SISSA
Scuola Internazionale Superiore di Studi Avanzati

Mathematics Area

PhD in Geometry and Mathematical Physics

Painlevé II, anharmonic oscillators, & degenerate orthogonal polynomials

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Academic Year 2021-22

Abstract

In this thesis we study a surprising connection relating the second Painlevé transcendent, anharmonic oscillators and degenerate orthogonal polynomials. This connection arose from the investigations into the similarity of two sets of points in the complex plane. On one side is the set of zeroes of the Vorob'ev-Yablonskii polynomials, that is, the poles of rational solutions of the second Painlevé transcendent

$$\frac{\mathrm{d}^2u}{\mathrm{d}t^2} = 2u^3 + tu + \alpha, \qquad \alpha \in \mathbb{C},$$

where $u = u(t)$ is a complex function of the complex variable t. On the other side is the values of $t \in \mathbb{C}$ for which the spectrum of the quartic anharmonic oscillator in the complex domain

$$\frac{\mathrm{d}^2y}{\mathrm{d}z^2} - \left(z^4 + tz^2 + 2Jz\right) = \Lambda y,$$

has eigenvalues Λ of algebraic multiplicity at least 2 (under suitable boundary conditions). The similarity between these two sets of points when $\alpha = n$ and $J = n + 1$, with $n \in \mathbb{N}$, was first observed by Shapiro and Tater in [ST22] and we give an explanation for this phenomenon.

The study of this problem naturally lead us to the notion of certain non-hermitian orthogonal polynomial polynomials P_n satisfying an excess of orthogonality conditions

$$\int_\Gamma P_n(z)z^k e^{\theta(z)}\mathrm{d}z = 0, \quad k = n, n+1, \dots, n+\ell-1,$$

where θ is such that θ' is a rational functions as below and the contour Γ depends on θ.

These *degenerate* orthogonal polynomials are in one-to-one correspondence to the solutions of the *Stieltjes-Fekete* equilibrium problem

$$\sum_{k \neq j} \frac{1}{z_j - z_k} = \frac{A(z)}{2B(z)}, \qquad \theta'(z) = -\frac{A(z) + B'(z)}{B(z)},$$

where A, B are relatively prime polynomials. This generalises the famous result of Stieltjes, which relates the zeroes of the classical orthogonal polynomials to the configuration of points on the line that minimize a suitable potential with logarithmic interactions under an external field. We study the case when the derivative of the external field is an arbitrary rational complex function. When the differential of the external field is of degree 3 on the Riemann sphere our result reproduces Stieltjes original findings and, for more than a century after the original result, provides a direct generalisation for higher degree.

Acknowledgements

First and foremost I am very grateful to my advisors Tamara Grava, Marco Bertola and Francesco Mezzadri for introducing me to the kaleidoskopic world of integrable systems, Painlevé equations, orthogonal polynomials and random matrices. Your passion and enthusiasm for mathematics is incredibly inspiring and it has been a pleasure to learn from you. In particular I want to thank Tamara for the opportunity to pursue a co-tutelle PhD joint between the University of Bristol and SISSA and for her continued support and patience throughout this programme. Her efforts to build an international community of mathematicians around integrable systems and SISSA have caused a deep impression in me and I will always remember fondly being part of it.

I also really appreciated the many useful and productive discussion with Davide Guzzetti, Alexander Its, Pieter Roffelsen and Ahmad Barhoumi.

I am indebted to my friends Emilia Alvarez and Jordan Frost, both of which embarked on a PhD at the same time as me, and to the support they provided me during this academic journey, from the beginning until the end. Without you I would have not made it this far, so thank you!

I am also incredibly thankful to my cohort at Bristol: Akshat Mudgal, Valeriya Kovaleva, Bhargavi Jonnadula, Alex Little, Javier Pliego García and Emilia Alvarez (aka the 6th floor gang), as well as Harry Petyt, Ayesha Hussain and Peter Bradshaw for creating a fun and helpful enviroment in which to work, be curious and learn from each other.

A really big *grazie* also goes to the excellent people I met in SISSA: Massimo Gisonni, Felipe Lopez Reyes, Giulio Ruzza, Guido Mazzuca, Harini Desiraju, Sofia Tarricone, Mario de Marco, Michele Grafeo and Giuseppe Orsatti for the discussion on integrable systems and geometry from which I learnt so much, as well as for the shared conferences and workshops in which we enjoyed together.

From my time in Trieste I also want to thank my friends from Villa Spiga: Daniele, Anna and Peppe. Your support during the pandemic was stellar, it was a difficult time for all of us but we made it through together. Without you my time in Trieste would have been very different.

I also want to thank my friends Anastasia, Charlie, Kenji who were always there to cheer me up and also to Lottie for being there in some of my most stressful moments.

Finally I am eternally grateful to my family for their unconditional support, not only during this PhD, but always.

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Chapter 1

Introduction and statement of results

1.1 Overview

This thesis consists of two distinct but related pieces of work. The first concerns the second Painlevé equation, which famously appears in many applications in mathematical physics and integrable systems. In particular we look at the similarity between the poles $t = a$ of its rational solutions and certain degenerations of the *Exactly Solvable* spectrum (t, Λ) of a particular linear ODE with quasi-polynomial solutions $y(z) = p_n(z)e^{\theta(z)}$. This resemblance was first noticed by Shapiro and Tater back in 2014, but only recently published in [ST22], when studying the quasi-exactly solvable spectrum of a quartic potential. An illustrative picture of this similarity can be seen in Fig. 1.1. A priori, there is no connection between these two sets of points, since one is coming from a linear ODE and another from a non-linear ODE. Using the isomondromic representation of the Painlevé equations and the exact WKB method, we are able to explain the similarity of these two sets of points in the large n limit.

Along the way we discover that the polynomials $p_n(z)$ appearing in the quasi-polynomial solution satisfy an excess of orthogonality conditions, and are thus called *degenerate* orthogonal polynomials. Furthermore, we show that the roots of this degenerate orthogonal polynomials satisfy certain equations analogous to the electrostatic equilibrium of the classical orthogonal polynomials. This is the motivation for the second part of this thesis, where we introduce the concept of degenerate orthogonal polynomials, and we show they solve a linear ODE with rational potential and their roots satisfy the Stieltjes-Fekete equilibrium, thus generalising some of the well known properties of the classical orthogonal polynomials.

The structure of this thesis is the following.

Chapter 2 contains a review of the isomonodromic theory of ODEs in the complex plane, and its relation with the Painlevé equations.

Chapter 3 is an overview of the relevant results in the theory of orthogonal polynomials and the exact WKB method which will be used to prove our main theorems.

Chapter 4 is dedicated to the proof of the aforementioned phenomenon first noticed by Shapiro and Tater. The result of this chapter are taken from our preprint "*Exactly solvable anharmonic oscillator, degenerate orthogonal polynomials and Painlevé II*", written in collaboration with T. Grava and M. Bertola [BCG22-1]

Chapter 5 introduces the notion of degenerate orthogonal polynomials in the complex plane and provides necessary and sufficient conditions for their existence based on a electrostatic equilibrium of their roots. The result of this chapter are taken from our preprint "*The Stieltjes–Fekete problem and degenerate orthogonal polynomials*", also written in collaboration with T. Grava and M. Bertola [BCG22-2]

We will now describe our results in more detail.

1.2 The Shapiro-Tater conjecture

The second Painlevé equation

$$\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} = 2u^3 + tu + \alpha \tag{1.1}$$

admits rational solutions when $\alpha = n \in \mathbb{Z}$ in the shape

$$u_n(t) = \frac{\mathrm{d}}{\mathrm{d}t} \log \frac{Y_{n-1}(t)}{Y_n(t)}. \tag{1.2}$$

Here Y_j denotes the recursively defined polynomials

$$Y_{j+1}(t)Y_{j-1}(t) = tY_j^2(t) - 4\left[Y_j''(t)Y_j(t) - \left(Y_j'(t)\right)^2\right], \qquad j \geq 1, \tag{1.3}$$

with $Y_0(t) = 1$ and $Y_1(t) = t$. This was recognized by Vorob'ev and Yablonskii in two separate papers [Vor65, Yab59], and for this reason the polynomials Y_j are called Vorob'ev-Yablonskii polynomials. For $\alpha = \pm n$ there are n^2 poles of the rational solution, of which $n(n-1)/2$ correspond to poles residue $+1$ and the remaining $n(n+1)/2$ correspond to poles residue -1. The poles of negative and positive resiude are the zeros $Y_n(t)$ and $Y_{n-1}(t)$ respectively. The roots of the Vorob'ev-Yablonski follow a "triangular" pattern that has been studied extensively in the community of researchers of Painlevé equations [BB15, BM12, BM14, BM15].

A seemingly unrelated problem is the study of the quasi-exactly solvable spectrum of the quartic oscillator

$$\frac{\mathrm{d}^2 y}{\mathrm{d}z^2} - (z^4 + tz^2 + 2Jz + \Lambda)y = 0 \tag{1.4}$$

$$y(s\mathrm{e}^{k\pi i/3}) \to 0, \quad s \to +\infty, \ k = -1, 1, 3. \tag{1.5}$$

In their paper [ST22] [1] Shapiro and Tater expressed the spectrum associated to quasi-polynomial solutions $y(z) = p_n(z)\mathrm{e}^{\theta(z)}$ when $J = n$ explicitly in terms of a determinant. Thus they were able to obtain its branching locus in terms of a discriminant i.e. they obtained the points $t \in \mathbb{C}$ for which the spectrum Λ is repeated. Upon inspection of this discriminant, they found a remarkable similarity between these points and the roots of the Vorob'ev-Yablonskii polynomials which lead them to make the conjecture which we now explain full detail.

The conjecture. We start by looking at the boundary value problem of the anharmonic oscillator where t and J are in general complex parameters and z and $y(z)$ are the complex independent and dependent variables, respectively.

Consider *quasi-polynomial* solutions to (1.4), that it, solutions of the form

$$y(z) = p(z)\mathrm{e}^{\theta(z)}, \qquad \theta(z) = \frac{z^3}{3} + \frac{tz}{2}, \tag{1.6}$$

where $p(z)$ is a polynomial. We may sometimes use the notation $\theta(z;t)$ to emphasize the dependence of θ on the parameter t. A simple substitution leads us to the following differential equation satisfied by $p(z)$:

$$\frac{\mathrm{d}^2 p}{\mathrm{d}z^2} + 2\left(z^2 + \frac{t}{2}\right)\frac{\mathrm{d}p}{\mathrm{d}z} - 2(J-1)zp = \left(\Lambda - \frac{t^2}{4}\right)p. \tag{1.7}$$

Upon setting $J = n + 1$, one readily notices the ODE preserves the finite-dimensional linear space of polynomials of degree at most n. Interpreting the left hand side of (1.7) as a linear operator $\mathcal{L}_n$ acting on this space, we can find the associated spectrum by considering the matrix eigenvalue problem

$$M_n(t)\mathrm{v} = \lambda\mathrm{v} \tag{1.8}$$

[1] This anharmonic oscillator differs from the one in Shapiro and Tater in that it has three boundary conditions instead of two. This ensure that all the solutions are quasi-polynomial, as will be shown Proposition 4.7.

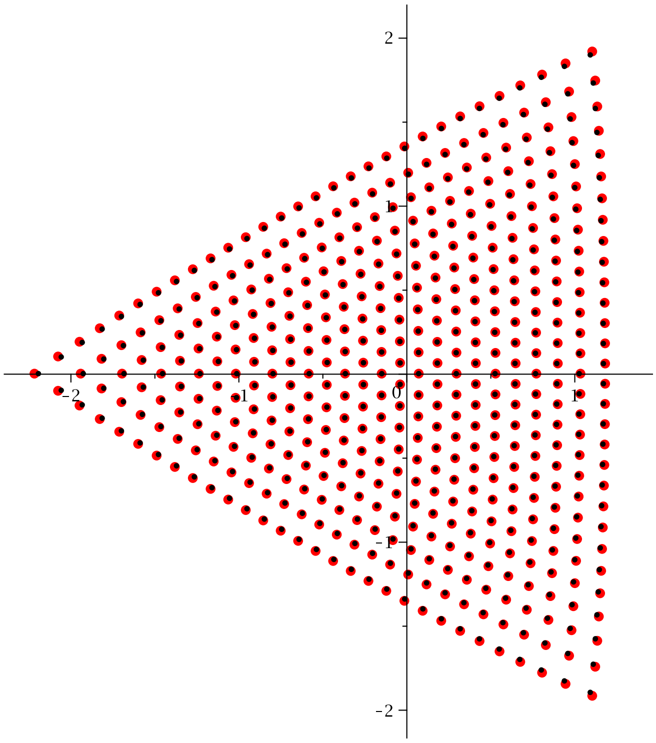


Figure 1.1: Scaled roots of the Vorob'ev-Yablonsky polynomials $Y_n(n^{2/3}s)$ in red, and roots of the discriminant $D_n(n^{2/3}s)$ in black, for $n = 30$. This particular scaling was conjectured by Shapiro and Tater in [ST22].

where $v = [v_0, v_1, \ldots, v_n]^\top$ is the vector of coefficients of the polynomial p and $M_n(t) = \left[m_n^{(i,j)}\right]_{i,j=0}^{n}$ is the matrix associated to $\mathcal{L}_n$ with respect to the standard basis $\{1, z, z^2, \ldots, z^n\}$. In other words $\mathcal{L}_n(z^j) = \sum_{i=0}^{n} m_n^{(i,j)} z^i$. A straight-forward computation shows that $M_n(t)$ is a band matrix with zeroes along the diagonal

$$M_n(t) = \begin{bmatrix} 0 & -2n & & & & & \\ t & 0 & -2(n-1) & & & & \\ 2 & 2t & 0 & & & & \\ & 6 & 3t & 0 & \ddots & & \\ & & 12 & \ddots & \ddots & -4 & \\ & & & \ddots & & 0 & -2 \\ & & & & n(n-1) & nt & 0 \end{bmatrix} \tag{1.9}$$

This way the associated spectrum can be computed from the characteristic polynomial

$$\mathrm{C}_n(t, \lambda) := \det\left[M_n(t) - \lambda \mathbb{I}\right]. \tag{1.10}$$

We call this the *Exactly Solvable* spectrum.

In turn, the points $t \in \mathbb{C}$ where the spectrum becomes repeated can be obtained from the discriminant of $\mathrm{C}_n(t)$ with respect to λ

$$\mathrm{D}_n(t) := \mathrm{Disc}_\lambda \mathrm{C}_n(t, \lambda). \tag{1.11}$$

In other words, the roots of $\mathrm{D}_n(t) = 0$ are precisely the values of t such that the matrix $M_n(t)$ has eigenvalues of algebraic multiplicity greater than 1.

The conjecture made by Shapiro and Tater in [ST22] relates the zeroes of $\mathrm{D}_n(t)$ to the poles of rational solutions $u_n(t)$ of the second Painlevé equation 1.1, as in (2.36). They noticed that the zeroes of the

n	$\mathrm{D}_n(t)$
1	t
2	$t^3 + \frac{27}{8}$
3	$t^6 + \frac{35}{2}t^3 - \frac{243}{4}$
4	$t^{10} + \frac{215}{4}t^7 + \frac{89}{8}t^4 + \frac{4084101}{512}t$
5	$t^{15} + \frac{255}{2}t^{12} + \frac{76211}{32}t^9 + \frac{3730405}{64}t^6 - \frac{8700637815}{4096}t^3 - \frac{125005275}{32}$

n	$Y_n(t)$
1	t
2	$t^3 + 4$
3	$t^6 + 20t^3 - 80$
4	$t^{10} + 60t^7 + 11200t$
5	$t^{15} + 140t^{12} + 2800t^9 + 78400t^6 - 3136000t^3 - 6272000$

Table 1.2: The first five monic Vorob'ev–Yablonskii polynomials $Y_n(t)$ and discriminant polynomials $D_n(t)$.

discriminant $\mathrm{D}_n(t)$ and the Vorob'ev-Yablonskii polynomials $Y_n(t)$ form a coinciding triangular pattern in the complex plane, as can be observed in Fig. 1.1. This observation can be summarized in the next *loosely* formulated conjecture.

Conjecture 1.1. *The roots of the scaled discriminant* $\mathrm{D}_n(n^{\frac{2}{3}}s)$ *in* (1.11)*, and the roots of the scaled Vorob'ev-Yablosnkii polynomials* $Y_n(n^{\frac{2}{3}}s)$ *form two coinciding lattices as* $n \to \infty$.

The evidence leading Shapiro and Tater to make their conjecture was mostly numerical, and partly suggested by some estimates for the rate of growth of both sets of roots. However, there was a priori no understanding why the two collection of points may be related to each other. The numerical picture is so precise that one may be tempted to compare the polynomials for Y_n and D_n; however a simple computation of these using a computer algebra package shows that the coefficients of these polynomials are not close to each other, as see in Table 1.2.

Strategy and main results. Our approach to this conjecture is contained in Chapter 4 and can be summarised in four steps.

1. Finding the connection between Painlevé II 1.1 and the Shapiro-Tater (ST) problem (1.4). We obtain an anharmonic oscillator equivalent to the Jimbo-Miwa (JM) Lax pair near the pole of the transcendent, which turns out to be almost identical to the ST anharmonic oscillator.

2. Understanding the Stokes phenomenon of the quasi-polynomials solutions of the ST anharmonic oscillator. We characterise the Stokes phenomenon of the Exactly Solvable spectrum, and we find explicit integrals (4.87) of the quasi-polynomial solution which determine the degeneracy of the ES spectrum.

3. Obtaining quantisation equations determining the location of both the zeroes of Vorob'ev-Yablonskii polynomials (the JM case) and the Exactly Solvable spectrum (the ST case). This is done by matching the Stokes phenomenon of both anharmonic oscillator using the exact WKB method. However, in the ST case it is necessary to additionally impose the repeated eigenvalue condition (4.87) in the WKB expansion in order to obtain leading order quantisation conditions.

4. Comparing the quantisation conditions. We show that both sets of points are determined by the same geometry to order $\mathcal{O}\left(\hbar^2\right)$ near the origin, and they match to order $\mathcal{O}\left(\hbar\right)$ in the bulk, thus explaining their similarity.

We now explain each step in more detail.

STEP 1 is covered in section 4.1. We obtain an anharmonic oscillator which is equivalent to the Jimbo-Miwa Lax pair (2.43) of Painlevé II 1.1 near one of its poles. This Lax pair is reduced to a scalar ODE by means of the gauge transformation (4.5). The resulting ODE has a potential which is *almost* identical to the ST potential, differing only in the linear term. This idea was originally used in [IN86] to study certain real-valued solution of Painlevé II and [Mas10b, Mas10b] to study the poles of the tritronqueé solution of Painlevé I. The key results in this section can summarised as follows:

Proposition 1.2. *Fix $\alpha \in \mathbb{C}$ and let $t = a$ be a pole with residue -1 of the second Painlevé transcendent* (PII) *function $u(t)$ with parameter α. The Jimbo-Miwa Lax pair* (2.43) *is equivalent in the limit $t \to a$ to the the following scalar ODE*

$$\begin{aligned} \frac{\mathrm{d}^2 y}{\mathrm{d}z^2} - V_{\mathrm{JM}}(z; a, b, \alpha) y = 0, \\ V_{\mathrm{JM}}(z; a, b, \alpha) = z^4 + az^2 + (2\alpha+1)z + \left(\frac{7a^2}{36} + 10b\right). \end{aligned} \tag{1.12}$$

Furthermore, the Stokes phenomenon of this ODE is the same as the original Stokes phenomenon of the Jimbo-Miwa Lax pair (2.43)*.*

STEP 2 is contained in section 4.2. In this section we study the eigenvalue problem (1.4), (1.5) and we compute explicit expression for the Stokes phenomenon corresponding to quasi-polynomial solutions. Furthermore, we characterise the repeated ES spectrum (t, Λ)in terms of integrals of the square of the quasi-polynomial. These results can be summarised as follows:

Theorem 1.3. *The point $(t, \Lambda) \in \mathbb{C}^2$ is in the ES spectrum of the ST eigenvalue problem* (1.4) *with the boundary conditions* (1.5) *and $J = n + 1 \in \mathbb{N}$ if and only if* (1.4) *admits a quasi-polynomial solution $y(z) = p(z)\mathrm{e}^{\theta(z)}$. Furthermore, these quasi-polynomial solutions have the following Stokes phenomenon:*

$$\mathbb{S}_{2j} = \mathbb{I}, \quad \mathbb{S}_{2j+1} = \begin{bmatrix} 1 & s_{2j+1} \\ 0 & 1 \end{bmatrix}, \tag{1.13}$$

where the s_{2j+1} are given explicitly in (4.33)*.*

Finally, if (t, Λ) corresponds to the repeated ES spectrum, that is $t \in \mathbb{C}$ is such that Λ is a repeated eigenvalue, then we have the vanishing of the following integrals (4.87)*:*

$$\int_{\infty_1}^{\infty_3} \left(p_n(z)\mathrm{e}^{\theta(z;t)}\right)^2 \mathrm{d}z = 0, \quad \int_{\infty_3}^{\infty_5} \left(p_n(z)\mathrm{e}^{\theta(z;t)}\right)^2 \mathrm{d}z = 0. \tag{1.14}$$

Here and below ∞_k indicates that the contour of integration extends to infinity along the direction $\arg(z) = k\frac{\pi}{3}$.

Along the way, we also discover that $y = p_n \mathrm{e}^{\theta}$ is a quasi-polynomial solution of (1.4) if and only if p_n is an orthogonal polynomials satisfying an excess of orthogonality conditions along a certain weighted contour. See Corollary 4.14 for details. In Chapter 5 this result is generalised to the case when $\theta(z)$ has a rational derivative. We also show that the roots of such *degenerate* orthogonal polynomials solve a Stieltjes-Fekete equilibrium problem.

STEP 3 is contained in sections 4.3 and 4.4. We introduce the following scaled variables for the ST and JM anharmonic oscillators:

$$\begin{aligned} &\text{Shapiro-Tater:} \quad \zeta = \hbar^{1/3} z, \quad s = \hbar^{2/3} t, \quad E = \hbar^{4/3}\Lambda, \quad \hbar^{-1} = n + 1, \\ &\text{Jimbo-Miwa:} \quad \zeta = \hbar^{1/3} z, \quad s = \hbar^{2/3} a, \quad \widehat{b} = \hbar^{4/3} b, \quad \hbar^{-1} = n + \frac{1}{2}. \end{aligned} \tag{1.15}$$

The reason for the difference in scaling is that it yields the same WKB-type equation with small parameter $\hbar$ and a n-independent quartic potential

$$\hbar^2 \frac{\mathrm{d}^2 y}{\mathrm{d}\zeta^2} - Q(\zeta; s, E) y = 0, \quad Q(\zeta; s, E) = \zeta^4 + s\zeta^2 + 2\zeta + E. \tag{1.16}$$

This puts both the ST and JM anharmonic oscillators on the same footing and allows us to use the exact WKB method to compute asymptotic expressions for the Stokes phenomenon of both systems simultaneously. This is done in Theorem 4.24. We then use these expressions to obtain leading order quantisation conditions which implicitly determine the position of both sets of points in question. We summarise the consequences of Theorem 4.27 (JM case) and the Theorems 4.28, 4.29 (ST case) as follows:

Theorem 1.4. *Suppose that $(a, b) \in \mathbb{C}$ determine a rational solution of the Painlevé II equation1.1 with $\alpha = n \in \mathbb{N}$ and Laurent expansion (4.3). Then in the scaled plane (1.15) they implicitly satisfy the following quantisation equations to leading order $\mathcal{O}(\hbar)$:*

$$\begin{aligned} (2n+1)\int_{\tau_1}^{\tau_0} \sqrt{Q(\zeta_+; s, E)}\mathrm{d}\zeta &= -i\pi - 2\pi \mathrm{i} k_1, \\ (2n+1)\int_{\tau_2}^{\tau_0} \sqrt{Q(\zeta_+; s, E)}\mathrm{d}\zeta &= -i\pi - 2\pi \mathrm{i} k_2, \\ (2n+1)\int_{\tau_3}^{\tau_0} \sqrt{Q(\zeta_+; s, E)}\mathrm{d}\zeta &= -i\pi - 2\pi \mathrm{i} k_3, \end{aligned} \tag{1.17}$$

where k_1, k_2, k_3 are three positive integers such that $k_1 + k_2 + k_3 = n - 1$.

Similarly, suppose (t, Λ) determine a point in the Exactly Solvable spectrum of the ST eigenvalue problem (1.4) with Λ a repeated eigenvalue. Then in the scaled plane (1.15) they implicitly satisfy the following quantisation equations to leading order $\mathcal{O}(\hbar)$:

$$\begin{aligned} 2(n+1)\int_{\tau_1}^{\tau_0} \sqrt{Q(\zeta_+; s, E)}\mathrm{d}\zeta &= \log\left(\frac{-1}{1+\boldsymbol{\tau}(s,E)}\right) - 2i\pi(m_1+1) \\ 2(n+1)\int_{\tau_2}^{\tau_0} \sqrt{Q(\zeta_+; s, E)}\mathrm{d}\zeta &= \log\left(-1-\frac{1}{\boldsymbol{\tau}(s,E)}\right) - 2i\pi(m_2+1) \\ 2(n+1)\int_{\tau_3}^{\tau_0} \sqrt{Q(\zeta_+; s, E)}\mathrm{d}\zeta &= \log\left(\boldsymbol{\tau}(s,E)\right) - 2i\pi(m_3+1) \end{aligned} \tag{1.18}$$

where m_1, m_2, m_3 are positive integers such that $m_1 + m_2 + m_3 = n - 1$ and $\boldsymbol{\tau}(s, E)$ is the ratio of periods of holomorphic differentials as expressed in (4.165).

Despite the similar nature of both results, the ST and JM cases are proved in a very different ways.

The JM case is proved in a simple manner by matching the Stokes data corresponding to the rational solutions of Painlevé II 1.1 with the expressions for the Stokes matrices in Theorem 4.24. This Stokes data has been studied before in [BM12, BM14, BM15].

The ST case is more complicated, since matching the Stokes data corresponding to quasi-polynomial solution in Theorem (4.8) is not enough to determine quantisation equations for a point in the repeated Exactly Solvable spectrum of the ST anharmonic oscillator. Instead, this only determines a one parameter family of points in the ES spectrum (Type D), or quantisation equations that are later ruled out (Type E). See Theorem 4.28. In order to obtain the desired quantisation conditions, we need to impose the repeated eigenvalue condition (1.14), which we then estimate using the WKB method in Theorem 4.29. This rules out the quantisation equations of Type E and together with the parametrisation (4.157), yield the correct quantisation conditions as above.

STEP 4 is contained in section 4.5. We study the lattices given by both quantisation conditions. We show that both of the lattices in the s-plane form a slowly modulated hexagonal lattice, see Proposition 4.31. These lattices are determined by essentially the same geometry, namely the periods of the same holomorphic differentials (4.205). Then we show that both lattices match to order $\mathcal{O}(\hbar^2)$ near the origin in the (s, E)-plane, see Theorem 4.32. Together these results explain the similarity of the ST and JM set of points as $n \to \infty$.

It is worth noting that in the Exactly Solvable spectrum the scaling we use differs slightly from the one conjectured by Shapiro and Tater in [ST22]. We believe this difference to be inessential and we justify this in Appendix B.

1.3 Degenerate orthogonal polynomials

The motivation for the introduction of *degenerate orthogonal polynomials* stems from the characterisation they provide to the solutions of the *Stieljes-Fekete equilibrium problem*.

Given a smooth real-valued *external potential* $\mathcal{Q}(z)$ the problem consists of finding a critical configuration of n points $\mathscr{Z}_n = \{z_1, \dots, z_n\} \subset \mathbb{C}$, called the *weighted Fekete points*, providing a minimum to the energy functional

$$\mathcal{E}(z_1, \dots, z_n) = -\sum_{\substack{j,k=1 \\ j\neq k}}^{n} \log|z_j - z_k| + \sum_{j=1}^{n} \mathcal{Q}(z_j). \tag{1.19}$$

These weighted Fekete points can be given an electrostatic interpretation: they are identical point-charges in $\mathbb{C}$ that interact via a logarithmic potential under the influence of an external field $\mathcal{Q}$. The problem of finding the critical configurations of (1.19) is referred to as the Stieltjes-Fekete problem. For an introduction to the weighted Fekete problem and its connection with logarithmic potentials in an external field we refer the reader to [ST97].

Depending on the context under consideration, the external potential may satisfy suitable additional assumptions and the Fekete points may belong to some fixed domain $\mathcal{D}$. However for our purposes we will be interested in the case where $\mathcal{Q}$ is a harmonic function such that

$$\mathcal{Q}(z) = 2\,\mathrm{Re}\,\widehat{\theta}(z) \tag{1.20}$$

where $\widehat{\theta}(z)$ is analytic, except for a finite number of singularity and branch cuts and has a rational derivative $\widehat{\theta}'(z)$. If, under suitable assumptions, a configuration $\mathscr{Z}_n = \{z_1, \dots, z_n\}$ forms an *equilibrium configuration*, it then realises the zero gradient of the energy functional $\mathcal{E}$, from which we obtain the *Stieltjes-Bethe equations*:

$$\sum_{\substack{j,k=1 \\ j\neq k}}^{n} \frac{1}{z_j - z_k} = \partial_z \mathcal{Q}(z_j) = \frac{A(z_j)}{2B(z_j)}. \tag{1.21}$$

Here A, B are two relatively prime polynomials (with B monic) related to the fact that $\widehat{\theta}$ has a rational derivative, and derivative $\partial_z = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$ for the complex variable $z = x + iy$.

It is worth noting that specific choices of $\widehat{\theta}(z)$ and the domain $\mathcal{D}$ are associated with the equilibrium satisfied by the zeroes of the classical orthogonal polynomials:

- Hermite: $\widehat{\theta}(z) = z^2$ and $\mathcal{D} = \mathbb{R}$;
- Laguerre: $\widehat{\theta}(z) = z - (\alpha+1)\log z$, $\alpha > -1$, and $\mathcal{D} = (0, \infty)$;
- Jacobi: $\widehat{\theta}(z) = -(\alpha+1)\log(1-z) - (\beta+1)\log(1+z)$, $\alpha, \beta > -1$, and $\mathcal{D} = (-1, 1)$.

The electrostatic interpretation of the zeroes of the classical orthogonal polynomials is a classical result dating back to Stieltjes, although it was also studied by Bochner, Heine, Van Vleck, and Polya. For a review of this classical result we refer the reader to [MMM07].

Equations of similar nature as (1.21) are also sometimes referred to as *Stieltjes-Bethe* equations due of their appearance in Bethe-Ansatz for spin-chains [Gau76, HW95] and can also be considered on Riemann surfaces of higher genus [Kor18].

In this thesis we extend these results and show that the criticality condition of the energy functional (1.19) also applies to a larger family of orthogonal polynomials. Our results answer some of the questions posed in the excellent review [MMM07] around this circle of ideas, such as:

- Are there generalizations of electrostatic models to other families of polynomials?
- Why necessarily the global minimum of the energy E should be considered? Which other types of equilibria described above could be linked to the zeros of the polynomials?

- What is the appropriate model for the complex zeros (when they exist)?

In Chapter 5 we address and answer precisely the above three questions and the result can be described concisely by the following statement:

There is a one-to-one correspondence between the solutions of the Stieltjes–Bethe equations (1.21) *and the* **maximally degenerate** *orthogonal polynomials of degree n for a* **semiclassical moment functional** *of type (A, B).*

In order to understand what this means and formulate our main Theorem 1.8 below, we first need to define a some of the notions. We start with a *semiclassical moment functional*, which is a particular case of a moment functional explained in section 3.1

Definition 1.5. A moment functional $\mathcal{M} : \mathbb{C}[z] \to \mathbb{C}$ is called *semiclassical* if there exist two relatively prime polynomials $A(z), B(z)$ of degree a, b, respectively such that

$$\mathcal{M}\big[A(z)p(z)\big] = \mathcal{M}\big[B(z)p'(z)\big], \qquad \forall p(z) \in \mathbb{C}[z]. \tag{1.22}$$

Furthermore, we say that such a semiclassical functional $\mathcal{M}$ is of type (A, B).

The concept of semiclassical moment functional originated in [Mar87, MR98], see also the previously mentioned review [MMM07]. The main result of [IMR91, MR98, Mar87] is that a semiclassical moment functional $\mathcal{M}$ can be represented by the moment functional $\mathcal{M}_{\Gamma,\theta}$ defined by the expression

$$\mathcal{M}_{\Gamma,\theta} : \mathbb{C}[z] \to \mathbb{C}, \tag{1.23}$$

$$z^j \mapsto \mathcal{M}_{\Gamma,\theta}[z^j] = \mu_j = \int_\Gamma z^j \mathrm{e}^{\theta(z)} \mathrm{d}z, \tag{1.24}$$

where the *symbol* θ of the exponential weight satisfies

$$\theta'(z) = -\frac{A(z) + B'(z)}{B(z)}\,, \tag{1.25}$$

and the integral is taken over a "weighted contour" $\Gamma = \sum_{j=1}^d s_j \gamma_j$ defined so that the integral over Γ is

$$\int_\Gamma f(z)\mathrm{d}z := \sum_{j=1}^d s_j \int_{\gamma_j} f(z)\mathrm{d}z. \tag{1.26}$$

The weighted contour Γ is expressed in terms of contours γ_j in the complex plane that extend from a zero of B to another (or to infinity) described in Section 5.2.1; the complex parameters $s_j,\ j = 1, \dots d$ parametrize the space of semiclassical moment functionals of a given type (A, B).

The maximum number d of contours γ_j is the degree of the pole divisor of $\theta'(z)\mathrm{d}z$ on the Riemann surface minus 2. In keeping the notation as simple as possible, we omit the explicit dependence of the moment functional on the Γ and θ and we will simply write $\mathcal{M}$ instead of $\mathcal{M}_{\Gamma,\theta}$ where the choice is obvious but implicit.

Given a semiclassical moment functional $\mathcal{M}$, the corresponding orthogonal polynomials, when they exist, are a sequence $\{P_j(z)\}_{j\in\mathbb{N}}$ of polynomials, each of degree at most j, satisfying

$$\mathcal{M}\big[P_j(z)P_k(z)\big] = \delta_{jk}h_k, \quad j, k \in \mathbb{N}, \tag{1.27}$$

for some constants $h_j \in \mathbb{C}$. In this manner we can obtain the classical orthogonal polynomials from the semiclassical moment functionals shown in Table 1.3.

Now we introduce the concept of *degenerate orthogonal polynomials*.

Definition 1.6. A polynomial P_n of degree n is called **ℓ–degenerate orthogonal**, with $\ell = 0, 1, 2, \dots$, if it satisfies the following excess of orthogonality conditions

$$\mathcal{M}\big[P_n(z)z^k\big] = \sum_{j=1}^d s_j \int_{\gamma_j} P_n(z)z^k \mathrm{e}^{\theta(z)}\mathrm{d}z = 0, \quad k = 0, 1 \dots, n + \ell - 1. \tag{1.28}$$

The polynomial $P_n(z)$, is called maximally degenerate if $\ell = d - 1$ with $d = \max\{\deg A, \deg B - 1\}$.

Name	Symbol $\theta(z)$	Type (A, B)	Contour Γ
Jacobi	$\alpha \log(1-z) + \beta \log(1+z)$	$-\big((\beta+\alpha+2)z - (\alpha-\beta), z^2-1\big)$	$[-1, 1]$
Hermite	$-z^2$	$(-2z, 1)$	$\mathbb{R}$
Laguerre	$-z + \alpha \log z$	$(z-\alpha-1, z)$	$\mathbb{R}_+$

Table 1.3: The classical orthogonal polynomials (Jacobi, Hermite, Laguerre) can be obtained from a semi-classical moment functional $\mathcal{M}_{\Gamma,\theta}$ of type (A, B) with symbol $\theta(x)$ and contour Γ as indicated. The common feature is that $\mathrm{d}\theta = \theta' \mathrm{d}x$ is a differential on the Riemann sphere $\mathbb{P}^1$ with total degree of poles $= 3$.

Remark 1.7 (Connection to Painlevé equations)**.** These types of moment functionals were analyzed in [BEH06] where it was shown that the Hankel determinant of the moments, when considered as function of the coefficients of A, B, are "isomonodromic tau functions" in the sense of [JMU-I]. In particular this means that specializing the symbol θ one can connect the theory of orthogonal polynomials to certain solutions of the Painlevé equations II,. . . , VI, as well as many integrable generalizations thereof. In this perspective the maximal degeneration of a polynomial implies that the tau-function must vanish and hence, in the cases that overlap with the theory of Painlevé transcendents, we are considering poles of the corresponding transcendent.

For a given type of moment functional and given degree $n \in \mathbb{N}$, we consider the conditions of degeneracy as (homogeneous) constraints on the parameters $s_1, \ldots, s_d$. For this reason in general we expect a maximal degeneracy of $d-1$. If $d = 1$ then any orthogonal polynomial is maximally degenerate (i.e. 0–degenerate) by default. This applies to all the classical moment functionals giving rise to the classical orthogonal polynomials (see Table 1).

We now have all the ingredients to formulate our main theorem (modulo some technical assumptions that will be treated later).

Theorem 1.8. *Let $\mathscr{Z} = \{z_1, \ldots, z_n\}$ be a critical configuration of weighted Fekete points satisfying the Stieltjes-Bethe equilibrium equations*

$$\sum_{k \neq j} \frac{1}{z_j - z_k} = \frac{A(z_j)}{2B(z_j)}, \quad j = 1, \ldots, n, \tag{1.29}$$

where $A(z), B(z)$ are relatively prime arbitrary polynomials with B monic. Then

(1) *the polynomial $P_n(z) = \prod_{j=1}^{n}(z - z_j)$ is a (non-hermitean) maximally degenerate orthogonal polynomial for a semiclassical moment functional $\mathcal{M}$ of type (A, B).*

(2) *The quasi-polynomial $y(z) = P_n(z)\sqrt{B(z)}\mathrm{e}^{\frac{1}{2}\theta(z)}$, with*

$$\theta'(z) = -\frac{A(z) + B'(z)}{B(z)} \tag{1.30}$$

satisfies the differential equation

$$y(z)'' - V(z)y(z) = 0 \tag{1.31}$$

where the function V is a rational function of the form

$$V(z) = \frac{1}{2}\theta'' + \frac{1}{4}(\theta')^2 + \frac{B''}{2B} - \left(\frac{B'}{2B}\right)^2 + \frac{B'}{2B}\theta' + \frac{Q}{B}, \tag{1.32}$$

where Q is a polynomial of degree $\deg Q \leq d - 1$, see formula (5.58). Equivalently P_n solves

$$B(z)P_n'' - A(z)P_n' - Q(z)P_n(z) = 0. \tag{1.33}$$

Vice versa, if P_n is a maximally degenerate orthogonal polynomial of degree n for a semiclassical moment functional $\mathcal{M}$ of type (A, B) then its zeroes z_j satisfy (1.29) *and the quasi-polynomial $P_n(z)\sqrt{B(z)}\mathrm{e}^{\theta(z)/2}$ satisfies an ODE as in* (1.31) *with an appropriate rational function V.*

Chapter 2

Painlevé equations and Isomonodromic deformations

2.1 Monodromy theory

In this section we review the monodromy theory of linear ordinary differential equations (ODEs) in the complex plane. We mainly follow the exposition of [FIKN06]; for more details and proofs we refer the reader to the aforementioned book and the references therein.

2.1.1 The local picture

Consider a $N \times N$ matrix-valued rational differential $A(\lambda)\mathrm{d}\lambda$ in the Riemann sphere $\lambda \in \mathbb{CP}^1$. We're interested in the solutions to the linear system

$$\frac{\mathrm{d}\Phi}{\mathrm{d}\lambda} = A(\lambda)\Phi. \tag{2.1}$$

In a neighbourhood of a point $\lambda_0 \in \mathbb{CP}^1$ the behaviour the solutions Φ of (2.1) is determined by the type of singular point at $\lambda = \lambda_0$ of the matrix differential form $A(\lambda)\mathrm{d}\lambda$. There are three distinct possibilities depending whether λ_0 is a:

1. **Regular point.** The matrix differential $A(\lambda)\mathrm{d}\lambda$ is holomorphic at λ_0.
2. **Fuchsian point.** The matrix differential $A(\lambda)\mathrm{d}\lambda$ has a simple pole at λ_0.
3. **Non-Fuchsian point.** The matrix differential $A(\lambda)\mathrm{d}\lambda$ has a multiple pole at λ_0.

In each of these cases the condition of being holomorphic or having a pole at λ means that in a local parameter near λ_0 (i.e. $\zeta = \lambda - \lambda_0$ if $\lambda \in \mathbb{C}$ is in the finite complex plane or $\zeta = 1/\lambda$ if $\lambda_0 = \infty$) we can write the following local representation

$$A(\lambda)\mathrm{d}\lambda = \sum_{j=k}^{\infty} A_j \zeta^j \mathrm{d}\zeta, \quad A_k \neq 0, \quad \lambda \in D_{\lambda_0} \tag{2.2}$$

where D_{λ_0} is a small disk centered (punctured in the case of a singularity) at λ_0 and with $k \geq 0$ if λ_0 in the regular case, $k = -1$ in the Fuchsian case, and $k \leq -2$ in the non-Fuchsian case. In the non-Fuchsian case we call the positive integer integer $r = -(k+1)$ the *Poincaré rank* of the singularity. In other words, if $\lambda = \lambda_0$ is a pole of order 2 then we obtain a non-Fuchsian singularity of Poincaré rank $r = 1$, if the pole has order 3 we obtain a singularity of Poincaré rank $r = 2$, etc.

The unassuming reader will not notice anything uncomfortable with this explanation. However the experienced one will be certain to point out that the definition of the coefficients A_j in (2.2) is in conflict with the notation used in [FIKN06]. We will of course follow the expedient convention, since there is a good reason for it that will become apparent in the following. However we choose to leave the equation (2.2) in print as a stepping stone for the more naïve and easily confused readers such as my previous self. In fact the conventional representation near a *singular point* λ_0 is

$$A(\lambda)\mathrm{d}\lambda = \sum_{j=-r-1}^{\infty} A_{j+1}\zeta^j \mathrm{d}\zeta, \quad A_k \neq 0, \quad \lambda \in D_{\lambda_0} \setminus \{\lambda_0\}, \quad r = 0, 1, 2, \dots \tag{2.3}$$

It is clear that if $r = 0$ we are in the Fuchsian case, and if $r \geq 1$ we're in the non-Fuchsian case. The convention is such that the $r+1$ polar coefficients are labelled from 0 to r i.e. $A_j, j = 0, -1, \dots, -(r+1)$.

The behaviour of solutions $\Phi(\lambda)$ locally near a point λ_0 is classified in the following theorems, depending whether λ_0 is a regular, Fuchsian or non-Fuchsian point. The proofs can be found in [FIKN06].

Theorem 2.1 (Regular case)**.** *Let $A(\lambda)\mathrm{d}\lambda$ be $N \times N$ matrix-valued holomorphic differential form in the neighbourhood of the point $\lambda_0 \in \mathbb{CP}^1$, and let Φ_0 be a constant $N \times N$ matrix. Then there exists a unique solution $\Phi(\lambda)$ of the equation 2.1 which is holomorphic in the same neighbourhood and such that $\Phi(\lambda_0) = \Phi_0$*

For the Fuchsian case we make the additional assumption that the leading coefficient A_0 in the expansion (2.3) is diagonalisable and non-resonant, namely we can write

$$A_0 = P\Lambda_0 P^{-1}, \quad \det P \neq 0, \tag{2.4}$$

where Λ_0 is a diagonal matrix with (possibly repeated) eigenvalues whose difference is not a (non-zero) integer. The following statement can be adapted to the resonant and non-diagonalisable cases. See [Was87, Sib90] for further details.

Theorem 2.2 (Fuchsian case)**.** *Let $A(\lambda)\mathrm{d}\lambda$ be $N \times N$ matrix-valued differential form in a punctured neighbourhood of singularity $\lambda_0 \in \mathbb{CP}^1$ of Fuchsian type (i.e. simple pole). Suppose that the leading coefficient A_0 in the Laurent expansion (2.3) is diagonalisable as in (2.4) and non-resonant. Then the linear ODE 2.1 has a fundamental solution in the punctured neighbourhood of λ_0 of the form*

$$\Phi(\lambda) = P\left(\mathbb{I} + \sum_{j=1}^{\infty} \Phi_j \zeta^j\right)\zeta^{\Lambda_0}. \tag{2.5}$$

where the power series is convergent and ζ is a local variable near λ_0.

We shall call the matrix Λ_0 the *formal monodromy exponents*.

In the case of a non-Fuchsian singularity, also known as an *irregular singularity*, we make the following assumption: the leading coefficient coefficient A_{-r} in (2.3) is diagonalisable with distinct eigenvalues, namely

$$A_{-r} = P\Lambda_{-r}P^{-1}, \quad \det P \neq 0, \quad \Lambda_{-r} = \mathrm{diag}(d_1, \dots, d_N) \tag{2.6}$$

and $d_j \neq d_i$ whenever $i \neq j$.

