$C_{K,D} := \begin{cases} 1 & K \ge 0, \\ e^{-KD^2/2} & K < 0. \end{cases}$$

Proof. Theorem 2.2.2 and Lemma 2.3.1 imply the claim. □

Using Theorem 2.2.4, we obtain the following analogous statement for spaces verifying the weaker $\mathsf{MCP}(K, N)$ condition with dimension-dependent constant appearing. The proof, being completely the same is omitted.

Theorem 2.3.3. *Let $(X, \mathsf{d}, \mathfrak{m})$ be a bounded essentially non-branching m.m.s. verifying $\mathsf{MCP}(K, N)$.*

Let f be an eigenfunction of the Laplacian of eigenvalue $\lambda > 0$ accordingly to Definition 1.4.1 and assume moreover the existence of $x_0 \in X$ such that $\int_X |f(x)|\,\mathsf{d}(x, x_0)\,\mathfrak{m}(\mathrm{d}x) < \infty$.

Then the following estimate on the size of its nodal set holds true:

$$\mathsf{Per}(\{x \in X : f(x) > 0\}) \ge \frac{\sqrt{\lambda}}{8C_{K,N,D}\sqrt{\mathfrak{m}(X)}} \cdot \frac{\|f\|_{L^1}^2}{\|f\|_{L^2}\|f\|_{L^\infty}},$$

where $D = \mathrm{diam}\,(X)$ and

$$C_{K,N,D} := \begin{cases} 2^{N-1} & K \ge 0, \\ 2^{N-1}e^{\sqrt{-K(N-1)}\frac{D}{2}} & K < 0. \end{cases}$$

2.3.2 The infinitesimally Hilbertian case

Assuming the heat flow to be linear yields a more sophisticated argument and sharper estimates. In particular we have at disposal the following result.

Theorem 2.3.4 (Theorem 3 of [55])**.** *Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space verifying $\mathsf{RCD}(K, N)$, then for any $\mu, \nu \in \mathcal{P}_2(X)$ and $s, t > 0$*

$$W_2(\mathfrak{H}_t\mu, \mathfrak{H}_s\nu)^2 \leq e^{-K\tau(s,t)} W_2(\mu, \nu)^2 + 2N \frac{1 - e^{-K\tau(s,t)}}{K\tau(s,t)} (\sqrt{t} - \sqrt{s})^2, \tag{2.19}$$

where $\tau(s, t) = 2(t + s + \sqrt{ts})/3$.

For the definition of $\mathfrak{H}_t\mu$ see Section 1.4.2.

We include the following technical lemma that will be useful for the proof of the next proposition.

Lemma 2.3.5. *Let $(X, \mathsf{d}, \mathfrak{m})$ be a m.m.s. with* $\operatorname{diam}(X) < D$. *Let $f, g : X \to [0, \infty)$ be functions with $\|f\|_{L^1} = \|g\|_{L^1}$. Then*

$$W_1(f\,\mathfrak{m}, g\,\mathfrak{m}) \leq D\|f - g\|_{L^1}.$$

Proof. Construct an admissible plan $\bar{\pi} \in \Pi(f\,\mathfrak{m}, g\,\mathfrak{m})$, with $\bar{\pi} = \pi_1 + \pi_2$ by defining

$$\pi_1 := (Id, Id)_\sharp \left(g\,\mathfrak{m}_{|\{g \leq f\}}\right) + (Id, Id)_\sharp \left(f\,\mathfrak{m}_{|\{g > f\}}\right)$$

and considering any $\pi_2 \in \Pi((f - g)^+\mathfrak{m}, (f - g)^-\mathfrak{m})$. Then it is straightforward to check that

$$W_1(f\,\mathfrak{m}, g\,\mathfrak{m}) \leq \int_{X \times X} \mathsf{d}(x, y)\,\pi_2(dxdy) \leq D\,\pi_2(X \times X) = D \int_X (f - g)^+\,\mathfrak{m}(\mathrm{d}x),$$

proving the claim. □

We are now ready to state our refined upper bound for W_1. We follow the proof in [91], where the result has been proved in the smooth setting.

Proposition 2.3.6. *Let $(X, \mathsf{d}, \mathfrak{m})$ be a m.m.s. verifying $\mathsf{RCD}(K, N)$ and such that* $\operatorname{diam}(X) = D < \infty$. *Let f be an eigenfunction of eigenvalue $\lambda > 2$. Then*

$$W_1(f^+\mathfrak{m}, f^-\mathfrak{m}) \leq C(K, N, D)\sqrt{\frac{\log\lambda}{\lambda}}\|f\|_{L^1},$$

with $C(K, N, D)$ growing linearly in D and as square root in N.

Proof. We define

$$\mu_0^\pm := f^\pm\,\mathfrak{m}, \qquad \mu_t^\pm := \mathfrak{H}_t\mu_0^\pm,$$

and by triangular inequality

$$W_1(\mu_0^+, \mu_0^-) \leq W_1(\mu_0^+, \mu_t^+) + W_1(\mu_t^+, \mu_t^-) + W_1(\mu_t^-, \mu_0^-),$$

notice indeed that $\mu_t^+(X) = \mu_0^+(X) = \mu_0^-(X) = \mu_t^-(X)$. Then by Theorem 2.3.4 we deduce that

$$\begin{aligned} W_1(\mu_t^\pm, \mu_0^\pm) &= \left(\int_X f^+ \,\mathfrak{m}\right) W_1(\mu_t^\pm/\mu_t^\pm(X), \mu_0^\pm/\mu_0^\pm(X)) \\ &\le \|f\|_{L^1(X,\mathfrak{m})} W_2(\mu_t^\pm/\mu_t^\pm(X), \mu_0^\pm/\mu_0^\pm(X)) \\ &\le \sqrt{t}\|f\|_{L^1(X,\mathfrak{m})} C(t,K,N), \end{aligned}$$

where $C(t,K,N) := \left(2N\frac{1-e^{-K2t/3}}{K2t/3}\right)^{1/2}$, (with $C(t,K,N) \le \sqrt{2N}$ if $K \ge 0$).

To bound $W_1(\mu_t^+, \mu_t^-)$ we use Lemma 2.3.5. Call g_t the evolution of a function g through the heat flow ($g_t = H_t g$), by the identification (1.23), it follows that (recall that $f \in W^{1,2}(X, \mathsf{d}, \mathfrak{m})$ by definition)

$$\mu_t^\pm = (H_t f^\pm)\,\mathfrak{m} = f_t^\pm\,\mathfrak{m}.$$

Notice that by infinitesimal Hilbertianity

$$f_t^+ - f_t^- = H_t(f^+ - f^-) = H_t(f) = e^{-\lambda t} f,$$

where the last identity is a consequence of f being an eigenfunction (see 1.19). Then we have that

$$W_1(\mu_t^+, \mu_t^-) \le D\|f_t^+ - f_t^-\|_{L^1} = D\|f_t\|_{L^1(X)} = De^{-\lambda t}\|f\|_{L^1}.$$

So finally

$$W_1(\mu_0^+, \mu_0^-) \le \left(\sqrt{t}C(t,K,N) + De^{-\lambda t}\right)\|f\|_{L^1}.$$

Choosing $t = \frac{1}{\lambda}\log(\lambda)$ we obtain

$$W_1(f^+\,\mathfrak{m}, f^-\,\mathfrak{m}) \le C(K,D,N)\sqrt{\frac{\log\lambda}{\lambda}}\|f\|_{L^1},$$

with $C(K,N,D)$ growing linearly in D and as square root in N. □

The estimate is not sharp in λ. The sharp version where the logarithmic factor does not appear has been proved for smooth closed Riemannian manifolds in [30].

Once that we have the upper bound of the previous proposition, we can state our first theorem on the size of the nodal set in this setting.

Theorem 2.3.7 (Nodal set RCD-spaces I). *Let $K, N \in \mathbb{R}$ with $N > 1$. Let $(X, \mathsf{d}, \mathfrak{m})$ be a m.m.s. satisfying $\mathsf{RCD}(K,N)$. Assume moreover $\operatorname{diam}(X) = D < \infty$. Let f be an eigenfunction of the Laplacian of eigenvalue $\lambda > 2$. Then the following estimate is valid:*

$$\mathsf{Per}(\{x \in X \colon f(x) > 0\}) \ge \sqrt{\frac{\lambda}{\log\lambda}} \cdot \frac{\|f\|_{L^1}}{\bar{C}_{K,D,N}\|f\|_{L^\infty}}, \tag{2.20}$$

where $\bar{C}_{K,D,N}$ grows linearly in D if $K \ge 0$ and exponentially if $K < 0$ and grows with power $1/2$ in N.

Proof. Since $\operatorname{diam}(X) < \infty$, it follows that $\mathfrak{m}(X) < \infty$ and therefore $f \in L^1(X)$, it has zero mean and satisfies the growth conditions and regularity needed to invoke Theorem 2.2.2. Hence Theorem 2.2.2 implies that

$$W_1(f^+\mathfrak{m}, f^-\mathfrak{m}) \cdot \mathsf{Per}(\{x \in X \colon f(x) > 0\}) \ge \frac{\|f\|_{L^1}^2}{8C_{K,D}\|f\|_{L^\infty}},$$

that together with Proposition 2.3.6 implies that

$$\mathsf{Per}\left(\{x \in X : f(x) > 0\}\right) \geq \sqrt{\frac{\lambda}{\log \lambda}} \frac{\|f\|_{L^1}}{C(K,N,D)C_{K,D}\|f\|_{L^\infty}},$$

giving therefore the claim. □

From the previous theorem follows an explicit lower bound on the size of the nodal set of an eigenfunction thanks to an estimate of the L^∞ norm available in the $\mathsf{RCD}(K,N)$ setting.

Theorem 2.3.8 (Nodal sets on RCD spaces II)**.** *Let $K, N \in \mathbb{R}$ with $N > 1$. Let $(X, \mathsf{d}, \mathfrak{m})$ be a m.m.s. verifying $\mathsf{RCD}(K,N)$, and with $\operatorname{diam}(X) = D < \infty$; finally pose $\mathfrak{m}(X) = 1$. Let f_λ be an eigenfunction of the Laplacian of eigenvalue $\lambda > \max\{2, D^{-2}\}$. Then the following estimate is valid:*

$$\mathsf{Per}\left(\{x \in X : f_\lambda(x) > 0\}\right) \geq \frac{1}{\bar{C}_{K,D,N}} \frac{1}{\sqrt{\log \lambda}} \lambda^{\frac{1-N}{2}}. \tag{2.21}$$

Proof. It is a straightforward consequence of Theorem 2.3.7 and of the following observation: given an eigenfunction f of eigenvalue λ, there exists a constant $C = C(K,N,D)$ such that

$$\|f\|_{L^\infty} \leq C\lambda^{\frac{N}{2}} \|f\|_{L^1},$$

provided $\lambda \geq D^{-2}$. Indeed from [12, Proposition 7.1] and assuming $\mathfrak{m}(X) = 1$, one has that

$$\|f\|_{L^\infty} \leq C\lambda^{\frac{N}{4}} \|f\|_{L^2} \leq C\lambda^{\frac{N}{4}} \|f\|_{L^\infty}^{\frac{1}{2}} \|f\|_{L^1}^{\frac{1}{2}},$$

from which the claim follows dividing by the L^∞ norm and squaring both sides. □

2.4 Linear combinations of eigenfunctions

We now consider functions obtained as linear combination of eigenfunctions. As expected, for the following results it will be necessary to assume the linearity of the Laplacian, i.e. infinitesimal Hilbertianity.

We however present two different upper bounds for the W_1 distance between the positive and the negative part of the function, one following the lines of Proposition 2.3.6 valid for RCD spaces and one following Lemma 2.3.1 valid for MCP spaces.

Proposition 2.4.1. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an essentially non-branching m.m.s. verifying $\mathsf{MCP}(K,N)$ with $\operatorname{diam}(X) = D < \infty$; assume moreover $(X, \mathsf{d}, \mathfrak{m})$ to be infinitesimally Hilbertian.*

Let f be a continuous function or, alternatively, $f \in W^{1,2}(X, \mathsf{d}, \mathfrak{m})$, such that it satisfies in L^2 sense $f = \sum_{\lambda_k \geq \lambda} a_k f_{\lambda_k}$, $k \in \mathbb{N}$, where $\{f_{\lambda_k}\}_{k\in\mathbb{N}}$ are mutually L^2 orthogonal eigenfunctions of eigenvalue λ_k.

Then the following estimate on the size of the nodal set of f holds true:

$$\mathsf{Per}(\{x \in X : f(x) > 0\}) \geq \frac{\sqrt{\lambda}}{\sqrt{\mathfrak{m}(X)}8C_{K,N,D}} \cdot \frac{\|f\|_{L^1}^2}{\|f\|_{L^2}\|f\|_{L^\infty}},$$

where $C_{K,N,D}$ is given by Theorem 2.2.4.

Proof. From diam $(X) < \infty$, it follows that $\mathfrak{m}(X) < \infty$ and therefore $f \in L^1(X, \mathfrak{m})$, it has zero mean and it satisfies the growth conditions needed to apply Theorem 2.2.4. To prove the claim it will be therefore sufficient to obtain an upper bound for $W_1(f^+\,\mathfrak{m}, f^-\,\mathfrak{m})$. Let for simplicity $\bar{k}$ be the minimum k such that $\lambda_{\bar{k}} \geq \lambda$. Using the Kantorovich formulation, there exists a 1-Lipschitz function $u \in W^{1,2}(X)$ such that $W_1(f^+\mathfrak{m}, f^-\mathfrak{m}) = \int_X f u\,\mathfrak{m}$. Therefore we have the following chain of inequalities that we explain at the end:

$$
\begin{aligned}
W_1(f^+\mathfrak{m}, f^-\mathfrak{m}) &= \int_X f u\,\mathfrak{m} = \int_X \sum_{k=\bar{k}}^{+\infty} a_k f_{\lambda_k} u\,\mathfrak{m} = \sum_{k=\bar{k}}^{+\infty} a_k \int_X f_{\lambda_k} u\,\mathfrak{m} \\
&= \lim_{n\to+\infty} \sum_{k=\bar{k}}^{n} a_k \int_X f_{\lambda_k} u\,\mathfrak{m} = \lim_{n\to+\infty} \sum_{k=\bar{k}}^{n} a_k \int_X -\frac{\Delta f_{\lambda_k}}{\lambda_k} u\,\mathfrak{m} \qquad (2.22)\\
&= \lim_{n\to+\infty} \sum_{k=\bar{k}}^{n} \int_X \frac{a_k}{\lambda_k} \nabla f_{\lambda_k} \cdot \nabla u\,\mathfrak{m} = \lim_{n\to+\infty} \int_X \sum_{k=\bar{k}}^{n} \frac{a_k}{\lambda_k} \nabla f_{\lambda_k} \cdot \nabla u\,\mathfrak{m} \qquad (2.23)\\
&\leq \limsup_{n\to+\infty} \| |D \sum_{k=\bar{k}}^{n} \frac{a_k}{\lambda_k} f_{\lambda_k}|_w \|_{L^2} \| |Du|_w \|_{L^2} \qquad (2.24)\\
&\leq \limsup_{n\to+\infty} \sqrt{\mathfrak{m}(X)} \| |D \sum_{k=\bar{k}}^{n} \frac{a_k}{\lambda_k} f_{\lambda_k}|_w \|_{L^2} \qquad (2.25)\\
&= \limsup_{n\to+\infty} \sqrt{\mathfrak{m}(X)} \sqrt{\sum_{k=\bar{k}}^{n} \frac{|a_k|^2}{\lambda_k^2} \| |D f_{\lambda_k}|_w \|_{L^2}^2} \qquad (2.26)\\
&= \limsup_{n\to+\infty} \sqrt{\mathfrak{m}(X)} \sqrt{\sum_{k=\bar{k}}^{n} \frac{|a_k|^2}{\lambda_k} \| f_{\lambda_k} \|_{L^2}^2} \\
&\leq \limsup_{n\to+\infty} \frac{\sqrt{\mathfrak{m}(X)}}{\sqrt{\lambda}} \sqrt{\sum_{k=\bar{k}}^{n} \| a_k f_{\lambda_k} \|_{L^2}^2} = \frac{\sqrt{\mathfrak{m}(X)}}{\sqrt{\lambda}} \| f \|_{L^2},
\end{aligned}
$$

where in (2.22) we used that f_{λ_k} is an eigenfunction of the Laplacian, $u \in W^{1,2}(X)$, the space is Infinitesimally Hilbertian and therefore (1.17) holds. The inequality between (2.23) and (2.24) follows from (1.14). The equality between (2.25) and (2.26) follows by observing that for $i \neq j$, thanks to the L^2-orthogonality of $\{f_{\lambda_k}\}_{k \geq \bar{k}}$, one has

$$\int_X \nabla f_{\lambda_i} \cdot \nabla f_{\lambda_j} \mathfrak{m} = 0,$$

for $i \neq j$ which together with (1.15) and the fact that $f \mapsto \nabla f \cdot \nabla g$ is linear, $\nabla f \cdot \nabla g = \nabla g \cdot \nabla f$ for any f and g in $W^{1,2}(X)$, and (1.15), gives for any $n \geq \bar{k}$

$$\| |D \sum_{k=\bar{k}}^{n} \frac{a_k}{\lambda_k} f_{\lambda_k}|_w \|_{L^2}^2 = \sum_{k=\bar{k}}^{n} \frac{|a_k|^2}{\lambda_k^2} \| |D f_{\lambda_k}|_w \|_{L^2}^2.$$

Finally we used that $\| |D f_{\lambda_k}|_w \|_{L^2}^2 = \lambda_k \| f_{\lambda_k} \|_{L^2}^2$, and in the last equality that $\{f_{\lambda_k}\}_{k \geq \bar{k}}$ satisfy

$$\sum_{k=\bar{k}}^{+\infty} \| a_k f_{\lambda_k} \|_{L^2}^2 = \| f \|_{L^2}.$$

□

Lemma 2.4.2. *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be a m.m.s. verifying* $\mathsf{RCD}(K,N)$ *and such that* $\operatorname{diam}(X) = D < \infty$ *and* $K \geq 0$*. Let* $f : X \to \mathbb{R}$ *be a continuous function or, alternatively,* $f \in W^{1,2}(X, \mathsf{d}, \mathfrak{m})$*, such that*

$$f = \sum_{\lambda_k \geq \lambda} \langle f, f_{\lambda_k} \rangle f_{\lambda_k}, \qquad \{\lambda_k\}_{k \in \mathbb{N}},$$

where $\{f_{\lambda_k}\}_{k\in\mathbb{N}}$ *are eigenfunctions of the Laplacian with eigenvalue* λ_k*,* $\langle f, f_{\lambda_k}\rangle$ *is the scalar product of* $L^2(X, \mathfrak{m})$*,* $\langle f_{\lambda_j}, f_{\lambda_k}\rangle = \delta_{j,k}$ *and the convergence of the series is in* $L^2(X, \mathfrak{m})$*. Then if* $\lambda \geq 2\sqrt{\mathfrak{m}(X)}$

$$W_1(f^+\mathfrak{m}, f^-\mathfrak{m}) \leq C(K, N, D, \mathfrak{m}(X)) \left(\frac{1}{\lambda} \log \left(\lambda \frac{\|f\|_{L^2}}{\|f\|_{L^1}} \right) \right)^{\frac{1}{2}} \|f\|_{L^1},$$

with $C(K, N, D, \mathfrak{m}(X))$ *an explicit constant.*

Proof. Following the approach and the same notation of the proof of Proposition 2.3.6 we have

$$W_1(\mu_0^+, \mu_0^-) \leq W_1(\mu_0^+, \mu_t^+) + W_1(\mu_t^+, \mu_t^-) + W_1(\mu_t^-, \mu_0^-),$$

and deduce from Theorem 2.3.4 that

$$W_1(\mu_t^\pm, \mu_0^\pm) \leq \sqrt{t}\|f\|_{L^1(X,\mathfrak{m})} C(t, K, N),$$

where $C(t, K, N) := \left(2N \frac{1-e^{-K2t/3}}{K2t/3}\right)^{1/2}$. Then to bound $W_1(\mu_t^+, \mu_t^-)$, again using Lemma 2.3.5, by orthonormality of $\{f_{\lambda_k}\}_k$ it follows that

$$\begin{aligned} \|f_t\|^2_{L^1(X,\mathfrak{m})} &= \left\| \sum_{\lambda_k \geq \lambda} e^{-\lambda_k t} \langle f, f_{\lambda_k}\rangle f_{\lambda_k} \right\|^2_{L^1(X,\mathfrak{m})} \leq \mathfrak{m}(X) \left\| \sum_{\lambda_k \geq \lambda} e^{-\lambda_k t} \langle f, f_{\lambda_k}\rangle f_{\lambda_k} \right\|^2_{L^2(X,\mathfrak{m})} \\ &= \mathfrak{m}(X) \sum_{\lambda_k \geq \lambda} e^{-2\lambda_k t} |\langle f, f_{\lambda_k}\rangle|^2 \leq \mathfrak{m}(X) e^{-2\lambda t} \|f\|^2_{L^2(X,\mathfrak{m})}. \end{aligned} \tag{2.27}$$

So finally

$$W_1(\mu_0^+, \mu_0^-) \leq \sqrt{t}\|f\|_{L^1(X,\mathfrak{m})} C(t, K, N) + D\sqrt{\mathfrak{m}(X)} e^{-\lambda t} \|f\|_{L^2(X,\mathfrak{m})}.$$

Using that $K \geq 0$ (so $C(t, K, N) \leq \sqrt{2N}$) and choosing $t = \frac{1}{\lambda} \log \left(\frac{\lambda \|f\|_{L^2(X,\mathfrak{m})}}{\|f\|_{L^1(X,\mathfrak{m})}} \right)$, it holds

$$W_1(\mu_0^+, \mu_0^-) \leq C(K, N, D, \mathfrak{m}(X)) \left(\frac{1}{\lambda} \log \left(\lambda \frac{\|f\|_{L^2}}{\|f\|_{L^1}} \right) \right)^{\frac{1}{2}} \|f\|_{L^1},$$

proving the claim. □

The following result is then a straightforward consequence

Corollary 2.4.3. *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be a m.m.s. verifying* $\mathsf{RCD}(K,N)$ *with* $K \geq 0$ *and such that* $\operatorname{diam}(X) = D < \infty$*.*

Let $f : X \to \mathbb{R}$ *be a continuous or, alternatively,* $f \in W^{1,2}(X, \mathsf{d}, \mathfrak{m})$ *such that*

$$f = \sum_{\lambda_k \geq \lambda} \langle f, f_{\lambda_k} \rangle f_{\lambda_k}, \qquad \{\lambda_k\}_{k \in \mathbb{N}},\ \lambda > 0,$$

where $\{f_{\lambda_k}\}_{k\in\mathbb{N}}$ *are eigenfunctions of the Laplacian with eigenvalue* λ_k, $\langle f, f_{\lambda_k}\rangle$ *is the scalar product of* $L^2(X,\mathfrak{m})$, $\langle f_{\lambda_j}, f_{\lambda_k}\rangle = \delta_{j,k}$ *and the convergence of the series is in* $L^2(X,\mathfrak{m})$.

Then the following estimate on the size of the nodal set of f *holds true:*

$$\mathsf{Per}(\{x \in X : f(x) > 0\}) \geq \frac{\sqrt{\lambda}}{8C(K,N,D,\mathfrak{m}(X))} \log\left(\lambda \frac{\|f\|_{L^2}}{\|f\|_{L^1}}\right)^{-1/2} \cdot \frac{\|f\|_{L^1}}{\|f\|_{L^\infty}},$$

with $C(K,N,D,\mathfrak{m}(X))$ *the same constant of Lemma 2.4.2, provided that* $\lambda \geq 2\sqrt{\mathfrak{m}(X)}$.

