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Compound Du Val singularities, five-dimensional SCFTs and GV invariants

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[1]Please take a look to the remark below, that obviusly does not refer to my personal experience with Roberto, Andrés or Alessandro.

[2]Fu cosí che l' ordine alfabetico saltó al secondo nome della lista.

[3]Alla fine viene 7, come é ben spiegato sotto.

the blowup magician[4], Red, Riccardo Ruggeri, Raurui, Riccardo Ciccone, Shani, Simona la mia sistarz, Ticcolin, Teresa, Uitto (Srappiu), Yonathan pelican, Ziruo my lawyer.
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A bitter remark I would like to conclude with a bitter but necessary note about the narration, in our society, of the academic world (at least talking about the high-energy/mathematical physics sector). When I started my bachelor degree, I got immediately fascinated by the idea of becoming a researcher. I was thrilled about the idea of working in a world full of passionate people, where no authority principle holds and whether something is true or false is decided just by the rules of mathematics and logic: if I count 2875 lines on a quintic of $\mathbb{P}^4$ I am right, whether I am a full professor or a Ph.D. student. Undoubtedly, years later I am still in love with mathematics and physics, but my judgment on the academic world went through a harsh reality check.

Academia is that world in which you, on average, have to go through at least 3-4 postdocs before getting (in your **late 30s or early 40s**) a permanent position (at least in the high-energy physics and mathematical physics sector). Academia is also the world in which you have to ask your previous employers to write you reference letters to apply for new positions. This gives them immense power over you, and you become extremely "blackmailable". This is reinforced by the fact that the average young researcher is extremely passionate about her/their/his work and hence, again, blackmailable. My experience in academia taught me, in this sense, a harsh lesson[5]:

[4] Che si merita un ringraziamento speciale, dato che é anche riuscito a spiegare qualcosa di matematica addirittura a me.

[5] My experience in academia and my friend Beatrice, during a Flixbus trip to Rome :).

passions are something wonderful, as long as they do not possess you. This means that they are wonderful as long as they do not force you to work 10 hours per day, as long as they do not force you to work during the weekends, or to live with the constant fear of not finding a contract for the next year. Creating a "mythology" that leads us to identify with our careers (and also with our passions, if they coincide with our careers) is the most effective way for this capitalist society to make us forget about something crucial[6]: we are human first, before being researchers/employees. This is something that we have to remind to people around us, but first of all to ourselves: to change the disgusting situation in the academic world, the first political action we have to do is to deconstruct this toxic narration by loving ourselves just for being "us" and not for being "researchers". If this is not the case then

- when we drop out of academia we do not just lose our job, but also our identities;
- we can not find the strength to fight against this disgusting state of precarious work that affects the academic world.

I hence apologize, to myself and to all my colleagues, for every time I overworked, every time I accepted any misbehavior by a professor, and every time I forgot to make my employee rights respected.

[6] By far, a carrot is way most effective than a stick to control people.

Contents

Abstract

In this thesis work we introduce a new method to study the dynamics of M-theory on compound Du Val threefold singularities (cDV). Incidentally, this also furnishes a new way to systematically count the Gopakumar-Vafa invariants (GV) of these geometries and, reversely, to produce threefolds whose GV invariants display required properties. Our construction is inspired by the type IIA limit of M-theory on the considered singularities and rephrases the data of the threefolds in the language of seven-dimensional super Yang-Mills theory. This, more deeply, creates a connection between the algebraic properties of the ADE algebras and the geometric properties of the cDVs. We focused our analysis on two interesting classes of compound Du Val: the simple flops and the quasihomogeneous cDVs, obtaining in both cases complete information on the GV invariants (or, equivalently, on the Higgs Branch of M-theory reduced on these singularities). We also elucidate, during this procedure, the role of exotic type IIA branes bound states, called T-branes, that lack a clear interpretation in terms of the geometry of the threefold.

Notation

We sum up briefly the notation and the abbreviations that we will use in this thesis.

- We will denote, for a supersymmetric field theory with eight supercharges, with HB the Higgs Branch of the theory, with CB the Coulomb Branch and with ECB the Extended Coulomb branch.

- We will denote, for a gauge theory, with RG flow the Renormalization Group flow of the theory, with UV the UltraViolet regime, with IR the InfraRed regime.

- We will denote with $H_{n,\text{cpct}}(X, \mathbb{Z})$ the compact support homology with integer coefficients of a manifold X. We will denote with $H^n_{\text{cpct}}(X, \mathbb{Z})$ its compact support cohomology. If the subscript "cpct" is omitted, then we are considering both compact and non-compact homology and cohomology.

- For a Calabi-Yau n-fold X, we will denote with $\tilde{X}$ its smoothing, and with $\hat{X}$ its crepant resolution. For a hypersurface singularity X, we will denote with $M(X)$ its Milnor number.

- In the context of Lie algebras, we will denote with $\mathcal{G}$ a simple Lie algebra. We will denote with $\mathcal{L}$ a Levi subalgebra of $\mathcal{G}$, and with $\mathcal{M}$ a maximal-rank maximal subalgebra of $\mathcal{L}$. We will denote with $\mathfrak{t}$ the Cartan subalgebra of $\mathcal{G}$.

We will denote with $\mathcal{M}_{\text{s.s.}} \equiv \bigoplus_h \mathcal{M}_h$ and $\mathcal{L}_{\text{s.s.}} \equiv \bigoplus_h \mathcal{L}_h$ the semisimple part of, respectively, $\mathcal{M}$ and $\mathcal{L}$. In this notation, the summands $\mathcal{M}_h$ and $\mathcal{L}_h$ will be simple Lie algebras. We will denote with $\mathcal{W}_\mathcal{M}$, $\mathcal{W}_\mathcal{L}$, $\mathcal{W}_\mathcal{G}$ the Weyl groups of respectively, $\mathcal{M}$, $\mathcal{L}$ and $\mathcal{G}$. We will call t_i the coordinates of $\mathfrak{t}$. We will denote with ϱ (up to subscripts and superscripts) the $\mathcal{W}_\mathcal{M}$-invariants coordinates (built from the t_i) on $\mathfrak{t}/\mathcal{W}_\mathcal{M}$ and with μ_i the $\mathcal{W}_\mathcal{G}$ invariant coordinates on $\mathfrak{t}/\mathcal{W}_\mathcal{G}$. If $\mathcal{M} = \mathcal{L}$, we will denote always with ϱ the $\mathcal{W}_\mathcal{M} = \mathcal{W}_\mathcal{L}$ invariant coordinates.

- We will denote with C_{ij} the Cartan matrix of a Lie algebra $\mathcal{G}$, with α its roots, and with $\langle \alpha_i^* \rangle$ the dual roots of the simple roots $\alpha_i = 1, ..., r \equiv \text{rank}(\mathcal{G})$. We will denote with e_α the root vector associated to the root α.
- The subscript x in $\mathbb{C}_x$ (and similarly for $\mathbb{C}_w, \mathbb{C}_z, \mathbb{C}^2_{w,z}$...) denotes the fact that we are using x as coordinate on $\mathbb{C}$. Similarly, the subscript "789" in $\mathbb{R}^3_{789}$ denotes the fact that we are calling x_7, x_8, x_9 the coordinates of $\mathbb{R}^3$. In general, we use subscripts if we want to indicate the names of the coordinates spanning a certain affine space.

Chapter 1

Introduction

1.1 General introduction

One of the main open questions in modern theoretical physics is to understand strongly-coupled quantum field theories (QFTs). The question is motivated by the fact that one of the four fundamental forces in nature, the strong nuclear force, is described by a theory, the quantum chromodynamics (QCD), that is strongly coupled in the IR regime. If a QFT is strongly coupled, perturbative techniques can not be applied and it is in general very hard to get quantitative results. In this sense, one is led to consider simplified models of QFTs, where there exist techniques to perform non-perturbative computations. One of the most powerful tools introduced in this sense is supersymmetry, a symmetry relating the bosonic and fermionic field content of a *supersymmetric* quantum field theory (SQFT), that permits to obtain exact results valid also at strong-coupling. Among the many possible quantities that we can compute in a SQFT we have the BPS spectrum. This is the spectrum of particles fitting in short representations of the supersymmetry algebra and is protected against quantum corrections.

Supersymmetry might sound, as presented here, just as a mere tool to perform computations in simplified models. However, its appearance is motivated by (supersymmetric) string theory, one of the most promising candidates for the quantization of gravity. String theory and its non-perturbative completions called M-theory and F-theory have an intrinsically geometric nature as they predict the existence of compact space-time dimensions (called internal dimensions), invisible at the energy scales of the currently performed experiments, whose geometry constrains the four-dimensional physics. Phrased differently, our universe is a fibration of an internal geometry X over the four-dimensional space-time that we visualize at low energies.

In certain regimes (in which we take a decompactification limit of the internal dimensions), string theory reduces to a supersymmetric quantum field theory. In this sense, string theory relates two apparently distant languages: the dynamics of supersymmetric quantum field theories and the geometry of the internal dimensions. In particular, the study of the BPS objects in string theory is rephrased in terms of the topological invariants of X (see, e.g., [1, 2]). Given its relation with the geometry of X, it then makes sense to regard a SQFT as a *mathematical object with its own dignity*, investigating also SQFTs that do not live in four dimensions. For example, in this thesis work we are going to focus on the relation between the five-dimensional physical theories arising from compactification of M-theory on a Calabi-Yau three-fold X and the topological invariants of X.

Whenever a new class of mathematical objects is introduced, one has two immediate temptations: to find a way to characterize isomorphism classes of such objects and to classify them. In the context of supersymmetric quantum field theories, simply taking a look at the space of all possible couplings one understands that achieving a classification of SQFTs is an unreachable target. A better-posed and more meaningful question is phrased requiring the SQFT to display also conformal symmetry, trying to classify all the superconformal field theories (SCFTs). For example, from the gauge theory perspective, the SCFTs make up a subset, preserved by the RG flow, of the space of all the possible supersymmetric theories. Many different supersymmetric gauge theories may flow to the same SCFT or, viceversa, can be understood as different relevant deformations of the same superconformal point. In this thesis work, we concentrate on the case of five-dimensional supersymmetric quantum field theories, intensively studied in the recent years [3–63]. In dimension five it is a known fact [64] that superconformal manifolds are isolated points: every five-dimensional SCFT does not admit any exactly marginal deformation.

Apart from the classification of such objects, the second interesting question is finding a way to describe and study the dynamics of such SCFTs. Nowadays, the dynamics of supersymmetric gauge theory at finite coupling is well understood in various dimensions and permits to answer to many questions of modern theoretical physics. However, in five dimensions, the Yang-Mills coefficient $\frac{1}{g_{YM}^2} = m_I$ is a relevant coupling of mass dimension one. This has two immediate consequences: we can reach a superconformal point just in the UV, where m_I is sent to zero, and this UV point is reached at infinite coupling. The well-known claim that some computable quantities of a supersymmetric gauge theory (such as the Higgs Branch (HB)) are protected along the RG flow is not true anymore when we take an infinite coupling limit. Indeed, m_I is linked to the conserved topological current $J_I \equiv \star\mathrm{Tr}(F \wedge F)$ and can be interpreted as the mass of instantonic particles. When m_I is sent to zero the local degrees of freedom of the particles become massless and can modify coarse features of the HB (like the quaternionic dimension of the HB). Indeed, it might happen that a five-dimensional SCFT, e.g. the E_0 theory [11], does not even admit a deformation that makes it flow to a gauge theory phase. These facts impose the scientific community the urge to find different methods to investigate such strongly coupled fixed points. In this sense, string theory furnishes a great source of inspiration: stringy constructions, involving five-branes webs in type IIB [6, 9, 33, 58–63, 65] or geometric engineering of M-theory on threefold Calabi-Yau (CY) singularities [5, 8, 10, 14–18, 21, 24, 25, 39, 40, 50, 52–54, 66–68] can be used to study and classify five-dimensional superconformal fixed points. This latter M-theoretic geometric approach is well understood for toric CY [5], but still less studied for other classes of CY.

In this thesis work we are going to focus on the last approach: we are going to study isolated non-toric hypersurface singularities (IHS) of a special subclass, the so-called compound Du Val (cDV) threefolds, which are complex one-parameter families of deformed Du Val singularities. One can characterize this class of CY threefolds also noticing that these singularities are "terminal", in the sense that they either do not admit any crepant resolution, or they just admit a small crepant resolution, with exceptional locus of codimension two. From the physics viewpoint, this means that the Coulomb Branch (CB) of the theory is empty, and we can have, at most, background flavor multiplets associated to the non-compact divisors of the resolved threefold geometry. Stated differently, if the singularity admits a small crepant resolution, we can still have a non-empty Extended Coulomb Branch (ECB), parametrized by the Kähler volumes of the compact holomorphic curves (possibly) inflated in the resolution. These Kähler parameters play the role of five-dimensional real masses and are analogous to $\frac{1}{g^2_{Y.M.}}$ in gauge theory. In the framework of M-theory geometric engineering, the degrees of freedom of the five-dimensional SCFTs analogous to the instantonic particles in gauge theory descend from the M-theory M2-branes, wrapped on the curves contracted by the crepant resolution $\pi : \hat{X} \twoheadrightarrow X$ of the CY singularity X. The theory is effectively five-dimensional as a consequence of the fact that the eleven-dimensional profile describing the M2-brane state is peaked on the point where the curves have been contracted, namely on the singular point of X.

The simplest example of this setup is the conifold singularity, which can be thought of as a family of deformed A_1 singularities over a complex parameter w:

$$x^2 + y^2 + z^2 + w^2 = 0, \quad (x, y, w, z) \in \mathbb{C}^4. \tag{1.1.1}$$

The crepant resolution of the conifold blows-up a single $\mathbb{P}^1$, hence being a small resolution, and supports a single M2-brane state that yields a five-dimensional hypermultiplet. The divisor dual to the inflated $\mathbb{P}^1$ is associated with a non-normalizable two-form, that, upon reduction of the M-theory three-form C_3, gives the Cartan of the $Sp(2)$ flavor symmetry associated with a single five-dimensional hypermultiplet.

