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On the minimum problem for non-quasiconvex vectorial functionals

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Abstract. We consider functionals of the form

$$\mathcal{F}(u) = \int_\Omega f(x, u(x), Du(x))\, dx, \quad u \in u_0 + W_0^{1,r}(\Omega, \mathbb{R}^m),$$

where the integrand $f : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R}$ is assumed to be non-quasiconvex in the last variable and $u_0 \in W^{1,r}(\Omega, \mathbb{R}^m)$ is an arbitrary boundary value. We study the minimum problem by the introduction of the lower quasiconvex envelope $\overline{f}$ of f and of the relaxed functional

$$\overline{\mathcal{F}}(u) = \int_\Omega \overline{f}(x, u(x), Du(x))\, dx, \quad u \in u_0 + W_0^{1,r}(\Omega, \mathbb{R}^m),$$

imposing standard differentiability and growth properties on $\overline{f}$. In addition we assume a suitable structural condition on $\overline{f}$ and a special regularity on the minimizers of $\overline{\mathcal{F}}$, showing that under such assumptions $\mathcal{F}$ attains its infimum. Futhermore, we study the minimum problem for a class of functionals with separate dependence on the gradients of competing maps by the use of integro-extremality method, proving an existence result inspired by analogous ones obtained in the scalar case ($m = 1$). This last argument does not require the special regularity assumption mentioned above but the usual notion of classical differentiability (almost everywhere).

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1. Introduction

This paper is a contribution to the study of the minimum problem for the classical vectorial functionals of the Calculus of Variations of the form

$$\mathcal{F}(u) = \int_\Omega f(x, u(x), Du(x))\, dx,$$

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where the competing maps $u : \Omega(\subseteq \mathbb{R}^n) \to \mathbb{R}^m$ are Sobolev functions subject to a prescribed boundary condition expressed by imposing that u belongs to the set $u_0 + W_0^{1,r}(\Omega, \mathbb{R}^m)$, for some $u_0 \in W_0^{1,r}(\Omega, \mathbb{R}^m)$, where the exponent $r \in]1, \infty[$ is related to standard growth properties at infinity of the integrand $f = f(x, p, \xi)$ with respect to the variables p, ξ. The lagrangian $f : \Omega \times \mathbb{R}^n \times \mathbb{M}^{m\times n}$ is assumed to be non-quasiconvex in the last variable and our aim is to treat the consequent non-(sequential weakly lower)-semicontinuity of the functional $\mathcal{F}$ in the Sobolev space of definition. Indeed it is well known that, dropping the quasiconvexity, the functional is not sequentially weakly lower semicontinuous and then it may have no minimizers. However many applications induce to consider functionals which do not satisfy such property, and we mention arguments in optimal design (see [1,14]), the problem of singular values ([11]), of potential wells ([7,13,15,16]) and, in general, studies in nonlinear elasticity, as described and discussed, for example, in [2,3].

In these mentioned papers, the authors treat specific non-quasiconvex problems adopting techniques developed for each singular case, since a general theory for non-semicontinuous variational problems is up to now lacking. In particular, the excellent and celebrated works [6,7], in the part which concerns with variational problems, transform them in implicit first order differential equations or differential inclusions, and this fact forces strong restrictions on the boundary datum u_0 and on the lagrangian f.

Futher efforts in the solution of non-quasiconvex problems have been made by various authors, and we mention papers [4,6,8–10,17,20]. Unfortunately, also in these studies, special assumption are imposed on the lagrangians f and on the boundary datum u_0, so that the need of a satisfactory theory remains unsatisfied.

These mentioned difficulties emerge clearly if we sketch the main approach to this class of variational problems. As shown in several papers and monographs (see for example [5]), it is based on the introduction of the relaxed functional, given by

$$\overline{\mathcal{F}}(u) = \int_\Omega \overline{f}(x, u(x), Du(x))\, dx,$$

where $\overline{f}$ is the lower quasiconvex envelope of f with respect to the last variable ξ, and in the search for a minimizer of $\overline{\mathcal{F}}$ solving the differential equation

$$\overline{f}(x, u(x), Du(x)) = f(x, u(x), Du(x)) \quad \text{for almost every } x \in \Omega. \tag{1.1}$$

The direct solution of (1.1) is very hard. First of all it is a fully nonlinear partial differential equation in several variables and in a vector valued unknown function $u = (u^1, \ldots, u^m)$; in addition the set of possible solutions is restricted to the family of minimizers of the relaxed functional. Due to these difficulties all papers mentioned above provide only partial results.

In paper [21] we try to build up a rather general method in order to solve the problem along the line just sketched, considering integrands of sum type

$$f(x,p,\xi) = g(x,\xi) + h(x,p), \qquad \overline{f}(x,p,\xi) = \overline{g}(x,\xi) + h(x,p),$$

where $\overline{g}$ is the lower quasiconvex envelope of g. In detail we adopt the integro-extremality method, which consists in the selection of a specific minimizer of the the relaxed functional $\overline{\mathcal{F}}$ which, by extremizing the integral of one component u^i, forces the competing map $u = (u^1, \dots, u^m)$ to solve equation (1.1). In order to reach the result, in such paper we need the following hypotheses:

(a) the map $\mathbb{M}^{m\times n} \ni \xi \mapsto \overline{g}(x,\xi)$ is quasiaffine on the set in which $g(x,\xi) > \overline{g}(x,\xi)$;

(b) the map $p^i \mapsto h(x,p)$ is monotone in one single component p^i of the vector $p = (p^1, \dots p^m)$.

(c) there exits at least one minimizer of $\overline{\mathcal{F}}$ which is piecewice C^1 on Ω.

Conditions *(a-b)* seem to be unavoidable in the treatment of these problems and integro-extremality method allows to solve equations (1.1) in a powerful way. Unfortunately the third regularity requirement *(c)* is very strong and difficult to be proved. This fact induces to explore further techniques in the declared goal of finding a general theory in this field.

Actually, in the present article, we propose a new way to face non-quasiconvex variational problems, following ideas firstly introduced in papers [22,23], which are devoted to one-dimensional non-convex problems, corresponding to dimensions $n = 1$, $m > 1$. This new approach is based on Euler-Lagrange equations and the relevant novelty of the results is that they provide almost necessary and sufficient conditions for existence of minimizers. Such conditions are suspected to be extendable also to the multidimensional case and, actually, in the present work, we made a first step in this direction. We sketch now the main hypotheses used in our present theory, so that the reader may easily and quickly compare them with those listed above, in relation to the quoted paper [21], and to those adopted in [22,23].

We assume standard growth properties ensuring the existence of minimizers of the relaxed functional $\overline{\mathcal{F}}$ and the validity of classical necessary conditions in weak form. In addition we impose structural conditions that can be summarized as follows:

(i) the function $\mathbb{M}^{m\times n} \ni \xi \mapsto \overline{f}(x,p,\xi)$ is quasiaffine on the set on which $f > \overline{f}$,

(ii) the map $\mathbb{R} \ni p^i \mapsto \overline{f}(x,p,\xi)$ is strictly monotone for at least one component p^i of the vector variable $p = (p^1, \dots, p^m)$.

Unfortunately, also in the present theory we have to impose a regularity property on the minimizers of $\overline{\mathcal{F}}$. More precisely, instead of piecewise continuity of the derivative Du of one minimizer u of $\overline{\mathcal{F}}$, as it happens in [21], we impose that

(iii) there exists at least one minimizer $\overline{u}$ of the relaxed functional $\overline{\mathcal{F}}$ which is locally Lipschitz-continuous and the derivative $D\overline{u}$ is strongly approximately continuous almost everywhere in Ω.