Chapter 3

Indeterminacy estimate and lower bound for the Wasserstein distances of eigenfunctions via heat flow

3.1 A crucial inequality

In this section we only present a crucial inequality which is due to Luise and Savaré on which the results in the two next sections are based. The inequality shows how the regularizing effect of the heat flow allows to get an estimate on the Hellinger distance in terms of the weaker W_2 distance. Also a more refined version for the Hellinger-Kantorovich distance is included. For the definition of Hellinger distance and Hellinger-Kantorovich distance see Section 1.3. We recall again the definition of $R_K(t)$ for $t > 0$, which will appear often in the sequel:

$$R_K(t) := \begin{cases} \frac{e^{2Kt}-1}{K} & \text{if } K \neq 0, \\ 2t & \text{if } K = 0. \end{cases} \tag{3.1}$$

Proposition 3.1.1. *[74, Theorem 5.2 and 5.4] Let $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathsf{RCD}(K, \infty)$ metric measure space for some $K \in \mathbb{R}$, and let $p \in [1, 2]$. For $\mu_0, \mu_1 \in \mathcal{P}_p(X)$ it holds*

$$W_p(\mu_0, \mu_1) \geq p(R_K(t))^{\frac{1}{2}} \mathsf{He}_p(H_t^*\mu_0, H_t^*\mu_1) \quad \forall\, t > 0. \tag{3.2}$$

Moreover, for every $\mu_0, \mu_1 \in \mathcal{M}(X)$ it holds

$$\mathbb{H}\mathbb{K}_{4R_K(t)}(\mu_0, \mu_1) \geq \mathsf{He}_2(H_t^*\mu_0, H_t^*\mu_1) \quad \forall\, t > 0. \tag{3.3}$$

3.2 Indeterminacy estimate via the use of heat flow

In this section we prove a version of the sharp (in the exponent) indeterminacy estimate valid in spaces which satisfy the $\mathsf{RCD}(K, \infty)$ condition of finite measure. The proof is completely different from the one of Section 2.2 and uses heat flow techniques. The Cheeger constant of the space plays a role in the estimate. So we first recall the definition of Cheeger constant of a metric measure space $(X, \mathsf{d}, \mathfrak{m})$:

$$h(X) := \inf \left\{ \frac{\mathsf{Per}(A)}{\mathfrak{m}(A)} \; : \; A \subset X \text{ Borel with } 0 < \mathfrak{m}(A) \leq \frac{\mathfrak{m}(X)}{2} \right\}. \tag{3.4}$$

Theorem 3.2.1. *Let $(X, \mathsf{d}, \mathfrak{m})$ be a space of finite measure satisfying the $\mathsf{RCD}(K, \infty)$ condition for some $K \in \mathbb{R}$. Let $f \in L^\infty(X, \mathfrak{m})$ be such that $\int_X f \, d\mathfrak{m} = 0$ and $\int_X \mathsf{d}(\bar{x}, x) \, |f(x)| \, d\mathfrak{m}(x) < +\infty$ for some $\bar{x} \in X$. Then one has*

$$W_1(f^+\mathfrak{m}, f^-\mathfrak{m}) \mathsf{Per}(\{f > 0\}) \geq C(K, h(X)) \left(\frac{\|f\|_{L^1}}{\|f\|_{L^\infty}} \right) \|f\|_{L^1}, \tag{3.5}$$

with

$$C(K, h(X)) := \begin{cases} \dfrac{\sqrt{\pi}}{27\sqrt{2}} & K \geq 0\,, \\ \left(1 - \dfrac{1}{(2\pi)^{\frac{1}{4}}}\right) \dfrac{h(X)}{8h(X) + 2\,|K|^{\frac{1}{2}}} & K < 0\,. \end{cases}$$

We start by a result which follows from the computations performed in Theorem 1.1 in [49]. For the convenience of the reader we report the proof. To state it we need the following:

$$J_K(t) := \int_0^t \sqrt{\frac{2}{\pi R_K(s)}} \, ds = \begin{cases} \sqrt{\frac{2}{\pi K}} \arctan\left(\sqrt{e^{2Kt} - 1}\right) & \text{if } K > 0, \\ \frac{2}{\sqrt{\pi}} \sqrt{t} & \text{if } K = 0, \\ \sqrt{-\frac{2}{\pi K}} \operatorname{arctanh}\left(\sqrt{1 - e^{2Kt}}\right) & \text{if } K < 0. \end{cases} \quad t > 0. \tag{3.6}$$

Proposition 3.2.2. *Let $(X, \mathsf{d}, \mathfrak{m})$ be a space of finite measure satisfying the $\mathsf{RCD}(K, \infty)$ condition for some $K \in \mathbb{R}$. Let $A \subseteq X$ be a Borel set. Then*

$$\int_{A^c} H_t(\chi_A) \, \mathfrak{m} \leq \frac{1}{2} J_K(t) \mathsf{Per}(A),$$

where $J_K(t)$ is defined in (3.6)*.*

Proof. By the above mentioned regularizing effect of the heat semigroup we know ([49, Proposition 3.1]) that for every function $f \in L^\infty(X)$ it holds

$$\| |D(H_t f)|_w \|_{L^\infty} \leq \sqrt{\frac{2}{\pi R_K(t)}} \|f\|_{L^\infty} \,. \tag{3.7}$$

By a duality argument, from (3.7) one easily derives

$$\|f - H_t(f)\|_{L^1} \leq J_K(t) \| |Df|_w \|_{L^1} \,, \tag{3.8}$$

say for $f \in \mathrm{Lip}_{bs}(X)$. Now, for any Borel set A we consider a sequence $f_n \in \mathrm{Lip}_{bs}(X)$, $f_n \to \chi_A$ in $L^1(X)$, recovery sequence for $\mathsf{Per}(A)$. By applying (3.8) to f_n and passing to the limit $n \to \infty$ we deduce

$$J_K(t) \mathsf{Per}(A) \geq \|\chi_A - H_t(\chi_A)\|_{L^1} = \int_A [1 - H_t(\chi_A)] \mathfrak{m} + \int_{A^c} H_t(\chi_A) \mathfrak{m} \tag{3.9}$$

$$= \int_A [1 - H_t(\chi_A)] \mathfrak{m} + \int_X H_t(\chi_A) \mathfrak{m} - \int_A 1 \, \mathfrak{m} + \int_{A^c} H_t(\chi_A) \mathfrak{m} = 2 \int_{A^c} H_t(\chi_A) \mathfrak{m}. \tag{3.10}$$

as desired. ☐

We now derive a sort of generalization of the previous Proposition 3.2.2 valid for bounded functions.

Proposition 3.2.3. *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be a metric measure space of finite measure satisfying the* $\mathsf{RCD}(K, \infty)$ *condition for some* $K \in \mathbb{R}$*, and let* $f \in L^\infty(X, \mathfrak{m})$*. Then*

$$\int_X \sqrt{H_t(f^+)H_t(f^-)}\,\mathfrak{m} \leq J_K(t)^{\frac{1}{2}} \mathsf{Per}(\{x \in X \mid f(x) > 0\})^{\frac{1}{2}} \|f\|_{L^1}^{\frac{1}{2}} \|f\|_{L^\infty}^{\frac{1}{2}},$$

where $J_K(t)$ *was defined in* (3.6)*.*

Proof. By taking advantage of the maximum principle for the heat semigroup, the Cauchy-Schwarz inequality and Proposition 3.2.2, one has

$$\begin{aligned}
&\int_{\{f>0\}} \sqrt{H_t(f^+)H_t(f^-)}\,\mathfrak{m} \leq \|f^-\|_{L^\infty}^{\frac{1}{2}} \int_{\{f>0\}} \sqrt{H_t(f^+)H_t(\chi_{\{f\leq 0\}})}\,\mathfrak{m} \\
&\leq \|f^-\|_{L^\infty}^{\frac{1}{2}} \|H_t(f^+)\|_{L^1}^{\frac{1}{2}} \left(\int_{\{f>0\}} H_t(\chi_{\{f\leq 0\}})\,\mathfrak{m}\right)^{\frac{1}{2}} \\
&\leq \frac{1}{\sqrt{2}} \|f^-\|_{L^\infty}^{\frac{1}{2}} \|f^+\|_{L^1}^{\frac{1}{2}} J_K(t)^{\frac{1}{2}} \mathsf{Per}(\{f > 0\})^{\frac{1}{2}},
\end{aligned}$$

where we have also used that the heat flow is mass preserving. Along the same lines, one also gets

$$\int_{\{f\leq 0\}} \sqrt{H_t(f^+)H_t(f^-)}\,d\mathfrak{m} \leq \frac{1}{\sqrt{2}} \|f^+\|_{L^\infty}^{\frac{1}{2}} \|f^-\|_{L^1}^{\frac{1}{2}} J_K(t)^{\frac{1}{2}} \mathsf{Per}(\{f \leq 0\})^{\frac{1}{2}}.$$

In particular splitting the integral in the statement of the proposition in an integral on the set where f is positive, and an integral on the set where f is non-negative, we deduce

$$\begin{aligned}
&\int_X \sqrt{H_t(f^+)H_t(f^-)}\,\mathfrak{m} = \int_{\{f>0\}} \sqrt{H_t(f^+)H_t(f^-)}\,\mathfrak{m} + \int_{\{f\leq 0\}} \sqrt{H_t(f^+)H_t(f^-)}\,\mathfrak{m} \\
\leq &\frac{1}{\sqrt{2}} \|f^-\|_{L^\infty}^{\frac{1}{2}} \|f^+\|_{L^1}^{\frac{1}{2}} J_K(t)^{\frac{1}{2}} \mathsf{Per}(\{f > 0\})^{\frac{1}{2}} + \frac{1}{\sqrt{2}} \|f^+\|_{L^\infty}^{\frac{1}{2}} \|f^-\|_{L^1}^{\frac{1}{2}} J_K(t)^{\frac{1}{2}} \mathsf{Per}(\{f \leq 0\})^{\frac{1}{2}}.
\end{aligned}$$

The conclusion follows by observing that $\mathsf{Per}(\{f > 0\}) = \mathsf{Per}(\{f \leq 0\})$, $\|f^\pm\|_{L^\infty} \leq \|f\|_{L^\infty}$ and $\|f^+\|_{L^1} + \|f^-\|_{L^1} = \|f\|_{L^1}$. □

In the course of the proof of Theorem 3.2.1 we also take advantage of the following easy Lemma.

Lemma 3.2.4. *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be a metric measure space of finite measure. Then, for every* $f \in L^\infty(X, \mathfrak{m})$ *of null mean we have*

$$\frac{\|f\|_{L^\infty} \mathsf{Per}(\{f > 0\})}{\|f\|_{L^1}} \geq \frac{h(X)}{2}, \tag{3.11}$$

where $h(X)$ *is the Cheeger constant of the space defined in* (3.4)*.*

Proof. We can suppose without loss of generality that $\mathfrak{m}(\{f > 0\}) \leq \mathfrak{m}(X)/2$ (since the left hand side of (3.11) does not change if we replace f with $-f$). We have

$$\|f\|_{L^1} = \int_X f^+\mathfrak{m} + \int_X f^-\mathfrak{m} = 2\int_{\{f>0\}} f^+\mathfrak{m} \leq 2\mathfrak{m}(\{f > 0\})\|f\|_{L^\infty}.$$

As a consequence,

$$\frac{\|f\|_{L^\infty}\mathsf{Per}(\{f>0\})}{\|f\|_{L^1}} \geq \frac{\mathsf{Per}(\{f>0\})}{2\mathfrak{m}(\{f>0\})} \geq \frac{h(X)}{2}$$

where in the last passage we have used the definition of Cheeger constant, since the set $\{f>0\}$ is a possible competitor in the right hand side of (3.4). □

We are now able to prove the indeterminacy estimate.

Proof of Theorem 3.2.1. We divide the proof in two steps.

Step 1: general estimate involving time.
Using the inequality (3.2) with $p=1$, the definition of H_t^* (1.22) and the inequality (1.8) we have that for every $t>0$

$$\begin{aligned} W_1(f^+\mathfrak{m}, f^-\mathfrak{m}) &\geq R_K(t)^{\frac{1}{2}}\mathsf{He}_1(H_t(f^+)\mathfrak{m}, H_t(f^-)\mathfrak{m}) \\ &\geq R_K(t)^{\frac{1}{2}}\mathsf{He}_2^2(H_t(f^+)\mathfrak{m}, H_t(f^-)\mathfrak{m}). \end{aligned} \tag{3.12}$$

Now we make use of the explicit expression of He_2, of the mass preservation property of the heat flow, and of Proposition 3.2.3 to obtain

$$\begin{aligned} &\mathsf{He}_2^2(H_t(f^+)\mathfrak{m}, H_t(f^-)\mathfrak{m}) = \int_X \Big(H_t(f^+) + H_t(f^-) - 2\sqrt{H_t(f^+)H_t(f^-)}\Big)\,d\mathfrak{m} \\ &\geq \|f\|_{L^1} - 2J_K(t)^{\frac{1}{2}}\mathsf{Per}(\{f(x)>0\})^{\frac{1}{2}}\|f\|_{L^1}^{\frac{1}{2}}\|f\|_{L^\infty}^{\frac{1}{2}}. \end{aligned} \tag{3.13}$$

By putting together (3.12) and (3.13) we thus obtain that for every $t>0$

$$W_1(f^+\mathfrak{m}, f^-\mathfrak{m}) \geq R_K(t)^{\frac{1}{2}}\|f\|_{L^1} - 2\Big(R_K(t)J_K(t)\mathsf{Per}(\{f(x)>0\})\|f\|_{L^1}\|f\|_{L^\infty}\Big)^{\frac{1}{2}}. \tag{3.14}$$

Step 2: optimizing in t.
In the case $K=0$ the right hand side of (3.14), that we denote with $g(t)$, has the following expression

$$g(t) = \sqrt{2}\|f\|_{L^1}\,t^{\frac{1}{2}} - \frac{4}{\pi^{\frac{1}{4}}}\|f\|_{L^1}^{\frac{1}{2}}\|f\|_{L^\infty}^{\frac{1}{2}}\mathsf{Per}(\{f>0\})^{\frac{1}{2}}\,t^{\frac{3}{4}}. \tag{3.15}$$

By choosing

$$\bar{t} = \frac{\pi}{324}\frac{\|f\|_{L^1}^2}{\|f\|_{L^\infty}^2\mathsf{Per}(\{f>0\})^2}$$

we maximize the function g and we obtain

$$\begin{aligned} W_1(f^+\mathfrak{m}, f^-\mathfrak{m}) \geq g(\bar{t}) &= \left(\sqrt{2}\sqrt{\frac{\pi}{324}} - \frac{4}{\pi^{\frac{1}{4}}}\frac{\pi^{\frac{3}{4}}}{324^{\frac{3}{4}}}\right)\frac{\|f\|_{L^1}^2}{\|f\|_{L^\infty}\mathsf{Per}(\{f>0\})} \\ &= \frac{\sqrt{\pi}}{27\sqrt{2}}\frac{\|f\|_{L^1}^2}{\|f\|_{L^\infty}\mathsf{Per}(\{f>0\})}. \end{aligned}$$

For $K<0$ we use again the notation $g(t)$ for the right hand side of (3.14) so that

$$g(t) = D_K(f)\sqrt{1-e^{2Kt}}\left[1 - \frac{2^{\frac{5}{4}}}{\pi^{\frac{1}{4}}}\Big(D_K(f)\,\mathrm{arctanh}(\sqrt{1-e^{2Kt}})\Big)^{\frac{1}{2}}\right]\frac{\|f\|_{L^1}^2}{\|f\|_{L^\infty}\mathsf{Per}(\{f>0\})}, \tag{3.16}$$

where we have denoted by $D_K(f)$ the quantity

$$D_K(f) := \frac{\|f\|_{L^\infty} \mathsf{Per}(\{f>0\})}{\|f\|_{L^1} |K|^{\frac{1}{2}}}.$$

We use the change of variable $(0,1) \ni s := \sqrt{1-e^{2Kt}}$ and we consider the function

$$g_1(s) := D_K(f) s\Big[1 - \frac{2^{\frac{5}{4}}}{\pi^{\frac{1}{4}}} (D_k(f)\, \mathrm{arctanh}(s))^{\frac{1}{2}}\Big] \qquad s \in (0,1).$$

We recall now the elementary inequality

$$\mathrm{arctanh}(s) \le \frac{s}{1-s} \qquad s \in (0,1),$$

so that

$$g_1(s) \ge D_K(f) s\Big[1 - \frac{2^{\frac{5}{4}}}{\pi^{\frac{1}{4}}} \Big(D_k(f) \frac{s}{1-s}\Big)^{\frac{1}{2}}\Big] =: g_2(s) \qquad s \in (0,1).$$

We finally take the admissible choice

$$\bar{s} := \frac{1}{8D_K(f)+1}$$

and, putting everything together, we obtain

$$\begin{aligned} W_1(f^+\mathfrak{m}, f^-\mathfrak{m}) &\ge g_2(\bar{s}) \frac{\|f\|_{L^1}^2}{\|f\|_{L^\infty} \mathsf{Per}(\{f>0\})} \\ &= \Big(1 - \frac{1}{(2\pi)^{\frac{1}{4}}}\Big) \frac{D_K(f)}{8D_K(f)+1} \frac{\|f\|_{L^1}^2}{\|f\|_{L^\infty} \mathsf{Per}(\{f>0\})}. \end{aligned} \tag{3.17}$$

Notice that, thanks to Lemma 3.2.4 we know that

$$D_K(f) \ge h(X)/(2|K|^{\frac{1}{2}}). \tag{3.18}$$

Moreover, the function

$$x \mapsto \frac{x}{8x+1} \qquad x > 0,$$

is increasing, so that we can bound from below the right hand side of (3.17) using (3.18) and obtain

$$W_1(f^+\mathfrak{m}, f^-\mathfrak{m}) \ge \Big(1 - \frac{1}{(2\pi)^{\frac{1}{4}}}\Big) \frac{h(X)}{8h(X) + 2|K|^{\frac{1}{2}}} \frac{\|f\|_{L^1}^2}{\|f\|_{L^\infty} \mathsf{Per}(\{f>0\})},$$

which concludes the proof. □

In the next corollary we show how to obtain an indeterminacy estimate for the p-Wasserstein distance as a simple consequence of the indeterminacy estimate for the 1-Wasserstein distance.

Corollary 3.2.5. *Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space of finite measure satisfying the $\mathsf{RCD}(K,\infty)$ condition for some $K \in \mathbb{R}$, and let $f \in L^\infty(X,\mathfrak{m})$ with null mean and satisfying $\int_X \mathsf{d}(\bar{x},x)|f_\lambda(x)|\mathfrak{m}(\mathrm{d}x)$ for some $\bar{x} \in X$. Then, for any $p > 1$*

$$W_p(f^+\mathfrak{m}, f^-\mathfrak{m}) \mathsf{Per}(\{f>0\}) \ge 2^{\frac{p-1}{p}} C(h(X),K) \left(\frac{\|f\|_{L^1}}{\|f\|_{L^\infty}}\right) \|f\|_{L^1}^{\frac{1}{p}}, \tag{3.19}$$

where $C(h(X),K)$ is the constant appearing in Theorem 3.2.1.

Proof. The result follows from Theorem 3.2.1 and the bound

$$W_p(f^+\mathfrak{m}, f^-\mathfrak{m})\frac{\|f\|_{L^1}^{1-\frac{1}{p}}}{2^{1-\frac{1}{p}}} \geq W_1(f^+\mathfrak{m}, f^-\mathfrak{m}) \tag{3.20}$$

which is a consequence of the Holder's inequality for the Wasserstein distance (see for instance [98, Remark 6.6] and recall that the measures here have total mass equal to $\|f^+\|_{L^1} = \|f^-\|_{L^1} = \frac{\|f\|_{L^1}}{2}$). □

Remark 3.2.6. We notice that one can recover an indeterminacy estimate involving the ∞-Wasserstein distance for example by taking the limit for $p \to +\infty$ in (3.19) and observing that the constant depending on p does not degenerate for $p \to +\infty$.

We conclude the section with an indeterminacy estimate for the Hellinger-Kantorovich distance. In analogy with the comparison between the estimates (3.2) and (3.3), we obtain an implicit but more refined result than Theorem 3.2.1. Another advantage of the following Theorem is that it is not restricted to functions f with null mean and bounded moment.

Theorem 3.2.7. *Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space of finite measure satisfying the $\mathsf{RCD}(K,\infty)$ condition for some $K \in \mathbb{R}$, and let $f \in L^\infty(X, \mathfrak{m})$. Then*

$$\mathrm{HK}_{4R_K(t)}(f^+\mathfrak{m}, f^-\mathfrak{m}) \geq \Big(\|f\|_{L^1} - 2J_K(t)^{\frac{1}{2}}\mathsf{Per}(\{f(x)>0\})^{\frac{1}{2}}\|f\|_{L^1}^{\frac{1}{2}}\|f\|_{L^\infty}^{\frac{1}{2}}\Big)^{\frac{1}{2}} \quad \forall\, t>0\,, \tag{3.21}$$

where $R_K(t)$ and $J_K(t)$ were defined in (3.1) *and* (3.6) *respectively.*

Proof. Using the inequality (3.3) and the definition of H_t^* (1.22) we have that for every $t > 0$

$$\mathrm{HK}^2_{4R_K(t)}(f^+\mathfrak{m}, f^-\mathfrak{m}) \geq \mathsf{He}_2^2(H_t(f^+)\mathfrak{m}, H_t(f^-)\mathfrak{m}). \tag{3.22}$$

With the same estimate as in (3.13) we can now bound from below the square of the 2-Hellinger distance and reach the desired conclusion. □

3.3 Lower bound on the Wasserstein distance between eigenfunctions

We recall a conjecture proposed by Steinerberger. Given a (M, g) smooth closed Riemannian manifold, for any $p \geq 1$ there exists a constant C, depending only on p and on the manifold, such that for every non-constant eigenfunction f_λ, of eigenvalue λ, it holds

$$\frac{C}{\sqrt{\lambda}}\|f_\lambda\|_{L^1(M)}^{\frac{1}{p}} \geq W_p(f_\lambda^+\mathfrak{m}, f_\lambda^-\mathfrak{m}) \geq \frac{1}{C\sqrt{\lambda}}\|f_\lambda\|_{L^1(M)}^{\frac{1}{p}}\,. \tag{3.23}$$

In this section we prove the lower bound in the Steinerberger's conjecture (3.23) for any $p \geq 1$. We actually prove a more general result valid for a class of spaces which includes closed Riemannian manifolds. The case $p = 1$ is stated in the following Theorem, while the case $p > 1$ is be derived from the case $p = 1$ and it is stated in Corollary 3.3.5.

Theorem 3.3.1. *Let $M > 0$, $K \in \mathbb{R}$ and $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathsf{RCD}(K, \infty)$ space of finite measure. Then for any non-constant eigenfunction f_λ of the Laplacian, of eigenvalue $\lambda \geq M$ and satisfying $\int_X \mathsf{d}(\bar{x}, x)|f_\lambda(x)|\,\mathfrak{m}(\mathrm{d}x) < +\infty$ for some $\bar{x} \in X$, it holds*

$$W_1(f_\lambda^+\mathfrak{m}, f_\lambda^-\mathfrak{m}) \geq C(K,M)\frac{1}{\sqrt{\lambda}}\|f_\lambda\|_{L^1}\,,$$

where

$$C(K,M) := \begin{cases} e^{-\frac{1}{2}} & \text{if } K \geq 0\,, \\ \left(1 - \frac{K}{M}\right)^{\frac{M}{2K}-\frac{1}{2}} & \text{if } K < 0\,. \end{cases} \tag{3.24}$$

Remark 3.3.2. Notice that in Theorem 3.3.1 we are not requiring any compactness of the space (X, d), nor are we assuming that the spectrum of the metric measure space is discrete. The assumptions $\mathfrak{m}(X) < \infty$ and $\int_X \mathsf{d}(\bar{x}, x)|f_\lambda(x)|\,\mathfrak{m}(\mathrm{d}x) < +\infty$, trivially satisfied for compact spaces, are requested here to ensure that the measures $f_\lambda^+\mathfrak{m}$, $f_\lambda^-\mathfrak{m}$ have the same total mass and finite 1-moment.

From the above result, in the case of Riemannian manifolds, together with the upper bound obtained in [30, Theorem 3], follows the full conjecture (3.23) for $p = 1$ and an equivalent formulation of Yau's conjecture:

Corollary 3.3.3. *Let (M, g) be a smooth, closed, Riemannian manifold. Then there exists a constant C, depending only on the manifold, such that for any non-constant eigenfunction f_λ, of eigenvalue λ, the following inequality is satisfied*

$$\frac{C}{\sqrt{\lambda}}\|f_\lambda\|_{L^1(M)} \geq W_1(f_\lambda^+\mathfrak{m}, f_\lambda^-\mathfrak{m}) \geq \frac{1}{C\sqrt{\lambda}}\|f_\lambda\|_{L^1(M)}\,.$$

As a consequence, Yau's conjecture holds if and only if there exists a constant C, depending only on the manifold, such that for any eigenfunction f_λ the following inequality is satisfied

$$C\|f_\lambda\|_{L^1(M)} \geq W_1(f_\lambda^+\mathfrak{m}, f_\lambda^-\mathfrak{m})\mathcal{H}^{n-1}\left(\{x : f_\lambda(x) = 0\}\right) \geq \frac{\|f_\lambda\|_{L^1(M)}}{C}\,.$$

Proof of Theorem 3.3.1. As in the case of Theorem 3.2.1, we divide the proof in two steps.

Step 1: general estimate involving time.