We now have the tools to be more precise in stating how the relation between the enumerative invariants of X and the dynamics of the five-dimensional SCFT appears in our analysis. The M2 branes states are the BPS states of the five-dimensional SCFT and are linked to some topological invariants of the threefold X, the Gopakumar-Vafa (GV) invariants [69, 70]. These invariants count, from a physical viewpoint, the number of M2 states wrapped on compact holomorphic curves appearing in the resolved geometry $\hat{X}$. Mathematically, the GV can be obtained resumming the Gromov-Witten generating series and hence contain the same information of the Gromov-Witten invariants. Furthermore, descending from the Gromov-Witten theory, the GV are related to other invariants of X, e.g. the Donaldson-Thomas invariants [71] and the Pandharipande-Thomas invariants [72]. All these kind of invariants are deeply interconnected between each other, and are currently investigated by the mathematical community. In other words, finding a new method to extract M-theory dynamics on CY threefold singularities X forces us to find a new way to compute the topological invariants of X. This is just another chapter of a long love story, started more than 50 years ago, with string theory being a flourishing source of inspiration for mathematics. Indeed, as we will briefly explain, string dualities provide unforeseen links between completely different mathematical objects. In our case, for example, string dualities will permit us to link the counting of GV invariants on a threefold to the dynamics of a certain Hitchin system canonically associated with the threefold geometry. In this sense, inspired by string theory, we can build up a detailed vocabulary between two apparently unrelated areas of mathematics.

1.2 Analyzed cases

In this thesis work, we propose a new method to study the dynamics of M-theory on cDV singularities. We will study two particular meaningful subclasses of cDV singularities: the simple flops of any length and the quasi-homogeneous cDV. The first class consists of hypersurface singularities whose small crepant resolution $\pi : \hat{X} \twoheadrightarrow X$ blow-ups, as exceptional locus, just a single $\mathbb{P}^1$. These singularities and their topological invariants are currently studied both from a mathematical [73–82] and a physical viewpoint [83–85]. The length invariant corresponds to the topological intersection between the inflated $\mathbb{P}^1$ and a dual non-compact divisor, Weil non-Cartier before the resolution, that becomes Cartier after the blowup process. The second class of singularities, studied intensively in the type IIB context [86–100], requires the presence of a quasi-homogeneous $\mathbb{C}^*$ action on the equation of X.

In all the studied cases, we compute the Higgs Branches (HBs) of the 5d $\mathcal{N}=1$ SCFTs arising from M-theory on all these kinds of singularities. To achieve this result, we regard X as a deformation of a trivial fibration of $\mathcal{G}$-type ADE singularities, with $\mathcal{G} \in A, D, E$. M-theory on a trivial ADE fibration over $\mathbb{C}_w$ gives origin to a seven-dimensional $\mathcal{N}=1$ gauge theory on $\mathbb{R}^5 \times \mathbb{C}_w$, with gauge algebra $\mathcal{G}$ [101]. We now break half of the supercharges of the seven-dimensional theory as follows: We re-organize the three real scalars into a complex adjoint scalar $\Phi = \phi_1 + i\phi_2$ and a real adjoint scalar ϕ_3. *We then switch on a vev for* Φ *that depends holomorphically on the complex coordinate* w. As a result we obtain:

- The 7d gauge algebra $\mathcal{G}$ is broken to the commutant $\mathcal{H}$ of Φ. The 7d vector boson resides now in 5d background vector multiplets that support the 5d flavor group. There can also be a discrete part of the 7d gauge group that survives the Higgsing: this leads to 5d discrete gauging.

- The zero modes of Φ are deformations in $\mathcal{G}$ that cannot be gauge fixed to zero; in particular one obtains zero modes that are localized at $w=0$, i.e. they are 5d modes. These organize in 5d hypermultiplets and correspond, in M-theory, to the M2 brane states. The total number of hypermultiplets gives the dimension of the HB (as there is no continuous 5d gauge group).

- With our method, one can easily derive the charges of the hypermultiplets with respect to the continuous flavor group and the discrete symmetry.[1]

One can physically interpret the seven-dimensional theory, and the fields Φ, ϕ_3, as describing the dynamics of type IIA brane systems. In the case of classical Lie algebras, this is simply the type IIA limit of M-theory on $\mathbb{C}^*$-fibered threefolds. The E_6, E_7, E_8 singularities are instead elliptically fibered: we can then consider F-theory on them, reducing to type IIB with seven-branes, and T-dualize the system to produce the seven-dimensional gauge theory with E_6, E_7, E_8 gauge algebra.

From a geometric point of view, the Casimir invariants of the vevs Φ and ϕ_3 (for fixed w) control, respectively, the complex structure and the Kähler moduli of the ALE fiber over the point w. In other words, introducing a w-dependence on the vev of Φ deforms the trivial ADE fibration to a non-trivially fibered threefold X. In this fashion we can realize all the cDV threefolds.

Our method explicitly creates a link between the geometry of the considered CY threefolds and a Hitchin system. For classical $\mathcal{G}$, the Hitchin system is supported on the D6 brane-locus and arises from the BPS equations describing supersymmetric

[1] This allows, in principle, to compute refined quantities, such as the Hilbert series of the Higgs branch.

configurations of the branes systems that preserve a five-dimensional Poincaré group. This relation, suggested for classical $\mathcal{G}$ by the type IIA limit of M-theory, is an explicit example of how string dualities can create connections between distant fields of mathematics. The counting of the GV invariants, which appeared as a difficult task from the CY geometry viewpoint, is powerfully rephrased, with our method, in an easy linear algebra problem in the Hitchin system context. The main problem, in our case, will be to find rules to associate to a given threefold the right Higgs field background Φ (and viceversa). Given the Higgs, we will recover the threefold describing it in terms of the Casimir invariants of Φ. Given the threefold, we will be able, using the quasi-homogeneity assumption, to give a recipe to determine Φ.

For what concerns the simple flops, among all the possible lengths ($\ell = 1, 2, 3, 4, 5, 6$), just examples of length $1, 2, 3$ were already studied (from the viewpoint of the GV invariants) in the literature [80]. Our method permitted us to produce examples of flops of any length fitting them into families of singular threefolds sharing the same GV invariants[2].

In our analysis of the quasi-homogeneous cDV, the main technical problem we had to solve is finding the explicit Higgs background Φ corresponding to a given quasi-homogeneous cDV singularity: indeed, once it is in our hands, we can compute, with our method, all the relevant quantities needed to characterize the 5d Higgs Branch. In this thesis we will report a novel method that we used in [102] to directly identify the Higgs backgrounds, solely by looking at the quasi-homogeneous cDV threefold equations and relying on the homogeneity of the coefficients of the versal deformation of the ADE singularities.

As one could expect, in all the analyzed cases there is an ambiguity: several different Φ's (giving different 5d symmetries and modes) can lead to the same CY equation. This phenomenon is common in the F-theory literature, in the context of T-brane backgrounds [103–122]. Indeed, very roughly speaking, our method describes the cDV as spectral varieties of a complex-valued matrix Φ, whose entries depend on the basespace coordinate w of the cDV family. The ambiguity arises from the simple fact that the eigenvalues of a matrix (or, more precisely, its Casimir invariants) do not determine the matrix uniquely. This means that the geometry is not able to capture all the information of the 5d theory; one needs to add more, and our claim is that the field Φ, that we specify, does the job.

This thesis work is structured as follows.

In Chapter 2 we will review some known general aspects of cDV (along the lines of [123, 124]), we will review the theory of simultaneous and partial simultaneous resolution of cDV [125] and we will refine it introducing our description of the cDV

[2]This, as we will see, holds true everywhere apart from particular divisors of the base space of the family, of which we can easily furnish an explicit expression

of type $\mathcal{G}$ in terms of the Casimir invariants of a certain matrix Φ, valued in a certain representation of $\mathcal{G}$.

In Chapter 3 we will review some relevant aspects of the physics required to explain our results. We will review aspects of seven-dimensional Super Yang-Mills (SYM) theory and two of its string theory realizations: as the theory describing the dynamics of D-branes stacks, and as M-theory on Du Val singularities of type $\mathcal{G}$. We will in particular focus on showing how the moduli space of BPS vacua of the SYM theory can be identified with the moduli space of the hyperkähler ALF metric describing the resolved/deformed Du Val singularity. We will conclude the chapter with a short handbook on M-theory geometric engineering on CY threefolds X and how this is related to the Gopakumar-Vafa invariants of X.

In Chapter 4, we will explain how to count the GV invariants (together with their degrees) of the threefold once we have the associated Higgs background Φ and we will use this to study famous examples (as the (A_1, A_{2k-1}) singularities, or the Laufer's singularities) of cDV already studied in mathematical literature [80, 126, 127].

In Chapter 5 we will study the M-theory dynamics (by counting the GV invariants) for the simple flops of any length $\ell = 1, 2, 3, 4, 5, 6$.

In Chapter 6, we will deal with the sub-class of quasi-homogeneous cDV singularities. We will completely scan this class of singularities, giving closed formulas for their GV invariants and specifying the GV invariants degrees (in physical terms, the charges of the M2 branes states with respect to the discrete and flavor five-dimensional symmetries). We will also do a "reverse engineering" of the method presented in Chapter 2, giving a rule to build explicitly, starting from the threefold equation, the corresponding Higgs background.

In Chapter 7, we will deal with some open questions that arose in this thesis work. We will begin analyzing the so-called T-branes backgrounds: open string states displaying less five-dimensional zero-modes (or less symmetries) than what we expected from the corresponding threefold X. Using our technique, we will be able to partly organize them according to the Hasse diagrams of the Lie algebra $\mathcal{G}$ associated to X. Furthermore, we will give a nice geometric interpretation of the five-dimensional modes in terms of the transverse directions of a nilpotent orbit that we can associate, in every different case, to the Higgs field Φ. We will conclude this chapter trying to address two puzzling open problems. First, in Section 7.2 we will comment on the mismatch between the rank of the five-dimensional flavor group dictated by the geometry of the resolved threefold and the one expected for five-dimensional free-hypers (whenever no discrete gauging groups are present), interpreting it in terms of non-trivial corrections of the Higgs Branch hyperkähler metric. Secondly, in Section 7.3 we will comment on the R-charges assignments of the five-dimensional zero-modes with respect to the five-dimensional R-symmetry $SU(2)_R$.

In Chapter 8 we will draw our conclusions on this thesis work, pointing out possible future directions.

In the Appendices, we will gather some auxiliary results that are needed to present the thesis work. We will review some algebraic and geometric aspects of the structure theory of simple Lie algebras, we will give some auxiliary formulas for the description of E_k fibered cDV in terms of the Casimir invariants of Φ, we will give tables for the GV invariants for some of the (A_k, D_n) quasi-homogeneous singularities and auxiliary results to proceed in the GV computation for the (A, D) series. Finally, we will present the Mathematica code that can be downloaded at *this* arXiv webpage that permits to extract, given the Higgs background Φ the corresponding five-dimensional matter spectrum, together with the explicit charges of the matter modes under the flavor and discrete gauging symmetries.

1.2.1 Original contributions

The main part of the original contributions contained in this thesis are taken from the papers

- Andrés Collinucci, Mario De Marco, Andrea Sangiovanni and Roberto Valandro. "Higgs branches of 5d rank-zero theories from geometry". In: *JHEP* 10.18 (2021), p. 018. DOI: 10.1007/JHEP10(2021)018. arXiv: 2105.12177 [hep-th];

- Mario De Marco and Andrea Sangiovanni. "Higgs Branches of rank-0 5d theories from M-theory on $(\mathrm{A}_j,\ \mathrm{A}_l)$ and $(\mathrm{A}_k,\ \mathrm{D}_n)$ singularities". In: *JHEP* 03 (2022), p. 099. DOI: 10.1007/JHEP03(2022)099. arXiv: 2111.05875 [hep-th];

- Andrés Collinucci, Mario De Marco, Andrea Sangiovanni and Roberto Valandro. "Flops of any length, Gopakumar-Vafa invariants and 5d Higgs branches". In: *JHEP* 08 (2022), p. 292. DOI: 10.1007/JHEP08(2022)292. arXiv: 2204.10366 [hep-th];

- Mario De Marco, Andrea Sangiovanni and Roberto Valandro. "5d Higgs branches from M-theory on quasi-homogeneous cDV threefold singularities". In: *JHEP* 10 (2022), p. 124. DOI: 10.1007/JHEP10(2022)124. arXiv: 2205.01125 [hep-th].

The work has been realized in collaboration with Andrés Collinucci, Andrea Sangiovanni and Roberto Valandro.

The results obtained in these papers are contained in Chapter 4, Chapter 5, Chapter 6 and in Section 7.1 of Chapter 7. The contents of Section 7.2 and Section 7.3 of Chapter 7 are unpublished and are intended, in the mind of the author, more as open questions rather than grounded results. Finally, we gathered other ancillary original results in Appendix B, Appendix C, Appendix D, Appendix E and Appendix F.

Chapter 2

The geometry of compound Du Val singularities

In this chapter we will first review, in Section 2.1, general known aspects of Du Val and compound Du Val (cDV) singularities. In Section 2.2 we will present our method to describe the cDV singularities in terms of the Casimir invariants of a certain matrix Φ that we will physically interpret as the adjoint Higgs field of a seven-dimensional Super Yang-Mills theory in Chapter 4.

2.1 General concepts on cDV

We are going to work over the field of complex numbers, we are interested in studying threefold Calabi-Yau (CY) singularities X.