The notion of *strong approximate continuity* for a measurable maps is firstly introduced in this paper and we anticipate now the definition. Let $f : \mathbb{R}^d \to \mathbb{R}^\delta$ be a measurable function and $x_0 \in \mathbb{R}^d$ a Lebesgue point of f; we say that f is strongly approximately continuous at x_0 if, for every $\epsilon > 0$, there exists a map $\omega = \omega_\epsilon : [0, +\infty[\to [0, +\infty[$ with $\omega(t) \to 0$ as $t \to 0+$, such that we have

$$\frac{\mathrm{m}\left(\{x \in B(x_0, \rho) : |f(x) - f(x_0)| \geq \epsilon\}\right)}{\mathrm{m}(B(x_0, \rho))} = \rho\, \omega(\rho). \tag{1.2}$$

The difference with respect to classical approximate continuity (see for example section 1.7 in [12]) is that here, at the right hand side of (1.2), we have the extra factor ρ multiplying $\omega(\rho)$. It is then evident that continuity implies strong approximate continuity which, in turn, implies approximate continuity, and we stress that the reason of this assumption is merely technical, since, without it, we are not able to carry on the proof of our main result. In Sect. 2 (remark 2.4) we give a simple example explaining the relation between continuity and (strong) approximate continuity.

Under conditions *(i)*-*(iii)* we are able to prove the existence of minimizers of the non-quasiconvex functional, and the main novelties of the present work are two: we remove the piecewice C^1 requirement on the minimizers of $\overline{\mathcal{F}}$; we are going towards a theory, inspired by the analogous one in the one-dimensional case treated in [22,23], which is claimed to provide almost necessary and sufficient conditions for the minimization of non-semicontinuous functionals of the Calculus of Variations.

As a simple explicative example (see remark 3.3), we exhibit the functional

$$\mathcal{F}(u) = \int_\Omega \left((|Du|^2 - 1)^2 + \psi \cdot u\right) dx$$

defined for $u \in u_0 + W_0^{1,4}(\Omega, \mathbb{R}^m)$ with a boundary datum $u_0 \in W^{1,4}(\Omega, \mathbb{R}^m)$. Assuming that at least one component of the function $\psi = (\psi^1, \ldots, \psi^m) \in C^0(\overline{\Omega}, \mathbb{R}^m)$ is strictly positive on Ω, and invoking classical results on relaxation (see remark 3.3 in Sect. 3), we show that the above conditions *(i)*-*(ii)* are satisfied.

In the last section of the paper we treat a class of functionals which do not fit properties *(i)*-*(ii)*. We assume separate dependence with respect to the rows ξ^i of the matrix $\xi = (\xi^1, \ldots, \xi^m)$, imposing that the lagrangian has the following form:

$$f(x, p, \xi) = \sum_{i=1}^m g_i(x, \xi^i) + h(x, p).$$

In such situation we may apply the integro-extremality method introduced for scalar functional, along the lines that the reader can find, for example, in [19]. In this case we do not need strong approximate continuity *(iii)*, but classical differentiability almost everywhere, as it happens in the scalar case (see [18,19]).

2. Notations

In this paper $\mathbb{R}^d$ is the d-dimensional euclidean space ($d = m$ or n) and $|p| \doteq (\sum_{i=1}^d p_i^2)^{\frac{1}{2}}$ is the euclidean norm in $\mathbb{R}^d$, while the inner product of the vectors $p, q \in \mathbb{R}^d$ is written $p \cdot q \doteq \sum_{i=1}^d p_i q_i$. Given $x_0 \in \mathbb{R}^d$ and $\rho > 0$ we call $B(x_0, \rho)$ the open ball in $\mathbb{R}^d$ of center x_0 and radius ρ; for $E \subseteq \mathbb{R}^d$ we denote by $\mathrm{m}(E)$ the Lebesgue measure and by χ_E the characteristic function.

By $\mathbb{M}^{m\times n}$ we denote the space of $m \times n$ real matrices (with m rows and n columns) and an element $\xi \in \mathbb{M}^{m\times n}$ is written as

$$\xi = \left(\xi_j^i\right)_{\binom{i=1,\dots,m}{j=1,\dots,n}} = \begin{pmatrix} \xi_1^1 & \dots & \xi_n^1 \\ \vdots & \ddots & \vdots \\ \xi_1^m & \dots & \xi_n^m \end{pmatrix}.$$

Identifying $\mathbb{M}^{m\times n}$ with $\mathbb{R}^{mn}$, given $\xi, \eta \in \mathbb{M}^{m\times n}$, we write

$$\xi \cdot \eta \doteq \sum_{\binom{i=1,\dots,m}{j=1,\dots,n}} \xi_j^i \eta_j^i$$

and

$$|\xi|^2 \doteq \sum_{\binom{i=1,\dots,m}{j=1,\dots,n}} (\xi_j^i)^2.$$

Given a differentiable function $F : \mathbb{R}^m \to \mathbb{R}$, $F = F(p)$, we denote its gradient by $F_p = \nabla F = (F_{p_1}, \dots F_{p_m})$, where $F_{p_i} = \partial F / \partial p_i$. Analogoulsy, for a differentiable function $F : \mathbb{M}^{m\times n} \to \mathbb{R}$, $F = F(\xi)$, we write

$$F_\xi = D_\xi F = \begin{pmatrix} F_{\xi_1^1} & \dots & F_{\xi_n^1} \\ \vdots & \ddots & \vdots \\ F_{\xi_1^m} & \dots & F_{\xi_n^m} \end{pmatrix}.$$

For every $s = 2, \dots m \wedge n \doteq \min\{m, n\}$ we introduce the matrix of all $s \times s$ subdeterminants of a matrix $\xi \in \mathbb{M}^{m\times n}$:

$$(m_s(\xi))_{(j_1,\dots,j_s)}^{(i_1,\dots,i_s)} \doteq \det \begin{pmatrix} \xi_{j_1}^{i_1} & \dots & \xi_{j_s}^{i_1} \\ \vdots & \ddots & \vdots \\ \xi_{j_s}^{i_s} & \dots & \xi_{j_s}^{i_s} \end{pmatrix}, \tag{2.1}$$

where $1 \le i_1 < \dots < i_s \le m$, $1 \le j_1 < \dots < j_s \le n$. In order to clarify the notation we remark that, if $m = n$ and $s = n-1$, $m_s(\xi)$ describes all cofactors or order $n-1$ of the matrix ξ. More precisely, let $I = (1, 2, \dots, n)$ and, for $j \in I$, call I_j the $(n-1)$-ple obtained from I by suppressing the element j. Then we have (see Sect. 5.2 in [5] for notations and more details):

$$\left(\mathrm{Adj}_{n-1}(\xi)\right)_j^i = (-1)^{i+j} \left(m_{n-1}(\xi)\right)_{I_j}^{I_i}$$

and, in particular,

$$\det \xi = \left\langle \left(\mathrm{Adj}_{n-1}(\xi)\right)^i, \xi^i \right\rangle$$
$$= \sum_{j=1}^{n} \left(\mathrm{Adj}_{n-1}(\xi)\right)^i_j \xi^i_j \quad \forall i \in I. \tag{2.2}$$

Throughout the paper Ω is an open bounded subset of $\mathbb{R}^n$ with Lipschitz boundary, and we consider vector valued maps

$$u = \begin{pmatrix} u^1 \\ \vdots \\ u^m \end{pmatrix} : \Omega \to \mathbb{R}^m, \tag{2.3}$$

writing $D_j = \partial/\partial x_j$, for $j = 1, \dots, n$, and

$$Du = \begin{pmatrix} \nabla u^1 \\ \vdots \\ \nabla u^m \end{pmatrix} = \begin{pmatrix} D_1 u^1 & \dots & D_n u^1 \\ \vdots & \ddots & \vdots \\ D_1 u^m & \dots & D_n u^m \end{pmatrix} : \Omega \to \mathbb{M}^{m \times n}.$$

In particular, recalling (2.1), we have

$$(m_s(Du))^{(i_1,\dots,i_s)}_{(j_1,\dots,j_s)} = \frac{\partial(u^{i_1}, \dots, u^{i_s})}{\partial(x_{j_1}, \dots, x_{j_s})}.$$

We use the spaces $C^k(\Omega, \mathbb{R}^m)$, $C^k_c(\Omega, \mathbb{R}^m)$ $(k \in \mathbb{N} \cup \{0, \infty\})$, $L^r(\Omega, \mathbb{R}^m)$ and the Sobolev spaces $W^{1,r}(\Omega, \mathbb{R}^m)$, $W^{1,r}_0(\Omega, \mathbb{R}^m)$, for $1 \le r \le \infty$, with their usual (strong and weak) topologies. By a Sobolev map in $W^{1,r}(\Omega, \mathbb{R}^m)$ we mean its precise representative and, given a real valued function v, by v^+ we denote its positive part.