Using the inequality (3.2) with $p = 1$ and the definition of H_t^* (1.22) we bound from below the cost W_1 in terms of the total variation:

$$W_1(f_\lambda^+\mathfrak{m}, f_\lambda^-\mathfrak{m}) \geq (R_K(t))^{\frac{1}{2}}\mathsf{He}_1(H_t(f_\lambda^+)\mathfrak{m}, H_t(f_\lambda^-)\mathfrak{m})\,. \tag{3.25}$$

We observe that

$$\mathsf{He}_1(H_t(f_\lambda^+)\mathfrak{m}, H_t(f_\lambda^-)\mathfrak{m}) = \|H_t(f_\lambda^+) - H_t(f_\lambda^-)\|_{L^1(X)} = \|H_t(f_\lambda)\|_{L^1(X)} = e^{-\lambda t}\|f_\lambda\|_{L^1(X)}, \tag{3.26}$$

using the linearity of the heat flow and recalling that $H_t(f_\lambda) = e^{-\lambda t}f_\lambda$.

So inequality (3.25) reads as

$$W_1(f_\lambda^+\mathfrak{m}, f_\lambda^-\mathfrak{m}) \geq (R_K(t))^{\frac{1}{2}}e^{-\lambda t}\|f_\lambda\|_{L^1(X)} \quad \forall\, t > 0.$$

Step 2: optimizing in t.
In the case $K = 0$ the result follows by choosing $\bar{t} = \frac{1}{2\lambda}$ in the previous inequality.

For $K < 0$ we choose instead $\bar{t} = \frac{1}{2K}\log(\frac{\lambda}{\lambda - K})$ in order to obtain

$$W_1(f_\lambda^+\mathfrak{m}, f_\lambda^-\mathfrak{m}) \geq \frac{1}{\sqrt{\lambda}}\sqrt{-\frac{\lambda}{K}\left(e^{(-\frac{\lambda}{K})\log\frac{\lambda}{\lambda-K}} - e^{(1-\frac{\lambda}{K})\log\frac{\lambda}{\lambda-K}}\right)}\|f_\lambda\|_{L^1(X)}.$$

The result follows by standard computations, setting $x = -\frac{\lambda}{K} \geq -\frac{M}{K} > 0$ and noticing that the function

$$x \mapsto \sqrt{x\left(e^{x\log\frac{x}{x+1}} - e^{(1+x)\log\frac{x}{x+1}}\right)} = \left(\frac{x}{x+1}\right)^{\frac{x+1}{2}}$$

is increasing.

□

Remark 3.3.4. We notice that in the proof of Theorem 3.3.1 we have avoided using fine properties of Laplace eigenfunctions, exploiting only the equality $H_t(f_\lambda) = e^{-\lambda t} f_\lambda$ in the last passage of (3.26).

Along the same lines of Corollary 3.2.5, one can easily prove the following:

Corollary 3.3.5. *Let $M > 0$, $K \in \mathbb{R}$ and $(X, \mathsf{d}, \mathfrak{m})$ be an $\mathsf{RCD}(K, \infty)$ space of finite measure. Then for any non-constant eigenfunction f_λ of the Laplacian, of eigenvalue $\lambda \geq M$ and satisfying $\int_X \mathsf{d}(\bar{x}, x)|f_\lambda(x)|\,\mathfrak{m}(\mathrm{d}x) < +\infty$ for some $\bar{x} \in X$, it holds for any $p > 1$*

$$W_p(f_\lambda^+\mathfrak{m}, f_\lambda^-\mathfrak{m}) \geq 2^{\frac{p-1}{p}} C(K, M)\frac{1}{\sqrt{\lambda}}\|f_\lambda\|_{L^1}^{\frac{1}{p}},$$

where $C(K, M)$ was defined in (3.24).

Remark 3.3.6. One can recover a lower bound for the ∞-Wasserstein distance as in Remark 3.2.6.

Chapter 4

Monge problem for distance cost on an infinite product

This chapter contains the results obtained in [34].

4.1 Setting

In this section we give the definition of the space in which we are going to work and we show the properties that we are going to use.

4.1.1 Definition of the space

For any $k \in \mathbb{N}$, let $(Y_k, \bar{d}_k, \eta_k)$, with $\eta_k \in \mathcal{P}(Y_k)$, be non-branching metric measure spaces satisfying the $\mathsf{CD}(K, M_k)$ condition for a $K \in \mathbb{R}$ and $M_k \geq 1$. Assume that

$$\sum_{k \in \mathbb{N}} \operatorname{diam}(Y_k)^2 < +\infty. \tag{4.1}$$

For any $n \in \mathbb{N}$, define the finite product spaces $(\tilde{X}_n, \tilde{\mathsf{d}}_n, \tilde{\mathfrak{m}}_n)$ where $\tilde{d}_n$ is the distance on the product of the first n spaces given by the Pythagorean Theorem and $\tilde{\mathfrak{m}}_n$ is the product measure, namely

$$\tilde{X}_n := \prod_{k=1}^{n} Y_k; \quad \tilde{\mathsf{d}}_n^2(x, y) := \sum_{k=1}^{n} \bar{\mathsf{d}}_k^2(x_k, y_k); \quad \tilde{\mathfrak{m}}_n := \otimes_{k=1}^{n} \eta_k,$$

with x_i, $y_i \in Y_i$, for $i \in \{1, \dots, n\}$. We assume in addition that for any $n \in \mathbb{N}$, the space $(\tilde{X}_n, \tilde{\mathsf{d}}_n, \tilde{\mathfrak{m}}_n)$ satisfies the $\mathsf{CD}(K, N_n)$ condition with $N_n < +\infty$.

Infinite product

Define

$$\begin{aligned} X &:= \prod_{k=1}^{+\infty} Y_k; \\ \mathsf{d}^2(x, y) &:= \sum_{k=1}^{+\infty} \bar{\mathsf{d}}_k^2(x_k, y_k) \quad x = (x_k)_k,\ y = (y_k)_k \in X; \\ \mathfrak{m} &:= \bigotimes_{k \in \mathbb{N}} \eta_k, \end{aligned} \tag{4.2}$$

where $\mathfrak{m}$ is the infinite product measure defined on the following σ-algebra generated by cylinders $\mathcal{E}_c(X)$ (see [23, Vol 1, Sec. 3.5]),

$$\mathcal{E}_c(X) := \sigma \left\{ \prod_{n=1}^{+\infty} B_n \text{ with } B_n \in \mathcal{B}(Y_n),\ B_n = Y_n\ \forall\, n \in \mathbb{N} \setminus J \text{ with } J \subset \mathbb{N} \text{ finite} \right\}.$$

We say that a set $C \subseteq X$ is a cylinder of base B, $C = C(B)$ for short, with $B \subseteq \tilde{X}_n$ for some $n \in \mathbb{N}$, if $C = B \times \prod_{k=n+1}^{+\infty} Y_n$.

We mention that this type of product was considered as an example for the tensorization property of the RCD curvature bounds in [10, Section 6.4].

Proposition 4.1.1. $(X, \mathsf{d}, \mathfrak{m})$ *is a compact probability metric measure space and the topology induced by* d *on* X *is the product topology.*

Proof. From (4.1) it is easily checked that d is a distance on X.

Compactness follows once we prove that d induces the product topology since countable product of compact spaces is compact. From compactness, completeness and separability follow.

To prove that the topology induced by d on X is the product topology we call τ_{d} the topology induced by d and τ_p the product topology, namely:

$$\tau_{\mathsf{d}} := \tau(\{B_r^{\mathsf{d}}(x) \colon x \in X, r \in \mathbb{R}\}),$$
$$\tau_p := \tau(\{\prod_{n=1}^{+\infty} B_n \colon B_n = B_{r_n}^{\bar{\mathsf{d}}_n}(x_n) \subseteq Y_n,\ \forall\, n \in J,\ B_n = Y_n\,\forall\, n \in \mathbb{N} \setminus J \text{ with } |J| < +\infty\}),$$

where with the symbol $\tau(A)$ we mean the topology generated by the sets in A, and we show that the two inclusions hold.
$\tau_p \subseteq \tau_{\mathsf{d}}$:
let $A_p \in \tau_p$ and $x \in A_p, x = (x_1, x_2, \dots)$. We have to prove that there exists an open set $A_d \in \tau_d$ with $x \in A_d \subset A_p$. W.l.o.g. we can assume $A_p = \prod_{n=1}^{+\infty} A_n$ with $A_n = B_{r_n}^{\bar{\mathsf{d}}_n}(x_n) \subseteq Y_n$, $\forall\, n \in J$, $A_n = Y_n\,\forall\, n \in \mathbb{N} \setminus J$ with $|J| < +\infty$. Then for $r := \min\{r_n \colon n \in J\}$ one has that $x \in B_r^{\mathsf{d}}(x_1, x_2, \dots) \subseteq A_p$.
$\tau_p \supseteq \tau_{\mathsf{d}}$:
let $A_d \in \tau_d$, $\bar{x} \in A_d$, with $\bar{x} = (\bar{x}_1, \dots, \bar{x}_n, \dots)$. We need to prove that there exists an open set $A_p \in \tau_p$ with $\bar{x} \in A_p \subset A_d$. W.l.o.g. we can assume that $A_d = B_r^{\mathsf{d}}(\bar{x})$. We fix $N \in \mathbb{N}$ such that $\sum_{n=N+1}^{+\infty} \operatorname{diam}(Y_n)^2 < \frac{r^2}{2}$. Let

$$A_p := \left\{ x = (x_1, x_2, \dots) \in X \colon \sup_{1 \le n \le N} \bar{\mathsf{d}}_n^2(x_n, \bar{x}_n) < \frac{r}{2N} \right\}.$$

Then A_p is open by definition of product topology. Clearly $\bar{x}$ is in A_p and any $x \in A_p$ satisfies

$$\mathsf{d}^2(x, \bar{x}) = \sum_{n=1}^{N} \bar{\mathsf{d}}_n^2(x_n, \bar{x}_n) + \sum_{n=N+1}^{+\infty} \bar{\mathsf{d}}_n^2(x_n, \bar{x}_n) < \sum_{n=1}^{N} \frac{r^2}{2N} + \frac{r^2}{2} = r^2. \tag{4.3}$$

Finally the equality of the two topologies implies that $\mathfrak{m}$ is a Borel measure on X since $\mathcal{E}_c(X)$ contains the Borel sets generated by τ_p. $\square$

Immersions and projections

Fix $\bar{y} := (\bar{y}_1, \dots, \bar{y}_n, \dots) \in X$ and define for any $n \in \mathbb{N}$ the isometric embeddings $I_n : \tilde{X}_n \to X$, $I_n(x) := (x_1, \dots, x_n, \bar{y}_{n+1}, \dots)$. We call

$$\begin{aligned} X_n &:= I_n(\tilde{X}_n), \\ \mathsf{d}_n &:= \mathsf{d}_{|X_n \times X_n}, \\ \mathfrak{m}_n &:= (I_n)_\sharp \tilde{\mathfrak{m}}_n. \end{aligned} \tag{4.4}$$

In particular $(X_n, \mathsf{d}_n, \mathfrak{m}_n)$ is isomorphic as a m.m.s. to $(\tilde{X}_n, \tilde{\mathsf{d}}_n, \tilde{\mathfrak{m}}_n)$. Define in addition for any $n \in \mathbb{N}$ the map

$$P_n : X \to X_n \subset X, \tag{4.5}$$

$$P_n(x) := (x_1, \dots, x_n, \bar{y}_{n+1}, \dots) \text{ for } x = (x_1, \dots x_n, x_{n+1}, \dots). \tag{4.6}$$

We observe that for any $x \in X$, $\lim_{n\to+\infty} \mathsf{d}(P_n(x), x) = 0$.

Lemma 4.1.2. *For any $n \in \mathbb{N}$ it holds $\mathfrak{m}_n = (P_n)_\sharp \mathfrak{m}$. In particular $(X_n, \mathsf{d}_n, (P_n)_\sharp \mathfrak{m})$ is a non-branching $\mathsf{CD}(K, N_n)$ m.m.s..*

Proof. We first show that $\mathfrak{m}_n = (P_n)_\sharp \mathfrak{m}$. Let A be a Borel set in X. Then

$$\begin{aligned} &\mathfrak{m}_n(A) = (I_n)_\sharp \tilde{\mathfrak{m}}_n(A) = \tilde{\mathfrak{m}}_n(I_n^{-1}(A)) = \tilde{\mathfrak{m}}_n(\{(x_1, \dots, x_n) \colon I_n(x_1, \dots, x_n) \in A\}) \\ &= \tilde{\mathfrak{m}}_n(\{(x_1, \dots, x_n) \colon (x_1, \dots, x_n, \bar{y}_{n+1}, \bar{y}_{n+2}, \dots) \in A\}), \end{aligned}$$

and analogously

$$\begin{aligned} &(P_n)_\sharp \mathfrak{m}(A) = \mathfrak{m}((P_n)^{-1}(A)) = \mathfrak{m}(\{(x_1, \dots, x_n) \colon (x_1, \dots, x_n, \bar{y}_{n+1}, \bar{y}_{n+2}, \dots) \in A\} \times \prod_{k=n+1}^{+\infty} Y_k) \\ &= \tilde{\mathfrak{m}}_n \otimes_{k=n+1}^{+\infty} \eta_k(\{(x_1, \dots, x_n) \colon (x_1, \dots, x_n, \bar{y}_{n+1}, \bar{y}_{n+2}, \dots) \in A\} \times \prod_{k=n+1}^{+\infty} Y_k) \\ &= \tilde{\mathfrak{m}}_n(\{(x_1, \dots, x_n) \colon (x_1, \dots, x_n, \bar{y}_{n+1}, \bar{y}_{n+2}, \dots) \in A\}). \end{aligned}$$

The second claim follows from the fact that isomorphisms preserve the CD condition (see e.g. [94]). □

Remark 4.1.3. Let $\gamma : [0, 1] \to X$, $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n, \dots)$ be an absolutely continuous curve in (X, d). Then for any $n \in \mathbb{N}$, $\gamma_n : [0, 1] \to Y_n$ is absolutely continuous in $(Y_n, \bar{\mathsf{d}}_n)$. Indeed for any $s < t$,with s, $t \in [0, 1]$ we have

$$\bar{\mathsf{d}}_n(\gamma_n(s), \gamma_n(t)) \le \mathsf{d}(\gamma(s), \gamma(t)) \le \int_{(s,t)} |\dot{\gamma}_r| \, \mathrm{d}r.$$

Moreover it can be checked that $\sum_{n=1}^{+\infty} \left|(\dot{\gamma_n})_t\right|^2 \le |\dot{\gamma}_t|^2$ for a.e. $t \in (0, 1)$.

Lemma 4.1.4. *Let $\gamma : [0, 1] \to X$, $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n, \dots)$. Then γ is a geodesic of (X, d) if and only if for any $n \in \mathbb{N}$, $\gamma_n : [0, 1] \to Y_n$ is a geodesic of $(Y_n, \bar{\mathsf{d}}_n)$.*

Proof. Assume first that $\gamma[0,1] \to X$ is a geodesic. We know that by the characterization in (1.3) that γ satisfies

$$\sum_{n=1}^{+\infty} \bar{\mathsf{d}}_n^2(\gamma_n(0),\gamma_n(1)) = \mathsf{d}(\gamma(0),\gamma(1))^2 = \int_{(0,1)} |\dot{\gamma}_t|^2 \, \mathrm{d}t \geq \sum_{n=1}^{+\infty} \int_{(0,1)} \left|(\dot{\gamma_n})_t\right|^2 \mathrm{d}t, \tag{4.7}$$

where $\left|(\dot{\gamma_n})_t\right| = \lim_{h\to 0} \frac{\mathsf{d}_n(\gamma_n(t+h),\gamma_n(t))}{|h|}$ is the metric speed of the curve γ_n in the space (Y_n, d_n). Since γ_n is an absolutely continuous curve, as explained in (1.2)

$$\bar{\mathsf{d}}_n^2(\gamma_n(0),\gamma_n(1)) \leq \left(\int_{(0,1)} \left|(\dot{\gamma_n})_t\right| \mathrm{d}t\right)^2 \leq \int_{(0,1)} \left|(\dot{\gamma_n})_t\right|^2 \mathrm{d}t. \tag{4.8}$$

By combining (4.7) and (4.8) we get that for any $n \in \mathbb{N}$, γ_n satisfies

$$\bar{\mathsf{d}}_n(\gamma_n(0),\gamma_n(1))^2 = \int_{(0,1)} \left|(\dot{\gamma_n})_t\right|^2 \mathrm{d}t. \tag{4.9}$$

which is precisely the characterization of geodesic in (1.3).
Assume now that γ_n is a geodesic for any $n \in \mathbb{N}$.
Then $\bar{\mathsf{d}}_n(\gamma_n(t+h),\gamma_n(t)) = |h|\bar{\mathsf{d}}_n(\gamma_n(0),\gamma_n(1))$ and

$$\left|(\dot{\gamma})_t\right|^2 = \lim_{h\to 0} \frac{1}{h^2} \sum_{n=1}^{\infty} \bar{\mathsf{d}}_n(\gamma_n(t+h),\gamma_n(t))^2 = \sum_{n=1}^{\infty} \bar{\mathsf{d}}_n(\gamma_n(0),\gamma_n(1))^2 = \mathsf{d}(\gamma(0),\gamma(1))^2,$$

from which we conclude again by the characterization in (1.3). □

By using the notion of $\mathbb{D}$ convergence (see Definition 1.4.7), one can obtain that the infinite product space satisfies some curvature dimension condition.

Proposition 4.1.5. *The m.m.s.* $(X, \mathsf{d}, \mathfrak{m})$ *is the* $\mathbb{D}$*-limit of* $(\tilde{X}_n, \tilde{\mathsf{d}}_n, \tilde{\mathfrak{m}}_n)$ *and it satisfies the* $\mathsf{CD}(K,\infty)$ *condition.*

Proof. $(X, \mathsf{d}, \mathfrak{m})$ is the $\mathbb{D}$-limit of $(\tilde{X}_n, \tilde{\mathsf{d}}_n, \tilde{\mathfrak{m}}_n)$: it follows considering the definition of $\mathbb{D}-$convergence and taking $(\hat{M}, \hat{d}) := (X, d)$, the isometric embeddings I_n in (4.4) and the coupling $(Id, P_n)_\sharp \mathfrak{m} \in \mathrm{Adm}(\mathfrak{m}, \mathfrak{m}_n)$. The convergence follows from condition (4.1).
The $\mathsf{CD}(K,\infty)$ condition follows from the fact that every $(\tilde{X}_n, \tilde{\mathsf{d}}_n, \tilde{\mathfrak{m}}_n)$ satisfies the $\mathsf{CD}(K,\infty)$ condition and the $\mathsf{CD}(K,\infty)$ condition is stable under $\mathbb{D}-$convergence, see [94]. □

Remark 4.1.6. The space $(X, \mathsf{d}, \mathfrak{m})$ is the $\mathbb{D}$ limit of $(X_n, \mathsf{d}_n, \mathfrak{m}_n)$, since $(X_n, \mathsf{d}_n, \mathfrak{m}_n)$ are isomorphic to $(\tilde{X}_n, \tilde{\mathsf{d}}_n, \tilde{\mathfrak{m}}_n)$.

Proposition 4.1.7. *The space* $(X, \mathsf{d}, \mathfrak{m})$ *is geodesic and non-branching.*

Proof. The fact that X is geodesic follows from Lemma 4.1.4. We pass to the non-branching property. Let $\gamma^1 = (\gamma_1^1, \gamma_2^1, \dots)$ and $\gamma^2 = (\gamma_1^2, \gamma_2^2, \dots)$ be two geodesics, and $t \in (0,1)$ such that $\gamma^1(s) = \gamma^2(s)$ for any $s \in [0,t]$. Then for any $n \in \mathbb{N}$, $\gamma_n^1(s) = \gamma_n^2(s)$ and since γ_n^1 and γ_n^2 are geodesic (again by Lemma 4.1.4) in the non-branching space $(Y_n, \bar{\mathsf{d}}_n)$, $\gamma_n^1(s) = \gamma_n^2(s)$ for any $s \in [0,1]$ which implies that $\gamma^1(s) = \gamma^2(s)$ for any $s \in [0,1]$.
□

4.1.2 Approximation of measures in the infinite product space

Proposition 4.1.8. *Let* $(X, \mathsf{d}, \mathfrak{m})$ *as in* (4.2)*. Let* μ, $\nu \in \mathcal{P}(X)$ *with* $\mu := \rho_\mu \mathfrak{m}$, $\nu := \rho_\nu \mathfrak{m}$, ρ_μ, $\rho_\nu \leq C$*. Let for any* $n \in \mathbb{N}$, $(X_n, \mathsf{d}_n, \mathfrak{m}_n)$ *be as in* (4.4)*. Set*

$$\mu_n := (P_n)_\sharp \mu;$$
$$\nu_n := (P_n)_\sharp \nu.$$

Then

1. μ_n *and* ν_n *have supports contained in* X_n, $\mu_n = \rho_{\mu_n} \mathfrak{m}_n$, $\nu_n = \rho_{\nu_n} \mathfrak{m}_n$ *with* $\rho_{\mu_n}, \rho_{\nu_n} \leq C$, *and* μ_n *and* ν_n *converge weakly respectively to* μ *and* ν *in the sense of* (1.1)*;*

2. *for any sequence* $\pi_n \in \mathrm{Opt}_1(\mu_n, \nu_n)$ *there exists a subsequence converging to some* $\pi \in \mathrm{Opt}_1(\mu, \nu)$*.*

Proof. $\mu_n, \nu_n \leq C\mathfrak{m}_n$ for any n. Indeed from $\mu \leq C\mathfrak{m}$, we get for any $A \in B(X)$, $\mu_n(A) = (P_n)_\sharp \mu(A) = \mu(P_n^{-1}(A)) \leq C\mathfrak{m}(P_n^{-1}(A)) = C(P_n)_\sharp \mathfrak{m}(A) = C\mathfrak{m}_n(A)$. The same holds for ν_n. In addition $\{\mu_n\}_n$ converges weakly to μ and $\{\nu_n\}_n$ converges weakly to μ. Indeed take $\varphi \in C_b(X)$, from (1.5) we have

$$\lim_{n \to +\infty} \int_X \varphi \mu_n(\mathrm{d}x) = \lim_{n \to +\infty} \int_X \varphi \circ P_n(x) \mu(\mathrm{d}x) = \int_X \varphi(x) \mu(\mathrm{d}x), \tag{4.10}$$

since $\varphi \circ P_n$ are uniformly bounded and $\lim_{n \to +\infty} P_n(x) = x$. Finally $\pi \in \mathrm{Adm}(\mu, \nu)$, since the weak convergence is preserved by the pushforward via the projections on the first and second component. π is optimal since the weak convergence preserves optimality (see e.g. [98, Proposition 5.20]). □

The aim of the next sections is to prove that in the space $(X, \mathsf{d}, \mathfrak{m})$ one can solve the Monge problem between μ and ν are good enough.

4.2 Optimal map via disintegration

In this section we heavily use the notations and the objects introduced in Section 1.5.2. In particular we show how one can define an optimal map for the L^1-optimal transport problem, by reducing the problem to L^1-optimal transport problem in the real line. The results are well known, but we report here the versions that we need with detailed proof to keep the exposition self contained (see [21] or [33] for references). We consider φ a 1-Lipschitz function in a proper non-branching geodesic metric space (X, d). We recall that we defined the set Γ_φ as

$$\Gamma_\varphi := \{(x, y) \in X \times X \colon \varphi(x) - \varphi(y) = \mathsf{d}(x, y)\}$$

and the set $\mathcal{T}_\varphi^{nb}$

$$\mathcal{T}_\varphi^{nb} := \mathcal{T}_\varphi^e \setminus (A_\varphi^+ \cup A_\varphi^-).$$

where $\mathcal{T}_\varphi^e := P_1(R_\varphi \setminus \{(x, x) \colon x \in X\})$ and A_φ^+, A_φ^- are the set of branching points (see Section 1.5.2 for detailed definitions), which in the non-branching case are contained respectively in $\mathfrak{a}_\varphi$ and $\mathfrak{b}_\varphi$ (see Remark 1.5.7). The set Γ_φ gives a partition of $\mathcal{T}_\varphi^{nb}$ (see Proposition 1.5.13 and (1.31)) into equivalence classes $\{X_q\}_{q \in \mathcal{Q}}$. We also have that in the space $(\mathcal{T}_\varphi^{nb}, \mathcal{B}(\mathcal{T}_\varphi^{nb}))$, Proposition 1.5.18 holds. Therefore we can disintegrate Borel measures $\mathfrak{m}$ if restricted to $\mathcal{T}_\varphi^{nb}$:

$$\mathfrak{m}_{|\mathcal{T}_\varphi^{nb}} = \int_{\mathcal{Q}} \mathfrak{m}_q \mathfrak{q}(dq)$$

given a Borel section of the partition $\mathcal{Q}$ and a Borel quotient map f_φ (which exists thanks to Lemma 1.5.17), with $\mathfrak{q} = (f_\varphi)_\sharp \mathfrak{m}_{|\mathcal{T}_\varphi^{nb}}$.