Theorem 2.3 (non-Fuchsian case)**.** *Let $A(\lambda)\mathrm{d}\lambda$ be $N \times N$ matrix-valued differential form in a punctured neighbourhood of a non-Fuchsian singularity $\lambda_0 \in \mathbb{CP}^1$ of Poincaré rank r (i.e. pole of order r+1). Suppose that the leading coefficient A_{-r} in the Laurent expansion (2.3) is diagonalisable as in (2.6) with distinct eigenvalues. Then the linear ODE (2.1) has* ***formal*** *fundamental solution in the punctured neighbourhood of λ_0 of the form*

$$\Phi_{\mathrm{formal}}(\lambda) = P\left(\mathbb{I} + \sum_{j=1}^{\infty} \Phi_j \zeta^j\right)\exp\left(\frac{\Lambda_{-r}}{-r}\zeta^{-r} + \dots + \frac{\Lambda_{-1}}{-1} + \Lambda_0 \log\zeta\right) \tag{2.7}$$

where all the matrices $\Lambda_{-j}, j = 0, 1, \dots, r$ are all diagonal. Furthermore, the coefficients Φ_j and the exponents Λ_j can all be determined recursively as polynomials in the coefficients A_j in the expansion (2.3).

The series solution (2.7) is called formal because it is not convergent in general. This is the main difference between the Fuchsian and non-Fuchsian case. To understand the connection between the formal solutions and actual solutions in the non-Fuchsian case we need to introduce the *Stokes sectors*.

Fix a pair of eigenvalues α_j, α_i of the leading coefficient matrix A_{-r} in the expansion (2.3). Define the following $2r$ rays emanating from the origin in the ζ-plane:

$$l_n^{(i,j)} = \left\{ \zeta \in \mathbb{C} : |\zeta| < \rho, \quad \arg \zeta = \frac{1}{r} \arg(\alpha_j - \alpha_i) + \frac{\pi}{r}\left(n + \frac{1}{2}\right) \right\}, \quad n = 0, 1, \ldots, 2r-1. \tag{2.8}$$

The rays $l_n^{(i,j)}$ are called *Stokes rays*. We will say a sector Ω around the point λ_0 is a **Stokes sector** if for each pair (i,j) with $i < j$ it contains exactly one Stokes ray from the family $\{l_n^{(i,j)}\}_{n=0}^{2r-1}$.

This definition is rather clunky and obtuse, however, it is the correct one since it ensures that in each Stokes sector the fundamental solution Φ_{formal} has a mixture of dominance and recessiveness in each sector. Fortunately for us, there is a canonical way of choosing Stokes sectors [FIKN06]. Indeed, any sector of opening $\pi/r + \delta$, for a sufficiently small δ, is a Stokes sector. Let us then define the following collection of $2r+1$ sectors $\{\Omega_k\}_{k=0}^{2r+1}$ covering the punctured disk $D_{\lambda_0} \setminus \lambda_0$:

$$\begin{gathered} \Omega_k := e^{i\frac{\pi}{r}k}\Omega_\star, \quad k = 0, 1, \ldots, 2r, \\ \Omega_\star := \left\{ \zeta \in \mathbb{C} : 0 < |\zeta| < \rho, \quad \theta < \arg\zeta < \theta + \frac{\pi}{r} + \delta \right\} \end{gathered} \tag{2.9}$$

where θ is arbitrary and ρ, δ are sufficiently small. Note that $\Omega_{2r} = \Omega_0 = \Omega_\star$. We are now ready to formulate the following foundational theorem.

Theorem 2.4. *Let $A(\lambda)\mathrm{d}\lambda$ be $N \times N$ matrix-valued differential form in a punctured neighbourhood of a non-Fuchsian singularity $\lambda_0 \in \mathbb{CP}^1$ of Poincaré rank r. Suppose that the leading coefficient A_{-r} in the Laurent expansion (2.3) is diagonalisable as in (2.6) with distinct eigenvalues. Additionally suppose that $\Omega_k, k = 0, 1, \ldots, 2r$ are canonical Stokes sectors at the singular point λ_0. Then there exist unique solutions Φ_k of the linear ODE (2.1) which are asymptotic [1] to the formal solution Φ_{formal} in each Stokes sector Ω_k:*

$$\begin{gathered} \Phi_k(\lambda) \simeq \Phi_{\text{formal}}(\lambda), \quad k = 0, 1, \ldots, 2r \\ \lambda \to \lambda_0, \quad \lambda \in \Omega_k \end{gathered} \tag{2.10}$$

*where $\Phi_{\text{formal}}(\lambda)$ is given in (2.7) and the branch of the logarithm in that formula is chosen appropriately. The solutions $\Phi_k, k = 0, \ldots, 2r$ are termed the **canonical solutions**.*

We are now ready to understand the following key concepts.

Characteristic exponent matrices. Owing to Theorem 2.2, the local solutions of the system (2.1) near a Fuchsian singular point are multivalued due to the presence of the formal monodromy factor

$$\zeta^{\Lambda_0} := e^{\Lambda_0 \log(\zeta)}. \tag{2.11}$$

Suppose we fix a determination of branch cuts. If we perform a loop [2] starting at a point $\lambda_\star$ and going around the singular point λ_0, it is clear that the multivalued function (2.11) attains an extra factor of $e^{2\pi i \Lambda_0}$. We denote this by

$$\zeta^{\Lambda_0} \underset{\gamma}{\rightsquigarrow} \zeta^{\Lambda_0} e^{2\pi i \Lambda_0} \tag{2.12}$$

where γ is a closed contour starting and ending at $\lambda_\star$

Stokes phenomenon. Note that Φ_k are in general different solution all with the same asymptotics in different sectors. From this result it is straight forward to understand the concept of Stokes phenomenon. Around each non-Fuchsian singular point λ_0 there exist $2r+1$ canonical solutions to (2.1). Any two solutions of the same linear ODE are related by a invertible linear transformation, so that we can write

$$\Phi_{k+1}(\lambda) = \Phi_k(\lambda)\mathbb{S}_k, \quad \det \mathbb{S}_k \neq 0, \quad k = 0, \ldots, 2r-1. \tag{2.13}$$

[1] in the sense of Poincaré.
[2] a.k.a. analytic continuation along a closed contour

These matrices are called **Stokes matrices** and their non-trivial entries are called *Stokes parameters or multipliers*. These matrices can be shown to be upper or lower triangular with ones along the diagonal. Furthermore, these matrices together with the datum of exponents $\Lambda_{-r}, r = 0, \ldots, r$ completely determine the linear system (2.1) up to local holomorphic gauge equivalent, see [FIKN06, Thm 1.5] for more details. It is worth noting that although $\Omega_0 = \Omega_{2r}$, in general the canonical solutions $\Phi_0(\lambda) \neq \Phi_{2r}(\lambda)$. This is due to the logarithm in the formal solution (2.7), and so we obtain

$$\Phi_{2r}(\lambda) = \Phi_0 \mathrm{e}^{2\pi \mathrm{i} \Lambda_0}. \tag{2.14}$$

Furthermore, around an irregular singular point λ_0 the Stokes matrices are related by

$$\mathbb{S}_0 \mathbb{S}_1 \ldots \mathbb{S}_{2r-1} = \mathrm{e}^{2\pi \mathrm{i} \Lambda_0} \tag{2.15}$$

where Λ_0 is the *formal monodromy exponent*.

2.1.2 The global picture

In this section we consider the global behaviour of solutions of the linear ODE (2.1). The following theorem will be essential for what follows.

Theorem 2.5 (Monodromy theorem). *Consider the singular points $a_j \in \mathbb{CP}^1, j = 1, \ldots, m$ of the coefficient matrix $A(\lambda)$ of the linear system (2.1), and suppose that γ is a contour in $\mathbb{CP}^1$ avoiding all the singular points*

$$\gamma : [0, 1] \to \mathbb{CP}^1 \setminus \{a_1, \ldots, a_m\}\,. \tag{2.16}$$

with endpoints $\lambda_0 = \gamma(0)$ and $\lambda_1 = \gamma(1)$. Then any germ of a solution $\Phi(\lambda)$ at the (regular) point λ_0 can be analytically continued along the contour γ to another local solution at the point λ_1. Furthermore, this analytic continuation depends only on the homotopy class (with fixed endpoint at λ_0 and λ_1) of the contour γ.

This theorem gives us a rather disappointing answer to the problem of obtaining global solutions of the linear ODE (2.1): take any local solution and "simply" perform analytic continuation! This is however more easily said than done, for performing analytic continuation is not as straight forward as one may hope. [3] This approach however allows us to define the key theoretical concept of *monodromy representation*.

Take the singular points $a_j \in \mathbb{CP}^1, j = 1, \ldots, m$ of the coefficient matrix $A(\lambda)$ of the linear system (2.1) and fix a base point $\lambda_\star \in \mathbb{CP}^1 \setminus \{a_1, \ldots, a_m\}$. Consider the fundamental group of the punctured Riemann sphere, namely the homotopy classes of loops starting at $a_\star$ and avoiding the singular points

$$\pi_1 = \left\{ [\gamma] : \begin{array}{c} \gamma : [0,1] \to \mathbb{CP}^1 \setminus \{a_1, \ldots, a_m, \} \\ \gamma(0) = \gamma(1) = \lambda_\star \end{array} \right\}. \tag{2.17}$$

Fix a solution $\Phi(\lambda)$ with initial condition $\Phi(\lambda_\star) = \Phi_0$. The analytic continuation of this solution along a loop $\gamma \in \pi_1$ yields another solution $\Phi_\gamma(\lambda)$ differing from the initial solution $\Phi(\lambda)$ by an invertible linear transformation

$$\Phi_\gamma(\lambda) = \Phi(\lambda) \mathbb{M}_\gamma. \tag{2.18}$$

We shall call $\mathbb{M}_\gamma$ the **monodromy matrix**. It is clear from the Monodromy Theorem 2.5 that $\mathbb{M}_\gamma$ only depends on the homotopy class of the contour γ. This defines a representation of the fundamental group of the punctured Riemann sphere

$$\begin{aligned} \mathcal{M} : \pi_1\Big(\mathbb{CP}^1 1 \setminus \{a_1, \ldots, a_m\}\Big) &\to \mathbf{GL}_{\mathrm{N}}(\mathbb{C}), \\ \gamma &\mapsto \mathbb{M}_\gamma. \end{aligned} \tag{2.19}$$

[3] Ask any student analytic number theory how to derive the functional equation of the Riemann zeta function.

It is possible to relate the monodromy matrices, Stokes matrices, characteristic exponents in the following fashion

$$\mathbb{M}_1\mathbb{M}_2\dots\mathbb{M}_m = \mathbb{I}, \tag{2.20a}$$

$$\mathcal{E}_j^{-1}\mathrm{e}^{2\pi\mathrm{i}\Lambda_0^{(j)}}\mathcal{E}_j = \mathbb{M}_j, \quad \text{(if } a_j \text{ is Fuchsian sing.)}, \tag{2.20b}$$

$$\mathcal{E}_j^{-1}\mathrm{e}^{2\pi\mathrm{i}\Lambda_0^{(j)}}\left(\mathbb{S}_0^{(j)}\dots\mathbb{S}_{2r_j-1}^{(j)}\right)^{-1}\mathcal{E}_j = \mathbb{M}_j \quad \text{(if } a_j \text{ is a non-Fuchsian sing. of Poincaré rank } r_j) \tag{2.20c}$$

where througout $j = 1,\dots,m$ and $\mathbb{M}_j$ is the monodromy matrix corresponding to a loop around a_j, $\mathbb{S}_k^{(j)}$ with $k = 0,\dots,2r_j-1$ is a Stokes matrix of the irregular singular point a_j of Poincaré rank r_j and $\Lambda_0^{(j)}$ is the formal monodromy matrix of the singular point a_j. Furthermore $\mathcal{E}_j$ is the connection matrix relating solutions at the base point $\lambda_\star$ to solutions near the singular points a_j. For more details see [FIKN06, Ch. 2.0.3, 2.0.4].

It is worth noting that the characteristic exponents, monodromy matrices and Stokes matrices are highly transcendental quantities and they are only known explicitly in very specific cases such as the hypergeometric case.

2.1.3 The inverse monodromy problem

We shall now describe what is known as the *extended monodromy data*, which describes the linear system globally. Suppose that the linear system (2.1) has m singularities a_j, where p of them are Fuchsian for $j = 1,\dots,p \le m$ and $m-p$ of them are non-Fuchsian for $j = p+1,\dots,m$.

We now collect all the important matrices that we've considered in the previous section. For the Fuchsian points corresponding to $j = 1,\dots,p$ take the characteristic exponents $\Lambda_0^{(j)}$ as defined in (2.4). For the non-Fuchsian points corresponding to $j = p+1,\dots,m$, each of them with Poincaré rank r_j, take the *"extended"* characteristic exponents $\Lambda_{r_j}^{(j)},\dots,\Lambda_0^{(j)}$, as well as the associated Stokes matrices $\mathbb{S}_0^{(j)},\dots,\mathbb{S}_{2r_j-1}^{(j)}$. So far all these matrices provide only local information. For this reason we also need to consider, regardless of the nature of the singular point, the connection matrices $\mathcal{E}_j$ which give rise to the global monodromy matrices, as defined in formulas (2.20a)–(2.20c). Together with the location of the poles, this datum forms the so called ***extended* monodromy data** of the rational coefficient $A(\lambda)$, and we denote it

$$\mathfrak{M} := \left\{ \underset{(j=1,\dots,m)}{\overset{\text{Singular points}}{a_j}}, \underset{(j=1,\dots,p)}{\overset{\text{Formal monodromy of Fuchsian pts.}}{\Lambda_0^{(j)}}}, \underset{(j=p+1,\dots,m)}{\overset{\textit{Extended}\text{ formal monodromy of non-Fuchsian pts.}}{\Lambda_{r_j}^{(j)},\dots,\Lambda_0^{(j)}}}, \underset{(j=p+1,\dots,m)}{\overset{\text{Stokes matrices of non-Fuchsian pts.}}{\mathbb{S}_0^{(j)},\dots,\mathbb{S}_{2r_j-1}^{(j)}}}, \underset{(j=1,\dots,m)}{\overset{\text{Connection matrices}}{\mathcal{E}_j}} \right\}. \tag{2.21}$$

It is shown in [FIKN06, Ch.2.0.5] that the extended monodromy data $\mathfrak{M}$ uniquely determines the linear ODE (2.1) with rational coefficient $A(\lambda)$ with a fixed number m of points of given multiplicities r_j. In other words, the map $A(\lambda) \mapsto \mathfrak{M}$ is one to one.

This map however, is highly transcendental and there is no easy way of realising it. This setting is also highly rigid since the bijective nature leaves no room for deformations. For this reason, we consider a more unconstrained version of the extended monodromy data $\mathfrak{M}$ known as the ***essential* monodromy data**:

$$\mathfrak{m} := \left\{ \underset{(j=1,\dots,m)}{\overset{\text{Formal monodromy of singular pts.}}{\Lambda_0^{(j)}}}, \underset{(j=p+1,\dots,m)}{\overset{\text{Stokes matrices of non-Fuchsian pts.}}{\mathbb{S}_0^{(j)},\dots,\mathbb{S}_{2r_j-1}^{(j)}}}, \underset{(j=1,\dots,m)}{\overset{\text{Connection matrices}}{\mathcal{E}_j}} \right\} \tag{2.22}$$

Thus the monodromy theory of linear ODEs in the complex domain with rational coefficients attempts to understand the ***direct*** *monodromy map*

$$A(\lambda) \longmapsto \mathfrak{m} \tag{2.23}$$

and the ***inverse*** *monodromy map*

$$\mathfrak{M} \longmapsto A(\lambda). \tag{2.24}$$

The inverse monodromy problem, also known as the Riemann-Hilbert problem, has a long and illustrious history. It started as one of Hilbert's 21st problem [Hil00], consisting of proving the existence of a linear system with Fuchsian singularities at prescribed locations with prescribed monodromy. In Hilbert's own words (or rather the words of Hilbert's translator) [Hil02]:

> *" In the theory of linear differential equations with one independent variable z, I wish to indicate an important problem, one which very likely Riemann himself may have had in mind. This problem is as follows: to show that there always exists a linear differential equation of the Fuchsian class, with given singular points and monodromic group. The problem requires the production of n functions of the variable z, regular throughout the complex z plane except at the given singular points ; at these points the functions may become infinite of only finite order, and when z describes circuits about these points the functions shall undergo the prescribed linear substitutions. The existence of such differential equations has been shown to be probable by counting the constants, but the rigorous proof has been obtained up to this time only in the particular cases* [...]. *The theory of linear differential equations would evidently have a more finished appearance if the problem here sketched could be disposed of by some perfectly general method. "*

The Riemann-Hilbert problem was initially thought to be solved by J. Plemelj in 1908, but a counterexample of was found. This original setting has been generalised Birkhoff to the case of a linear system with arbitrary singularities (i.e. admitting non-Fuchsian points) and prescribed Stokes phenomena, and depending on the specific formulation has been resolved by Bolibruch [AB94].

2.1.4 Isomonodromic deformations

Since the essential monodromy data (2.22) does not uniquely determine the linear system (2.1) one may wonder what kind of perturbations keep the essential monodromy data $\mathfrak{m}$ intact. This is what is known as isomonodromic deformations.

Consider a family linear ODE with rational coefficients

$$\frac{\partial \Phi}{\partial \lambda} = A(\lambda, t)\Phi \tag{2.25}$$

parametrised by $t = (t_1, \ldots, t_q) \in \mathbb{C}^q$, where $A(\lambda, t)$ is rational in t and holomorphic in t.

Definition 2.6 (Isomonodromic deformation)**.** Suppose that the holomorphic family of linear ODEs with rational coefficients (2.25) satisfies the conditions:

(I) The number m of poles $a_j, j = 1, \ldots, m$ is independent of the deformation parameters t. Additionally, as t varies the poles do not become arbitrarily close.

(II) The spectrum of the leading coefficient A_{-r_j} of the Laurent expansion (2.3) of the coefficient matrix $A(\lambda)\mathrm{d}\lambda$ near the singular point a_j is independent of t and satisfies the usual conditions for Fuchsian and non-Fuchsian points.

(III) At each irregular singular point $a_j(t)$ with Poincaré rank $r_j \geq 1$, the set of Stokes sectors can be chosen unambiguously so that it is t independent and invariant under the map $\lambda \mapsto \lambda - a_j(t)$.

(IV) Canonical solutions (2.5) near Fuchsian points are holomorphic in t and and the asymptotics of canonical solutions (2.10) near irregular points holds uniformly in t.

If the *essential monodromy data* $\mathfrak{m} = \left\{\Lambda_0^j, \mathbb{S}_k^j, \mathcal{E}_j\right\}$ is independent of t then we say the linear system (2.25) is **isomonodromic**.

Assuming that one of the poles is always at infinity (i.e. $a_m(t) = \infty$) and other additional technical assumptions (see [FIKN06, p.134]) it is possible to describe isomonodromic deformations in terms of *nonlinear* differential equations. This was first done in [JMU-I] who considered the differential form

$$\Theta(\lambda, t) = \mathrm{d}\Phi \cdot \Phi^{-1} = \left(\sum_{j=1}^{q} \frac{\partial \Phi}{\partial t_j} \mathrm{d}t_j \right) \Phi^{-1}. \tag{2.26}$$

Due to the isomonodromic property of the system, this differential 1-form is a single-valued in the punctured Riemann sphere $\mathbb{CP}^1 \setminus \{a_1, \ldots, a_m\}$. In fact, more can be stated about this differential form.

Theorem 2.7. *The differential form $\Theta(\lambda, t)$ is rational in λ with poles at $a_1, \ldots, a_{m-1}, a_m = \infty$ with the same Poincaré rank r_j of the corresponding poles of $A(\lambda)\mathrm{d}\lambda$. Furthermore $\Theta(\lambda, t)$ is determined by the coefficients of the partial fraction decomposition of the rational function $A(\lambda)$:*

$$\Theta(\lambda, t) = \Theta(\lambda; \{A_k^{(j)}\}, \{a_j\}) \tag{2.27}$$

where

$$A(\lambda, t) = A^{(\infty)}(\lambda) + \sum_{j=1}^{m-1} A^{(j)}(\lambda), \tag{2.28}$$

$$A^{(j)}(\lambda) = \sum_{k=1}^{r_j+1} \frac{A_{-k+1}^{(j)}}{(\lambda - a_j)^k}, \tag{2.29}$$

$$A^{(\infty)} = \begin{cases} -\displaystyle\sum_{k=0}^{r_\infty - 1} A_{-k-1}^{(\infty)} \lambda^k & \text{if } r_\infty > 0, \\ 0 & \text{if } r_\infty = 0, \end{cases} \tag{2.30}$$

where the $A_k^{(j)}$ coincide with the Laurent expansion in (2.3).

The determination of these coefficients $A_k^{(j)}$ and consequently of the differential form $\Theta(\lambda, t)$ can be done recursively and explicitly, see [FIKN06, p.135] for more details. The importance of this differential is that it allows us to write an auxiliary linear system in the parameters t, so that

$$\mathrm{d}\Phi = \Theta(\lambda, t)\Phi, \quad \text{i.e.} \quad \frac{\partial \Phi}{\partial t_j} = \Theta_j(\lambda, t)\Phi \quad (j = 1, \ldots, q) \tag{2.31}$$

where we use $\Theta(\lambda, t) = \sum_j \Theta_j(\lambda, t)\mathrm{d}t_j$. Therefore the function Φ (2.25) satisfies the overdetermined linear system

$$\begin{cases} \dfrac{\partial \Phi}{\partial \lambda} = A(\lambda, t)\Phi, \\ \mathrm{d}\Phi = \Theta(\lambda, t)\Phi. \end{cases} \quad \text{i.e.} \quad \begin{cases} \dfrac{\partial \Phi}{\partial \lambda} = A(\lambda, t)\Phi, \\ \dfrac{\partial \Phi}{\partial t_j} = \Theta_j(\lambda, t)\Phi, \quad (j = 1, \ldots, q) \end{cases} \tag{2.32}$$

The imposing the condition of equality of mixed partial derivatives we obtain the following *compatibility conditions*

$$\mathrm{d}A - \frac{\partial \Theta}{\partial \lambda} + [A, \Theta] = 0, \tag{2.33}$$

which hold identically in Λ. Equating both sides of the equations yields a system of non-linear differential equations in t for the matrix coefficient $A_k^{(j)}(t)$. In their seminal papers Jimbo, Miwa and Ueno realised all of the Painlevé transcendents in this manner: as the compatibility condition of an isomonodromic system. Furthermore, they showed that the compatibility condition 2.33 is not only necessary, but also sufficient to ensure isomonodromy. See the original papers [JMU-I, JM-II, JM-III] or [FIKN06] for a more comprehensive approach.

2.2 Painlevé equations

The six Painlevé transcendents are non-linear second order ordinary differential equations (ODEs) satisfying an equation

$$\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} = F\left(t, u, \frac{\mathrm{d}u}{\mathrm{d}t}\right) \tag{2.34}$$

where F is meromorphic in t and rational in u, u_t. They were introduced at the turn of the century by Paul Painlevé and his followers as a result of the classification problem of second order ODEs [4] satisfying the

Painlevé property. All the solutions are free from *movable critical points*, i.e. the locations of *branch points* and *essential singularities* does not depend on initial conditions.

This property is enjoyed by all linear ODEs, however even the simplest non-linear differential equation $y_t = (2y)^{-1}$, which has solution $y(t) = (t + c)^{1/2}$, fails the Painlevé property movable since it has a branch point at $t = c$. This is critical point is movable since it depends on the initial conditions $y(0) = c^{1/2}$. The motivation for this classification problem is now clear. Any solutions of a Painlevé transcendent can be analytically continued to the universal covering the punctured Riemann sphere, which is determined uniquely by the equation. Therefore they bridge the gap between linear and non-linear equations, and they provide important examples of non-linear special functions, as opposed to the classical special functions (Airy, Bessel, Hypergeometric, etc.) which are all given in terms of linear ODEs.

The following are the canonical form of the six Painlevé equations:

$$u'' = 6u^2 + t, \tag{PI}$$

$$u'' = 2u^3 + tu + \alpha, \tag{PII}$$

$$u'' = \frac{1}{u}(u')^2 - \frac{u'}{t} + \frac{1}{t}(\alpha u^2 + \beta) + \gamma u^3 + \frac{\delta}{u}, \tag{PIII}$$

$$u'' = \frac{1}{2u}(u')^2 + \frac{3}{2}u^3 + 4tu^2 + 2(t^2 - \alpha)u + \frac{\beta}{u}, \tag{PIV}$$

$$u'' = \frac{3u-1}{2u(u-1)}(u')^2 - \frac{u'}{t} + \frac{(u-1)^2}{t^2}\left(\alpha u + \frac{\beta}{u}\right) + \frac{\gamma u}{t} + \frac{\delta u(u+1)}{u-1}, \tag{PV}$$

$$\begin{aligned} u'' &= \frac{1}{2}\left(\frac{1}{u} + \frac{1}{u-1} + \frac{1}{u-t}\right)(u')^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{u-t}\right)u' + \\ &+ \frac{u(u-1)(u-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{u^2} + \gamma\frac{t-1}{(u-1)^2} + \delta\frac{t(t-1)}{(u-t)^2}\right). \end{aligned} \tag{PVI}$$

where $' = \frac{\mathrm{d}}{\mathrm{d}t}$ and $\alpha, \beta, \gamma, \delta$ are arbitrary complex parameters.

2.2.1 Special solutions of PII

In general Painlevé equations cannot be solved explicitly, this is why they are often called Painlevé *transcendents*. However, for special choice of the parameters it is possible to obtain explicit solutions that can be written explicitly in terms of previously known special functions. To exemplify this point, we will focus on the special solutions of the second Painlevé transcendent.

It was recognised by Vorob'ev and Yablonskii in [Yab59, Vor65] that the PII equation admits rational solutions when $\alpha = n \in \mathbb{Z}$. They are explicitly given in terms of the homonymous polynomials defined in (1.3)

A priori the recursion only determines a rational function of t. However it can be shown, as the name suggests, that $Y_n(t)$ is in fact a polynomial and additionally that the polynomials $Y_n(t)$ have degree $n(n+1)/2$. It is worth noting that the arrangement of the roots of $Y_n(t)$ follows a rather symmetric triangular pattern, as can be seen in Fig. 2.1. Surprisingly, these polynomials also admit a representation

[4] The classification of *first order* ODEs with the aforementioned property was established by Poincaré and L. Fuchs who showed that they must be either the Weierstrass $\wp$–function or a solution of the Riccati equation.

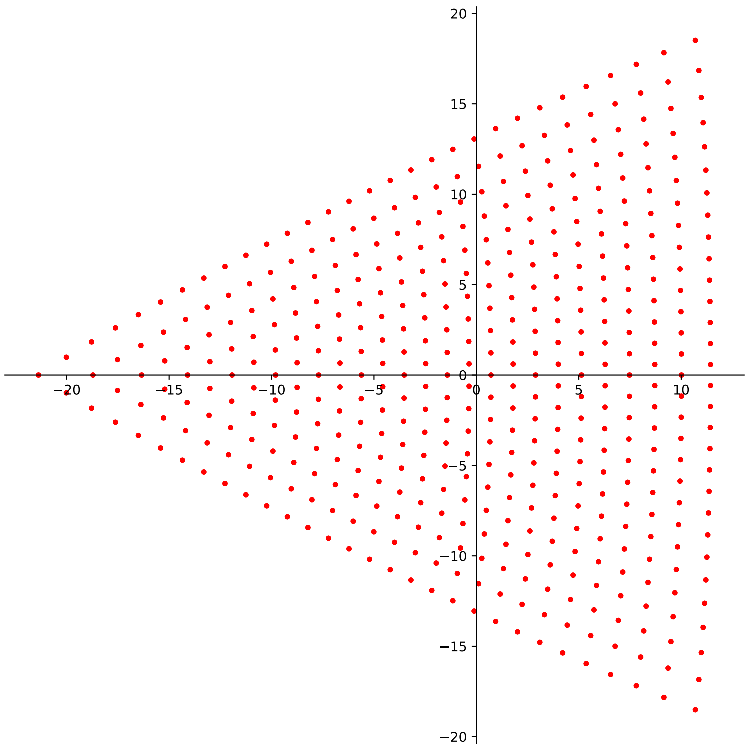


Figure 2.1: Roots of the Vorob'ev-Yablonskii polynomial $Y_n(t)$ with $n = 30$.

in terms of Schur polynomials indexed with the "staircase partition" [KO96]

$$Y_n(t) = \left(-\frac{4}{3}\right)^{\frac{n(n+1)}{6}} \left(\prod_{k=1}^{n}(2k-1)!!\right) S_{(n,n-1,\cdots,1)}\left(\left(-\frac{3}{4}\right)^{\frac{1}{3}} t, 0, 1, 0, 0, \dots\right). \tag{2.35}$$

These polynomials give an explicit formula for the rational solutions of PII.

Theorem 2.8. *The second Painlevé equation* (PII) *admits rational solutions if and only if $\alpha = n \in \mathbb{Z}$. We denote these solutions $u_n(t)$. In particular, if $\alpha = 0$ the solution is trivial $u_0(t) = 0$; if $n \geq 1$ the solution is given in terms of the Vorob'ev-Yablonskii polynomials:*

$$u_n(t) = \frac{\mathrm{d}}{\mathrm{d}t} \log \frac{Y_{n-1}(t)}{Y_n(t)} = \frac{Y'_{n-1}(t)}{Y_{n-1}(t)} - \frac{Y'_n(t)}{Y_n(t)}. \tag{2.36}$$

For a negative integer $\alpha = -n$ the corresponding rational solution is given by $u_{-n}(t) = -u_n(t)$.

We remark that the poles of solutions to PII are always simple and have residue ± 1 [GLS02]. In the case of rational solutions $u_n(t)$ this can be deduced from the formula (2.36) and the fact that consecutive Vorob'ev–Yablonskii polynomials do not share any roots [Tan00]. Consequently the poles of residue $+1$ correspond to the zeroes of $Y_{n-1}(t)$ and the poles of residue -1 correspond to the zeroes of $Y_n(t)$.

In light of the Painlevé property, the study of the location of poles of solutions of any of the Painlevé transcendents remains an interestingly remarkable challenge . The problem has received significant attention for the cases of the *tritronquée solution* of the first Painlevé equation [Mas10a, Mas10b], the rational solutions of the second Painlevé transcendent [BM12, BM12, BM14, BM15, BB15], its hierarchy [CM03, BBB16] and the fourth Painlevé transcendent [MR18, BM20].

2.2.2 Lax pairs of Painlevé II

Following the isomonodromic deformation theory outlined in Section 2.1 we introduce two distinct rank-2 Lax pair representations for the second Painlevé transcendent (PII) which will be of crucial importance in the upcoming computations. The isomonodromic theory of Painlevé equations has been developed in [JMU-I, JM-II, JM-II]. They are known as the Flaschka-Newell and the Jimbo-Miwa Lax pairs and they're given as an overdetermined system of the shape

$$\begin{cases} \dfrac{\partial \Phi}{\partial z} = A(z,t)\Phi, \\ \dfrac{\partial \Phi}{\partial t} = B(z,t)\Phi. \end{cases} \tag{2.37}$$

The relation with the framework in Section 2.1.4 is that $z \in \mathbb{C}$ is a local coordinate of $\lambda \in \mathbb{CP}^1$, and that there is only one parameter $t = t_1 \in \mathbb{C}$. In this case the compatibility condition (2.33) becomes the equality of the partial derivatives $\Phi_{xt} = \Phi_{tx}$, which is equivalent to

$$\frac{\partial A}{\partial t} - \frac{\partial B}{\partial t} + [A, B] = 0, \tag{2.38}$$

gives a system of non-linear ODEs which is equivalent to the second Painlevé equation, different in each case. Throughout we denote the Pauli matrices

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{2.39}$$

and additionally we define

$$\sigma_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \sigma_- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. \tag{2.40}$$

Flaschka-Newell Lax pair

Theorem 2.9 ([FN80]). *The Painlevé II equation* (PII) *for the function $u(t)$ is determined from the compatibility condition of the following rank-2 overdetermined linear system* (2.37) *with the coefficients matrices*

$$\begin{aligned} A(z,t) &= -i(4z^2 + t + 2u^2)\sigma_3 + \left(4zu + \frac{\alpha}{z}\right)\sigma_1 - 2v\sigma_2, \\ B(z,t) &= -iz\sigma_3 + u\sigma_1, \end{aligned} \tag{2.41}$$

where $u = u(t)$ and $v = v(t)$ are meromorphic functions of t.

This system describes an isomonodromic deformation of a rank-2 linear ODE with a Fuchsian singularity at the $z = 0$ and an non-Fuchsian singularity of Poincaré rank 3 at $z = \infty$. The compatibility condition of the partial derivatives (2.38) yields the non-linear system

$$\frac{\mathrm{d}u}{\mathrm{d}t} = v, \quad \frac{\mathrm{d}v}{\mathrm{d}t} = 2u^3 + tu + \alpha, \tag{2.42}$$

which undoubtedly will be recognised as (PII) for $u(t)$.

Jimbo-Miwa Lax pair

Theorem 2.10 ([JM-II]). *The second Painlevé equation* (PII) *for the function $u(t)$ is determined from the compatibility condition of the rank-2 overdetermined linear system* (2.37) *with the coefficients matrices*

$$\begin{aligned} A(z,t) &= \left(z^2 + w + \frac{t}{2}\right)\sigma_3 + (z-u)v\sigma_+ - \frac{2}{v}\left(zw + uw - \alpha + \frac{1}{2}\right), \\ B(z,t) &= \frac{z}{2}\sigma_3 + \frac{v}{2}\sigma_+ - \frac{w}{v}\sigma_-, \end{aligned} \tag{2.43}$$

where $u = u(t), v = v(t), w = w(t)$ are meromorphic functions of t.

This system describes the isomonodromic deformation of a rank-2 linear ODE with a non-Fuchsian singularity of Poincaré rank 3 at $\lambda = \infty$. The compatibility condition of the partial derivatives (2.38) gives the non-linear system for u, v, w

$$\frac{\mathrm{d}u}{\mathrm{d}t} = u^2 + w + \frac{t}{2}, \quad \frac{\mathrm{d}v}{\mathrm{d}t} = -uv, \quad \frac{\mathrm{d}w}{\mathrm{d}t} = -2uw + \alpha - \frac{1}{2}. \tag{2.44}$$

Eliminating w from this system one finds that $u(t)$ satisfies (PII) with parameter α. In a similar manner getting rid of u one discovers that $w(t)$ satisfies the the thirty-fourth equation of the original Painlevé classification,

$$\frac{\mathrm{d}^2w}{\mathrm{d}t^2} = \frac{1}{2w}\left(\frac{\mathrm{d}w}{\mathrm{d}t}\right)^2 + 4aw^2 - tw - \frac{1}{2w} \tag{PXXXIV}$$

as listed in [Inc44, p.340].

Chapter 3

Orthogonal Polynomials and the exact WKB method

3.1 Speedy introduction to Orthogonal polynomials

In this section we give a quick introduction the some of the facts of the theory of orthogonal polynomials. For more detail we refer the reader to the classic book on the topic [Sze75] as well as [Chi78] for the moment functional approach, as well as the more modern approach [Sim15].

A moment functional is a linear map on the space of polynomials $\mathcal{M} : \mathbb{C}[z] \to \mathbb{C}$, associating to the monomial powers z^j, to a sequence of *moments* μ_j and then extended by linearity:

$$\mathcal{M}[z^j] = \mu_j, \quad j \in \mathbb{N}, \tag{3.1}$$

$$\mathcal{M}\left[aP(z) + bQ(z)\right] = a\mathcal{M}\left[P(z)\right] + b\mathcal{M}\left[Q(z)\right] \tag{3.2}$$

for all polynomials $P(z), Q(z) \in \mathbb{C}[z]$.

Definition 3.1. We say that $P_n \in \mathbb{C}[z]$ is an *orthogonal polynomial* of degree n if it satisfies

$$\langle P_n, z^m \rangle = \mathcal{M}\left[P_n(z) z^m\right] = \delta_{nm} h_n \quad h_n, \neq 0, \quad m = 1, 2, \ldots, n. \tag{3.3}$$

These orthogonal polynomials are unique up to a non-zero multiplicative factor and their existence is guaranteed by the non-vanishing of the Hankel determinants

$$\Delta_n = \det\left[\mu_{i+j}\right]_{i,j=0}^{n} = \det \begin{bmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \cdots & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{bmatrix} \neq 0. \tag{3.4}$$

Furthermore the orthogonal polynomials have the following determinantal expression:

$$P_n(z) = \det \begin{bmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \cdots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-1} \\ 1 & z & \cdots & z^n \end{bmatrix} \neq 0. \tag{3.5}$$

One of the key properties of orthogonal polynomials is that they satisfy a 3-term recurrence relation.

Theorem 3.2 (3-term recurrence relation). *Suppose that $\{P_n\}_{n=1}^{\infty}$ is a sequence of orthogonal polynomials defined with respect to the moment functional $\mathcal{M}$. Then there exist sequences a_n and b_n such that*

$$zP_n(z) = b_n P_{n+1}(z) + a_n P_n(z) + b_{n-1} P_{n-1}(z). \tag{3.6}$$

We remark that the orthogonality (3.3) is sometimes called "non-Hermitean" because the inner product is complex and bilinear rather than sesquilinear. This will be the setting of the orthogonal polynomials considered in Chapter 5. However, the sesquilinear case is the most studied and it leads us to the *classical orthogonal polynomials*.

3.1.1 Classical orthogonal polynomials

Definition 3.3. The *Hermite polynomials* H_n are defined by

$$H_n(z) = (-1)^n e^{z^2} \frac{d^n}{dz^n} e^{-z^2}. \tag{3.7}$$

The *Laguerre polynomials* $L_n^{(\alpha)}$ with parameter $\alpha > -1$ are defined by

$$L_n^{(\alpha)}(z) = \frac{1}{n!} z^{-\alpha} e^z \frac{d^n}{dz^n} \left(e^{-z} z^{n+\alpha} \right). \tag{3.8}$$

The *Jacobi polynomials* with parameter $\alpha, \beta > -1$ are defined by

$$J_n^{(\alpha,\beta)}(z) = \frac{(-1)^n}{n! 2^n} (1-z)^{-\alpha} (1+z)^{-\beta} \frac{d^n}{dz^n} \left[(1-z)^{n+\alpha} (1+z)^{n+\beta} \right]. \tag{3.9}$$

Collectively, these polynomials are known as the *classical orthogonal polynomials* because they satisfy the following orthogonality conditions:

$$\int_{-\infty}^{+\infty} H_n(z) H_m(z) e^{-z^2} dz = \sqrt{\pi} 2^n n! \delta_{nm} \tag{3.10}$$

$$\int_0^{+\infty} L_n^{(\alpha)}(z) L_m(z) z^\alpha e^{-z} dz = \frac{\Gamma(n+\alpha+1)}{n!} \delta_{nm} \tag{3.11}$$

$$\int_{-1}^{+1} J_n^{(\alpha,\beta)}(z) J_m^{(\alpha,\beta)}(z) (1-z)^\alpha (1+z)^\beta dz = \kappa_n \delta_{nm} \tag{3.12}$$

where the normalisation constant κ_n in the Jacobi case is

$$\kappa_n = \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+2)}{n!\Gamma(n+\alpha+\beta+1)}, \tag{3.13}$$

and the parameters satisfy $\alpha, \beta \geq -1$.

The formulæ in Definition 3.3 is known as the Rodriges formula. Although we take it as a definition, the classical orthogonal polynomials can also be defined in other ways (such as from equations (3.10)) from which the Rodriges formulæ can be rederived.

However, we will be consider them as arising from a moment functional given by the integral

$$\mathcal{M}[z^j] = \int_\Gamma z^j e^{\theta(z)} dz, \tag{3.14}$$

where in each of the cases, Hermite, Jacobi and Laguerre the symbol θ and contour Γ is chosen according to:

$$\text{Hermite}: \quad \theta(z) = -z^2, \quad \Gamma = \mathbb{R}, \tag{3.15}$$

$$\text{Jacobi}: \quad \theta(z) = \alpha \log(1-z) + \beta \log(1+z), \quad \Gamma = [-1, 1], \tag{3.16}$$

$$\text{Laguerre}: \quad \theta(z), = -z + \alpha \log(z), \quad \Gamma = [0, \infty). \tag{3.17}$$

Many things can be said about these polynomials but a noteworthy feature of all of them is that they all satisfy a second order ODEs

Theorem 3.4. *The Hermite polynomials H_n satisfy the differential equation:*

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}z^2}+2z\frac{\mathrm{d}}{\mathrm{d}z}\right)H_n(z)=2nH_n(z). \tag{3.18}$$

The Laguerre polynomials $L_n^{(\alpha)}$ satisfy the differential equation:

$$\left(-x\frac{\mathrm{d}^2}{\mathrm{d}z^2}+(x+1-\alpha)\frac{\mathrm{d}}{\mathrm{d}z}\right)L_n^{(\alpha)}(z)=nL_n^{(\alpha)}(z). \tag{3.19}$$

The Jacobi polynomials $J_n^{(\alpha,\beta)}(z)$ satisfy the differential equation:

$$\left(-(1-z)^2\frac{\mathrm{d}^2}{\mathrm{d}z^2}+[(\alpha+\beta+2)z+\alpha-\beta]\frac{\mathrm{d}}{\mathrm{d}z}\right)P_n^{(\alpha,\beta)}(z)=n(n+\alpha+\beta+1)P_n^{(\alpha,\beta)}(z) \tag{3.20}$$

Furthermore, the converse of this theorem holds, and which is name after Bochner and Brenke.

Theorem 3.5 ([Sim15])**.** *Suppose that $\{P_n\}_{n=0}^{\infty}$ is a family of orthogonal polynomials in the real line that obey a differential equation of the form*

$$f(x)\frac{\mathrm{d}^2P_n}{\mathrm{d}x^2}+g(x)\frac{\mathrm{d}P_n}{\mathrm{d}x}+h(x)P_n(x)=\lambda_nP_n(x) \tag{3.21}$$

where only λ_n is dependent on n. Then, up to a change of variables $x\mapsto ax+b$ with $a\in\mathbb{R}\setminus\{0\}, b\in\mathbb{R}$, the polynomial P_n is one of the classical orthogonal polynomials $H_n, L_n^{(\alpha)}, J_n^{(\alpha,\beta)}$.

In Chapter 5 we show that the degenerate orthogonal polynomials share many properties with the classical orthogonal polynomials. For example, they are proved to be as solutions to a second order ODE in the complex plane with rational coefficients. More importantly we show the distribution of the roots of degenerate orthogonal polynomials can be given an electrostatic interpretation, similar to the one of the classical orthogonal polynomials (1.21).

3.1.2 Riemann-Hilbert problem for Orthogonal Polynomials

Orthogonal polynomials have a characterisation in terms of a Riemann-Hilbert problem, which we now explain. We overlook some of the finer analytic issues,[1] opting instead to point the reader to the excellent introduction [Dei99].

Let μ be a measure which is absolutely continuous with respect to the Lebesgue measure so that we can write $\mathrm{d}\mu(z)=w(z)\mathrm{d}z$. We also assume that that $\mathrm{d}\mu=w\mathrm{d}z$ has finite moments. Let us denote by $\pi_n(z)$ the **monic** orthogonal polynomial of degree n satisfying

$$\int_\Gamma \pi_n(z)\pi_m(z)w(z)\mathrm{d}z=h_n\delta_{n,m}. \tag{3.22}$$

Consider the following Riemann-Hilbert problem.

Riemann-Hilbert Problem 3.6. *Fix an positive integer $n\in\mathbb{N}$ and the contour $\Gamma=\mathbb{R}$, oriented from $-\infty$ to $+\infty$, and the weight function $w(z)$. Find a 2×2 matrix-valued function $Y=Y_n(z)$ such that:*

(1) *The function $Y^{(n)}(z)$ is analytic in $\mathbb{C}\setminus\Gamma$.*

(2) *It satisfies the jump condition $Y_+^{(n)}(z)=Y_-^{(n)}(z)\begin{bmatrix}1 & w(z)\\ 0 & 1\end{bmatrix}$ for $z\in\Gamma$.*

(3) *It is normalised at infinity $Y^{(n)}(z)=\left(\mathbb{I}+\mathcal{O}\left(z^{-1}\right)\right)\begin{bmatrix}z^n & 0\\ 0 & z^{-n}\end{bmatrix}$ as $z\to\infty$.*

[1] This is a speedy introduction after all

The celebrated results of Its, Fokas and Kapaev [FIK91, FIK92] characterises the solutions of this Riemann-Hilbert problem in terms of the orthogonal polynomials and their Cauchy transform.