Definition 2.1.1 (CY manifold)**.** Let X be a Kähler manifold, then X is a Calabi-Yau manifold (CY) if X is simply-connected and has a trivial canonical bundle K_X.

Remark 2.1.1. We will often assume the complex algebraic viewpoint, regarding X as a complex algebraic variety. In all the analyzed cases, we will deal with quasi-projective varieties, that naturally inherit a Kähler structure. Hence, it will be always sufficient to check the triviality of K_X for X to be CY.

We first need the notions of canonical and terminal singularities [124].

Definition 2.1.2 (Canonical and Terminal sigularities)**.** A variety X has *canonical singularities* iff the following two conditions hold:

- for some integer $r \geq 1$ the Weil divisor rK_X is Cartier (K_X is $\mathbb{Q}$-Cartier);
- if $\pi : \hat{X} \twoheadrightarrow X$ is a resolution, and $\{E_i\}_{i=1}^n$ are the exceptional prime divisors of π, then

$$rK_{\hat{X}} = \pi^*(rK_X) + \sum_{i=1}^{n} a_i E_i, \qquad a_i \geq 0 \ \forall\ i. \tag{2.1.1}$$

X has a *terminal singularity* iff we allow, in (2.1.1), strictly positive coefficients a_i.

Remark 2.1.2. Note that, according to the previous definition, if a CY threefold singularity X is terminal then it either does not admit a crepant resolution, or it admits just a *small* crepant resolution (namely, the exceptional locus of the resolution is in codimension two). Indeed, in that case there are no exceptional divisors, and the condition (2.1.1) is empty. The conifold is the most famous example of a variety admitting a small crepant resolution, with exceptional locus isomorphic to $\mathbb{P}^1$.

First, let us consider the case of complex dimension two.

Theorem 2.1.1 ([124])**.** Let X be a canonical CY surface singularity. Then X either is smooth or is locally isomorphic to one of the following hypersurface singularities of $\mathbb{C}^3_{x,y,z}$

$$x^2 + P_{\mathcal{G}}(y, z) = 0, \tag{2.1.2}$$

with $P_{\mathcal{G}}$ being one of the following polynomials:

$$\begin{aligned} P_{A_r} &= y^2 + z^{r+1}, \\ P_{D_r} &= zy^2 + z^{r-1}, \\ P_{E_6} &= y^3 + z^4, \\ P_{E_7} &= y^3 + yz^3, \\ P_{E_8} &= y^3 + z^5. \end{aligned} \tag{2.1.3}$$

The singularities appearing in (2.1.3) are known in literature as “Du Val singularities” or as “ADE” singularities. Their (full) crepant resolution $\pi : \hat{S} \twoheadrightarrow S$ has, as exceptional locus, a bunch of $\mathbb{P}^1$s intersecting according to (minus) the Cartan matrix of the corresponding ADE algebra $\mathcal{G}$. We will denote as $X_{\mathcal{G}}$ the Du Val singularity of type $\mathcal{G}$.

Apart from the resolution pattern, we want to understand also the complex deformations of the Du Val singularities. In this thesis we will deal only with affine hypersurfaces singularities. We will then recall the definitions of deformation, versal deformation and miniversal deformation[1] [131] in this particular case.

Definition 2.1.3 (Deformation)**.** A *deformation* of a hypersurface singularity defined as the zero locus of a polynomial $f : \mathbb{C}^{n+1} \to \mathbb{C}$, is the hypersurface defined by a polynomial $F \in \mathbb{C}[x_1, ..., x_{n+1}] \otimes \mathbb{C}[\nu_1, ..., \nu_h]$, such that $F|_{\nu=0} = f$. We will call ν the restriction of the canonical projection on the $\mathbb{C}^h_\nu$ factor of $\mathbb{C}^{n+1}_x \times \mathbb{C}^h_\nu$ to the zero locus of $\mathcal{F}$. We will denote the deformation as the pair (F, ν).

[1]We will adapt the definition of [131] to the case in which the singularity is defined as the zero locus of a polynomial (rather than a more general holomorphic function).

There are many possible deformations of a hypersurface singularity, among them we are interested in those enjoying a particular universal property.

Definition 2.1.4 (Versal Deformation)**.** A deformation (F, ν) of a singularity f is versal if, for any other deformation (G, η) with $\eta \in \mathbb{C}^m$, there exist two morphism of algebraic varieties $\psi : \mathbb{C}^m \to \mathbb{C}^h$, and $g : \mathbb{C}^{n+1} \times \mathbb{C}^m \to \mathbb{C}^{n+1}$ such that

(i) $g|_{\eta=0}$ is the identity map;

(ii) $G(g(x, \eta), \eta) = F(x, \psi(\eta))$.

In other words, a deformation is versal if any other deformation (G, η) can be obtained from (F, ν) with a base-change ψ that sends $\nu \to \nu(\eta)$, up to automorphisms of the fibers (induced, at each fixed η, by the map g).

In the case of hypersurface singularities, there exists an easy construction of a particular kind of versal deformations of the singularity: the miniversal deformations. First, we define the Jacobian ring of the singularity.

Definition 2.1.5 (Jacobian ring)**.** Let X be an hypersurface singularity of $\mathbb{C}[x_1, ..., x_n]$ defined as the zero locus of $f \in \mathbb{C}[x_1, ..., x_n]$. Then, the Jacobian ring $\mathcal{R}_X$ of X is

$$\mathcal{R}_X \equiv \frac{\mathbb{C}[x_1, ..., x_n]}{\left(f, \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\right)}. \tag{2.1.4}$$

The dimension of $\mathcal{R}_X$, as complex vector space[2], is called the "Milnor number" M of X.

As stated above, we are interested in a particular kind of versal deformations, called *miniversal deformations*.

Definition 2.1.6 (Miniversal deformations)**.** A versal deformation $(\mathcal{F}, \mu)$ is called a *minversal deformation* if the dimension of the basespace $\mu \in \mathbb{C}^M_\mu$ equals the Milnor number M of the singularity.

We can then use the generators of the Jacobian ring to explicitly build the miniversal deformations of the singularity.

Proposition 2.1.1. Let X be an isolated hypersurface singularity of complex dimension n, let M be its Milnor number, let $(\mathcal{F}, \mu)$ be the miniversal deformations of X. Then,

[2]The ring $\mathcal{R}_X$ is a $\mathbb{C}$-algebra by the identification of $\mathbb{C}$ with the equivalence classes in $\mathcal{R}_X$ represented by constant polynomials. Hence, if we forget about the polynomial multiplication, we can regard $\mathcal{R}_X$ as a vector space over $\mathbb{C}$.

(i) $\mathcal{F}$ is isomorphic, as algebraic variety, to the following hypersurface of $\mathbb{C}_x^{n+1} \times \mathbb{C}_\mu^M$

$$0 = F(x_1, ..., x_{n+1}) + \sum_{j=1}^{M} \mu_j h_j(x_1, ..., x_{n+1}) \tag{2.1.5}$$

with h_j the generators of $\mathcal{R}_X$.

(ii) The projection on the basespace is the restriction to (2.1.5) of the canonical projection of $\mathbb{C}_x^{n+1} \times \mathbb{C}_\mu^M$ on the factor $\mathbb{C}_\mu^M$.

The computation of the generators of $\mathcal{R}_X$ is easy and can be obtained, e.g., with [132]. It is worthful for future use, using Proposition 2.1.1, to explicitly write down the miniversal deformations of the Du Val singularities.

Corollary 2.1.1. Let $X_\mathcal{G}$ be a Du Val singularity, let r be the rank of $\mathcal{G}$. Let $(\mathcal{F}_\mathcal{G}, \mu)$ be the miniversal deformations of $X_\mathcal{G}$, then

(i) The Milnor number M of a Du Val singularity of type $\mathcal{G}$ equals the rank r of $\mathcal{G}$.

(ii) $\mathcal{F}_\mathcal{G}$ is isomorphic, as algebraic variety, to one of the following hypersurfaces of $\mathbb{C}^3_{x,y,z} \times \mathbb{C}^r_\mu$

$$\begin{aligned}
A_r :& \quad x^2 + y^2 + z^{r+1} + \sum_{i=2}^{r+1} z^{r+1-i}\mu_i = 0,\\
D_r :& \quad x^2 + zy^2 + z^{r-1} + \prod_{i=1}^{r-1} \mu_{2i} z^{r-1-i} + 2\tilde{\mu}_r y,\\
E_6 :& \quad x^2 + z^4 + y^3 + \mu_2 yz^2 + \mu_5 yz + \mu_6 z^2 + \mu_8 y + \mu_9 z + \mu_{12} = 0,\\
E_7 :& \quad x^2 + y^3 + yz^3 + \mu_2 y^2 z + \mu_6 y^2 + \mu_8 yz + \mu_{10} z^2 + \mu_{12} y + \mu_{14} z + \mu_{18} = 0,\\
E_8 :& \quad x^2 + y^3 + z^5 + \mu_2 yz^3 + \mu_8 yz^2 + \mu_{12} z^3 + \mu_{14} yz + \mu_{18} z^2 + \mu_{20} y + \mu_{24} z + \mu_{30} = 0,
\end{aligned} \tag{2.1.6}$$

(iii) the projection μ is realized as the restriction to $\mathcal{F}_\mathcal{G}$ of the canonical projection on the factor $\mathbb{C}^r_\mu$.

Remark 2.1.3. The Du Val singularities and their resolutions admit the structure of hyperkähler varieties. Consequently, we can understand their resolutions as deformations, according to which complex structure we pick on the twistor line. This is also the reason why the Milnor number of a $\mathcal{G}$ type Du Val singularity equals the rank of $\mathcal{G}$. This remark will be important in the following chapters, in which the hyperkähler rotation on the twistor line will be matched, in the type IIA setup, with the R-symmetry of a $\mathcal{N} = 1, \mathcal{D} = 7$ supersymmetric gauge theory canonically associated with the Du Val singularity.

Among all the possible Calabi-Yau threefolds, we are going to focus on the subclass of terminal singularities. Let's first recall the definition [124] of compound Du Val singularity (cDV).

Definition 2.1.7 (Compound Du Val (cDV))**.** Let X be an affine threefold hypersurface singularity defined as the zero locus of $F \in \mathbb{C}[x, y, w, z]$. Then, X is called compound Du Val (cDV) singularity (of type $\mathcal{G}$) iff

$$F(x, y, w, z) = x^2 + P_{\mathcal{G}}(y, z) + wg(x, y, w, z), \tag{2.1.7}$$

with $g \in \mathbb{C}[x, y, w, z]$, and $P_{\mathcal{G}}$ one of the polynomials appearing in the list (2.1.3).

Remark 2.1.4. We can fix, up to affine coordinate redefinition, the singular point P to be the origin of $\mathbb{C}^4$. We are interested in the local geometry of X around the origin, and we can hence always pick g to be a linear combination, with w-dependent coefficients, of the monomials forming the basis of the Jacobian ring of the Du Val singularity $x^2 + P_{\mathcal{G}} = 0$. In particular, the restriction of the projection on the w coordinate to X endows X of the structure of a family of deformed Du Val singularities over the parameter w. This is the meaning of the word "compound" in the name cDV.

We are ultimately interested in studying terminal threefold singularities. In this sense, the definition of cDV turns out to be very useful. Indeed, let X be a rational Gorenstein[3] CY threefold, then the following theorem [124] holds.

Theorem 2.1.2. A rational Gorenstein threefold singularity X is terminal iff [124] locally is isomorphic to a cDV singularity.

In other words, if we want to study rational Gorenstein terminal threefold singularities, then it is enough to study the cDVs. It is natural to ask how the geometry of the fibers X_w and of the threefold X are related.

Definition 2.1.8. Let X be a cDV singularity, with the fiber X_0 on $w = 0$ displaying a Du Val singularity of type $\mathcal{G}$. A resolution $\pi : \hat{X} \twoheadrightarrow X$ is called a (partial) simultaneous resolution of X iff the $\pi|_{w=0} : \pi^{-1}(X_0) \twoheadrightarrow X_0$ is a (partial) resolution of X_0.

In the next section we will present a nice construction [125] of such (partial) simultaneous resolution, in a language that we will use in this thesis.

[3]In the algebraic geometry context, the Gorenstein condition rephrases (roughly speaking) the triviality of the canonical bundle. For the proper definition of Gorenstein ring and Gorenstein scheme see [133, 134].

2.2 cDV as spectral varieties

In this subsection we will describe the miniversal deformations of the Du Val singularities $X_{\mathcal{G}}$ in terms of the Casimir invariants of a certain matrix $\Phi \in \mathrm{End}(R^{\mathcal{G}})$, with $R^{\mathcal{G}}$ being

- the fundamental representation for $\mathcal{G} = A_r$;
- the **27** for $\mathcal{G} = E_6$;
- **133** for $\mathcal{G} = E_7$;
- **248** for $\mathcal{G} = E_8$.

For the D_r case, we can use the standard embedding $D_r \hookrightarrow A_{2r-1}$ defined in the Appendix A by (A.1.8) and (A.1.9).

We will show how the Casimir invariants of Φ will parametrize the basis of the miniversal deformations of $X_{\mathcal{G}}$. At the moment, this might seem to overcomplicate the setup, but it will turn out to be crucial in deriving the results presented in this thesis. We can, however, already get a taste of how this construction is useful in the context of cDV singularities giving, at the end of the section, an explicit criterium to extract cDV threefolds whose *small* crepant resolution displays particular properties. Indeed, the fact that the resolution pattern of the Du Val singularities resembles one of the Dynkin diagrams of the ADE Lie algebras hides a deep interconnection between the geometry of the Du Val (and of cDV) varieties and the algebraic structure of the ADE Lie algebras.