We use the notion of quasiconvex and quasiaffine function and of quasiconvex envelope, defined and discussed in chapters 5 and 6 of [5], recalling that a quasiaffine function $F : \mathbb{M}^{m \times n} \to \mathbb{R}$ is an affine function of all subdeterminants of the matrix ξ, more precisely we have the following representation.

Proposition 2.1. *Let $F : \mathbb{M}^{m \times n} \to \mathbb{R}$ be a quasiaffine function. Then there exist $A \in \mathbb{M}^{m \times n}$, real numbers $b^{(i_1,\dots,i_s)}_{(j_1,\dots,j_s)}$, for $s = 2, \dots m \wedge n$, $1 \le i_1 < \dots < i_s \le m$, $1 \le j_1 < \dots < j_s \le n$, and c such that*

$$F(\xi) = \sum_{\substack{i = 1, \dots, m \\ j = 1, \dots, n}} A^i_j \xi^i_j$$
$$+ \sum_{s=2}^{m \wedge n} \sum_{\substack{(i_1,\dots,i_s) \\ 1 \le i_1 < \dots < i_s \le m}} \sum_{\substack{(j_1,\dots,j_s) \\ 1 \le j_1 < \dots < j_s \le n}} b^{(i_1,\dots,i_s)}_{(j_1,\dots,j_s)} (m_s(\xi))^{(i_1,\dots,i_s)}_{(j_1,\dots,j_s)} + c.$$

It is evident by this proposition that a quasiaffine function $F = F(\xi)$ defined on $\mathbb{M}^{m\times n}$ is a polynomial of degree k, where k is an integer $0 \le k \le \min\{n, m\}$, and we say that k is the degree of F. Hence, in particular, for every $r \ge k$ and $u \in W^{1,r}(\Omega, \mathbb{R}^m)$ we have $F(Du) \in L^1(\Omega, \mathbb{R})$. We recall the following well known property (corollary 5.22 in [5]).

Proposition 2.2. *Let $F : \mathbb{M}^{m\times n} \to \mathbb{R}$ be a quasiaffine function of order k and $r \in [k, \infty]$ be a real index. Then, for every $u, v \in W^{1,r}(\Omega, \mathbb{R}^m)$ such that $u - v \in W^{1,r}_0(\Omega, \mathbb{R}^m)$, we have*

$$\int_\Omega F(Du(x))\, dx = \int_\Omega F(Dv(x))\, dx.$$

We shall use the following notion.

Definition 2.3. Let $f : \mathbb{R}^d \to \mathbb{R}^\delta$ be a measurable function and $x_0 \in \mathbb{R}^d$ a Lebesgue point of f. We say that f is strongly approximately continuous at x_0 if, for every $\epsilon > 0$, there exists a map $\omega = \omega_\epsilon : [0, +\infty[\to [0, +\infty[$ with $\omega(t) \to 0$ as $t \to 0+$, such that we have

$$\frac{\mathrm{m}\left(\{x \in B(x_0, \rho) : |f(x) - f(x_0)| \ge \epsilon\}\right)}{\mathrm{m}(B(x_0, \rho))} = \rho\, \omega(\rho). \tag{2.4}$$

Remark 2.4. The above definition differs from the classical notion of approximate continuity by the presence of the factor ρ at the right hand side of (2.4). Removing it, indeed, we obtain the classical notion, and it is well known that any measurable function is approximately continuous almost everywhere. In addition, it is evident that continuity implies strong approximate continuity which, in turn, implies approximate continuity.

In order to give a simple and pictorial image of this notion, we exhibit the following example for functions defined on the space $\mathbb{R}^2$.

Consider a parameter $\alpha \ge 0$, define the set

$$E_\alpha \doteq \left\{(x_1, x_2) : x_1 > 0,\ 0 < x_2 \le x_1^{1+\alpha}\right\}$$

and introduce the map

$$u_\alpha(x_1, x_2) \doteq \chi_{E_\alpha}(x_1, x_2).$$

Call B_r the ball of center zero and radius r in $\mathbb{R}^2$, and observe that there exist two positive constant a, b, with $0 < a < b$, depending on α but independent on r, such that, for any $t \in]0, 1[$, we have

$$ar^\alpha \le \frac{\mathrm{m}(B_r \cap \{u_\alpha \ge t\})}{\mathrm{m}(B_r)} \le br^\alpha.$$

Remarking that $u_\alpha(0, 0) = 0$, it is immediate to see that u_α is not continuous at the point $(0, 0)$ for every $\alpha \ge 0$. Then we immediately see that

- for $\alpha > 1$, u_α is strongly approximately continuous at the point $(0, 0)$;
- for $0 < \alpha \le 1$, u_α is approximately continuous, but not strongly approximately continuous, at the point $(0, 0)$;
- for $\alpha = 0$, u_α is not approximately continuous at the point $(0, 0)$.

The reason and the use of this notion is merely technical, and in remark 3.5 below we point out the passage of the proof of our main theorem in which it is required and necessary to impose the existence of a minimizer $\overline{u}$ of the functional $\overline{\mathcal{F}}$ such that its gradient $D\overline{u}$ is strongly approximately continuous at almost everypoint of Ω. Clearly this property is intermediate between (almost everywhere) approximate continuity, which is enjoyed by the gradient Du of any Sobolev map, and (piecewise) C^1-regularity of a map u. In this sense, our theory marks a step from existing results (see paper [21] and points *(a,b,c)* of the introduction) towards an existence theory for the class of problems considered in this paper.

In section 4 we shall need the following tools (see [19]).

Definition 2.5. We set

$$\mathcal{M}^-(\Omega,\mathbb{R}^n) \doteq \left\{\gamma \in W^{1,\infty}(\Omega,\mathbb{R}^n) : -\mathrm{div}\ \geq 0 \text{ a.e.}\right\},$$

where $-\mathrm{div}(\cdot)$ denotes the divergence operator. In particular, for every $\gamma \in \mathcal{M}^-(\Omega,\mathbb{R}^n)$ and $w \in W_0^{1,1}(\Omega)$ such that $w \geq 0$ in Ω, we have

$$\int_\Omega \gamma \cdot \nabla w \, dx \geq 0.$$

Lemma 2.6. *Let Ω be an open subset of $\mathbb{R}^n$, $r \in [1,\infty]$, $v \in W^{1,r}(\Omega,\mathbb{R})$. Let $x_0 \in \Omega$ be a point at which v is classically differentiable with differential $\nabla v(x_0)$, $t > 0$ and $\rho > 0$ such that $B(x_0,\rho) \subseteq \Omega$. Then there exists a map $\check{v} \in W^{1,r}(\Omega,\mathbb{R})$ with the following properties:*

$$\check{v} - v \in W_0^{1,r}(\Omega,\mathbb{R}), \tag{2.5}$$

$$\check{v}(x) \leq v(x) \text{ for a.e. } x \in U, \tag{2.6}$$

$$\check{\Lambda} \doteq \{x \in \Omega\colon \check{v}(x) < v(x)\} \text{ is nonempty and } \check{\Lambda} \subseteq B(x_0,\rho), \tag{2.7}$$

$$|\nabla\check{v}(x) - \nabla v(x_0)| = t, \quad \text{for a.e. } \ x \in \check{\Lambda}, \tag{2.8}$$

$$\nabla\check{v}(x) = \nabla v(x), \quad \text{for a.e. } \ x \in \Omega \setminus \check{\Lambda} \tag{2.9}$$

$$\int_\Omega \check{v}\, dx < \int_\Omega v\, dx, \tag{2.10}$$

$$\int_{\check{\Lambda}} \gamma \cdot (\nabla\check{v} - \nabla v)\, dx \leq 0 \quad \forall \gamma \in \mathcal{M}^-(\Omega,\mathbb{R}^n). \tag{2.11}$$

This last statement is a simpler version of lemma 2 in [19] and we refer to such paper for the proof.