Theorem 3.7 (Its-Kitaev-Fokas)**.** *Let $Y^{(n)}$ be the solution of the Riemann-Hilbert problem 3.6. Then*

$$Y^{(n)}(z) = \begin{bmatrix} \gamma_n^{-1}\pi_n(z) & \dfrac{1}{2\pi\mathrm{i}}\displaystyle\int_\Gamma \frac{\pi_n(x)w(x)}{x-z}\mathrm{d}x \\ c_n\pi_{n-1}(z) & \dfrac{1}{2\pi\mathrm{i}}\displaystyle\int_\Gamma \frac{\pi_{n-1}(x)w(x)}{x-z}\mathrm{d}x \end{bmatrix} \tag{3.23}$$

where c_n is the constant

$$c_n = -2\pi\mathrm{i}h_{n-1}^{-1} \tag{3.24}$$

Moreover, the coefficients in the three term recurrence relation are given by:

$$a_n = \left(Y_1^{(n)}\right)_{1,1} - \left(Y_1^{(n)}\right)_{1,1} \tag{3.25}$$

$$b_n^2 = \left(Y_1^{(n)}\right)_{1,2}(Y_1^{(n)})_{2,1} \tag{3.26}$$

where we express

$$Y^{(n)}(z) = \left(\mathbb{I} + \frac{Y_1^{(n)}}{z} + \mathcal{O}\left(z^{-2}\right)\right)\begin{bmatrix} z^n & 0 \\ 0 & z^{-n}\end{bmatrix}. \tag{3.27}$$

3.2 Brief introduction to exact WKB analysis

In this section we review the theory of the exact WKB analysis, as initiated by [Vor83] and further developed by the Japanese school [KT05]. We follow the exposition in [IN14]

Consider the following Schrödinger equation with small parameter $\hbar$

$$\hbar^2\frac{\mathrm{d}^2\psi}{\mathrm{d}z^2} - Q(z)\psi = 0 \tag{3.28}$$

We assume the potential $Q(z)$ is a meromorphic function and that $Q(z)$ it is $\hbar$-independent. For the general case in which the potential depends on $\hbar$ we refer the reader to [IN14].

The poles of the potential $Q(z)$ determine the singular points of the differential equation; we denote the set of poles of $Q(z)$ by $\mathrm{P} = \{p_0, p_1, p_2, \dots\}$. In the exact WKB analysis the zeroes also of critical importance.

Definition 3.8. A zero of $Q(z)$ is called a *turning point*. We say a turning point is *simple* if it is a simple zero of $Q(z)$ and we denote the set of zeroes by $\mathrm{T} = \{\tau_0, \tau_1, \tau_2, \dots\}$

3.2.1 Construction of WKB solutions

To construct the WKB solutions of (3.28) we make the following ansatz

$$\psi(z,\hbar) = \exp\left(\int^z S(\zeta,\hbar)\mathrm{d}\zeta\right) \tag{3.29}$$

where $S(z,\hbar)$ is a formal series in $\hbar$

$$S(z,\hbar) = \sum_{n=-1}^{\infty}\hbar^n S_n(z) = \frac{1}{\hbar}S_{-1}(z) + S_0(z) + S_1(z) + \cdots. \tag{3.30}$$

This formal series satisfies the *Ricatti equation*

$$\frac{\mathrm{d}S}{\mathrm{d}z} + S^2 = \hbar^{-2}Q(z), \tag{3.31}$$

which means the coefficient functions $S_n(z)$ satisfy the recursion

$$S_{-1}^2 = Q(z), \tag{3.32}$$

$$2S_{-1}S_{n+1} = -\frac{\mathrm{d}S_n}{\mathrm{d}z} - \sum_{i+j=n} S_i S_j \quad (n \geq -1). \tag{3.33}$$

Depending on the sign chosen for $S_{-1}(z) = \pm\sqrt{Q(z)}$ we obtain two sequences of coefficients $\{S_j^{(\pm)}\}_{j=1}^{\infty}$. If we change the sign in S_{-1} then all the odd S_{2k+1} flip their sign while the even S_{2k} remain unchanged. With the choice $\mathbb{S}_{-1} = \sqrt{Q}$, the first three coefficients are:

$$S_{-1} = \sqrt{Q(z)}, \qquad S_0 = -\frac{1}{4}\frac{Q'(z)}{Q(z)}, \qquad S_1 = \frac{4Q(z)Q(z)'' - 5(Q(z)')^2}{32Q(z)^{\frac{5}{2}}}. \tag{3.34}$$

The choice of initial sign for S_{-1} gives two formal series solutions $S_{\pm}(z,\hbar)$ to the Ricatti equation (3.31). In turn, these correspond to two linearly independent (formal) solutions to (3.28)

$$\psi(z,\hbar) = \exp\left(\int^z S_{\pm}(\zeta,\hbar)\mathrm{d}\zeta\right). \tag{3.35}$$

These solutions however, can be simplified further as we now explain. Let us define

$$S_{\mathrm{odd}}(z,\hbar) := \frac{1}{2}\left(S_+(z,\hbar) - S_-(z,\hbar)\right), \tag{3.36}$$

$$S_{\mathrm{even}}(z,\hbar) := \frac{1}{2}\left(S_+(z,\hbar) + S_-(z,\hbar)\right). \tag{3.37}$$

Since the potential $Q(z)$ is independent of $\hbar$ it follows that S_{odd} only contains odd powers of $\hbar$

$$S_{\mathrm{odd}}(z,\hbar) = \frac{1}{\hbar}\sqrt{Q(z)} + \hbar S_1 + \hbar^3 S_3 + \dots. \tag{3.38}$$

It was shown in [KT05] that

$$S_{\mathrm{even}}(z,\hbar) = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\log S_{\mathrm{odd}}(z,\hbar). \tag{3.39}$$

This allows us to rewrite the formal solution in terms of S_{odd}.

Definition 3.9 (WKB solutions)**.** The *WKB solutions* to (3.28) are formal power series in $\hbar$ given in terms of S_{odd} in (3.38). We give two different normalizations that will be used throughout this paper.

- **Near a turning point** $\tau \in \mathrm{T}$ of the potential $Q(z)$ we define the *normalized WKB solutions* to be:

$$\psi_{\pm}^{(\tau)}(z,\hbar) := \frac{1}{\sqrt{S_{\mathrm{odd}}(z,\hbar)}}\exp\left(\pm\int_\tau^z S_{\mathrm{odd}}(\zeta,\hbar)\mathrm{d}\zeta\right) \tag{3.40}$$

- **Near a pole** $p \in \mathrm{P}$ we define the *normalized WKB solutions* to be:

$$\psi_{\pm}^{(\infty)}(z,\hbar) := \frac{1}{\sqrt{S_{\mathrm{odd}}(z,\hbar)}}\exp\Big(\pm R(z;\hbar)\Big) \tag{3.41}$$

  where

$$R(z;\hbar) := \frac{1}{\hbar}\int^z \left[\sqrt{Q(w)}\right]_- \mathrm{d}w + \sum_{j\geq 0}\hbar^{2j+1}\int_p^z S_{2j+1}(w)\mathrm{d}w \tag{3.42}$$

where we denote by $[Q(w)\mathrm{d}w]_-$ denotes the polar part of the differential $\sqrt{Q(w)}\mathrm{d}w$ at $w = p$, i.e. the strictly negative powers in the Puiseux expansion at infinity. Notice that $R(x;\hbar)$ is the anti-derivative of $S_{\mathrm{odd}}(x,\hbar)$ that does not have a constant term in the expansion as $|x| \to \infty$.

Remark 3.10 (Geometry of the WKB solutions)**.** The integral in the exponent of $\psi_+^{(\tau)}$ is to be understood term by term in each coefficient of the powers of $\hbar$. Additionally, $S_{\mathrm{odd}}(z,\hbar)$ is multivalued in $\mathbb{C}$ with branch points at the zeroes T of the potential $Q(z)$. Thus the integrals should be considered on the the compact Riemann surface $\overline{\Sigma}$ obtain from the curve

$$\Sigma = \Big\{(w,z) \in \mathbb{C}^2 : w^2 = Q(z)\Big\}. \tag{3.43}$$

For our purposes the potential $Q(z)$ will be a quartic polynomial, which means that $\overline{\Sigma}$ will be a compact elliptic curve of genus 1 obtained from Σ by adding two points at infinity $P_\infty^{\pm}$.

The projection $\pi : \overline{\Sigma} \mapsto \overline{\mathbb{C}}$ from the Riemann surface to the extended complex plane mapping $(w,z) \mapsto z$ realizes $\overline{\Sigma}$ as a double cover of $\overline{\mathbb{C}}$ ramified at the turning points of the ODE (3.28) (i.e. at the zeroes of $Q(z)$). The pre-image of any point $x \in \overline{\mathbb{C}}$ are the two points

$$\pi^{-1}(z) = \{(z,+w),(z,-w)\} \in \overline{\Sigma} \tag{3.44}$$

on the two sheets of the Riemann surface where the numbering of the sheets is such that $(x, w = +\sqrt{Q(x)})$ belongs to the first sheet. Choosing branch cuts and the first sheet of the Riemann surface, we can talk about the integral from a turning point τ to x by defining:

$$\int_\tau^z S_{\mathrm{odd}}(z,\hbar)\mathrm{d}z := \frac{1}{2}\int_{\gamma(z)} S_{\mathrm{odd}}(z,\hbar)\mathrm{d}z \tag{3.45}$$

where the contour $\gamma(z)$ lives in the Riemann surface $\overline{\Sigma}$ joining the points in the pre-image of $\pi^{-1}(z)$, oriented from the second sheet to the first sheet. This integral is a well defined convergent integral.

Remark 3.11. It is shown in [IN14] that, under certain conditions (which are met our case of a quartic polynomial), the higher order terms in S_{odd}, namely $S_{2j+1}(z)\mathrm{d}z$ for $j \geq 0$ is integrable at any pole $p \in \mathrm{P}$.

3.2.2 Stokes curves and Stokes graph

The WKB series are asymptotic to actual solutions of (3.28) in certain regions of the complex plane that we presently define. We begin by introducing the notion of Stokes' curve, which we will use to build Stokes' graphs.

Definition 3.12 (Stokes curve)**.** A **Stokes curve** of the potential $Q(z)$ is a horizontal trajectory of the quadratic differential $Q(x)\mathrm{d}x^2$ where one of the end-points is a turning point. In other words, in a local coordinate z of $\overline{\Sigma}$, it is a curve emanating from a turning point $\tau \in \mathrm{T}$ and satisfying

$$\mathrm{Im}\int_\tau^z \sqrt{Q(w)}\mathrm{d}w = 0. \tag{3.46}$$

The support of the curve is independent of choice of determination of $\sqrt{Q}$. Furthermore the **orientation of a Stokes curve** is defined by the direction in which

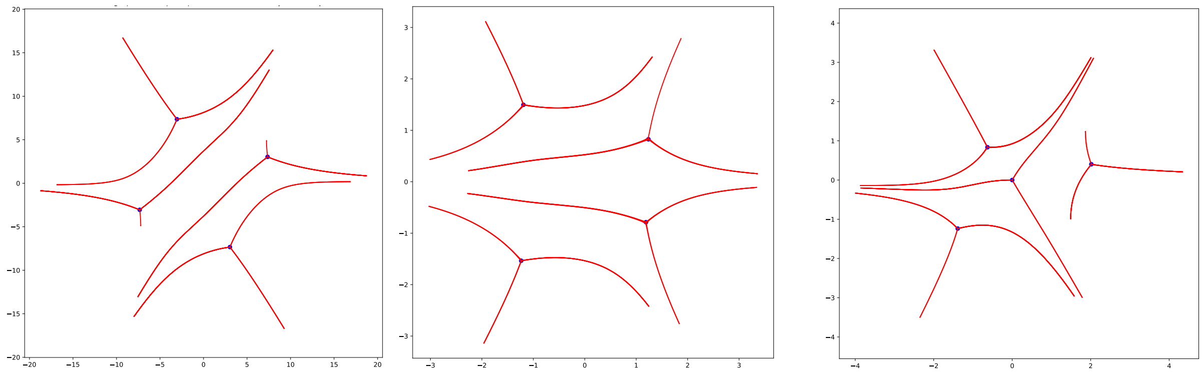
$$\mathrm{Re}\int_\tau^z \sqrt{Q(w)}\mathrm{d}w \tag{3.47}$$

is *increasing*. The orientation of a Stokes curve near a pole $p \in \mathrm{P}$ is denoted by the symbol $\oplus$ or $\ominus$ depending whether (3.47) is increasing or decreasing, respectively, along the z in the Stokes curve. We say the Stokes curve is oriented towards $\oplus$ and away from $\ominus$, respectively.

The Stokes curves always end at either a pole $p \in \mathrm{P}$ or at a turning point $\tau \in \mathrm{T}$ of the potential $Q(z)$ [Str84]. This means we can consider the turning points and points $\mathrm{T} \cup \mathrm{P}$ as vertices and the Stokes' curves as edges of a graph embedded in the Riemann surface $\overline{\Sigma}$.

Definition 3.13 (Stokes graph)**.** The **Stokes graph** $\mathcal{G} = (V,E)$ associated to the potential $Q(z)$ is the graph embedded in $\mathbb{CP}^1$ where the vertices are the turning points and poles $V = \mathrm{T} \cup \mathrm{P}$ and the edges are the Stokes curves.

In addition, call a **decorated** Stokes graph a Stokes graph with the following additional contours:



(a) $(s, E) = (0.00001, 1000\mathrm{i})$ (b) $(s, E) = (0.1\mathrm{i}, 2)$ (c) $(s, E) = (1 + \mathrm{i}, 0)$

Figure 3.1: Generic Stokes graph for the quartic polynomial $Q(z) = z^4 + sz^2 + 2z + E$ for various choices of (s, E). We remark that under small perturbations of $(s + \epsilon, E + \delta)$ the geometry of the Stokes graph varies continuously (provided (s, E) are generic), however, the underlying graph structure remains the same.

- **Branch cuts.** We draw arbitrary (smooth) paths called *branch-cuts* between pairs of turning points in such a way they do not intersect any Stokes curve. Their orientation is fixed in an arbitrary way that we shall specify in each case.
- **Ideal paths.** We draw arbitrary smooth paths connecting the different poles P in all possible ways that do not intersect any of the Stokes curves.

  We say a Stokes graph is simple if all of its

Definition 3.14 (Stokes regions)**.** Consider the connected components of the complement of the Stokes embedded graph in $\mathbb{CP}^1$ (i.e. the faces of the graph). Amongst them, the components with at least one Stokes curve on the boundary will be called *Stokes regions*. The remaining ones will be called *external regions*.

The construction of Stokes regions is crafted so that, under the genericity Assumptions 3.15, each Stokes region has precisely exactly one turning point on its boundary.

Assumption 3.15. The following assumptions shall prevail.

- *Simplicity.* The zeroes of the potential $Q(z)$ are all simple.
- *Genericity.* There are no saddle trajectories, i.e. there are no Stokes curves connecting two turning points. Saddle trajectories can only occur if there is a contour $\gamma \in H_1(\overline{\Sigma})$ in the homology group of the Riemann surface for which

$$\mathrm{Im} \oint_\gamma \sqrt{Q(z)}\mathrm{d}z = 0. \tag{3.48}$$

3.2.3 Normalised solutions and connection formulæ

Near a simple turning point τ there are three Stokes curves emanating from it, and so there are three different Stokes regions surrounding τ.

In each Stokes region $\mathscr{D}$ one can select a fundamental basis of solutions of the ODE (3.28) as we now explain. Consider a Stokes curve γ originating at the turning point τ extending to a pole $p \in \mathrm{P}$ and oriented towards $\oplus$ (with similar considerations for the $\ominus$ curves): since the real part of $\int_\tau^z \sqrt{Q(x)}dx$ is increasing, there is a region around γ near p where the real part is positive; then the formal solution $\psi_-^{(\tau)}$ is *recessive* (i.e. exponentially small as $\hbar \to 0_+$). Then there is a unique solution $\Psi(z;\hbar)$ which is asymptotic to this $\psi_-^{(\tau)}$ in both Stokes regions on the two sides of γ. If $\mathscr{D}$ is one of these regions, we will denote this first selected solution by $\Psi_-^{(\mathscr{D})}$.

To uniquely determine the other solution $\Psi_+^{(\mathscr{D})}$, it is not sufficient to examine its asymptotic behaviour near γ because its asymptotics there is *dominant* (i.e. exponentially large as $\hbar \to 0_+$). However, the same Stokes region must be bounded also by either a $\ominus$ trajectory or a branch-cut. In the former case, in a neighbourhood of the $\ominus$ trajectory the formal solution $\psi_+^{(\tau)}$ is now recessive and this allows to uniquely determine $\Psi_+^{(\mathscr{D})}$. If, instead, the other boundary is a branch-cut, we need to consider the Stokes region, $\mathscr{D}'$ on the other side of the cut: in this region, due to having crossed the branch-cut, the formal solution $\psi_+^{(\tau)}$ is now somewhere recessive and this allows to fix $\Psi_+^{(\mathscr{D})}$ uniquely. We refer to [BK21, §5] for more details.

Theorem 3.16 (Existence of solutions)**.** *Let $\mathscr{D}$ be a Stokes' region. Then, under the Assumptions 3.15, there exist unique solutions $\Psi_\pm^{(\mathscr{D})}$ of (3.28), that we refer to as* **normalized solutions**, *that are asymptotic to the WKB solutions in Def. 3.9 uniformly in x in the Stokes region, that is:*

$$\Psi_\pm^{(\mathscr{D})}(x) \sim \psi_\pm^{(\tau)}(x;\hbar) \quad \hbar \to 0, \quad x \in \mathscr{D}. \tag{3.49}$$

Remark 3.17. In the exact WKB literature, this result is usually stated in terms of Borel resummation of of the asymptotic series $\psi_\pm^{(\tau)}$, and it is typically attributed to the work of Koike and Schäfke in 2014. Their work concerned the Borel summability of WKB solutions of Schrödinger-type equations with polynomial potentials (although other accounts such as [IN14] assume the potential is meromorphic). Unfortunately, this paper was never published [2] and only a sketch of their ideas can be found [Tak17]. However a different, more geometrical, account by Nikolaev [Nik20, Nik21] puts their results on rigorous footing.

The following theorem of Vöros [Vor83, KT05] relates the WKB solutions of different Stokes regions near the same turning point.

Theorem 3.18 (Voros connection formulæ)**.** *Let $\mathscr{D}$ be a Stokes' region. Then, under the Assumptions 3.15, there exist unique solutions $\Psi_\pm^{(\mathscr{D})}$ of (3.28), that we refer to as* **normalized solutions**, *that are asymptotic to the WKB solutions in Def. 3.9 uniformly in x in the Stokes region, that is:*

$$\Psi_\pm^{(\mathscr{D})}(x) \sim \psi_\pm^{(\tau)}(x;\hbar) \quad \hbar \to 0, \quad x \in \mathscr{D}. \tag{3.50}$$

Furthermore, let $\mathscr{D}_\ell, \mathscr{D}_r$ be two adjacent Stokes regions separated by the Stokes curve γ oriented as in Def. 3.12 with $\mathscr{D}_\ell$ on the left and $\mathscr{D}_r$ on the right of γ. Then the corresponding solutions $\Psi_\pm^{\{\mathscr{D}_\ell\}}, \Psi_\pm^{\{\mathscr{D}_r\}}$ are related by:

$$\left[\Psi_+^{(\mathscr{D}_\ell)}\Psi_-^{(\mathscr{D}_\ell)}\right] = \left[\Psi_+^{(\mathscr{D}_r)}\Psi_-^{(\mathscr{D}_r)}\right] \begin{cases} B := \begin{bmatrix} 1 & 0 \\ -i & 1 \end{bmatrix} & \text{if } \gamma \text{ is oriented towards } \oplus \\ R := \begin{bmatrix} 1 & i \\ 0 & 1 \end{bmatrix} & \text{if } \gamma \text{ is oriented away from } \ominus. \end{cases} \tag{3.51}$$

We can formulate a result similar to Thm. 3.18 but relating two regions of a decorated Stokes graph separated by an ideal path. As we see below, the relationship between such WKB solutions is a simple scaling.

Proposition 3.19. *Let $\mathscr{D}_\ell, \mathscr{D}_r$ be two Stokes regions separated by an ideal path, with $\mathscr{D}_\ell$ on the left and $\mathscr{D}_r$ on the right of σ with the orientation given in Def. 3.13. Let τ_ℓ, τ_r be the (unique) turning points on the boundaries of $\mathscr{D}_\ell, \mathscr{D}_r$, respectively. Then we have the following connection formula*

$$[\psi_+^{(\tau_\ell)}, \psi_-^{(\tau_\ell)}] = [\psi_+^{(\tau_r)}, \psi_-^{(\tau_r)}] \exp(\sigma_3 v_{\ell r}) \tag{3.52}$$

where

$$v_{\ell r} = v_{\ell r}(\hbar) := \int_{\tau_\ell}^{\tau_r} S_{\mathrm{odd}}(\zeta_+, \hbar)\mathrm{d}\zeta \tag{3.53}$$

[2] The author thanks Nikita Nikolaev for explaining this issue to me, and saving me many headaches in the process.

with the integration along the branch cut according to Def. 4.18.

The corresponding actual solutions $\Psi_\pm^{(\mathscr{D}_\ell)}, \Psi_\pm^{(\mathscr{D}_r)}$ *given in Thm. 3.18 are similarly related:*

$$[\Psi_+^{(\mathscr{D}_\ell)}, \Psi_-^{(\mathscr{D}_\ell)}] = [\Psi_+^{(\mathscr{D}_r)}, \Psi_-^{(\mathscr{D}_r)}] \exp(\sigma_3 \hat{v}_{\ell r}) \tag{3.54}$$

where now $\hat{v}_{\ell r}(\hbar)$ *is a function of* $\hbar$ *that is asymptotic, in the Poincaré sense, to* $v_{\ell r}(\hbar)$ *in* (3.53).

Chapter 4

The Shapiro-Tater conjecture

This chapter is dedicated to the to the proof of the conjecture made by Shapiro and Tater regarding the similarity of two set of points in the complex plane. For full details of the conjecture and the approach we follow we refer the reader to section 1.2.

4.1 From Lax pair to scalar potential

In this section we outline how we obtain a scalar ODE from the Lax pairs in Section 2.2.2. In what follows we will need local expressions near a pole of the solutions to (PII) and of the system (2.44). The Laurent series of all of the Painlevé transcendents is well known and can be found in the encyclopedic book [GLS02].

Proposition 4.1 ([GLS02]). *Let $t = a$ be a pole of residue $\epsilon = \pm 1$ of the second Painlevé transcendent (PII). Then $u(t)$ has the following Laurent series expansion near $t = a$:*

$$u(t) = \frac{\epsilon}{t-a} - \frac{\epsilon a}{6}(t-a) - \frac{\alpha+\epsilon}{4}(t-a)^2 + b(t-a)^3 + \mathcal{O}\left((t-a)^4\right) \tag{4.1}$$

where $b \in \mathbb{C}$ is arbitrary and the higher coefficients are polynomial in a and b.

Starting from this Laurent series it is easy to determined expansion for the functions $v(t), w(t)$ appearing in the system (2.44). It is a simple (though tedious) computation to check the following (We seriously advise the reader the use of a computer algebra program for this purpose.)

Proposition 4.2. *Let $t = a$ be a pole of residue $+1$ of the Painlevé II function $u(t)$ with parameter α. Then the system (2.44) has the following Laurent series expansion near $t = a$*

$$\begin{aligned} u(t) &= \frac{1}{t-a} - \frac{a}{6}(t-a) - \frac{\alpha+1}{4}(t-a)^2 + b(t-a)^3 + \mathcal{O}\left((t-a)^4\right), \\ w(t) &= -\frac{2}{(t-a)^2} - \frac{a}{3} - \frac{1}{2}(t-a) + \left(-\frac{a^2}{36} + b\right)(t-a)^2 + \mathcal{O}\left(((t-a)^3)\right), \\ v(t) &= c\left(\frac{1}{t-a} + \frac{a}{12}(t-a) + \frac{\alpha+1}{12}(t-a)^2 + \mathcal{O}\left((t-a)^3\right)\right), \end{aligned} \tag{4.2}$$

where b is arbitrary and $c \neq 0$ is a constant of integration. Similarly, if $t = a$ is a pole of residue -1, then the system has a Laurent expansion near $t = a$

$$\begin{aligned} u(t) &= \frac{-1}{t-a} + \frac{a}{6}(t-a) - \frac{\alpha-1}{4}(t-a)^2 + b(t-a)^3 + \mathcal{O}\left((t-a)^3\right), \\ w(t) &= \left(\frac{1}{2} - \alpha\right)(t-a) + \left(5b - \frac{a^2}{36}\right)(t-a)^2 + \frac{a(2\alpha-1)}{6}(t-a)^3 + \mathcal{O}\left((t-a)^4\right), \\ v(t) &= c\left((t-a) - \frac{a}{12}(t-a)^3 + \mathcal{O}\left((t-a)^4\right)\right), \end{aligned} \tag{4.3}$$

where again b is arbitrary and $c \neq 0$ is a constant of integration.

With a particular gauge transformation we can convert *matrix* ODE to a *scalar* ODE. The procedure results in adding an *apparent singularity* in the scalar equation at the position $z = u(t)$. Thus when the independent variable t tends to one of the poles of the Painlevé transcendent $u(t)$ the location of the singularity "escapes" to infinity resulting in a *polynomial* ODE. This procedure was originally used in [Mas10a] to obtain asymptotic quantisation conditions for the poles of the tritronquée solution of the first Painlevé transcendent. We now explain it in detail and apply it to both the Flaschka-Newell (2.41) and Jimbo-Miwa (2.43) Lax pairs.

Consider a traceless rank-2 system

$$\frac{\partial \Phi}{\partial z} = A(z,t)\Phi, \quad A(z,t) := \begin{bmatrix} a_{11}(z,t) & a_{12}(z,t) \\ a_{21}(z,t) & a_{11}(z,t) \end{bmatrix}. \tag{4.4}$$

The gauge transformation $\mathcal{W} = G(z,t)\Phi$ with

$$G(z,t) := \begin{bmatrix} a_{12}^{-\frac{1}{2}} & 0 \\ -\frac{a_{12}'}{2a_{12}^{\frac{3}{2}}} + \frac{a_{11}}{a_{12}^{\frac{1}{2}}} & a_{12}^{\frac{1}{2}} \end{bmatrix} \tag{4.5}$$

(where $' = \frac{\partial}{\partial z}$) converts the original matrix ODE into $\frac{\partial}{\partial z}\mathcal{W} = \widetilde{A}(z,t)\mathcal{W}$ with the coefficient matrix

$$\widetilde{A}(z,t) = \frac{\partial G}{\partial z}G^{-1} + GAG^{-1} = \begin{bmatrix} 0 & 1 \\ V(z,t) & 0 \end{bmatrix} \tag{4.6}$$

and the function $V(z,t)$ is given in term of the entries of the initial coefficient matrix $A(z,t)$:

$$V(z,t) = a_{12}a_{21} + a_{11}^2 + a_{11}' - a_{11}\frac{a_{12}'}{a_{12}} - \frac{a_{12}''}{2a_{12}} + \frac{3}{4}\left(\frac{a_{12}'}{a_{12}}\right)^2, \tag{4.7}$$

where again $' = \frac{\partial}{\partial z}$. Therefore the matrix ODE $\frac{\partial}{\partial z}\mathcal{W} = \widetilde{A}(z,t)\mathcal{W}$ is equivalent to the Wronskian matrix of pair of independent solutions of the scalar ODE

$$\frac{\partial^2 y}{\partial z^2} - V(z,t)y = 0. \tag{4.8}$$

It is remarkable that applying this transformation to the Jimbo-Miwa (2.43) and Flaschka-Newell (2.41) Lax pairs, the limit of the potential $V(z,t)$ as t approaches a pole a of Painlevé has a neat formula.

Proposition 4.3 (FN potential). *Fix $\alpha \in \mathbb{C}$ and let $t = a$ be a pole with residue $+1$ of the second Painlevé transcendent (PII) with parameter α. Then the Flaschka-Newell Lax pair (2.41) is equivalent in the limit as $t \to a$ to the scalar ODE*

$$\begin{gathered} \frac{\mathrm{d}^2 y}{\mathrm{d}\zeta^2} - V_{\mathrm{FN}}(\zeta; s, r, \alpha)y = 0, \\ V_{\mathrm{FN}}(\zeta; s, r, \alpha) := \zeta^4 + s\zeta^2 + \left(\frac{s^2}{18} + 10r\right) + \frac{\alpha(\alpha-1)}{\zeta^2}, \end{gathered} \tag{4.9}$$

in the scaled variables $z = i2^{-2/3}\zeta, a = -2^{-1/3}s, b = 2^{-2/3}r$. Furthermore, for a pole of residue -1 the associated scalar ODE has potential $V_{\mathrm{FN}}(\zeta; s, -r, \alpha - 1)$.

Proof. We apply the gauge tranformation (4.5) to the matrix ODE $\frac{\partial}{\partial z}\Phi = A(z,t)\Phi$ of the Flaschka-Newell Lax pair (2.41). This gives us a potential $V(z,t)$ which is a a rational function of z and t, as well as the functions $v(t)$ and $u(t)$. The explicit expression of this potential is obtained using symbolic algebra software and we choose not to include it here as it is a very long and complicated formula. The key point is that substituting the series (4.1) for $u(t)$ and $v(t) = \frac{\mathrm{d}}{\mathrm{d}t}u(t)$, and taking limits we obtain

$$\lim_{t\to a} V(z,t) = -16z^4 - 8z^2 a - \frac{2a^2}{9} - 40b + \frac{\alpha(\alpha-1)}{z^2} \tag{4.10}$$

so that the associated ODE becomes

$$\frac{\mathrm{d}^2y}{\mathrm{d}z^2}-\left(-16z^4-8z^2a-\frac{2a^2}{9}-40b+\frac{\alpha(\alpha-1)}{z^2}\right)y=0. \tag{4.11}$$

After the scaling

$$z=i2^{-2/3}\zeta,\quad a=-2^{-1/3}s,\quad b=2^{-2/3}r, \tag{4.12}$$

we obtain the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}\zeta^2}-\left(\zeta^4+s\zeta^2+\left(\frac{s^2}{18}+10r\right)+\frac{\alpha(\alpha-1)}{\zeta^2}\right)y=0. \tag{4.13}$$

This is the claimed equation (4.9). ■

A similar result holds for the Jimbo-Miwa Lax pair.

Proposition 4.4 (JM potential). *Fix $\alpha\in\mathbb{C}$ and let $t=a$ be a pole with residue -1 of the second Painlevé transcendent* (PII) *function $u(t)$ with parameter α. The Jimbo-Miwa Lax pair* (2.43) *is equivalent in the limit $t\to a$ to the the following scalar ODE*

$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}z^2}-V_{\mathrm{JM}}(z;a,b,\alpha)y=0,\\ V_{\mathrm{JM}}(z;a,b,\alpha)=z^4+az^2+(2\alpha+1)z+\left(\frac{7a^2}{36}+10b\right).\end{aligned} \tag{4.14}$$

For a pole $t=a$ of residue $+1$ we obtain the same ODE with the potential $V_{\mathrm{JM}}(z;a,-b,\alpha-1)$.

Proof. We apply the gauge transformation (4.5) to the matrix ODE $\frac{\partial}{\partial z}\Phi=A(z,t)\Phi$ of the Jimbo-Miwa Lax pair (2.43). This gives us a potential $V(x,t)$ which is a function of both z and t, via the functions $u(t)$ and $v(t)$.

$$\begin{aligned}V(z,t)=&\frac{1}{(z-u(t))^2}\Bigg[2\,w\,(t)\,u(t)^4+(-4\,zw\,(t)-2\,\alpha+1)\,u(t)^3\\ &+\left(w(t)^2+\left(2\,z^2+\,t\right)w(t)+z^4+tz^2+(6\alpha-1)z+\frac{t^2}{4}\right)u(t)^2\\ &+\left(-2\,zw(t)^2+(-2\,tz+1)w(t)-2\,z^5-2\,tz^3-\frac{1}{2}\,t^2z-6\,\alpha\,z^2+\frac{t}{2}\right)u(t)\\ &+z^2w(t)^2+\left(tz^2-z\right)w(t)+z^6+tz^4+\frac{t^2z^2}{4}+2\,\alpha\,z^3-\frac{tz}{2}+\frac{3}{4}\Bigg].\end{aligned} \tag{4.15}$$

Substituting the Laurent series expansions (4.3) of $u(t),v(t),w(t)$ in the (complicated) expression above, and taking the limit as $t\to a$, we obtain:

$$\lim_{t\to a}V(z,t)=z^4+az^2+(2\alpha+1)z+\left(\frac{7a^2}{36}+10b\right). \tag{4.16}$$

As claimed, the associated scalar ODE becomes

$$\frac{\mathrm{d}^2y}{\mathrm{d}z^2}-\left(z^4+az^2+(2\alpha+1)z+\left(\frac{7a^2}{36}+10b\right)\right)y=0. \tag{4.17}$$

A similar computation can be done for the case with residue $+1$. ■

The transformation from a matrix ODE to a scalar ODE is important to us for the following reason: the gauge transformation (4.5) introduces a singularity at the zeroes of $a_{1,2}(z;t)$, which in the Jimbo-Miwa case happens at $x=u(t)$, namely, at the value of the Painlevé transcendent solution $u(t)$. The singularity is of square-root type with local monodromy -1. Other than this, the Stokes' phenomenon of the differential equation $y''(z) - V(z;t)y(z) = 0$ remains unchanged and independent of t by construction.

The key observation has the consequence that as t approaches one of the poles of the given solution $u(t)$ of the Painlevé transcendent, the additional singularity $x = u(t)$ moves off to infinity. Therefore we have the following simple but essential statement that we formalise in the proposition below.

Proposition 4.5. *Let $u(t)$ be a solution of the Painlevé II equation corresponding to a particular set of Stokes data for the Jimbo-Miwa Lax pair (2.43). Let $t = a$ be a pole of $u(t)$ with residue -1 and b the coefficient in the Laurent expansion (4.3). Then the Stokes phenomenon of the ODE*

$$y''(z) - \left(z^4 + az^2 + (2\alpha + 1)z + \Lambda\right) y(z) = 0, \quad \Lambda := \frac{7a^2}{36} + 10b \tag{4.18}$$

is the same as the original Stokes phenomenon of the Jimbo-Miwa Lax pair (2.43).

4.2 A study in quasi-polynomials

In this section we find a characterization of the (quasi)-polynomials corresponding to a repeated eigenvalue for the operator , namely, the Shapiro-Tater eigenvalue problem (1.4) We will call the set (t, J, λ) for which there is a solution of the problem (1.4)-(1.5), the *Exactly Solvable* (ES) spectrum. Our setting is slightly different from [ST22], where the authors considered a modified eigenvalue problem (1.4) with only *two* boundary conditions. In this case only a finite portion of the spectrum can be computed explicitly (in terms of a determinant), and for this reason it is called a *Quasi Exactly Solvable* (QES) spectrum, owing the name to [BB98].

We will see below that with three boundary conditions at infinity as in (1.5), the whole spectrum can be characterised by the vanishing of a finite determinant, and it is therefore *Exactly Solvable*.

The following lemma characterises the quasi-polynomials as the solutions to (1.4) with the three boundary conditions (1.5).

Lemma 4.6. *The equation (1.4) admits quasi–polynomial solutions of the form*

$$y(z) = p(z)e^{\theta(z;t)} \quad \text{where } \theta(z;t) = \frac{z^3}{3} + \frac{tz}{2} \tag{4.19}$$

with $p(x)$ a polynomial of degree n if and only if $J = n + 1$ and $\lambda = \Lambda - \frac{t^2}{4}$ is an eigenvalue of the operator

$$\mathcal{L}_J := \frac{\mathrm{d}^2}{\mathrm{d}z^2} + 2\left(z^2 + \frac{t}{2}\right)\frac{\mathrm{d}}{\mathrm{d}z} - 2(J-1)z \tag{4.20}$$

acting on the space of polynomials of degree up to n.

Proof. Substituting $y(z) = p(z)\mathrm{e}^{\theta(z)}$ in the ODE (1.4) gives an equivalent differential equation for the function $p(z)$:

$$\mathcal{L}_J[p(z)] = \lambda p(z) \quad \text{where} \quad \lambda = \Lambda - \frac{t^2}{4}, \tag{4.21}$$

and $\mathcal{L}_J$ as in (4.20). One can readily see that if $J = n + 1$ then the operator $\mathcal{L}_{n+1}$ in (4.21) preserves the space of polynomials of degree at most n and then Λ is, by definition, an eigenvalue of (1.4).

Viceversa, if $p(z)$ is a polynomial of degree n and solves (4.21) then one finds by inspection that the l.h.s is a polynomial of degree $n + 1$ whose leading coefficient is $2(n - J + 1)$ while the r.h.s. is a polynomial of degree n. Thus $J = n + 1$. Then $\mathcal{L}_{n+1}$ preserves the space of polynomials of degree n and Λ (and the corresponding λ as per (4.21)) is an eigenvalue of the corresponding finite dimensional operator. ∎

4.2.1 Stokes phenomenon of quasi-polynomials

Proposition 4.7. *The eigenvalue problem* (1.4) *with the boundary conditions* (1.5) *requires that* $J = n+1,\ n = 0, 1, \ldots$ *and that* λ *in* (4.21) *is an eigenvalue of the operator* $\widehat{L_J}$. *In particular the solutions are quasipolynomials as in Lemma 4.6.*

Proof. The equation (1.4) can be written as a first order system for the Wronkstian matrix $\mathcal{W}(x)$:

$$\frac{\mathrm{d}\mathcal{W}}{\mathrm{d}z} = \begin{bmatrix} 0 & 1 \\ z^4 + tz^2 + 2Jz + \Lambda & 0 \end{bmatrix} \mathcal{W}(z), \quad \Lambda = \lambda + \frac{t^2}{4}. \tag{4.22}$$

This differential equation has a singularity at infinity with Poincaré rank 3. Following the ordinary asymptotic analysis [Was87] we see that we have formal-series solutions of the form

$$\mathcal{W}_{\text{form}}(z) \sim z^{-\sigma_3}(\mathbb{I} - \sigma_+ + \sigma_-)\left(\mathbb{I} + \mathcal{O}\left(z^{-1}\right)\right) z^{J\sigma_3} e^{\theta(z;a)\sigma_3}, \quad \theta(z;t) = \frac{z^3}{3} + \frac{tz}{2} \tag{4.23}$$

$$= \begin{bmatrix} z^{J-1} + \mathcal{O}\left(z^{J-2}\right) & -z^{-J-1} + \mathcal{O}\left(z^{-J-2}\right) \\ z^{J+1} + \mathcal{O}\left(z^{J}\right) & z^{-J+1} + \mathcal{O}\left(z^{-J}\right) \end{bmatrix} \mathrm{e}^{\theta(z;t)\sigma_3}, \quad z \to \infty. \tag{4.24}$$

We define the Stokes sectors as shown in Figure 4.1, i.e. $\mathcal{S}_k$ is the sector of opening $\pi/3$ centered around the rays of argument $\pi/6 - k\pi/3$:

$$\mathcal{S}_k = \left\{ z \in \mathbb{C} : \left| \arg(z) - \frac{\pi}{6} + k\frac{\pi}{3} \right| < \frac{\pi}{6} \right\}, \quad k = 0, 1, 2, 3, 4, 5. \tag{4.25}$$

Each open sector $\Omega_k = \left\{ z \in \mathbb{C} : \left|\arg(z) - \frac{\pi}{6} + k\frac{\pi}{3}\right| < \frac{\pi}{6} + \delta \right\}$ with $\delta > 0$, strictly contains a Stoke sector S_k. Then there exists a unique solution $\mathcal{W}^{(k)}(z)$ of (4.22) in Ω_k asymptotic to $\mathcal{W}_{\text{form}}(z)$, namely

$$\mathcal{W}^{(k)}(z) \simeq \mathcal{W}_{\text{form}}(z), \quad z \to \infty, \quad z \in \Omega_k, \quad k = 0, \ldots, 5. \tag{4.26}$$

Since $\mathcal{W}^{(k)}(z)$ and $\mathcal{W}^{(k+1)}(z)$ have the same asymptotic expansion in $\Omega_k \cap \Omega_{k+1}$, there is a constant matrix $\mathbb{S}_k$ called Stokes multiplier, such that

$$\mathcal{W}^{(k+1)}(z) = \mathcal{W}^{(k)}(z)\mathbb{S}_k, \quad z \in \Omega_k \cap \Omega_{k+1}, \quad k = 0, 1, \ldots, 5, \tag{4.27}$$

where the Stokes multipliers are of the form

$$\mathbb{S}_{2k} = \begin{bmatrix} 1 & 0 \\ s_{2k} & 1 \end{bmatrix}, \quad \mathbb{S}_{2k+1} = \begin{bmatrix} 1 & s_{2k+1} \\ 0 & 1 \end{bmatrix}, \quad k = 0, 1, 2. \tag{4.28}$$

By standard methods one can see that the Stokes phenomenon consists of the following relation

$$\mathcal{W}^{(6)}(z) = \mathcal{W}^{(0)}(x)\mathbb{S}_0\mathbb{S}_1 \ldots \mathbb{S}_5, \quad \mathcal{W}^{(6)}(z) = \mathcal{W}^{(0)}(z)e^{2\pi i J\sigma_3} \tag{4.29}$$

which gives the relation

$$\begin{bmatrix} 1 & 0 \\ s_0 & 1 \end{bmatrix} \begin{bmatrix} 1 & s_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ s_2 & 1 \end{bmatrix} \begin{bmatrix} 1 & s_3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ s_4 & 1 \end{bmatrix} \begin{bmatrix} 1 & s_5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{e}^{-2\mathrm{i}\pi J} & 0 \\ 0 & \mathrm{e}^{2\mathrm{i}\pi J} \end{bmatrix} = \mathbb{I} \tag{4.30}$$

The first six matrices are the Stokes matrices associated with the directions $\arg(z) = k\frac{\pi}{3}$, $k = 0, 1, .., 5$ and the last matrix is the formal monodromy matrix. The boundary conditions (1.4) imply that $s_0 = s_2 = s_4 = 0$ because it means that the recessive solution along the direction $\arg(x) = \frac{\pi}{3}$ is also recessive along the directions $\arg(z) = k\frac{\pi}{3}$, $k = 3, 5$. But then the matrix equation (4.30) implies that

$$s_1 + s_3 + s_5 = 0 \tag{4.31}$$

and $\mathrm{e}^{2\mathrm{i}\pi J} = 1$ and hence J must be an integer.

To show that $J = n + 1$ is a *positive* integer and that λ is an eigenvalue of $\mathcal{L}_J$ we proceed as follows. Given that now the Stokes matrices are all upper triangular, the first column of the solution is an entire function which is asymptotic to the first column of the formal-series solution (4.23) along all directions.

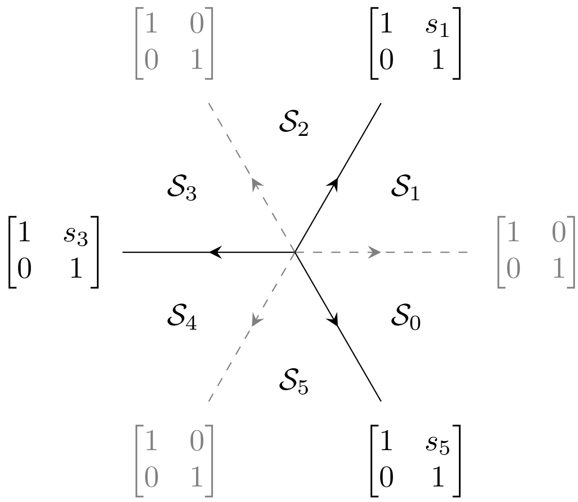


Figure 4.1: Stokes matrices and Stokes sectors for the Shapiro-Tater eigenvalue problem (1.4) with quasi-polynomial solutions. The Stokes matrices $\mathbb{S}_0, \mathbb{S}_2, \mathbb{S}_4$ are all the identity.

But then the asymptotic (4.23) implies that the $(1,1)$ entry is of the form $p(z)\mathrm{e}^{\theta(z;t)}$ with $p(z)$ entire and bounded at infinity by z^{J-1}. Then Liouville's theorem implies that if $J \geq 1$ then $p(z)$ is a polynomial, and for $J = 0, -1, -2, \ldots$ $p(z)$ should vanish at infinity and hence it should be identically zero, leading to a contradiction.

We have now established that the only solutions of the eigenvalue problem (1.4) are quasipolynomials and therefore the hypothesis of Lemma 4.6 prevail, thus showing that λ is the claimed eigenvalue. ■

Theorem 4.8. *Let $J = n+1$, $n \in \mathbb{N}$ and Λ be an eigenvalue of the boundary value problem (1.4)-(1.5) with eigenfunction the quasi-polynomial $F = p_n(z)\mathrm{e}^{\theta(z)}$. Let G_k be the linearly independent solutions of the ODE (1.4), which can be expressed as*

$$G_k(z) = F(z) \int_{\infty_k}^{z} \frac{\mathrm{d}\zeta}{F(\zeta)^2}, \quad k = 0, 2, 4. \tag{4.32}$$

Here ∞_k indicates that the contour of integration extends to infinity along the direction $\arg(z) = k\frac{\pi}{3}$.