Here we assume φ to be a Kantorovich potential for an L^1-optimal transport problem between two probability measures and we analyse the links between disintegration and optimal transport. The next result shows two things. First that we can disintegrate measures μ and ν and $\pi \in \mathrm{Opt}_1(\mu, \nu)$ using the same common measure on the quotient set (that we call $\mathfrak{q}_c$). Second that the marginal plans are optimal plans between the marginal measures.

Lemma 4.2.1. *Let (X, d) be a proper geodesic space. Let μ and ν be two probability measures in $\mathcal{P}_1(X)$. Let φ be a Kantorovich potential for the L^1-optimal transport problem. Let $\mathcal{T}_\varphi^{nb}$ be the set defined above and $\mathcal{Q}$ Borel quotient set and $f_\varphi : \mathcal{T}_\varphi^{nb} \to \mathcal{Q}$ Borel quotient map. Assume that $\mu(\mathcal{T}_\varphi^{nb}) = \nu(\mathcal{T}_\varphi^{nb})$. Then*

$$\mathfrak{q}_c := (f_\varphi)_\sharp \mu_{|\mathcal{T}_\varphi^{nb}} = (f_\varphi)_\sharp \nu_{|\mathcal{T}_\varphi^{nb}}. \tag{4.11}$$

In particular if we consider the $\mathfrak{q}_c$ unique disintegration of μ and ν given by Proposition 1.5.18, we get

$$\mu_{|\mathcal{T}_\varphi^{nb}} = \int_Q \mu_q \mathfrak{q}_c(\mathrm{d}q), \quad \nu_{|\mathcal{T}_\varphi^{nb}} = \int_Q \nu_q \mathfrak{q}_c(\mathrm{d}q). \tag{4.12}$$

Moreover if $\pi \in \mathrm{Opt}_1(\mu_{|\mathcal{T}_\varphi^{nb}}, \nu_{|\mathcal{T}_\varphi^{nb}})$, then there exists a weakly measurable map $q \mapsto \pi_q \in \mathcal{P}(X \times X)$ satisfying

$$\pi = \int_{\mathcal{Q}} \pi_q \, \mathfrak{q}_c(\mathrm{d}q),$$

with $\pi_q \in \mathrm{Opt}_1(\mu_q, \nu_q)$, $\mathfrak{q}_c$ a.e. $q \in \mathcal{Q}$.

With $q \mapsto \pi_q \in \mathcal{P}(X \times X)$ weakly measurable, we mean that for any $B \subseteq X \times X$ Borel, the map $\mathcal{Q} \ni q \mapsto \pi_q(B)$ is measurable (with $(\mathcal{Q}, \mathcal{B}(\mathcal{Q}))$).

Proof. Let $\pi \in \mathrm{Opt}_1(\mu_{|\mathcal{T}_\varphi^{nb}}, \nu_{|\mathcal{T}_\varphi^{nb}})$. Consider the space $(\mathcal{T}_\varphi^{nb} \times \mathcal{T}_\varphi^{nb}, \mathcal{B}(\mathcal{T}_\varphi^{nb} \times \mathcal{T}_\varphi^{nb}))$, with the partition $\{X_q \times \mathcal{T}_\varphi^{nb}\}_{q \in \mathcal{Q}}$. Then we apply Theorem 1.5.3 and we have an essentially unique disintegration consistent with the partition: $q \mapsto \pi_q$

$$\pi = \int_{\mathcal{Q}} \pi_q \mathfrak{q}_\pi(\mathrm{d}q),$$

with $\mathfrak{q}_\pi := \psi_\sharp \pi$ where $\psi := f_\varphi \circ P_1$ is a quotient map with quotient set $\mathcal{Q} \subset X$ (see Remark 1.5.2). Fix a point $\bar{x} \in \mathcal{T}_\varphi^{nb}$. Consider the Borel section $\mathcal{Q} \times \{\bar{x}\}$ and the Borel quotient map $(f_\varphi, \bar{x})$, we can apply Remark 1.5.4 to get that our disintegration is strongly consistent. Now we show that $\mathfrak{q}_\pi = (f_\varphi)_\sharp \mu_{|\mathcal{T}_\varphi^{nb}} = (f_\varphi)_\sharp \nu_{|\mathcal{T}_\varphi^{nb}}$ For any $A \in \mathcal{B}(Q)$ we have

$$(f_\varphi)_\sharp \mu_{|\mathcal{T}_\varphi^{nb}}(A) = \mu_{|\mathcal{T}_\varphi^{nb}}((f_\varphi)^{-1}(A)) = \pi((f_\varphi)^{-1}(A) \times X) = \pi(\psi^{-1}(A)) = \psi_\sharp \pi(A),$$

which shows that $\mathfrak{q}_\pi = \psi_\sharp \pi = (f_\varphi)_\sharp \mu_{|\mathcal{T}_\varphi^{nb}}$. Moreover $(f_\varphi)_\sharp \mu_{|\mathcal{T}_\varphi^{nb}} = (f_\varphi)_\sharp \nu_{|\mathcal{T}_\varphi^{nb}}$, since for any $A \in \mathcal{B}(Q)$ we have

$$\begin{aligned} \mu_{|\mathcal{T}_\varphi^{nb}}((f_\varphi)^{-1}(A)) &= \pi((f_\varphi)^{-1}(A) \times X) = \pi((f_\varphi)^{-1}(A) \times X \cap \Gamma_\varphi) \\ &= \pi(X \times (f_\varphi)^{-1}(A) \cap \Gamma_\varphi) = \nu_{|\mathcal{T}_\varphi^{nb}}((f_\varphi)^{-1}(A)), \end{aligned}$$

where we have used that π is concentrated on Γ_φ by optimality condition and $(f_\varphi)^{-1}(A) \times \mathcal{T}_\varphi^{nb} \cap \Gamma_\varphi = \mathcal{T}_\varphi^{nb} \times (f_\varphi)^{-1}(A) \cap \Gamma_\varphi$. To show this last fact we observe that

$$\begin{aligned}((f_\varphi)^{-1}(A) \times \mathcal{T}_\varphi^{nb}) \cap \Gamma_\varphi &= \cup_{q\in A}\left((X_q \times \mathcal{T}_\varphi^{nb}) \cap \Gamma_\varphi\right) = \cup_{q\in A}\left((X_q \times X_q) \cap \Gamma_\varphi\right) \\ &= \cup_{q\in A}\left((\mathcal{T}_\varphi^{nb} \times X_q) \cap \Gamma_\varphi\right) = \mathcal{T}_\varphi^{nb} \times (f_\varphi)^{-1}(A) \cap \Gamma_\varphi.\end{aligned}$$

From now on we set $\mathfrak{q}_c := (f_\varphi)_\sharp \mu_{|\mathcal{T}_\varphi^{nb}} = (f_\varphi)_\sharp \nu_{|\mathcal{T}_\varphi^{nb}} = \psi_\sharp \pi$.

It remains to show that $\pi_q \in \mathrm{Opt}_1(\mu_q, \nu_q)$ for $\mathfrak{q}_c$-a.e. $q \in \mathcal{Q}$. We first show that it is admissible. To prove that for $\mathfrak{q}_c$ a.e. $q \in \mathcal{Q}$, $\mu_q = (P_1)_\sharp \pi_q$, we show that for any $A \in \mathcal{B}(\mathcal{T}_\varphi^{nb})$

$$\int_{\mathcal{Q}} (P_1)_\sharp \pi_q(A) \mathfrak{q}_c(\mathrm{d}q) = \int_{\mathcal{Q}} \mu_q(A) \mathfrak{q}_c(\mathrm{d}q),$$

from which we deduce the result by the uniqueness of the disintegration.

$$\begin{aligned}\int_{\mathcal{Q}} (P_1)_\sharp \pi_q(A) \mathfrak{q}_c(\mathrm{d}q) &= \int_{\mathcal{Q}} \pi_q(A \times X) \mathfrak{q}_c(\mathrm{d}q) \\ = \pi(A \times X) = \mu_{|\mathcal{T}_\varphi^{nb}}(A) &= \int_{\mathcal{Q}} \mu_q(A) \mathfrak{q}_c(\mathrm{d}q).\end{aligned}$$

The fact that $\mathfrak{q}_c$ a.e. $q \in \mathcal{Q}$, $\nu_q = (P_2)_\sharp \pi_q$ can be proved analogously. It remains to show that $\mathfrak{q}_c$ a.e. $q \in \mathcal{Q}$, $\pi_q \in \mathrm{Opt}_1(\mu_q, \nu_q)$. This follows from the fact that π is concentrated on Γ_φ, so

$$\pi(X \times X) = \pi(\Gamma_\varphi) = \int_{\mathcal{Q}} \pi_q(\Gamma_\varphi) \mathfrak{q}_c(\mathrm{d}q),$$

therefore $\pi_q(\Gamma_\varphi) = \pi_q(X \times X)$, $\mathfrak{q}_c$ a.e. $q \in \mathcal{Q}$ and the conclusion follows by the Fundamental Theorem of Optimal Transport 1.2.2. □

Remark 4.2.2. Let (X, d), μ, ν, φ be as in the previous Lemma. Let $\mathfrak{m}$ be a finite Borel measure on X. Assume in addition that $\mu = \rho_\mu \mathfrak{m}$, $\nu = \rho_\nu \mathfrak{m}$. Let $q \mapsto \mathfrak{m}_q$ be the disintegration of $\mathfrak{m}_{|\mathcal{T}_\varphi^{nb}}$ as in (1.32)

$$\mathfrak{m}_{|\mathcal{T}_\varphi^{nb}} = \int_{\mathcal{Q}} \mathfrak{m}_q \mathfrak{q}_{\mathfrak{m}}(\mathrm{d}q),$$

with $\mathfrak{q}_{\mathfrak{m}} = f_\sharp \mathfrak{m}$. Then for $\mathfrak{q}_{\mathfrak{m}}$ a.e. $q \in \mathcal{Q}$

$$\int \rho_\mu \mathfrak{m}_q = \int \rho_\nu \mathfrak{m}_q. \tag{4.13}$$

This is immediate by combining (4.11) with (1.29).

Gluing monotone rearrangements

In the following we show that if the partition into transport rays is good enough, namely if the branching points have zero measure and the marginal measures are non atomic, then one can glue together the monotone rearrangements along the rays to get an optimal transport map. Since we will heavily use this construction in the next section, we decided to present here the whole construction and proof (see [21] and [33] for references).

Lemma 4.2.3. *Let (X, d) be a proper geodesic space. Let μ and ν be two probability measures in $\mathcal{P}_1(X)$. Let φ be a Kantorovich potential for the L^1-optimal transport problem. Let $\mathcal{T}_\varphi^{nb}$ as in* (1.30)*. Let $\mathcal{Q}$ and f_φ be a Borel quotient set and a Borel quotient map of the partition $\{X_q\}_{q\in\mathcal{Q}}$ of $\mathcal{T}_\varphi^{nb}$ induced by φ. Consider the disintegrations $q \mapsto \mu_q$, $q \mapsto \nu_q$ of $\mu_{|\mathcal{T}_\varphi^{nb}}$ and $\nu_{|\mathcal{T}_\varphi^{nb}}$ strongly consistent with the partition. Assume that*

1. *$\mu(A_\varphi^+) = \nu(A_\varphi^-) = 0$,*
2. *there exists $\bar{\mathcal{Q}} \subseteq \mathcal{Q}$ Borel with $\mathfrak{q}_c(\mathcal{Q} \setminus \bar{\mathcal{Q}}) = 0$, such that for any $q \in \bar{\mathcal{Q}}$, μ_q has no atoms,*

where $\mathfrak{q}_c := (f_\varphi)_\sharp \mu_{|\mathcal{T}_\varphi^{nb}} = (f_\varphi)_\sharp \nu_{|\mathcal{T}_\varphi^{nb}}$ (recall Lemma 4.2.1).

Then there exists $T : X \to X$ Borel optimal map for the L^1-transport problem between μ and ν. Moreover T can be taken as

$$T(x) := \begin{cases} T_{mon,X_q}(x) & x \in \mathcal{T}_\varphi^{nb},\ q = f_\varphi(x), \\ x & x \in X \setminus \mathcal{T}_\varphi^{nb}, \end{cases} \tag{4.14}$$

where $T_{mon,X_q} : \mathrm{supp}(\mu_q) \to X_q$ is the monotone rearrangement *on X_q between μ_q and ν_q if $q \in \bar{\mathcal{Q}}$ defined as*

$$\begin{aligned} &x \mapsto T_{mon,X_q}(x), \\ &x \mapsto g^{-1}(x) = (q, t) \mapsto (q, T_{mon,q}(t)) \mapsto g(q, T_{mon,q}(t)), \end{aligned}$$

with g the ray map defined in Definition 1.5.21 and where $T_{mon,q}(t)$ is the monotone rearrangement between $\tilde{\mu}_q := g_\sharp^{-1}\mu_q$ and $\tilde{\nu}_q := g_\sharp^{-1}\nu_q$. If $q \in \mathcal{Q} \setminus \bar{\mathcal{Q}}$ we set $T_{mon,X_q}(x) := x$.

Proof. **Step 1**
We notice that $\mu(\mathcal{T}_\varphi^{nb}) = \nu(\mathcal{T}_\varphi^{nb})$ and if $\bar{T}$ is an optimal transport map between $\mu_{|\mathcal{T}_\varphi^{nb}}$ and $\nu_{|\mathcal{T}_\varphi^{nb}}$, then the map

$$T(x) := \begin{cases} \bar{T}(x) & x \in \mathcal{T}_\varphi^{nb}, \\ x & x \in X \setminus \mathcal{T}_\varphi^{nb}, \end{cases}$$

is an optimal transport map between μ and ν. To prove this it is enough to show the following
claim: if π is in $\mathrm{Opt}_1(\mu, \nu)$, then it is concentrated on the set

$$(\mathcal{T}_\varphi^{nb} \times \mathcal{T}_\varphi^{nb}) \cup D \tag{4.15}$$

with $D \coloneqq \{(x, x) \in X\}$ the diagonal. If the claim is true then

$$\begin{aligned} \mu(\mathcal{T}_\varphi^{nb}) &= \pi(\mathcal{T}_\varphi^{nb} \times X) = \pi((\mathcal{T}_\varphi^{nb} \times X) \cap (\mathcal{T}_\varphi^{nb} \times \mathcal{T}_\varphi^{nb}) \cup D) \\ &= \pi(\mathcal{T}_\varphi^{nb} \times \mathcal{T}_\varphi^{nb}) = \pi((X \times \mathcal{T}_\varphi^{nb}) \cap (\mathcal{T}_\varphi^{nb} \times \mathcal{T}_\varphi^{nb}) \cup D) = \pi(X \times \mathcal{T}_\varphi^{nb}) \\ &= \nu(\mathcal{T}_\varphi^{nb}), \end{aligned}$$

from which we also get that $\pi_{|\mathcal{T}_\varphi^{nb} \times \mathcal{T}_\varphi^{nb}} \in \mathrm{Adm}(\mu_{|\mathcal{T}_\varphi^{nb}}, \nu_{|\mathcal{T}_\varphi^{nb}})$. We can then show the claim as follows. Let $\bar{\pi} := (Id, T)_\sharp \mu$. It is in $\mathrm{Adm}(\mu, \nu)$ since $\mu_{|(\mathcal{T}_\varphi^{nb})^c} = \nu_{|(\mathcal{T}_\varphi^{nb})^c}$ which follows again by (4.15). Moreover taking $\pi \in \mathrm{Opt}_1(\mu,\nu)$, then

$$\begin{aligned}\int_{X\times X} \mathsf{d}(x,y)\bar\pi(\mathrm{d}x,\mathrm{d}y) &= \int_{\mathcal{T}_\varphi^{nb}\times\mathcal{T}_\varphi^{nb}} \mathsf{d}(x,y)\bar\pi(\mathrm{d}x,\mathrm{d}y)\\ \leq \int_{\mathcal{T}_\varphi^{nb}\times\mathcal{T}_\varphi^{nb}} \mathsf{d}(x,y)\pi(\mathrm{d}x,\mathrm{d}y) &\leq \int_{X\times X} \mathsf{d}(x,y)\pi(\mathrm{d}x,\mathrm{d}y),\end{aligned}$$

where we have used that $\pi_{|\mathcal{T}_\varphi^{nb}\times\mathcal{T}_\varphi^{nb}} \in \mathrm{Adm}(\mu_{|\mathcal{T}_\varphi^{nb}},\nu_{|\mathcal{T}_\varphi^{nb}})$. Hence $\bar\pi$ is also optimal. Now we show (4.15). We observe that

- $X\times X = (\mathcal{T}_\varphi^e\times\mathcal{T}_\varphi^e)\cup(\mathcal{T}_\varphi^e\times\mathcal{T}_\varphi^e)^c$

We know that any optimal transport plan for the L^1 optimal transport problem between μ and ν is concentrated on Γ_φ. We have that

- $(\mathcal{T}_\varphi^e\times\mathcal{T}_\varphi^e)^c\cap\Gamma_\varphi\subseteq D$;
- $\mathcal{T}_\varphi^e\times\mathcal{T}_\varphi^e\cap\Gamma_\varphi\subseteq A_\varphi^+\times\mathcal{T}_\varphi^e\cup\mathcal{T}_\varphi^{nb}\times\mathcal{T}_\varphi^{nb}\cup\mathcal{T}_\varphi^e\times A_\varphi^-\cup A_\varphi^-\times A_\varphi^+\cap\Gamma_\varphi$;

since $(A_\varphi^-\times\mathcal{T}_\varphi^{nb})\cap\Gamma_\varphi=\emptyset$, $(\mathcal{T}_\varphi^{nb}\times A_\varphi^+)\cap\Gamma_\varphi=\emptyset$. In addition let π be any plan in $\mathrm{Opt}_1(\mu,\nu)$, we have

- $\pi(A_\varphi^+\times\mathcal{T}_\varphi^e)\leq\pi(A_\varphi^+\times X)=\mu(A^+)=0$;
- $\pi(\mathcal{T}_\varphi^e\times A_\varphi^-)\leq\pi(X\times A_\varphi^-)=\nu(A_\varphi^-)=0$.

Finally $A_\varphi^-\times A_\varphi^+\cap\Gamma_\varphi\subset D$. Therefore let π be in $\in\mathrm{Opt}_1(\mu,\nu)$, we just proved that it is concentrated on

$$(\mathcal{T}_\varphi^{nb}\times\mathcal{T}_\varphi^{nb})\cup D.$$

Step 2
So let $T: X\to X$ be the map in (4.14). We only need to prove that $T_{|\mathcal{T}_\varphi^{nb}}$ is an optimal map between $\mu_{|\mathcal{T}_\varphi^{nb}}$ and $\nu_{|\mathcal{T}_\varphi^{nb}}$. Then the statement follows by Step 1. We first observe that the map $T_{|\mathcal{T}_\varphi^{nb}}$ is Borel if the map

$$\begin{aligned}\bar T: \cup_{q\in\mathcal{Q}}\{q\}\times I_q &\to \cup_{q\in\mathcal{Q}}\{q\}\times I_q\\ (q,t)&\mapsto(q,T_{\mathrm{mon},q}(t))\end{aligned}$$

is Borel (see Definition 1.5.21 for the definition of I_q), where $T_{\mathrm{mon},q}(t)$ is the monotone rearrangement between $\tilde\mu_q$ and $\tilde\nu_q$ if $q\in\bar{\mathcal{Q}}$ and the identity map for $q\in\mathcal{Q}\setminus\bar{\mathcal{Q}}$. Indeed $T_{|\mathcal{T}_\varphi^{nb}}=g\circ\bar T\circ g^{-1}$, where we are using that g and g^{-1} are Borel (see Proposition 1.5.22). We prove that $\bar T$ is Borel. Let for any $q\in\bar{\mathcal{Q}}$

$$\begin{aligned}F_{\tilde\mu_q}(t)&:=\tilde\mu_q((-\infty,t)),\\ F_{\tilde\nu_q}(t)&:=\tilde\nu_q((-\infty,t)),\end{aligned}$$

be the repartition functions of $\tilde\mu_q$ and $\tilde\nu_q$. Then $T_{\mathrm{mon},q}(t)=F_{\tilde\nu_q}^{-1}\circ F_{\tilde\mu_q}$, by defining $F_{\tilde\nu_q}^{-1}(s):=\sup\{t\colon F_{\tilde\nu_q}(t)\leq s\}$. The maps $(q,t)\mapsto F_{\tilde\mu_q}(t)$ and $(q,t)\mapsto F_{\tilde\nu_q}^{-1}(t)$ are Borel. We show that $(q,t)\mapsto F_{\tilde\mu_q}(t)$ is Borel by showing that the counterimage of $(-\infty,a)$ is a Borel set for any $a\in\mathbb{R}$. This follows from the fact that since $q\mapsto\tilde\mu_q$ is continuous and non decreasing, the following equality holds

$$\{(q,t)\colon F_{\tilde\mu_q}(t)<a\}=\cup_{s\in\mathbb{Q}}\{(q,t)\colon F_{\tilde\mu_q}(s)<a,\,t\leq s\},$$

where any term of the union is Borel, by recalling that $q \mapsto \tilde{\mu}_q(A)$ is Borel for any A Borel. Analogously one can prove that $(q,t) \mapsto F^{-1}_{\tilde{\nu}_q}(t)$ is Borel. It follows that $\cup_{q\in\bar{\mathcal{Q}}}\{q\} \times I_q \ni (q,t) \mapsto (q, T_{\mathrm{mon},q}(t))$ is Borel and therefore $\tilde{T}$ is Borel. It remains to show that $T_{|\mathcal{T}^{nb}_\varphi}$ is an optimal transport map between $\mu_{|\mathcal{T}^{nb}_\varphi}$ and $\nu_{|\mathcal{T}^{nb}_\varphi}$. We first show that $(T_{|\mathcal{T}^{nb}_\varphi})_\sharp\mu_{|\mathcal{T}^{nb}_\varphi} = \nu_{|\mathcal{T}^{nb}_\varphi}$. Take $A \in \mathcal{B}(X)$,

$$
\begin{aligned}
&(T_{|\mathcal{T}^{nb}_\varphi})_\sharp\mu_{|\mathcal{T}^{nb}_\varphi}(A) = \mu_{|\mathcal{T}^{nb}_\varphi}(T^{-1}_{|\mathcal{T}^{nb}_\varphi}(A)) = \int_{\mathcal{Q}} \mu_q(T^{-1}_{|\mathcal{T}^{nb}_\varphi}(A))\mathfrak{q}_c(\mathrm{d}q) = \int_{\mathcal{Q}} \mu_q(T^{-1}_{|\mathcal{T}^{nb}_\varphi}(A) \cap X_q)\mathfrak{q}_c(\mathrm{d}q) \\
&= \int_{\mathcal{Q}} \mu_q(T^{-1}_{\mathrm{mon},X_q}(A \cap X_q))\mathfrak{q}_c(\mathrm{d}q) = \int_{\mathcal{Q}} \nu_q(A \cap X_q))\mathfrak{q}_c(\mathrm{d}q) = \nu_{|\mathcal{T}^{nb}_\varphi}(A).
\end{aligned}
$$

To see that it is optimal we take $\pi \in \mathrm{Opt}_1(\mu_{|\mathcal{T}^{nb}_\varphi}, \nu_{|\mathcal{T}^{nb}_\varphi})$. We have a strongly consistent disintegration as explained in Lemma 4.2.1 $q \mapsto \pi_q$, $\pi = \int_{\mathcal{Q}} \pi_q \psi_\sharp \mathfrak{q}(\mathrm{d}q)$. Now

$$
\begin{aligned}
&\int_{X\times X} \mathsf{d}(x,y)\pi(\mathrm{d}x,\mathrm{d}y) = \int_{\mathcal{Q}}\int_{X\times X} \mathsf{d}(x,y)\pi_q(\mathrm{d}x,\mathrm{d}y)\,\mathfrak{q}(\mathrm{d}q) \\
&= \int_{\mathcal{Q}}\int_{X\times X} \mathsf{d}(x,y)(Id, T_{\mathrm{mon},X_q})_\sharp\mu_q\,\mathfrak{q}(\mathrm{d}q) = \int_{\mathcal{Q}}\int_{X\times X} \mathsf{d}(x, T_{\mathrm{mon},X_q}(x))\mu_q\,\mathfrak{q}(\mathrm{d}q) \\
&= \int_{\mathcal{Q}}\int_X \mathsf{d}(x, T_{|\mathcal{T}^{nb}_\varphi}(x))\mu_q\,\mathfrak{q}(\mathrm{d}q) = \int_X \mathsf{d}(x, T_{|\mathcal{T}^{nb}_\varphi}(x))\mu(\mathrm{d}x) = \int_X \mathsf{d}(x,y)(Id, T_{|\mathcal{T}^{nb}_\varphi})\sharp\mu(\mathrm{d}x,\mathrm{d}y),
\end{aligned}
$$

where the second equality follows from the fact that $\pi_q \in \mathrm{Opt}_1(\mu_q, \nu_q)$ as shown in Lemma 4.2.1, and $(Id, T_{\mathrm{mon},X_q}))_\sharp\mu_q$ is optimal since T_{mon,X_q} is isometric image of $T_{\mathrm{mon},q}$ which is optimal between $\tilde{\mu}_q$ and $\tilde{\nu}_q$ (see Theorem 1.2.4). □

4.3 Regularity of the disintegration

In this section we prove that our partition and our disintegration of the measure μ are good enough to construct an optimal transport map by gluing monotone rearrangements along the rays.