We end this section by an elementary property.

Lemma 2.7. *Let $x_0 \in \mathbb{R}^n$ and $\rho > 0$. Then there exists $\beta > 0$ such that, setting*

$$\varphi_\rho(x) \doteq \frac{\beta}{\rho^n}\left(1 - \frac{|x - x_0|}{\rho}\right)^+, \tag{2.12}$$

we have $\varphi_\rho \in W_0^{1,\infty}(\mathbb{R}^n,\mathbb{R})$, $\mathrm{supp}\,\varphi_\rho = \overline{B(x_0,\rho)}$, $\varphi_\rho \geq 0$ almost everywhere,

$$\int_{\mathbb{R}^n} \varphi_\rho\, dx = \int_{B(x_0,\rho)} \varphi_\rho\, dx = 1; \tag{2.13}$$

and

$$|\nabla \varphi_\rho| = \frac{\beta}{\rho^{n+1}} \chi_{B(x_0,\rho)}, \quad a.e.\ in\ \ \mathbb{R}^n. \tag{2.14}$$

Proof. Straightforward. □

3. Main results

We consider a continuous function $f : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R}$, where Ω is a bounded open subset of $\mathbb{R}^n$ with Lipschitz boundary and $f = f(x, p, \xi)$ is assumed to be non-quasiconvex in the last variable ξ. The lower quasiconvex envelope with respect to ξ, denoted by $\overline{f} = \overline{f}(x, p, \xi)$, is also assumed to be a continuous function $\overline{f} : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R}$.

We devote our study to the minimization of the functional

$$\mathcal{F}(u) = \int_\Omega f(x, u(x), Du(x))\, dx, \quad u \in \mathcal{W}.$$

The set $\mathcal{W}$ of competing maps is defined by

$$\mathcal{W} \doteq u_0 + W_0^{1,r}(\Omega, \mathbb{R}^m), \tag{3.1}$$

where $r \in]1, \infty[$ is an index related to the growth of f at infinity (see Hypothesis 1 below) and the boundary datum u_0 is a given map in $W^{1,r}(\Omega, \mathbb{R}^m)$. We introduce the relaxed functional

$$\overline{\mathcal{F}}(u) = \int_\Omega \overline{f}(x, u(x), Du(x))\, dx, \quad u \in \mathcal{W}$$

and, for convenience, we set $\mathcal{F}(u) = \overline{\mathcal{F}}(u) = +\infty$ for every $u \in W^{1,r}(\Omega, \mathbb{R}^m) \setminus \mathcal{W}$.

Our theory requires three hypotheses on the lagrangians f and $\overline{f}$. The first one is the classical growth conditions which ensure the existence of minimizers for the relaxed functional $\overline{\mathcal{F}}$, while the second one consists in the regularity assumpions usually imposed in order to guarantee that such minimizers satisfy Euler-Lagrange equations in weak form. Finally we have to impose the structural conditions which are necessary to treat the non quasiconvexity of the lagrangian f.

Hypothesis 1. *There exist $r > 1$, $c_1 > 0$, $c_2 \geq c_1$, $c_3 \in \mathbb{R}$ and two maps $\gamma_1, \gamma_2 \in L^1(\Omega, \mathbb{R})$ such that for a.e. $x \in \Omega$, $\forall p \in \mathbb{R}^m$, $\forall \xi \in \mathbb{M}^{m\times n}$, we have*

$$c_1|\xi|^r + c_3|p|^{r_1} + \gamma_1(x) \leq \overline{f}(x, p, \xi) \ \leq\ c_2(|\xi|^r + |p|^r) + \gamma_2(x), \tag{3.2}$$

where $1 \leq r_1 < r$.

Remark 3.1. By classical results, condition (3.2) ensure the existence of a minimizer of $\overline{\mathcal{F}}$ on $\mathcal{W}$, that is to say an element $u \in \mathcal{W}$ such that $\overline{\mathcal{F}}(u) \leq \overline{\mathcal{F}}(v)$ for every $v \in \mathcal{W}$.

Hypothesis 2. *We assume that $\overline{f} : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R}$ is differentiable with respect to the variables p, ξ and that the gradients $\overline{f}_p, \overline{f}_\xi : \Omega\times\mathbb{R}^m\times\mathbb{M}^{m\times n} \to \mathbb{R}$ are continuous functions satisfying the following growth properties: there exist $c_4 \geq 0$ and $\gamma_3 \in L^1(\Omega, \mathbb{R})$ such that, for a.e. $x \in \Omega$, $\forall p \in \mathbb{R}^m$, $\forall \xi \in \mathbb{M}^{m\times n}$, we have*

$$|\overline{f}(x,p,\xi)|, |\overline{f}_p(x,p,\xi)|, |\overline{f}_\xi(x,p,\xi)| \leq \gamma_3(x) + c_4(|p|^r + |\xi|^r) \tag{3.3}$$

where $r > 1$ is the same index of Hypothesis 1.

Remark 3.2. Inequalities (3.3) guarantee that any minimizer of $\overline{\mathcal{F}}$ satisfies Euler-Lagrange equations in the following form:

$$\int_\Omega \left[\overline{f}_\xi(x, u(x), Du(x)) \cdot D\phi(x) + \overline{f}_p(x, u(x), Du(x)) \cdot \phi(x)\right] dx = 0 \tag{3.4}$$

for every $\phi \in W_0^{1,\infty}(\Omega, \mathbb{R}^m)$.

It is well known (See Sect. 3.4.2 in [5]), that the conditions expressed in Hypothesis 1 and 2 can be weakened in relation to the dimension n of the space. We leave to the reader the faculty of checking such weakened conditions directly in the quoted section of [5]. What is needed here is the existence of minimizers of $\overline{\mathcal{F}}$ and the validity of equations (3.4).

The next hypothesis contains the structural conditions on f and $\overline{f}$ which allow us to manage the non-(s.w.l)semicontinuity of the functional $\mathcal{F}$, that is to say the non-quasiconvexity of f.

Hypothesis 3. There exist an index $i \in \{1, \ldots, m\}$ and a positive constant $\alpha > 0$ such that

$$\overline{f}_{p^i}(x, p, \xi) \geq \alpha \quad \forall (x, p, \xi) \in \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n}. \tag{3.5}$$

For every $(x, p, \xi) \in \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n}$ such that

$$f(x, p, \xi) > \overline{f}(x, p, \xi) \tag{3.6}$$

we have

$$\overline{f}_{\xi^i}(x, p, \xi) = 0. \tag{3.7}$$

Remark 3.3. While Hypotheses 1 and 2 are classical requirements in order to ensure the existence of minimizer an the validity of Euler equations, Hypothesis 3 is the structural assumption on the lagrangians f an $\overline{f}$ that we impose in order to manage the non-semicontinuity (i.e. the non-quasiconvexity) of our problem. In particular, condition (3.7) is an affinity assumption with respect to a single component ξ^i of the matrix $\xi = (\xi^1, \ldots, \xi^m)$ on the relaxed lagrangian $\overline{f}$ in the detachment set on which the inequality (3.6) holds true.