Then the Stokes phenomenon for the solutions $[F, G_k]$ is given by the following equations:

$$\begin{aligned}
[F, G_2] =& [F, G_0] \begin{bmatrix} 1 & s_1 \\ 0 & 1 \end{bmatrix}, \qquad s_1 := \int_{\infty_2}^{\infty_0} \frac{\mathrm{d}\zeta}{F(\zeta)^2} \\
[F, G_4] =& [F, G_2] \begin{bmatrix} 1 & s_3 \\ 0 & 1 \end{bmatrix}, \qquad s_3 := \int_{\infty_4}^{\infty_2} \frac{\mathrm{d}\zeta}{F(\zeta)^2} \\
[F, G_0] =& [F, G_4] \begin{bmatrix} 1 & s_5 \\ 0 & 1 \end{bmatrix}, \qquad s_5 := \int_{\infty_0}^{\infty_4} \frac{\mathrm{d}\zeta}{F(\zeta)^2}.
\end{aligned} \tag{4.33}$$

Furthermore the Stokes parameters s_j satisfy (4.31).

Proof. Let $p_n(z)$ be a polynomial solution of degree n of (4.21) with $J = n+1$. We can obtain a second linearly independent solution q of (4.21) using the Wronskian identity:

$$\frac{\mathrm{d}p_n}{\mathrm{d}x} q - p_n \frac{\mathrm{d}q}{\mathrm{d}x} = e^{-2\theta}. \tag{4.34}$$

The solution is written as:

$$q(z) := p_n(z) \int_{z_0}^{z} \left(p_n(\zeta)\mathrm{e}^{\theta(\zeta;t)}\right)^{-2} \mathrm{d}\zeta \tag{4.35}$$

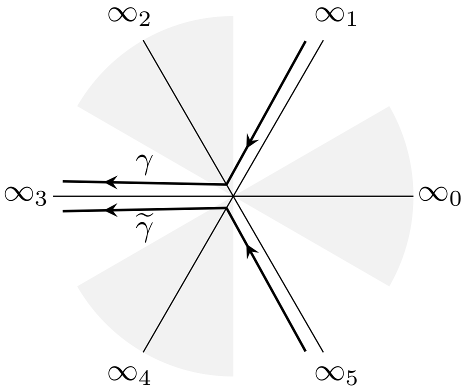


Figure 4.2: Directions at infinity ∞_k of argument $k\frac{i\pi}{3}$. The shaded regions denote the sectors of dominance of $e^{\theta(x;t)}$ where $\theta(z,t) = \frac{z^3}{3} + \frac{tz}{2}$. The unshaded regions denote the sectors of recessiveness.

with z_0 is arbitrary. We have thus found two linearly independent solutions of (4.21), and in turn we found two particular solutions of the eigenvalue problem (1.4), namely:

$$F(z) := p_n(z)e^{\theta(z;t)}, \tag{4.36}$$

$$G_{2k}(z) := F(z)\int_{\infty_{2k}}^{z} \frac{d\zeta}{F(\zeta)^2}, \quad k = 0, 1, 2. \tag{4.37}$$

Note that $e^{\theta(z;t)} \to 0$ along rays $\arg(z) = \pi/3, \pi, 5\pi/3$ (see Fig. 4.2), and so the function $F(z)$ satisfies the boundary conditions (1.5), while G_{2k} is unbounded as $z \to \infty$ along the same directions.

We can split the integral representation of G_{2k} in (4.32) as follows: for $k = 0, 1, 2$ we have

$$G_{2k+2}(z) = F(z)\int_{\infty_{2k+2}}^{z} \frac{d\zeta}{F(\zeta)^2} \tag{4.38}$$

$$= F(z)\left(\int_{\infty_{2k+2}}^{\infty_{2k}} \frac{d\zeta}{F(\zeta)^2} + \int_{\infty_{2k}}^{z} \frac{d\zeta}{F(\zeta)^2}\right) \tag{4.39}$$

$$= s_{2k+1}F(z) + G_{2k}(z) \tag{4.40}$$

were the indices are taken mod 6 and we have defined

$$s_1 := \int_{\infty_2}^{\infty_0} F(\zeta)^{-2}d\zeta, \quad s_3 := \int_{\infty_4}^{\infty_2} F(\zeta)^{-2}d\zeta, \quad s_5 := \int_{\infty_0}^{\infty_4} F(\zeta)^{-2}d\zeta, \tag{4.41}$$

owing to the observation the contour of integration in $s_{2k+1}, k = 0, 1, 2$ crosses the Stokes line of argument $(2k+1)\pi/3$ with $k = 0, 1, 2$.

The specific contour of integration of the Stokes parameters s_{2k+1} defined in (4.33) does not matter as long as it avoids the poles of the integrand $F(z)^{-2}dz$. Indeed the integrand $F(z)^{-2}dz$ has zero residue in the finite complex plane because if there was a non-zero residue at a pole, the function $G_{2k}(z)$ would have non-trivial monodromy around that pole, but this cannot be the case since G_{2k} is a solution to the linear ODE system (1.4) which has analytic coefficients in the finite complex plane $\mathbb{C}$. By an application of Cauchy's theorem we conclude that the sum of the Stokes parameters vanishes:

$$s_1 + s_3 + s_5 = 0. \tag{4.42}$$

■

4.2.2 Quasi-polynomials as degenerate orthogonal polynomials

In this section we introduce the concept of degenerate orthogonality of polynomials, and we show that it is satisfied by the polynomial part of the quasi-polynomial solutions of (1.4).

In the following we will consider *weighted contours* of the type

$$\Gamma = \sum_{j=1}^{n} s_j \gamma_j \tag{4.43}$$

where $s_j \in \mathbb{C}$ are parameters and $\gamma_j \subset \mathbb{C}$ are contours in the complex plane. For an arbitrary complex function we will denote

$$\int_\Gamma f(z)\mathrm{d}z := \sum_{j=1}^{n} s_j \int_{\gamma_j} f(z)\mathrm{d}z. \tag{4.44}$$

provided the integrals of $f(z)\mathrm{d}z$ along γ_j exist and are well defined. Using this notation, $p_n(z)$ is a **non-hermitian orthogonal polynomial** with respect to the weighted contour Γ and the weight $w(z) = \mathrm{e}^{\theta(z)}\mathrm{d}z$ if it satisfies:

$$\langle p_n(z), z^k\rangle := \int_\Gamma p_n(w) w^k \mathrm{e}^{2\theta}\mathrm{d}w = 0, \quad k = 0, 1, 2, \ldots, n-1. \tag{4.45}$$

In this classical definition, the orthogonality condition of $p_n(z)$ affects all the polynomials of degree $d < n$. We now introduce a the concept of *degenerate* orthogonality, in which additionally to the conditions (4.45), $p_n(z)$ is orthogonal to polynomials of degree $d \le \ell$ and $\ell \ge n$.

Definition 4.9 (Naïve degenerate orthogonality). Let $\theta(z)$ be a fixed polynomial of degree $d+1$ and positive leading coefficient, and let $\Gamma = \sum_{j=1}^{d} s_{2j-1}\gamma_{2j-1}$, where γ_{2j-1} are the *wedge* contours extending from ∞_{2j-1} to ∞_{2j+1} and ∞_k denotes the point at infinity in the directions $\arg(z) = \frac{k\pi}{d+1}$.

We say the polynomial $p_n(z)$ is **ℓ-degenerate orthogonal** if we additionally to the orthogonality conditions (4.45), it satisfies

$$\langle p_n(z), z^{n+k}\rangle = 0 \quad k = 0, 1, \ldots, \ell - 1. \tag{4.46}$$

In Chapter 5 this concept is generalised to the case when $\theta'(z)$ is a rational function, for this reason we call it here *naïve degenerate orthogonality*.

To prove that the quasi-polynomials solving (1.4) are degenerate orthogonal polynomials we will interpret the Stokes phenomenon in Theorem 4.8 as the jumps of a Riemann-Hilbert problem for orthogonal polynomials [Dei99].

Theorem 4.10. *Supppose that $F(z) = p_n(z)e^{\theta(z;t)}$ is a quasi-polynomial solution of the ODE (1.4). Then $p_n(z)$ is a 1-degenerate non-hermitian orthogonal polynomial with respect to the weight $w(z) = e^{2\theta(z,t)}\mathrm{d}z$ on the contour*

$$\Gamma = \varkappa\gamma + \widetilde{\varkappa}\widetilde{\gamma} \tag{4.47}$$

where $\varkappa = s_1, \widetilde{\varkappa} = s_5$ as defined in (4.33). The contour γ is the wedge contour from ∞_1 to ∞_3 and $\widetilde{\gamma}$ is the wedge contour from ∞_5 to ∞_3 (see Figure 4.2).

Proof. Consider the quasi-polynomial $F(z) := p_n(z)\mathrm{e}^{\theta(z;t)}$ which solves (1.4) with $J = n+1$. Define $G_k(z), k = 0, 2, 4$ as in (4.32), which also solves the same ODE, and let us set

$$\Psi_k(z) := [F(z), G_k(z)], \quad k = 0, 2, 4 \tag{4.48}$$

to be a fundamental solution of (1.4). We interpret Theorem 4.8 as an 2×2 Riemann-Hilbert problem solved by $\Psi_k(z)$ in the sectors of opening $2\pi/3$:

$$\mathcal{S}_k \cup \mathcal{S}_{k+1}, \quad k = 0, 2, 4. \tag{4.49}$$

The jump conditions (4.33) imply that G_k satisfies:

$$\mathrm{e}^{\theta(z;t)}G_{k+2}(z) = \mathrm{e}^{\theta(z;t)}G_k(z) + s_{k+1}p_n(z)\mathrm{e}^{2\theta(z;t)}, \quad k = 0, 2, 4. \tag{4.50}$$

The three functions $G_k, k = 0, 2, 4$ define a piecewise–analytic function $G(z)$ with discontinuities along the three rays $[0, \infty_j], j = 1, 3, 5$ where it satisfies

$$e^{\theta(z;t)}G_+(z) = e^{\theta(z;t)}G_-(z) + s_j p_n(z)e^{2\theta(z;t)}, \quad z \in [0, \infty_j], \quad j = 1, 3, 5. \tag{4.51}$$

By the Sokhotski–Plemelj formula we can express G as the following Cauchy transform over a weighted contour Γ:

$$G(z) := \frac{e^{-\theta(z;t)}}{2\pi i}\int_\Gamma \frac{p_n(w)e^{2\theta(w;t)}}{w-z}dw, \qquad \Gamma := s_1\gamma + s_5\widetilde{\gamma} \tag{4.52}$$

where the contours $\gamma, \widetilde{\gamma}$ are as in Fig.4.2. Note that the two contours overlap with the same orientation on the ray $[0, \infty_3]$. The function G defined by (4.52) coincides with G_k in each appropriate sector, since they satisfy the same Riemann-Hilbert jumps and both $G(z)e^{\theta(z)}$ and $G_k(z)e^{\theta(z)}$ are normalised with behaviour $\mathcal{O}\left(z^{-1}\right)$ at infinity.

To prove the normalisation it is sufficient to use the formula (4.52) for G and the asymptotics (4.23) for G_k. Indeed, writing

$$\frac{1}{w-z} = -\frac{1}{z}\left(1 + \frac{w}{z} + \mathcal{O}\left(z^{-2}\right)\right) \tag{4.53}$$

we find

$$G(z)e^{\theta(z;t)} = -\frac{1}{2\pi i z}\int_\Gamma p_n(w)we^{2\theta(w;t)}dw + \mathcal{O}\left(z^{-2}\right), \tag{4.54}$$

which shows that $G(z) = \mathcal{O}\left(z^{-1}\right)$ as $z \to \infty$. We obtain similar asymptotics for $G_k(z)$ from the asymptotics (4.23). Taking $J = n+1$ we see the Wronskian matrix of the ODE (4.22) for the pair of solutions (F, G_k) has the following asymptotic expansion (using $' = \frac{d}{dz}$):

$$\begin{bmatrix} F(z) & G_k(z) \\ F'(z) & G_k'(z) \end{bmatrix} \sim \begin{bmatrix} z^n e^{\theta(z)} & -z^{-n-2}e^{-\theta(z)} \\ z^{n+2}e^{\theta(z)} & z^{-n}e^{-\theta(z)} \end{bmatrix}\left(\mathbb{I} + \mathcal{O}\left(z^{-1}\right)\right), \tag{4.55}$$

$$z \to \infty, \quad z \in \mathcal{S}_k \cup \mathcal{S}_{k+1}, \quad j = 0, 2, 4$$

The $(1, 2)$ entry of the Wronskian shows that $G_k(z)e^{\theta(z;t)} = \mathcal{O}\left(z^{-1}\right)$. Therefore $G(z)$ and $G_k(z)$ coincide in each appropriate sector as claimed.

These considerations imply the degenerate orthogonality of the polynomials $p_n(z)$. Taking F and G given as above, it follows that (4.55) is the asymptotic expansion of F and G in each of the sectors (4.49). Thus the asymptotic

$$G(z) = \frac{e^{-\theta(z)}}{2\pi i}\int_\Gamma \frac{p_n(w)e^{2\theta(w)}}{w-z}dw \tag{4.56}$$

$$= -\frac{1}{z}\frac{e^{-\theta(z)}}{2\pi i}\int_\Gamma p_n(w)e^{2\theta(w)}\left(1 + \frac{w}{z} + \cdots + \frac{w^{n+1}}{z^{n+1}} + \dots\right)dw \sim -z^{-n-2}e^{-\theta(z)} \tag{4.57}$$

implies the vanishing of the integrals:

$$\int_\Gamma p_n(z)z^k e^{2\theta(z)}dz, \quad k = 0, 1, \dots, n-1, n. \tag{4.58}$$

Identifying $\varkappa = s_1$ and $\widetilde{\varkappa} = s_5$ we find Therefore the polynomials p_n are *degenerate* orthogonal as claimed. ■

In order to prove the converse of the previous theorem we need the following lemma, which makes use of the basic properties of the indicial equation associated to an ODE; we refer the reader to [Olv97].

Lemma 4.11. *Consider the second order ODE*

$$\frac{d^2y}{dz^2} - V(z)y = 0. \tag{4.59}$$

Suppose that $z = z_*$ is a (possible) singularity of the potential $V(z)$ and it is a pole of order at most 2. Assume that it is an apparent singularity, in the sense that the two linearly independent solutions to the ODE are actually analytic at the point $z = z_*$. Then, $V(z)$ is locally analytic at z_*.

Proof. We argue by contradiction and consider separately the cases when $V(z)$ has a double pole and when $V(z)$ has a simple pole.

- *Case 1: simple pole.* Suppose that near the singular point $z = z_\star$ the potential is of the form

$$V(z) = \frac{a}{\zeta} + b + \mathcal{O}(\zeta) \tag{4.60}$$

with $a \neq 0$ and $\zeta = z - z_\star$. The indicial equation of the ODE

$$d(d-1) = 0 \tag{4.61}$$

has two solutions $d_1 = 1$ and $d_2 = 0$ differing by a non-zero integer $d_1 - d_2 = 1$, meaning that that there are two linearly independent solutions $y_1(z), y_2(z)$ such that

$$y_1(z) = \mathcal{O}\left(\zeta^{d_1}\right), \tag{4.62}$$
$$y_2(z) = \mathcal{O}\left(\zeta^{d_2}\right) \tag{4.63}$$

as $z \to z_\star$. But it is evident that $y_2(z) = 1 + \mathcal{O}(\zeta)$ cannot solve the differential equation (4.59) since the left of the equation is analytic at $z = z_\star$ and the right side has a pole at $z = z_\star$ This is a contradiction, and so $V(z)$ cannot have a simple pole.

- *Case 2: double pole.* Suppose that near the singular point $z = z_\star$ the potential has the shape

$$V(z) = \frac{a}{\zeta^2} + \frac{b}{\zeta} + \mathcal{O}(1) \tag{4.64}$$

with $a \neq 0$ and again we denote $\zeta = z - z_\star$. The indicial equation now gives

$$a = d(d-1)\,. \tag{4.65}$$

Let d_1, d_2 be the two solutions of the indicial equation (4.65). By the assumption that the solutions to the ODE are all analytic, we must have that d_1 and d_2 must be non-negative integer and not equal to each other (if $d_1 = d_2$ then one of the solutions has a logarithmic singularity). Note that $d_1 = d_2$ if and only if $d_1 = d_2 = 1/2$, so we may simply assume that d_1, d_2 are non-negative integers. Rewriting the equation $d_1(d_1 - 1) = d_2(d_2 - 1)$ as

$$(d_1 - d_2)(d_1 + d_2 - 1) = 0 \tag{4.66}$$

we see that the only non-negative integer solutions are $d_1 = 0$ and $d_2 = 1$ and vice versa. In either case from the indicial equation we find that $a = 0$, which is a contradiction. Therefore $V(z)$ cannot have a double pole.

■

We now are ready to prove the converse of Theorem 4.10, namely that the degenerate orthogonal polynomials are precisely the polynomial part of the quasi-polynomial solutions to the Shapiro-Tater eigenvalue problem (1.4).

Theorem 4.12. *Suppose that $p_n(z)$ is a 1-degenerate orthogonal polynomial with respect to the weight $w(z) = e^{2\theta(z;t)}dz$ on the contour $\Gamma := \varkappa\gamma + \tilde{\varkappa}\tilde{\gamma}$ introduced in Theorem 4.10. Then $F(z) = p_n(z)e^{\theta(z;t)}$ is a quasi-polynomial solution of the boundary problem (1.4)-(1.5) with $J = n + 1$.*

Proof. Suppose that $p_n(z)$ satisfies

$$\int_\Gamma p_n(w) w^k e^{2\theta(w;t)} dw = 0, \quad k = 0, 1, \ldots, n, \tag{4.67}$$

namely, $p_n(z)$ is a degenerate orthogonal polynomial with respect to the measure $\mathrm{d}\mu(z) = \mathrm{e}^{2\theta(z;t)}\mathrm{d}z$ on the weighted contour Γ as in (4.47). Let us define the functions

$$F(z) := p_n(z)\mathrm{e}^{\theta(z;t)}, \tag{4.68}$$

$$G(z) := \frac{\mathrm{e}^{-\theta(z;t)}}{2\pi\mathrm{i}} \int_\Gamma \frac{p_n(w)\mathrm{e}^{2\theta(w;t)}\mathrm{d}w}{w-z}. \tag{4.69}$$

We claim the Wronskian $W = \mathrm{Wr}\{F, G\} = FG' - F'G$ is constant. Indeed, the degenerate orthogonality condition implies that

$$G(z) \sim \left(h_n z^{-n-2} + \mathcal{O}\left(z^{-n-3}\right)\right) \mathrm{e}^{-\theta(z;t)}, \quad z \to \infty \tag{4.70}$$

where the leading factor is

$$h_n := -\frac{1}{2\pi i} \int_\Gamma p_n(w)\mathrm{e}^{2\theta(w;t)} w^{n+1}\mathrm{d}w. \tag{4.71}$$

Furthermore, by differentiating (denoting $' = \frac{\mathrm{d}}{\mathrm{d}x}$) we find

$$G'(z) = -\theta'(z)G(z) + \frac{\mathrm{e}^{-\theta(z;t)}}{2\pi\mathrm{i}} \int_\Gamma \frac{p_n(w)\mathrm{e}^{2\theta(w;t)}}{(w-z)^2}\mathrm{d}w \tag{4.72}$$

$$= -\theta'(z)G(z) + \frac{1}{z^2}\frac{\mathrm{e}^{-\theta(z;t)}}{2\pi\mathrm{i}} \int_\Gamma p_n(w)\mathrm{e}^{2\theta(w;t)} \left(1 + 2\frac{w}{z} + 3\frac{w^2}{z^2} + \dots\right)\mathrm{d}w \tag{4.73}$$

$$= \mathrm{e}^{-\theta(z;t)}\left(-h_n z^{-n} + \mathcal{O}\left(z^{-n-1}\right)\right) + \frac{1}{z^2}\frac{\mathrm{e}^{-\theta(z;t)}}{2\pi\mathrm{i}}\mathcal{O}\left(z^{-n-1}\right) \tag{4.74}$$

$$= \mathrm{e}^{-\theta(z;t)}\left(-h_n z^{-n} + \mathcal{O}\left(z^{-n-1}\right)\right). \tag{4.75}$$

Additionally, since $p_n(z)$ is a monic polynomial, we have that $F(z) \sim z^n \mathrm{e}^{\theta(z;t)}$ and $F'(z) \sim z^{n+2}\mathrm{e}^{\theta(z;t)}$, which means that the Wronskian is bounded at infinity, i.e.

$$W(z) = F'(z)G(z) - F(z)G'(z) = -2h_n + \mathcal{O}\left(z^{-1}\right), \quad z \to \infty. \tag{4.76}$$

Using the fact that the derivative of the Cauchy transform $G(z)$ satisfies the jump condition

$$G'_+(z) - G'_-(z) = s_k F'(z), \quad z \in \gamma, \widetilde{\gamma}, \tag{4.77}$$

we can see that the Wronskian has no jump-discontinuities since on each contour $z \in \gamma, \widetilde{\gamma}$ we have

$$W_+ = F'G_+ - FG'_+ = F'(s_k F + G_-) - F(s_k F' + G'_-) = F'G_- - FG'_- = W_-. \tag{4.78}$$

Finally, since W is built from locally analytic functions, it also follows that $W(x)$ has no poles and so W must be an entire function. We observe that $W(z)$ is bounded at infinity since $F'G, FG'$ are bounded in the complement of the contour. By the theorem of Liouville we conclude that $W(x)$ must be a constant, i.e.

$$W(z) = F'(z)G(z) - F(z)G'(z) \equiv -2h_n. \tag{4.79}$$

Differentiating this equation gives that $F''/F \equiv G''/G$. Let us denote by $V(z)$ this ratio; then both F and G satisfy a 2nd order linear ODE of the form:

$$y'' - V(z)y = 0 \quad \text{with potential } V(z) := \frac{F''(z)}{F(z)}. \tag{4.80}$$

We can rewrite the potential using the defining expression $F(z) = p_n(z)\mathrm{e}^{\theta(z)}$ in terms of the polynomial $p_n(z)$, which gives us:

$$V(z) = \theta''(z) + \theta'(z)^2 + 2\theta'(z)\frac{p_n'(z)}{p_n(z)} + \frac{p_n''(z)}{p_n(z)}. \tag{4.81}$$

Let z_0 be one of the zeros of p_n of multiplicity d and write $p_n(z) = (z-z_0)^d h(z)$. Expand the potential (4.81) near $z = z_0$:

$$V(z) = \frac{d(d-1)}{(z-z_0)^2} + \frac{2d}{z-z_0}\left(\frac{h'(z_0)}{h(z_0)} + \theta'(z_0)\right) + \mathcal{O}(1), \quad z \to z_0. \tag{4.82}$$

This shows that $V(z)$ may have at most a second-order pole, depending on the multiplicity of the zero z_0. But both $F(z)$ and $G(z)$ are analytic near $z = c$ and satisfy (4.80), and so we deduce that all the singularities of the ODE are apparent. We can thus apply Lemma 4.11 to the ODE (4.80), from which it follows $V(z) = F''/F$ is an entire function.

We conclude that $d = 1$, namely all the zeros $z_1, \dots, z_n$ of the polynomial $p_n(x)$ are simple and we obtain the following singular expression for the potential:

$$V(z) = \theta''(z) + (\theta'(z))^2 + 2nz + 2\sum_{j=1}^{n} z_j + \sum_{j=1}^{n} \frac{1}{z - z_j}\left[\theta'(z_j) + \sum_{\substack{k=1\\k\neq j}}^{n} \frac{1}{z_j - z_k}\right]. \tag{4.83}$$

Since $V(z)$ must be entire, the zero residue condition implies that

$$\theta'(z_j) = \sum_{\substack{k=1\\k\neq j}}^{n} \frac{1}{z_k - z_j}, \qquad j = 1, \dots, n. \tag{4.84}$$

This gives the potential

$$V(z) = z^4 + tz^2 + 2z(n+1) + \frac{t^2}{4} + 2\sum_{j=1}^{n} z_j \tag{4.85}$$

which coincides with the potential in (1.4) after identifying $\Lambda = \frac{t^2}{4} + 2\sum_{j=1}^{n} z_j$ ■

Remark 4.13. This proof shows that $V(z)$ is a quartic polynomial if and only if the zeroes $z_1, \dots, z_n$ of $p_n(z)$ are simple and satisfy a Fekete type equilibrium property (4.84). This curious observation is generalised to the zeroes of all non-hermitian degenerate orthogonal polynomials and it is the main issue under consideration in Chapter 5.

Corollary 4.14. *The following statements are equivalent:*

1. *The Shapiro-Tater boundary problem* (1.4) *admits quasi-polynomial solutions* $F(z) = p_n(z)\mathrm{e}^{\theta(z)}$ *where* $p_n(z)$ *is a polynomial of degree* n.
2. *The polynomial* $p_n(z)$ *is a* 1*-degenerate orthogonal polynomial of degree* n *on the weighted contour* $\Gamma = \varkappa\gamma + \widetilde{\varkappa}\widetilde{\gamma}$ *with the weight* $w(z) = \mathrm{e}^{\theta(z)}$*; and where* $\varkappa = s_1, \widetilde{\varkappa} = s_5$ *are explicitly given in terms of* $p_n(z)$ *via the formulæ* (4.33).

4.2.3 Degenerate spectrum

In this section we prove a key theorem that will be necessary for what follows. It characterises the degenerate spectrum, i.e. the repeated eigenvalues, of the Shapiro-Tater eigenvalue problem (1.4) in terms of the vanishing of the square of the quasi-polynomials.

Theorem 4.15. *The following statements are equivalent:*

1. *The value of* $t \in \mathbb{C}$ *is such that the Exactly Solvable spectrum of* (1.4) *has a repeated eigenvalue.*
2. *The values of* J, t, Λ *in* (1.4) *satisfy*

$$\mathrm{C}_n(t,\lambda) = 0, \quad \frac{\mathrm{d}}{\mathrm{d}\lambda}\mathrm{C}_n(t,\lambda) = 0, \quad J = n+1 \in \mathbb{N}, \tag{4.86}$$

   where $C_n(t,\lambda)$ *is the characteristic polynomial defined in* (1.10) *and* $\lambda = \Lambda - \frac{t^2}{4}$.
3. *There is a quasi-polynomial solution* $p_n(x)\mathrm{e}^{\theta(x;t)}$ *of* (1.4) *that satisfies*

$$\int_\gamma \left(p_n(z)e^{\theta(z;t)}\right)^2 \mathrm{d}z = 0, \qquad \int_{\widetilde{\gamma}} \left(p_n(z)e^{\theta(z;t)}\right)^2 \mathrm{d}z = 0 \tag{4.87}$$

   where γ *and* $\widetilde{\gamma}$ *are defined as in Theorem* 4.10 *(wedge contours from* $\infty_1 \to \infty_3$ *and* $\infty_3 \to \infty_5$ *respectively).*

Proof.

(1⇒2) Suppose that $y(z)$ is a solution to the eigenvalue problem (1.4) with an associated eigenvalue Λ of algebraic multiplicity greater than 1. Then $y(z) = p(z)\mathrm{e}^{\theta(z;t)}$ must be a quasi-polynomial according to Proposition 4.7, and we must have that $J = n + 1$.

It is clear that Λ is a repeated eigenvalue of (1.4) if and only if $\lambda = \Lambda - \frac{t^2}{4}$ is an eigenvalue of the polynomial differential equation (1.7), and this happens precisely when the characteristic polynomial $\mathrm{C}_n(t,\lambda)$ in (1.10) vanishes.

Moreover, this eigenvalue is repeated precisely then the derivative of the characteristic polynomial $\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathrm{C}_n(t,\lambda)$ also vanishes.

(2⇒1) This is immediate consequence of Lemma 4.6 together with the fact that the derivative of a polynomial vanish at each root of multiplicity higher than one.

(2⇒3) Suppose that the statement 2 is satisfied. This means the parameter $t \in \mathbb{C}$ must be such that the eigenvalue Λ has algebraic multiplicity at least 2 and so we can consider the generalized eigenvector.

We kindly remind the reader that if $\mathrm{v} \in \mathbb{C}^{n+1}$ is the eigenvector of the matrix $M_n(t)$ in (1.9) with eigenvalue λ, then a generalized eigenvector $\mathrm{w} \in \mathbb{C}^{n+1}$ satisfies the equation

$$(M - \lambda I)\mathrm{w} = \mathrm{v}. \tag{4.88}$$

Thus the generalized eigenvector equation associated to the linear operator (1.7) takes the form of the following differential equation for a *polynomial* $r(x)$ of degree at most n:

$$\frac{\mathrm{d}^2 r}{\mathrm{d}z^2} + 2\left(z^2 + \frac{t}{2}\right)\frac{\mathrm{d}r}{\mathrm{d}z} - (2(J-1)z + \lambda)r = p_n(z). \tag{4.89}$$

The homogeneous part of this differential equation is precisely (1.7), therefore two linearly independent are given by $p_n(z)$ (a polynomial of degree $n = J - 1$) and

$$q_k(z) = p_n(z)\int_{\infty_k}^{z} F(w)^{-2}\mathrm{d}w, \tag{4.90}$$

where $F(z) := p_n(z)\mathrm{e}^{\theta(x;a)}$ is the a quasi-polynomial and we can choose any $k = 0, 2, 4$. Then we can find a particular solution $r_0(z)$ of (4.89) by "variation of parameters" as follows:

$$r_0(z) := p_n(z)\underbrace{\int_{\infty_0}^{z} F(s)^{-2}\left(\int_{\infty_1}^{s} F(w)^2\mathrm{d}w\right)\mathrm{d}s}_{H(z)}. \tag{4.91}$$

This solution of (4.89) is defined up to addition of a linear combination of $p_n(z), q_0(z)$, so we mus verify that we can choose constants A, B such that $r_0(z) + Ap_n(z) + Bq_0(z)$ is a polynomial of degree at most n. Clearly here only the value of B is relevant (since p_n is already a polynomial). Thus the issue boils down to whether $r_0 + Bq_0$ is a polynomial for some value of B.

We first observe that B must be zero. Indeed, consider the asymptotic behaviour as $z \to \infty_1$: the inner integral defining $H(z)$ tends to zero at exponential rate as as $z \to \infty_1$ and hence one sees that $H(z)$ is bounded. However, $q_k(z)$ has dominant exponential growth in this direction. Thus we necessarily have $B = 0$ for otherwise $r = r_0 + Bq_0$ is not polynomially bounded in the direction of ∞_1.

We deduce that r_0 must itself be a polynomial. Now we consider the behaviour of r_0 near ∞_3, ∞_5. We can write, for example for ∞_3,

$$\begin{aligned} r_0(z) &= p_n(z)\int_{\infty_0}^{z} F(w)^{-2}\left(\int_{\infty_1}^{\infty_3} F(s)^2\mathrm{d}s + \int_{\infty_3}^{w} F(s)^2\mathrm{d}s\right)\mathrm{d}w \\ &= \left(\int_{\infty_1}^{\infty_3} F(s)^2\mathrm{d}s\right)q_0(z) + p_n(z)\int_{\infty_0}^{z} F(w)^{-2}\left(\int_{\infty_3}^{w} F(s)^2\mathrm{d}s\right)\mathrm{d}w \end{aligned} \tag{4.92}$$

The second term above is polynomially bounded near ∞_3, by the same argument used to show that H is bounded near ∞_1. But since q_0 is exponentially dominant also near ∞_1 we deduce that the $\int_{\infty_1}^{\infty_3} F(s)^2 \mathrm{d}s = 0$. One similarly deduces $\int_{\infty_5}^{\infty_3} F(s)^2 \mathrm{d}s = 0$, which establishes (4.87).

(3⇒2) Consider the expression (4.91). It is easy to see directly that it satifies the generalized eigenvector equation (4.89); we must only verify that the conditions (4.87) guarantee that $r_0(z)$ is a polynomial. But this follows again from the Liouville theorem and using (4.92).

■

Remark 4.16. The Theorem 4.15 seems at first sight nothing short of a miracle; indeed once we fix $J = n+1 \in \mathbb{N}$, then the ODE (1.4) has only two continuous parameters t, λ. However the multiple eigenvalue condition apparently involves now three equations which are:

(i) the existence of a quasi-polynomial solution which determines λ as a function of t as per Lemma 4.6;

(ii) the two equations involving the vanishing of the integral of the square of quasi-polynomials (4.87) for the parameter t.

However, the system is actually not overdetermined due to the following reasoning: for a given $J = n+1 \in \mathbb{N}$, if the pair (t, Λ) is in the *Exactly Solvable* spectrum then Theorem 4.10 demonstrates that $p_n(z)$ is a *degenerate* orthogonal polynomial, namely

$$\varkappa \int_\gamma p_n^2(z) \mathrm{e}^{2\theta(z;t)} \mathrm{d}z + \widetilde{\varkappa} \int_{\widetilde{\gamma}} p_n^2(z) \mathrm{e}^{2\theta(z;t)} \mathrm{d}z = 0, \tag{4.93}$$

with $\gamma, \widetilde{\gamma}, \varkappa, \widetilde{\varkappa}$ defined in the same theorem. The coefficients $\varkappa, \widetilde{\varkappa}$, which are explicitly given by the Stokes multipliers as in Theorem 4.10, cannot be both vanishing for otherwise the Stokes phenomenon of the ODE would be trivial by (4.31), which is not possible. Then the two equations (4.87) yield only one additional constraint.

4.3 Stokes phenomenon via exact WKB

In this section we make use of the exact WKB analysis to study the properties of the Shapiro-Tater eigenvalue problem (1.4) and the Jimbo-Miwa anharmonic oscillator (4.14).

To begin, we see that after scaling appropriately, both ODEs become of the form (3.28), and so we are able to apply the exact WKB method to study these ODEs. The small parameter $\hbar$ is dependent on each case. These scalings work as follows.

Shapiro-Tater potential.

$$\begin{gathered} V_{\mathrm{ST}}(z;t,\Lambda) = z^4 + tz^2 + 2(n+1)z + \Lambda, \\ \zeta = (n+1)^{-\frac{1}{3}} z, \qquad s = (n+1)^{-\frac{2}{3}} t, \qquad E = (n+1)^{-\frac{4}{3}} \Lambda, \end{gathered} \tag{4.94}$$

where (t, Λ) are part of the Exactly Solvable spectrum of the eigenvalue problem (1.4)-(1.5).

Jimbo-Miwa potential.

$$\begin{gathered} V_{\mathrm{ST}}(z;a,b) = z^4 + az^2 + (2n+1)z + \left(\frac{7a^2}{36} + 10b\right), \\ \zeta = \left(n+\frac{1}{2}\right)^{-\frac{1}{3}} z, \qquad s = \left(n+\frac{1}{2}\right)^{-\frac{2}{3}} a, \qquad \widehat{b} = \left(n+\frac{1}{2}\right)^{-\frac{4}{3}} b, \end{gathered} \tag{4.95}$$

where a is a pole of residue -1 of a solution to PII with $\alpha = n \in \mathbb{N}$ and b is the coefficient in the Laurent expansion, as in (4.3).

In either case the scaling yields an ODE with the same potential, but with different small parameter $\hbar$ depending on each case:

$$\hbar^2 \frac{\mathrm{d}^2 y}{\mathrm{d}\zeta^2} - \left(\zeta^4 + s\zeta^2 + 2\zeta + E\right) y = 0, \quad \hbar = \begin{cases} (n+1)^{-1} \text{ in Shapiro-Tater case} \\ (n+\frac{1}{2})^{-1} \text{ in Jimbo-Miwa case} \end{cases} \tag{4.96}$$

The potential, which we denote

$$Q(\zeta; s, E) := \zeta^4 + s\zeta^2 + 2\zeta + E, \tag{4.97}$$

is independent of n. Crucially, this would not be the case if we took $\hbar = n^{-1}$ as our large parameter for the Jimbo-Miwa potential V_{JM}, instead giving a potential dependent on $h = n^{-1}$.

4.3.1 Exact WKB applied to the quartic potential

When we apply the exact WKB method to the potential (4.97) we will encounter the differential $\sqrt{Q(\zeta; s, E)}\mathrm{d}\zeta$, which is multivalued in the $\mathbb{C}$ with branch points at the zeroes of the potential $Q(\zeta; s, E)$. For this reason we make the following definitions to avoid ambiguity in the upcoming calculations.

Definition 4.17 (Square root of Q)**.** Choosing the branch cuts $\mathcal{B} \subset \mathbb{C}$ of $\sqrt{Q(\zeta)}$ in the *finite part* of the complex plane in an arbitrary way, the function $\sqrt{Q(z)}$ becomes single valued in the complement of $\mathcal{B}$. We fix a determination of $\sqrt{Q(\zeta)}$ in such a way that $\sqrt{Q(\zeta)} \sim \zeta^2$ as $|\zeta| \to \infty$. This choice identifies the first sheet of the Riemann surface $\overline{\Sigma}$. The second sheet correspond to the choice $\sqrt{Q(\zeta)} \sim -\zeta^2$ as $|\zeta| \to \infty$.

To minimize confusion when performing integrations along the branch cuts we will make the following explicit definition.

Definition 4.18 (Branch cut integration)**.** Let τ and $\widehat{\tau}$ be two zeroes of $Q(z; s, E)$ joined by a branch cut. We denote by

$$\int_\tau^{\widehat{\tau}} \sqrt{Q(\zeta_+; s, E)}\mathrm{d}\zeta \tag{4.98}$$

to be the integral of $\sqrt{Q(z; s, E)}$ along the $+$ side when the branch cut is *oriented from τ to $\widehat{\tau}$*. Similarly, we denote by

$$\int_\tau^{\widehat{\tau}} \sqrt{Q(\zeta_-; s, E)}\mathrm{d}\zeta \tag{4.99}$$

the corresponding integral along the $-$ side of the branch cut. As usual, the $+$ and $-$ sides correspond to left side and right side of the oriented contour, respectively.

With this definition in mind, we have the following relation between periods and branch cut integration:

$$\oint_\gamma \sqrt{Q(\zeta; s, E)}\mathrm{d}\zeta = 2\int_\tau^{\widehat{\tau}} \sqrt{Q(\zeta_+; s, E)}\mathrm{d}\zeta = -2\int_\tau^{\widehat{\tau}} \sqrt{Q(\zeta_-; s, E)}\mathrm{d}\zeta \tag{4.100}$$

where γ is an **clockwise** contour surrounding τ and $\widehat{\tau}$.

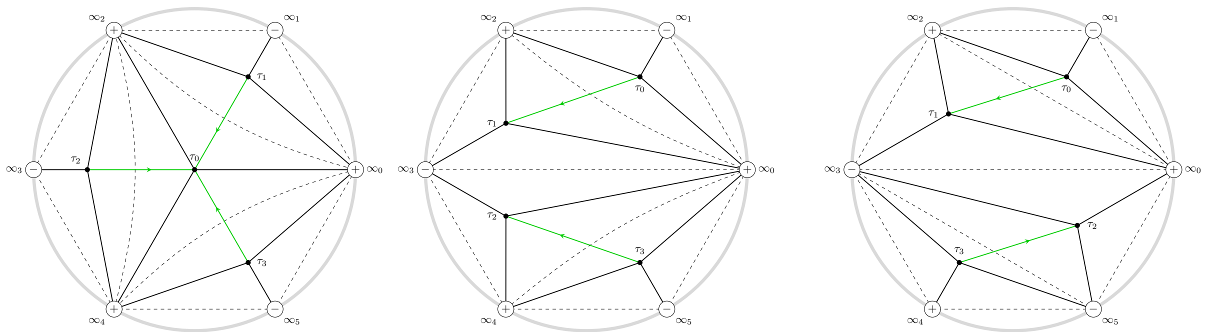
It is useful to relate the periods of S_1 to the periods of S_{-1}. To this end we have the following

Proposition 4.19. *Let us denote*

$$I(s, E) = \oint_\gamma S_{-1}(\zeta; s, E)\mathrm{d}\zeta \tag{4.101}$$

where γ is a closed contour and $S_{-1}(\zeta; s, E) = \sqrt{Q(\zeta; s, E)}$ as in (3.34)*. Then corresponding period of S_1 in* (3.34) *is given in terms of $I(s, E)$ by*

$$\oint_\gamma S_1(\zeta; s, E)\mathrm{d}\zeta = \left(-\frac{\partial^2}{\partial s \partial E} - \frac{s}{6}\frac{\partial^2}{\partial E^2}\right) I(s, E) \tag{4.102}$$



(a) Graph of type D (b) Graph of type E (c) Graph of type Z

Figure 4.3: Generic decorated Stokes graph configurations for a quartic polynomial $Q(x)$. Solid lines depict the Stokes curves emanating from the turning points τ_j, dashed lines denote the triangulation of the hexagon and green curvy lines correspond to our choice of branch cuts. We remark that for various $(s, E) \in \mathbb{C}^2$ the potential $Q(\zeta; s, E)$ may have a different different Stokes *lines* in $\overline{\mathbb{C}}$ but the underlying Stokes graph will be topologically identical to one of the types depicted.

Proof. From the expression of the quartic potential (4.97) we see that

$$\partial_E \sqrt{Q(x; s, E)} = \frac{1}{2\sqrt{Q(x; s, E)}}, \tag{4.103}$$

where we denote $\partial_E = \frac{\partial}{\partial E}$. Now, dropping the explicit dependence on ζ, s, E, we write

$$S_1 = \frac{1}{48} \frac{Q''}{Q^{\frac{3}{2}}} - \frac{5}{24} \left(\frac{1}{\sqrt{Q}} \right)'', \tag{4.104}$$

where $' = \frac{\partial}{\partial z}$ The periods of the second term in (4.104) vanish because this gives an exact differential; the first term reads

$$\frac{1}{48} \frac{Q''}{Q^{\frac{3}{2}}} = \frac{6x^2 + s}{24 Q^{\frac{3}{2}}} = -(6\partial_s + s\partial_E) \left(\frac{1}{12 Q^{\frac{1}{2}}} \right) = -(6\partial_s + s\partial_E)\partial_E \left(\frac{1}{6} Q^{\frac{1}{2}} \right). \tag{4.105}$$

Integrating (4.104) along γ and using the identity (4.105) completes the proof. ■

In the quartic case we will work under the assumption that all the turning point are simple and that there are no *saddle trajectories*, as specified below.

Assumption 4.20. The following assumptions shall prevail.

- *Simplicity.* The roots of the potential are all simple. In the case of the potential $Q(x; s, E)$, there are no repeated roots if and only if (s, E) satisfy:

$$E s^4 - 8E^2 s^2 + 16E^3 - s^3 + 36Es - 27 \neq 0. \tag{4.106}$$

- *Genericity.* There are no saddle trajectories i.e. there are no Stokes' curves connecting two turning points. Saddle trajectories can only occur if there is a contour γ in the homology group of the Riemann surface $\overline{\Sigma}$ for which

$$\operatorname{Im} \oint_\gamma \sqrt{Q(z)} \mathrm{d}z = 0. \tag{4.107}$$

The assumption of simplicity means there are exactly three Stokes' curves emanating from each turning point. The assumption of genericity implies that all the Stokes' curves must extend to ∞. With these assumptions we can classify all the possible Stokes graphs.

Proposition 4.21 (Classification of generic Stokes graph)**.** *Under the Assumptions 4.20 and with determination of $\sqrt{Q}$ in Definition 4.17 (with the branch cuts chosen as in Fig. 4.3), the Stokes graphs are in one-to-one correspondence with the triangulations of the hexagon, so there are 14 such configurations. Three of them are topologically distinct (as graphs), they're depicted in Fig. 4.3 and named E, D and Z. The remaining configurations can be obtained from the graphs of types E, D and Z shown in Fig. 4.3 by a $\mathbb{Z}_6$ rotation and by a reflection along the line $\infty_3 \to \infty_0$.*

Proof. Assuming simplicity, from each turning point there are exactly three Stokes curves emanating from it. These Stokes curves end either at infinity or at another turning point. Assuming genericity the latter cannot happen, therefore the Stokes curves determine an ideal triangulation of the Riemann sphere with a small disk around infinity removed and with 6 marked points in the boundary (corresponding to the asymptotic directions at infinity). These triangulations correspond to triangulations of the hexagon, and there are 14 such triangulations.

Indeed, we can view these 6 marked points as determining a (topological) hexagon. Furthermore, each of the turning points τ_j determines a triangle inside the hexagon by connecting the asymptotic directions at infinity with a Stokes curve originating from τ_j. Up to rotations and reflections of the hexagon there are 3 distinct such triangulations, named E, D and Z as depicted in Fig. 4.3. The 14 possible configuration can be obtained as follows. From configuration D we obtain one other configuration rotating by $2\pi/6$. From configuration Z we obtain 6 distinct configurations, 3 of them corresponding to a $\mathbb{Z}_3$ (i.e. $2\pi/3$) rotation, and another 3 corresponding to a reflection followed by a $\mathbb{Z}_3$ rotation. Finally from configuration E_+ we obtain 6 distinct configurations corresponding to a $\mathbb{Z}_6$ (i.e. $2\pi/6$) rotation. Finally, $2 + 3 + 3 + 6 = 14$ as claimed. ■

4.3.2 WKB Riemann-Hilbert problem

In this section we coordinate the classification of Stokes graphs in Proposition 4.21 with the connection formulæ in Theorem 3.18 and Proposition 3.19 in order to formulate a Riemann-Hilbert problem satisfied by the WKB solutions.