Let's make this concrete for the A_r case, we can write the miniversal deformations of the A_r singularity as

$$x^2 + y^2 + \Delta(z) = 0, \quad \Delta(z) = z^{r+1} + \mu_2 z^{r-1} + ... + \mu_{r+1}. \tag{2.2.1}$$

We can write $\Delta(z)$ in terms of its roots t_i, $i = 1, ..., r+1$ obtaining

$$x^2 + y^2 + \prod_{i=1}^{r+1} (z + t_i) = 0, \tag{2.2.2}$$

where, due to the lack of the z^r term in $\Delta(z)$, we have

$$\sum_{i=1}^{r+1} t_i = 0. \tag{2.2.3}$$

We can then consider a diagonal traceless matrix[4] $\Phi \in \mathfrak{su}(r+1)$, such that the t_i

[4]Unless differently specified, we are always considering the complexified version of the real Lie algebras. In this case, for example, $\mathfrak{su}(r+1)$ indicates the traceless elements of $\mathrm{Mat}(r+1, \mathbb{C})$.

are the eigenvalues of Φ and $\Delta(z)$ is its characteristic polynomial. It is not a chance that we had to pick Φ exactly in the fundamental representation of the ADE algebra associated to the $\mathfrak{su}(r+1) = A_r$ singularity.

A similar construction also applies to all the other types of Du Val singularities. Let's first assume $\Phi \in \mathfrak{t}$ (with $\mathfrak{t}$ the Cartan subalgebra of $\mathcal{G}$), and call t_i the eigenvalues of Φ, then we have

$$
\begin{aligned}
A_r : &\quad x^2 + y^2 + \prod_{i=1}^{r+1} (z + t_i) = 0 \qquad \sum_{i=1}^{r+1} t_i = 0 \\
D_r : &\quad x^2 + zy^2 + \frac{\prod_{i=1}^{r} (z + t_i^2) - \prod_{i=1}^{r} t_i^2}{z} + 2\prod_{i=1}^{r} t_i y = 0 \\
E_6 : &\quad x^2 + z^4 + y^3 + \mu_2 y z^2 + \mu_5 y z + \mu_6 z^2 + \mu_8 y + \mu_9 z + \mu_{12} = 0 \\
E_7 : &\quad x^2 + y^3 + y z^3 + \mu_2 y^2 z + \mu_6 y^2 + \mu_8 y z + \mu_{10} z^2 + \mu_{12} y + \mu_{14} z + \mu_{18} = 0 \\
E_8 : &\quad x^2 + y^3 + z^5 + \mu_2 y z^3 + \mu_8 y z^2 + \mu_{12} z^3 + \mu_{14} y z + \mu_{18} z^2 + \mu_{20} y + \mu_{24} z + \mu_{30} = 0,
\end{aligned}
\tag{2.2.4}
$$

where the μ_i are known functions of $t_i \in \mathfrak{t}$ (see [125] for the explicit expressions of μ_i in E_6, E_7 and an algorithm to compute them for E_8).

We notice that the eigenvalues of Φ are not good coordinates on the deformation space of the Du Val singularity. Indeed, taking the case of A_r as an example, the coefficients of the z expansion of the polynomial $\Delta(z) = \prod_{i=1}^{r+1}(z + t_i)$ are symmetric polynomials of the t_i. In other words, in this case, the t_i coordinates cover the basespace of miniversal deformations of A_r. The covering map is realized as the quotient via the action of the symmetric group $\mathfrak{S}_{r+1}$ of $r+1$ elements (the eigenvalues of Φ).

The previous statement is a trivial algebraic fact, but we can rephrase it the language of ADE Lie algebras. Indeed, the eigenvalues t_i parametrize the elements of the Cartan subalgebra $\Phi \in \mathfrak{t} < \mathcal{G}$, the symmetric group $\mathfrak{S}_{r+1}$ acts exactly as the Weyl group $\mathcal{W}_{\mathcal{G}}$ of $\mathcal{G} = A_r$ and the basespace of $(\mathcal{F}_{\mathcal{G}}, \mu)$ is realized as $\mathbb{C}^r_\mu \cong \mathfrak{t}/\mathcal{W}_{\mathcal{G}}$. The map $\mathfrak{t} \twoheadrightarrow \mathfrak{t}/\mathcal{W}_{\mathcal{G}}$ is called the *Weyl cover*. We can picture the previous construction with the following commutative diagram:

$$
\begin{tikzcd}
\mathcal{F}_{\mathfrak{t}} \arrow[r] \arrow[d, "t"] & \mathcal{F} \arrow[d, "\mu"] \\
\mathfrak{t} \cong \mathbb{C}^r_t \arrow[r, "/\mathcal{W}_{\mathcal{G}}"] & \frac{\mathfrak{t}}{\mathcal{W}_{\mathcal{G}}} \cong \mathbb{C}^r_\mu,
\end{tikzcd}
\tag{2.2.5}
$$

with $\mathcal{F}_{\mathfrak{t}}$ the hypersurface of $\mathbb{C}^3_{x,y,z} \times \mathbb{C}^r_t$ defined by one of the equation of (2.2.4).

Let's see how we can use this construction to build cDV with specific resolution patterns. First, as we saw in the previous section, the miniversal deformations enjoy universal properties, in particular we have the following proposition.

Proposition 2.2.1. Let X be a cDV singularity of type $\mathcal{G}$. Then, there exists a base-change $\iota_X : \mathbb{C}_w \hookrightarrow \mathbb{C}^r_\mu$ that closes the following commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "w"] & \mathcal{F}_{\mathcal{G}} \arrow[d, "\mu"] \\ \mathbb{C}_w \arrow[r, "\iota_X"] & \frac{\mathfrak{t}}{\mathcal{W}_{\mathcal{G}}} \cong \mathbb{C}^r_\mu \end{tikzcd} \tag{2.2.6}$$

In other words, the geometric data of X are stored in the base-change map ι_X. We can now ask ourselves if the base-change ι_X factors through the Weyl cover:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "w"] & \mathcal{F}_{\mathfrak{t}} \arrow[r] \arrow[d, "t"] & \mathcal{F}_{\mathcal{G}} \arrow[d, "\mu"] \\ \mathbb{C}_w \arrow[r, red] & \mathfrak{t} \arrow[r, blue, "/\mathcal{W}_{\mathcal{G}}"] & \frac{\mathfrak{t}}{\mathcal{W}_{\mathcal{G}}} \cong \mathbb{C}^r_\mu, \end{tikzcd} \tag{2.2.7}$$

with ι_X being the composition of the red and the blue arrow. Let's check this in an explicit example: the well-known conifold case. The conifold singularity can be described as a family of deformed A_1 over the parameter w:

$$x^2 + y^2 + z^2 - w^2 = 0. \tag{2.2.8}$$

In fact, the miniversal deformations of the A_1 singularity read

$$x^2 + y^2 + z^2 + \mu_2 = 0, \tag{2.2.9}$$

with μ_2 the miniversal deformation parameter. The Weyl cover is, for the A_1 case,

$$\begin{aligned} \mathbb{C} \cong \mathfrak{t} &\longrightarrow \frac{\mathbb{C}}{\mathcal{W}_{\mathcal{G}}} \\ t_1 &\longrightarrow \mu_2(t_1) = -t_1^2, \end{aligned} \tag{2.2.10}$$

We see that the base-change ι_X, defined by $\mu_2(w) = w^2$ factors through the Weyl cover, with the red map in Proposition 2.2.7 being the identity map:

$$t_1(w) = w \tag{2.2.11}$$

and ι_X being the composition of the red and the blue map. The fact that ι_X factors through the Weyl cover tells us something about the resolution of the conifold.

Indeed, the space $\mathcal{F}_{\mathfrak{t}}$ for the A_1 case is known as the *universal flop of length one.* It is worthful to be quantitative, as we will come back to these concepts in Chapter 5, and to precisely define what does it mean for a cDV X to admit a *simple flop* and what is the *length* invariant of a simple flop (first introduced in [135]).

Definition 2.2.1 (Simple flop of length ℓ)**.** Let X be an isolated cDV singularity, with singular point $p \in X$, admitting a small crepant resolution $\pi : \hat{X} \twoheadrightarrow X$ whose exceptional locus $\mathcal{C}$ is a (possibly non-reduced) $\mathbb{P}^1$. In this case, we say that X admits a simple flop.
Let $\mathcal{O}$ be the structure sheaf of X and let $\mathcal{O}_p$ be the stalk of $\mathcal{O}$ at the point $p \in X$. In general, for a simple flop, we will have $\pi^* \mathcal{O}_p \cong \mathcal{O}_{\mathbb{P}^1}^{\oplus \ell}$, with ℓ the *length* of the simple flop.

Remark 2.2.1. It turns out that Definition 2.2.1 can be understood also in terms of the intersection number between the exceptional locus of $\pi : \hat{X} \twoheadrightarrow X$ and the divisors of $\hat{X}$. The fact that the flop is of length ℓ means that the exceptional $\mathbb{P}^1$ is intersected by the divisors of the resolved geometry with intersection number at most ℓ, and at least once with intersection number ℓ. In what follows, we will often call a cDV threefold X admitting a simple flop just "simple flop".

The space $\mathcal{F}_{\mathfrak{t}}$ for the A_1 case is a universal space (the *universal flop of length one*) in the sense that any threefold enjoying a length one simple flop is a base-change of $\mathcal{F}_{\mathfrak{t}}$. It is not a chance that we found this universal space performing the Weyl cover of the A_1 singularity miniversal deformation: as we will see in the next theorem, the factorization property of the map ι_X strongly characterizes the resolved geometry.

Theorem 2.2.1 (Resolution pattern via Weyl cover)**.** Let X be a cDV threefold, with DV singularity X_0 of type $\mathcal{G}$ on the $w = 0$ fiber. Then, the small resolution of X restricts to the complete resolution of the X_0 if, and only if, the base change ι_X factors through the Weyl cover $\mathfrak{t} \twoheadrightarrow \frac{\mathfrak{t}}{\mathcal{W}_{\mathcal{G}}}$.

There exists a similar story for cDV threefold that admit a *partial simultaneous resolution.* Let $\mathcal{L}$ be a Levi subalgebra of $\mathcal{G}$, we note that, by definition, $\mathfrak{t} \subseteq \mathcal{L}$ and then there is a well-defined action of the Weyl group of $\mathcal{L}$ on $\mathfrak{t}$. Then, we can define a *partial Weyl cover*:

Definition 2.2.2 (partial Weyl cover)**.** Let $\mathcal{L}$ be a Levi subalgebra of $\mathcal{G}$. We define the partial Weyl cover associated to $\mathcal{L} \subseteq \mathcal{G}$ to be the map

$$\begin{aligned} \mathbb{C}^r_t \cong \mathfrak{t} &\longrightarrow \frac{\mathbb{C}^r}{\mathcal{W}_{\mathcal{L}}} \\ t &\longrightarrow \varrho = \varrho(t), \end{aligned} \tag{2.2.12}$$

with $\mathcal{W}_{\mathcal{L}}$ the Weyl group of[5] $\mathcal{L}$.

Remark 2.2.2. We note that, for all $\mathcal{L} \subseteq \mathcal{L}'$, we have a holomorphic map $\Psi_{\mathcal{L},\mathcal{L}'}$: $\mathfrak{t}/\mathcal{W}_{\mathcal{L}} \twoheadrightarrow \mathfrak{t}/\mathcal{W}_{\mathcal{L}'}$. In other words, if we have $\mathcal{L} \subseteq \mathcal{L}' \subseteq \mathcal{G}$, then the $\mathcal{W}_{\mathcal{L}'}$-invariants of $\mathcal{L}'$ are holomorphic functions of the $\mathcal{W}_{\mathcal{L}}$-invariants of $\mathcal{L}$. We will prove the existence of $\Psi_{\mathcal{L},\mathcal{L}'}$ in the Section 2.2.1. To conclude this remark it is worthful to notice that the Cartan subalgebra $\mathfrak{t}$ is itself a Levi subalgebra (defined to be the maximal commutant[6] of the subalgebra $\mathcal{H} \subset \mathcal{G}$ generated by all the dual roots α_i^*, $i = 1, ..., r$). The partial Weyl group of $\mathfrak{t}$ regarded as a Levi subalgebra is the trivial one, the Weyl invariants are the t_i and we have $\mathcal{W}_{\mathcal{G}} = \Psi_{\mathfrak{t},\mathcal{G}}$.

From Remark 2.2.2 we see that it is also a well-posed question to ask if the map ι_X factors through the map $\Psi_{\mathcal{L},\mathcal{G}}$ associated to $\mathcal{L}$: the Weyl invariants of $\mathcal{G}$ are always holomorphic functions of the Weyl invariants of $\mathcal{L}$). In this case, the Theorem 2.2.1 extends to the powerful Theorem 2.2.2 [125]. To formulate the theorem, we first need to settle down our convention for the labeling of $\mathcal{L}$:

Definition 2.2.3 (Convention for the labelling of Levi subalgebras)**.** Our convention is the following: the coloured Dynkin diagram associated to a $\mathcal{L}$ defined as the commutant of a set of roots $\left\{\alpha_{i_1}, ..., \alpha_{i_f}\right\}$ will be the Dynkin diagram with the nodes at positions $\mathcal{S}_{\text{black}} = \{i_1, ..., i_f\}$ coloured in black. By an abuse of language, we will ofter refer to the colored Dynkin diagram mentioning its associated set of coloured roots $\mathcal{S}_{\text{black}}$.

We will denote with $\mathcal{H}$ the abelian algebra associated to the resolved roots:

$$\mathcal{H} \equiv \langle \alpha_{i_1}^*, ..., \alpha_{i_f}^* \rangle, \tag{2.2.13}$$

with $\alpha_{i_j}^*$ the dual root vector of the root α_{i_j} (as defined in Appendix A).