4.3.1 Positive evolution

Our main objective is to construct a positive evolution along the rays. This is done in three steps. The first of them is the following Lemma. We give a lover bound for the evolution on a weighted interval of a set of positive measure, when restricted to the set where the optimal plan is a graph. This technical assumption allow us to use the change of variable formula.

Lemma 4.3.1. *Let h be a $\mathsf{CD}(K,\infty)$ density on an interval $I \subseteq \mathbb{R}$ (recall Definition 1.4.9). Let*

$$\tilde{\mathfrak{m}} := h\mathcal{L}^1_{|I}.$$

Let $\tilde{\mu}$, $\tilde{\nu} \in \mathcal{P}(I)$, compactly supported, with $\tilde{\mu} := \rho_{\tilde{\mu}}\tilde{\mathfrak{m}}$, $\tilde{\nu} := \rho_{\tilde{\nu}}\tilde{\mathfrak{m}}$, $\rho_{\tilde{\mu}}, \rho_{\tilde{\nu}} \leq C$ for some positive constant C. Let $\tilde{T}$ be the monotone rearrangement between $\tilde{\mu}$ and $\tilde{\nu}$ defined in Theorem 1.2.4, and $\pi := (Id, \tilde{T})_\sharp\tilde{\mu} \in \mathrm{Opt}_1(\tilde{\mu}, \tilde{\nu})$. Let $\Gamma \subseteq \mathrm{graph}(\tilde{T})$ be a Borel set such that $\pi(\Gamma) = 1$. Then for any $t \in [0,1]$ and any Borel set and $A \in \mathcal{B}(I)$ the following estimate holds

$$\tilde{\mathfrak{m}}(T_t(\Gamma \cap A \times I)) \geq \frac{C_{K,M}}{C}\tilde{\mu}(A), \tag{4.16}$$

where

$$C_{K,M} := \begin{cases} 1 & K \geq 0, \\ e^{\frac{K}{8}M^2} & K < 0; \end{cases}$$

$M \geq \sup\{d(x,y) \colon x \in \mathrm{supp}(\tilde{\mu}), y \in \mathrm{supp}(\tilde{\nu})\}$ *and* $T_t : I \times I \to \mathbb{R}$ *is given by* $T_t(x,y) := (1-t)x + ty$.

Proof. Since $\tilde{T}$ is monotone it is differentiable at $\tilde{\mu}$-a.e. point and by Theorem 1.2.4 we have that $\tilde{T}'$ positive at $\tilde{\mu}$-a.e. point. Define $\tilde{T}_t : \mathrm{supp}(\tilde{\mu}) \to I$, $\tilde{T}_t(x) = (1-t)x + t\tilde{T}(x)$. We have that

$$(\tilde{T}_t)'(x) = (1-t) + t\tilde{T}'(x) > 0, \quad \tilde{\mu} - \text{a.e.}$$

so we can take the logarithms on both sides and we have

$$\log((\tilde{T}_t)'(x)) = \log((1-t) + t\tilde{T}'(x)) \geq t\log(\tilde{T}'(x)) \quad \tilde{\mu} - \text{a.e.},$$

where we used the concavity of the logarithm. We get that

$$(\tilde{T}_t)'(x) \geq \tilde{T}'(x)^t \quad \tilde{\mu} - \text{a.e.} \tag{4.17}$$

Now we observe that $\tilde{T}_t : \mathrm{supp}(\tilde{\mu}) \to \mathbb{R}$ is $\tilde{\mu}$-essentially injective (see Remark 1.2.5), so we can consider its inverse $\tilde{T}_t^{-1}$. In order to avoid measurability issues, we fix $\varepsilon > 0$ and take $\Gamma_\varepsilon \subseteq \Gamma$ compact in $I \times I$ with $\pi(\Gamma_\varepsilon) \geq 1-\varepsilon$ and $A_\varepsilon \subseteq A$ compact such that $\tilde{\mu}(A_\varepsilon) \geq \tilde{\mu}(A) - \varepsilon$. So we have that $(T)_t(\Gamma_\varepsilon \cap A \times I)$ is a Borel set (is compact), moreover $P_1(\Gamma_\varepsilon)$ is Borel. Note that $T_t(\Gamma_\varepsilon \cap A_\varepsilon \times I) = \tilde{T}_t(P_1(\Gamma_\varepsilon) \cap A_\varepsilon)$ because $\Gamma_\varepsilon \subseteq \mathrm{graph}(\tilde{T})$. Then

$$\tilde{\mathfrak{m}}(T_t(\Gamma_\varepsilon \cap A_\varepsilon \times I)) = \int_{\tilde{T}_t(P_1(\Gamma_\varepsilon)\cap A_\varepsilon)} \tilde{\mathfrak{m}}(dx) \geq \int_{P_1(\Gamma_\varepsilon)\cap A_\varepsilon} (\tilde{T}_t)'(x) h \circ \tilde{T}_t(x)\, dx \tag{4.18}$$

where the second inequality follows by applying Theorem 1.2.7 to $f = \tilde{T}_t$ and $\psi = h \circ \tilde{T}_t$ and by observing that $\tilde{T}_t$ is injective and differentiable $\tilde{\mu}$ a.e. Now combining (4.18) and (4.17) we get

$$\tilde{\mathfrak{m}}((T)_t(\Gamma_\varepsilon \cap A_\varepsilon \times I)) \geq \int_{P_1(\Gamma_\varepsilon)\cap A_\varepsilon} \tilde{T}'(x)^t h \circ \tilde{T}_t(x)\, dx = \tag{4.19}$$

$$\int_{P_1(\Gamma_\varepsilon)\cap A_\varepsilon} \left(\frac{\rho_{\tilde{\mu}}(x)h(x)}{\rho_{\tilde{\nu}} \circ \tilde{T}(x) h \circ \tilde{T}(x)} \right)^t h \circ \tilde{T}_t(x)\, dx \tag{4.20}$$

$$\geq \int_{P_1(\Gamma_\varepsilon)\cap A_\varepsilon} \frac{\rho_{\tilde{\mu}}(x)^{t-1}}{(\rho_{\tilde{\nu}} \circ \tilde{T}(x))^t} \frac{h(x)^{t-1}}{(h \circ \tilde{T}(x))^t} h \circ \tilde{T}_t(x)\, \tilde{\mu}(dx),$$

where we used that

$$\tilde{T}'(x) = \frac{\rho_{\tilde{\mu}}(x)h(x)}{\rho_{\tilde{\nu}} \circ \tilde{T}(x) h \circ \tilde{T}(x)} \qquad \mathcal{L}^1 \text{ a.e. in } \mathrm{supp}(\tilde{\mu})$$

combining Theorem 1.2.6 and the fact that h is positive in $\mathring{I}$ and $\rho_{\tilde{\nu}} \circ \tilde{T} > 0$ $\mathcal{L}^1$ a.e. in $\mathrm{supp}(\tilde{\mu})$. Now we observe that by the fact that h is a $\mathsf{CD}(K,\infty)$ density we have

$$\begin{aligned} \log h(\tilde{T}_t(x)) &= \log h((1-t)x + t\tilde{T}(x)) \\ &\geq (1-t)\log h(x) + t\log h(\tilde{T}(x)) + \frac{K}{2}t(1-t)(x - \tilde{T}(x))^2 \\ &= \log\left(h(x)^{1-t}(h(\tilde{T}(x))^t\right) + \frac{K}{2}t(1-t)(x-\tilde{T}(x))^2 \end{aligned}$$

which gives by taking the exponential of both sides

$$h(\tilde{T}_t(x)) \geq h(x)^{1-t}(h(\tilde{T}(x))^t e^{\frac{K}{2}t(1-t)(x-\tilde{T}(x))^2} \geq C_{K,M} h(x)^{1-t}(h(\tilde{T}(x))^t \tag{4.21}$$

with

$$C_{K,M} := \begin{cases} 1 & K \geq 0, \\ e^{\frac{K}{8}M^2} & K < 0. \end{cases}$$

So by (4.21) we get

$$h(\tilde{T}_t(x)) \frac{h(x)^{t-1}}{(h(\tilde{T}(x))^t} \geq C_{K,M}. \tag{4.22}$$

In addition

$$\frac{(\rho_{\tilde{\mu}}(x))^{t-1}}{(\rho_{\tilde{\nu}}(\tilde{T}(x)))^t} \geq \frac{1}{C}, \tag{4.23}$$

since $\rho_{\tilde{\mu}} \leq C$ and $\rho_{\tilde{\nu}} \leq C$. We combine (4.19) with (4.22) and (4.23) and we get

$$\tilde{\mathfrak{m}}(T_t(\Gamma \cap A \times I)) \geq \tilde{\mathfrak{m}}((T)_t(\Gamma_\varepsilon \cap A_\varepsilon \times I)) \geq \frac{C_{K,M}}{C} \tilde{\mu}(A_\varepsilon \cap P_1(\Gamma_\varepsilon)) \geq \frac{C_{K,M}}{C} (\tilde{\mu}(A_\varepsilon) - 2\varepsilon). \tag{4.24}$$

The result follows by letting $\varepsilon \to 0$. □

After having proved a one dimensional result, the second step is to show that it can be used together with the localization theorem, to prove that an analogous result can be used in any essentially non-branching $\mathsf{CD}(K,N)$ space with finite N.

Corollary 4.3.2. *Let $(X, \mathsf{d}, \mathfrak{m})$ be an essentially non-branching m.m.s. satisfying the $\mathsf{CD}(K,N)$ condition for some $N < +\infty$. Let $\mu, \nu \in \mathcal{P}(X)$ be two compactly supported probability measures absolutely continuous with respect to $\mathfrak{m}$: $\mu = \rho_\mu \mathfrak{m}$ and $\nu = \rho_\nu \mathfrak{m}$, with ρ_μ, $\rho_\nu : X \to \mathbb{R}$, ρ_μ, $\rho_\nu \leq C$. Then there exists $\pi \in \mathrm{Opt}_1(\mu, \nu)$, with $\pi = (Id, T)_\sharp \mu$ such that for any $\Gamma \subseteq \mathrm{graph}(T)$, Γ Borel with $\pi(\Gamma) = 1$, for any $t \in [0,1]$ and $A \in \mathcal{B}(X)$, one has*

$$\mathfrak{m}(T_t(\Gamma \cap A \times X)) \geq \frac{C_{K,M}}{C} \mu(A), \tag{4.25}$$

with

$$C_{K,M} := \begin{cases} 1 & K \geq 0, \\ e^{\frac{K}{8}M^2} & K < 0, \end{cases}$$

$M \geq \sup\{d(x,y) \colon x \in \mathrm{supp}(\mu), y \in \mathrm{supp}(\nu)\}$ and T_t is the possibly multivalued midpoint map defined as $T_t : X \times X \to X$, $T_t(x,y) := \{z \colon \mathsf{d}(x,z) = (1-t)\mathsf{d}(x,y), \mathsf{d}(y,z) = t\mathsf{d}(x,y)\}$.

Proof. We observe that we are under the hypotheses of Lemma 4.2.3. Indeed let φ be a Kantorovich potential for the L^1-optimal transport problem between μ and ν. Then from Proposition 1.6.1 $\mathfrak{m}(A^+_\varphi) = \mathfrak{m}(A^-_\varphi) = 0$. Let $\mathcal{T}^{nb}_\varphi$ as in (1.30). Let $\mathcal{Q}$ and f_φ be Borel quotient map and quotient set of the partition $\{X_q\}_{q\in\mathcal{Q}}$ of $\mathcal{T}^{nb}_\varphi$ induced by φ. Consider the disintegrations $q \mapsto \mu_q$, $q \mapsto \nu_q$, $q \mapsto \mathfrak{m}_q$ of $\mu_{|\mathcal{T}^{nb}_\varphi}$, $\nu_{|\mathcal{T}^{nb}_\varphi}$ and $\mathfrak{m}_{|\mathcal{T}^{nb}_\varphi}$ respectively, strongly consistent with the partition (as in Proposition 1.5.18). Recall that $\mathfrak{q}_c := (f_\varphi)_\sharp \mu_{|\mathcal{T}^{nb}_\varphi} = (f_\varphi)_\sharp \nu_{|\mathcal{T}^{nb}_\varphi}$ (by Lemma 4.2.1). We have in particular (see Lemma 1.5.5)

$$\mathfrak{m}_{|\mathcal{T}^{nb}_\varphi} = \int_Q \mathfrak{m}_q \mathfrak{q}_\mathfrak{m}, \quad \mu_{|\mathcal{T}^{nb}_\varphi} = \int_Q \mu_q \mathfrak{q}_c, \quad \nu_{|\mathcal{T}^{nb}_\varphi} = \int_Q \nu_q \mathfrak{q}_c.$$

By (4.13) we have that $\int_X \rho_\mu \mathfrak{m}_q = \int_X \rho_\nu \mathfrak{m}_q$ for $\mathfrak{q}_\mathfrak{m}$ a.e. $q \in \mathcal{Q}$. Therefore we call $l(q) := \int_X \rho_\mu \mathfrak{m}_q$ and we have by Lemma 1.5.5

$$\mathfrak{q}_c = l(q)\mathfrak{q}_\mathfrak{m}, \quad \mu_q = \frac{\rho_\mu}{l(q)}\mathfrak{m}_q, \quad \nu_q = \frac{\rho_\mu}{l(q)}\mathfrak{m}_q \tag{4.26}$$

$\mathfrak{q}_c$ a.e. $q \in \mathcal{Q}$. By Theorem 1.6.2 and Lemma 1.4.10 $\mathfrak{q}_\mathfrak{m}$ a.e. $q \in \mathcal{Q}$, $\mathfrak{m}_q \ll \mathcal{H}^1_{|X_q}$. Therefore $\mu_q \ll \mathcal{H}^1$ $\mathfrak{q}_c$ a.e. $q \in \mathcal{Q}$ which implies that μ_q are non atomic. Hence we are in the hypotheses of Lemma 4.2.3 and we can take the optimal map T given by the statement. Let $\pi := (Id, T)_\sharp \mu$. Let $\Gamma \subseteq \text{graph}(T)$ Borel with $\pi(\Gamma) = 1$. We first get the estimate for $\mathfrak{m}(T_t(\Gamma \cap (A \cap \mathcal{T}_\varphi^{nb}) \times X))$. We have

$$\begin{aligned}
&\mathfrak{m}(T_t(\Gamma \cap (A \cap \mathcal{T}_\varphi^{nb}) \times X)) \geq \mathfrak{m}_{|\mathcal{T}_\varphi^{nb}}(T_t(\Gamma \cap ((A \cap \mathcal{T}_\varphi^{nb}) \times X))) \\
&= \int_\mathcal{Q} \mathfrak{m}_q \left((T_t(\Gamma \cap ((A \cap \mathcal{T}_\varphi^{nb}) \times X))) \right) \mathfrak{q}_\mathfrak{m}(dq) \\
&= \int_\mathcal{Q} \mathfrak{m}_q \left(\left(T_t(\Gamma \cap (A \cap \mathcal{T}_\varphi^{nb}) \times X \right) \cap X_q^\varphi \right) \mathfrak{q}_\mathfrak{m}(dq) \\
&= \int_\mathcal{Q} \mathfrak{m}_q(T_t(\Gamma \cap (A \times X) \cap (X_q^\varphi \times X_q^\varphi)) \cap X_q^\varphi)\mathfrak{q}_\mathfrak{m}(dq).
\end{aligned}$$

where the last equality follows from the definition of the map T and $\Gamma \subseteq \text{graph}(T)$. Before going on with the estimate, we need to fix some notations and make some observations. Let $g : \text{Dom}\,(g) \to X$ be the Ray Map defined in 1.5.21. We set for $\mathfrak{q}_c$ a.e. $q \in \mathcal{Q}$, following the notations already used,

$$\begin{aligned}
&\tilde{\mathfrak{m}}_q := g_\sharp^{-1}\mathfrak{m}_q, \quad \tilde{\mu}_q := g_\sharp^{-1}\mu_q, \quad \tilde{\nu}_q := g_\sharp^{-1}\nu_q. \\
&I_q := g^{-1}(X_q).
\end{aligned}$$

We also set

$$\begin{aligned}
A_q &:= A \cap X_q^\varphi, & \tilde{A}_q &:= g^{-1}(A_q), \\
\Gamma_q &:= \Gamma \cap (X_q^\varphi \times X_q^\varphi), & \tilde{\Gamma}_q &:= (g^{-1}, g^{-1})(\Gamma_q).
\end{aligned}$$

We prove that $(I_q, |\cdot|, \tilde{\mathfrak{m}}_q)$, $\tilde{\mu}_q$, $\tilde{\nu}_q$, $\tilde{\Gamma}_q$, $\tilde{A}_q$ satisfy the hypotheses of Lemma 4.3.1. We first observe that by Theorem 1.6.2 and Remark 1.6.5 we have that $\mathfrak{q}_c$ a.e. $q \in \mathcal{Q}$, $(\bar{I}_q, |\cdot|, \tilde{\mathfrak{m}}_q)$ are $\mathsf{CD}(K, N)$ spaces and $\tilde{\mathfrak{m}}_q = h_q \mathcal{L}^1$ with h_q being $\mathsf{CD}(K, N)$ densities and consequently $\mathsf{CD}(K, \infty)$ densities on I_q. Then we observe that from Disintegration Theorem we also have that $\mu_q(X_q^\varphi) = \nu_q(X_q^\varphi) = 1$ $\mathfrak{q}_c$ a.e. $q \in \mathcal{Q}$. In addition, from (4.26) the densities of μ_q and ν_q with respect to $\mathfrak{m}_q$ are bounded from above by $\frac{C}{l(q)}$ for $\mathfrak{q}_c$ a.e. $q \in \mathcal{Q}$, and since $g(q, \cdot)$ is an invertible isometry, $\tilde{\mu}_q, \tilde{\nu}_q \ll \tilde{\mathfrak{m}}_q$ with densities $\tilde{\rho}_\mu^q, \tilde{\rho}_\nu^q \leq \frac{C}{l(q)}$. Let $T_{\text{mon},q}$ be the monotone rearrangement between $\tilde{\mu}_q$ and $\tilde{\nu}_q$. Then $\tilde{\Gamma}_q$ is Borel and $\tilde{\Gamma}_q \subseteq \text{graph}(T_{\text{mon},q})$, by definition of T (see Lemma 4.2.3) and from the fact that $\Gamma \subseteq \text{graph}(T)$. Finally it remains to verify that for $\mathfrak{q}_c$ a.e. $q \in \mathcal{Q}$, calling $\tilde{\pi}_q := (Id, T_{\text{mon},q})_\sharp \tilde{\mu}_q$,

$$\tilde{\pi}_q(\tilde{\Gamma}_q) = 1. \tag{4.27}$$

To prove this we notice that by definition of μ_q

$$\begin{aligned}\mu_{|\mathcal{T}_\varphi^{nb}}((Id,T)^{-1}(\Gamma)) &= \int_{\mathcal{Q}} \mu_q((Id,T)^{-1}(\Gamma))\mathfrak{q}_c(\mathrm{d}q) = \int_{\mathcal{Q}} \mu_q((Id,T)^{-1}(\Gamma)\cap X_q)\mathfrak{q}_c(\mathrm{d}q)\\ &= \int_{\mathcal{Q}} \mu_q((Id,T)^{-1}(\Gamma\cap X_q\times X_q)\cap X_q)\mathfrak{q}_c(\mathrm{d}q) = \int_{\mathcal{Q}} \mu_q((Id,T)^{-1}(\Gamma_q)\cap X_q)\mathfrak{q}_c(\mathrm{d}q)\\ &= \int_{\mathcal{Q}} \mu_q((Id,T_{\mathrm{mon},X_q})^{-1}(\Gamma_q)\cap X_q)\mathfrak{q}_c(\mathrm{d}q) = \int_{\mathcal{Q}} \mu_q((Id,T_{\mathrm{mon},X_q})^{-1}(\Gamma_q))\mathfrak{q}_c(\mathrm{d}q).\end{aligned}$$

In addition by hypotheses

$$\mu_{|\mathcal{T}_\varphi^{nb}}((Id,T)^{-1}(\Gamma)) = \int_{\mathcal{Q}} \mu_q(X)\mathfrak{q}_c(\mathrm{d}q),$$

indeed

$$(Id,T)^{-1}(\Gamma)\cap\mathcal{T}_\varphi^{nb} = (Id,T)^{-1}\left(\Gamma\cap(\mathcal{T}_\varphi^{nb}\times X)\right),$$

and $\mu_q(X)=1$, $\mathfrak{q}_c$ a.e. $q\in\mathcal{Q}$. It follows that

$$\mu_q((Id,T_{\mathrm{mon},X_q})^{-1}(\Gamma_q)) = 1 \quad \mathfrak{q}_c \text{ a.e. } q\in\mathcal{Q},$$

and by definition of T_{mon,X_q},

$$\tilde{\mu}_q((Id,T_{\mathrm{mon},q})^{-1}(\tilde{\Gamma}_q)) = 1 \quad \mathfrak{q}_c \text{ a.e. } q\in\mathcal{Q}.$$

The last one is precisely (4.27).

Therefore we can apply Lemma 4.3.1 to obtain that for $\mathfrak{q}_c$ a.e. $q\in\mathcal{Q}$

$$\tilde{\mathfrak{m}}_q(T_t(\tilde{\Gamma}_q\cap(\tilde{A}_q\times I_q))\geq\frac{C_{K,M}}{C}l(q)\tilde{\mu}_q(\tilde{A}_q).$$

Notice that inequality

$$\tilde{\mathfrak{m}}_q(T_t(\tilde{\Gamma}_q\cap(\tilde{A}_q\times I_q))\geq\frac{C_{K,M}}{C}l(q)\tilde{\mu}_q(\tilde{A}_q)$$

is true also $\mathfrak{q}_{\mathfrak{m}}$ a.e. (recall that $\mathfrak{q}_c = l(q)\mathfrak{q}_{\mathfrak{m}}$) since if $l(q)=0$ it is trivially verified. We now observe that

$$T_t(\Gamma\cap(A\times X)\cap(X_q^\varphi\times X_q^\varphi))\cap X_q^\varphi = T_t(\Gamma\cap(A\times X)\cap(X_q^\varphi\times X_q^\varphi)),$$

and

$$T_t(\Gamma\cap(A\times X)\cap(X_q^\varphi\times X_q^\varphi))\cap X_q^\varphi = g(q,\cdot)(T_t(\tilde{\Gamma}_q\cap(\tilde{A}_q\times I_q))),$$

as a consequence of the definition of $\tilde{\Gamma}_q,\tilde{A}_q$ and of the fact that $g(q,\cdot)$ is an isometry. It follows that

$$\begin{aligned}\int_{\mathcal{Q}} \mathfrak{m}_q(T_t(\Gamma\cap(A\times X)\cap(X_q^\varphi\times X_q^\varphi))\cap X_q^\varphi)\mathfrak{q}_{\mathfrak{m}}(dq) &= \int_{\mathcal{Q}} \mathfrak{m}_q(g(q,\cdot)(T_t(\tilde{\Gamma}_q\cap(\tilde{A}_q\times I_q)))\mathfrak{q}_{\mathfrak{m}}(dq)\\ &= \int_{\mathcal{Q}} \tilde{\mathfrak{m}}_q((T_t(\tilde{\Gamma}_q\cap(\tilde{A}_q\times I_q)))\mathfrak{q}_{\mathfrak{m}}(dq) \geq \frac{C_{K,M}}{C}\int_{\mathcal{Q}} l(q)\tilde{\mu}_q(\tilde{A}_q)\mathfrak{q}_{\mathfrak{m}}(dq)\\ &= \frac{C_{K,M}}{C}\int_{\mathcal{Q}} l(q)\mu_q(g(q,\cdot)(\tilde{A}_q))\mathfrak{q}_{\mathfrak{m}}(dq) = \frac{C_{K,M}}{C}\int_{\mathcal{Q}} \mu_q(A_q)l(q)\mathfrak{q}_{\mathfrak{m}}(dq)\\ &= \frac{C_{K,M}}{C}\int_{\mathcal{Q}} \mu_q(A_q)\mathfrak{q}_c(dq) = \frac{C_{K,M}}{C}\mu_{|\mathcal{T}_\varphi^{nb}}(A).\end{aligned}$$

So what we get is

$$\mathfrak{m}(T_t(\Gamma \cap (A \cap \mathcal{T}_\varphi^{nb}) \times X)) \geq \frac{C_{K,M}}{C} \mu_{|\mathcal{T}_\varphi^{nb}}(A). \tag{4.28}$$

Now we get the analogous estimate for $\mathfrak{m}(T_t(\Gamma \cap (A \cap (\mathcal{T}_\varphi^{nb})^c) \times X))$. Set $D := \{(x,x) \colon x \in X\}$, we can observe that since $T = Id$ in $(\mathcal{T}_\varphi^{nb})^c$ and $\Gamma \subseteq \text{graph}(T)$,

$$T_t(\Gamma \cap (A \cap (\mathcal{T}_\varphi^{nb})^c) \times X) = T_t(\Gamma \cap D \cap (A \cap (\mathcal{T}_\varphi^{nb})^c \times X))$$

and so recalling that $\mu \leq C\mathfrak{m}$,

$$\begin{aligned} &\mathfrak{m}(T_t(\Gamma \cap (A \cap (\mathcal{T}_\varphi^{nb})^c) \times X)) = \mathfrak{m}(P_1(\Gamma) \cap (A \cap (\mathcal{T}_\varphi^{nb})^c) \\ &\geq \frac{1}{C}\mu(P_1(\Gamma) \cap (A \cap (\mathcal{T}_\varphi^{nb})^c)) = \frac{1}{C}\mu((A \cap (\mathcal{T}_\varphi^{nb})^c)). \end{aligned} \tag{4.29}$$

So by putting together (4.28) and (4.29) we get

$$\mathfrak{m}(T_t(\Gamma \cap (A \cap \times X))) \geq \frac{C_{K,M}}{C}\mu(A).$$

□

Remark 4.3.3. Under the same assumptions on the preceding Corollary, for $\hat{\Gamma} \subseteq \text{graph}(T)$, one has that

$$\mathfrak{m}(T_t(\hat{\Gamma} \cap A \times X)) \geq \frac{C_{K,M}}{C}\mu(P_1(\hat{\Gamma}) \cap A),$$

for any $t \in [0,1]$ and $A \in \mathcal{B}(X)$. Indeed

$$\hat{\Gamma} = (P_1(\hat{\Gamma}) \times X) \cap \text{graph}(T). \tag{4.30}$$

So from (4.25) one has

$$\mathfrak{m}(T_t(\hat{\Gamma} \cap A \times X)) = \mathfrak{m}(T_t(\text{graph}(T) \cap (A \cap P_1(\hat{\Gamma}) \times X)) \geq \frac{C_{K,M}}{C}\mu(P_1(\hat{\Gamma}) \cap A).$$

We are finally ready for the last step which is the main result of this section. We prove that in the infinite product space we have a positive evolution constructed through an optimal plan which is limit of good approximating plans. These plans lie on finite products spaces and are given by Corollary 4.3.2.