Riemann-Hilbert Problem 4.22 (Quartic WKB jumps)**.** *Fix a quartic polynomial potential $Q(z)$ and suppose that its Stokes graph is of type D, E or Z as indicated in Fig. 4.3. The Riemann-Hilbert problem for the vector valued function Ψ such that $\Psi|_{\mathscr{D}} = (\Psi_+^{(\mathscr{D})}, \Psi_-^{(\mathscr{D})})$ with $\Psi|_{\mathscr{D}}$ defined in Theorem 3.18, consists of the following oriented contours in $\overline{\mathbb{C}}$ with their associated jump matrices.*

1. ***Square root branch cuts****: To each square-root branch cut (coloured green in Fig. 4.4), with the orientation indicated in Fig. 4.4, we associate the jump matrix*

$$G := \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}. \tag{4.108}$$

2. ***Fourth root branch cuts****: in configurations E and Z there is an extra jump contour corresponding to the fact that*

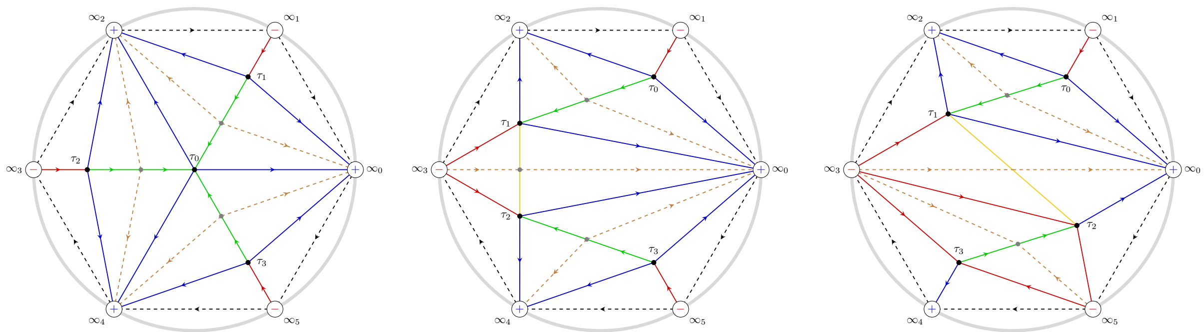
$$\sqrt{S_{\text{odd}}(x,\hbar)} \sim h^{-1/2} Q(x)^{1/4} + \mathcal{O}\left(\hbar^{1/2}\right). \tag{4.109}$$

To each fourth-root branch cut we associate the jump matrix $Y = G^2 = -\mathbf{1}$. The fourth-root branch cuts are coloured in yellow in Fig. 4.4. Note there is no need to specify the orientation of these contours.

3. ***Stokes curves****: along each Stokes curve oriented towards $\oplus$ and away from $\ominus$ we assign the jump matrices B and R (respectively)*

$$B := \begin{bmatrix} 1 & 0 \\ -i & 1 \end{bmatrix}, \qquad R := \begin{bmatrix} 1 & i \\ 0 & 1 \end{bmatrix}. \tag{4.110}$$

The Stokes curves are coloured blue or red, respectively, in Fig. 4.4.



(a) RHP for graphs of type D. (b) RHP for graphs of type E. (c) RHP for graphs of type Z.

Figure 4.4: Generic WKB Riemann-Hilbert problem corresponding to each Stokes graph configuration of the quartic polynomial potential $Q(x)$.

4. ***Inner ideal paths:*** *along the inner ideal paths with the orientation in Def. 3.13 (and indicated in Fig. 4.4) we associate the following jump matrix corresponding to the connection formula in Proposition 3.19:*

$$V_{jk} := \exp\left(\sigma_3 v_{jk}\right) = \begin{bmatrix} \mathrm{e}^{v_{jk}} & 0 \\ 0 & \mathrm{e}^{-v_{jk}} \end{bmatrix}, \quad v_{jk}(\hbar) = \int_{\tau_j}^{\tau_k} S_{\mathrm{odd}}(z_+, \hbar)\mathrm{d}z \tag{4.111}$$

where $j, k \in \{0, 1, 2, 3\}$ and τ_j, τ_k are turning points. These contours are denoted by a brown dashed line in Fig. 4.4.

5. ***Outer ideal paths:*** *along the outer ideal paths separating the external regions from the Stokes regions we associate the jump matrix corresponding to the connection formula between turning points and infinity:*

$$W_j := \exp(\sigma_3 w_j) = \begin{bmatrix} \mathrm{e}^{w_j} & 0 \\ 0 & \mathrm{e}^{-w_j} \end{bmatrix}, \quad w_j(\hbar) := R(z;\hbar) - \int_{\tau_j}^{z} S_{\mathrm{odd}}(w, \hbar)\mathrm{d}w. \tag{4.112}$$

where $j \in \{0, 1, 2, 3\}$, τ_j is a turning point and $R(x;\hbar)$ is defined in (3.42) and in the quartic polynomial case it has the following asymptotics

$$R(z;\hbar) := \frac{1}{\hbar} \lim_{p\to\infty} \left[\int_p^z \sqrt{Q(z)}\mathrm{d}z - \left(\frac{p^3}{3} + \frac{t}{2}p + \log p \right) \right] + \sum_{j\geq 0} \hbar^{2j+1} \int_\infty^z S_{2j+1}(z)\mathrm{d}z \tag{4.113}$$

$$= \frac{1}{\hbar}\left(\frac{z^3}{3} + \frac{t}{2}z + \log(z) \right) + \mathcal{O}\left(z^{-1}\right)\mathcal{O}\left(\hbar\right), \quad z \to \infty. \tag{4.114}$$

The outer ideal paths are denoted by a black dashed line in Fig. 4.4.

As a consequence of Proposition 4.21, the construction above produces, up to rotations and reflections, three distinct WKB Riemann-Hilbert problem, which are shown in Fig. 4.4.

This Riemann-Hilbert problem allows us to express the normalised solutions $\Psi^{(\mathscr{D})}$ in a particular Stokes region $\mathscr{D}$ as a linear combination of the normalised solutions $\Psi^{(\mathscr{D}')}$ in any other region $\mathscr{D}'$. This linear combination is dictated by the jump matrices in the Riemann-Hilbert problem above.

Doing this one can verify that the monodromy is trivial around any of the *finite* vertices of the Riemann-Hilbert Problems, i.e. around any of the vertices of the graphs in Fig. 4.4 that are not an asymptotic direction at infinity. This is a consequence (or rather a *requirement* when building the WKB Riemann-Hilbert problem) of the fact that any solution to the Schrödinger equation (3.28) with the polynomial potential $Q(\zeta; s, E)$ is entire so that solutions have no monodromy in the finite complex plane. Indeed, a simple computation of each possible configurations gives

- Type 1: $RB^{-1}G^{-1}B^{-1} = \mathbb{I}$,
- Type 2: $GB^{-1}RYB^{-1} = \mathbb{I}$,

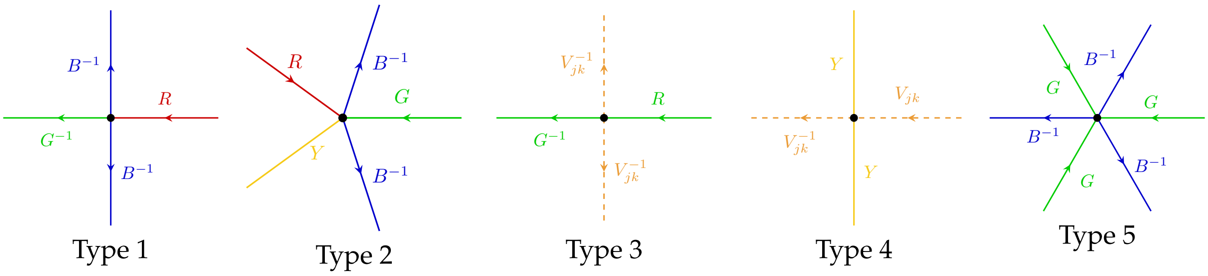


Figure 4.5: Possible configuration of edges incident in each finite vertex of the decorated Stokes graph. Around each of these points the monodromy is trivial. Note that the configurations of type 1,2 and 5 correspond to edges emerging from a turning point $\tau \in \mathrm{T}$, where as the vertex in configurations of type 3 and 4 are the intersection point of a branch cut and inner ideal path.

- Type 3: $GV_{jk}^{-1}G^{-1}V_{jk}^{-1} = \mathbb{I}$,
- Type 4: $YV_{jk}YV_{jk}^{-1} = \mathbb{I}$,
- Type 5: $GB^{-1}GB^{-1}GB^{-1} = \mathbb{I}$,

where each type refers to a possible configuration of edges near a vertex in Fig. 4.4, as indicated in the Fig. 4.5

In order to simplify the upcoming computations we will make some notational definitions. Given turning points τ_j and τ_k of the potential $Q(\zeta; s, E)$ connected by a branch cut as indicated in Fig. 4.4, we will denote

$$\xi_{jk} := \exp(2v_{jk}) = \exp\left(2\int_{\tau_j}^{\tau_k} S_{\mathrm{odd}}(\zeta_+, \hbar)\mathrm{d}\zeta\right) \tag{4.115}$$

with the determination of S_{odd} given in Def. 4.17 and the boundary value ζ_+ in accordance to Def. 4.18. Note that ξ_{jk} is in fact an asymptotic series in $\hbar$, however, for convenience we drop the dependence on $\hbar$. We will call the above parameter the *Fock-Goncharov* parameters, in reference to the works [GMN13, IN14, All19]. Confusingly, we will abuse the same notation to indicate the "exact" Fock-Goncharov parameters, namely, the result of the Borel resummation of the asymptotic series ξ_{jk}. To phrase it differently, we will not distinguish in the notation the Borel resummation from its asymptotic expansion in $\hbar$.

In each configuration of a Stokes graph, the Fock-Goncharov parameters satisfy certain relations corresponding to the homological nature of the integrals through which they are defined.

Proposition 4.23. *Suppose that $(s, E) \in \mathbb{C}^2$ determine a Stokes graph of type D. Then the Fock-Goncharov parameters satisfy the following equations:*

$$\xi_{10}\xi_{20}\xi_{30} = \mathrm{e}^{\frac{2\pi\mathrm{i}}{\hbar}}, \tag{4.116}$$

$$\mathrm{e}^{w_j - w_k} = \xi_{k0}\xi_{j0}, \tag{4.117}$$

where $(j, k) \in \{(1, 2), (2, 3), (3, 1)\}$. Similarly, if $(s, E) \in \mathbb{C}^2$ determine a Stokes graph of type E or type Z, then the Fock-Goncharov parameters satisfies:

$$\xi_{01}\xi_{23} = \mathrm{e}^{\frac{2\pi\mathrm{i}}{\hbar}}. \tag{4.118}$$

Proof. According to Definition 4.18 we express the Fock-Goncharov in terms of the contour integrals on the Riemann surface $\overline{\Sigma}$:

$$\xi_{jk} = \exp\left(\oint_{\gamma_{jk}} S_{\mathrm{odd}}(\zeta; \hbar)\mathrm{d}\zeta\right) \tag{4.119}$$

where γ_{jk} is a contour surrounding the turning points τ_j and τ_k in the clock-wise direction.

Consider a Stokes graph of type D. One can readily verify that the contour [1] $\gamma_{10}+\gamma_{20}+\gamma_{30}$ is homologous to the sum of two contours

$$\gamma_{10}+\gamma_{20}+\gamma_{30}=\Gamma_{\infty}+\Gamma_{\tau_0}. \tag{4.120}$$

Here Γ_{∞} is a contour which surrounds infinity with winding number 1 and Γ_{τ_0} a contour surrounding τ_0 with winding number 2. Due to the winding of Γ_{τ_0} and the location of the branch cuts, the following integral vanishes:

$$\oint_{\Gamma_{\tau_0}} S_{\rm odd}(\zeta,\hbar)\mathrm{d}\zeta=0. \tag{4.121}$$

Thus the only contribution comes from the residue at infinity:

$$\left(\oint_{\gamma_{01}}+\oint_{\gamma_{02}}+\oint_{\gamma_{03}}\right)S_{\rm odd}(\zeta,\hbar)\mathrm{d}\zeta=\operatorname*{Res}_{\zeta=\infty}\hbar^{-1}\sqrt{Q(\zeta;s,E)}\mathrm{d}\zeta=\frac{2\pi\mathrm{i}}{\hbar}. \tag{4.122}$$

Exponentiating this expression we obtain the first relation$\xi_{01}\xi_{02}\xi_{03}=\mathrm{e}^{\frac{2\pi\mathrm{i}}{\hbar}}$. The second relation in configuration D follows from the definition of w_j in (4.112):

$$\begin{aligned}
w_j-w_k&=\int_{\tau_k}^{z}S_{\rm odd}(\zeta,\hbar)\mathrm{d}\zeta-\int_{\tau_j}^{z}S_{\rm odd}(\zeta,\hbar)\mathrm{d}\zeta &(4.123)\\
&=\int_{\tau_k}^{\tau_j}S_{\rm odd}(\zeta,\hbar)\mathrm{d}\zeta &(4.124)\\
&=\int_{\tau_k}^{\tau_0}S_{\rm odd}(\zeta_+,\hbar)\mathrm{d}\zeta+\int_{\tau_0}^{\tau_j}S_{\rm odd}(\zeta_-,\hbar)\mathrm{d}\zeta &(4.125)\\
&=\int_{\tau_k}^{\tau_0}S_{\rm odd}(\zeta_+,\hbar)\mathrm{d}\zeta-\int_{\tau_j}^{\tau_0}S_{\rm odd}(\zeta_-,\hbar)\mathrm{d}\zeta &(4.126)\\
&=\int_{\tau_k}^{\tau_0}S_{\rm odd}(\zeta_+,\hbar)\mathrm{d}\zeta+\int_{\tau_j}^{\tau_0}S_{\rm odd}(\zeta_+,\hbar)\mathrm{d}\zeta. &(4.127)
\end{aligned}$$

Here we have made use of the orientation of the branch cuts of Stokes graphs of type D as in Fig. (4.4), and this imposes the condition $(j,k)\in\{(1,2),(2,3),(3,1)\}$. Exponentiating the previous expression gives the relation $\mathrm{e}^{w_j-w_k}=\xi_{k0}\xi_{j0}$.

Next, in a Stokes graph of type D or Z have the same the topology of the branch cuts so that we can consider both cases simultaneously. One readily verifies that the contour $\gamma_{01}+\gamma_{23}$ is homologous to a contour surrounding infinity. Thus

$$\left(\oint_{\gamma_{01}}+\oint_{\gamma_{23}}\right)S_{\rm odd}(\zeta,\hbar)\mathrm{d}\zeta=\operatorname*{Res}_{\zeta=\infty}\hbar^{-1}\sqrt{Q(\zeta;s,E)}\mathrm{d}\zeta=\frac{2\pi\mathrm{i}}{\hbar}. \tag{4.128}$$

Exponentiating the expression above we obtain the relation $\xi_{01}\xi_{23}=\mathrm{e}^{\frac{2\pi\mathrm{i}}{\hbar}}$. ■

We are now ready to formulate the main theorem of this section, where we will compute the Stokes matrices of the WKB ODE (3.28) in terms of the Fock-Goncharov parameters.

Theorem 4.24. *The Stokes matrices*

$$\mathbb{S}_j=\begin{bmatrix}1&0\\s_j&1\end{bmatrix},\quad j=0,2,4,\qquad \mathbb{S}_j=\begin{bmatrix}1&s_j\\0&1\end{bmatrix},\quad j=1,5,\qquad \mathbb{S}_3=\begin{bmatrix}1&s_3\\0&1\end{bmatrix}\mathrm{e}^{\frac{2i\pi}{\hbar}\sigma_3}, \tag{4.129}$$

in each of the WKB Riemann-Hilbert problems in Fig. 4.4 are expressed in terms of the contour integrals v_{jk}, w_j

[1] Rather, their class in the homology group $H^1(\overline{\Sigma})$ of the Riemann surface $\overline{\Sigma}$.

in (4.111), (4.112) *and the Fock-Goncharov parameters* ξ_{jk} *in* (4.115) *as follows:*

Type D	*Type E*	*Type Z*
$s_0 = -i\mathrm{e}^{2w_1}(\xi_{10}\xi_{30}+\xi_{10}+1)$,	$s_0 = -i\mathrm{e}^{2w_0}(\xi_{01}\xi_{12}\xi_{32} + \xi_{01}\xi_{12}+\xi_{01}+1)$,	$s_0 = -i\mathrm{e}^{2w_0}(\xi_{01}\xi_{12}+\xi_{01}+1)$,
$s_1 = -i\mathrm{e}^{-2w_1}$	$s_1 = -i\mathrm{e}^{-2w_0}$	$s_1 = -i\mathrm{e}^{-2w_0}$
$s_2 = -i\mathrm{e}^{2w_2}(\xi_{20}\xi_{10}+\xi_{20}+1)$,	$s_2 = -i\mathrm{e}^{2w_1}(\xi_{01}+1)$,	$s_2 = -i\mathrm{e}^{2w_1}(\xi_{01}+1)$,
$s_3 = -i\mathrm{e}^{-2w_2}$	$s_3 = -i\mathrm{e}^{-2w_2}(\xi_{12}+1)$	$s_3 = -i\mathrm{e}^{-2w_3}(\xi_{32}\xi_{12}+\xi_{32}+1)$
$s_4 = -i\mathrm{e}^{2w_3}(\xi_{30}\xi_{20}+\xi_{30}+1)$,	$s_4 = -i\mathrm{e}^{2w_3}(\xi_{32}+1)$,	$s_4 = -i\mathrm{e}^{2w_3}$
$s_5 = -i\mathrm{e}^{-2w_3}$.	$s_5 = -i\mathrm{e}^{-2w_3}$.	$s_5 = -i\mathrm{e}^{-2w_2}(\xi_{32}+1)$.

(4.130)

Proof. We compute the Stokes matrices associated to Configuration D of the Riemann-Hilbert problem in Fig. 4.4. Consider the vertex ∞_0 in Configuration D, there are seven edges incident on it. The Stokes matrix $\mathbb{S}_0$ is given by the clockwise (about the vertex ∞_0) product of all the jump matrices corresponding to the edges incident to ∞_0 as follows

$$\mathbb{S}_0 = W_3^{-1}BV_{30}BV_{10}BW_1 \tag{4.131}$$

Notice that the jump matrix V_{30} is in the correct order since we consider the negative side of the branch cut and

$$\exp\left(2\int_{\tau_0}^{\tau_3} S_{\mathrm{odd}}(z_-,\hbar)\mathrm{d}z\right) = \exp\left(2\int_{\tau_3}^{\tau_0} S_{\mathrm{odd}}(z_+,\hbar)\mathrm{d}z\right), \tag{4.132}$$

where z_+ and z_- is the boundary values of $S_{\mathrm{odd}}(z,\hbar)$ on the left and right side, respectively, of the segment $[\tau_0,\tau_3]$ oriented as in Fig. 4.4. Therefore we have

$$\mathbb{S}_0 = \exp\left[\sigma_3(w_1 - w_3 + v_{10} + v_{30})\right]\begin{bmatrix}1 & 0\\ s_0 & 1\end{bmatrix}, \quad s_0 = -ie^{2w_1}(\xi_{10}\xi_{30}+\xi_{10}+1). \tag{4.133}$$

We observe that

$$w_1 - w_3 = -\int_{\tau_1}^{x} S_{\mathrm{odd}}(z,\hbar)\mathrm{d}z + \int_{\tau_3}^{x} S_{\mathrm{odd}}(z,\hbar)\mathrm{d}z = \int_{\tau_3}^{\tau_1} S_{\mathrm{odd}}(z,\hbar)\mathrm{d}z. \tag{4.134}$$

and

$$v_{10} + v_{30} = \int_{\tau_1}^{\tau_0} S_{\mathrm{odd}}(z_+,\hbar)\mathrm{d}z + \int_{\tau_3}^{\tau_0} S_{\mathrm{odd}}(z_+,\hbar)\mathrm{d}z = \int_{\tau_1}^{\tau_3} S_{\mathrm{odd}}(z,\hbar)\mathrm{d}z \tag{4.135}$$

where the integral from τ_1 to τ_3 is with the determination in Def. 4.17 (note there is not branch cut joining τ_1 and τ_3 so there is no need to specify boundary values of $z_\pm$). Therefore we obtain

$$w_1 - w_3 + v_{10} + v_{30} = \int_{\tau_3}^{\tau_1} S_{\mathrm{odd}}(z,\hbar)\mathrm{d}z + \int_{\tau_1}^{\tau_3} S_{\mathrm{odd}}(z,\hbar)\mathrm{d}z = 0, \tag{4.136}$$

and so we obtain

$$\exp\left(\sigma_3[w_1 - w_3 + v_{10} + v_{30}]\right) = \mathbb{I}, \tag{4.137}$$

and consequently:

$$\mathbb{S}_0 = \begin{bmatrix}1 & 0\\ s_0 & 1\end{bmatrix}, \qquad s_0 = s_0 = -ie^{2w_1}(\xi_{10}\xi_{30}+\xi_{10}+1) \tag{4.138}$$

In a similar manner one can compute the remaining Stokes matrices for each possible configuration. We omit them here no to bore the reader with tedious calculations.

The extra $\mathrm{e}^{2i\pi\hbar^{-1}\sigma_3}$ in the form of $\mathbb{S}_3$ is due to our choice of branch-cut for the logarithm in (4.113). The fact that the product of all Stokes matrices is trivial follows by construction. ■

Remark 4.25 (Gauge arbitrariness)**.** We remind the reader that the Stokes phenomenon is not intrinsically defined because we can conjugate the fundamental matrix by an arbitrary diagonal matrix. This freedom translates to the following scaling equivalence for the Stokes parameters $s_0, \dots, s_5$:

$$s_{2j+1} \mapsto \chi s_{2j+1}, \quad s_{2j} \mapsto \chi^{-1} s_{2j}, \quad \chi \in \mathbb{C}^\times \tag{4.139}$$

Using this freedom we can rewrite the Stokes parameters completely in terms of the Fock-Goncharov parameters ξ_{jk} as follows

Configuration D	Configuration E	Configuration Z
$\chi = -e^{2w_1}$	$\chi = -e^{2w_0}$	$\chi = -e^{2w_0}$
$s_0 = \mathrm{i}(\xi_{10}\xi_{30} + \xi_{10} + 1)$,	$s_0 = \mathrm{i}(\xi_{01}\xi_{12}\xi_{32} + \xi_{01}\xi_{12} + \xi_{01} + 1)$,	$s_0 = \mathrm{i}(\xi_{01}\xi_{12} + \xi_{01} + 1)$,
$s_1 = \mathrm{i}$	$s_1 = \mathrm{i}$	$s_1 = \mathrm{i}$
$s_2 = \mathrm{i}\frac{\xi_{20}}{\xi_{10}}(\xi_{20}\xi_{10} + \xi_{20} + 1)$,	$s_2 = \mathrm{i}\xi_{01}(\xi_{01} + 1)$,	$s_2 = \mathrm{i}\xi_{01}(\xi_{01} + 1)$,
$s_3 = \mathrm{i}\frac{\xi_{10}}{\xi_{20}}$	$s_3 = \mathrm{i}\xi_{02}^{-1}(\xi_{12} + 1)$	$s_3 = \mathrm{i}\xi_{03}^{-1}(\xi_{32}\xi_{12} + \xi_{32} + 1)$
$s_4 = \mathrm{i}\frac{\xi_{30}}{\xi_{10}}(\xi_{30}\xi_{20} + \xi_{30} + 1)$,	$s_4 = \mathrm{i}\xi_{03}(\xi_{32} + 1)$,	$s_4 = \mathrm{i}\xi_{03}$
$s_5 = \mathrm{i}\frac{\xi_{10}}{\xi_{30}}$.	$s_5 = \mathrm{i}\xi_{03}^{-1}$	$s_5 = \mathrm{i}\xi_{02}^{-1}(\xi_{32} + 1)$.

(4.140)

We chose to leave the factor of i intact in the Stokes parameters for reasons that will become apparent in the following section.

4.4 Quantization conditions

In this section we derive quantisation conditions for both the zeroes of the Vorob'ev-Yablonskii polynomials (the Jimbo-Miwa case) and for the points of the Exactly Solvable spectrum corresponding to the repeated eigenvalues in the eigenvalue problem (1.4) (the Shapiro-Tater case).

These quantisation conditions are given in terms Fock-Goncharov coordinates ξ_{jk} (or alternatively in terms of S_{odd}), and are therefore asymptotic expressions in $\hbar$. They can be turned into "exact" quantisation equations by the process of Borel resummation, but that is beyond the scope of this thesis. To leading order these quantization conditions yield a system of equations describing both sets of points in terms of contour integral of $\sqrt{Q(\zeta; s, E)}$.

In the Jimbo-Miwa case this is achieved by matching the Stokes' phenomenon of the Jimbo-Miwa Lax pair representation of (PII) with the Stokes' phenomenon of the quartic WKB Riemann-Hilbert problem (4.24).

In the Shapiro-Tater case it is not enough matching the Stokes phenomenon obtained in Theorem 4.8 as can be seen from Theorem 4.28. For this reason we additionally impose the condition (4.87), which yields the correct quantisation equations (4.194) from an application of Theorem (4.29).

4.4.1 The Jimbo-Miwa case

The parameters s, E in the potential $Q(\zeta; s, E)$ i.e. the parameters t, Λ corresponding to the pole with residue -1 of the rational solution u_n and "eigenvalue" of the potential (4.14) are determined by the implicit requirement that the Stokes phenomenon for the ODE matches the one indicated below.

Indeed it was shown in [BM14] that rational solutions of the Painlevé II equation correspond to a particular Stokes phenomenon as shown in Fig. 4.6. We recall that the map to the Stokes' data for general solution of the Painlevé II transcendent was obtained originally in [IN86], see also [FIKN06].

Theorem 4.26. *([BM14]) The matrices $\mathbb{S}_0, \dots, \mathbb{S}_5$ in Fig. 4.6 form the monodromy data of the Jimbo-Miwa Lax pair (2.43) corresponding to the rational solutions of PII.*

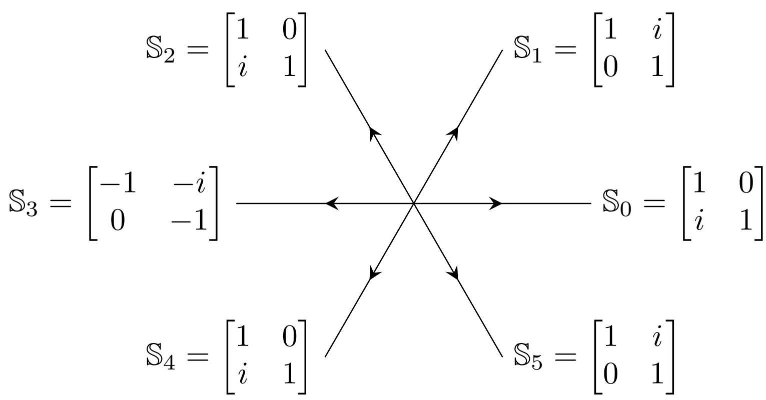


Figure 4.6: Stokes data for the Lax pair (2.43) corresponding to rational solutions of Painlevé II

Instead of studying the Stokes' phenomenon of the Jimbo-Miwa isomonodromic system, we can instead leverage Proposition 4.5 and study instead the scalar system (4.14). To see this in more detail consider a rational solution $u_n(t)$ (2.36) of the PII equation with $\alpha = n \in \mathbb{Z}$. Take a pole a of residue -1, namely a zero of the Vorob'ev-Yablonskii polynomial $Y_n(t)$. Let b be the coefficient in the Laurent expansion as in (4.3), and set $\Lambda = \frac{7a^2}{36} + 10b$ as in (4.14). Then, according to Proposition 4.5, the Stokes phenomenon of the ODE (4.18) must coincide with the one in Fig. 4.6. Viceversa, if the anharmonic potential (4.18) exhibits the Stokes phenomenon as in Fig. 4.6, then the pair of values a, b characterize uniquely the solution $u(t)$ of the Painlevé II equation with a pole at $t = a$ through the Laurent expansion (4.3). Furthermore, this solution must necessarily be a rational of PII, owing to its Stokes parameters. This idea was first utilised by Masoero in [Mas10a, Mas10b] to study the poles of the tritronquée solution of the first Painlevé transcendent, which were associated to a cubic anharmonic oscillator.

Thus, to find the positions of a pole, we can find for which values of a, b in (4.18) (with $\alpha = n$) the Stokes phenomenon matches Fig. 4.6. Of course the map that associates to the parameters a, b in (4.18) the Stokes data is highly transcendental.

It is the nature of our problem, however, that we are interested in the behaviour when n is large and also in the re-scaled plane. Thus we can apply our exact WKB analysis to obtain asymptotic information on the Stokes parameters using Theorem 4.24 and set up an implicit equation for a, b, or rather their rescaled counterparts s, E as in (4.97) with the identifications (4.95).

In order to apply the exact WKB method, we set our large parameter to be $\hbar^{-1} = (n + 1/2)$ in the Jimbo-Miwa case.. By the general theory in Section 3.2, we construct the WKB solutions associated to (4.96) as formal power series in $\hbar$:

$$\psi_{\pm}^{(\tau)}(\zeta, \hbar) = \frac{1}{\sqrt{S_{\text{odd}}(\zeta; \hbar)}} \exp\left\{ \left(\pm \int_{\tau}^{\zeta} S_{\text{odd}}(w; \hbar) \mathrm{d}w \right) \right\} \tag{4.141}$$

normalized near a turning point τ (i.e. a root of the potential $Q(\zeta; s, E)$).

Theorem 4.27 (Vorob'ev-Yablonskii quantisation)**.** *Suppose that $(a, b) \in \mathbb{C}^2$ determine a rational solution of PII with $\alpha = n$ and Laurent expansion (4.3). Let us consider the scaled (s, E)-plane*

$$s = \hbar^{2/3} a, \quad E = \frac{7s^2}{36} + \hbar^{4/3} b, \quad \hbar^{-1} = n + \frac{1}{2} \tag{4.142}$$

in accordance to (4.95).

Then the Stokes' graph for the potential $Q(z; s, E)$ must be of type D and the corresponding Fock–Goncharov parameters (4.115) must satisfy

$$\xi_{10} = \xi_{20} = \xi_{30} = -1. \tag{4.143}$$

Alternatively, in terms of $S_{\rm odd}$, these exact quantisation equations become

$$\pi i(2k_1+1) = 2\int_{\tau_1}^{\tau_0} S_{\rm odd}(w_+,\hbar){\rm d}w, \tag{4.144}$$

$$\pi i(2k_2+1) = 2\int_{\tau_2}^{\tau_0} S_{\rm odd}(w_+,\hbar){\rm d}w, \tag{4.145}$$

$$\pi i(2k_3+1) = 2\int_{\tau_3}^{\tau_0} S_{\rm odd}(w_+,\hbar){\rm d}w. \tag{4.146}$$

Proof. The strategy is very simple: simply take the Stokes data for the Jimbo for the Jimbo-Miwa Lax pair corresponding to rational solutions of PII in Fig. 4.6 and equate it to the Stokes matrices from the WKB Riemann-Hilbert problem in each configuration of Theorem 4.24. Note that in this case $\hbar^{-1} = n+1/2$ so that the extra factor in $\mathbb{S}_3$ in (4.129) becomes

$$\exp\left(\frac{2\pi {\rm i}}{\hbar}\sigma_3\right) = \exp\Big(\pi {\rm i}\,(2n+1)\,\sigma_3\Big) = -\mathbb{I}. \tag{4.147}$$

Therefore the Stokes parameters must satisfy

$$s_0 = s_1 = s_2 = s_3 = s_4 = s_5 = {\rm i}. \tag{4.148}$$

From each configuration of the Stokes graph we obtain a system of 6 equations involving the exponentials of the periods v_{jk}. For example, in **configuration D** we obtain from (4.140) the following system:

$${\rm i} = {\rm i}(\xi_{10}\xi_{30}+\xi_{10}+1) \tag{4.149}$$

$${\rm i} = {\rm i}\frac{\xi_{20}}{\xi_{10}}(\xi_{20}\xi_{10}+\xi_{20}+1), \tag{4.150}$$

$${\rm i} = {\rm i}\frac{\xi_{30}}{\xi_{10}}(\xi_{30}\xi_{20}+\xi_{30}+1), \tag{4.151}$$

$${\rm i} = {\rm i}\frac{\xi_{10}}{\xi_{20}}, \tag{4.152}$$

$${\rm i} = {\rm i}\frac{\xi_{10}}{\xi_{20}}. \tag{4.153}$$

There are 5 equations instead of 6 because $s_2 = i$ with the Stokes parameter as in (4.140) does not yield any meaningful conditions.

One can readily verify that the only solutions to this system is

$$\xi_{10} = \xi_{20} = \xi_{30} = -1. \tag{4.154}$$

Indeed, the last two equations imply that the $\xi_{10}, \xi_{20}, \xi_{30}$ must be all equal, and the first three equations reduce their value to the solutions of the polynomial equation $x^2+x=0$. Finally note the Fock-Goncharov coordinates cannot be zero since due to their exponential nature.

Direct inspection of the formulas in Theorem 4.24 shows that it is impossible to satisfy the constraints (4.148) in configurations E and Z. ∎

4.4.2 Shapiro-Tater case

Unlike in the previous section the matching of the Stokes matrices (4.129) to the Stokes' phenomenon in Theorem 4.8 is not enough to determine the parameters s, E in the potential $Q(\zeta; s, E)$ corresponding to the degenerate Exactly Solvable spectrum, that is, the parameters t, λ in (1.11) such that λ is an eigenvalue of algebraic multiplicity at least 2. This is made precise in the following theorem:

Theorem 4.28. *Suppose that* $(t, \Lambda) \in \mathbb{C}^2$ *belong to the ES spectrum, i.e. they determine a quasi-polynomial solution* $y = p_n(z)e^{\theta(z;t)}$ *to the boundary problem* (1.4)-(1.5) *with* $J = n + 1$. *Consider the scaled* (s, E)*-plane*

$$s = \hbar^{-\frac{2}{3}} t, \qquad E = \hbar^{-\frac{4}{3}} \Lambda, \qquad \hbar^{-1} = n + 1, \tag{4.155}$$

in accordance to (4.94).

Then the Stokes' graph for the potential $Q(\zeta; s, E)$ *must be of type D or E; type Z cannot occur. In the case of configurations D and E the "exact" Fock–Goncharov parameters* ξ_{jk} *in* (4.115) *must satisfy one of the following systems*

- *Type D (Fig. 4.3a):*

$$\begin{cases} \xi_{10}\xi_{30} + \xi_{10} + 1 = 0, \\ \xi_{20}\xi_{10} + \xi_{20} + 1 = 0, \\ \xi_{30}\xi_{20} + \xi_{30} + 1 = 0. \end{cases} \tag{4.156}$$

  This systems cuts an affine rational curve in $\mathbb{C}^3$ *given by:*

$$\xi_{10} = \rho, \quad \xi_{20} = -\frac{1}{\rho + 1}, \quad \xi_{30} = -\frac{\rho + 1}{\rho}. \tag{4.157}$$

- *Type E (Fig. 4.3c) or its* $\mathbb{Z}_3$ *rotations:*

$$\begin{cases} \xi_{01}\xi_{12}(\xi_{23} + 1) + \quad \xi_{01} + 1 = 0, \\ \qquad\qquad\qquad\quad \xi_{01} + 1 = 0, \\ \qquad\qquad\qquad\quad \xi_{23} + 1 = 0. \end{cases} \tag{4.158}$$

  This system has the following solution:

$$\xi_{01} = -1, \quad \xi_{23} = -1 \quad \xi_{12} = 1. \tag{4.159}$$

Proof. In Proposition 4.7 it was shown that if $J = n + 1$ and (t, Λ) belong to the ES spectrum (i.e. there is a solution of the boundary problem (1.4) and (1.5)) then the Stokes parameters s_0, s_2, s_4 all vanish simultaneously. In Theorem 4.24 we have expressed the Stokes parameters in terms of the exact Fock-Goncharov parameters ξ_{jk}. Thus we only have to see which configurations are compatible with the three equations $0 = s_0 = s_2 = s_4$.

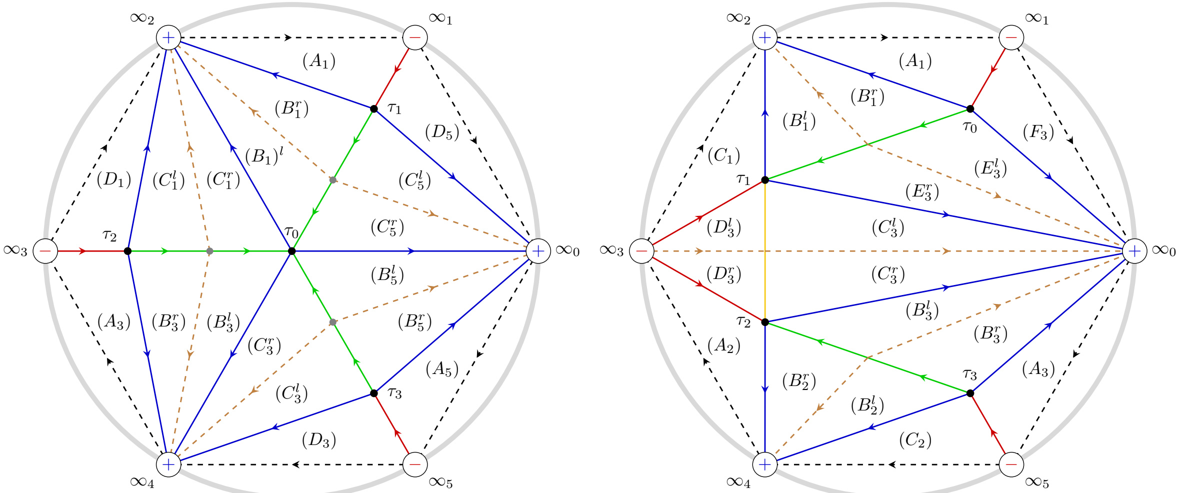
Note that $\hbar^{-1} = (n + 1)$ and hence $e^{\frac{2i\pi}{\hbar}} = 1$ in the aforementioned Theorem. Taking the even-numbered Stokes parameters (4.140) and equating them to zero gives the system (4.156) for Stokes configuration of type D and the system (4.158) for Stokes configuration of type E. Note that the type E system (4.158) only determines ξ_{01}, ξ_{23}. The Fock-Goncharov parameter ξ_{12} is determined by using the relations in Proposition 4.23. ■

4.4.3 Repeated eigenvalue condition

Theorem 4.28 establishes the conditions for the Vöros symbols to yield a point in the ES spectrum; together with Theorem 4.28 the conditions are equivalent to the statement that the Stokes' graph is either of D or E type and the Fock-Goncharov parameters ξ_{jk} satisfy the corresponding conditions specified in Theorem 4.28.

In addition we must now impose the condition that the eigenvalue is a repeated one: as proved in Theorem 4.15 this requires that all the integrals of $p_n^2 e^{2\theta}$ between ∞_{2k+1} vanish. It was also explained in the theorem that it suffices to impose one of the three vanishing conditions and the other two will follow. Thus the strategy now is to compute the integrals

$$I_{13} = \int_{\infty_1}^{\infty_3} p_n(\zeta)^2 e^{2\theta(\zeta;t)} d\zeta, \quad I_{35} = \int_{\infty_3}^{\infty_5} p_n(\zeta)^2 e^{2\theta(\zeta;t)} d\zeta, \tag{4.160}$$



(a) Configuration D (b) Configuration E

Figure 4.7: Labelled regions in the WKB Riemann-Hilbert problem.

using the asymptotic expansion in terms of formal WKB solutions obtained so far.

In order to simplify the upcoming computations, we will label the regions in the WKB Riemann-Hilbert problem according to Fig. 4.7. This will help us distinguish between the solutions to (3.28) which are asymptotic to WKB solutions $\psi_\pm^{(\tau_j)}$ in different regions. We will denote

$$\Psi_\pm^{(\mathscr{R})}(\zeta) \sim \psi_\pm^{(\tau_j)}(\zeta;\hbar) = \frac{1}{\sqrt{S_{\mathrm{odd}}(\zeta,\hbar)}} \exp\left(\pm \int_{\tau_j}^{\zeta} S_{\mathrm{odd}}(w,\hbar)\mathrm{d}w\right) \qquad \hbar \to 0, \zeta \in \mathscr{R}, \tag{4.161}$$

where $\mathscr{R}$ is one of the labelled regions in Fig. 4.7, and τ_j is one of the turning points in its boundary.

Note that the function $\Psi_+^{(A_1)}$ is proportional to the quasi-polynomial $p_n(x)^2 \mathrm{e}^{2\theta(x;t)}$, since they are both recessive on the direction ∞_1. Using this particular solution, instead of working with the integrals (4.160) we can equivalently compute the integral of $\Psi_+^{(A_1)}$ because we are only interested in the vanishing of I_{13}, I_{35}. We recall that $\Psi_+^{(A_1)}$ is the entire solution of the differential equation (3.28) that is asymptotic to the WKB solution $\psi_+^{(\tau)}$ in Def. 3.40 in the region (A_1), in accordance with Thm. 3.18. Thus, the main aim of this section is to prove the following Theorem.

Theorem 4.29. *Suppose that $(t,\Lambda) \in \mathbb{C}^2$ belong to the ES spectrum, i.e. they determine a quasi-polynomial solution $y = p_n(z)e^{\theta(z;t)}$ to the boundary problem (1.4)-(1.5) with $J = n+1$. Consider the scaled (s,E)-plane*

$$s = \hbar^{-\frac{2}{3}} t, \qquad E = \hbar^{-\frac{4}{3}}\Lambda, \qquad \hbar^{-1} = n+1, \tag{4.162}$$

in accordance to (4.94). Then we have the following WKB asymptotic estimates for the integral I_{13} as $\hbar \to 0$.

1. *In configuration D as in Fig. (4.7) we have:*

$$\int_{\infty_1}^{\infty_3} \Psi_+^{(A_1)}(\zeta;\hbar)^2 \mathrm{d}\zeta \sim 2\mathrm{i}\hbar \left(\mathrm{e}^{2v_{12}} \int_{\tau_2}^{\tau_0} \frac{\mathrm{d}\zeta}{\sqrt{Q(\zeta_+;s,E)}} - \int_{\tau_1}^{\tau_0} \frac{\mathrm{d}\zeta}{\sqrt{Q(\zeta_+;s,E)}} \right) + \mathcal{O}(\hbar^2). \tag{4.163}$$

Furthermore, the rescaled parameters (s,E) of the ES spectrum correspond to a repeated eigenvalue whenever

$$\exp\left(\frac{2}{\hbar}\int_{\tau_1}^{\tau_2} \sqrt{Q(\zeta;s,E)}\mathrm{d}\zeta\right) = \boldsymbol{\tau}(s,E) + \mathcal{O}(\hbar), \tag{4.164}$$

where

$$\boldsymbol{\tau}(s,E) = \frac{\displaystyle\int_{\tau_1}^{\tau_0} \frac{\mathrm{d}\zeta}{\sqrt{Q(\zeta_+;s,E)}}}{\displaystyle\int_{\tau_2}^{\tau_0} \frac{\mathrm{d}\zeta}{\sqrt{Q(\zeta_+;s,E)}}}, \qquad \mathrm{Im}\,\big(\boldsymbol{\tau}(s,E)\big) > 0. \tag{4.165}$$

2. *In configuration E as in Fig. (4.4) we have*

$$\int_{\infty_1}^{\infty_3} \Psi_+^{(A_1)}(\zeta;\hbar)^2 \mathrm{d}\zeta \simeq 2\mathrm{i} \int_{\tau_2}^{\tau_0} \frac{\hbar \mathrm{d}\zeta}{\sqrt{Q(\zeta_+;s,E)}} + \mathcal{O}(\hbar^2). \tag{4.166}$$

Furthermore, the rescaled parameters (s,E) of the ES spectrum cannot correspond to a double eigenvalue for large n.

The proof of this theorem is relatively straightforward but significantly technical and delicate. For this reason, we first give here the heuristics of the naïve approach which is appealing for its simplicity and we defer the technicalities for the later section. In fact this approach yields the correct result, but over-estimates the error term.

Proof idea. The idea is to replace $\Psi_+^{(A_1)}$ with the appropriate linear combinations of the formal WKB solutions along the pieces of the contour of integration that traverse each Stokes region indicated in Fig. 4.7.