If we consider the scalar case, corresponding to $m = 1$, this condition reduces to affinity in plain sense, that is to say that on the set on which $f(x, p, \xi) > \overline{f}(x, p, \xi)$, the relaxed lagrangian has the following form:

$$\overline{f}(x, p, \xi) = m \cdot \xi,$$

where $m \in \mathbb{R}^n$ is a fixed vector and ξ is the variable ranging in $\mathbb{R}^n$.

This situation is clear if we consider a model scalar functional of sum type corresponding to a lagrangian $f : \Omega \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ of the form

$$f(x, p, \xi) = g(x, \xi) + h(x, p),$$

with

$$g(x, \xi) = a(x)(|\xi|^2 - 1)^2,$$

with a factor $a \in C^0(\overline{\Omega}, \mathbb{R})$ such that $a(x) \geq \overline{a} > 0$ for every $x \in \Omega$. In the scalar case quasiconvexity reduces to convexity, hence it is immediate to see that the convex envelope with respect the variable ξ of the function $g = g(x, \xi)$ is given by

$$\overline{g}(x, \xi) = \begin{cases} g(x, \xi) & |\xi| \geq 1 \\ 0 & |\xi| < 1. \end{cases}$$

Then we have

$$\overline{f}(x, p, \xi) = \overline{g}(x, \xi) + h(x, p).$$

Observe that the condition (3.6) is equivalent to require $|\xi| < 1$ and for such vaues of the vector ξ we have $\overline{g}(\xi) = 0$. The affinity condition (3.7) is then obviously satisfied, and, in addition, we may assert that it holds true also for a lagrangian of the form

$$f(x, p\, \xi) = g(x, \xi) + h(x, p) + m \cdot \xi,$$

where $m \in \mathbb{R}^n$.

This example is quite instructive, since it can be translated to the vectorial setting. Take now a lagrangian $f : \Omega \times \mathbb{R}^n \times \mathbb{M}^{m \times n} \to \mathbb{R}$ of the same form

$$f(x, p, \xi) = g(x, \xi) + h(x, p), \tag{3.8}$$

where

$$g(x, \xi) = a(x)(|\xi|^2 - 1)^2 \tag{3.9}$$

with a factor $a(\cdot)$ as above. By the results exposed in Sect. 6.6.7 p. 309 of [5], we see that, also in this case, the quasiconvex envelope with respect the variable ξ of the function $g = g(x, \xi)$ coincides with the convex envelope and is given by

$$\overline{g}(x, \xi) = \begin{cases} g(x, \xi) & |\xi| \geq 1 \\ 0 & |\xi| < 1. \end{cases} \tag{3.10}$$

Then we have

$$\overline{f}(x, p, \xi) = \overline{g}(x, \xi) + h(x, p). \tag{3.11}$$

Maintaining all notations, we set $r = 4$ and assume that the term $h : \Omega \times \mathbb{R}^m \to \mathbb{R}$ is a continuous function, differentiable in the variable $p \in \mathbb{R}^m$, that $h_p : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$ is itself a continuous function and that the following growth conditions are satisfied for every $(x, p) \in \Omega \times \mathbb{R}^m$:

$$C_3 |p|^{r_1} + \beta_1(x) \leq h(x, p) \leq C_4 |p|^4 + \beta_2(x),$$

$$|h_p(x,p)| \le \beta_3(x) + C_5|p|^4,$$

where $C_3 \in \mathbb{R}$, $C_4, C_5 \ge 0$ are suitable constants, $\beta_1, \beta_2, \beta_3$ are $L^1(\Omega, \mathbb{R})$-functions and $1 \le r_1 < 4$.

It is easy to see that Hypotheses 1, 2 are satisfied. The monotonicity assumption (3.5) assume the form

$$h_{p^i}(x,p) \ge \alpha \quad \forall (x,p) \in \Omega \times \mathbb{R}^m, \tag{3.12}$$

while the affinity condition (3.7) can be easlily verified by a direct inspection of formulas (3.8), (3.9) and (3.10).

Hence we conclude that Hypotheses 1, 2, 3 are satisfied and we may apply Theorem 3.4 below with $r = 4$ and $u_0 \in W^{1,4}(\Omega, \mathbb{R}^m)$.

Theorem 3.4. *Assume Hypotheses 1, 2, 3 and that there exists a minimizer $\overline{u}$ of $\overline{\mathcal{F}}$ belonging to $\mathcal{W} \cap W^{1,\infty}_{loc}(\Omega, \mathbb{R}^m)$ such that $D\overline{u}$ is strongly approximately continuous almost everywhere in Ω. Then $\overline{u}$ is a minimizer of $\mathcal{F}$.*

Proof. We claim that

$$f(x, \overline{u}(x) D\overline{u}(x)) = \overline{f}(x, \overline{u}(x), D\overline{u}(x)) \quad \text{for a.e. } x \in \Omega. \tag{3.13}$$

Assume, by contradiction, that there exists a measurable $E \subseteq \Omega$, with $\mathrm{m}(E) > 0$, such that

$$f(x, \overline{u}(x) D\overline{u}(x)) > \overline{f}(x, \overline{u}(x), D\overline{u}(x)) \quad \text{for a.e. } x \in E, \tag{3.14}$$

so that we have

$$\overline{f}_{\xi^i}(x, u(x), Du(x)) = 0 \quad \text{for a.e. } x \in E. \tag{3.15}$$

Take x_0 point of density of the set E and Lebesgue point of $D\overline{u}$ at which $D\overline{u}$ is strongly approximately continuous. Since, by assumption $\overline{u}$ lies in $W^{1,\infty}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$, we may assume that it is continuous at x_0. Take $\overline{\rho} > 0$ such that $B(x_0, \overline{\rho}) \subseteq \Omega$, $\epsilon > 0$, and, for every $\rho \in \,]0, \overline{\rho}]$ consider the sets

$$S_{\rho,\epsilon} \doteq \{x \in B(x_0, \rho) : \ |D\overline{u}(x) - D\overline{u}(x_0)| \ge \epsilon\}. \tag{3.16}$$

Choosing $\overline{\rho}$ and ϵ sufficiently small, by the continuity of f, $\overline{f}$ and $\overline{u}$, for every $\rho \in \,]0, \overline{\rho}]$, we have

$$f(x, \overline{u}(x) D\overline{u}(x)) > \overline{f}(x, \overline{u}(x), D\overline{u}(x)) \quad \text{for a.e. } x \in B(x_0, \rho) \setminus S_{\rho,\epsilon}, \tag{3.17}$$

and then, by (3.6)–(3.7) in Hypothesis 3,

$$\overline{f}_{\xi^i}(x, \overline{u}(x), D\overline{u}(x)) = 0 \quad \text{for a.e. } x \in B(x_0, \rho) \setminus S_{\rho,\epsilon}. \tag{3.18}$$

In addition, recalling definition 2.3, there exists a map $\omega : [0, +\infty[\,\to [0, +\infty[$ such that

$$\lim_{t \to 0+} \omega(t) = 0 \tag{3.19}$$

and

$$\frac{\mathrm{m}(S_{\rho,\epsilon})}{\mathrm{m}(B(x_0, \rho))} = \rho\omega(\rho). \tag{3.20}$$