For an example of coloured Dynkin diagram, with $\mathcal{G} = A_3$, $f = 1$ and $\alpha_{i_1} = \alpha_2$ (according to the labelling of Figure F.1), see Figure 2.1. We can now formulate the following theorem [125].

Theorem 2.2.2 (Resolution pattern via *partial* Weyl cover)**.** Let X be a cDV threefold, with DV singularity X_0 of type $\mathcal{G}$ on the $w = 0$ fiber. Then, X admits a partial simultaneous resolution blowing-up *at least* a subset $\mathcal{S}_{\text{black}}$ of the nodes of the $\mathcal{G}$ Dynkin diagram if, and only if, the associated base-change ι_X factors through the map $\Psi_{\mathcal{L},\mathcal{G}}$ associated to $\mathcal{L}$, with $\mathcal{L}$ the Levi subalgebra of $\mathcal{G}$ associated to[7] the

[5] Note that the Weyl group of a reductive Lie algebra (as $\mathcal{L}$ is) acts trivially on the $\mathfrak{u}(1)$ factors and is isomorphic to the Weyl group of the semisimple parte of $\mathcal{L}$.

[6] We say that a subalgebra $\mathcal{A}_1 \subset \mathcal{G}$ is in the commutant of another subalgebra $\mathcal{A}_2 \subset \mathcal{G}$ if and only if, for all $g_1 \in \mathcal{A}_1$ and $g_2 in \mathcal{A}_2$, $[g_1, g_2] = 0$.

[7] As we will see in Appendix A, given $\mathcal{S}_{\text{black}}$, the associated Levi subalgebra is, in our convention, the maximal commutant of the duals of the black roots in $\mathcal{S}_{\text{black}}$.

nodes in $\mathcal{S}_{\text{black}}$. We can depict this setup in the following commutative diagram:

$$
\begin{tikzcd}
X \arrow[r] \arrow[d, "w"] & \mathcal{F}_{\mathcal{L}} \arrow[r] \arrow[d, "\varrho"] & \mathcal{F}_{\mathcal{G}} \arrow[d, "\mu"] \\
\mathbb{C}_w \arrow[r] & \mathcal{L} \arrow[r, "\Psi_{\mathcal{L},\mathcal{G}}"] & \frac{\mathfrak{t}}{\mathcal{W}_{\mathcal{G}}} \cong \mathbb{C}^r_{\mu},
\end{tikzcd}
\tag{2.2.14}
$$

with $\Psi_{\mathcal{L},\mathcal{G}}$ the holomorphic map defined in Remark 2.2.2 expressing $\mu = \mu(\varrho)$.

Remark 2.2.3. Note that the words "at least" are crucial in stating the previous theorem. We saw that if we have $\mathcal{L} \subseteq \mathcal{L}' \subseteq \mathcal{G}$ the partial Casimir invariants of $\mathcal{L}'$ are holomorphic functions of ones of $\mathcal{L}$. Consequently, if ι_X factors through the $\mathcal{L}$ partial Weyl cover, then it factors through all the partial Weyl covers of subalgebras of $\mathcal{G}$ containing $\mathcal{L}$. We can get the full information on the resolution pattern by selecting the minimal Levi subalgebra $\mathcal{L}$ such that ι_X factors through the $\mathcal{L}$ partial Weyl cover.

Let's see how the previous result can be used in a concrete example: the (A_1, A_3) singularity[8]:

$$x^2 + y^2 + z^4 - w^2 = 0. \tag{2.2.15}$$

The projection over w describes the (A_1, A_3) singularity as a family of deformed A_3, with singular $w = 0$ fiber $X_0 \cong A_3$. For the (A_1, A_3) the resolution $\hat{X}$ contains just one $\mathbb{P}^1$ [123]. It turns out that the (small) resolution of the (A_1, A_3) is a partial simultaneous resolution of the A_3 singularity, and inflates the central $\mathbb{P}^1$ of the $X_0 \cong A_3$ Dynkin diagram. Indeed, we can check this result verifying that the base-change ι_X factors through the map $\Psi_{\mathcal{L},\mathcal{G}}$ associated to the Levi subalgebra that commutes with the dual root α_2^* of the resolved node of the A_3 Dynkin diagram. Explicitly, we can write all the elements $v \in \mathfrak{t}$ as:

$$v = t_1\alpha_1^* + t_2\alpha_2^* + t_3\alpha_3^*. \tag{2.2.16}$$

The Weyl group of the subalgebra $\mathcal{L} = A_1 \oplus A_1 \oplus \langle \alpha_2^* \rangle$ associated to the external nodes of the A_3 diagram is $\mathfrak{S}_2 \times \mathfrak{S}_2$. We can label $\mathcal{L}$, following Definition 2.2.3, with the following coloured Dynkin diagram: In Figure 2.1, the central node α_2 is

$\boldsymbol{A_3}$ 

Figure 2.1: *Partial resolution of A_3*

[8]This singularity is known, in the mathematical literature, as the *Reid's Pagoda of width two*. We will return on this singularity (and its higher width analogous) in Section 4.3.

coloured because $\mathcal{L}$ is defined to be the (maximal) commutant of α_2^*.

Let's concentrate on the action of the Weyl group $\mathcal{W}_{\mathcal{L}}$. Let's call ϵ_1 and ϵ_3 its generators, where ϵ_i acts sending $t_j \to t_j$ for $i \neq j$ and $t_i \to -t_i$. The partial Weyl cover is given by

$$\varrho_1 = t_1^2, \quad \tilde{\varrho}_2 = t_2, \quad \varrho_3 = t_3^2, \tag{2.2.17}$$

where we note that the ϱ_i are invariants coordinates w.r.t. the ϵ_1, ϵ_2 action, and hence are well defined functions on $\mathfrak{t}/\mathcal{W}_{\mathcal{L}}$. The coefficients of the A_3 miniversal deformations in terms of the partial Casimirs are

$$\mu_4 = \tilde{\varrho}_2^4 - \varrho_1\varrho_2^2 - \varrho_3\tilde{\varrho}_2^2 + \varrho_1\varrho_3, \quad \mu_3 = 2\tilde{\varrho}_2\varrho_3 - 2\varrho_1\tilde{\varrho}_2, \quad \mu_2 = -2\varrho_2^2 - \varrho_1 - \varrho_3. \tag{2.2.18}$$

(2.2.18) can be regarded as the equation that defines the holomorphic map $\Psi_{\mathcal{L},\mathcal{G}}$, namely the blue map of Theorem 2.2.2, in the (A_1, A_3) example. The base-change ι_X is $\mu_4 = -w^2, \mu_3 = \mu_2 = 0$. We see that ι_X factors through the map $\Psi_{\mathcal{L},\mathcal{G}}$ (2.2.18) choosing

$$\varrho_1 = w, \quad \tilde{\varrho}_2 = 0, \quad \varrho_3 = -w. \tag{2.2.19}$$

It is worthful to remark here that, instead, ι_X does not factor through $\mathcal{W}_{\mathcal{G}}$ as is evident from (2.2.19) and (2.2.17). The map (2.2.19) (that corresponds to the red map of Theorem 2.2.2), together with Theorem 2.2.2, tells us that the (A_1, A_3) singularity admits a small partial simultaneous resolution inflating just the $\mathbb{P}^1$ associated to the central node of the A_3 Dynkin diagram. We recognized that this was the inflated node since the $\mathcal{L}$ we picked was defined as the (maximal) commutant of corresponding dual root α_2^*.

At this point, one can already get a taste of how effective is the Weyl cover method (namely Theorem 2.2.2) to understand the resolved geometry $\hat{X}$. However, the partial Weyl cover (2.2.18) might seem to be pulled out of the hat. We now give our prescription to construct all the partial Weyl covers. This will permit us to check if a particular base-change ι_X factors through the associated $\Psi_{\mathcal{L},\mathcal{G}}$ map, and hence the associated threefold enjoys a particular resolution pattern.

2.2.1 cDV resolutions from matrix algebras

We saw that the miniversal deformations of a Du Val singularity of type $\mathcal{G}$ can be reproduced as the spectral algebraic varieties of an element of $\mathcal{G}$, in a particular representation $R^{\mathcal{G}}$. We showed indeed that, for each $\mathcal{G}$, there is a holomorphic covering map (the Weyl cover) that expresses the coefficients of the miniversal deformation of $X_{\mathcal{G}}$ in terms of the eigenvalues of Φ. In other words, the coefficients of the miniversal deformation of $X_{\mathcal{G}}$ are generators of the coordinate ring of $\mathfrak{t}/\mathcal{W}_{\mathcal{G}}$. Indeed, if one adopts the Lie-algebraic viewpoint, there are more natural coordinates on $\mathfrak{t}/\mathcal{W}_{\mathcal{G}}$: the Casimir invariants of Φ (whose definition is reviewed in Appendix A). We then expect a biholomorphic change of coordinates, on $\mathfrak{t}/\mathcal{W}_{\mathcal{G}}$, to exist, relating the coefficients μ of the miniversal deformation with the Casimir invariants of Φ. Indeed, this is the case:

- for families of A- and D-type one obtains the hypersurface equations

$$\begin{array}{ll} \boldsymbol{A_r}: & x^2+y^2+\det(z\mathbb{1}-\Phi)=0 \\ \boldsymbol{D_r}: & x^2+zy^2-\frac{\sqrt{\det(z\mathbb{1}+\Phi^2)}-\mathrm{Pfaff}^2(\Phi)}{z}+2y\,\mathrm{Pfaff}(\Phi)=0 \end{array}, \tag{2.2.20}$$

  that manifestly depend on the Casimirs invariants of Φ.

- One can show [130] analogous formulae for the exceptional cases, where one can write the deformation parameters μ_i in terms of the Casimir invariants of E_r, that can easily be computed once one has the explicit form of Φ. We report the formulae in Appendix B.

First of all, using Casimir invariants permits to reconstruct the cDV associated to a certain Higgs profile Φ in a much straightforward and simple way than (2.2.4). Indeed, the first step required for (2.2.4) would be to diagonalize Φ, a computation that quickly runs out of hand. Instead, using the Casimir invariants, we can limit ourselves to just computing traces of powers of a matrix (rather than solving spectral equations of increasing degree).

We now report an auxiliary result [136], relating the elements g along the Slodowy slice through the regular nilpotent orbit[9] of a semisimple Lie algebra $\mathcal{G}_{\text{s.s.}}$ and the Casimir invariants of g. For a simple Lie algebra $\mathcal{G}$, the Slodowy slice through x associated to the standard $\mathfrak{su}(2)$ triple $\{\mathrm{x},\mathrm{y},[\mathrm{x},\mathrm{y}]\}$ is defined as all the elements $g\in\mathcal{G}$ of the following shape:

$$g=\mathrm{x}+v$$

with v an element of the Kernel of $[y,\cdot]$ and it models the transverse space (at the point x) to the nilpotent orbit containing x. We will return to the concept of Slodowy slice in a more detailed way in Definition A.2.3. For a semi-simple Lie algebra $\mathcal{G}_{\text{s.s.}}=\bigoplus\mathcal{G}_h$ (with $\mathcal{G}_h$ simple Lie algebras), we say that $g\in\mathcal{G}_{\text{s.s.}}$ is in the Slodowy slice through the regular nilpotent orbit of $\mathcal{G}_{\text{s.s.}}$ if and only if each component $g|_{\mathcal{G}_h}\in\mathcal{G}_h$ is in the Slodowy slice through the regular nilpotent orbit of $\mathcal{G}_h$. In our

[9] We say that an element x of $\mathcal{G}_{\text{s.s.}}$ is in the regular nilpotent orbit of the semisimple subalgebra $\mathcal{G}_{\text{s.s.}}$ if and only if its component along each simple factor $\mathcal{G}_h$ of $\mathcal{G}_{\text{s.s.}}=\bigoplus\mathcal{G}_h$ is in the regular nilpotent orbit of $\mathcal{G}_h$.

context, $\mathcal{G}_{\text{s.s}}$ will always be the semisimple part $\mathcal{L}_{\text{s.s}}$ of the Levi subalgebra $\mathcal{L}$, or (as we will mention in Remark 2.2.1) the semisimple part $\mathcal{M}_{\text{s.s.}}$ of a maximal-rank maximal subalgebra of a Levi subalgebra $\mathcal{M} \subseteq \mathcal{L} \subseteq \mathcal{G}$.

Proposition 2.2.2. Let $\Phi_{\text{Slod}}(\vec{\varrho})$ be an element of the Slodowy slice through an element $\mathrm{x} \in \mathcal{N}$ of the regular nilpotent orbit of[10] $\mathcal{G}_{\text{s.s.}}$. Let y be the nilnegative of a standard triple $\{\mathrm{x}, \mathrm{y}, [\mathrm{x}, \mathrm{y}]\}$ associated to x. Then

(i) We can always decompose $\Phi_{\text{Slod}}(\vec{\varrho})$ as

$$\Phi_{\text{Slod}}(\vec{\varrho}) = \mathrm{x} + \sum_{i=1}^{\text{rank}(\mathcal{G})} \varrho_i v_i, \tag{2.2.21}$$

with $\{v_i\}_{i=1}^{\text{rank}(\mathcal{G}_{\text{s.s.}})}$ a basis of the kernel of $[\mathrm{y}, \cdot]$ (namely, of the commutants of y) and ϱ_i being the components of a complex vector $\vec{\varrho} \in \mathbb{C}^{\text{rank}(\mathcal{G}_{\text{s.s.}})}$.

(ii) There exists a biholomorphic change of coordinates relating the Casimir invariants of g and the $\{\varrho_i\}_{i=1}^{\text{rank}(\mathcal{G}_{\text{s.s.}})}$.