We will only consider the case of a Stokes graph of type D as it will transpire that this is the only meaningful case. First we deform the path of integration between the infinities ∞_1 and ∞_3 to run along the branch cuts joining the the turning points τ_1, τ_0, τ_2, and we split it into four segments along these three points. The integrals over the unbounded paths are then along steepest descent paths for the integrand and can be neglected. In the integrations along $[\tau_1, \tau_0]$ and $[\tau_0, \tau_2]$ we observe that, as a consequence of the Riemann-Hilbert problem 4.22 we can express $\Psi_+^{(A_1)}$ as suitable linear combinations of $\psi_\pm^{(\tau_j)}$ in different regions:

$$\Psi_+^{(A_1)}(\zeta;\hbar) \simeq \begin{cases} \psi_+^{(\tau_1)}(\zeta;\hbar) + i\psi_-^{(\tau_1)}(\zeta;\hbar) & \zeta \in B_1^r \cup B_1^\ell \\ \mathrm{e}^{v_{12}} \left(\psi_+^{(\tau_2)}(\zeta;\hbar) - i\psi_-^{(\tau_2)}(\zeta;\hbar)\right) & \zeta \in C_1^r \cup C_1^\ell \end{cases} \tag{4.167}$$

where we have assumed that the (implicit) scaled parameters $(s,E) \in \mathbb{C}^2$ correspond to the ES spectrum so that Theorem 4.28 applies and the even-numbered Stokes matrices $\mathbb{S}_j, j = 0,2,4$ in (4.24) are trivial. Computing the square of $\Psi_+^{(A_1)}$, we have that the cross–products yield non-oscillatory functions that contribute to the leading order while the squares of the "pure" WKB solutions give oscillatory integrals which can be neglected to leading order.

Thus one is lead to the rough estimate

$$\int_{\infty_1}^{\infty_3} \left(\Psi_+^{(A_1)}(\zeta)\right)^2 \mathrm{d}\zeta \simeq 2i \int_{\tau_1}^{\tau_0} \psi_+^{(\tau_1)}(\zeta)\psi_-^{(\tau_1)}(\zeta)\mathrm{d}\zeta - 2i\mathrm{e}^{2v_{12}} \int_{\tau_0}^{\tau_2} \psi_+^{(\tau_2)}(\zeta)\psi_-^{(\tau_2)}(\zeta)\mathrm{d}\zeta \tag{4.168}$$

$$\simeq 2i \int_{\tau_1}^{\tau_0} \frac{\hbar \mathrm{d}\zeta}{\sqrt{Q(\zeta_-;s,E)}} - 2i\mathrm{e}^{2v_{12}} \int_{\tau_0}^{\tau_2} \frac{\hbar \mathrm{d}\zeta}{\sqrt{Q(\zeta_+;s,E)}} \tag{4.169}$$

where, confusingly, the boundary values of $\sqrt{Q(z;s,E)}$ are due to our choice of orientations for the branch-cuts in Fig. 4.7 (i.e. not according to Def. 4.18 [2]). Rearranging the endpoints and boundary values yields (4.163).

The reason why the above reasoning is defective is that it replaces the formal WKB expansions also in the neighbourhoods of the turning points, where the formal WKB solutions have a singularity. One may still make sense of the resulting integrals because they involve a singularity of type $(\zeta-\tau)^{-\frac{1}{2}}$ which is integrable. However, approaching the integrals in this way and using a (formal) application of the Laplace method would suggest that the subleading order is $\mathcal{O}(\hbar^{\frac{4}{3}})$. The careful analysis, instead, of the contribution near the turning points reveals that the subleading correction is of order is $\mathcal{O}(\hbar^2)$.

Unfortunately we could not find a reference in the vast literature on exact WKB analysis that helps us in this analysis and the details become quite awkward. For this reason we have postponed this part of the proof to the next section, with most of the technical details buried in the Appendix A

[2]This is done to keep consistency with the WKB Riemann-Hilbert problems and so that the reader is able to navigate the computations with the help of the map in Fig. 4.7

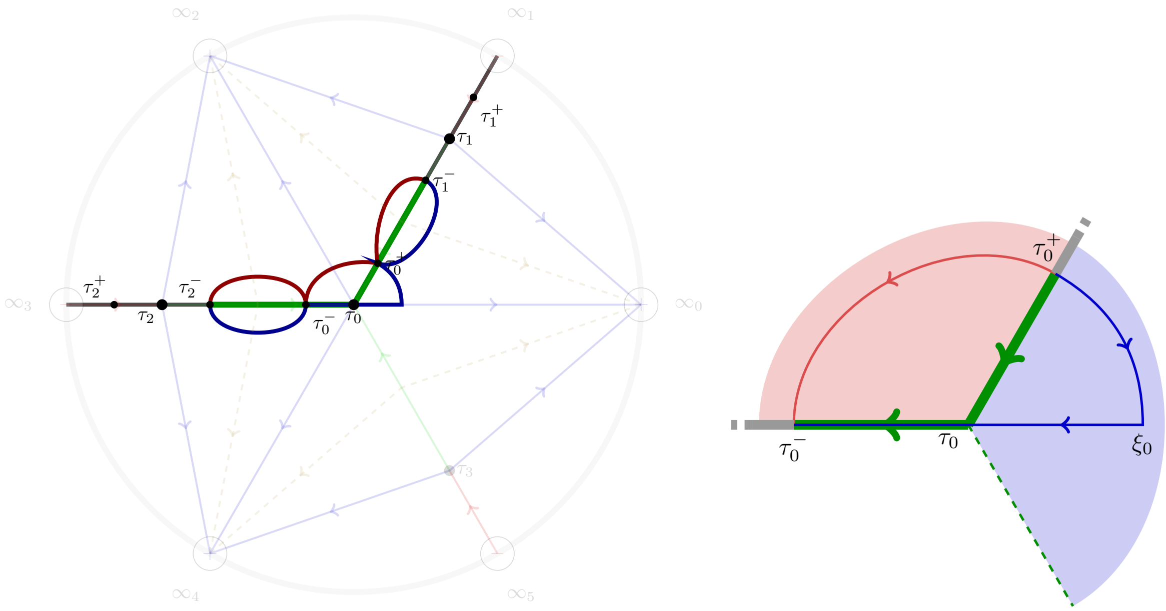


Figure 4.8: The splitting of the integration of (4.182) along the three contours for each of the three integrands near τ_0.

4.4.4 Proof of Theorem 4.29

We will consider in detail only the case of a Stokes graph of type D because it contains all the intricacies.

Since we want to impose that the integral I_{13} in (4.160) vanishes, it suffices to determine the integral of a function that is proportional to $y_n = p_n \mathrm{e}^{2\theta}$. Keeping this in mind we then observe that in the region (A_1) along the Stokes curve (τ_1, ∞_1) the function is *recessive* as $n \to \infty$ and as $x \to \infty_1$. Hence it must be proportional to $\Psi_+^{(A_1)}(z;\hbar)$ and thus asymptotic to $\psi_+^{(\tau_1)}$. We will then forget about y_n and integrate instead the function $\Psi_+^{(A_1)}(z;\hbar)^2$ along a contour γ going from ∞_1 to ∞_3. Recall that this function is entire since it is the solution of a WKB-type ODE (4.96). with a quartic polynomial potential.

Let us refer to Fig. 4.7. We choose the contour γ to follow the steepest descent from ∞_1 to τ_1, then the branch-cut to τ_0, then to τ_2 and the descent path to ∞_3. The branch-cut will be chosen to follow the anti–Stokes curve [3] in a neighbourhood of each turning point.

We then split our contour $\gamma = \bigcup_{j=1}^{7} \gamma_j$ in several pieces:

$$\infty_1 \xrightarrow{\gamma_1} \tau_1^+ \xrightarrow{\gamma_2} \tau_1^- \xrightarrow{\gamma_3} \tau_0^+ \xrightarrow{\gamma_4} \tau_0^- \xrightarrow{\gamma_5} \tau_2^+ \xrightarrow{\gamma_6} \tau_2^- \xrightarrow{\gamma_7} \infty_3, \tag{4.170}$$

where the $\tau_j^\pm$ are determined according to Fig. 4.8.

In order to simplify notation, in the following we will drop the dependence on $\zeta, \hbar$ as well as the specification of the Stokes region A_1 and simply write

$$\Psi := \Psi_+^{(A_1)}(\zeta;\hbar). \tag{4.171}$$

Observe that Ψ is recessive both on the Stokes line $\infty_1 \to \tau_1$ as well as on the Stokes line $\tau_2 \to \infty_3$ because we are already on a point of the ES spectrum, where the Stokes matrices satisfy $\mathbb{S}_0 = \mathbb{S}_2 = \mathbb{S}_4 = \mathbf{1}$. More precisely it follows that

$$\Psi := \Psi_+^{(A_1)}(\zeta;\hbar) = \Psi_+^{(D_5)}(\zeta;\hbar) = \mathrm{e}^{v_{12}}\Psi_+^{(D_1)}(\zeta;\hbar) = \mathrm{e}^{v_{12}}\Psi_+^{(A_3)}(\zeta;\hbar). \tag{4.172}$$

[3] i.e. the curve emanating from the turning point τ determined by $z \in \mathbb{C}$ such that $\operatorname{Re} \int_\tau^z \sqrt{Q(\zeta)}\mathrm{d}\zeta = 0$

Contributions of γ_1, γ_7. On both these contours the integral is exponentially small as $\hbar \to 0_+$ since $\Psi = \Psi_+^{(A_1)}$ is recessive on γ_1 and $\Psi = \mathrm{e}^{v_{12}}\Psi_+^{(D_1)}$ is recessive on γ_7.

Contributions of γ_2, γ_6. These are estimated by the use of Theorem A.3

$$\int_{\tau_1^+}^{\tau_1^-} \Psi^2 \mathrm{d}\zeta = \int_{\tau_1}^{\tau_1^-} \frac{2\mathrm{i}\hbar \mathrm{d}z}{\sqrt{Q(\zeta_-; s, E)}} + \mathcal{O}(\hbar^2), \tag{4.173}$$

$$\int_{\tau_2^+}^{\tau_2^-} \Psi^2 \mathrm{d}\zeta = \mathrm{e}^{2v_{12}} \int_{\tau_2^+}^{\tau_2^-} \left(\Psi_+^{(D_1)}\right)^2 \mathrm{d}\zeta = \mathrm{e}^{2v_{12}} \left(\int_{\tau_2^+}^{\tau_2} \frac{2i\hbar\, \mathrm{d}\zeta}{\sqrt{Q(\zeta_-; s, E)}} + \mathcal{O}(\hbar^2) \right). \tag{4.174}$$

The boundary value is the $-$ because the orientation of the branch-cut that we have chosen is the one leaving from the turning point, while in Theorem A.3 it was the one towards the turning point (and the boundary value was the $+$ one).

Contributions of γ_3, γ_5. Along either paths we are away from the turning points and can substitute the asymptotic expressions in terms of the WKB solutions. We can use the Riemann–Hilbert problem 4.22 to see that the function $\Psi = \Psi_+^{(A_1)}$ has the asymptotic behaviour

$$\Psi_+^{(A_1)}(\zeta;\hbar) \simeq \begin{cases} \psi_+^{(\tau_1)}(\zeta;\hbar) + \mathrm{i}\psi_-^{(\tau_1)}(\zeta;\hbar) & \zeta \in B_1^r \\ \mathrm{e}^{-v_{10}}\psi_+^{(\tau_0)}(\zeta;\hbar) + \mathrm{i}\mathrm{e}^{v_{10}}\psi_-^{(\tau_0)}(\zeta;\hbar) & \zeta \in B_1^\ell \\ \mathrm{e}^{v_{12}}\left(\mathrm{e}^{v_{20}}\psi_+^{(\tau_0)}(\zeta;\hbar) - \mathrm{i}\mathrm{e}^{-v_{20}}\psi_-^{(\tau_0)}(\zeta;\hbar)\right) & \zeta \in C_1^r \\ \mathrm{e}^{v_{12}}\left(\psi_+^{(\tau_2)}(\zeta;\hbar) - \mathrm{i}\psi_-^{(\tau_2)}(\zeta;\hbar)\right) & \zeta \in C_1^\ell \end{cases} \tag{4.175}$$

The different expressions in the regions B_1^ℓ and B_1^r are due simply to the different choice of normalization point for the WKB formal solutions, but we can use either of the two expressions also as asymptotic expansion in the other. The same occurs for the pair C_1^ℓ and C_1^r. Consequently we can estimate the contribution of the whole γ_3 using the asymptotic expression for B_1^r as follows:

$$\int_{\tau_1^-}^{\tau_0^+} \Psi^2 \mathrm{d}\zeta \simeq \int_{\tau_1^-}^{\tau_0^+} \left((\psi_+^{(\tau_1)})^2 - (\psi_-^{(\tau_1)})^2\right) \mathrm{d}\zeta + 2\mathrm{i} \int_{\tau_1^-}^{\tau_0^+} \psi_+^{(\tau_1)}\psi_-^{(\tau_1)} \mathrm{d}\zeta, \tag{4.176}$$

with similar expression for the integral along γ_5. The contribution of the first integral in the right side of (4.176) is of order $\mathcal{O}(\hbar^2)$ as we show a few lines below. The main contribution comes instead from the last integral and it yields

$$2\mathrm{i} \int_{\tau_1^-}^{\tau_0^+} \psi_+^{(\tau_1)}\psi_-^{(\tau_1)} \mathrm{d}\zeta = 2i \int_{\tau_1^-}^{\tau_0^+} \frac{\hbar \mathrm{d}\zeta}{\sqrt{Q(\zeta_-, s, E)}} + \mathcal{O}(\hbar^2), \tag{4.177}$$

where the $-$ boundary value in the last integral is due to the fact that the regions B_1^ℓ and B_1^r lie on the $-$ side of the branch-cut, with the orientation that we have chosen. In particular this contribution together with the contribution (4.173) yield a single integration from τ_1 to τ_0^+. Similar considerations apply to the integral along γ_5 which yield

$$\int_{\tau_0^-}^{\tau_2^+} \Psi^2 \mathrm{d}\zeta \simeq \xi_{12} \left(\int_{\tau_0^-}^{\tau_2^+} \frac{2\mathrm{i}\hbar \mathrm{d}\zeta}{\sqrt{Q(\zeta_+; s, E)}} + \mathcal{O}(\hbar^2) \right) \tag{4.178}$$

To see that the first integral in (4.176) is sub-dominant, for example consider the integration of $(\psi_+^{(\tau_1)})^2$. We can deform the path of integration (at fixed endpoints) into the region C_3 where the formal solution is recessive. Then a simple estimate shows that the only contribution come from the neighbourhood of the endpoints of integration (at which points the real part of the exponential is zero). Then we can use Laplace's method to easily estimate their contribution of order[4] $\mathcal{O}(\hbar^2)$. A similar reasoning applies to the integration of $(\psi_-^{(\tau_1)})^2$ where instead we deform the contour within the region B_1.

[4]Note that the function being integrated has already an $\hbar$ in front, and the integral contributes another order $\mathcal{O}(\hbar)$.

Contribution of γ_4. We are going to show that this integral is estimated as follows:

$$\int_{\tau_0^+}^{\tau_0^-} \Psi^2 \mathrm{d}\zeta \simeq -\int_{\tau_0^+}^{\tau_0} \frac{2\mathrm{i}\hbar}{\sqrt{Q(\zeta_+;s.E)}}(1+\mathcal{O}(\hbar))\mathrm{d}\zeta - \left(1+\mathrm{e}^{2v_{12}}\right)\int_{\tau_0}^{\tau_0^-} \frac{2\mathrm{i}\hbar}{\sqrt{Q(\zeta_+;s.E)}}(1+\mathcal{O}(\hbar))\mathrm{d}\zeta \quad (4.179)$$

In the regions B_1^l our function $\Psi = \Psi_+^{(A_1)}$ can be expressed as

$$\Psi = \Psi_+^{(B_1^l)} + i\Psi_-^{(B_1^l)}. \quad (4.180)$$

Near τ_0 the function Ψ is, along the contours of integration, an oscillatory solution. We express it in terms of the normalized solutions in the right side B_1^r:

$$\Psi_+^{(A_1)} = \mathrm{e}^{v_{10}}\Psi_+^{(B_1^r)} + \mathrm{i}\mathrm{e}^{-v_{10}}\Psi_-^{(B_1^r)} \quad (4.181)$$
$$\Downarrow$$
$$\left(\Psi_+^{(A_1)}\right)^2 = \mathrm{e}^{-2v_{10}}\underbrace{\left(\Psi_+^{(B_1^r)}\right)^2}_{(\mathbf{a})} - \mathrm{e}^{2v_{10}}\underbrace{\left(\Psi_-^{(B_1^r)}\right)^2}_{(\mathbf{b})} + 2\mathrm{i}\underbrace{\Psi_+^{(B_1^r)}\Psi_-^{(B_1^r)}}_{(\mathbf{c})}. \quad (4.182)$$

Refer now to Fig. 4.8: the term marked $(\mathbf{a})$ is recessive in the blue-shaded region, the term $(\mathbf{b})$ in the pink-shaded region while $(\mathbf{c})$ is oscillatory throughout. Consequently we will split the integration of Ψ^2 into three paths from τ_0^+ to τ_0^- and integrate $(\mathbf{a})$ along the blue path, $(\mathbf{b})$ along the red path and $(\mathbf{c})$ along the green path.

The whole contribution of $(\mathbf{b})$ is then sub-leading and of order $\mathcal{O}(\hbar^2)$ where the main contributions come from the neighbourhoods of $\tau_0^\pm$.

The contribution of $(\mathbf{a})$ near τ_0^+ and ξ_0 similarly is of order $\mathcal{O}(\hbar)^2$ (with the main contribution coming solely from the neighbourhood of τ_0^-). The integration from ξ_0 to τ_0^- is achieved by Theorem A.3 and we have

$$(\mathbf{a}) \mapsto \mathrm{e}^{-2v_{10}}\int_{\xi_0}^{\tau_0^-}\left(\Psi_+^{(B_1^r)}\right)^2\mathrm{d}\zeta = \mathrm{e}^{-2v_{10}}\int_{\tau_0}^{\tau_0^-}\frac{2i\hbar(1+\mathcal{O}(\hbar))\mathrm{d}\zeta}{\sqrt{Q(\zeta_+;s,E)}}. \quad (4.183)$$

We then have the contribution of $(\mathbf{c})$ along the green path; for this we need to use a similar reasoning as in the proof of Theorem A.3. The function $\Psi_+^{(B_1^r)}$ is recessive in the blue-shaded region of Fig. 4.8; thus in the coordinate $\xi(z;\hbar)$ along the direction $\arg\xi = 0$ and is therefore proportional to $\mathrm{Ai}(\hbar^{-\frac{2}{3}}\xi)/\sqrt{\xi'}$ after the change of coordinates. Similarly the function $\Psi_-^{(B_1^r)}$ is recessive in the pink-shaded region and hence proportional to $\mathrm{Ai}(\hbar^{-\frac{2}{3}}\omega^2\xi)/\sqrt{\xi'}$, with $\omega = \mathrm{e}^{2i\pi/3}$:

$$\int_{\tau_0^+}^{\tau_0^-}\Psi_+^{(B_1^r)}\Psi_-^{(B_1^r)}\mathrm{d}\zeta = C(\hbar)\int_{\xi_+}^{\xi_-}\frac{\mathrm{Ai}(\hbar^{-\frac{2}{3}}\xi)\mathrm{Ai}(\hbar^{-\frac{2}{3}}\omega^2\xi)\mathrm{d}\xi}{(\xi')^2} \simeq C(\hbar)\left(\int_{\tau_0^+}^{\tau_0^-}\frac{i\hbar^{\frac{1}{3}}\mathrm{d}\zeta}{4\pi\sqrt{Q(\zeta_+;s,E)}} + \mathcal{O}(\hbar^{\frac{4}{3}})\right). \quad (4.184)$$

The latter integral was estimated as in the first part of this proof using part 2 of Proposition A.2: it requires the case $j=0, k=2$ in (A.16). The proportionality constant $C(\hbar)$ is now estimated by matching the asymptotic behaviour;

$$\frac{\hbar}{\sqrt{Q(\zeta;s,E)_+}} \simeq \Psi_+^{(B_1^r)}\Psi_-^{(B_1^r)} \simeq C(\hbar)\frac{\hbar^{\frac{1}{3}}\mathrm{e}^{\frac{i\pi}{3}}}{4\pi\xi^{\frac{1}{2}}\,\xi'} \quad (4.185)$$

So, the coefficient $(1+\mathrm{e}^{2v_{12}})$ in (4.179) comes from the contribution of $(\mathbf{c})$ and $(\mathbf{a})$ while the first term in (4.179) is coming only from the integration of $(\mathbf{a})$ from (4.182). We have thus established (4.179). To complete the proof we now recall that the parameters $\xi_{jk} = \mathrm{e}^{2v_{jk}}$ satisfy the relation (4.157) so that $1+\xi_{10} = -\xi_{20}^{-1}$ so that all the integrals combine nicely. Indeed putting together all contributions we obtain

$$\int_\gamma \Psi^2 \mathrm{d}\zeta = \overset{(\gamma_2)}{\int_{\tau_1}^{\tau_1^-} \frac{2\mathrm{i}\hbar\,\mathrm{d}z}{\sqrt{Q(\zeta_-;s,E)}}} + \overset{(\gamma_3)}{\int_{\tau_1^-}^{\tau_0^+} \frac{2\mathrm{i}\hbar\mathrm{d}\zeta}{\sqrt{Q(\zeta_-,s,E)}}} - \overset{(\gamma_4)}{\int_{\tau_0^+}^{\tau_0} \frac{2\mathrm{i}\hbar(1+\mathcal{O}(\hbar))\mathrm{d}z}{\sqrt{Q(\zeta_+;s.E)}}} + \tag{4.186}$$

$$-\left(1+\mathrm{e}^{-2v_{10}}\right)\overset{(\gamma_4)'}{\int_{\tau_0}^{\tau_0^-} \frac{2\mathrm{i}\hbar(1+\mathcal{O}(\hbar))\mathrm{d}z}{\sqrt{Q(\zeta_+;s.E)}}} + \xi_{12}\overset{(\gamma_5)}{\int_{\tau_0^-}^{\tau_2^+} \frac{2\mathrm{i}\hbar(1+\mathcal{O}(\hbar))\mathrm{d}\zeta}{\sqrt{Q(\zeta_+;s,E)}}} + \tag{4.187}$$

$$+\xi_{12}\overset{(\gamma_6)}{\int_{\tau_2^+}^{\tau_2} \frac{2\mathrm{i}\hbar(1+\mathcal{O}(\hbar))\,\mathrm{d}z}{\sqrt{Q(\zeta_-;s,E)}}} \tag{4.188}$$

where above each term we have recalled with contribution it comes from. We now see that the contribution indicated with $(\gamma_4)'$ and (γ_5) combine to give a single integral: this is due to the fact that $\mathrm{e}^{-2v_{10}} = \xi_{10}^{-1}$ and from (4.157) we see

$$1+\xi_{10}^{-1} = -\xi_{30}.$$

Then, using the identity

$$\int_{\tau_3}^{\tau_0} S_{\mathrm{odd}}(z_+;\hbar)\mathrm{d}\zeta = 2\mathrm{i}\pi(n+1) - \int_{\tau_1}^{\tau_0} S_{\mathrm{odd}}(z_+;\hbar)\mathrm{d}\zeta - \int_{\tau_1}^{\tau_0} S_{\mathrm{odd}}(z_+;\hbar)\mathrm{d}\zeta = 2\mathrm{i}\pi(n+1) + \int_{\tau_1}^{\tau_2} S_{\mathrm{odd}}(z_+;\hbar)\mathrm{d}\zeta \tag{4.189}$$

we deduce that $\xi_{30} = (\xi_{10}\xi_{20})^{-1} = \xi_{12} = \mathrm{e}^{2v_{12}}$, which allows us to write the two terms (4.187) as a single integral. Adding all the contributions yields the final statement. ■

4.5 Comparison of quantization conditions

In this section we compare both of hte

The ST case. In view of Theorem 4.29 we can now express, to within the leading order, the quantization conditions that characterize those points (s,E) in the ES spectrum with a double eigenvalue. Due to the residue at infinity and that $\hbar^{-1} = n+1$, it follows that in a Stokes graph of type D the Fock-Goncharov parameters satisfy $\xi_{12} = \xi_{30}$. Thus the quantisation condition (4.164) becomes

$$\exp\left(\frac{2}{\hbar}\int_{\tau_3}^{\tau_0} \sqrt{Q(\zeta_+;s,E)}\mathrm{d}\zeta\right) = \boldsymbol{\tau}(s,E) + \mathcal{O}(\hbar) \tag{4.190}$$

In view of the parametrisation (4.157) it follows that

$$\xi_{10} = \frac{-1}{1+\boldsymbol{\tau}}, \qquad \xi_{20} = -1 - \frac{1}{\boldsymbol{\tau}}, \qquad \xi_{30} = \boldsymbol{\tau}. \tag{4.191}$$

Thus in terms of the Voros' symbols and to leading order $\mathcal{O}(\hbar)$:

$$2(n+1)\int_{\tau_1}^{\tau_0} \sqrt{Q(\zeta_+;s,E)}\mathrm{d}\zeta = \log\left(\frac{-1}{1+\boldsymbol{\tau}(s,E)}\right) - 2i\pi(m_1+1) \tag{4.192}$$

$$2(n+1)\int_{\tau_2}^{\tau_0} \sqrt{Q(\zeta_+;s,E)}\mathrm{d}\zeta = \log\left(-1 - \frac{1}{\boldsymbol{\tau}(s,E)}\right) - 2i\pi(m_2+1) \tag{4.193}$$

$$2(n+1)\int_{\tau_3}^{\tau_0} \sqrt{Q(\zeta_+;s,E)}\mathrm{d}\zeta = \log\left(\boldsymbol{\tau}(s,E)\right) - 2i\pi(m_3+1) \tag{4.194}$$

where the three integers satisfy $m_1+m_2+m_3 = n-1$ due to the fact that the sum of the three integrals on the left is $-2i\pi(n+1)$ while the sum of the three logarithms is $2i\pi$ (principal determination) due to the definition of $\boldsymbol{\tau}(s,E)$ as in (4.165).

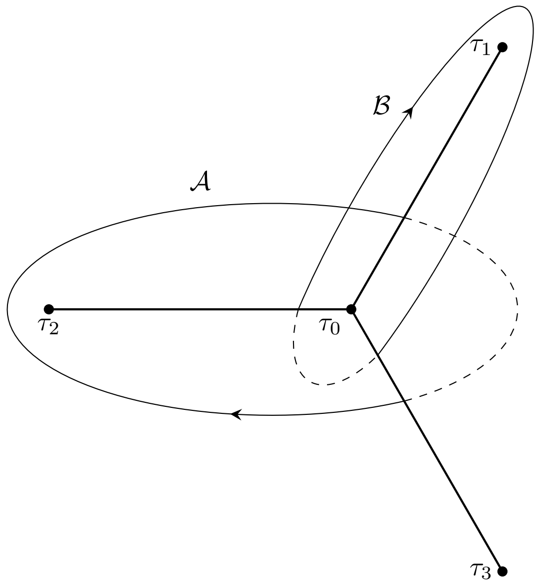


Figure 4.9: Canonical basis of cycles for the homology of the elliptic Riemann surface $\overline{\Sigma}$ with Stokes graph configuration of type D. The points τ_j are the branch points of $\sqrt{Q(\zeta; s, E)}$ and the lines connecting them are the branch cuts.

The JM case. On the other hand, the quantization conditions (4.144) of the zeroes of the Vorob'ev-Yablonskii polynomial give to leading order $\mathcal{O}(\hbar)$:

$$(2n+1)\int_{\tau_j}^{\tau_0} \sqrt{Q(\zeta_+; s, E)}\mathrm{d}\zeta = -i\pi - 2i\pi k_j, \tag{4.195}$$

$$k_1 + k_2 + k_3 = n - 1. \tag{4.196}$$

Both conditions (4.194), (4.195) involve a triple of positive integers adding to $n-1$ but they differ notably in the multiplicative factor $2(n+1)$ vs. $(2n+1)$ on the left side, and on the values on the right side. We now analyze the two lattices to explain their similarity which is apparent from the numerical experiments.

Analysis of the two lattices

Both lattices involve implicit equations for the parameters (s, E) via the periods of the differential $\sqrt{Q(\zeta; s, E)}\mathrm{d}\zeta$. We introduce a canonical basis of cycles $\mathcal{A}, \mathcal{B}$ of the elliptic Riemann surface $\overline{\Sigma}$ as in the Fig.4.9 so that

$$\boldsymbol{\tau}(s, E) = \frac{\displaystyle\int_{\tau_1}^{\tau_0} \frac{\mathrm{d}\zeta}{\sqrt{Q(z_+; s, E)}}}{\displaystyle\int_{\tau_2}^{\tau_0} \frac{\mathrm{d}\zeta}{\sqrt{Q(z_+; s, E)}}} = \frac{\displaystyle\int_{\mathcal{B}} \frac{\mathrm{d}\zeta}{\sqrt{Q(z; s, E)}}}{\displaystyle\int_{\mathcal{A}} \frac{\mathrm{d}\zeta}{\sqrt{Q(z; s, E)}}}. \tag{4.197}$$

Notice that the quantities v_{j0} defined in (4.111) can be written in the form

$$v_{10} = \frac{1}{2}\int_{\mathcal{B}} \sqrt{Q(\zeta; s, E)}\mathrm{d}\zeta, \quad v_{20} = \frac{1}{2}\int_{\mathcal{A}} \sqrt{Q(\zeta; s, E)}\mathrm{d}\zeta, \quad v_{30} = -\frac{1}{2}\int_{\mathcal{A}+\mathcal{B}} \sqrt{Q(\zeta; s, E)}\mathrm{d}\zeta \tag{4.198}$$

The Jacobian determinant of the map $(s, E) \mapsto (I_{\mathcal{A}}, I_{\mathcal{B}})$ is a constant as we prove in the next lemma.

Lemma 4.30. *Let $\mathcal{A}$ and $\mathcal{B}$ be the canonical homology basis as in Fig. 4.9 and consider the periods*

$$I_{\mathcal{A}} = \oint_{\mathcal{A}} \sqrt{Q(\zeta; s, E)}\mathrm{d}\zeta, \quad I_{\mathcal{B}} = \oint_{\mathcal{B}} \sqrt{Q(\zeta; s, E)}\mathrm{d}\zeta. \tag{4.199}$$

Then

$$\det\begin{bmatrix}\frac{\partial I_{\mathcal{A}}}{\partial s} & \frac{\partial I_{\mathcal{A}}}{\partial E}\\ \frac{\partial I_{\mathcal{B}}}{\partial s} & \frac{\partial I_{\mathcal{B}}}{\partial E}\end{bmatrix} = \mathrm{i}\pi. \tag{4.200}$$

Proof. The determinant gives

$$\det \mathbb{J} = \frac{\partial I_{\mathcal{A}}}{\partial s}\frac{\partial I_{\mathcal{B}}}{\partial E} - \frac{\partial I_{\mathcal{B}}}{\partial s}\frac{\partial I_{\mathcal{A}}}{\partial E}. \tag{4.201}$$

Since $Q(\zeta; s, E) = \zeta^4 + s\zeta^2 + 2\zeta + E$, the derivative w.r.t. E gives the holomorphic periods and the derivative in s gives second-kind periods. Thus we can compute the above expressions with the Riemann bilinear identity to give

$$\det \mathbb{J} = \frac{1}{4}\int_{\mathcal{A}} \frac{\zeta^2 \mathrm{d}\zeta}{\sqrt{Q(\zeta; s, E)}} \int_{\mathcal{B}} \frac{\mathrm{d}\zeta}{\sqrt{Q(\zeta; s, E)}} - \frac{1}{4}\int_{\mathcal{A}} \frac{\zeta^2 \mathrm{d}\zeta}{\sqrt{Q(\zeta; s, E)}} \int_{\mathcal{B}} \frac{\mathrm{d}\zeta}{\sqrt{Q(\zeta; s, E)}} \tag{4.202}$$

$$= 2\mathrm{i}\pi \frac{1}{2} \operatorname*{res}_{z=\infty^+} \frac{\zeta^2 \mathrm{d}\zeta}{\sqrt{Q(\zeta; s, E)}} \int_{\tau_1}^{z} \frac{\mathrm{d}w}{\sqrt{Q(w; s, E)}} = \mathrm{i}\pi \tag{4.203}$$

where we have used that

$$\frac{\zeta^2 \mathrm{d}\zeta}{\sqrt{Q(\zeta; s, E)}} = \left(1 + \mathcal{O}(\zeta^{-2})\right)\mathrm{d}\zeta \qquad \int_{\tau_1}^{\zeta} \frac{\mathrm{d}w}{\sqrt{Q(w; s, E)}} = \int_{\tau_1}^{\infty} \frac{\mathrm{d}w}{\sqrt{Q(w; s, E)}} - \frac{1}{\zeta} + \mathcal{O}(\zeta^{-2}). \tag{4.204}$$

and the contribution from the point ∞^- is the same as the point ∞^+. ■

We observe that

$$\omega := \frac{\partial I_{\mathcal{A}}}{\partial E} = \int_{\tau_2}^{\tau_0} \frac{\mathrm{d}\zeta}{\sqrt{Q(\zeta_+; s, E)}}, \quad \omega' := \frac{\partial I_{\mathcal{B}}}{\partial E} = \int_{\tau_1}^{\tau_0} \frac{\mathrm{d}\zeta}{\sqrt{Q(\zeta_+; s, E)}} \tag{4.205}$$

are the half periods of the holomorphic differential $\frac{\mathrm{d}\zeta}{\sqrt{Q(\zeta;s,E)}}$. The lemma is useful in that it allows us to explore the geometry of the quantization conditions, which is what we do next.

Proposition 4.31. *Let (s_0, E_0) correspond to the first-order quantization conditions (4.194) or (4.195) in the bulk, namely, $m_j/n \simeq c_j \neq 0$. Then the neighbour points in the s–plane form a slowly modulated hexagonal lattice in the sense that the six closest neighbours of s_0 are*

$$s_0 + 2\hbar\left(\omega\Delta m_1 - \omega'\Delta m_2\right) \tag{4.206}$$

where ω and ω' are the half periods of the holomorphic differentials in (4.205) and

$$\Delta m_j \in \{-1, 0, 1\}, \qquad |\Delta m_1 + \Delta m_2| \leq 1, \qquad |\Delta m_1| + |\Delta m_2| \geq 1. \tag{4.207}$$

Proof. Let (m_1, m_2, m_3) be a triple of quantization numbers for either (4.194) or (4.195). The neighbour points correspond to adding/subtracting 1 from each, subject to the constraints

$$\Delta m_1 + \Delta m_2 + \Delta m_3 = 0, \quad \Delta m_j \in \{-1, 0, 1\}. \tag{4.208}$$

There are six elementary possibilities

$$(\Delta m_1, \Delta m_2, \Delta m_3) \in \Big\{(1,-1,0), (1,0,-1), (0,1,-1), (-1,1,0), (-1,0,1), (0,-1,1)\Big\}. \tag{4.209}$$

The values of the periods $\int_{\tau_\ell}^{\tau_0} \sqrt{Q(z_+; s, E)}\mathrm{d}z$, $\ell = 1, 2, 3$, change by $\hbar\Delta m_\ell$, where $\hbar = (n+1)^{-1}$ in the ST case and $\hbar(n + 1/2)^{-1}$ in the VY case.

Let $(s,E)=(s_0,E_0)+(\Delta s,\Delta E)$ be a neighbour point in the lattice. We want to estimate $(\Delta s,\Delta E)$: we observe that

$$\int_{\tau_2}^{\tau_0}\sqrt{Q(\zeta_+;s,E)}\mathrm{d}\zeta=\frac{1}{2}I_{\mathcal{A}},\quad \int_{\tau_1}^{\tau_0}\sqrt{Q(\zeta_+;s,E)}\mathrm{d}\zeta=\frac{1}{2}I_{\mathcal{B}},\quad \int_{\tau_3}^{\tau_0}\sqrt{Q(\zeta_+;s,E)}\mathrm{d}\zeta=-\frac{1}{2}I_{\mathcal{A}}-\frac{1}{2}I_{\mathcal{B}} \tag{4.210}$$

where $I_{\mathcal{A}}$ and $I_{\mathcal{B}}$ are defined in Lemma 4.30. If we take for example the first two periods and expand (4.194) or (4.195) to linear order, we obtain

$$\begin{bmatrix}\Delta s\\ \Delta E\end{bmatrix}\simeq 2\hbar\begin{bmatrix}\dfrac{\partial I_{\mathcal{A}}}{\partial E} & -\dfrac{\partial I_{\mathcal{B}}}{\partial E}\\ -\dfrac{\partial I_{\mathcal{A}}}{\partial s} & \dfrac{\partial I_{\mathcal{B}}}{\partial s}\end{bmatrix}\begin{bmatrix}\Delta m_1\\ \Delta m_2\end{bmatrix} \tag{4.211}$$

so that, from the definition (4.205) one recovers (4.206). This local lattice generators ω and ω' are slowly modulated across the elliptic region. ■

Near the origin. If $(s,E)=\mathcal{O}(\hbar)$ then the elliptic surface is $w^2=z^4+\mathcal{O}(\hbar)z^2+2z+\mathcal{O}(\hbar)$ and then a direct computation shows that $\boldsymbol{\tau}=\mathrm{e}^{\frac{2i\pi}{3}}+\mathcal{O}(\hbar)$. Note that the $\mathbb{Z}_3$ symmetry of the limiting elliptic curve gives

$$\mathrm{e}^{2i\pi/3}=\boldsymbol{\tau}=\frac{-1}{1+\boldsymbol{\tau}}=-1-\frac{1}{\boldsymbol{\tau}}. \tag{4.212}$$

We are now going to show that two quantization conditions yield the same lattices to order $\hbar^2$.

Theorem 4.32. *The rescaled lattices of the zeroes of the VY Polynomials, and of the ST problem coincide to within order $\mathcal{O}(\hbar^2)=\mathcal{O}(n^{-2})$ in a $\mathcal{O}(\hbar)$ neighbourhood of the origin in the s–plane. More precisely the quantization conditions (4.194), (4.195) corresponding to the triples (m_1,m_2,m_3) satisfying $m_1+m_2+m_3=n-1$ and the triples (k_1,k_2,k_3) satisfying $k_1+k_2+k_3=n-1$ with $m_j=k_j$ single out values of s,E that differ by a discrepancy of order $\mathcal{O}(\hbar^2)$, provided that $m_j-\frac{n-1}{3}$ remain bounded as $n\to\infty$.*

Proof. Let $(s,E)=\mathcal{O}(\hbar)$. Then the two quantization conditions (4.194), (4.195) read, to order $\hbar$,

$$\begin{aligned}2(n+1)\int_{\tau_j}^{\tau_0}\sqrt{Q(\zeta_+;s,E)}\mathrm{d}\zeta&=-2i\pi\left(m_j+\frac{2}{3}\right)\\ (2n+1)\int_{\tau_j}^{\tau_0}\sqrt{Q(\zeta_+;s,E)}\mathrm{d}\zeta&=-2i\pi\left(k_j+\frac{1}{2}\right)\end{aligned} \tag{4.213}$$

By fixing $s=\Delta s$ and $E=\Delta E$, and setting $\Delta s=\hbar\delta s,\ \Delta E=\hbar\delta E$, we can use the linear approximation

$$\int_{\tau_j}^{\tau_0}\sqrt{Q(\zeta_+;s,E)}\mathrm{d}\zeta\simeq\int_{\tau_j}^{\tau_0}\sqrt{Q(\zeta_+;0,0)}\mathrm{d}\zeta+\hbar\delta s\int_{\tau_j}^{\tau_0}\frac{\zeta^2\mathrm{d}\zeta}{2\sqrt{Q(\zeta_+;0,0)}}+\hbar\delta E\int_{\tau_j}^{\tau_0}\frac{\mathrm{d}\zeta}{2\sqrt{Q(\zeta_+;0,0)}} \tag{4.214}$$

$$=-\frac{i\pi}{3}+\hbar\left(c_2\mathrm{e}^{\frac{2i\pi}{3}(j-1)}\delta s+c_0\mathrm{e}^{-\frac{2i\pi}{3}(j-1)}\delta E\right)\mathrm{e}^{\frac{7i\pi}{6}} \tag{4.215}$$

$$c_\ell=\int_{-2^{\frac{1}{3}}}^{0}\frac{\zeta^\ell\mathrm{d}\zeta}{2\sqrt{|\zeta^4+2z|}},\quad c_0\simeq 1.9276,\quad c_2\simeq 0.9409. \tag{4.216}$$

where $\hbar$ means either $(n+1)^{-1}$ or $(n+1/2)^{-1}$ depending on the case we are considering. Replacing the above expansions in (4.213) we obtain (all to within $\mathcal{O}(\hbar)$);

$$2(n+1)\left(-\frac{i\pi}{3}+\frac{1}{n+1}\left(c_2\mathrm{e}^{\frac{2i\pi}{3}(j-1)}\delta s+c_0\mathrm{e}^{-\frac{2i\pi}{3}(j-1)}\delta E\right)\mathrm{e}^{\frac{7i\pi}{6}}\right)=-2i\pi\left(m_j+\frac{2}{3}\right) \tag{4.217}$$

$$(2n+1)\left(-\frac{i\pi}{3}+\frac{1}{n+\frac{1}{2}}\left(c_2\mathrm{e}^{\frac{2i\pi}{3}(j-1)}\delta s+c_0\mathrm{e}^{-\frac{2i\pi}{3}(j-1)}\delta E\right)\mathrm{e}^{\frac{7i\pi}{6}}\right)=-2i\pi\left(k_j+\frac{1}{2}\right) \tag{4.218}$$

Simplifying we get the quantization rules for $\delta s, \delta E$ in identical form provided we identify $m_j = k_j$:

$$2\left(c_2 e^{\frac{2i\pi}{3}(j-1)}\delta s + c_0 e^{-\frac{2i\pi}{3}(j-1)}\delta E\right) e^{\frac{7i\pi}{6}} = -2i\pi\left(m_j - \frac{n-1}{3}\right) + \mathcal{O}(\hbar). \tag{4.219}$$

Thus, the two quantization conditions give two approximate lattices that differ by $\mathcal{O}(\hbar^2)$ as long as $m_j - \frac{n-1}{3}$ remain bounded as $n \to \infty$. ■

Remark 4.33. The differential equation (1.4) for $t = 0 = \Lambda$ can be solved in terms of Whittaker $\mathrm{W}_{\mu,\nu}, \mathrm{M}_{\mu,\nu}$ functions (i.e. confluent hypergeometric functions) [DLMF 13.14] as follows:

$$y'' - (\zeta^4 + 2J\zeta)y = 0, \quad y_1 = \frac{1}{\zeta}\mathrm{M}_{-\frac{J}{3},\frac{1}{6}}\left(\frac{2\zeta^3}{3}\right), \quad y_2 = \frac{1}{\zeta}\mathrm{W}_{-\frac{J}{3},\frac{1}{6}}\left(\frac{2\zeta^3}{3}\right). \tag{4.220}$$

Writing $J = n+1$ with $n = 0, 1, 2, \dots$ we have that for $n \equiv 1 \mod 3$ ($J \equiv 2 \mod 3$) the solution y_1 is our quasi-polynomial solution. For example

$$J = 2 \qquad y_1(z) = \left(\frac{2}{3}\right)^{\frac{2}{3}} \zeta\, e^{\zeta^3/3} \tag{4.221}$$

$$J = 5 \qquad y_1(\zeta) = \left(\frac{2}{3}\right)^{\frac{2}{3}} \frac{\zeta}{2}(\zeta^3 + 2)\, e^{\zeta^3/3} \tag{4.222}$$

$$J = 8 \qquad y_1(\zeta) = \left(\frac{2}{3}\right)^{\frac{2}{3}} \frac{\zeta}{7}(\zeta^6 + 7\zeta^3 + 7)\, e^{\zeta^3/3} \tag{4.223}$$

$$J = 11 \qquad y_1(\zeta) = \left(\frac{2}{3}\right)^{\frac{2}{3}} \frac{\zeta}{70}(2\zeta^9 + 30\zeta^6 + 105\zeta^3 + 70)\, e^{\zeta^3/3} \tag{4.224}$$

et cetera. This corresponds to the quantization conditions $m_j = \frac{n-1}{3}$, $j = 1, 2, 3$.

Chapter 5

The Stieljes-Fekete equilibrium problem

5.1 Examples

In order to illustrate and motivate the following results, we will begin by providing a some examples of degenerate orthogonal polynomials.

5.1.1 The Shapiro-Tater degenerate orthogonal polynomials

We have already encountered the first example of a 1-degenerate orthogonal polynomial.

In section 4.2.2 we showed that the polynomial part of the quasi-polynomial solution $y(z) = p_n(z)\mathrm{e}^{\theta(z;t)}$ is a 1-degenerate orthogonal polynomial. Here the symbol is $2\theta(z;t) = \frac{2z^3}{3} + tz$ and the contour is given by $\Gamma = \varkappa\gamma + \widetilde{\varkappa}\widetilde{\gamma}$ with the particular choices of $\varkappa = s_1$ and $\widetilde{\varkappa} = s_5$ as in (4.33). This means that

$$\mathcal{M}[Q] = \varkappa \int_\gamma Q(z)\mathrm{e}^{2\theta(z;t)}\mathrm{d}z + \widetilde{\varkappa} \int_{\widetilde{\gamma}} Q(z)\mathrm{e}^{2\theta(z;t)}\mathrm{d}z, \quad Q \in \mathbb{C}[z] \tag{5.1}$$

is the associated semiclassical moment functional is of type $(A, B) = (2z^2 + t, 1)$ so that $d = 2$ and $l = 1$.