Recalling that $\overline{u} \in W^{1,\infty}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$, we take $M > 0$ such that

$$\left|\overline{f}_{\xi^i}(x, \overline{u}(x), D\overline{u}(x))\right| \leq M \quad \text{for a.e. } \ x \in B(x_0, \overline{\rho}). \tag{3.21}$$

Now, for every $\rho \in]0, \overline{\rho}]$, we consider the map $\varphi_\rho \in W^{1,\infty}_0(\Omega, \mathbb{R})$ given by lemma 2.7 and $\phi \in W^{1,\infty}_0(\Omega, \mathbb{R}^m)$ defined by

$$\phi_j \doteq 0 \quad \text{for} \quad j \neq i, \quad \phi_i \doteq \varphi_\rho. \tag{3.22}$$

Then we write equations (3.4) for the minimizer $\overline{u}$, which take the form

$$\int_\Omega \left(\overline{f}_{\xi^i}(x, \overline{u}(x), D\overline{u}(x)) \cdot \nabla\varphi_\rho(x) + \overline{f}_{p^i}(x, \overline{u}(x), D\overline{u}(x))\varphi_\rho(x)\right) \, dx = 0. \tag{3.23}$$

Recalling condition (3.5) in Hypothesis 3, we have

$$\begin{aligned}
0 < \alpha = \alpha \int_\Omega \varphi_\rho(x)\, dx &\leq \int_\Omega \overline{f}_{p^i}(x, \overline{u}(x), D\overline{u}(x))\varphi_\rho(x)\, dx \\
&= -\int_\Omega \overline{f}_{\xi^i}(x, \overline{u}(x), D\overline{u}(x)) \cdot \nabla\varphi_\rho(x)\, dx \\
&\leq \int_{B(x_0,\rho)} |\overline{f}_{\xi^i}(x, \overline{u}(x), D\overline{u}(x))| \cdot |\nabla\varphi_\rho(x)|\, dx \\
&= \frac{\beta}{\rho^{n+1}} \int_{S_{\rho,\epsilon}} |\overline{f}_{\xi^i}(x, \overline{u}(x), D\overline{u}(x))|, dx \\
&\quad + \frac{\beta}{\rho^{n+1}} \int_{B(x_0,\rho)\setminus S_{\rho,\epsilon}} |\overline{f}_{\xi^i}(x, \overline{u}(x), D\overline{u}(x))|, dx \\
&\leq \frac{\beta}{\rho^{n+1}} \int_{S_{\rho,\epsilon}} M\, dx \\
&= \frac{\beta M}{\rho^{n+1}}\, \mathrm{m}\,(S_{\rho,\epsilon}) \\
&= \frac{\beta M}{\rho}\, \frac{\mathrm{m}\,(B(x_0,\rho))}{\rho^n}\, \frac{\mathrm{m}\,(B(x_0,\rho) \cap S_{\rho,\epsilon})}{\mathrm{m}\,(B(x_0,\rho))} \\
&\leq C\omega(\rho) \to 0, \quad \text{as} \quad \rho \to 0+,
\end{aligned} \tag{3.24}$$

where C is a positive constant and we have used all properties (3.18)-(3.21). The obtained nequality is the contradiction, coming from the assumption of the existence of the set E in (3.13), which concludes the proof. □

Remark 3.5. It is clear from computation (3.24) in the above proof, that assuming that $D\overline{u}$ is strongly approximately continuous is necessary for our argument. This situation is radically different with respect to the scalar case, as treated in the quoted papers [18, 19], since in that situation it possible to modify locally the scalar minimizer $\overline{u}$ without affecting the value of the functional, obtaining in such a way the existence result by integro-extremality method. In the present vectorial setting, being $\overline{u}$ a vector, a simultaneous modification of two or more components of the minimizer $\overline{u}$ would destroy the contradictory argument used in the scalar situation. The *mixture* of the components of the competing map $u = (u^1, \ldots, u^m)$ in the determination of the value of the functional $\overline{\mathcal{F}}(u)$ is the crucial difference which makes impossible, in general, to translate the integro-extremality method to the vectorial case. Anyway, such translation is possible if the contribution of the single components ∇u^i of the matrix Du to the value of $\overline{\mathcal{F}}(u)$ is not *mixed*, and this surely happens whenever the dependence of the functional $\overline{\mathcal{F}}(u)$ from the lines of the matrix Du is *separate*. Actually, this idea is developed in subsequent section 4, where we prove an existence theorem directly inspired by the quoted scalar results.

We stress that in the related one-dimensional theory developed in [22,23], it is not necessary to assume strong approximate continuity of the derivatives, due to the standard nice properties of maps depending on a scalar real variable, which allow to manage Euler equations in a more efficient way.

We now give a version of Theorem 3.4 in which the quasiconvex envelope $\overline{f}$ is quasiaffine on the set $f > \overline{f}$. We have to modify the structural Hypothesis 3.

Hypothesis 4. There exists an index $i \in \{1, \dots, m\}$ and a positive constant $\alpha > 0$ such that

$$\overline{f}_{p^i}(x, p, \xi) \geq \alpha \qquad \forall (x, p, \xi) \in \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n}. \tag{3.25}$$

There exist a quasiaffine function $G : \mathbb{M}^{m\times n} \to \mathbb{R}$ of order k and a function $\beta \in L^1(\Omega, \mathbb{R}) \cap C^0(\Omega, \mathbb{R})$ such that for every $(x, p, \xi) \in \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n}$ such that

$$f(x, p, \xi) > \overline{f}(x, p, \xi) \tag{3.26}$$

we have

$$\overline{f}(x, p, \xi) = \beta(x) + G(\xi). \tag{3.27}$$

Definition 3.6. Given a real index $r > k$, we define the lagrangians $f^1 : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R}$ and $\overline{f}^1 : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R}$ by setting

$$f^1(x, p, \xi) \doteq f(x, p, \xi) - [\beta(x) + G(\xi)], \tag{3.28}$$

and

$$\overline{f}^1(x, p, \xi) \doteq \overline{f}(x, p, \xi) - [\beta(x) + G(\xi)]. \tag{3.29}$$

Then we introduce the corresponding functionals

$$\mathcal{F}_1(u) = \int_I f^1(x, u(x), Du(x))\, dx, \quad u \in \mathcal{W}$$

and

$$\overline{\mathcal{F}}_1(u) = \int_I \overline{f}^1(x, u(x), Du(x))\, dx, \quad u \in \mathcal{W},$$

where, as above, $\mathcal{W} = u_0 + W_0^{1,r}(\Omega, \mathbb{R}^m)$, u_0 is a given map in $W^{1,r}(\Omega, \mathbb{R}^m)$ and $r > k$.

For convenience, we set also

$$\mathcal{G}(u) = \int_I [\beta(x) + G(Du(x))]\, dx, \quad u \in \mathcal{W}.$$

Lemma 3.7. *Assume Hypotheses 1, 2, 4 and let $r > k$. Then*

(i) the function $\overline{f}^1$ is the lower quasiconvex envelope of f^1 with respect to the last variable.

(ii) The functional $\mathcal{G}$ is constant on $\mathcal{W}$.

(iii) By a suitable re-definition of the constants c_l, $l = 1, \ldots, 5$ and of the functions γ_l, $l = 1, 2, 3$, the lagrangian $\overline{f}^1$ satisfies all conditions in Hypotheses 1 and 2.