Remark 2.2.4. At this point it might sound disturbing that we used the same letter, ϱ, to denote

- the invariants under the Weyl group $\mathcal{W}_{\mathcal{L}}$ of a Levi subalgebra $\mathcal{L} \subseteq \mathcal{G}$,
- the coordinates along the Slodowy slice through the regular orbit of $\mathcal{G}_{\text{s.s.}}$ in Proposition 2.2.2.

Indeed, a consequence of Proposition 2.2.2 is that there are holomorphic changes of coordinates that permit us to write the $\mathcal{W}_{\mathcal{L}}$ invariants of a Levi subalgebras in terms of the coordinates along the Slodowy slice through the regular nilpotent orbit of the semisimple part $\mathcal{L}_{s.s.}$ and of the coefficient along the Cartan generators dual to the roots $\left\{\alpha_{i_1}, ..., \alpha_{i_f}\right\}$ that commute with $\mathcal{L}$. To avoid the introduction of too many letters that would make the notation cumbersome, we use this slight abuse of notation.

Using Proposition 2.2.2 we can give an easy prescription[11] to obtain the maps $\Psi_{\mathcal{L},\mathcal{G}}$ introduced in Remark 2.2.2. In other words, we are able to find the expression of the Casimir invariants of Φ in terms of the partial Casimir invariants of Φ (or, analogously, in terms of ϱ_i). Given a Levi subalgebra $\mathcal{L}$ defined as the commutant of a certain set $\mathcal{S}_{\text{black}} = \{i_1, ..., i_f\}$ of dual roots $\alpha^*_{i_j}$, the procedure to construct $\Psi_{\mathcal{L},\mathcal{G}}$ is:

[10] In our notation, an element g belongs to the regular nilpotent orbit of a semisimple Lie algebra $\mathcal{G}_{\text{s.s.}} = \bigoplus_h \mathcal{G}_h$, with $\mathcal{G}_h$ the simple Lie algebras summands, if and only if each $\mathcal{G}_h$ component $g|_{\mathcal{G}_h}$ belongs to the corresponding regular nilpotent orbit.

[11] Incidentally, this prescription also shows the existence of such holomorphic maps.

1. consider the following element $\Phi_\varrho \in \mathcal{L}$,

$$\Phi_\varrho = \Phi_{\text{Slod}}(\vec{\varrho}) + \sum_{j=1}^{f} \varrho_1^{i_j} \alpha_{i_j}^*, \tag{2.2.22}$$

with $\varrho_1^{i_j}$ the coordinates along the abelian part $\mathcal{H} = \langle \alpha_{i_1}, ..., \alpha_{i_f} \rangle_{\mathbb{C}}$ of $\mathcal{L}$, $\Phi_{\text{Slod}}(\vec{\varrho})$ being the generic element (2.2.21) of the Slodowy slice through the regular nilpotent orbit of the semisimple part $\mathcal{L}_{\text{s.s.}}$ of $\mathcal{L}$ and $\vec{\varrho} \in \mathbb{C}^{\text{rank}(\mathcal{L}_{\text{s.s.}})}$ the coordinates along the Slodowy slice.

2. Compute the Casimir invariants of Φ_ϱ. They will be holomorphic functions of the coordinates $(\vec{\varrho}, \varrho_1^{i_j})$ and, being the Casimir invariants good coordinates on $\mathfrak{t}/\mathcal{W}_\mathcal{G}$, their expression will define the map $\Psi_{\mathcal{L},\mathcal{G}}$. If we then want the coefficients of the miniversal deformations of the considered Du Val singularity as functions of $(\vec{\varrho}, \varrho_1^{i_j})$, we just need to use the relations (2.2.20) and the ones in Appendix B. These expressions relate the Casimir invariants of Φ to the Du Val deformation coefficients, and permit, together with $\Psi_{\mathcal{L},\mathcal{G}}$, to obtain the deformation coefficients in terms of $(\vec{\varrho}, \varrho_1^{i_j})$.

Let's see how the construction works for $\mathcal{L} = A_1^{(1)} \oplus A_1^{(3)} \oplus \langle \alpha_2^* \rangle$, that we used to compute (2.2.18) in the previous (A_1, A_3) example . Φ_ϱ is, in this case,

$$\Phi_\varrho = \begin{pmatrix} \tilde{\varrho}_2 & 1 & 0 & 0 \\ \varrho_1 & \tilde{\varrho}_2 & 0 & 0 \\ 0 & 0 & -\tilde{\varrho}_2 & 1 \\ 0 & 0 & \varrho_3 & -\tilde{\varrho}_2 \end{pmatrix}, \tag{2.2.23}$$

with, in the language of (2.2.22), $\vec{\varrho} = (\varrho_1, \varrho_3)$ and $\tilde{\varrho}_2$ being the coordinate along the abelian factor of $\mathcal{L} = A_1^{(1)} \oplus A_1^{(3)} \oplus \langle \alpha_2^* \rangle$. We note that, killing the coordinates $\varrho_1, \tilde{\varrho}_2, \varrho_3$ we recover an element of the regular nilpotent orbit of the semisimple part of $\mathcal{L}$, as expected by the definition of Slodowy slice. Then, we can simply compute the Casimir invariants of Φ_ϱ, obtaining

$$\mu_4 = \varrho_1\varrho_3 - \varrho_1\tilde{\varrho}_2^2 - \varrho_3\tilde{\varrho}_2^2 + \tilde{\varrho}_2^4, \quad \mu_3 = 2\varrho_3\tilde{\varrho}_2 - 2\varrho_1\tilde{\varrho}_2, \quad \mu_2 = -\varrho_1 - \varrho_3 - 2\tilde{\varrho}_2^2, \tag{2.2.24}$$

that coincides with (2.2.18).

We can finally give an evaluative criterium to understand if a certain cDV displays a determined simultaneous resolution pattern. The following result is the backbone of many of the results contained in this thesis.

Theorem 2.2.3. Let X be a threefold cDV singularity of type $\mathcal{G}$, with basespace coordinate w. Then, the following statements are equivalent:

(i) there exists a partial simultaneous resolution $\pi : \hat{X} \twoheadrightarrow X$, inflating the nodes $\mathcal{S}_{\text{black}}$ of a certain coloured Dynkin diagram.

(ii) there exists a global section $\Phi \in \Gamma\left(\mathbb{C}_w, \mathcal{L} \otimes \mathbb{C}[w]\right)$, with $\mathcal{L}$ the Levi subalgebra associated to $\mathcal{S}_{\text{black}}$, such that the Casimir invariants of Φ reproduce (via (2.2.20 and the formulas in Appendix B) the equation of X.

Proof. We first prove that (ii) implies (i). If such Φ exists, the *partial* Casimir invariants (namely, the $\mathcal{W}_{\mathcal{L}}$-invariants) of Φ are holomorphic functions of the entries of the matrix Φ, hence they holomorphically depend on w. We can use the partial Casimir invariants as coordinates on $\mathfrak{t}/\mathcal{W}_{\mathcal{L}}$ and then the map ι_X appearing in (2.2.6) factors through the map $\Psi_{\mathcal{L},\mathcal{G}}$ associated to $\mathcal{L}$. Consequently, using Theorem 2.2.2, we have the existence of the map π.

Let's prove the other implication. Using again Theorem 2.2.2, if a resolution $\pi : \hat{X} \twoheadrightarrow X$ inflates the $\mathbb{P}^1$s associated to $\mathcal{S}_{\text{black}}$, then this means that the coordinates of $\mathfrak{t}/\mathcal{W}_{\mathcal{L}}$ are holomorphic in w (because of the factorization property of the map ι_X). Since, by Proposition 2.2.2, there exists a biholomorphic change of coordinates between the entries ϱ of Φ_ϱ introduced in (2.2.22), and any set of coordinates that we are using on $\mathfrak{t}/\mathcal{W}_{\mathcal{L}}$, the matrix entries of Φ_ϱ depend holomorphically on w and we can always take $\Phi = \Phi_\varrho$. ∎

We end this section with two subtle remarks that will become relevant when applying this mathematical machinery to physics.

Remark 2.2.5. In particular situations that will be relevant in the following chapters, the Φ associated with the threefold can be constructed using the Slodowy slice of a nilpotent orbit $\mathcal{O}_0$ of $\mathcal{L}_{s.s.}$ tinier than the regular nilpotent orbit of $\mathcal{L}_{s.s.}$.

The coordinates along the Slodowy slice through the regular nilpotent orbit are always holomorphic functions of the coordinates along the Slodowy slice of $\mathcal{O}_0$. Indeed, the Casimir invariants of g in Proposition 2.2.2 are biholomorphic to the coordinates of the Slodowy slice of $\mathcal{O}_{\text{reg.}}$ by Proposition 2.2.2 and are holomorphic functions of the entries of Φ. Consequently, the coordinates along the Slodowy slice of the regular nilpotent orbit of $\mathcal{L}_{\text{s.s.}}$ are holomorphic functions of the coordinates along the Slodowy slice of $\mathcal{O}_0$. Hence, a Higgs field Φ_{naive} of the form (2.2.22) always exists also for these particular threefolds. Indeed, the resolution pattern in these cases will be the one associated to Φ_{naive}, but more refined geometric aspects, such as the GV invariants of X can be computed only using Φ (and not using Φ_{naive}).

As we will see later, for physical reasons it will be relevant to consider Φ in a proper maximal subalgebra $\mathcal{M}$ of maximal rank of $\mathcal{L}$.

Remark 2.2.6 (Maximal subalgebras of maximal rank)**.** Let $\mathcal{L}$ be a Levi subalgebra of $\mathcal{G}$. Let $\mathcal{M} \subseteq \mathcal{L}$ be a maximal subalgebra of maximal rank. Obviously, the subalgebra $\mathcal{M}$ commutes with the abelian algebra $\mathcal{H}$ associated to the simple roots that commute with $\mathcal{L}$. Being $\mathcal{M}$ a maximal-rank subalgebra of $\mathcal{L}$, we also have that $\mathcal{H}$ is the bigger vector subspace of $\mathfrak{t}$ whose elements commute with all the elements of $\mathcal{M}$. This implies that a Higgs field of the form (2.2.22) in $\mathcal{M}$ will produce a threefold whose partial simultaneous resolution inflates just the nodes associated to $\mathcal{L}$.

Indeed, $\mathfrak{t} \subseteq \mathcal{M}$ and the $\mathcal{W}_{\mathcal{L}}$ invariants are holomorphic function of the $\mathcal{W}_{\mathcal{M}}$ invariants. Consequently, using Theorem 2.2.2, we can still resolve all the $\mathbb{P}^1$ that we could resolve with a Φ of the form (2.2.22) in $\mathcal{L}$. Furthermore, we can not resolve other $\mathbb{P}^1$s as this would imply the presence of another Levi subalgebra $\mathcal{L}' < \mathcal{L} \subseteq \mathcal{G}$ such that $\mathcal{M} < \mathcal{L}'$. However, in this case we would have $\text{rank}(\mathcal{M}) \leq \text{rank}(\mathcal{L}') < \text{rank}(\mathcal{L}')$, but this can not be true as $\mathcal{M}$ is required to be a *maximal-rank* subalgebra of $\mathcal{L}$.

In the remaining part of this thesis, we will (almost always) refer to $\mathcal{M}$, since this is the true general prescription (rather than using $\mathcal{L}$). Very often, however, $\mathcal{M} = \mathcal{L}$ will be the trivial maximal subalgebra of $\mathcal{L}$.

Summing up, in this section we learned how to express the coefficients of the miniversal deformations of a Du Val singularity in terms of the matrix entries of $\Phi_\varrho \in \mathcal{G}$ defined in (2.2.22). This permits us to associate a Higgs field to a cDV X once we have access to the resolution pattern $\pi : \hat{X} \to X$, or equivalently, by Theorem 2.2.2 and Remark 2.2.6, to the minimal Levi subalgebra $\mathcal{L}$ of $\mathcal{G}$ where Φ resides. In that case, Φ is obtained from Φ_ϱ giving a polynomial w dependence to the matrix entries ϱ (with w the coordinate on the basespace of the cDV) of Φ_ϱ. The correspondence between the threefold and the Higgs field can be formally stated in Theorem 2.2.3 (up to the subtleties in Remark 2.2.5 and Remark 2.2.6) and will be used to obtain all the results discussed in this thesis.

Chapter 3

M-theory compactification and type IIA limit of M-theory on Du Val singularities

In this chapter we will present the two main characters of this thesis. First, we will review the most relevant aspects of $\mathcal{N}=1, \mathcal{D}=7$ Super-Yang-Mills (SYM) theory with gauge algebra $\mathcal{G}$. This class of theories will play an important role in the next chapters, being, for classical $\mathcal{G}$, the theory describing stacks of parallel D6 branes and parallel D6 branes on orientifolds. Indeed, a key ingredient for the result we obtained in [102, 128–130] is the type IIA limit of M-theory, in which D6 branes appear.

The second section of this chapter contains a quick review of five-dimensional superconformal field theories. We will recall the field content of the five-dimensional vector multiplet and hypermultiplet, and we will recall the well-known supersymmetric gauge theory perspective on 5d SCFTs.

The last part of this chapter will review M-theory compactification[1]. We will describe in details the compactification on Du Val singularities, reproducing in this language both the description of Du Val as spectral algebraic varieties we saw in Section 2.2 and the aforementioned type IIA limit in this controlled setup. We will also briefly comment on M-theory compactification on a generic CY threefold, and on the role played by some topological invariants, the Gopakumar-Vafa invariants [69, 70], in the M-theory compactification dynamics.

[1]As we will mention in the Section 3.3, the word "compactification" might be slightly misleading in this case: all the geometries over which we will reduce M-theory will be non-compact as topological spaces. However, we will use the word "compactification" meaning that we distinguish between "internal" directions along the CY variety and "external" directions spanning the space-time of a lower-dimensional theory.