Proposition 4.3.4. *Let $(X, \mathsf{d}, \mathfrak{m})$ be a $\mathsf{CD}(K,\infty)$ space as in 4.2. Let $\mu, \nu \in \mathcal{P}(X)$ be two probability measures absolutely continuous with respect to $\mathfrak{m}$: $\mu = \rho_\mu \mathfrak{m}$ and $\nu = \rho_\nu \mathfrak{m}$, with ρ_μ, $\rho_\nu : X \to \mathbb{R}$, ρ_μ, $\rho_\nu \leq C$ on their supports.*

Then there exists $\pi_\infty \in \mathcal{P}(X \times X)$ such that for every $\Gamma_\infty \subseteq X \times X$ Borel such that $\pi_\infty(\Gamma_\infty) = 1$, every closed set $A \subseteq X$, for any $t \in [0,1]$ it holds

$$\mathfrak{m}(T_t(\Gamma_\infty \cap A \times X)) \geq \frac{C_{K,M}}{C}\mu(A), \tag{4.31}$$

where M is a constant such that $M \geq \sup\{d(x,y) \colon x \in \text{supp}(\mu), y \in \text{supp}(\nu)\}$, T_t is the possibly multivalued midpoint map defined as $T_t : X \times X \to X$, $T_t(x,y) := \{z \colon \mathsf{d}(x,z) = (1-t)\mathsf{d}(x,y), \mathsf{d}(y,z) = t\mathsf{d}(x,y)\}$ and

$$C_{K,M} := \begin{cases} 1 & K \geq 0, \\ e^{\frac{K}{8}M^2} & K < 0. \end{cases}$$

Proof. **Step 0**
Let $\mu_n := (P_n)\sharp\mu$, $\nu_n := (P_n)\sharp\nu$, $\mathfrak{m}_n := (P_n)\sharp\mathfrak{m}$. Then by Proposition 4.1.8, $\mu_n \leq C\mathfrak{m}_n$, $\nu_n \leq C\mathfrak{m}_n$. Recall from Lemma 4.1.2 that the space $(X_n = P_n(X), \mathfrak{m}_n, \mathsf{d})$ is a $\mathsf{CD}(K, M_n)$ non-branching m.m.s. Hence, for any n, we are in the hypotheses of Corollary 4.3.2, and we call $\pi_n = (Id, T_n)_\sharp\mu_n \in \mathrm{Opt}_1(\mu_n, \nu_n)$ the one given by the statement. By Proposition 4.1.8 up to a subsequence π_n converges to a $\pi_\infty \in \mathrm{Opt}_1(\mu, \nu)$. Let Γ_∞ be such that $\pi_\infty(\Gamma_\infty) = 1$.

Step 1
We fix $\delta > 0$. Let $\Gamma_\delta \subseteq \Gamma_\infty$ be a compact set satisfying $\pi_\infty(\Gamma_\delta) \geq 1 - \delta$. Let Γ_n be $\mathrm{supp}(\pi_n)$. We want to approximate Γ_δ with subsets of Γ_n. For any fixed $\varepsilon > 0$, we construct a subsequence n_d, compact sets $\{\Gamma_{n_d,\varepsilon}\}_{d\in\mathbb{N}}$ and Γ_ε satisfying:

1. $\Gamma_{n_d,\varepsilon} \subseteq \Gamma_{n_d}$ and $\pi_{n_d}(\Gamma_{n_d,\varepsilon}) \geq 1 - \varepsilon - \delta$;
2. $\Gamma_{n_d,\varepsilon}$ converges Hausdorff to Γ_ε for $d \to +\infty$;
3. $\Gamma_\varepsilon \subseteq \Gamma_\delta$, $\pi_\infty(\Gamma_\varepsilon) \geq 1 - \varepsilon - \delta$.

We can proceed as follows. For any $d \in \mathbb{N}$ we consider the $\frac{1}{d}$-enlargement of Γ_δ, namely $(\Gamma_\delta)^{\frac{1}{d}}$ which is an open set. Thanks to the lower semicontinuity of the weak convergence for open sets one has that

$$\liminf_{n\to+\infty} \pi_n((\Gamma_\delta)^{\frac{1}{d}}) \geq \pi_\infty((\Gamma_\delta)^{\frac{1}{d}}) \geq \pi_\infty(\Gamma_\delta) \geq 1 - \delta;$$

so in particular for any fixed $\varepsilon > 0$ there exists $n_d = n_d(\varepsilon)$ depending on ε such that

$$\pi_{n_d}\left(cl((\Gamma_\delta)^{\frac{1}{d}})\right) \geq \pi_{n_d}((\Gamma_\delta)^{\frac{1}{d}}) \geq 1 - \varepsilon - \delta;$$

where $cl((\Gamma_\delta)^{\frac{1}{d}})$ is the closure of $(\Gamma_\delta)^{\frac{1}{d}}$.
Then we can define the compact set $\Gamma_{n_d,\varepsilon} := cl((\Gamma_\delta)^{\frac{1}{d}}) \cap \Gamma_{n_d}$. Point (1) clearly holds true. The convergence in (2) follows up to subsequences from the fact that they are all closed sets of a compact space (recall Theorem 1.1.3). We call Γ_ε the limit. To prove point (3) we observe that given $x \in \Gamma_\varepsilon$ then $x \in \bigcap_{d\in\mathbb{N}} cl((\Gamma_\delta)^{\frac{1}{d}}) = \Gamma_\delta$. The fact that $\pi_\infty(\Gamma_\varepsilon) \geq 1 - \varepsilon - \delta$ follows from Lemma 1.1.4. From now on the sequence $\{n_d\}_{d\in\mathbb{N}}$ will be denoted for convenience only by $\{n\}_{n\in\mathbb{N}}$.

Step 2 Let $A \subseteq X$ be a cylinder with base B, with B closed subset of $X_{\bar{n}}$ for some $\bar{n} \in \mathbb{N}$: $A = C(B)$ of positive μ measure. We claim that the family $\{T_t(\Gamma_{n,\varepsilon} \cap A \times X)\}_{d\in\mathbb{N}}$ converges up to subsequences for $n \to +\infty$ in the Hausdorff distance to a subset C_ε of $T_t(\Gamma_\varepsilon \cap A \times X)$. To prove the claim we first observe that for any n, $T_t(\Gamma_{\varepsilon,n} \cap A \times X)$ is closed. Let $\{x_k\}_k$ be a sequence of points in $T_t(\Gamma_{n,\varepsilon} \cap A \times X)$ converging to some c. Then each x_k satisfies

$$\mathsf{d}(x_k, a_k) = t\mathsf{d}(a_k, b_k), \tag{4.32}$$
$$\mathsf{d}(x_k, b_k) = (1-t)\mathsf{d}(a_k, b_k), \tag{4.33}$$

for some $a_k \in P_1(\Gamma_{n,\varepsilon}) \cap A$ and $b_k \in P_2(\Gamma_{n,\varepsilon})$ compact sets. So up to subsequences $\lim_{k\to+\infty} a_k = a \in P_1(\Gamma_{n,\varepsilon}) \cap A$ and $\lim_{k\to+\infty} b_k = b \in P_2(\Gamma_{n,\varepsilon})$. So passing to the limit in (4.32) and (4.33) we have

$$\mathsf{d}(a, c)) = t\, \mathsf{d}(a, b),$$
$$\mathsf{d}(b, c)) = (1-t)\, \mathsf{d}(a, b).$$

Hence $\lim_{k\to+\infty} x_k = c \in T_t(\Gamma_{n,\varepsilon} \cap A \times X)$. Since $\{T_t(\Gamma_{n,\varepsilon} \cap A \times X)\}_n$ are closed in a compact space, they converge up to subsequences for $n \to +\infty$ in the Hausdorff distance to a closed set C_ε. It remains to show that C_ε is a subset of $T_t(\Gamma_\varepsilon \cap A \times X)$. Let c be a limit point of points $c_n \in T_t(\Gamma_{n,\varepsilon} \cap A \times X)$. As before

$$\mathsf{d}(c_n, a_n) = t\mathsf{d}(a_n, b_n), \tag{4.34}$$
$$\mathsf{d}(c_n, b_n) = (1-t)\mathsf{d}(a_n, b_n), \tag{4.35}$$

for some $(a_n, b_n) \in \Gamma_{n,\varepsilon} \cap A \times X$. From compactness of the space $\{(a_n, b_n)\}_n$ converges up to subsequences to (a, b), and $(a, b) \in \Gamma_\varepsilon \cap A \times X$ since $\Gamma_{n,\varepsilon} \cap A \times X$ Hausdorff converges to $\Gamma_\varepsilon \cap A \times X$. So by passing to the limit in (4.34) and (4.35) we have that

$$\begin{aligned} \mathsf{d}(a, c)) &= t\, \mathsf{d}(a, b), \\ \mathsf{d}(b, c)) &= (1-t)\, \mathsf{d}(a, b). \end{aligned}$$

Therefore $c \in T_t(\Gamma_\varepsilon \cap A \times X)$.

Step 3 Note that as above the set $T_t(\Gamma_\varepsilon \cap A \times X)$ is closed since Γ_ε and A are both closed, in particular it is $\mathfrak{m}$ measurable. It follows that

$$\begin{aligned} \mathfrak{m}(T_t(\Gamma_\varepsilon \cap A \times X)) &\geq \mathfrak{m}(C_\varepsilon) \geq \limsup_{n\to+\infty} \mathfrak{m}_n(T_t(\Gamma_{n,\varepsilon} \cap A \times X)) \\ &\geq \frac{C_{M,K}}{C} \limsup_{n\to+\infty} \mu_n(P_1(\Gamma_{n,\varepsilon} \cap \operatorname{graph}(T_n)) \cap A) \\ &\geq \frac{C_{M,K}}{C} \left(\limsup_{n\to+\infty} \mu_n(A) - \varepsilon - \delta \right) = \frac{C_{M,K}}{C} \left(\mu(A) - \varepsilon - \delta\right); \end{aligned}$$

where the second inequality follows from Lemma 1.1.4, the third inequality follows from Remark 4.3.3 and the fourth inequality from (2) of Step 1. The last equality is a consequence of the fact that $A = C(B)$ with $B \subseteq X_{\bar{n}}$ which implies that $\mu_n(A) = (P_n)_\sharp \mu(A) = \mu(P_n^{-1}(A)) = A$ for any $n \geq \bar{n}$.

Step 4 Let A be any compact set. We consider the following sequence of sets $A_n = C(P_n(A))$. We observe that A_n satisfies the hypotheses of the previous step. In addition A_n is a decreasing sequence of sets $A_n \supseteq A_{n+1} \supseteq A$ with

$$\cap_{n\in\mathbb{N}} A_n = A. \tag{4.36}$$

The inclusion $\cap_{n\in\mathbb{N}} A_n \supset A$ is trivial. For the other inclusion, let x be in $\cap_{n\in\mathbb{N}} A_n$. Then for any n

$$\mathsf{d}(x, A) \leq \mathsf{d}(x, P_n(x)) + \mathsf{d}(P_n(x), P_n(A)) + \mathsf{d}(P_n(A), A). \tag{4.37}$$

First we observe that $\lim_{n\to+\infty} P_n(x) = x$ which implies that the first and the third terms go to zero for $n \to +\infty$. In addition we observe that by the definition of A_n, $P_{\tilde{n}}(\cap_{n\in\mathbb{N}} A_n) = P_{\tilde{n}}(A)$ for any $\tilde{n}$, therefore $P_{\tilde{n}}(x) \in P_{\tilde{n}}(A)$, which implies that the second term is zero for any n. Therefore by passing to the limit in (4.37) we see that $x \in A$ since A is closed. For any n, by the previous step we have

$$\mathfrak{m}(T_t(\Gamma_\varepsilon \cap A_n \times X)) \geq \frac{C_{M,K}}{C} \left(\mu(A_n) - \varepsilon - \delta\right) \geq \frac{C_{M,K}}{C} \left(\mu(A) - \varepsilon - \delta\right). \tag{4.38}$$

Finally we notice that also $\{T_t(\Gamma_\varepsilon \cap A_n \times X)\}_n$ is a decreasing sequence and

$$\cap_{n\in\mathbb{N}} T_t(\Gamma_\varepsilon \cap A_n \times X) = T_t(\Gamma_\varepsilon \cap A \times X).$$

To prove this last equality, we observe that $\supseteq$ is trivial being $A \subseteq A_n$ for any n. For the other inclusion, we take a point $x \in \cap_{n\in\mathbb{N}} T_t(\Gamma_\varepsilon \cap A_n \times X)$, then there exists $(a_n, b_n) \in \Gamma_\varepsilon$ with $a_n \in A_n$ such that

$$\mathsf{d}(x, a_n) = t\mathsf{d}(a_n, b_n), \tag{4.39}$$

$$\mathsf{d}(x, b_n) = (1-t)\mathsf{d}(a_n, b_n). \tag{4.40}$$

Up to subsequences $\lim_{n\to+\infty} a_n = a$, $\lim_{n\to+\infty} b_n = b$ with $(a,b) \in \Gamma_\varepsilon$ (since Γ_ε is closed) and $a \in A$ from (4.36) and the closedness of A. Therefore $x \in T_t(\Gamma_\varepsilon \cap A \times X)$ by passing to the limit in (4.39).

Passing to the limit for $n \to +\infty$ in (4.38) we get

$$\mathfrak{m}(T_t(\Gamma_\varepsilon \cap A \times X))) = \lim_{n\to+\infty} \mathfrak{m}(T_t(\Gamma_\varepsilon \cap A_n \times X)) \geq \frac{C_{M,K}}{C}\left(\mu(A) - \varepsilon - \delta\right).$$

Since $\Gamma_\varepsilon \subseteq \Gamma_\delta \subseteq \Gamma_\infty$, we get

$$\mathfrak{m}(T_t(\Gamma_\infty \cap A \times X)) \geq \frac{C_{M,K}}{C}\left(\mu(A) - \varepsilon - \delta\right),$$

for any $\varepsilon > 0$ and $\delta > 0$, which gives (4.31) letting ε and δ go to 0 . $\square$

4.3.2 Regularity of the conditional measures

In this section we use the positive evolution built in the previous section to prove that if φ is a Kantorovich potential between two probability measures μ and ν, and we consider the partition of the transport set induced by φ, then the set of initial and final points of all transport rays has measure zero.

Lemma 4.3.5. *Let $(X, \mathsf{d}, \mathfrak{m})$ be as in (4.2). Let $\mu, \nu \in \mathcal{P}(X)$ be two probability measures absolutely continuous with respect to $\mathfrak{m}$: $\mu = \rho_\mu \mathfrak{m}$ and $\nu = \rho_\nu \mathfrak{m}$, with $\rho_\mu, \rho_\nu : X \to \mathbb{R}$, $\rho_\mu, \rho_\nu \leq C$ and $\mu \perp \nu$. Let φ be a Kantorovich potential associated to the L^1 optimal transport problem of transporting μ into ν. Then $\mu(\mathfrak{a}_\varphi) = 0$.*

Proof. Assume by contradiction that there exists $A \subseteq \mathfrak{a}_\varphi$ Borel of positive μ measure. Then by inner regularity of μ there exist $\delta > 0$ and a closed set $\hat{A} \subseteq A$ of positive μ measure with $\rho_\mu \geq \delta$ $\mathfrak{m}$-a.e. in $\hat{A}$. Let $\pi_\infty \in \mathcal{P}(X \times X)$ be given in Proposition 4.3.4. Let Γ be a Borel set such that $\pi_\infty(\Gamma) = 1$, $\Gamma \subseteq \partial^c\varphi = \Gamma_\varphi$, where Γ_φ is defined in Section 1.5.2 (such a Γ exists thanks to (1.6)) and $\Gamma \cap D = \emptyset$, indeed $\pi(\Gamma \setminus D) = \pi(\Gamma)$ since $\pi(D) = 0$. By Proposition 4.3.4 one has

$$\mathfrak{m}(T_t(\Gamma \cap \hat{A} \times X)) \geq C\mu(\hat{A}),$$

with T_t the t midpoint map defined in Proposition 4.3.4 and C independent on t. Call $\hat{A}_t := T_t(\Gamma \cap \hat{A} \times X)$. Then $\hat{A}_t \cap \hat{A} = \emptyset$ for any $t > 0$. Indeed let $x \in \hat{A}_t$, by construction there exists a couple of distinct points $(x_0, x_1) \in \Gamma \subseteq \Gamma_\varphi$ such that $\mathsf{d}(x_0, x) = t\mathsf{d}(x_0, x_1)$ and $\mathsf{d}(x_1, x) = (1-t)\mathsf{d}(x_0, x_1)$ since $\Gamma \cap D = \emptyset$. It follows by Lemma 1.5.12 (since the space is geodesic) that $(x_0, x) \in \Gamma_\varphi$. Then $x \notin \mathfrak{a}_\varphi$. Fix $\varepsilon > 0$, then for any $t < \frac{\varepsilon}{\operatorname{diam}(X)}$, $\hat{A}_t \subseteq \hat{A}^\varepsilon$ where $\hat{A}^\varepsilon$ is the ε-enlargement of $\hat{A}$. So

$$\mathfrak{m}(\hat{A}^\varepsilon) \geq \mathfrak{m}(\hat{A}) + \mathfrak{m}(\hat{A}_t) \geq \mathfrak{m}(\hat{A}) + C\mu(\hat{A}) \geq (1 + C\delta)\mathfrak{m}(\hat{A}),$$

which gives a contradiction by sending ε to 0 and observing that $\lim_{\varepsilon\to0} \mathfrak{m}(\hat{A}^\varepsilon) = \mathfrak{m}(\hat{A})$ being $\hat{A}$ closed. $\square$

Another consequence of the construction of the positive evolution is that we can prove that the conditional measures are non atomic.

Lemma 4.3.6. *Let $(X, \mathsf{d}, \mathfrak{m})$ be as in (4.2). Let $\mu, \nu \in \mathcal{P}(X)$ be two probability measures absolutely continuous with respect to $\mathfrak{m}$: $\mu = \rho_\mu \mathfrak{m}$ and $\nu = \rho_\nu \mathfrak{m}$, with ρ_μ, $\rho_\nu : X \to \mathbb{R}$, ρ_μ, $\rho_\nu \leq C$ and $\mu \perp \nu$. Let φ be a Kantorovich potential associated to it. Let $\mathcal{T}_\varphi^{nb}$ be the set defined in Section 1.5.2. Consider the disintegration of $\mu_{|\mathcal{T}_\varphi^{nb}}$ given by Proposition 1.5.18*

$$\mu_{|\mathcal{T}_\varphi^{nb}} = \int_{\mathcal{Q}} \mu_q \mathfrak{q}_\mu(dq). \tag{4.41}$$

For $\mathfrak{q}$ a.e. $q \in \mathcal{Q}$ the measure μ_q has no atoms.

We recall here two results of Lusin that are useful in the next proof: the first one is the classical *Lusin theorem*, the second one is a result on Borel sets (see [87, Theorem 5.8.11] for more details).

Theorem 4.3.7 (Lusin Theorem)**.** *Let (X, τ_X) be a topological space with a finite Borel measure $\mathfrak{m}$. Let (Y, τ_Y) be a second countable topological space. Let $f : X \to Y$ be a measurable function. Then for any $\varepsilon > 0$ there exists a compact $K \subseteq X$ such that $\mathfrak{m}(X \setminus K) < \varepsilon$ and $f : K \to Y$ is continuous.*

Theorem 4.3.8. *Let X and Y be two Polish spaces. Let $B \subset X \times Y$ be a Borel set. Assume that for any $x \in P_1(B)$ the x-section of B, $B(x) := \{y : (x, y) \in B\}$, is countable, then B is a countable union of Borel graphs.*

Lemma 4.3.9. *Consider $(\mathcal{Q}, \mathcal{B}(\mathcal{Q}))$ a measure space , $(X, \mathcal{B}(X))$. Let $q \mapsto \mathfrak{m}_q \in \mathcal{P}(X)$ be a map. Assume that for any $B \in \mathcal{B}(X)$, $q \mapsto \mathfrak{m}_q(B)$ is a Borel map $\mathcal{Q} \to \mathbb{R}$. Then the map*

$$\begin{aligned} (\mathcal{Q}, \sigma(\mathcal{Q}), \mathfrak{q}_\mu) &\to (\mathcal{P}(X), \tau_w) \\ q &\mapsto \mu_q \end{aligned}$$

is Borel.