(iv) For every $(x, p, \xi) \in \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n}$ such that

$$f^1(x, p, \xi) > \overline{f}^1(x, p, \xi) \tag{3.30}$$

we have

$$\overline{f}^1_{\xi^i}(x, p, \xi) = 0. \tag{3.31}$$

Proof. Property *(i)* is a direct consequence of (3.26) and (3.27) in Hypothesis 4, while *(ii)* follows directly from proposition 2.2, and we stress that for this reason we need $r > k$, because otherwise the functional $\mathcal{G}$ would be undefined. Point *(iii)* follows from the fact that G is a polynomial in the variables ξ^i_j of degree $k < r$. The last property *(iv)* is immediate, since on the open set $f^1 > \overline{f}^1$ we have $\overline{f}^1 \equiv 0$ □

Theorem 3.8. *Assume Hypotheses 1, 2, 4 and $r > k$. Suppose in addition that there exists a minimizer $\overline{u}$ of $\overline{\mathcal{F}}$ belonging to $\mathcal{W} \cap W^{1,\infty}_{loc}(\Omega, \mathbb{R}^m)$ such that $D\overline{u}$ is strongly approximately continuous almost everywhere in Ω. Then $\overline{u}$ is a minimizer of $\mathcal{F}$.*

Proof. By virtue of point *(ii)* in lemma 3.7, the functionals $\mathcal{F}$ and $\mathcal{F}_1$, as well as the functionals $\overline{\mathcal{F}}$ and $\overline{\mathcal{F}}_1$, when defined on $\mathcal{W}$, differ by a constant. Hence the minimization of $\mathcal{F}$ and $\mathcal{F}_1$, as well as the minimization of $\overline{\mathcal{F}}$ and $\overline{\mathcal{F}}_1$, are equivalent. By the properties listed in lemma 3.7, we see that the integrands f^1 and $\overline{f}^1$ satisfy the Hypotheses of Theorem 3.4. Hence the conclusion follows immediately. □

In the proof of Theorem 3.4 we have used the properties expressed in Hypothesis 3 only on the solution $\overline{u}$. This fact induces to formulate the following consequence.

Corollary 3.9. *Assume Hypotheses 1, 2 and suppose in addition that there exists a minimizer $\overline{u}$ of $\overline{\mathcal{F}}$ belonging to $\mathcal{W} \cap W^{1,\infty}_{loc}(\Omega, \mathbb{R}^m)$ such that $D\overline{u}$ is strongly approximately continuous almost everywhere in Ω and the following properties hold true:*

(i) there exist an index $i \in \{1, \ldots, m\}$ and a positive constant $\alpha > 0$ such that

$$f_{p^i}(x, \overline{u}(x), D\overline{u}(x) \geq \alpha \quad \textit{for a.e. } x \in \Omega;$$

(ii) for almost every $x \in \Omega$ such that

$$f(x, \overline{u}(x), D\overline{u}(x)) > \overline{f}(x, \overline{u}(x), D\overline{u}(x)),$$

we have

$$f_{\xi^i}(x, \overline{u}(x), D\overline{u}(x)) = 0.$$

Then $\overline{u}$ is a minimizer of $\mathcal{F}$ too.

Proof. Straightforward consequence of Theorem 3.4. □

We leave to the reader the statement of the analogous corollary under the assumptions of Theorem 3.8.

4. The case of separate variables

In this section we consider a class of functionals not covered by previous theory, corresponding to lagrangians with a separate dependence on the single rows ∇u^i of the jacobian matrix Du. In such situation Hypothesis 3 must be modified, since the index i corresponding to the null derivative (condition (3.7)) may depend on the point (x, p, ξ) for which (3.6) holds true. On the other side, the separation of the gradient variables allows us to adopt the approach used in [19] for scalar functionals, so that we do not need the strong approximate continuity of the minimizers of $\overline{\mathcal{F}}$ and of their derivatives Du. The regularity condition required in this case is classical differentiability almost everywhere, which holds true, for example, by assuming that $r > n$.

We introduce a lagrangian $f : \Omega \times \mathbb{R}^n \times \mathbb{M}^{m\times n} \to \mathbb{R}$ of the following form

$$f(x, p, \xi) = \sum_{i=1}^{m} g_i(x, \xi^i) + h(x, p), \tag{4.1}$$

where $g_i : \Omega\times\mathbb{R}^m \to \mathbb{R}$, $i = 1, \dots, m$ are continuous functions and $h : \Omega\times\mathbb{R}^m \to \mathbb{R}$ is a Caratheodory function. Denoting by $\overline{g}_i : \Omega \times \mathbb{R}^m \to \mathbb{R}$, $i = 1, \dots, m$ the lower convex envelopes of the maps g_i with respect to the variables ξ^i, we set

$$\overline{f}(x, p, \xi) = \sum_{i=1}^{m} \overline{g}_i(x, \xi^i) + h(x, p), \tag{4.2}$$

and impose growth conditions analogous to the ones of Hypothesis 1:

Hypothesis 5. *There exist $r > 1$, $c_1 > 0$, $c_2 \geq c_1$, $c_3 \in \mathbb{R}$, $c_4 \geq 0$ and two maps $\gamma_1, \gamma_2 \in L^1(\Omega, \mathbb{R})$ such that for a.e. $x \in \Omega$, $\forall p \in \mathbb{R}^m$, $\forall \xi \in \mathbb{M}^{m\times n}$, $\forall i = 1, \dots, m$, we have*

$$c_1|\xi|^r + \gamma_1(x) \leq \overline{g}_i(x, \xi) \ \leq \ c_2|\xi|^r + \gamma_2(x), \tag{4.3}$$

$$c_3|p|^{r_1} + \gamma_1(x) \leq h(x, p) \ \leq c_4|p|^r + \gamma_2(x), \tag{4.4}$$

where $1 \leq r_1 < r$.

As above, these growth conditions may be adapted in relation with the dimension of the space, according to the remark of previous section.

Hypothesis 6. *We assume the following structural conditions on the integrand $\overline{f}$.*

(i) There exists $k \in \{1,\dots,m\}$ such that for every $i \in \{1,\dots,k\}$ we have

$$\overline{g}_i(x,\eta^i) < g_i(x,\eta^i) \tag{4.5}$$

for some point $(x,\eta^i) \in \Omega \times \mathbb{R}^m$ and $g^i \equiv \overline{g}^i$ for $i \in \{k+1,\dots,m\}$.

(ii) For every $i \in \{1,\dots,k\}$ there exist a field $\gamma_i \in \mathcal{M}^-(\Omega,\mathbb{R}^n)$ and a function $q_i \in C^0(\Omega)$ such that, for every point $(x_0,\xi_0^i) \in \Omega \times \mathbb{R}^m$ for which

$$\overline{g}_i(x_0,\xi_0^i) < g(x_0,\xi_0^i),$$

there exist a neighbourhood U of x_0 and $\tau \in \mathbb{R}^+$ such that

$$\overline{g}_i(x,\xi^i) = \gamma_i(x)\cdot\xi^i + q_i(x) \quad \forall (x,\xi) \in U \times \overline{B(\xi_0^i,\tau)} \tag{4.6}$$

and

$$\overline{g}_i(x,\xi^i) \geq \gamma_i(x)\cdot\xi + q_i(x) \quad \forall (x,\xi) \in U \times \mathbb{R}^m. \tag{4.7}$$

(iii) For every $i \in \{1,\dots,k\}$, the map

$$\mathbb{R} \ni p^i \mapsto h(x,p^1,\dots,p^m) = h(x,p) \tag{4.8}$$

is monotone non decreasing for almost every $x \in \Omega$.

We consider the functionals

$$\mathcal{F}(u) = \int_\Omega f(x,u,Du)\,dx = \int_\Omega \left(\sum_{i=1}^m g_i(x,\nabla u^i) + h(x,u)\right)dx, \tag{4.9}$$

and

$$\overline{\mathcal{F}}(u) = \int_\Omega \overline{f}(x,u,Du)\,dx = \int_\Omega \left(\sum_{i=1}^m \overline{g}_i(x,\nabla u^i) + h(x,u)\right)dx, \tag{4.10}$$

defined for $u \in \mathcal{W} = u_0 + W_0^{1,r}(\Omega,\mathbb{R}^m)$, with the same extension of previous section.