3.1 Supersymmetric gauge theories in seven dimensions

We will first recall what is a $\mathcal{N}=1, \mathcal{D}=7$ super Yang-Mills theory (SYM) with gauge algebra $\mathcal{G}$. For $\mathcal{G}=A_r$ this theory is known to describe the dynamics of $r+1$ D6 branes, while for $\mathcal{G}=D_r$ it describes $2r$ branes on the top of an $O6^-$ plane. We will consider the case in which the spacetime over which the SYM theory lives is $\mathbb{R}^{1,4}\times\mathbb{C}_w$. The bosonic field content consists of a gauge connection A_ν (with $\nu=0,...,6$) in the Lie algebra $\mathcal{G}$ and three adjoint scalars Φ^a, with $a=1,2,3$. Furthermore, we have a pair of adjoint symplectic Majorana spinors Ψ_I, satisfying

$$\Psi_I=\epsilon_{IJ}C\left(\overline{\Psi}^J\right)^T, \tag{3.1.1}$$

with C the seven-dimensional charge conjugation matrix[2] and $\overline{\Psi}\equiv\Psi^\dagger\Gamma_0$, with Γ_0 the Dirac gamma matrix along the time direction. The Lagrangian is

$$\begin{aligned}\mathscr{L} &= -\frac{1}{4g_{YM}^2}\mathrm{Tr}\left(F\wedge\star F\right)-\frac{1}{2}\mathrm{Tr}\left(D_\nu\Phi^a D^\nu\Phi_a\right)+\\ &\quad \frac{1}{4}\mathrm{Tr}\left([\Phi_a,\Phi_b]\left[\Phi^a,\Phi^b\right]\right)-\frac{i}{2}\mathrm{Tr}\left(\overline{\Psi}^J\Gamma^\nu D_\nu\Psi_I\right)-\frac{i}{2}\mathrm{Tr}\left(\overline{\Psi}^I\left[\Phi_a\left(\sigma^a\right)_I^J,\Psi_J\right]\right),\end{aligned} \tag{3.1.2}$$

with σ^a the Pauli matrices, $a,b=1,2,3$, $I,J=1,2$ and the traces taken over the adjoint representation indices. $\mathscr{L}$ enjoys $\mathcal{N}=1,\mathcal{D}=7$ supersymmetry and we have a $SO(3)$ R-symmetry action that rotates the Φ_a as a triplet. The fields of the theory transform as follows [137] under the action of the $\mathcal{N}=1,\mathcal{D}=7$ superalgebra:

$$\begin{aligned}\delta A_M &= \frac{i}{2}\bar{\epsilon}^J\Gamma_M\Psi_J,\\ \delta\Phi_a &= \frac{1}{2}\bar{\epsilon}^J(\sigma_a)_J^I\Psi_I,\\ \delta\Psi_I &= -\frac{1}{4}F_{MN}\Gamma^{MN}\epsilon_I+\frac{i}{2}\Gamma^M D_M(\Phi_a\sigma^a)_I^J\epsilon_J+\frac{1}{4}\epsilon^{abc}[\Phi_a,\Phi_b](\sigma_c)_I^J\epsilon_J,\end{aligned} \tag{3.1.3}$$

where a,b,c run from one to three and ϵ^I are the infinitesimal parameters (susy parameters) parametrizing infinitesimal supersymmetry transformations.

[2] We can pick C to be

$$C=\Gamma_0\Gamma_2\Gamma_5,$$

with Γ_M the generators of the seven-dimensional Clifford algebra [137].

3.1.1 Supersymmetric configurations

We are interested in describing the moduli space of supersymmetric vacua of the $\mathcal{N}=1, \mathcal{D}=7$ SYM theory. By a supersymmetric vacuum we mean a field configuration of the $\mathcal{N}=1, \mathcal{D}=7$ SYM theory being invariant under the full supersymmetry algebra. In the $\mathcal{D}=7$ case, the minimal supersymmetry algebra is generated by sixteen real supercharges. The supersymmetric vacua have to preserve all the supercharges and are invariant under the seven-dimensional Poincaré group. The moduli space of vacua is spanned by the (constant) expectation values of the Φ^a. Each Φ^a is an element of $\mathcal{G}$ that we can take to be, up to (constant) gauge transformations, in[3] the Cartan subalgebra $\mathfrak{t} \cong \mathbb{R}^r$. Consequently the moduli space of vacua is simply $\mathcal{M}_{SYM}(\mathcal{G}) = \frac{\mathfrak{t}^3}{\mathcal{W}_\mathcal{G}} = \frac{(\mathbb{R}^r)^3}{\mathcal{W}_\mathcal{G}}$, with the Weyl group $\mathcal{W}_\mathcal{G}$ acting diagonally on each $\mathfrak{t}$ factor and accounting for the residual gauge-action[4] on $\mathfrak{t}$. We will review in the next sections that, interestingly, $\mathcal{M}_{SYM}(\mathcal{G})$ coincides with the moduli space of an hyperkähler metric on the corresponding $X_\mathcal{G}$ Du Val singularity, with $SO(3)_R$ being identified with the hyperkähler rotation. Apart from the coordinates on the moduli space of vacua, we also have some photons coming from the commutants A^i_μ, with $i=1,...,r$, of the Φ^a. These are the vector bosons that remain unhiggsed in the considered vacuum.

Let's now consider another type of supersymmetric configurations that we will study in this thesis. We are interested in studying $\mathcal{D}=5$ theories, hence we consider supersymmetric configurations of the $\mathcal{N}=1, \mathcal{D}=7$ theory preserving the five-dimensional Poincaré group and half of the supercharges of the $\mathcal{D}=7, \mathcal{N}=1$ supersymmetry algebra. To preserve half of the supercharges of the seven-dimensional theory we apply the following procedure [102, 128–130]. We split $\mathbb{R}^{1,6} = \mathbb{R}^{1,4} \times \mathbb{C}_w$, where $\mathbb{C}_w$ is parametrized by a complex coordinate w. We give a vev to the complex adjoint scalar $\Phi = \phi_1 + i\phi_2$, we pick the vev to be holomorphic in w. The vev for Φ breaks the 7d Poincaré group to the 5d one and preserves half of the supersymmetries. We consider only cases when the 5d symmetry is abelian, say $U(1)^\ell$. The zero modes around such a background organize then in 5d supermultiplets:

- The zero modes of A_ν ($\nu = 0,...,4$) and ϕ_3 propagate in 7d and are collected into $U(1)$ background vector multiplets, giving rise to the $U(1)^\ell$ flavor group.
- There are 7d zero modes of Φ that are collected together with zero modes of $A_5 + iA_6$ into background hypermultiplets, that are neutral under the flavor $U(1)$'s.

[3] In the gauge theory context, we consider the real algebra associated to $\mathcal{G}$. In the A_r case, e.g., we consider $\mathfrak{su}(r+1)$.

[4] We can regard this, in the D6s picture (for the $\mathcal{G} = A_r$ case), as the freedom of relabelling the D6 branes floating in the transverse $\mathbb{R}^3$ directions.

- There are 5d zero modes of Φ that are localized at $w = 0$. They are collected into 5d massless hypermultiplets that are charged under the $U(1)^{\ell}$ flavor group.

The moduli space of these solutions will reproduce the moduli space of vacua of five-dimensional SCFTs defined as the theory of the zero-modes around these field configurations.

3.1.2 D-branes interpretation

String theory displays two maximally supersymmetric versions, distinguished by the chirality of the spinors generating the supersymmetry transformations. If one chooses the generators to be Majorana-Weil spinors with the opposite (same) chirality one obtains, respectively, type IIA (resp. type IIB) string theory. In both type IIA and IIB theories, open and closed strings propagate in a ten-dimensional spacetime M_{10}. Apart from open and closed strings, the list of dynamical objects of a superstring theory includes also extended $p+1$ real-dimensional dynamical membranes called Dp-branes (with p even for type IIA, odd for type IIB). Naively, we can regard D-branes as submanifolds of M_{10} where open strings can end (namely, Dirichlet boundary conditions for the open strings). It was then realized that [138] D-branes are not just fixed boundary conditions (as they are charged under Ramond-Ramond fields) and hence they behave as (heavy) dynamical objects. In this thesis we are interested in D6-brane systems of type IIA string theory, whose dynamics is described, at low energies, by $\mathcal{N} = 1, \mathcal{D} = 7$ SYM with classical gauge algebra $\mathcal{G}$. Let's recap quickly the dictionary relating the gauge theory with the D-branes degrees of freedom.

First of all, the degrees of freedom of the effective theory describing the dynamics of the D6 branes can be understood both from an open and a closed string perspective. Let's start considering the A_r case, realized in type IIA by a stack of $r+1$ coincident D6s.

From an open-string viewpoint, a string stretching from the j-th to the k-th brane (with $j, k = 1, ..., r+1$) will be represented by hermitian matrices Φ^a, Ψ^I, A_ν with non-zero jk and kj entries.

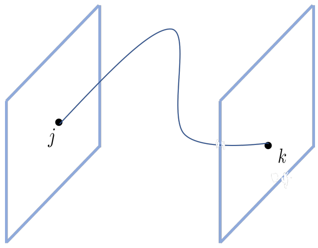

Figure 3.1: *Open string stretching between the j-th and k-th brane.*

More precisely, the massless spectrum of an open string propagating between $r+1$ parallel coincident D6 branes displays two kinds of massless states, labelled by the $(r+1)^2$ choices of Dirichlet boundary conditions. Denoting with $j, k = 1, ..., r+1$ the endpoints of each of these massless states, we can mentally organize them as follows:

- the states with $k = j$ represent fields[5] $(A_\nu)^j_j$, identified with the elements of the abelian algebra $\mathfrak{t} + \mathfrak{u}_{\text{c.m.}}(1) \subset \mathfrak{u}(r+1)$ (with $\mathfrak{t}$ the Cartan subalgebra of A_r);
- the states with $j \neq k$ are bifudamentally charged w.r.t. to $(A_\nu)^j_j$ and $(A_\nu)^k_k$.

We see that the charges of these states under the diagonal elements $(A_\nu)^j_j$ organize into the root system of the A_r algebra, together with an overall decoupled $U_{\text{c.m.}}(1)$ field representing the rigid translations of the brane system. When the coincident branes are separated, the bifundamental strings states aquire a mass proportional to the separtion between the j-th and the k-th brane, while the $(A_\nu)^j_j$ fields remain massless. This matches the field theory expectation, since the massless theory on a generic point of the moduli space of vacua of $\mathcal{N} = 1, \mathcal{D} = 7$ SYM with gauge algebra A_r consists of r abelian vector multiplets whose scalar components parametrize $\mathfrak{t}^3/\mathcal{W}_{\mathcal{G}}$. In the open string picture, the action of $\mathcal{W}_{\mathcal{G}}$ is nothing but the freedoom of permuting the labels j, k of the D6 branes.

The effective massless abelian theory on the moduli space of vacua of $\mathcal{N} = 1, \mathcal{D} = 7$ SYM has a clear interpretation also in terms of closed strings. From a closed-string viewpoint the D6 branes stack is seen as a closed string BPS configuration. The D-branes are extended in the $x_0, ..., x_6$ directions and are points in the $\mathbb{R}^3_{789} \cong \mathbb{C}_w \times \mathbb{R}_9$ as depicted in Table 3.1. The eigenvalues of the IR Φ^a_i describe the displacement of the D6 in the transverse directions $\mathbb{R}^3_{789}$. This description of the D6-branes stack as a closed string BPS solution makes clear the stringy interpretation of the $SO(3)$ R-symmetry. Indeed, the backreacted metric on the $\mathbb{R}^3_{789}$ develops a singularity at

[5]Indeed, they are associated to the full abelian vector multiplet $((\Phi^a)^j_j, (\Psi^I)^j_j, (A_\nu)^j_j$.

	$\mathbb{R}^{1,6}_{0123456}$	$\mathbb{R}^3_{789}$
D6	$\times$	$\cdot$

Table 3.1: *IIA setup dual to M-th. on* $\mathbb{R}^{1,6} \times ADE$*.*

the D6 brane position. When the positions of all the D6 branes in the transverse directions coincide, the transverse backreacted metric loses translational invariance but retains a rotational $SO(3)$ invariance around the location of the D6 branes. When the D6 positions do not coincide anymore, we are on a point of $\mathcal{M}_{SYM}(\mathcal{G})$ with spontaneously broken R-symmetry: the isometry rotates the branes in $\mathbb{R}^3_{789}$. As we will see in Section 3.3.1, the $\mathcal{N}=1, \mathcal{D}=7$ SYM theory with gauge algebra $\mathcal{G}$ can be obtained compactifying M-theory on $X_{\mathcal{G}}$. In that context, the $SO(3)$ R-symmetry action will be re-interpreted as the $SO(3)$ hyperkähler rotation that moves us in the space of complex structures of $X_{\mathcal{G}}$ compatible with the same hyperkähler metric on $X_{\mathcal{G}}$.

We conclude this section considering D6 branes on top of an $O6^-$ plane. To define orientifold planes, we first pick a spacetime involution $\sigma : M_{10} \to M_{10}$. In our case $M_{10} = \mathbb{R}^{1,6} \times \mathbb{R}^3_{789}$ and, if we pick the involution to be the reflection around the origin of $\mathbb{R}^3_{789}$, the fixed locus of this involution is called "O6 plane". The O6 plane projects out of the spectrum all the ten-dimensional fields that are not invariant w.r.t.

$$\Sigma \equiv \mathcal{P} \cdot \sigma^*, \tag{3.1.4}$$

with $\mathcal{P}$ the composition of the string worldsheet parity and the action of the worldsheet fermion number operator $(-1)^{N_f}$, and σ^* the pullback via the map σ. There is still a subtlety to fix: the space transverse to the O6-plane can host [139, 140] $\mathbb{Z}_2$-torsion fluxes for the NS field dB_{NS}. If the flux dB_{NS} is the trivial element of the $\mathbb{Z}_2$ transverse torsion homology, the O6-plane is called an $O6^-$ plane. It turns out [140] that the effective theory on a D6-stack placed on the top of a $O6^-$ plane is the seven-dimensional $SO(2n)$ SYM theory.