Proof. From the fact that $q \mapsto \mathfrak{m}_q(B)$ is Borel for any $q \in \mathcal{Q}$ it follows that for any $\varphi \in C_b(X)$, the map $q \mapsto \int \varphi \, \mathfrak{m}_q$ is Borel. Then we can fix $\bar{f} \in C_b(X)$ and $\nu \in \mathcal{P}(X)$ and $\varepsilon > 0$. The set

$$\left\{ q \colon \left| \int \bar{f} \nu - \int \bar{f} \mu_q \right| < \varepsilon \right\}$$

is Borel. It follows by using the definition of open neighbourhood for the weak topology that the preimage of open neighbourhoods is Borel, from which we can conclude that the map is Borel. □

Proof of Lemma 4.3.6. We consider the two following spaces $(\mathcal{Q}, \sigma(\mathcal{Q}), \mathfrak{q}_\mu)$ and $(\mathcal{P}(X), \tau_w)$. Then as explained in Remark 1.5.19 $\mathfrak{q}_\mu$ is a Borel probability measure on $\mathcal{Q}$ and as a consequence of 1.1.1, the space $(\mathcal{P}(X), \tau_w)$ is second countable. We observe in addition that by the Definition 1.5.1, point (1) we know that for any Borel set $B \in \mathcal{B}(X)$ the map $q \mapsto \mu_q(B)$ is Borel and so as said in Lemma 4.3.9 the map

$$\begin{aligned} (\mathcal{Q}, \sigma(\mathcal{Q}), \mathfrak{q}_\mu) &\to (\mathcal{P}(X), \tau_w) \\ q &\mapsto \mu_q \end{aligned}$$

is Borel. Therefore we can apply Lusin Theorem 4.3.7. In particular for any $n \in \mathbb{N}$ we have a compact set $Q_n \subseteq \mathcal{Q}$ such that $\mathfrak{q}(\mathcal{Q} \setminus Q_n) \leq \frac{1}{n}$ and $q \mapsto \mu_q$ is continuous on Q_n. We define

$$A_n := \{(q,x) \colon q \in Q_n,\, x \in \mathcal{T}_\varphi^{nb},\, \mu_q(\{x\}) > 0\},$$

and we call $A_n(q) := \{x \colon (q,x) \in A_n\}$, the q-section of A_n. We claim that $\mu(P_2(A_n)) = 0$. If the claim is true then

$$0 = \mu(P_2(A_n)) = \int_Q \mu_q(P_2(A_n))\mathfrak{q}_\mu(\mathrm{d}q) = \int_{Q_n} \mu_q(A_n(q))\mathfrak{q}_\mu(\mathrm{d}q),$$

and so $\mu_q(A_n(q)) = 0$ $\mathfrak{q}_\mu$ a.e. $q \in \mathcal{Q}_n$ which means that $\mathfrak{q}_\mu$ a.e. $q \in \mathcal{Q}_n$, $\mathrm{card}(A_n(q)) = 0$. If the claim is true for any n we have that $\mathfrak{q}$ a.e. $q \in \cup_{n\in\mathbb{N}}\mathcal{Q}_n$, μ_q has no atoms and $\mathfrak{q}_\mu(\mathcal{Q} \setminus \cup_{n\in\mathbb{N}}\mathcal{Q}_n) = 0$.
We prove the claim for a fixed $n \in \mathbb{N}$. We first observe that A_n is a σ closed set and therefore Borel. Indeed $A_n = \cup_{k\in\mathbb{N}} A_{n,k}$ with

$$A_{n,k} := \{(q,x) \colon q \in Q_n,\, x \in X,\, \mu_q(\{x\}) \geq 2^{-k}\}.$$

$A_{n,k}$ is closed indeed if $\{(q_j,x_j)\}_{j\in\mathbb{N}}$ is a sequence in $A_{n,k}$ converging to a (q,x), (in the product topology on $\mathcal{Q} \times X$ induced by (Q_n, d), (X, d)) then $q \in Q_n$ since Q_n is compact, and by Lemma 1.1.4

$$\mu_q(\{x\}) \geq \limsup_{j\to+\infty} \mu_{q_j}(\{x_j\}) \geq 2^{-k}.$$

In addition $A_n(q)$ is countable since μ_q is a finite measure. So we can apply Theorem 4.3.8 to (Q_n, d), (X, d), A_n and we get that $A_n = \cup_{j\in\mathbb{N}} G_j$ with G_j Borel graph. Now we assume by contradiction that there exists $\bar{j}$ such that $\mu(P_2(G_{\bar{j}})) > 0$. Then by disintegration formula (4.41)

$$0 < \mu(P_2(G_{\bar{j}})) = \int_{\mathcal{Q}} \mu_q(P_2(G_{\bar{j}}))\mathfrak{q}_\mu(\mathrm{d}q) = \int_{P_1(G_{\bar{j}})} \mu_q(P_2(G_{\bar{j}}))\mathfrak{q}_\mu(\mathrm{d}q),$$

and so $\mathfrak{q}_\mu(P_1(G_{\bar{j}})) > 0$. Again by inner regularity there exists $\hat{A} \subseteq P_2(G_{\bar{j}})$ such that $\mu(\hat{A}) > 0$ and $\hat{A} \subseteq \{\rho_\mu > \delta\}$ for a positive δ. Let π_∞ be a plan as in Proposition 4.3.4 and Γ be a Borel set such that $\pi_\infty(\Gamma) = 1$, $\Gamma \subseteq \Gamma_\varphi$ (see Section 1.5.2 for the definition) and $\Gamma \cap D = \emptyset$. Then for $t < \frac{\varepsilon}{\mathrm{diam}\,(X)}$ we have that $T_t(\Gamma \cap \hat{A} \times X) \subseteq \hat{A}^\varepsilon$. Moreover $T_t(\Gamma \cap \hat{A} \times X) \cap \hat{A} = \emptyset$. Indeed suppose that $y \in T_t(\Gamma \cap \hat{A} \times X) \cap \hat{A}$, then y belongs to a non trivial (since $\Gamma \cap D = \emptyset$) geodesic γ such that $(\gamma_0, \gamma_1) \in \Gamma \subseteq \Gamma_\varphi$ and $\gamma(0) \in \hat{A}$. Then by Lemma 1.5.12 $(\gamma(0), y) \in \Gamma_\varphi$. This implies that $\gamma(0)$ and y belong to the same X_q^φ equivalence class for some $q \in \mathcal{Q}$ and this is a contradiction since it implies that $(q, \gamma(0))$, $(q, y) \in G_{\bar{j}}$, which is a graph. Therefore

$$\mathfrak{m}(\hat{A}^\varepsilon) \geq \mathfrak{m}(\hat{A}) + \mathfrak{m}(T_t(\Gamma \cap \hat{A} \times X)) \geq \frac{C_{K,M}}{C}\mu(\hat{A}) + \mathfrak{m}(\hat{A}) \geq \left(\frac{C_{K,M}}{C} + \delta\right)\mathfrak{m}(\hat{A}),$$

by using Proposition 4.3.4. Since the inequality holds for any $\varepsilon > 0$ we get a contradiction. $\square$

In the next proposition we sum up the regularity for μ and prove the analogous for ν.

Proposition 4.3.10. *Let $(X, \mathsf{d}, \mathfrak{m})$ be as in (4.2). Let $\mu, \nu \in \mathcal{P}(X)$ be two probability measures absolutely continuous with respect to $\mathfrak{m}$: $\mu = \rho_\mu \mathfrak{m}$ and $\nu = \rho_\nu \mathfrak{m}$, with ρ_μ, ρ_ν : $X \to \mathbb{R}$, ρ_μ, $\rho_\nu \leq C$ and $\mu \perp \nu$. Let φ be a Kantorovich potential associated to it. Assume that $\mu(\mathfrak{b}_\varphi) = 0$. Let $\mathcal{T}_\varphi^{nb}$ be the set defined in Section 1.5.2. Let in addition $\mathcal{Q}$ be a Borel section and $f : (\mathcal{T}_\varphi^{nb}, \mathcal{B}(\mathcal{T}_\varphi^{nb})) \to (Q, \sigma(Q))$ be a Borel quotient map.*

Set $\mathfrak{q}_c := f_\sharp \mu = f_\sharp \mu$ and consider the disintegrations $q \mapsto \mu_q$, $q \mapsto \nu_q$, of $\mu_{|\mathcal{T}_\varphi^{nb}}$ and $\nu_{|\mathcal{T}_\varphi^{nb}}$ respectively:

$$\mu_{|\mathcal{T}_\varphi^{nb}} = \int_{\mathcal{Q}} \mu_q \mathfrak{q}_c(dq), \quad \nu_{|\mathcal{T}_\varphi^{nb}} = \int_{\mathcal{Q}} \nu_q \mathfrak{q}_c(dq). \tag{4.42}$$

Then

1. *$\mu(\mathfrak{a}_\varphi) = 0$;*
2. *$\nu(\mathfrak{b}_\varphi) = 0$;*
3. *for $\mathfrak{q}_c$ a.e. $q \in \mathcal{Q}$, the marginal distributions μ_q and ν_q have no atoms.*

Proof. Point 1 follows immediately from Lemma 4.3.5. To show point 2 we know that since φ is a Kantorovich potenstial for the L^1 optimal transport problem between μ and ν, then $-\varphi$ s kantorovich potential for the problem between ν and μ. In addition $\Gamma_{-\varphi} = \Gamma_\varphi^{-1}$. From which it follows that $\mathfrak{a}_{-\varphi} = \mathfrak{b}_\varphi$. Therefore we can apply Lemma 4.3.5 with swapped measures and $-\varphi$ Kantorovich potential. We get that $\nu(a_{-\varphi}) = 0$ which is precisely point 2. Analogously 3 follows by applying Lemma 4.3.6 for μ_q by considering as Kantorovich potential φ for ν_q by considering as Kantorovich potential $-\varphi$. □

4.4 Solution to the Monge problem

We finally prove that the Monge problem between two mutually singular absolutely continuous and bounded measures in our $(X, \mathsf{d}, \mathfrak{m})$ $\mathsf{CD}(K, \infty)$ product space has a solution. In the following with $\mu \perp \nu$ we mean that μ and ν are mutually singular.

Theorem 4.4.1. *Let $(X, \mathsf{d}, \mathfrak{m})$ be a m.m.s as in (4.2). Let $\mu, \nu \in \mathcal{P}(X)$ be two probability measures absolutely continuous with respect to $\mathfrak{m}$: $\mu = \rho_\mu \mathfrak{m}$ and $\nu = \rho_\nu \mathfrak{m}$, with ρ_μ, ρ_ν : $X \to \mathbb{R}$, ρ_μ, $\rho_\nu \leq C$ and $\mu \perp \nu$. Then there exists a solution to the Monge problem (MP), i.e. a Borel map $T : X \to X$ such that $T_\sharp \mu = \nu$ and*

$$\int_X \mathsf{d}(x, T(x)) \mu(\mathrm{d}x) = \inf_{T_\sharp \mu = \nu} \int_X \mathsf{d}(x, T(x)) \mu(dx).$$

Proof. To prove the Theorem we need to verify that the hypotheses of Lemma 4.2.3 are satisfied. We observe that our space is compact geodesic and non-branching by construction as shown in Proposition 4.1.1. μ, ν are in $\mathcal{P}_1(X)$ since the space is bounded. The existence of a Kantorovich potential φ is guaranteed (see Remark 1.2.3). Hypothesis (1) is satisfied: in Proposition 4.3.10 we prove that $\mu(\mathfrak{a}_\varphi) = 0$ and since the space is non-branching, as shown in Remark 1.5.7 we have that $A^+ \subseteq \mathfrak{a}_\varphi$ which gives $\mu(A^+) \leq \mu(\mathfrak{a}_\varphi) = 0$. Analogously $\nu(A^-) \leq \nu(\mathfrak{b}_\varphi) = 0$. Hypothesis (2) is satisfied as proven in Proposition 4.3.10. Therefore we can apply the Lemma and we get the existence of the desired map. □

Appendix A

Wasserstein-Hellinger inequality: the case $p > 2$

We remark that what is present in this appendix is obtained in collaboration with Nicolò De Ponti and Luca Tamanini and will be present in a joint future work.

Our result shows that the regularizing effect of the heat semigroup H_t allows to control the stronger p-Hellinger distance in terms of the weaker p-Wasserstein distance. We recall that in the case $p \in [1,2]$ it has been firstly proved in [74, Theorem 5.2].

As short preliminaries we first recall two properties of the heat flow evolution that we use in the sequel and are valid in the setting of $\mathsf{RCD}(K,\infty)$ spaces. Given $f \in L^\infty(X)$, since $H_t f \in \mathrm{Lip}_b(X)$ (recall Section 1.4.2), its slope $|DH_t f|$ is well defined and for all $t > 0$ (see [11, Theorem 3.17]),

$$|DH_t f|_w = |DH_t f| \quad \mathfrak{m}\,\text{a.e.} \ \in X. \tag{A.1}$$

Additionally the following improved version of the Bakry-Emery inequality (recall [11]) proved in [83, Corollary 3.5] holds:

$$|DH_t f|_w^{2\alpha} \le e^{-2\alpha K t} H_t(|Df|_t^{2\alpha}), \quad \mathfrak{m}\text{-a.e.}, \tag{A.2}$$

for any $f \in W^{1,2}(X)$ and $\alpha \in [\frac{1}{2}, 1]$.

To state the first result, we recall the following definition:

$$R_K(t) := \begin{cases} \frac{e^{2Kt}-1}{K} & \text{if } K \neq 0, \\ 2t & \text{if } K = 0. \end{cases} \tag{A.3}$$

Proposition A.1. *Let $(X,\mathsf{d},\mathfrak{m})$ be an $\mathsf{RCD}(K,\infty)$ metric measure space with $\mathfrak{m}(X) < +\infty$, $K \in \mathbb{R}$. For $p > 2$ and $\mu_0, \mu_1 \in \mathcal{P}_p(X)$ it holds*

$$\frac{p}{\sqrt{p-1}}(R_K(t))^{\frac{1}{2}} \mathsf{He}_p(H_t^*\mu_0, H_t^*\mu_1) \le W_p(\mu_0,\mu_1) \quad \forall\, t > 0. \tag{A.4}$$

To prove Proposition A.1 we take advantage of a new functional inequality for the heat flow.

Proposition A.2. *Let $(X,\mathsf{d},\mathfrak{m})$ be an $\mathsf{RCD}(K,\infty)$ metric measure space, $K \in \mathbb{R}$ and $\mathfrak{m}(X) < +\infty$. Then for every $1 < q < 2$ and $f \in L^\infty(X)$ it holds*

$$(q-1) R_K(t) |DH_t(f)|_w^2 \le (H_t(|f|^q))^{\frac{2}{q}} - (H_t(f))^2, \quad \mathfrak{m}\text{-a.e. in } X, \text{ for any } t > 0. \tag{A.5}$$

The strategy of the proof of the previous inequality is based on a classical semigroup interpolation argument which has been extensively used in the last decades, (see e.g. [17, 18]). We follow [49, Proposition 3.1] in which similar results have been obtained also in the context of RCD spaces, where some additional technical arguments are needed to perform the computations (see also [11, Corollary 2.3]).

Proof of Proposition A.2. We approximate the function $t \mapsto |t|^q$, which second derivative is not defined a $t = 0$, with the following function

$$\phi^q_\varepsilon(t) := (t^2 + \varepsilon^2)^{\frac{q}{2}} - \varepsilon^q,$$

which is $\mathcal{C}^\infty(\mathbb{R})$, $\phi^q_\varepsilon(0) = 0$. We consider the map $s \mapsto (H_s(\phi^q_\varepsilon(H_{t-s}f)))^{\frac{2}{q}}$ and we have that it is locally Lipschitz from $(0, t)$ to L^2. This follows from the very definition of heat flow, that is a locally Lipschitz map from $(0, +\infty)$ to L^2 and the fact that its extension to $L^p(X)$ preserves this property, together with the fact that ϕ^q_ε is smooth. So in particular using the fundamental theorem of calculus for the Bochner integral in the space $L^2(X)$ and computing the derivative inside the integral, we have that for any fixed and sufficiently small $\delta > 0$ one has

$$\begin{aligned}
&(H_{t-\delta}(\phi^q_\varepsilon(H_\delta f)))^{\frac{2}{q}} - (H_\delta(\phi^q_\varepsilon(H_{t-\delta}f)))^{\frac{2}{q}} = \int_\delta^{t-\delta} \frac{d}{\mathrm{d}s}(H_s(\phi^q_\varepsilon(H_{t-s}f)))^{\frac{2}{q}}\,\mathrm{d}s \\
&= \frac{2}{q}\int_\delta^{t-\delta} (H_s(\phi^q_\varepsilon(H_{t-s}f)))^{\frac{2}{q}-1}\Big[\Delta H_s(\phi^q_\varepsilon(H_{t-s}f)) + H_s((\phi^q_\varepsilon)'(H_{t-s}(f))\frac{d}{ds}H_{t-s}f)\Big]\,\mathrm{d}s \\
&= \frac{2}{q}\int_\delta^{t-\delta} (H_s(\phi^q_\varepsilon(H_{t-s}f)))^{\frac{2}{q}-1}\Big[H_s\Big(\Delta(\phi^q_\varepsilon(H_{t-s}(f))) - (\phi^q_\varepsilon)'(H_{t-s}(f))\Delta(H_{t-s}(f))\Big)\Big]\,\mathrm{d}s \\
&= \frac{2}{q}\int_\delta^{t-\delta} (H_s(\phi^q_\varepsilon(H_{t-s}f)))^{\frac{2}{q}-1}\Big[H_s\Big((\phi^q_\varepsilon)''(H_{t-s}(f))|D(H_{t-s}f)|^2_w)\Big)\Big]\,\mathrm{d}s, \qquad \text{(A.6)}
\end{aligned}$$

where the last equality follows from the chain rule of the Laplacian thanks to the fact that $(\phi^q_\varepsilon)''$ is locally bounded. Now recalling that H_t is an integral operator, we apply Holder's inequality

$$H_t(hg) \le H_t(|h|^{p'})^{\frac{1}{p'}} H_t(|g|^{q'})^{\frac{1}{q'}} \qquad \frac{1}{p'} + \frac{1}{q'} = 1$$

with exponents $p' = \frac{2}{q}$ and $q' = \frac{2}{2-q}$, namely

$$\begin{aligned}
&\Big[H_s\Big(|D(H_{t-s}f)|^{qp'}((\phi^q_\varepsilon)''(H_{t-s}f))^{\frac{q}{2}p'}\Big)\Big]^{\frac{1}{p'}}\Big[H_s\Big((\phi^q_\varepsilon)(H_{t-s}f)^{(1-\frac{q}{2})q'}\Big)\Big]^{\frac{1}{q'}} \qquad \text{(A.7)} \\
&\qquad \ge H_s\Big(|D(H_{t-s}f)|^q_w((\phi^q_\varepsilon)''(H_{t-s}f))^{\frac{q}{2}}(\phi^q_\varepsilon)(H_{t-s}f)^{(1-\frac{q}{2})}\Big).
\end{aligned}$$

We observe that the left hand side of (A.7) to the power $\frac{2}{q}$ is exactly the integrand of (A.6), so we get that

$$\begin{aligned}
&(H_{t-\delta}(\phi^q_\varepsilon(H_\delta f)))^{\frac{2}{q}} - (H_\delta(\phi^q_\varepsilon(H_{t-\delta}f)))^{\frac{2}{q}} \\
&\qquad \ge \frac{2}{q}\int_\delta^{t-\delta}\Big(H_s\Big(|D(H_{t-s}f)|^q_w((\phi^q_\varepsilon)''(H_{t-s}f))^{\frac{q}{2}}(\phi^q_\varepsilon)(H_{t-s}f)^{(1-\frac{q}{2})}\Big)\Big)^{\frac{2}{q}}\,\mathrm{d}s.
\end{aligned}$$

Now we take $\psi \in L^\infty(X)$, $\psi \geq 0$. We integrate the previous inequality against ψ in order to be able to tackle the limit $\varepsilon \to 0$:

$$\begin{aligned}
&\int_X (H_{t-\delta}(\phi_\varepsilon^q(H_\delta f)))^{\frac{2}{q}} - (H_\delta(\phi_\varepsilon^q(H_{t-\delta}f)))^{\frac{2}{q}}\psi\,\mathfrak{m} \\
&\geq \frac{2}{q}\int_\delta^{t-\delta}\int_X \left(H_s\Big(|D(H_{t-s}f)|_w^q((\phi_\varepsilon^q)''(H_{t-s}f))^{\frac{q}{2}}(\phi_\varepsilon^q)(H_{t-s}f)^{(1-\frac{q}{2})}\Big)\right)^{\frac{2}{q}}\psi\,\mathfrak{m}\,\mathrm{d}s.
\end{aligned} \tag{A.8}$$

Noticing that $f \in L^\infty(X)$, $\phi_\varepsilon^q(t) \in \mathcal{C}^\infty(\mathbb{R})$ and using (1.21), we can infer that the integrands in both sides of (A.8) are uniformly integrable with respect to ε.

So we can take the limit for $\varepsilon \to 0$ in both sides of the inequality using the dominated convergence theorem, obtaining

$$\begin{aligned}
&\int_X (H_{t-\delta}(|(H_\delta f)|^q))^{\frac{2}{q}} - (H_\delta(|H_{t-\delta}f|^q))^{\frac{2}{q}}\psi\,\mathrm{d}\mathfrak{m} \\
&\geq \frac{2}{q}q(q-1)\int_\delta^{t-\delta}\int_X H_s(|D(H_{t-s}f)|_w^q)^{\frac{2}{q}}\psi\,\mathfrak{m}\,\mathrm{d}s \\
&\geq 2(q-1)\int_\delta^{t-\delta}\int_X |DH_s(H_{t-s}f)|_w^2 e^{2Ks}\,\mathfrak{m}\,\mathrm{d}s = 2(q-1)\int_\delta^{t-\delta} e^{2Ks}\,\mathrm{d}s\int_X |D(H_t f)|_w^2\psi\,\mathfrak{m},
\end{aligned}$$

where the last inequality follows from (A.2) with $\alpha = \frac{q}{2}$. Now we can send $\delta \to 0$ in the previous inequality and get

$$\int_X (H_t(|f|^q))^{\frac{2}{q}} - (H_t(f))^2\psi\,\mathfrak{m} \geq (q-1)\frac{e^{2Kt-1}}{K}\int_X |D(H_t f)|_w^2\psi\,\mathfrak{m},$$

and observing that inequality holds for any non-negative $\psi \in L^\infty(X)$, the result is proved. □

Proof of Proposition A.1. We closely follow the proof of [74, Theorem 5.2], to which we refer for details. The main difference is that we deduce (A.12) from Proposition A.2, while in [74] the equivalent estimate is deduced by combining the classical Bakry-Émery inequality (which corresponds to (A.5) with $q = 2$) with Jensen's inequality.

The proof makes use of the dual dynamic formulations of both the p-Wasserstein and the p-Hellinger distances, stated respectively in (1.7) and (1.9), that we report here for the convenience of the reader:

$$\frac{1}{p}W_p^p(\mu_0,\mu_1) = \sup\left\{\int_X \xi_1\,\mu_1 - \int_X \xi_0\,\mu_0 \colon \xi \in \mathcal{C}^1([0,1],\mathrm{Lip}_b(X)),\ \partial_s\xi_s + \frac{1}{q}|D\xi_s|^q \leq 0\right\}, \tag{A.9}$$

$$\begin{aligned}
&\mathsf{He}_p^p(H_t^*\mu_0, H_t^*\mu_1) = \\
&\quad = \sup\left\{\int_X \xi_1\,(H_t^*\mu_1) - \int_X \xi_0\,(H_t^*\mu_0) \colon \xi \in \mathcal{C}^1([0,1],B_b(X)),\ \partial_s\xi_s + (p-1)|\xi_s|^q \leq 0\right\} \\
&\quad = \sup\left\{\int_X H_t\xi_1\,\mu_1 - \int_X H_t\xi_0\,\mu_0 \colon \xi \in \mathcal{C}^1([0,1],B_b(X)),\ \partial_s\xi_s + (p-1)|\xi_s|^q \leq 0\right\},
\end{aligned}$$

where $B_b(X)$ is the space of bounded Borel functions on X and q is the conjugate exponent of p:

$$\frac{1}{p} + \frac{1}{q} = 1.$$

In particular we use the dual formulation of the p-Wasserstein distance in the following rescaled version valid for $a > 0$

$$\frac{1}{a}W_p^p(\mu_0,\mu_1) = \sup\left\{\int_X \xi_1\,\mu_1 - \int_X \xi_0\,\mu_0 \colon \xi \in C^1([0,1],\mathrm{Lip}_b(X)),\ \partial_s\xi_s + \frac{a^{q-1}}{qp^{q-1}}|D\xi_s|^q \le 0\right\},$$

which can be derived from (A.9) as follows:

$$\begin{aligned}
&\frac{1}{a}W_p^p(\mu_0,\mu_1) = \frac{p}{a}\frac{1}{p}W_p^p(\mu_0,\mu_1)\\
=&\frac{p}{a}\sup\left\{\int_X \xi_1\,\mu_1 - \int_X \xi_0\,\mu_0 \colon \xi \in C^1([0,1],\mathrm{Lip}_b(X)),\ \partial_s\xi_s + \frac{1}{q}|D\xi_s|^q \le 0\right\}\\
=&\sup\left\{\int_X \frac{p}{a}\xi_1\,\mu_1 - \int_X \frac{p}{a}\xi_0\,\mu_0 \colon \xi \in C^1([0,1],\mathrm{Lip}_b(X)),\ \partial_s\xi_s + \frac{1}{q}|D\xi_s|^q \le 0\right\}\\
=&\sup\left\{\int_X \phi_1\,\mu_1 - \int_X \phi_0\,\mu_0 \colon \phi \in C^1([0,1],\mathrm{Lip}_b(X)),\ \partial_s\phi_s + \frac{a^{q-1}}{qp^{q-1}}|D\phi_s|^q \le 0\right\}.
\end{aligned}$$

The last inequality follows observing that $\xi \in C^1([0,1],\mathrm{Lip}_b(X))$ satisfies

$$\partial_s\xi_s + \frac{1}{q}|D\xi_s|^q \le 0,$$

if and only if $\phi := \frac{p}{a}\xi \in C^1([0,1],\mathrm{Lip}_b(X))$ and satisfies

$$\frac{a}{p}\partial_s\phi_s + \frac{1}{q}\left(\frac{a}{p}\right)^q |D\phi_s|^q \le 0,$$

so in particular

$$\partial_s\phi_s + \frac{1}{q}\left(\frac{a}{p}\right)^{q-1} |D\phi_s|^q \le 0.$$

To get the result it is enough to prove that any function $\xi \in \mathcal{C}^1([0,1],B_b(X))$ satisfying

$$\partial_s\xi_s + (p-1)|\xi_s|^q \le 0, \tag{A.10}$$

is such that $H_t(\xi_s)$ satisfies

$$\partial_s H_t(\xi_s) + \frac{a(t)^{q-1}}{qp^{q-1}}|DH_t(\xi_s)|^q \le 0, \qquad t > 0, \tag{A.11}$$

with

$$a(t) := \left(\frac{1}{p-1}\right)^{\frac{p}{2}} p^p R_K(t)^{\frac{p}{2}},$$

so that $H_t(\xi_s)$ is an admissible competitor in the definition of $\frac{1}{a(t)}W_p^p(\mu_0,\mu_1)$. The fact that (A.10) implies (A.11) follows by applying H_t to the inequality (A.10), recalling the fact that H_t is sign preserving and then by using the key inequality

$$(q-1)^{\frac{q}{2}}(R_K(t))^{\frac{q}{2}}|DH_t(\xi_s)|^q \le H_t(|\xi_s|^q), \tag{A.12}$$

which is a consequence of (A.5) neglecting the negative term in the right hand side and taking the $\frac{q}{2}$-power. We remark that in (A.12) we use the slope of $H_t(\xi_s)$ in place of the weak upper gradient thanks to equality (A.1), since $\xi_s \in L^\infty(X)$. □

Appendix B

More regularity for the conditional measures

Definition B.1. Let $\Gamma \subseteq \mathbb{R}^2$. We say that Γ is monotone if for any couple $(x_1, y_1), (x_2, y_2) \in \Gamma$,

$$x_1 < x_2 \Rightarrow y_1 \leq y_2.$$

We fix in addition the following notation: for a set $\Gamma \subseteq \mathbb{R}^2$ we define for any $x \in \mathbb{R}^2$ the x-section of Γ, $\Gamma(x) := \{y \colon (x, y) \in \Gamma\}$.