Indeed, the quasi-polynomial $y(z) = p_n(z)\mathrm{e}^{\theta(z;t)}$ solves the Shapiro-Tater differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}z^2} - V_{\mathrm{ST}}(z)y, \qquad V_{\mathrm{ST}}(z) = z^4 + az^2 + 2(n+1)z + \Lambda \tag{5.2}$$

as predicted by Theorem 1.8. Furthermore, it was shown in the proof of Theorem (4.12) that the roots of $p_n(z)$ satisfy the Stieltjes-Bethe equations

$$\sum_{\substack{k=1\\k\neq j}}^{n} \frac{1}{z_k - z_j} = \frac{A(z_j)}{2B(z_j)} = \theta(z_j;t) = z_j^2 + \frac{t}{2}, \qquad j = 1, 2, \ldots, n. \tag{5.3}$$

5.1.2 Semiclassical functional with Freud weight.

Motivated by [Fre76], we consider the symbol $\theta(z) = -\frac{z^4}{2}$ and the weighted contour $\Gamma = s_1\gamma_1 + s_2\gamma_2 + s_3\gamma_3$ where γ_j is the contour extending to infinity from the directions $\arg(z) = (j-1)\frac{\pi}{2}$ to $\arg(z) = \frac{j\pi}{2}$, $j = 1, 2, 3$, and the complex parameters s_ℓ are not all simultaneously zero. Next we define the moment functional

$$\mathcal{M}[z^j] = \left(s_1 \int_{\gamma_1} + s_2 \int_{\gamma_2} + s_3 \int_{\gamma_3}\right) z^j \mathrm{e}^{-z^4/2}\mathrm{d}z. \tag{5.4}$$

According to [MR98], a characterization of such a moment functional is that it satisfies the semiclassical condition

$$\mathcal{M}[2z^3\, p(z)] = \mathcal{M}[p'(z)], \qquad \forall p \in \mathbb{C}[z]. \tag{5.5}$$

which can be verified by integration by parts. Thus $\mathcal{M}$ is a semiclassical moment functional of type $(A(z) = 2z^3, B(z) = 1)$. The parameters s_ℓ can be thought of as parametrizing the space of solutions of (5.5).

The corresponding orthogonal polynomials, when they exist, are a sequence $\{P_j(z)\}_{j\in\mathbb{N}}$ of polynomials of degree $\leq j$ that satisfy the orthogonality relation (3.3). Clearly only the ratios of the parameters s_ℓ are relevant for the definition of orthogonal polynomials, so that we can think of the family of functionals (5.4) with the same symbol $\theta(z) = -\frac{z^4}{2}$ as parametrized by a point $[s_1 : s_2 : s_3] \in \mathbb{P}^2$ in projective space. Having the freedom to choose the point $[s_1 : s_2 : s_3] \in \mathbb{P}^2$, one can impose an excess of orthogonality. In this case the notion of *maximally degenerate* orthogonal polynomial is the following; for any $n \in \mathbb{N}$ there is $[s_1 : s_2 : s_3] \in \mathbb{P}^2$ such that a (nontrivial) polynomial $P_n(z)$ of degree n exists with the properties

$$\mathcal{M}[z^j P_n(z)] = 0, \qquad j = \underbrace{0, 1, 2, \dots, n-1}_{\text{orthogonality}},\ \underbrace{n, n+1}_{\substack{\text{degenerate}\\ \text{orthogonality}}}\ . \tag{5.6}$$

We have emphasized that the orthogonality of P_n extends beyond the range of powers that characterizes an ordinary orthogonal polynomial. The reader with some prior experience will quickly conclude that the two extra conditions can be fulfilled if and only if two certain determinants of size $n+1$ that involve the moments vanish simultaneously: this is indeed the case, see Lemma 5.7. This places two homogeneous polynomial conditions on the parameters s_1, s_2, s_3. In this case our theorem states that the zeros of such a maximally degenerate polynomial will satisfy eq. (1.21) with $A(z) = 2z^3 = -\theta'(z)$ and $B = 1$.

Remark 5.1. The function $\widehat{\theta}(z) = \int \frac{A}{B}$ appearing in the energy functional (1.19) is just $\widehat{\theta}(z) = z^4/2 = -\theta(z)$, and it is real for $z \in \mathbb{R}$. It is simple to see from the electrostatic interpretation that there is an optimal configuration with $z_j \in \mathbb{R}$. Our theorem says that the corresponding polynomial $P_n(z) = \prod_{j=1}^n (z - z_j)$ is indeed an orthogonal polynomial, but *not* for the orthogonality chosen on the real axis. This could not be because then the moment functional is strictly positive definite and the Hankel determinant of the moments cannot vanish. Instead, the moment functional is

$$\begin{aligned} \mathcal{M}[z^{2j+1}] &= \int_{\mathbb{R}} z^{2j+1} \mathrm{e}^{-\frac{z^4}{2}} \mathrm{d}z + s \int_{\mathrm{i}\mathbb{R}} z^{2j+1} \mathrm{e}^{-\frac{z^4}{2}} \mathrm{d}z = 0 \\ \mathcal{M}[z^{2j}] &= \int_{\mathbb{R}} z^{2j} \mathrm{e}^{-\frac{z^4}{2}} \mathrm{d}z + s \int_{\mathrm{i}\mathbb{R}} z^{2j} \mathrm{e}^{-\frac{z^4}{2}} \mathrm{d}z = (1 + (-1)^j \,\mathrm{Im}(s)) 2^{\frac{2j-3}{2}} \Gamma\left(\frac{2j+1}{4}\right) \end{aligned} \tag{5.7}$$

where Γ is the standard gamma-function, and $s \in \mathrm{i}\mathbb{R}$ has been chosen pure imaginary in order to obtain real moments.

We observe that, using our previous notation, the integration over the real line $\mathbb{R}$ is homotopic to the path $-\gamma_1 - \gamma_2$, while the integration along the imaginary axis is homotopic to the path $-\gamma_2 - \gamma_3$. The degeneracy condition on the polynomial of degree $n = 10$, gives $s = 0.000013495\,95\mathrm{i}$ and for $n = 11$ one obtains $s = -3.79352745 \times 10^{-6}\mathrm{i}$. The zeros of the corresponding degenerate orthogonal polynomials give a solution of the Stieltjes–Bethe equation (1.21), and are plotted in Fig. 5.1.

5.1.3 Contours and dual contours for Freud-type symbols.

In order to motive the dual contours introduced in the next section, we first consider the case where $B(z) = 1$ and $A(z)$ is a polynomial of degree a. This means that $\theta(z) = -\int A\mathrm{d}z$ is a polynomial of degree $a+1$ and $d = a$. Without major loss of generality we assume that $A(z) = z^a + \dots$ is a monic polynomial.

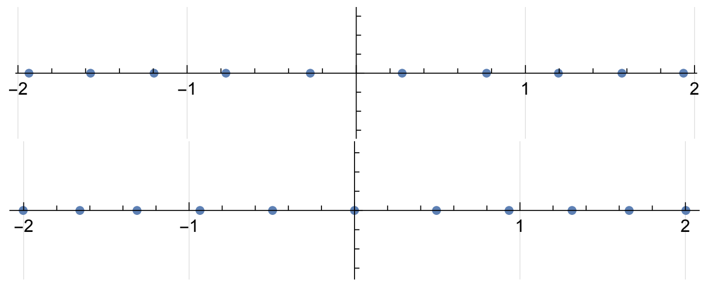


Figure 5.1: The numerically computed Fekete points with $n = 10$ (top) and $n = 11$ (bottom) on the real axis for the Freud weight $\mathrm{e}^{-x^4/2}$.

The linear space of moment functionals is in one-to-one correspondence with the solutions of the Pearcey–like ODE

$$A\left(\frac{\partial}{\partial\lambda}\right)\Phi(\lambda) = \lambda\Phi(\lambda). \tag{5.8}$$

Its solutions are described as follows. We denote by $\infty^{(j)}$ the asymptotic directions $\arg(z) = \frac{j\pi}{a+1}$ for $j = 0, \dots, 2a+1$ and by γ_{j+1} the oriented contour connecting $\infty^{(2j)}$ to $\infty^{(2j+2)}$ for $j = 0, \dots a$ and by γ_{a+1} the oriented contour from $\infty^{(2a)}$ to $\infty^{(0)}$, as seen in Fig. 5.2. Then the space of solutions to the ODE (5.8) is spanned by

$$\Phi_j(\lambda) = \int_{\infty^{(2j)}}^{\infty^{(2j+2)}} \mathrm{e}^{\theta(z)+\lambda z}\mathrm{d}z, \quad j = 1, \dots, a. \tag{5.9}$$

Furthermore we associate solutions of the Pearcey equation to moment functionals by

$$\Phi_j(\lambda) \quad\longleftrightarrow\quad \mathcal{M}_{\gamma_j}[p] = \int_{\gamma_j} p(z)\mathrm{e}^{\theta(z)}\mathrm{d}z, \;\; p \in \mathbb{C}[z], \tag{5.10}$$

where throughout, the contour integrals are seen to be absolutely convergent on account of the fact that $\theta(z) = -\frac{z^{a+1}}{(a+1)} + \mathcal{O}(z^a)$. Due to the Cauchy residue theorem the $a+1$ moment functionals defined in the above formula satisfy the linear relation

$$\mathcal{M}_{\gamma_1} + \cdots + \mathcal{M}_{\gamma_{a+1}} \equiv 0 \tag{5.11}$$

and hence the general moment functional of type $(A, 1)$ is expressed as

$$\mathcal{M}[p] = \sum_{j=1}^{a} s_j \mathcal{M}_{\gamma_j}[p] = \sum_{j=1}^{a} s_j \int_{\gamma_j} p(z)\mathrm{e}^{\theta(z)}\mathrm{d}z, \;\; \forall p \in \mathbb{C}[z]. \tag{5.12}$$

The *dual contours* $\widehat{\gamma}_j$ are the contours extending from $\infty^{(2a+1)}$ to $\infty^{(2j-1)}$, $j = 1, \dots, a$, see Fig. 5.2. They have the property that the intersection number is

$$\widehat{\gamma}_j \circ \gamma_\ell = \delta_{j\ell}. \tag{5.13}$$

5.1.4 Bessel degenerate orthogonal polynomials

In order to illustrate Theorem 1.8 let us consider the Bessel case, namely $\theta(z) = -z^{-1} + \nu \log z$. The only contour of integration is in Fig. 5.3 and a for dual we can take the positive real axis. The piecewise analytic remainder function $G(z)$ defined by (5.35), outside of the cardioid coincides with the definite integral (formula (5.65))

$$G_\infty(z) := F(z)\int_\infty^z \frac{\mathrm{d}w}{F^2(w)} \tag{5.14}$$

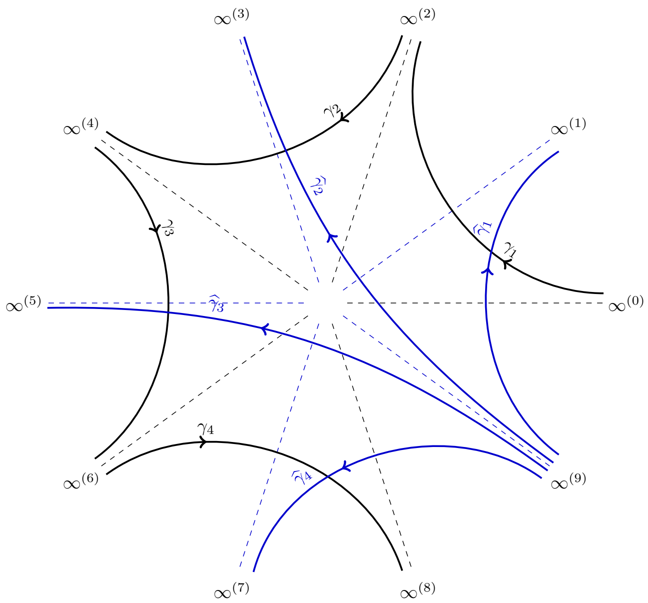


Figure 5.2: The contours, and dual contours, for the Freud-type symbol $\theta(z) = -z^5$, here with $d = a = 4$. In the figure $\infty^{(j)}$ are the asymptotic directions $\arg(z) = \frac{j\pi}{5}$ for $j = 0, \ldots, 9$.

Here the direction of integration at ∞ is immaterial since as long as $2n + \nu > -1$ we have

$$F(z)^2 \simeq z^{2n+\nu+2}. \tag{5.15}$$

Inside the cardioid $G(z)$ must be recessive near $z = 0$ along the positive direction and hence it must coincide with the definite integral

$$G_{0^{(1)}}(z) := F(z) \int_{0^{(1)}}^{z} \frac{\mathrm{d}w}{F^2(w)}, \tag{5.16}$$

where we recall that the notation $0^{(1)}$ denotes the steepest ascent direction of the symbol θ near the point $z = 0$. Thus we easily conclude that, for $z \in \gamma$ (with γ the cardioid, oriented as indicated in Fig. 5.3),

$$G(z_+) = G(z_-) + sF(z), \qquad s = \int_0^\infty \frac{\mathrm{d}w}{F(w)^2} \tag{5.17}$$

as claimed.

5.2 Contours for semiclassical moment functionals

We now provide a quick summary of the definition and some properties of semiclassical moment functionals. The notion originates in the works of Maroni [Mar87] and Marcellán-Rocha [MR98] and has also been extended to the bi-orthogonal case in [Ber03].

Recall now that for two relatively prime polynomials, $A(z), B(z)$ of degree a, b, respectively, the *semiclassical moment functional* of type (A, B) are those satisfying Definition 1.5. The main result of [IMR91, MR98, Mar87] (see also the introduction of [Ber03]) is that any such moment functional can be represented in a similar form as (5.4):

$$\mathcal{M}[p] = \sum_{\ell=1}^{d} s_\ell \int_{\gamma_\ell} p(z) \mathrm{e}^{\theta(z)} \mathrm{d}z \qquad \theta'(z) = -\frac{A(z) + B'(z)}{B(z)} \tag{5.18}$$

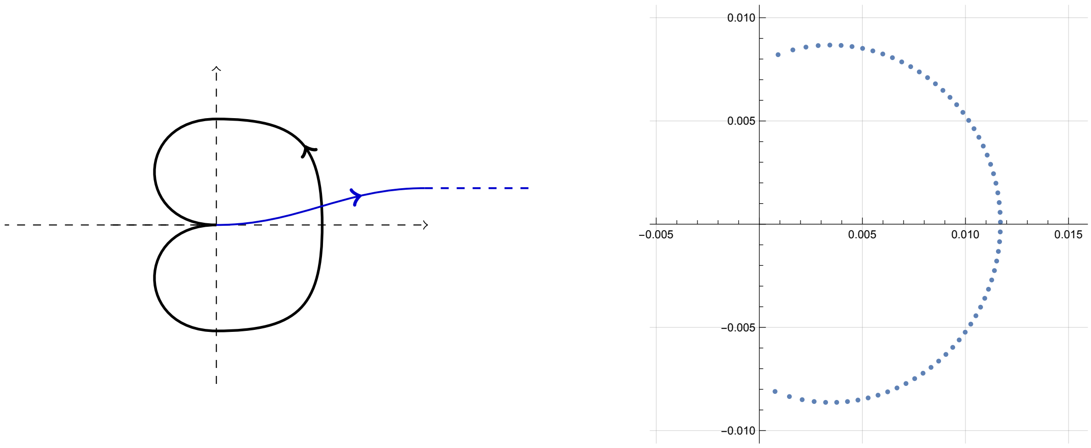


Figure 5.3: Left: the contour and dual contour for the Bessel functional of Table 1.3, for the case $\nu \notin \mathbb{Z}$; for the case $\nu \in \mathbb{Z}$ one can choose simply a circle. Right: plot of $n = 64$ points in critical configuration for the Bessel case with $A = 1 - (2+i)z$, $B = z^2$.

and γ_ℓ are suitable contours that extend from a zero of B to another (or to infinity) and described in the next section. If follows from a judicious use of Cauchy integral theorem and counting that there are $d = \max\{b-1, a\}$ such linearly independent contours; this integer d is also the total degree of the poles of the meromorphic differential $\theta'(z)\mathrm{d}z$ on the Riemann sphere, minus 2.

For brevity we will denote by $\Gamma = \sum_{\ell=1}^{d} s_\ell \gamma_\ell$ the element of a suitable homology space and simply denote by $\int_\Gamma$ the operation $\sum s_\ell \int_{\gamma_\ell}$.

Remark 5.2. We warn the reader that the symbol θ as defined in (5.18) may be regular at some (or all) of the zeros of B if they simplify in the ratio that appears in (5.18); in particular this means that there may be different moment functional with the same symbols but a different number d of contours of integration. This happens when there are "hard-edge" contours of integration. An example is the case of the Jacobi polynomials with $\alpha = \beta = 0$; in this case the symbol is $\theta = 0$ and the moment functional $\mathcal{M}[p] = \int_{-1}^{1} p(x)\mathrm{d}x$ satisfies

$$\mathcal{M}\Big[(x^2-1)p'(x)\Big] = \mathcal{M}\Big[-2xp(x)\Big], \quad \forall p \in \mathbb{C}[x], \tag{5.19}$$

so that $B = x^2 - 1$ and $A = -2x$. This follows from a simple integration by parts, where the contribution of the boundary evaluation vanishes thanks to the fact that $B(\pm 1) = 0$.

5.2.1 Directions of steepest descent, contours and dual contours

To provide a complete and general description of the moment functionals we need to introduce a notion used in [Ber03] which defines dual homologies and a non-degenerate intersection pairing. The notion is, interestingly, related to the notion of *bilinear concomitant* of a pair of differential equations, one the (formal) adjoint of the other; details of this connection can be found in *loco citato*.

We have already encountered dual contours in the case of Freud-type symbols in section 5.1.3. The general case follows a similar logic. We report a canonical choice of contours $\gamma_1, \dots, \gamma_d$ as defined in [BEH06, MS61, Ber03]. We say that a zero of B is a *visible singularity* if it is a pole of θ'; the zeros of B that simplify in the ratio that defines the symbol (5.18) will be called *hard-edges*. Note that these latter are necessarily simple zeros of B due to the assumption that A, B are relatively prime. We write the partial fraction decomposition

$$\theta'(z) = -\frac{A(z) + B'(z)}{B(z)} = U'_\infty(z) + \sum_{j=1}^{k} U'_j(z) \tag{5.20}$$

where $U'_\infty(z)$ is a polynomial of degree $d_\infty - 1$ with $d_\infty := \deg A - \deg B + 1$, and $U'_j(z)$ are polynomials in $(z-c_j)^{-1}$ of degree at most $d_{c_j}+1 = \mathrm{ord}_{c_j} B$, where c_j are the visible singularities. If $\deg A < \deg B$, then the term $U'_\infty(z)$ is simply zero. The local behaviour of $\theta(z)$ for each visible singularity $c = c_j$ of order d_{c_j} is

$$\theta(z) = \frac{T_j}{(z-c_j)^{d_{c_j}}}(1 + \mathcal{O}(z-c_j)) + r_j \log(z-c_j), \quad z \to c_j, \tag{5.21}$$

where

$$r_j = \operatorname*{Res}_{z=c_j} \theta'(z)\mathrm{d}z. \tag{5.22}$$

Definition 5.3 (Local directions of steepest descent)**.** Let $c \in \mathbb{P}^1$ be a pole of $\theta'(z)\mathrm{d}z$.

- Suppose that c has order $d_c + 1 \geq 2$ so that

$$\theta'(z)\mathrm{d}z = \frac{T_c}{\zeta_c^{d_c+1}}(1 + \mathcal{O}(\zeta_c))\mathrm{d}\zeta_c \tag{5.23}$$

  where T_c is the coefficient of the leading singularity in the local parameter ζ_c. [1] We denote by

$$c^{(\ell)} := \left\{ \arg(\zeta_c) = \frac{-\arg T_c}{d_c} + \ell \frac{\pi}{d_c} \right\}, \quad \ell = 0, 1, \ldots, 2d_c - 1. \tag{5.24}$$

  Namely, $\mathrm{Re}\,\theta$ tends to $-\infty$ along the directions $c^{(2k)}$ and to ∞ along the odd directions $c^{(2k+1)}$. These will be called local directions of steepest descent, ascent (respectively). Note that $c^{(2d_c)} = c^{(0)}$.

- Suppose that c is a simple pole of $\theta'(z)\mathrm{d}z$. We will denote its residue by

$$r_c = \operatorname*{Res}_{c} \theta'(z)\mathrm{d}z. \tag{5.25}$$

  A simple pole with $\mathrm{Re}(r_c) > -\frac{1}{2}$ will be called "end-pole", and "flag-pole" if $\mathrm{Re}(r_c) < -\frac{1}{2}$. If $r_c = -\frac{1}{2}$ we can consider the pole "end" or "flag" depending on later convenience.

The motivation behind this definition is that we can integrate the weight function $w(z) = \mathrm{e}^{\theta(z)}\mathrm{d}z$ on contours that approach a pole c along the steepest descent directions (if $d_c \geq 1$) or along any direction in the case of end-poles ($d_c = 0$, $\mathrm{Re}\, r_c > -1$) while obtaining a well defined integrable integrand. Consequently we will denote by $\int_{c^{(\ell)}}^{s^{(m)}}$ an integration along a path that approaches the poles c and s along the specified directions $c^{(\ell)}$ and $s^{(m)}$.

For each pole $c \in \mathbb{P}^1$ we now describe certain contours emanating from it. We give first the description of these contours for the case $\deg A \geq \deg B$ so that $d_\infty \geq 1$ and U_∞ is a polynomial of degree at least 1.

Definition 5.4 (Contours γ_j)**.** For each pole $c \in \mathbb{P}^1$ of $\theta'(z)\mathrm{d}z$ we define a number of contours depending on the order of the pole.

- For each pole $c \neq \infty$ of order $d_c + 1 \geq 2$ we choose d_c contours ("petals") approaching c along the consecutive steepest descent directions $c^{(2k)}$, $c^{(2k+2)}$. We also pick a contour ("stem") extending from $c^{(0)}$ to a steepest descent direction at ∞.

- For the pole at $c = \infty$ we choose $d_\infty - 1$ contours between $[\infty^{(2k)}, \infty^{(2k+2)}]$, $k = 0, \ldots, d_\infty - 2$, with the definition of the steepest descent directions as in Def. 5.3. Note that there is no contour between $\infty^{(2d_\infty-2)}$ and $\infty^{(0)}$ (see Fig. 5.4) [2]. These contours will be taken so as to leave all zeros of B on their left region.

[1] To avoid lengthy case distinctions, the local parameter near a point in the finite complex plane $c \in \mathbb{C}$ is simply $\zeta_c = z - c$ while if $c = \infty$ the local parameter will be $\zeta_\infty = \frac{1}{z}$.

[2] Note that for the Freud weight $A(z) = z^a$ we have $d_\infty = a + 1 = d + 1$.

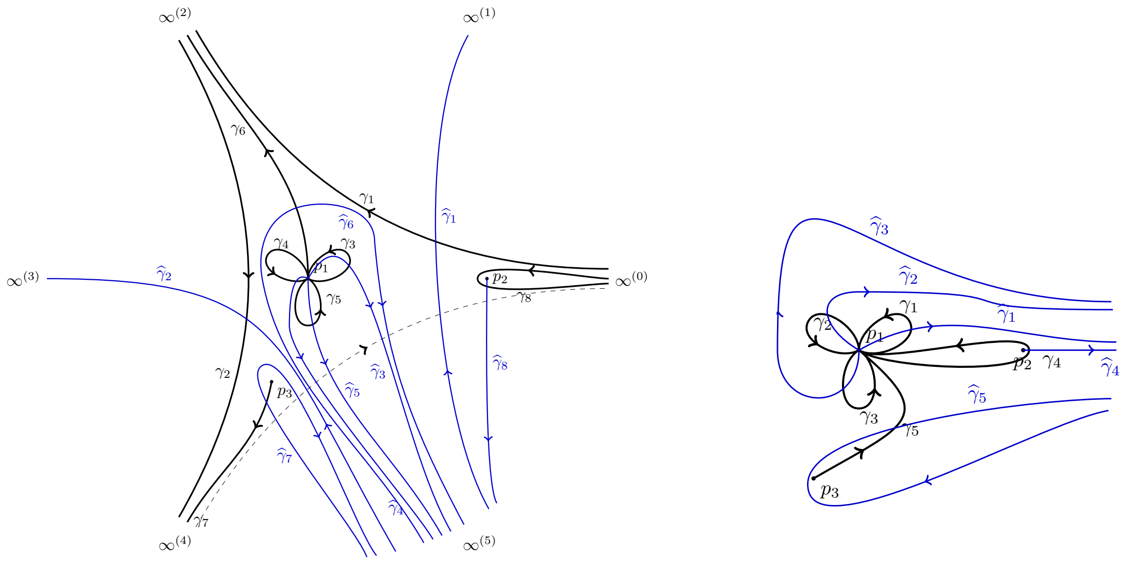


Figure 5.4: Left: The contours in the case with $d_\infty = 3$ and $d_1 = 3$, (the center of the "shamrock", p_1), $d_2 = d_3 = 0$. We also have a flag-pole (the point surrounded by the "lasso", p_2) and an end-pole (the endpoint of γ_7, p_3). In this example $B(z) = (z - p_1)^4(z - p_2)(z - p_3)$ and A is a relatively prime polynomial of degree 8 with positive leading coefficient. Right: the case where B is as above but A is a polynomial of degree less than that of B. The dual contours go to infinity in an arbitrary direction.

- For each end-pole c (including the hard-edges, i.e. the zeros of B that are not poles of θ') we pick a contour from c to ∞ along a steepest descent direction.
- For each flag-pole c with non-integer residue we choose a contour ("lasso") coming from $\infty^{(2k)}$, circling c in the counter-clockwise direction and returning to $\infty^{(2k)}$, where k is any choice. If the flag-pole has negative integer residue, we can replace this choice by a small circle. We will consider all poles with $r_c = -\frac{1}{2}$ to be flag-poles.

It is understood that the contours are chosen by avoiding the zeros of B except possibly at the endpoints, and are chosen so that they do not intersect each others except, possibly, at endpoints.

Dual contours and intersection pairing. For reasons that will become apparent but that already motivated a similar construction in [Ber03], we need to define "dual contours" and a notion of intersection pairing.

To give a sense of the motivation, we mention that the space of semiclassical moment functionals of type (A, B) is in duality with that of type $(-A - B', B)$: the weight functions are

$$(A, B) \mapsto \mathrm{e}^{\theta(z)}, \qquad \theta'(z) = -\frac{A(z) + B'(z)}{B(z)} \tag{5.26}$$

$$(-A - B', B) \mapsto \frac{\mathrm{e}^{-\theta(z)}}{B(z)} = \mathrm{e}^{\widehat{\theta}(z)}, \qquad \widehat{\theta}'(z) = -\theta'(z) - \frac{\mathrm{d}}{\mathrm{d}z}\log B(z) = \frac{A(z)}{B(z)}. \tag{5.27}$$

This duality maps the steepest descent directions of one symbol function into the ascent directions of the other, the end-poles (including hard-edges) of one into the flag-poles of the other, and viceversa. [3]

The simplest explanation of the duality is by considering a generating function $\Phi(\lambda) = \mathcal{M}[\mathrm{e}^{\lambda z}]$ of type (A, B) and a generating function $\Psi(\lambda) = \widehat{\mathcal{M}}[\mathrm{e}^{-\lambda z}]$ of the dual type defined similarly to $\mathcal{M}$ but with

[3] Since we have stipulated that the poles c with $r_c = -\frac{1}{2}$ are considered flag-poles for the functional $\mathcal{M}$, then for the dual functional they will be treated as end-poles.

the symbol $\widehat{\theta}$ (5.26) and integration along suitable contours (Def. 5.5). A simple exercise shows that they satisfy the two adjoint differential equations:

$$\left(\lambda B\left(\frac{\partial}{\partial\lambda}\right) - A\left(\frac{\partial}{\partial\lambda}\right)\right)\Phi(\lambda) = 0, \qquad \left(B\left(-\frac{\partial}{\partial\lambda}\right)\lambda - A\left(-\frac{\partial}{\partial\lambda}\right)\right)\Psi = 0. \tag{5.28}$$

The solution spaces of two adjoint equations are put in duality by the *bilinear concomitant* [Inc44]: for equations with linear coefficients in λ the bilinear concomitant has a homological interpretation as intersection pairing of the dual contours that we are describing here and are illustrated in Fig. 5.4.

Definition 5.5 (Dual contours $\widehat{\gamma}_j$)**.** For each pole $c \in \mathbb{P}^1$ $\theta'(z)\mathrm{d}z$ we define a number of contours dual to the contours γ_j in Def. 5.4.

- For each pole c of order $d_c + 1 \geq 2$ we choose d_c contours ("anti-petals") originating from c along the directions $c^{(2k+1)}, k = 0, \dots, d_c - 1$ and extending to $\infty^{(2d_\infty - 1)}$. We also pick a contour ("anti-stem") lassoing c from $\infty^{(2d_\infty-1)}$ and intersecting only the corresponding stem.
- For the pole at $c = \infty$ we choose $d_\infty - 1$ contours $[\infty^{(2d_\infty-1)}, \infty^{(2k+1)}]$, $k = 0, \dots, d_\infty - 2$. Note that they can be arranged so as to intersect only the corresponding steepest descent contours (see Fig. 5.4).
- For each end-pole c of the original functional (which is a flag-pole for the dual functional) we pick a lasso from $\infty^{(2d_\infty-1)}$. If the residue of c in $\theta'\mathrm{d}z$ is a positive integer (i.e. a zero of the weight e^{θ} (end-pole) and a pole of the dual weight $\mathrm{e}^{-\theta}/B$), we will choose a circle.
- For each flag-pole c we choose a contour $[c, \infty^{(2d_\infty-1)}]$ this includes the poles, if any, where $r_c = -\frac{1}{2}$ (which were considered as flag-poles).

The construction is such that for each contour γ_j there is exactly one dual contour $\widehat{\gamma}_j$ that intersects γ_j at a single point γ_j and has no intersections with any other contour. The orientations are chosen so that

$$\widehat{\gamma}_j \circ \gamma_k = \delta_{jk}. \tag{5.29}$$

5.2.2 The case $\deg A < \deg B$

In this case we assume that B has at least one zero, $z = p_0$, of multiplicity higher than 1. Note that in this case the condition that A and B are relatively prime prevents A to be a multiple of B'; this means also that the moment functional is of infinite rank. See Remark 5.6. Then the symbol θ has a pole at $z = p_0$ with at least one steepest descent direction and one steepest ascent direction. We keep the same terminology. The contours will be chosen similarly as in the previous case but with the "stems", "lassoes", and the contours to the flag-poles extending, instead to $p_0^{(1)}$ (the point p_0 along the first-steepest ascent direction) instead of $\infty^{(2d_\infty-1)}$. The dual contours are exactly as before, extending to ∞ (in any direction, for example the positive axis). The reason for this definition of dual contours in this case is motivated by the use that we need to make of them in the main Theorems 5.10, 5.12.

We will not treat the case where all the zeros of B are simple (the classical Heine-Stieltjes electrostatic problem); in this case the contours should be chosen, generically (i.e. for non-integer residues of θ') as Pochhammer contours (group-commutators of the generators in the fundamental group of the plane minus the zeros of B) There are case distinctions according to whether the residues of θ' at these poles are positive integers, negative integers or neither which complicate the description.

Remark 5.6. The case when $\deg A < \deg B$ allows in general the possibility that A is a multiple of B'. If A is an *integer* multiple of B', $A = kB'$ with $k = 1, 2, 3\dots$ then the weight of the semiclassical moment functional is

$$A = kB' \qquad \Rightarrow \qquad \mathrm{e}^{\theta(z)} = \frac{1}{B(z)^{k+1}}. \tag{5.30}$$

In this case the moment functionals $\mathcal{M}$ are of finite rank (they are linear combinations of derivatives of Dirac delta functions supported at the zeroes of B, with the order of derivative being equal to k times the multiplicity of said zero).

Then it follows that the orthogonal polynomials of degree $n \geq (k+1)\deg B$ are all the polynomials divisible by B^{k+1}, a type of solution that we could call "improper".

5.3 Characterisation of degenerate orthogonal polynomials

5.3.1 ℓ-degeneracy

It is well known that $P_n(z)$ is an orthogonal polynomial if and only if a certain Hankel determinant is non-vanishing [Sze75, Chi78, Dei99]. A similar result holds in the case of degenerate orthogonal polynomials, however, we require the vanishing of some additional determinants.

Lemma 5.7. *The orthogonal polynomial $P_n(z)$ is ℓ–degenerate ($\ell \geq 1$) if and only if the following determinants vanishes:*

$$D_{n+1,k}(\mathbf{s}) := \det H_{n+1,k} = 0, \quad k = 0, 1, \dots, \ell-1, \tag{5.31}$$

where $H_{n+1,k}$ are the matrices

$$H_{n+1,k} := \left[\begin{array}{cccc} \mu_0 & \mu_1 & \dots & \mu_n \\ \mu_1 & \mu_2 & \dots & \mu_{n+1} \\ \vdots & & & \\ \mu_{n-1} & \dots & & \mu_{2n-1} \\ \hline \mu_{n+k} & \mu_{n+k+1} & \dots & \mu_{2n+k} \end{array}\right]. \tag{5.32}$$

Proof. Using the determinantal expression (3.5) for the orthogonal polynomial $P_n(z)$ and distributing the integration over the last row we get:

$$\langle P_n, z^{n+k} \rangle = \int_\Gamma z^{n+k} \det \begin{bmatrix} \mu_0 & \mu_1 & \dots & \mu_n \\ \mu_1 & \mu_2 & \dots & \mu_{n+1} \\ \vdots & & & \\ \mu_{n-1} & \dots & & \mu_{2n-1} \\ 1 & z & \dots & z^n \end{bmatrix} \mathrm{d}z \tag{5.33}$$

$$= \det \begin{bmatrix} \mu_0 & \mu_1 & \dots & \mu_n \\ \mu_1 & \mu_2 & \dots & \mu_{n+1} \\ \vdots & & & \\ \mu_{n-1} & \dots & & \mu_{2n-1} \\ \int_\Gamma z^{n+k}\mathrm{d}z & \int_\Gamma z^{n+k+1}\mathrm{d}z & \dots & \int_\Gamma z^{2n+k}\mathrm{d}z \end{bmatrix} = D_{n+1,k}(\mathbf{s}) \tag{5.34}$$

Thus $Pn(z)$ is ℓ-degenerate if and only if $D_{n+1,k}(\mathbf{s})$ vanishes for all $k = 0, 1, \dots, \ell-1$. ∎

According to Definition 1.6, a 0–degenerate polynomial is just an orthogonal polynomial (no conditions are imposed) and generically it exists. We will say that P_n is **maximally** degenerate if it is $d-1$–degenerate, where $d = \max a, b-1$. The justification of the terminology is that the condition of ℓ–degeneracy imposes ℓ homogeneous polynomial equation constraints on the parameters $s_1, \dots, s_d$ and hence, generically, we can impose at most $d-1$ such constraints while expecting to have solutions.

The notion of ℓ-degeneracy is made relevant by the following proposition.

Proposition 5.8. *Let $\mathcal{M}$ be a semiclassical moment functional of type (A, B) with $\deg A = a, \deg B = b$ and let θ be its symbol according to (5.18). Given any polynomial $P_n(z)$ we set*

$$F(z) := \sqrt{B(z)} P_n(z) \mathrm{e}^{\frac{1}{2}\theta(z)}, \tag{5.35}$$

$$G(z) := \sqrt{B(z)} R_n(z) \mathrm{e}^{-\frac{1}{2}\theta(z)} \tag{5.36}$$

$$R_n(z) := \frac{1}{2i\pi} \int_\Gamma \frac{P_n(x)\mathrm{e}^{\theta(x)}\mathrm{d}x}{(x-z)}, \tag{5.37}$$

where $\Gamma = \sum_j^d s_j\gamma_j$ and the contours γ_j have been defined in Section 5.2.1.

Then the Wronskian

$$W = \mathrm{Wr}\{F, G\} = F'G - G'F \tag{5.38}$$

is a polynomial with $\deg W \le \max\{a+n-1, b+n-2\}$. Furthermore, if P_n is an ℓ–degenerate orthogonal polynomial, then W is a polynomial of degree $d-1-\ell$, with $d = \max\{a, b-1\}$.

Proof. We first show that W does not have jump discontinuities across the contours γ_j and extends to an entire function.

A direct computation shows:

$$W = \theta' B P_n R_n + B(P_n' R_n - P_n R_n') = -\widehat{A} P_n R_n + B(P_n' R_n - P_n R_n'), \tag{5.39}$$

where, for brevity, we have set $\widehat{A}(z) = A(z) + B'(z)$. Now, let $\Gamma = \sum_j^d s_j\gamma_j$ and take $z \in \gamma_j$. The Cauchy transform R_n satisfies

$$R_n(z_+) - R_n(z_-) = P_n(z)\mathrm{e}^{\theta(z)}, \tag{5.40}$$

$$R_n'(z_+) - R_n'(z_-) = \Big(P_n'(z) + \theta'(z)P_n(z)\Big)\mathrm{e}^{\theta(z)} \tag{5.41}$$

where we approach z in the oriented contour γ_j from the $\pm$ sides correspondingly. Thus, with $\Delta R_n = R_n(z_+) - R_n(z_-)$ the jump operator, we have

$$W(z_+) - W(z_-) = -\widehat{A}P_n\Delta R_n + B(P_n'\Delta R_n - P_n\Delta R_n') \tag{5.42}$$

$$= (BP_n' - \widehat{A}P_n)\Delta R_n - BP_n\Delta R_n' \tag{5.43}$$

$$= (BP_n' - \widehat{A}P_n)P_n\mathrm{e}^{\theta} - BP_n\Big(P_n' + \theta' P_n\Big)\mathrm{e}^{\theta} = 0, \tag{5.44}$$

where we have used the equation (5.18) for θ' in the last line. This concludes the proof of the absence of discontinuities.

The only possible singularities of θ' are the zeroes of B, although they may also be regular points of θ', as per Remark 5.2. Thus from the Wronskian expression (5.39) it is clear that the only possible singularities are at the endpoints of the contours γ_j. In the case this contours is petal or stem, then the integrand tends to zero exponentially and hence the singularity of R_n is at worst logarithmic. In the case of an end-pole then the integrand in the definition of R_n behaves as $(w-c)^{r_c}$ and hence the Cauchy transform has at worst growth bounded by

$$\max\{|z-c|^{\mathrm{Re}(r_c)}, |\log|z-c||\} \tag{5.45}$$

see [Gak90] for details. This shows that *a priori*, the expression (5.39) for the Wronskian may have at worst an isolated singularity at the zero $z = c$ with growth bounded by $|z-c|^{\mathrm{Re}(r_c)}$ (if $\mathrm{Re}(r_c) \in (-1, 0)$). So it actually must have a removable singularity and W extends analytically also at the zeroes of B. This shows that the Wronskian W is entire.

Noting that $R_n(z) = \mathcal{O}\left(z^{-1}\right)$ it follows that W is a polynomial of order at most $\max\{a+n-1, b+n-2\}$, as claimed.

Finally, suppose that P_n is an ℓ–degenerate polynomial. This means that the Cauchy transform R_n is of order $\mathcal{O}(z^{-n-1-\ell})$ as $|z| \to \infty$. Then, from (5.39) we see that we have

$$W(z) = \mathcal{O}(z^{\max\{a-\ell-1, b-2-\ell\}}), \qquad |z| \to \infty. \tag{5.46}$$

This completes the proof. ■

Counting the number of solutions

A naïve counting would suggest, based on Bézout's theorem, that there are $(n+1)^{d-1}$ solutions of the set of equations that characterize the maximal degeneracy:

$$D_{n+1,0}(\mathbf{s}) = 0, \quad \dots \quad , D_{n+1,d-2}(\mathbf{s}) = 0. \tag{5.47}$$

However, the conditions of Bézout's theorem are not satisfied due to the following result.

Proposition 5.9. *Suppose that $D_{n,0} = D_{n+1,0} = 0$. Then $P_n(z) \equiv 0$; more specifically, considering $P_n(z) \in \mathbb{C}[z] \otimes \mathbb{C}[\mu_0, \dots, \mu_{2n}]$ then $P_n(z)^2$ lies in the ideal generated by $D_{n,0}$ and $D_{n+1,0}$. In particular we also have $D_{n+1,k} = 0,\ \forall k \in \mathbb{N}$.*

Proof. Consider the expression for $P_n(z)$:

$$P_n(z) = \det \begin{bmatrix} \mu_0 & \mu_1 & \dots \mu_{n-1} & \mu_n \\ \mu_1 & \mu_2 & & \vdots \\ \vdots & & & \\ \mu_{n-1} & \mu_n & \dots \mu_{2n-2} & \mu_{2n-1} \\ 1 & z & \dots & z^n \end{bmatrix} = C_n z^n - C_{n-1} z^{n-1} + \dots + (-1)^n C_0. \tag{5.48}$$

We are going to show that $C_k C_r \in \langle D_{n,0}, D_{n+1,0} \rangle$ (the ideal generated by the two polynomials) for all $k, r = 0, \dots, n$; this immediately implies the two subsequent statements.

Recall the (general) Desnanot-Jacobi identity; if M is an $n \times n$ matrix and J, K are two subsets of $\{1, \dots, n\}$ of the same cardinality r (and listed in increasing order) then we denote by $M^{J;K}$ the $(n-r) \times (n-r)$ matrix obtained by deleting the rows indexed by J and the columns indexed by K. Then the identity reads

$$\det M \det M^{[a,b];[c,d]} = \det M^{[a];[c]} \det M^{[b];[d]} - \det M^{[b];[c]} \det M^{[a];[d]}. \tag{5.49}$$

Let H be the Hankel matrix of the moments of size $n+1$; then from (5.49) we obtain

$$\det H \det H^{[j,n+1];[k,n+1]} = \det H^{[j];[k]} \det H^{[n+1];[n+1]} - \det H^{[j];[n+1]} \det H^{[n+1];[k]}. \tag{5.50}$$

The last term in the right side of (5.50) is precisely $C_j C_k$ so that we have

$$C_j C_k = \left(\det H^{[j];[k]}\right) D_{n,0} - \left(\det H^{[j,n+1];[k,n+1]}\right) D_{n+1,0} \tag{5.51}$$

This proves that all the quadratic expressions belong to the claimed ideal. ∎

However, the conditions of Bézout's theorem are not satisfied because Proposition 5.9 implies that the equations (5.47) have a common component consisting of the intersection $D_{n+1,0} = 0 = D_{n,0}$. Hence there are infinitely many solutions of (5.47) in $[s_1 : \dots : s_d] \in \mathbb{P}^{d-1}$ as long as $d \geq 4$. On the other hand, as noted prior to the mentioned Proposition, this common component does not yield a meaningful solution to the Stieltjes–Fekete problem because $P_n(z)$ is the identically zero polynomial.

In principle we should only count the solutions outside this locus, i.e. with $D_{n,0} \neq 0$; a bit frustratingly, however, we cannot exclude that there are other common components, except that experiments suggest that this is not the case. This makes the application of general techniques of *Fulton's excess intersection formulas* difficult to implement. A semi-heuristic argument (to be formalized in [Mas]) would suggest that *generically* the number of solutions should be the number of solutions of the equation $k_1 + k_2 + \dots + k_d = n$ with k_j non negative integers, namely $(n+1)(n+2)\cdots(n+d-1)/(d-1)!$.

5.3.2 Degenerate OP ⇒ Stieltjes-Bethe equations

In view of Proposition 5.8 we deduce that if P_n is a maximally degenerate orthogonal polynomial, i.e. $d-1$-degenerate, then the Wronskian W of F and G in (5.37) is *constant*. We used this same argument in Theorem 4.12, see (4.80).

This crucial property, together with Lemma 4.11 will allow us to prove half of Theorem 1.8. We remind the reader the Lemma 4.11, which was established in section 4.2, guarantees that the potential of a 2nd order ODE is analytic provided both of its solutions are also analytic.

Theorem 5.10. *Suppose that P_n is a maximally degenerate polynomial of degree n for a semiclassical moment functional $\mathcal{M}$ of type (A, B) and with symbol θ. Then:*

(1) *The function* $F(z) = \sqrt{B(z)}P_n(z)e^{\frac{1}{2}\theta(z)}$ *solves the differential equation*

$$F''(z) - V(z)F(z) = 0 \tag{5.52}$$

where the potential $V(z)$ *is a rational function of the form:*

$$V(z) = \frac{1}{2}\theta'' + \frac{1}{4}(\theta')^2 + \frac{B''}{2B} - \left(\frac{B'}{2B}\right)^2 + \frac{B'}{2B}\theta' + \frac{Q}{B}, \quad \deg Q \le d-1 \tag{5.53}$$

with poles only at the zeros of $B(z)$ *which are of order at most twice the order of the zeroes of* $B(z)$. *Equivalently the polynomial* P_n *satisfies:*

$$B(z)P_n'' - A(z)P_n' - Q(z)P_n(z) = 0. \tag{5.54}$$

If $B \equiv 1$ *the potential* V *is a polynomial of degree* $2\deg A$.