Remark 4.1. The functional $\overline{\mathcal{F}}$ is clearly quasiconvex, hence the set of it minimizers is nonempty. Denoting by Qf the lower quasiconvex envelope of f with respect to the last variable ξ, clearly we have $\overline{f} \leq Qf$. Hence, if we are able to find a minimizer u of $\overline{\mathcal{F}}$ such that we have

$$\overline{f}(x,u(x,Du(x)) = f(x,u(x),Du(x) \quad \text{for a.e. } x \in \Omega,$$

we may conclude that u minimizes $\mathcal{F}$ too. In particular we are not required to show that $\overline{f}$ is the lower quasiconvex envelope of f.

In this section we do not use Euler-Lagrange equations, hence we do not require differentiability properties on $\overline{f}$.

As an example, we remark that a simple case satisfying our hypotheses is the following one:

$$f(x,p,\xi) = \sum_{i=1}^m \left(|\xi^i|^2 - 1\right)^2 + h(x,p), \tag{4.11}$$

where the map h satisfies the growth and monotonicity properties specified above.

Theorem 4.2. *Assume Hypotheses 5 and 6. Suppose, in addition, that all the minimizers of $\overline{\mathcal{F}}$ are classically differentiable almost everywhere in Ω. Then $\mathcal{F}$ admits a minimizer.*

Remark 4.3. By classical notions on Sobolev spaces, the differentiability almost everywhere required in the statement is automatically satisfied if we assume $r > n$.

Proof. The proof is an adaptation of the arguments performed in Theorem 1 of papers [19] or [21] to which we refer for details.

We call $\mathcal{S} \subseteq \mathcal{W}$ the nonempty set of minimizers of $\overline{\mathcal{F}}$ and observe that it is compact with respect to the strong topology of $L^1(\Omega, \mathbb{R}^m)$.

Then we define the map

$$\mathcal{S} \ni u \mapsto S(u) = \sum_{i=1}^{m} \int_{\Omega} u^i(x)\, dx \in \mathbb{R}, \tag{4.12}$$

which is clearly continuous with respect to $L^1(\Omega, \mathbb{R}^m)$-norm. Hence, by Weierstrass theorem, there exists an element $\overline{u} \in \mathcal{S}$ such that

$$S(\overline{u}) \leq S(u) \quad \forall u \in \mathcal{S}. \tag{4.13}$$

Claim. *The map $\overline{u}$ minimizes $\mathcal{F}$.*

As we have seen in previous sections, and taking into account remark 4.1, we have to show that

$$g_i(x, \nabla \overline{u}^i(x)) = \overline{g}_i(x, \nabla \overline{u}^i(x)) \quad \text{for a.e. } x \in \Omega,\ \forall i \in \{1, k\}. \tag{4.14}$$

Assume, by contradiction, that there exists $l \in \{1, k\}$ and $x_0 \in \Omega$ such that $\overline{u}$ is classically differentiable in x_0 and

$$g_l(x_0, \nabla \overline{u}^l(x_0)) > \overline{g}_l(x_0, \nabla \overline{u}^l(x_0)). \tag{4.15}$$

Since $\overline{u}$ is assumed to be classically differentiable almost everywhere in Ω, if inequality (4.16) provides a contradiction, the proof is achieved.

By the continuity of the map $\Omega \times \mathbb{R}^m \ni (x, \xi^l) \mapsto g_l(x, \xi^l)$, we deduce the existence of a neighbourhood $U \subseteq \Omega$ of the point x_0 and of a number $t > 0$ such that

$$g_l(x, \xi^l) > \overline{g}_l(x, \xi^l) \quad \forall x \in U,\ \forall \xi^l \in \overline{B(\nabla \overline{u}^l(x_0), t)}. \tag{4.16}$$

Applying lemma 2.6, we construct a map $\breve{\overline{u}}^l$ with properties (2.5)–(2.11). Then we define the map $\breve{u}$ by setting $\breve{u}^i = \overline{u}^i$ for $i \neq l$ and $\breve{u}^l = \breve{\overline{u}}^l$, that is to say $\breve{u} : \Omega \to \mathbb{R}^m$ given by

$$\breve{u} = \begin{pmatrix} \breve{u}^1 \\ \vdots \\ \breve{u}^l \\ \vdots \\ \breve{u}^m \end{pmatrix} = \begin{pmatrix} \overline{u}^1 \\ \vdots \\ \breve{\overline{u}}^l \\ \vdots \\ \overline{u}^m \end{pmatrix} \tag{4.17}$$

Property (2.5) implies that $\check{u} \in \mathcal{W}$. We claim that $\check{u} \in \mathcal{S}$, that is to say that is a minimizer of $\overline{\mathcal{F}}$.

By inequality (2.6) and the monotonicity of the map h expressed by (4.8) we have that

$$\int_\Omega h(x, \check{u}(x))\, dx \le \int_\Omega h(x, \overline{u}(x))\, dx. \tag{4.18}$$

Now we observe that, by (2.7), (2.9), (4.6) and (4.7), for almost every $x \in \check{\Lambda}$, we have

$$\overline{g}_l(x, \nabla \check{u}^l(x)) = \gamma_l(x) \cdot \nabla u^l(x) + q^l(x), \quad \text{for a.e. } x \in \check{\Lambda} \tag{4.19}$$

and

$$\overline{g}_l(x, \nabla \overline{u}^l(x)) \ge \gamma_l(x) \cdot \nabla \overline{u}^l(x) + q^l(x) \quad \text{for a.e. } x \in \check{\Lambda}. \tag{4.20}$$

Formulas (2.11) and (4.19)–(4.20) imply that

$$\begin{aligned} &\int_{\check{\Lambda}} \overline{g}_l(x, \nabla \check{u}^l(x))\, dx - \int_{\check{\Lambda}} \overline{g}_l(x, \nabla \overline{u}^l(x))\, dx \\ &\quad = \int_{\check{\Lambda}} \gamma_l(x) \cdot \left(\nabla \check{u}^l(x) - \nabla \overline{u}^l(x) \right)\, dx \le 0. \end{aligned} \tag{4.21}$$

Putting together (2.7), (2.9), and (4.21), we obtain that

$$\begin{aligned} \int_\Omega \overline{g}_l(x, \nabla \check{u}^l(x))\, dx &= \int_{\Omega\setminus\check{\Lambda}} g_l(x, \nabla \check{u}^l(x))\, dx \int_{\check{\Lambda}} \overline{g}_l(x, \nabla \check{u}^l(x))\, dx \\ &\le \int_{\Omega\setminus\check{\Lambda}} \overline{g}_l(x, \nabla \overline{u}^l(x))\, dx \\ &\quad + \int_{\check{\Lambda}} \overline{g}_l(x, \nabla \overline{u}^l(x))\, dx \int_{\Omega\setminus\check{\Lambda}} \overline{g}_l(x, \nabla \overline{u}^l(x))\, dx \\ &= \int_\Omega \overline{g}_l(x, \nabla \overline{u}^l(x))\, dx. \end{aligned}$$

Recalling definition (4.17), we conclude that

$$\int_\Omega \sum_{i=1}^m \overline{g}_i(x, \nabla \check{u}^i(x))\, dx \le \int_\Omega \sum_{i=1}^m \overline{g}_i(x, \nabla \overline{u}^i(x))\, dx. \tag{4.22}$$

Collecting (4.18) and (4.22), we obtain that $\overline{\mathcal{F}}(\check{u}) \le \overline{\mathcal{F}}(\overline{u})$ and this proves that $\check{u} \in \mathcal{S}$.

Now we observe that by definitions (4.12), (4.17) and by inequality (2.10), we have immediately that $S(\check{u}) < S(\overline{u})$, in contradiction with the definition (4.13) of $\overline{u}$. This ends the proof. □

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