Lemma B.2. *Let I be a real interval. Let $\tilde{\mathfrak{m}} \in \mathcal{P}(I)$. Let $\tilde{\mu} \in \mathcal{M}(I)$ be compactly supported. Assume that there exists a monotone set $\Gamma \subset I \times I$ Borel such that $d(\Gamma(x), x) > \delta$ for every x where $\Gamma(x) \neq \emptyset$ for a positive $\delta > 0$ and that for every set A Borel in I we have for any $t \in [0, 1]$,*

$$\tilde{\mathfrak{m}}(T_t(\Gamma \cap A \times I)) \geq C\tilde{\mu}(A), \tag{B.1}$$

for a constant $C > 0$, where $T_t : I \times I \to \mathbb{R}$, $T_t(x, y) = (1 - t)x + ty$.

Then $\tilde{\mu} \ll \mathcal{L}^1$.

Proof. The proof will be divided in three steps.
Step 1
Up to enlarging Γ, we can assume that there exists a non decreasing map $T : I \to I$ such that:

$$\Gamma = \{(x, y) \in I \times I \colon \lim_{z \to x^-} T(z) \leq y \leq \lim_{z \to x^+} T(z)\}, \tag{B.2}$$

and such that $T(x) \geq x + \delta$. We recall that a monotone map on the real line can have at most a countable number of discontinuities. We call

$$T(x^-) := \lim_{z \to x^-} T(z),$$
$$T(x^+) := \lim_{z \to x^+} T(z).$$

Note that $T(x^-)$ is left-continuous.

We take a point x in $\mathrm{supp}(\mu)$ and we want to get an estimate from above of $\mu([x-r, x+r])$ depending on x and r. We define for $t \in [0, 1]$

$$I_t(x, r) := [(1 - t)(x - r) + tT((x - r)^-), (1 - t)(x + r) + tT((x + r)^+)].$$

We choose $0 = t_1 < t_2 < \cdots < t_N < 1$, with $N = N(x,r)$, such that

$$\cup_{i=1}^N I_{t_i}(x,r) \supseteq [x, T((x-r)^-)], \tag{B.3}$$

and $I_{t_i}(x,r) \cap I_{t_{i+1}}(x,r)$ is a single point for any i. We do not write the dependence of N on x and r anymore. In particular $I_{t_i} \cap I_{t_j} = \emptyset$ whenever $|i-j| > 1$. This can be done by choosing $t_0 = 0$ and inductively

$$(1-t_i)(x+r) + t_i T((x+r)^+) = (1-t_{i+1})(x-r) + t_{i+1} T((x-r)^-).$$

Step 2

We give an estimate from below on N. We call $d_x := (T(x^-) - x) \geq \frac{\delta}{2} > 0$. Note that by left-continuity $d_{x,r} := \mathcal{L}^1(x, T((x-r)^-) \geq d_x$ if r is taken small enough (with respect to x), which we will assume from now on. Moreover $\lim_{r \to 0^+} d_{x,r} = d_x$. Hence we have that the length of the union of the sets we constructed, namely $\mathcal{L}^1\left(\cup_{i=1}^N I_{t_i}(x+r)\right)$ is more that d_x by (B.3).

Calling

$$Q_{x,r} := \frac{T((x+r)^+) - (x+r)}{T((x-r)^-) - (x-r)} > 0,$$

a direct computation gives

$$\mathcal{L}^1\left(I_{t_i}(x,r)\right) = 2rQ_{x,r}^{i-1} \quad \forall i \in \{1, \ldots, N\}.$$

It follows that

$$\mathcal{L}^1\left(\cup_{i=1}^N I_{t_i}\right) = 2r\frac{(Q_{x,r})^N - 1}{Q_{x,r} - 1} \geq d_{x,r},$$

from which we get

$$N \geq \frac{\log(\frac{d_{x,r}}{2r}(Q_{x,r} - 1) + 1)}{\log(Q_{x,r})}. \tag{B.4}$$

On the other hand we get an estimate from above of $\mu([x-r, x+r])$ in terms of N. To do this note that by (B.2)

$$T_t(\Gamma \cap [x-r, x+r] \times I) = I_t(x,r).$$

From hypothesis (B.1) we have that

$$\begin{aligned} 2 = 2\mathfrak{m}(I) &\geq 2\mathfrak{m}(\cup_{i=1}^N \mathring{I}_{t_i}(x,r) \cup \cup_{i=1}^N \partial I_{t_i}(x,r)) = 2\left(\mathfrak{m}(\cup_{i=1}^N \mathring{I}_{t_i}(x,r)) + \mathfrak{m}(\cup_{i=1}^N \partial I_{t_i}(x,r))\right) \\ &\geq \sum_{i=1}^N \mathfrak{m}(\mathring{I}_{t_i}(x,r)) + \sum_{i=1}^N \mathfrak{m}(\partial I_{t_i}(x,r)) = \sum_{i=1}^N \mathfrak{m}(I_{t_i}) = \sum_{i=1}^N \mathfrak{m}(T_{t_i}(\Gamma \cap [x-r, x+r] \times I)) \\ &\geq NC\mu([x-r, x+r]). \end{aligned} \tag{B.5}$$

By putting together (B.4) and (B.5) it follows that

$$\mu([x-r, x+r]) \leq \frac{2}{CN} \leq \frac{4}{C}\left(\frac{\log(Q_{x,r})}{\log(\frac{d_{x,r}}{2r}(Q_{x,r} - 1) + 1)}\right).$$

Step 3
We recall that a monotone function is a BV function. So let $DT \in \mathcal{M}(I)$ the measure derivative of T. We recall in addition that by the Lebesgue Decomposition Theorem DT can be decomposed as the sum of two mutually singular measures, one absolutely continuous w.r.t. $\mathcal{L}^1$ the other orthogonal to $\mathcal{L}^1$, namely

$$DT = T'\mathcal{L}^1 + T^{\perp}.$$

We define

$$\nu := \mathcal{L}^1 + T'\mathcal{L}^1 + T^{\perp}.$$

We prove that μ is absolutely continuous with respect to $\mathcal{L}^1$ by proving the following.
Claim 1 The Radon-Nykodym derivative of μ with respect to ν, namely $\frac{\mathrm{d}\mu}{\mathrm{d}\nu}$ when it exists is finite. We call $f_\mu(x) := \frac{\mathrm{d}\mu}{\mathrm{d}\nu}(x)$ when it exists. In addition from the computation of the derivative we have:
Claim 2 $f_\mu = 0$, $T^{\perp}$-a.e.

If the two claims are true we have that

$$\mu = f_\mu \nu = f_\mu(\mathcal{L}^1 + T'\mathcal{L}^1 + T^{\perp}) = f_\mu(1+T')\mathcal{L}^1 + f_\mu T^{\perp} = f_\mu(1+T')\mathcal{L}^1.$$

Proof of Claim 1: $\frac{\mathrm{d}\mu}{\mathrm{d}\nu}(x) < +\infty$ if it exists.

We take a sequence of $\{r_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to 0^+} \frac{\mu([x-r_n,x+r_n])}{\nu([x-r_n,x+r_n])} = \frac{\mathrm{d}\mu}{\mathrm{d}\nu}(x)$. Then up to extracting a subsequence we can distinguish three cases:

1. r_n is such that $\lim_{r_n\to 0} \frac{DT([x-r_n,x+r_n])}{2r_n} = l > 0$ and $l < +\infty$;
2. r_n is such that $\lim_{r_n\to 0} \frac{DT([x-r_n,x+r_n])}{2r_n} = +\infty$;
3. r_n is such that $\lim_{r_n\to 0} \frac{DT([x-r_n,x+r_n])}{2r_n} = 0$.

We want to estimate

$$\lim_{r_n\to 0^+} \frac{\mu([x-r_n,x+r_n])}{DT([x-r_n,x+r_n])}.$$

It will be useful to consider

$$\begin{aligned} Q_{x,r} - 1 &= \frac{T((x+r)^+) - T((x-r)^-) - 2r}{T((x-r)^-) - (x-r)} = \frac{DT([x-r,x+r]) - 2r}{T((x-r)^-) - (x-r)}, \\ \frac{Q_{x,r}-1}{2r} &= \left(\frac{DT([x-r,x+r])}{2r} - 1\right)\frac{1}{T((x-r)^-) - (x-r)}. \end{aligned} \tag{B.6}$$

In case (1) we can compute $\lim_{r_n\to 0} \frac{Q_{x,r_n}-1}{2r_n} = \frac{(l-1)}{d_x}$, from which follows that $\lim_{r_n\to 0} Q_{x,r_n} = 1$. So at the end using that $\nu \geq \mathcal{L}^1$, what we get is

$$\begin{aligned} f_\mu(x) &= \frac{\mathrm{d}\mu}{\mathrm{d}\nu}(x) = \lim_{r_n\to 0^+} \frac{\mu([x-r_n,x+r_n])}{\nu([x-r_n,x+r_n])} \leq \lim_{r_n\to 0^+} \frac{4}{2r_nC}\left(\frac{\log(Q_{x,r_n})}{\log(\frac{d_{x,r}}{r_n}(Q_{x,r_n}-1)+1)}\right) \\ &= \lim_{r_n\to 0^+} \frac{4}{2r_nC}\left(\frac{\log(1+(Q_{x,r_n}-1))}{\log(\frac{d_{x,r}}{r_n}(Q_{x,r_n}-1)+1)}\right) = \frac{4}{C}\lim_{r_n\to 0^+} \frac{(Q_{x,r_n}-1)}{2r_n}\frac{1}{\log(\frac{d_{x,r}}{2r_n}(Q_{x,r_n}-1)+1)} \\ &= \frac{(l-1)}{d_x}\frac{1}{\log l} < +\infty. \end{aligned}$$

In case (2) we can observe that

$$\frac{\mu([x-r_n,x+r_n])}{DT([x-r_n,x+r_n])} \le \frac{4}{C}\frac{1}{DT([x-r_n,x+r_n])}\left(\frac{\log(Q_{x,r_n})}{\log(\frac{d_{x,r}}{2r_n}(Q_{x,r_n}-1)+1)}\right)$$
$$= \frac{4}{C}\left(\frac{\log(1+(Q_{x,r_n}-1))}{\log(\frac{d_{x,r}}{2r_n}(Q_{x,r_n}-1)+1)}\right)\frac{1}{DT([x-r_n,x+r_n])}, \tag{B.7}$$

so

- if $\limsup_{r_n\to 0} DT([x-r_n,x+r_n]) = k > 0$, then by observing that by (B.6) we have $\lim_{r_n\to 0^+} Q_{x,r_n} - 1 = \frac{k}{d_x}$ and $\lim_{r_n\to 0^+} \frac{Q_{x,r_n}-1}{r_n} = +\infty$ and so using that $\nu \ge DT$,
$$f_\mu(x) = \frac{\mathrm{d}\mu}{\mathrm{d}\nu}(x) \le \lim_{r_n\to 0^+} \frac{\mu([x-r_n,x+r_n])}{DT([x-r_n,x+r_n])} = 0;$$
- $\limsup_{r_n\to 0} DT([x-r_n,x+r_n]) = 0$, then again from (B.7), by observing that by (B.6) we have $\lim_{r_n\to 0^+} Q_{x,r_n} - 1 = 0$ and $\lim_{r_n\to 0^+} \frac{Q_{x,r_n}-1}{r_n} = +\infty$ and so using that $\nu \ge DT$,
$$f_\mu(x) = \frac{\mathrm{d}\mu}{\mathrm{d}\nu}(x) \le \lim_{r_n\to 0^+} \frac{\mu([x-r_n,x+r_n])}{DT([x-r_n,x+r_n])} = 0;$$

In case (3) we can observe that $\lim_{r_n\to 0^+} Q_{x,r_n} - 1 = 0$ and $\lim_{r_n\to 0^+} \frac{Q_{x,r_n}-1}{2r_n} = -\frac{1}{d_x}$. So by following the computations that we did in case (1) we have

$$f_\mu(x) = \frac{\mathrm{d}\mu}{\mathrm{d}\nu}(x) \le \limsup_{r_n\to 0^+} \frac{\mu([x-r_n,x+r_n])}{\nu([x-r_n,x+r_n])} \le \frac{4}{C}\lim_{r_n\to 0^+} \frac{(Q_{x,r_n}-1)}{2r_n}\frac{1}{\log(\frac{d_{x,r}}{2r_n}(Q_{x,r_n}-1)+1)}$$
$$= 0.$$

Proof of Claim 2: we observe that $T^\perp((\{x\colon \frac{\mathrm{d}DT}{\mathrm{d}\mathcal{L}^1}(x) = +\infty\})^c) = 0$. Indeed we know that $\frac{\mathrm{d}T^\perp}{\mathrm{d}\mathcal{L}^1}(x) = +\infty$ $T^\perp$-a.e. (see e.g. [81, Theorem 7.15]). The claim follows because $f_\mu \neq 0$ only in case (1). ☐

Proposition B.3 (Regularity of the conditional measures)**.** *Let $(X,\mathsf{d},\mathfrak{m})$ be as in (4.2). Let $\mu,\nu \in \mathcal{P}(X)$ be two compactly supported probability measures absolutely continuous with respect to $\mathfrak{m}$: $\mu = \rho_\mu\mathfrak{m}$ and $\nu = \rho_\nu\mathfrak{m}$, with ρ_μ, $\rho_\nu : X \to \mathbb{R}$, ρ_μ, $\rho_\nu \le C$ for some positive C. Assume in addition that* $\mathrm{dist}(\mathrm{supp}(\mu),\mathrm{supp}(\nu)) > 0$. *Let φ be a Kantorovich potential for the problem and consider the partition of $\mathcal{T}^{nb}_\varphi$ into equivalence classes $\{X_q\}_{q\in\mathcal{Q}}$. Let $\mathcal{Q}$ be a Borel quotient set and f_φ be a Borel quotient map. Let $q \mapsto \mathfrak{m}_q$ be the disintegration of $\mathfrak{m}_{|\mathcal{T}^{nb}_\varphi}$, and define $\mu_q := \rho_\mu\mathfrak{m}_q$. Assume that there exists a closed $\Gamma \subseteq \Gamma_\varphi \cap (\mathcal{T}^{nb}_\varphi \times \mathcal{T}^{nb}_\varphi)$ with $P_1(\Gamma) \subset \mathrm{supp}(\mu)$, $P_2(\Gamma) \subset \mathrm{supp}(\nu)$, which satisfies the two following hypotheses:*

- *there exists $\tilde{C} > 0$ such that for any A Borel and $t > 0$,*
$$\mathfrak{m}(T_t(\Gamma \cap A \times X)) \ge C\mu(A); \tag{B.8}$$
- *for $\mathfrak{q}$-a.e. $q \in \mathcal{Q}$, $\Gamma \cap X_q \times X$ is d^2-cyclically monotone.*

Then for $\mathfrak{q}$ a.e. $q \in \mathcal{Q}$ the measure μ_q is absolutely continuous with respect to $\mathcal{H}^1_{X_q}$.

Remark B.4. *An important observation is that, given μ, ν as in the hypotheses, the rest of assumptions would be all satisfied if the plan $\pi_\infty \in \mathrm{Opt}_1(\mu,\nu)$ given by Proposition 4.3.4 was concentrated on a set $\tilde{\Gamma} \subset X \times X$ such that $\Gamma \cap X_q \times X$ is d^2-cyclically monotone $\mathfrak{q}$-a.e.. Indeed if this was true, then we could take*

$$\Gamma := \tilde{\Gamma} \cap \mathrm{supp}(\mu) \times \mathrm{supp}(\nu) \cap \Gamma_\varphi \cap (\mathcal{T}_\varphi^{nb} \times \mathcal{T}_\varphi^{nb}).$$

This is because by Proposition 4.3.4 hypothesis (7) *holds as soon as $\pi_\infty(\Gamma) = 1$, which is easily checked since $\pi_\infty(\{x = y\}) = 0$ (recall also that $\mu(\mathcal{T}_\varphi^e \setminus \mathcal{T}_\varphi^{nb}) = \nu(\mathcal{T}_\varphi^e \setminus \mathcal{T}_\varphi^{nb}) = 0$).*

Proof of Proposition B.3. The proof will be made in two steps: in the first step we start from the estimate in (B.8) and we show that it can be localized to any equivalence class X_q. In the second step we will show that for $\mathfrak{q}$ a.e. $q \in \mathcal{Q}$ the measure $\mathfrak{m}_q$ is absolutely continuous with respect to $\mathcal{H}^1$ by applying Lemma B.2.

Step 1. We need to prove that there exists a set $\bar{\mathcal{Q}}$ of $\mathfrak{q}$ full measure in $\mathcal{Q}$ such that for any $A \subseteq X$ Borel

$$\mathfrak{m}_q(T_t(\Gamma \cap (A \times X))) \geq \frac{C_{K,M}}{C} \mu_q(A) \tag{B.9}$$

holds for any $q \in \bar{\mathcal{Q}}$. We first claim that for a fixed A Borel, for $\mathfrak{q}$ a.e. $q \in \mathcal{Q}$

$$\mathfrak{m}_q(T_t(\Gamma \cap (A \times X))) \geq \frac{C_{K,M}}{C} \mu_q(A). \tag{B.10}$$

To prove the claim we show that for any $\tilde{\mathcal{Q}} \in \mathcal{B}(\mathcal{Q})$ and for any A Borel one has

$$\int_{\tilde{\mathcal{Q}}} \mathfrak{m}_q(T_t(\Gamma \cap (A \times X)))\mathfrak{q}(\mathrm{d}q) \geq \frac{C_{K,M}}{C} \int_{\tilde{\mathcal{Q}}} \mu_q(A)\mathfrak{q}(\mathrm{d}q).$$

Indeed fix $\tilde{\mathcal{Q}} \in \mathcal{B}(\mathcal{Q})$ and $A \subseteq X$ Borel. Define $S_{\tilde{\mathcal{Q}}} := \cup_{q \in \tilde{\mathcal{Q}}} X_q$ which is Borel since it is the preimage through the quotient map f_φ of $\tilde{\mathcal{Q}}$. Consider the Borel set $\tilde{A} := A \cap S_{\tilde{\mathcal{Q}}}$.

By the Disintegration theorem, by defining $\mu_q := \rho_\mu \mathfrak{m}_q$, we have

$$\int_{\mathcal{Q}} \mathfrak{m}_q(T_t(\Gamma \cap (\tilde{A} \times X)))\mathfrak{q}(\mathrm{d}q) \geq \frac{C_{K,M}}{C} \int_{\mathcal{Q}} \mu_q(\tilde{A})\mathfrak{q}(\mathrm{d}q),$$

and therefore

$$\begin{aligned}
\int_{\tilde{\mathcal{Q}}} \mathfrak{m}_q(T_t(\Gamma \cap (A \times X)))\mathfrak{q}(\mathrm{d}q) &= \int_{\tilde{\mathcal{Q}}} \mathfrak{m}_q(T_t(\Gamma \cap (\tilde{A} \times X)))\mathfrak{q}(\mathrm{d}q) \\
= \int_{\mathcal{Q}} \mathfrak{m}_q(T_t(\Gamma \cap (\tilde{A} \times X)))\mathfrak{q}(\mathrm{d}q) &\geq C \int_{\tilde{\mathcal{Q}}} \mu_q(A)\mathfrak{q}(\mathrm{d}q).
\end{aligned}$$

Now we can take $C := \{B_r(x_i) \colon r \in \mathbb{Q},\, i \in \mathbb{N}\}$, where $\{x_i\}_{i \in \mathbb{N}}$ is a dense subset of X. Then there exists $\bar{\mathcal{Q}}$ such that for $\mathfrak{q}$ a.e. $q \in \bar{\mathcal{Q}}$ estimate (B.10) holds for any $A \in C$ since it is a countable set. This implies that for any $q \in \bar{\mathcal{Q}}$ (B.10) holds for any open set A. By taking decreasing subsequence of open sets the estimate (B.10) holds for any $q \in \bar{\mathcal{Q}}$ for any A closed.

Step 2. Take $q \in \bar{Q}$ of the previous step. Let A_q be closed set of X_q. Then $A_q = A \cap X_q$ for some closed set A of X. We define $\Gamma_q := \Gamma \cap X_q \times X_q$, and we have

$$\mathfrak{m}_q(T_t(\Gamma_q \cap (A_q \times X_q))) = \mathfrak{m}_q(T_t(\Gamma \cap (A \times X))) \geq C\mu_q(A) = C\mu_q(A_q). \tag{B.11}$$

We call

$$
\begin{aligned}
&I_q := g^{-1}(X_q);\, \tilde{\Gamma}_q := (g^{-1}, g^{-1})(\Gamma_q); \quad \tilde{A}_q := g^{-1}(A_q);\\
&\tilde{\mathfrak{m}}_q := g^{-1}_\sharp \mathfrak{m}_q;\, \tilde{\mu}_q := g^{-1}_\sharp \mu_q;
\end{aligned}
$$

We claim that for $\mathfrak{q}$ a.e. $q \in \mathcal{Q}$: $(I_q, |\cdot|, \tilde{\mathfrak{m}}_q)$, $\tilde{\mu}_q, \tilde{\nu}_q$, $\tilde{\Gamma}_q$, satisfy the hypotheses of Lemma B.2. Indeed hypothesis (2) implies that $\tilde{\Gamma}_q$ is monotone for any q. Now from the previous step we have that for $\mathfrak{q}$ a.e. $q \in \mathcal{Q}$

$$
\tilde{\mathfrak{m}}_q(T_t(\tilde{\Gamma}_q \cap (I \times I_q))) \geq C\tilde{\mu}_q(I),
$$

for any I interval closed in I_q. Moreover $\mathrm{dist}(\tilde{\Gamma}_q(x), x) \geq \mathrm{dist}(\mathrm{supp}(\mu), \mathrm{supp}(\nu)) > 0$ for every $x \in I_q$ by hypotheses and since $g(q, .)$ is an isometry.

Therefore from Lemma B.2 $\tilde{\mu}_q \ll \mathcal{L}^1$, which gives the claim since $g(q, \cdot)$ is an isometry. □

Corollary B.5. *Under the same hypotheses of the previous proposition for $\mathfrak{q}$ a.e. $q \in \mathcal{Q}$, $(\mathfrak{m}_q)_{|\{\rho_\mu > 0\}}$ is absolutely continuous with respect to $\mathcal{H}^1_{|X_q}$.*

Proof. It follows immediately from the previous Proposition and the definition of $\mu_q = \rho_\mu \mathfrak{m}_q$. □

Remark B.6. *The proof that $(\mathfrak{m}_q)_{|\{\rho_\nu > 0\}}$ is absolutely continuous with respect to $\mathcal{H}^1_{|X_q}$ follows analogously.*

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