(2) *Let* $\{z_1, \ldots, z_n\}$ *be the roots of* P_n*; then they satisfy the Stieltjes-Bete equations:*

$$\sum_{j\neq\ell}^{n} \frac{1}{z_\ell - z_j} = \frac{A(z_\ell)}{2B(z_\ell)}, \qquad \ell = 1, \ldots, n. \tag{5.55}$$

Proof.

(1) By Proposition 5.8 the Wronskian of F, G defined in (5.35) is a constant (and necessarily non-zero). Then we can write (up to a rescaling)

$$W = FG' - F'G = 1 \qquad \Rightarrow \qquad W' = FG'' - F''G = 0. \tag{5.56}$$

This means that $\frac{F''}{F} = \frac{G''}{G}$ and thus we can recast the equation (5.56) as a differential equation of the form

$$y''(z) - V(z)y(z) = 0, \qquad V := \frac{F''}{F} = \frac{G''}{G}. \tag{5.57}$$

In principle the potential V in (5.57) is a rational function with poles at all the zeroes of F (i.e. the zeroes of P) as well as at the zeros of B. We will show that in fact V is analytic at the zeros of P.

To see this we observe that (5.57) has both F and G as solutions. The function G as presented in (5.36) has discontinuities across the contours γ_j's that are proportional to F so we can analytically continue G to the universal cover of the plane minus the zeros of B.

Let $P_n(z) = \prod_{j=1}^{n}(z - z_j)$; then both F and the analytic continuation of G are locally analytic near z_j, $j = 1, \ldots, n$, which is therefore an *apparent singularity* [4] of the equation (5.57). Then Lemma 4.11 says that V must be locally analytic at z_j. This holds for all the roots of the $P_n(z)$, so V must be analytic at all $z_j, j = 1, \ldots, n$. By definition of V we have

$$V(z) = \frac{F''(z)}{F(z)} = \frac{1}{2}\theta'' + \frac{1}{4}(\theta')^2 + \frac{B''}{2B} - \left(\frac{B'}{2B}\right)^2 + \frac{B'}{2B}\theta' + \frac{P_n''}{P_n} + \frac{\theta' P_n'}{P_n} + \frac{B' P_n'}{BP_n}. \tag{5.58}$$

Since V in (5.58) has no poles at the zeros of P_n, the last three terms in (5.58) are of the form

$$\frac{P_n''}{P_n} + \frac{\theta' P_n'}{P_n} + \frac{B' P_n'}{BP_n} = \frac{Q(z)}{B(z)}, \tag{5.59}$$

with $Q(z)$ a *polynomial* of degree at most $d - 1$. The ODE (5.54) for P_n follows then by straightforward manipulations.

[4] In the literature of Sturm–Liouville equations like (5.57) it is customary to call "apparent singularity" a pole of V such that both solutions have Puiseux series expansion in half–integer powers. Here we use the terminology "apparent singularity" in the strict sense that both solutions of the equation must be locally analytic.

(2) Since we have established that V is analytic at each z_j, it then follows that the zeros of P_n must be simple since F is a non-trivial solution to a second–order ODE. Then we express the zero residue condition $\operatorname*{Res}_{z=z_j} V(z)\mathrm{d}z = 0$ with the expression (5.58) for V and obtain

$$2\sum_{\ell\neq k}\frac{1}{z_k - z_\ell} = \operatorname*{Res}_{z=z_k}\frac{P_n''(z)}{P_n(z)}\mathrm{d}z = -\theta'(z_j) - \frac{B'(z_j)}{B(z_j)} = \frac{A(z_j)}{B(z_j)}. \tag{5.60}$$

■

Remark 5.11. The equation (5.54) for P_n

$$B(z)P_n'' - A(z)P_n' - Q(z)P_n(z) = 0 \tag{5.61}$$

is a (degenerate) Lamé equation in the terminology of [DS20].

5.3.3 Stieltjes-Bethe equations ⇒ Degenerate OP

Theorem 5.12. *Suppose that, for two relatively prime polynomials $A(z)$ and $B(z)$, a solution of the Stieltjes–Fekete equilibrium problem (5.55) consists of n (necessarily distinct) points $z_1, \ldots, z_n$. In the case $\deg A < \deg B$ we make the additional assumption that the polynomial B has at least one zero of higher multiplicity and that*

$$2n > \operatorname{Re}\Lambda - 1 - \deg B, \qquad \Lambda := -\operatorname*{Res}_{z=\infty}\theta'(z)\mathrm{d}z. \tag{5.62}$$

Then the polynomial $P_n(z) = \prod_{k=1}^n (z - z_k)$ is a maximally degenerate orthogonal polynomial for the pairing (1.28), with the parameters s_j given by

$$s_j = \int_{\widehat{\gamma}_j}\frac{\mathrm{e}^{-\theta(z)}}{B(z)P_n^2(z)}\frac{\mathrm{d}z}{2\pi\mathrm{i}}. \tag{5.63}$$

Here $\widehat{\gamma}_j$ is the dual path to γ_j in the homology as defined in Section 5.2.1.

Proof. The proof is mostly a back-tracking of the proof of Theorem 5.10. First of all the condition (5.55) is stating that the expression for $V(z)$ in (5.58) with $F(z) = \sqrt{B(z)}P_n(z)\mathrm{e}^{\frac{1}{2}\theta(z)}$, yields an analytic expression at all zeros of P_n. Now, with V given by (5.58) we are seeking the linearly independent solution of the differential equation:

$$y(z)'' - V(z)y(z) = 0, \qquad V(z) = \frac{F(z)''}{F(z)}. \tag{5.64}$$

Using Abel's theorem (stating that the Wronskian of two solutions of (5.64) is a constant) we can write the second linearly independent solutionv as:

$$G_q(z) = F(z)\int_q^z \frac{\mathrm{d}w}{F(w)^2}. \tag{5.65}$$

The basepoint q of integration in (5.65) can be chosen arbitrarily and different choices of basepoint amount to adding to G_q a multiple of F.

The differential $F(w)^{-2}\mathrm{d}w$ has double poles at the zeros of P_n but no residues. This is guaranteed by the fact that the potential $V(z)$ (5.58) is analytic at the zeroes of $P_n(z)$, precisely because of the Fekete equilibrium equations (5.55). Therefore the antiderivative has simple poles without logarithmic singularity. Upon multiplication by $F(z)$ (which has simple zeros) the poles will cancel and this provides the proof that $G(z)$ is locally analytic at all zeros of P_n.

Next we consider two different cases, depending on the degrees of A and B.

Case 1: $\deg A \geq \deg B$**.** Consider the connected components of $\mathbb{C} \setminus \Gamma = \coprod_\mu \mathscr{D}_\mu$ In the region $\mathscr{D}_0$ that contains $\infty^{(2d_\infty - 1)}$ we use the latter as basepoint of integration. According to our choice of contours γ_j (Sec. 5.2.1) in every other connected component $\mathscr{D}_\mu$, such that the boundary is $\gamma_j = \partial \mathscr{D}_\mu$, there is exactly one endpoint $c^{(\mu)}$ of a dual contour $\widehat{\gamma}_j$ (this also include the possibility of a particular direction of approach at infinity $c^{(\mu)} = \infty_\mu$). In those regions we therefore define $G_{c^{(\mu)}}(z)$ by using the basepoint $c^{(\mu)}$ for integration in the formula (5.65).

Then we define a piece-wise analytic function $G(z)$ by patching together these functions $G_{c^{(\mu)}}$ in each component $\mathscr{D}_\mu$

$$G(z) = F(z) \int_{c^{(\mu)}}^z \frac{\mathrm{d}w}{F^2(w)} =: G_{c^{(\mu)}}(z), \quad z \in \mathscr{D}_\mu. \tag{5.66}$$

If γ_j is the boundary of $\mathscr{D}_\mu$, elementary calculus shows that for $z \in \gamma_j$ we have

$$G(z_+) - G(z_-) = F(z) \int_{c^{(\mu)}}^{\infty^{(2d_\infty - 1)}} \frac{\mathrm{d}w}{F^2(w)} = F(z) \int_{\widehat{\gamma}_j} \frac{\mathrm{d}w}{F^2(w)} = s_j F(z). \tag{5.67}$$

Therefore if we define

$$R_n(z) := \frac{G(z)\mathrm{e}^{\frac{1}{2}\theta(z)}}{\sqrt{B(z)}} = P_n(z)\mathrm{e}^{\theta(z)} \int_{c^{(\mu)}}^z \frac{\mathrm{e}^{-\theta(w)}\mathrm{d}w}{B(w)P_n^2(w)}, \quad z \in \mathscr{D}_\mu, \tag{5.68}$$

we obtain the following jump conditions

$$R_n(z_+) - R_n(z_-) = s_j P_n(z)\mathrm{e}^{\theta(z)}, \qquad z \in \gamma_j. \tag{5.69}$$

This allows us to express R_n as a Cauchy transform. Indeed equation (5.69) implies that

$$R_n(z) = H(z) + \sum_{j=1}^d s_j \int_{\gamma_j} \frac{P_n(w)\mathrm{e}^{\theta(w)}}{(w-z)} \frac{\mathrm{d}w}{2\pi\mathrm{i}} \tag{5.70}$$

for some *entire* function $H(z)$, which we now show to be identically zero. To see this consider the component $\mathscr{D}_k$ that contains the steepest ascent direction $\infty^{(2k+1)}$. Then the integral representation (5.68) shows that $R_n(z)$ is bounded by $\mathcal{O}(z^{-n-b+1})$ within that sector and also within the two neighbouring steepest descent sectors. Since this holds for all the other regions that contain the steepest ascent directions, we see that $R_n(z)$ decays like z^{-n-b+1} in every direction. This then implies that H must be identically zero by Liouville theorem, since the Cauchy integrals in (5.70) are already bounded by $\mathcal{O}(z^{-1})$ a priori.

Now consider the Wronskian $W(G, F)$ which is identically 1 since F and G solve the ODE (5.64). Using the decay of $R_n(z) = \mathcal{O}(z^{-1-\ell}), \quad \ell \geq 0$ we deduce from the Wronskian expression (5.39) that

$$W(G, F) = -(A + B')P_n R_n + B(P_n' R_n - P_n R_n') = \mathcal{O}(z^{d+n-1-\ell}) \quad \text{as } |z| \to \infty. \tag{5.71}$$

Since $W \equiv 1$ we conclude that actually $\ell = d + n - 1$ and $R_n(z) = \mathcal{O}(z^{-n-d})$. This implies that P_n is maximally degenerate from inspection of the Cauchy integral representation (5.70) (with $H \equiv 0$).

Case 2: $\deg A < \deg B$**.** The only point in the proof where we have used the condition $\deg A \geq \deg B$ is in the choice of the base-point of integration $\infty^{(2d_\infty - 1)}$, as described in Section 5.2.1 for this case. We now suppose that $\deg A < \deg B$ and that B has at least one multiple zero, see Section 5.2.2. The modifications to handle this case are of minor nature; since none of the contours γ_j extends to infinity if $\deg A < \deg B$, we can choose dual contours $\widehat{\gamma}_k$ that have intersection δ_{jk} with the contours γ_j and extend to infinity (in any direction) as explained in Section 5.2.2. Note that the only use we make of the dual contours is in the reasoning from formula (5.67) to the end of the above proof.

Now, the integrand in the expression (5.68) (or equivalent (5.63)) is integrable at infinity under the following conditions:

$$-2n + \operatorname{Re}\Lambda - \deg B < -1 \quad \iff \quad 2n > \operatorname{Re}\Lambda - 1 - \deg B, \qquad \Lambda := -\operatorname*{Res}_{z=\infty} \theta'(z)\mathrm{d}z. \tag{5.72}$$

The condition is not automatically guaranteed for small $n \in \mathbb{N}$ because $\operatorname{Re}\Lambda$ can be arbitrarily large. If this inequality holds, however, then the integrand is $\mathcal{O}(z^{\Lambda-2n-\deg B})$ and hence the integral with base-point at infinity is $\mathcal{O}(z^{\Lambda-2n-\deg B+1})$: it then follows that the whole expression (5.68) is $\mathcal{O}(z^{-n-\deg B+1}) = \mathcal{O}(z^{-n-\deg B+1}) = \mathcal{O}(z^{-n-d})$. This implies the maximal degeneracy of P_n from the expression (5.70), where $H \equiv 0$ is established by the use of Liouville's theorem. We then see that the rest of the reasoning as well as the proof of the expressions (5.63) proceed unimpeded.

Remark 5.13. For the case $\deg A < \deg B$ in Theorem 5.12 we have assumed that B has a double zero, as well as the condition (5.62) on the degree n of the polynomial P_n. In general it would not be very complicated to lift the condition on the multiple zero of B, the only price being additional description of contours (Pochhammer contours in the generic case). However it is less clear how to lift the bound (5.62) on the degree n (i.e. the minimal number of points for our proofs to proceed unimpeded): we thus do not know whether the theorem fails in these circumstances or only the proof would need a modification.

■

Appendix A

Technical results

In order to estimate integrals $\int \Psi^2 \mathrm{d}\zeta$ and rigorously prove Theorem 4.29 near turning points we need, as a model, to estimate the corresponding integrals for the solutions of the normal form of the ODE near a simple turning point. Namely, we need to estimate integrals involving squares of solutions of the Airy equation, which is the result of Theorem A.3.

We will begin with the following lemma.

Lemma A.1. *Let $f = f(x)$ and $g = g(x)$ be two arbitrary solutions of the Airy equation*

$$y''(x) = xy(x). \tag{A.1}$$

There exists polynomials $A_n, B_n, C_n \in \mathbb{C}[x]$ of degrees $n+1, n, n-1$, respectively, such that

$$x^n f^2 = \frac{\mathrm{d}}{\mathrm{d}x}\left(A_n f^2 - B_n (f')^2 + C_n f f'\right), \tag{A.2}$$

$$x^n f g = \frac{\mathrm{d}}{\mathrm{d}x}\left(A_n\, f\, g - B_n\, f' g' + C_n \frac{f g' + g f'}{2}\right). \tag{A.3}$$

They are related to each other by the following equations:

$$A_n(x) = xB_n(x) - \frac{1}{2}B_n''(x), \qquad B_n(x) + 2xB_n'(x) - \frac{1}{2}B_n'''(x) = x^n, \qquad C_n(x) = B_n'(x), \tag{A.4}$$

where $' = \frac{\mathrm{d}}{\mathrm{d}x}$. The first few are:

n	A_n	B_n	C_n
0	x	1	0
1	$\frac{x^2}{3}$	$\frac{x}{3}$	$\frac{1}{3}$
2	$\frac{x^3}{5} - \frac{1}{5}$	$\frac{x^2}{5}$	$\frac{2x}{5}$
3	$\frac{x^4}{7}$	$\frac{x^3}{7} - \frac{3}{7}$	$\frac{3x^2}{7}$
4	$\frac{x^2(x^3-2)}{9}$	$\frac{x(x^3+4)}{9}$	$\frac{4x^3+4}{9}$

(A.5)

Finally, their leading coefficients are given explicitly by:

$$A_n(x) = \frac{x^{n+1}}{2n+1} + \mathcal{O}\left(x^n\right), \quad B_n = \frac{x^n}{2n+1} + \mathcal{O}\left(x^{n-1}\right), \quad C_n = \frac{nx^{n-1}}{2n+1} + \mathcal{O}\left(x^{n-2}\right) \tag{A.6}$$

Proof. We will first prove the identity (A.2). Let A, B, C be polynomials of degrees at most $n+1, n, n-1$ respectively and consider the ansatz

$$J = Af^2 - B(f')^2 + Cff'. \tag{A.7}$$

Differentiating and using that f is a solution of Airy's equation, we get

$$\mathcal{J}(f) := J' = \left(A' + xC\right)f^2 + \left(2A - 2xB + C'\right)ff' + \left(C - B'\right)(f')^2 \tag{A.8}$$

where we consider $\mathcal{J}$ to be a quadratic form in f and f'. Thus to satisfy the identity (A.2) $\mathcal{J}(f) = x^n f$, we seek polynomials A, B, C satisfying

$$A' + xC = x^n, \quad 2A - 2xB + C' = 0, \quad C = B'. \tag{A.9}$$

From this equations we obtain the third expression $C = B'$. Then it simplifies to

$$A' = x^n - xB', \quad A - xB + \frac{1}{2}B'' = 0, \tag{A.10}$$

from which we obtain that $A = xB - B''/2$. Finally, differentiating the second equation and substituting the first we obtain:

$$B + 2xB' - \frac{1}{2}B''' = x^n. \tag{A.11}$$

This last equation for B has a unique polynomial solution obtained by inverting the (finite-dimensional) linear operator on the space of polynomials of degree $\leq n$. The above holds for arbitrary $n \in \mathbb{N}$. Thus we define the A_n, B_n, C_n to be the the A, B, C obtained in this way accordingly for each n. The relationship of the leading coefficients follows by plugging into (A.2). Finally, we introduce the bilinear pairing

$$\mathcal{B}(f,g) = \frac{\mathrm{d}}{\mathrm{d}x}\left(A_n\, f\, g - B_n\, f'g' + C_n \frac{fg' + gf'}{2}\right). \tag{A.12}$$

Then second identity (A.3) follows from polarisation

$$\mathcal{B}(f,g) = \mathcal{J}(f+g) - \mathcal{J}(f) - \mathcal{J}(g) \tag{A.13}$$

since we have already established that the bilinear form $\mathcal{J}$ can be expressed using the pairing $\mathcal{J}(f) = \mathcal{B}(f,f) = x^n f^2$. ■

Now we need to establish certain results giving asymptotic estimates for particular integrals of square functions of the scaled Airy function $\mathrm{Ai}(\hbar^{-\frac{2}{3}}x)$.

Proposition A.2. *Let $f(x)$ be an analytic function in a neighbourhood of the origin.*

1. *Let $a, b > 0$. Then*

$$\int_{-a}^{b} \mathrm{Ai}^2(\hbar^{-\frac{2}{3}}x) f(x)\mathrm{d}x = i\hbar^{\frac{1}{3}} \int_{-a}^{0} \frac{f(x)\mathrm{d}x}{2\pi\sqrt{x}_+} + \mathcal{O}(\hbar^{\frac{4}{3}}), \tag{A.14}$$

   where $\sqrt{x}$ is analytic in $\mathbb{C}\backslash(-\infty, 0]$ and positive for $x > 0$ and $\sqrt{x}_+$ denotes the boundary value from the upper half plane.

2. *Let $a, b > 0$ and $j, k \in \{0, 1, 2\}$. Then*

$$\int_{-a}^{e^{i\pi/3}b} \mathrm{Ai}\left(\hbar^{-\frac{2}{3}}\omega^k x\right) \mathrm{Ai}\left(\hbar^{-\frac{2}{3}}\omega^j x\right) f(x)\mathrm{d}x = q_{jk} \int_{-a}^{0} \frac{i\hbar^{\frac{1}{3}} f(x)\mathrm{d}x}{2\pi\sqrt{x}_+} + r_{jk} \int_{0}^{e^{i\pi/3}b} \frac{i\hbar^{\frac{1}{3}} f(x)\mathrm{d}x}{2\pi\sqrt{x}} + \mathcal{O}(\hbar^{\frac{4}{3}}). \tag{A.15}$$

   Here $\omega = e^{\frac{2\pi i}{3}}$ and $\sqrt{x}$ and $\sqrt{x}_+$ are defined as above and the constants q_{jk} and r_{jk} are determined below:

$$q_{jk} = \begin{cases} 1 & j=0,\ k=0, \\ \frac{e^{\frac{i\pi}{3}}}{2} & j=0,\ k=1, \\ \frac{e^{-\frac{i\pi}{3}}}{2} & j=0,\ k=2, \\ 0 & j=1,\ k=1, \\ \frac{1}{2} & j=1,\ k=2, \\ 0 & j=2,\ k=2, \end{cases} \qquad r_{jk} = \begin{cases} 0 & j=0,\ k=0, \\ -\frac{e^{\frac{i\pi}{3}}}{2} & j=0,\ k=1, \\ \frac{e^{-\frac{i\pi}{3}}}{2} & j=0,\ k=2, \\ e^{-\frac{i\pi}{3}} & j=1,\ k=1, \\ \frac{1}{2} & j=1,\ k=2, \\ 0 & j=2,\ k=2. \end{cases} \tag{A.16}$$

Proof. We will consider $f(x) = x^n$, with the proof being completed simply by summing the Taylor series.

1. We need to expand the integral

$$\int_{-a}^{b} \mathrm{Ai}^2(\hbar^{-\frac{2}{3}}x)x^n \mathrm{d}x \tag{A.17}$$

We perform the change of variable $\xi = \hbar^{-\frac{2}{3}}$, and then use Lemma A.1 exploiting that $\xi^n \mathrm{Ai}^2(\xi)\mathrm{d}\xi$ is an exact differential:

$$\int_{-a}^{b} \mathrm{Ai}^2(\hbar^{-\frac{2}{3}}x)x^n \mathrm{d}x = \hbar^{\frac{2(n+1)}{3}} \int_{-\alpha}^{\beta} \mathrm{Ai}^2(\xi)\xi^n \mathrm{d}\xi \tag{A.18}$$

$$= \hbar^{\frac{2(n+1)}{3}} \left(A_n(\xi)\mathrm{Ai}^2(\xi) - B_n(\xi)\left(\mathrm{Ai}'(\xi)\right)^2 + C_n(\xi)\mathrm{Ai}(\xi)\mathrm{Ai}(\xi)' \right)\Bigg|_{-\alpha}^{\beta}, \tag{A.19}$$

where we denote the scaled limits of integration $\alpha = a\hbar^{-\frac{2}{3}}, \beta = b\hbar^{-\frac{2}{3}}$.

We now remind the reader of the asymptotic expansion of the Airy function along [DLMF]

$$\mathrm{Ai}(s) = \frac{\mathrm{e}^{-\frac{2}{3}s^{\frac{3}{2}}}}{2\sqrt{\pi}s^{\frac{1}{4}}}\left(1+\mathcal{O}\left(s^{-\frac{3}{2}}\right)\right), \qquad |s|\to\infty, \quad |\arg(s)| < \pi - \epsilon \tag{A.20}$$

$$\mathrm{Ai}(-s) = \frac{\sin\left(\frac{2}{3}s^{\frac{3}{2}} + \frac{\pi}{4}\right)}{\sqrt{\pi}s^{\frac{1}{4}}}\left(1+\mathcal{O}\left(s^{-3}\right)\right) - \frac{\cos\left(\frac{2}{3}s^{\frac{3}{2}} + \frac{\pi}{4}\right)}{\sqrt{\pi}s^{\frac{1}{4}}}\left(1+\mathcal{O}\left(s^{-\frac{3}{2}}\right)\right), |s|\to\infty, \quad |\arg(s)| < \frac{2\pi}{3}. \tag{A.21}$$

It is then clear that the evaluation at $\beta = \hbar^{-\frac{2}{3}}b$ yields exponentially small terms thanks to (A.20). For the evaluation at $-\alpha = -\hbar^{-\frac{2}{3}}a$ we use instead (A.21); thanks to the leading order estimates (A.6) we thus conclude that:

$$\hbar^{\frac{2(n+1)}{3}} \int_{-\alpha}^{\beta} \mathrm{Ai}^2(\xi)\xi^n \mathrm{d}\xi = \hbar^{\frac{2(n+1)}{3}} \frac{(-\alpha)^{n+1}}{2n+1}\frac{1}{\pi\sqrt{\alpha}}\left(1+\mathcal{O}(\alpha^{-\frac{3}{2}})\right) \tag{A.22}$$

$$= \hbar^{\frac{1}{3}} \frac{(-a)^{n+1}}{2n+1}\frac{1}{\pi\sqrt{a}}\left(1+\mathcal{O}(\alpha^{-\frac{3}{2}})\right) \tag{A.23}$$

$$= \hbar^{\frac{1}{3}} \int_{-a}^{0} \frac{x^n \mathrm{d}x}{2\pi\sqrt{-x}} + \mathcal{O}(\hbar^{\frac{4}{3}}). \tag{A.24}$$

Finally, we observe that $\sqrt{-x} = -\mathrm{i}\sqrt{x}_+$ for $x \in \mathbb{R}_{<0}$.

2. All these integrals involve oscillatory direction for each of the arguments. The computation is of entirely similar nature, where it is particularly important to pay great care at the relative phases.

Setting $\mathrm{Ai}_j(\xi) := \mathrm{Ai}(\omega^j\xi)$ and using Lemma A.1 we need to estimate integrals of the form

$$\int_{-a}^{\mathrm{e}^{\frac{i\pi}{3}}b} x^n \mathrm{Ai}_j(x\hbar^{-\frac{2}{3}})\mathrm{Ai}_k(x\hbar^{-\frac{2}{3}})\mathrm{d}x = \hbar^{\frac{2(n+1)}{3}} \int_{-\alpha}^{\mathrm{e}^{\frac{i\pi}{3}}\beta} \xi^n \mathrm{Ai}_j(\xi)\mathrm{Ai}_k(\xi)\mathrm{d}\xi = \hbar^{\frac{2(n+1)}{3}} H_{jk}(\xi)\Bigg|_{-\alpha}^{\mathrm{e}^{\frac{i\pi}{3}}\beta} \tag{A.25}$$

where we define

$$H_n^{(j,k)} := \mathcal{B}(\mathrm{Ai}_j, \mathrm{Ai}_k) = \left[A_n \mathrm{Ai}_j\mathrm{Ai}_k - B_n \mathrm{Ai}_j'\mathrm{Ai}_k' - C_n\frac{\mathrm{Ai}_j'\mathrm{Ai}_k + \mathrm{Ai}_j\mathrm{Ai}_k'}{2}\right] \tag{A.26}$$

At the point $\alpha = \hbar^{-\frac{2}{3}}a$ a direct computation using the asymptotic properties (A.20), (A.21) yields:

$$-\hbar^{\frac{2(n+1)}{3}} H_n^{(j,k)}(-\hbar^{-\frac{2}{3}}a) \simeq i\int_{-a}^{0} \frac{\hbar^{\frac{1}{3}}x^n \mathrm{d}x}{2\pi\sqrt{x_+}} \times q_{jk} + \mathcal{O}(\hbar^{\frac{4}{3}}) \tag{A.27}$$

Similarly one finds

$$\hbar^{\frac{2(n+1)}{3}} H_n^{(j,k)}(\hbar^{-\frac{2}{3}}b\mathrm{e}^{i\pi/3}) \simeq i\int_{0}^{b\mathrm{e}^{i\pi/3}} \frac{\hbar^{\frac{1}{3}}x^n \mathrm{d}x}{2\pi\sqrt{x}} \times r_{jk} + \mathcal{O}(\hbar^{\frac{4}{3}}) \tag{A.28}$$

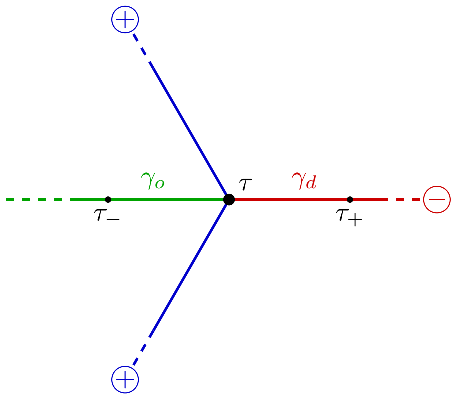


Figure A.1: Integration contours for Theorem A.3.

with q_{jk}, r_{jk} as in (A.16). Now the integral (A.15) with $f(x) = x^n$ becomes

$$\int_{-a}^{e^{i\pi/3}b} \mathrm{Ai}(\hbar^{-\frac{2}{3}}\omega^k x)\mathrm{Ai}(\hbar^{-\frac{2}{3}}\omega^j x)x^n \mathrm{d}x = \hbar^{\frac{2(n+1)}{3}} H_n^{(j,k)}(\xi)\Big|_{-\hbar^{-\frac{2}{3}}a}^{\hbar^{-\frac{2}{3}}e^{\frac{i\pi}{3}}b}. \tag{A.29}$$

Putting together the two terms we obtain the proof.

■

We now obtain the main technical estimate that will be needed to prove 4.29.

Theorem A.3. *Let τ be a simple turning point for the potential $Q(x)$, γ_d a steepest descent $\ominus$ path and γ_o the oscillatory path on the opposite side of τ. Choose two points τ_+, τ_- in the neighbourhood of τ at a finite positive distance from τ and intersecting the paths $\tau_+ \in \gamma_\mathrm{d}, \tau_- \in \gamma_\mathrm{o}$ as in Fig. A.1. Let Ψ be an entire solution of the WKB differential equation*

$$\hbar^2\Psi''(z) - Q(z)\Psi(z) = 0. \tag{A.30}$$

Suppose that Ψ is asymptotic to the recessive formal WKB solution $\psi_+^{(\tau)}$. Then

$$\int_{\tau_-}^{\tau_+} \Psi^2(z)\mathrm{d}z = \int_{\tau_-}^{\tau} \frac{2i\hbar\,\mathrm{d}z}{\sqrt{Q(z_+; s, E)}} + \mathcal{O}(\hbar^2). \tag{A.31}$$

with the branch-cut of $\sqrt{Q(z; s, E)}$ running along the oscillatory path and the determination the one that has negative real part on the descent path γ_d.

Proof. In [Vor83] it is shown that there is a conformal mapping $\xi(z; \hbar^2) \in \mathcal{O}(z) \otimes \mathbb{C}[[\hbar^2]]$ sending a turning point $z = \tau$ to $\xi(\tau; \hbar^2) \equiv 0$ identically in $\hbar$. This conformal mapping transforms the WKB-type Schrödinger equation (3.28) to the Airy equation:

$$\hbar^2\frac{\partial^2}{\partial z^2}\Psi(z) - Q(z)\Psi(z) = 0 \quad \mapsto \quad \hbar^2\frac{\partial^2}{\partial \xi^2}f(\xi) - \xi f(\xi) = 0. \tag{A.32}$$

The relationship between the two equations is as follows

$$Q(z) = \left(\frac{\mathrm{d}\xi}{\mathrm{d}z}\right)^2 \xi - \frac{\hbar^2}{2}\{\xi, z\}, \qquad \Psi(z) = \frac{f(\xi)}{\sqrt{\mathrm{d}\xi/\mathrm{d}z}} \tag{A.33}$$

where $\{\cdot,\cdot\}$ denotes the Schwarzian derivative

$$\{\xi, z\} = \frac{\xi'''}{\xi'} - \frac{3}{2}\left(\frac{\xi''}{\xi'}\right)^2, \tag{A.34}$$

where we use $' = \frac{\mathrm{d}}{\mathrm{d}z}$. The formal series $\xi(z;\hbar^2)$ is Borel–resummable to an analytic conformal mapping whose asymptotic expansion as $\hbar \to 0_+$ coincides with the formal series [KT05].

The arc $[\tau, \tau_+]$ on the contour γ_d is mapped to a segment $[0, \xi_+] \subset \mathbb{R}_+$ in the ξ–plane, and similarly the arc $[\tau_-, \tau] \subset \gamma_o$ maps to $[\xi_-, 0] \subset \mathbb{R}_-$.

The function Ψ is the recessive solution on γ_+ and hence it must map to a multiple of the Ai function within a whole neighbourhood of τ. Thus the integral to be estimated translates to

$$\int_{\tau_-}^{\tau_+} \Psi(z;\hbar)^2 \mathrm{d}z = C(\hbar) \int_{\xi_-}^{\xi_+} \frac{\mathrm{Ai}\left(\hbar^{-\frac{2}{3}}\xi\right)^2}{\xi'(z^{-1}(\xi);\hbar^2)^2} \mathrm{d}\xi, \tag{A.35}$$

where $C(\hbar)$ is an appropriate proportionality constant that depends on the chosen normalization of Ψ. Our choice is to normalize Ψ so that it is asymptotic to the recessive formal solution $\psi_+^{(\tau)}$. To fix $C(\hbar)$ we can consider the asymptotics of the Airy function at τ_+ (A.20) and compare it to the WKB solution. Indeed we find that

$$\frac{\mathrm{Ai}\left(\hbar^{-\frac{4}{3}}\xi_+\right)^2}{\xi'(z^{-1}(\xi_+);\hbar^2)} \simeq \frac{\hbar^{\frac{1}{3}} \mathrm{e}^{-\frac{4}{3\hbar}\xi_+^{\frac{3}{2}}}}{4\pi \xi'(z^{-1}(\xi_+;\hbar^2)\sqrt{\xi_+}} \simeq \frac{\hbar^{\frac{1}{3}} \exp\left(\frac{2}{\hbar}\int_{\tau}^{\tau_+} \sqrt{Q(z;s,E)}\mathrm{d}z\right)}{4\pi\sqrt{Q(\tau_+;s,E)}} \simeq \frac{1}{4\hbar^{\frac{2}{3}}\pi}\left(\psi_-^{(\tau)}\right)^2, \tag{A.36}$$

so that $C(\hbar) = 4\hbar^{\frac{2}{3}}\pi(1 + \mathcal{O}(\hbar^2))$. The integral involving the Airy function in (A.35) is of the general form of Proposition A.2 and hence, to within $\mathcal{O}(\hbar^{\frac{4}{3}})$ we have

$$\int_{\xi_-}^{\xi_+} \frac{\mathrm{Ai}\left(\hbar^{-\frac{2}{3}}\xi\right)^2}{\xi'(z^{-1}(\xi);\hbar^2)^2} \mathrm{d}\xi = \frac{i\hbar^{\frac{1}{3}}}{2\pi} \int_{\xi_-}^{0} \frac{\mathrm{d}\xi}{\xi'(z^{-1}(\xi);\hbar^2)^2 \sqrt{\xi(z^{-1}(\xi)_+;\hbar^2)}}. \tag{A.37}$$

Now from (A.33) we get

$$\xi'\sqrt{\xi} = \sqrt{Q(z)} + \mathcal{O}(\hbar^2), \tag{A.38}$$

and hence, to within $\mathcal{O}(\hbar^2)$, we have

$$\int_{\xi_-}^{0} \frac{\mathrm{d}\xi}{\xi'(z^{-1}(\xi);\hbar^2)^2\sqrt{\xi(z^{-1}(\xi);\hbar^2)^2}} = \int_{\xi_-}^{0} \frac{\mathrm{d}\xi}{\xi'(z^{-1}(\xi);\hbar^2)\sqrt{Q(z^{-1}(\xi))}} = \int_{\tau_-}^{\tau} \frac{\mathrm{d}z}{\sqrt{Q(z)}}. \tag{A.39}$$

Using the estimate of $C(\hbar)$ and substituting into (A.35) we obtain the statement of the theorem. ■

Appendix B

Overlapping patterns

In order to verify our result, we performed a series of numerical checks which validated our findings. These numerical checks worked in the following way.

For the JM case we proceed as indicated below.

1. Compute numerically in arbitrary precision the list of zeros, $\mathscr{Z}_n$, of the VY polynomial $Y_n(t)$, i.e., the poles of the rational solution u_n (2.36) of PII.

   For each of $a \in \mathscr{Z}_n$, compute the corresponding value of the parameter b appearing in (4.3). Thus for each $a \in \mathscr{Z}_n$ we can define the corresponding value of $\Lambda = \frac{7a^2}{36} + 10b$ as in (4.14). Let us call $\mathscr{L}_n$ the list of the corresponding values of Λ.

2. Let $\mathscr{S}_n := \mathscr{Z}_n/(n+\frac{1}{2})^{\frac{2}{3}}$ and $\mathscr{E}_n := \mathscr{L}_n/(n+\frac{1}{2})^{\frac{4}{3}}$ as per scaling (4.95). For each pair (s_ℓ, E_ℓ) in $\mathscr{S}_n, \mathscr{E}_n$ we construct the corresponding potential $Q(z; s_\ell, E_\ell) = z^4 + s_\ell z^2 + 2z + E_\ell$ and evaluate numerically the Voros symbols *including the subleading term*, using Prop. 4.19, along the relevant cycles of the Riemann surface of $\sqrt{Q(\zeta; s, E)}\mathrm{d}\zeta$. The numerical verification consists in checking

$$(2n+1)\oint \sqrt{Q(\zeta; s_\ell, E_\ell)}\mathrm{d}z + \frac{1}{2n+1}\oint S_1(z)\mathrm{d}z \simeq \mathrm{i}\pi + 2\pi\mathrm{i}m_\ell, \tag{B.1}$$

   i.e. an odd multiple of $\mathrm{i}\pi$. We tested up to $n = 26$ and the numerics indeed supports the formula. It is interesting to observe that the leading order computation yields (unsurprisingly) less accurate approximations of odd multiples of $\mathrm{i}\pi$, in particular in that it has a small but still non-negligible real part.

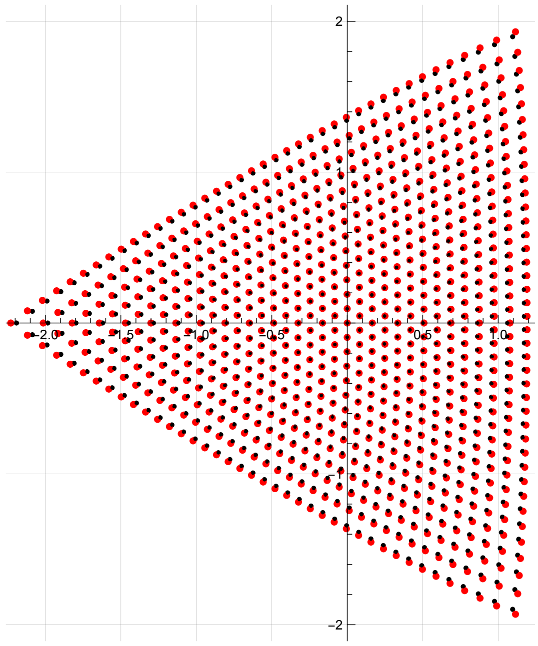
For the ST case we proceed similarly as follows.

1. Compute numerically in arbitrary precision the list of zeros, $\mathscr{Z}_n$, of the ST discriminant D_n (1.11).

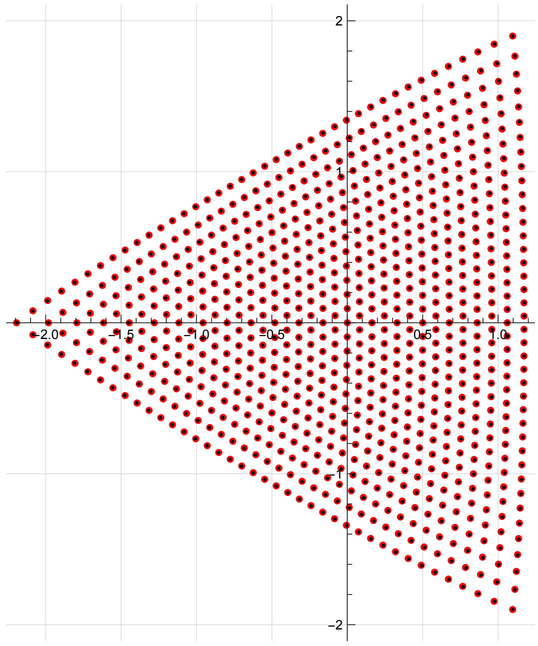
   For each of $t \in \mathscr{Z}_n$, compute the characteristic polynomial of the matrix $M_n(t)$ (1.9) and find the double eigenvalue λ: due to the numerical error in the evaluation of the zero of the dimathscriminant, one has to find the two roots that are very close to each other. Once we find λ we define $\Lambda = \lambda + \frac{t^2}{4}$ as per (4.21). We thus construct corresponding lists $\mathscr{Z}_n, \mathscr{L}_n$ as in the previous case, for the values in the ES spectrum.

2. We perform the appropriate scaling $\mathscr{S}_n := \mathscr{Z}_n/(n+1)^{\frac{2}{3}}$ and $\mathscr{E}_n := \mathscr{L}_n/(n+1)^{\frac{4}{3}}$ as per scaling (4.95). For each pair (s_ℓ, E_ℓ) in $\mathscr{S}_n, \mathscr{E}_n$ we construct the corresponding potential $Q(z; s_\ell, E_\ell) = z^4 + s_\ell z^2 + 2z + E_\ell$ and we compute the Voros symbols v_{jk} to subleading order as for the previous case. Let us remind the reader that ξ_{jk} are the Fock-Goncharov parameters defined in (4.115). Then the numerical verification consists in two checks:

   (a) We verify that the three Fock–Goncharov parameters $\xi_{10}, \xi_{20}, \xi_{30}$ satisfy the rational parametrization (4.157).



(a) Natural ST scaling $\hbar^{-1} = (n+1)^{\frac{2}{3}}$



(a) Conjectural ST scaling $\hbar^{-1} = n^{-\frac{2}{3}}$

Figure B.1: The Exactly Solvable spectrum (black) and the Vorob'ev-Yablonskii zeros (red) for $n = 40$. In both pictures the VY zeroes are scaled with $\hbar^{-1} = n + \frac{1}{2}$.

(b) We verify the equation (4.164), which returns correct within less than 1% for $n = 80$ with a higher accuracy of up to 0.01%, unsurprisingly, when we check zeros that lie away from the boundary of the elliptic region.

Discrepancy of scaling

According to our analysis, the natural scaling for the points in the ES spectrum is the one indicated in (4.94) for the ST case, and (4.95) for the VY polynomial case. In particular for the ST case the small parameter is $\hbar^{-1} = (n+1)$ and for the VY case $\hbar^{-1} = (n + \frac{1}{2})$.

If we plot the zeros with these exact scalings, we obtain the two lattices shown in Fig. B.1, on the left.

However, if we use the scaling $\hbar = n^{-\frac{2}{3}}$ for the ES spectrum, we obtain an almost perfect match as shown on the right pane of Fig. B.1.

We cannot find a mathematical justification of this coincidence, beyond what we already have proved; namely we can justify the coincidence of the two lattices in a $\mathcal{O}(\hbar)$ neighbourhood of the origin, as well as the matching, slowly modulated, local geometry of either lattices (Prop. 4.31). However we cannot quantify the reason why the "wrong" scaling seems to yield a much better match.

We have, however, verified that the discrepancy between the two (scaled) lattices decreases indeed like $\mathcal{O}(n^{-1})$ in the regions at finite distance from the origin, and as $\mathcal{O}(n^{-2})$ in the $\mathcal{O}(\hbar)$–region around the origin.

To effect this test, we choose a point, s_0 in the elliptic region and find the closest points of either lattices to s_0. Let $\Delta_n(s_0)$ be such difference; we call it the "local discrepancy". Then we plot $\ln \Delta_n(s_0)$ against $\ln n$. If we choose s_0 at finite positive distance from the origin, the slope of this graph is -1 while for $s_0 \simeq 0$ the slope is -2. This plot of local discrepancy $\Delta_n(s_0)$ is reported in Fig. B.2 in the two regimes of $s_0 \simeq 0$ and $s_0 \neq 0$, verifying that the decrease of the discrepancy is $\mathcal{O}(n^{-2})$ in the first case and $\mathcal{O}(n^{-1})$ in the second.

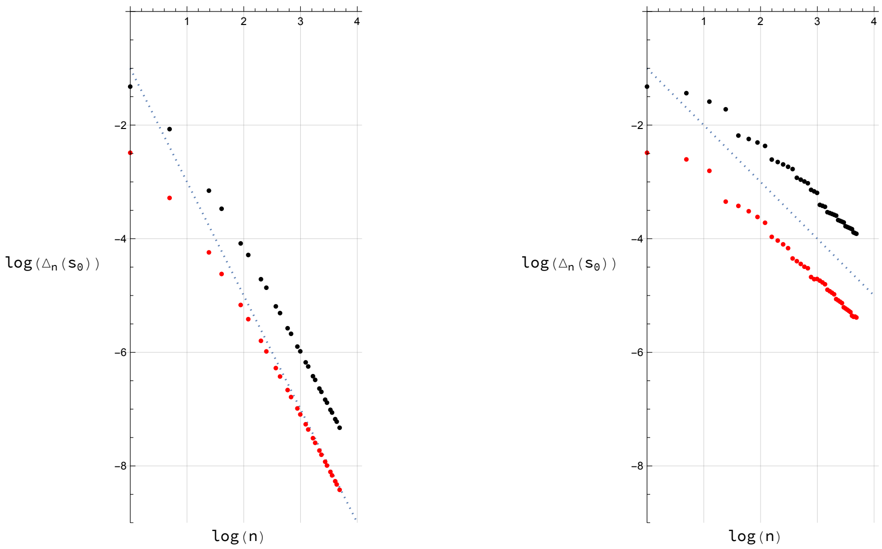


Figure B.2: The local discrepancies $\Delta_n(s_0)$ against n in a log-log plot for two values of s_0. The red dots correspond to the discrepancy with the "wrong" scaling $n^{-\frac{2}{3}}$. Although the "wrong" scaling is better, the rate of convergence is the same in the local lattices. In the left picture $s_0 = 0$ and the slope is -2 (the dotted line is plotted for reference). In the right picture $s_0 = 1 + i$ and the slope is -1 as expected.

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