Summing up, if we want to string-engineer $\mathcal{N}=1, \mathcal{D}=7$ theory with gauge algebra A_r we have to consider a stack of $r+1$ branes, for gauge algebra D_r we have to consider a stack of $2r$ branes on the top of a $O6^-$ plane. The $SO(3)$ R-symmetry, in both cases, is interpreted as an isometry of the transverse backreacted metric, (spontaneously) unbroken when all the D6 are coincident.

3.2 Five-dimensional SCFTs

In this section, we will quickly review the main aspects of $\mathcal{D} = 5$ SuperConformal Field Theories (SCFT). First of all, a conformal field theories in flat $\mathbb{R}^{1,4}$ space is a theory enjoying $SO(2,5)$ conformal invariance, obtained adding the dilatation invariance $x^\mu \to \lambda x^\mu$ to the five-dimensional Lorentz group.

The fields of a $\mathcal{N} = 1, \mathcal{D} = 5$ supersymmetric field theory are organized into vector multiplets and hypermultiplets.

A five-dimensional (on-shell[6]) hypermultiplet contains a $SU(2)_R$ doublet of[7] complex scalars $(Q, \tilde{Q})$ and a five-dimensional Dirac spinor Ψ.

A five-dimensional (on-shell) vector multiplet contains a vector A_μ, with $\mu = 0, ..., 4$, a real scalar σ and a five-dimensional Majorana spinor λ_I^α satisfying

$$\left(\lambda_\alpha^I\right)^* = \epsilon^{IJ} C_{\alpha\beta} \lambda_J^\beta, \tag{3.2.1}$$

with $I, J = 1, 2$ the indeces of the fundamental of the R-symmetry $SU(2)_R$, $\alpha, \beta = 1, ..., 4$ the spinor indeces of Spin(5) and $C_{\alpha\beta}$ the five-dimensional charge conjugation. A vector multiplet can be dynamical or part of a background flavor symmetry. In this thesis, we will just deal with background vector multiplets, with σ playing the role of a real-valued mass parameter for the background flavor group.

In general, the main targets in the study of $\mathcal{D} = 5$ SCFT are

- to obtain a classification of all the 5d SCFT;
- to study the dynamics of 5d SCFT.

To classify the $\mathcal{D} = 5$ SCFT is evaluable since SCFTs are, in many cases, the UV completion of five-dimensional supersymmetric gauge theories. In particular, many different gauge theories can be understood as mass-deformations of the same superconformal fixed point. Let's recap quickly how it works. In five-dimension, the $1/g^2_{Y.M.}$ has dimension of a mass. This can be seen by naive power counting, but remains true in the UV strongly coupled region of the supersymmetric gauge theory. Indeed $m_I \equiv 1/g^2_{Y.M.}$ can be seen as a mass term of a $U(1)$ background symmetry, denoted as the $U(1)_I$, generated by the following current

$$J_I \equiv \frac{1}{8\pi^2} \star \mathrm{Tr}(F \wedge F), \tag{3.2.2}$$

[6]In this thesis we will just consider on-shell supermultiplets. In particular, for the hypermultiplet the two complex-valued auxiliary fields will be integrated out to zero, and we have just four real independent (on-shell) components of the Dirac spinor.

[7]The true degrees of freedom are, indeed, Q and $\tilde{Q}^\dagger$.

with $\star$ the five-dimensional Hodge star, F the field-strength and the trace is taken over the adjoint representation indices. The charge operator is defined as

$$Q_I(\Sigma_4) \equiv \int_{\Sigma_4} \star J_I, \tag{3.2.3}$$

The local operators charged under J_I are called "instanton operators" and have to be understood as dynamical defect operators, enforcing the following boundary conditions on the bulk field:

$$Q_I(\mathbb{S}^4_x) = n_I, \tag{3.2.4}$$

with $\mathbb{S}^4_x$ a four-sphere surrounding the defect operator placed in $x \in \mathbb{R}^{1,4}$ and $n_I \in \mathbb{Z}$ the instanton number of the operator. These operators become massless once the $m_I \to 0$, namely at the infinite coupling limit of the gauge theory. This has a dramatic consequence on the moduli space of vacua of the theory, in particular on its Higgs Branch, as we can now give a non-zero vev to the instanton operators, spanning new directions of the "enhanced" Higgs branch.

As we said, many supersymmetric gauge theories can be obtained by different mass-deformation of the same isolated SCFT. If this happens, we say that the different gauge theories are UV dual. Consequently, classifying five-dimensional SCFT can help to classify five-dimensional gauge theories, collecting them accordingly to their UV completion.

As we just saw, we can not rely on the five-dimensional gauge theory analysis to study the SCFT. Indeed, the defect operators can not be described by fundamental fields of the theory and are difficult to treat in the Lagrangian setup. Furthermore, there exist theories, such the E_0 theory [11], that do not admit mass-deformations triggering a flow to a weakly coupled supersymmetric gauge theory. As we already mentioned, string theory furnishes (through either the five-branes construction and the geometric engineering approach), novel approaches to study five-dimensional SCFTs.

$\mathcal{D} = 5$ SCFTs can be characterized by their moduli space of vacua. Again, accordingly to the presence of a spontaneously broken (or unbroken) R-symmetry we can distinguish between HB and CB. We can also define an ECB if we include the Cartan mass-term of the SCFT (such as, e.g., the previously mentioned Yang-Mills kinetic terms $1/g^2_{Y.M.}$). A coarse characterization is given by the real-dimension of the CB, if a SCFT has a $\mathcal{R}$-dimensional CB we say it is "rank $\mathcal{R}$". In this thesis we are interested in the $\mathcal{R} = 0$ case.

3.3 M-theory geometric engineering

In this section we are going to concentrate on type IIA, and on its non-perturbative completion named M-theory[8]. Indeed, analyzing D6s BPS solutions of IIA, one discovers that the same physics can be described in purely geometric terms, by means of an eleven-dimensional theory of M-branes. The low-energy limit of M-theory can be captured by the eleven-dimensional supergravity theory. The fields involved in this theory are the eleven-dimensional metric, a spin $3/2$ gravitino field and a three-form gauge connection [143].

The minimal amount of supersymmetry in eleven dimensions is 32 real supercharges. Consequently, excluding the trivial Minkowski solution, in each non-trivial supersymmetric vacuum the preserved supercharges (and consequently the preserved diffeomorphism group) are those of a lower dimensional supergravity theory. This means that, to host a non-flat BPS solution, the eleven-dimensional spacetime M_{11} has to split (topologically) as a fibration of an *internal* $d \equiv 11 - \mathcal{D}$ real dimensional geometry X over a lower-dimensional Minkowski[9] spacetime $M_{\mathcal{D}}$:

$$\begin{tikzcd} X \arrow[r, hook] & M_{11} \arrow[d, two heads] \\ & M_{\mathcal{D}} \end{tikzcd} \tag{3.3.1}$$

The associated BPS equations have to be solved in terms of the metric and the M-theory three-form C_3, the gravitino being set to zero.

Remark 3.3.1. We remark again that our convention, in the context of geometric engineering, is different from the convention we used in gauge theory. Indeed, in the geometric engineering context, the only configuration invariant under the eleven-dimensional Poincaré group is the trivial Minkowski vacuum. We are not interested in this configuration and, for M-theory geometric engineering on M_{11}, we call the (3.3.1) solutions *supersymmetric vacua.*

It is well known that the number of preserved supercharges is associated with the holonomy group of the internal space metric g. We concentrate on the cases in which $d = 4$ or $d = 6$. In these cases, if we want to preserve at least eight real supercharges we have to impose the holonomy of g to be contained in $SU(\frac{d}{2})$, namely, g to be Ricci-flat. Furthermore, C_3 has to be a closed three-form: the flux dC_3 being turned off. These two conditions do not completely fix g and C_3 and we have a non-trivial moduli space of solutions of the BPS equations. The zero modes

[8]See [141–143] for classic and more modern reviews.

[9]Supersymmetry can also be preserved endowing M_D with an anti-De Sitter metric, but we will not treat this case in the following chapters.

spanning this moduli space become, in physical terms, part of the physical fields of the effective theory associated to (3.3.1).

Apart from the moduli of the BPS solution also the metric on $M_{\mathcal{D}}$ (together with the other fields of its $\mathcal{D}$-dimensional supermultiplet) is a dynamical field of the effective $\mathcal{D}$-dimensional theory. However, we are going to consider the so-called *geometric engineering limit*: we are going to take the metric volume $\mathrm{Vol}(g)$ of X to be infinite, decoupling physically the $\mathcal{D}$-dimensional gravity multiplet from the theory. Indeed, the matching of the Newton's gravitational constants dictates

$$G_{\mathrm{Newton},\mathcal{D}} = \frac{G_{\mathrm{Newton},11}}{\mathrm{Vol(g)}} \tag{3.3.2}$$

and, hence, the infinite volume limit decouples gravity[10] from the $\mathcal{D}$-dimensional effective theory. This, from the viewpoint of the metric-induced topology, corresponds to taking a decompactification limit of X: the *geometric-engineering limit* [144, 145].

In this thesis we are interested in two cases: $\mathcal{D}=7$ and $\mathcal{D}=5$, which correspond, respectively, to the case in which X is a CY surface and a CY threefold. Let's analyze them separately.

3.3.1 M-theory geometric engineering on complex surfaces

We are interested in M-theory compactification on a non-compact surface X. The minimal supersymmetry algebra in $\mathcal{D} = 11-4 = 7$ dimensions has sixteen real supercharges hence the metric on X has to be hyperkähler rather than simply Calabi-Yau. This is something that we have for free in $d = \dim_{\mathbb{R}} X = 4$ dimensions: to be hyperkähler, the holonomy group of g has to be contained in the symplectic group of rank one[11] $Sp(1)$, on the other hand Ricci-flatness ensures that the holonomy group is contained in $SU(2)$. We have that $Sp(1) \cong SU(2)$, hence any Ricci-flat metric on a complex surface is hyperkähler.

Apart from the flat space $\mathbb{C}^2$, hyperkähler affine varieties are affine patches inside K3, or, in other words, Du Val singularities $X = X_{\mathcal{G}}$. We are interested in the effective theory coming from compactification on $X_{\mathcal{G}}$, hence we study the moduli space $\mathcal{M}(g)$ of the metric g.

An hyperkähler metric can be described in terms of three Kähler structures[12]

[10]More precisely it decouples the $\mathcal{D}$-dimensional metric together with its full supersymmetry multiplet.

[11]We remark that other conventions denote the symplectic groups of rank r as Sp(2r).

[12]We remark that we used the same index a to label the three Kähler structures and the three real adjoint scalars of the $\mathcal{D}=7, \mathcal{N}=1$ theory. This is not a coincidence, as we will shortly identify these objects.

$(\omega_a, \mathcal{I}_a)$ with $a = 1, 2, 3$, subject to the following compatibility conditions:

$$\mathcal{I}_a^2 = -\mathbb{1}, \qquad \mathcal{I}_1\mathcal{I}_2\mathcal{I}_3 = -\mathbb{1}, \tag{3.3.3}$$

with $\mathbb{1}$ being the trivial endomorphism of the complexified tangent bundle of $X_{\mathcal{G}}$. Fixed a metric, we have a $\mathbb{P}^1$ of compatible complex structures, called "twistor line". Picking a point on the twistor lines corresponds to singling out a direction ω_1, whose periods will describe blowups Kähler moduli of the metric, and pairing the remaining two Kähler forms in an "holomorphic symplectic" form $\Omega_{2,0} \equiv \omega_2 + i\omega_3 \in H^2_{\text{cpct}}(\tilde{X}_{\mathcal{G}}, \mathbb{C})$. The periods of $\Omega_{2,0}$ on the generators of the $H_{2,\text{cpct}}(\tilde{X}_{\mathcal{G}}, \mathbb{Z})$ will describe the complex deformations of $X_{\mathcal{G}}$. One can see that this choice is purely arbitrary, and is modified by the action of the $SO(3)$ hyperkähler rotation under which the ω_a form a triplet.

To perform the period computation we can expand the ω_a along the generators B_j, with $j = 1, ...\text{rank}(\mathcal{G}) = r$, of $H^2_{\text{cpct}}(\tilde{X}_{\mathcal{G}}, \mathbb{Z})$. We will now review, studying the period description of g, or, equivalently, of $(\Omega_{2,0}, \omega_1)$ two results. First we will match the period description of the complex structure of $X_{\mathcal{G}}$ with the one we saw in Section 2.2. Then, we will show that $\mathcal{M}(g)$ is isomorphic to the moduli space of supersymmetric vacua of $\mathcal{N} = 1, \mathcal{D} = 7$ SYM theory with gauge algebra $\mathcal{G}$.

Torelli theorem, Picard-Lefschetz theory and Weyl cover

To clarify the relation between the Higgs field description of Section 2.2 and the period description[13] we start making explicit the generators of $H^2_{\text{cpct}}(X_{\mathcal{G}}, \mathbb{Z})$. Let's focus, for simplicity, on the A_r case [146]. The deformed/resolved A_r singularity can be described by the $r + 1$ multi-centered Taub-NUT metric[14]:

$$ds^2 = U(\mathbf{r})d\mathbf{r}^2 + \frac{1}{U(\mathbf{r})}(d\psi + \mathbf{p} \cdot d\mathbf{r}), \tag{3.3.4}$$

with $\mathbf{r} \in \mathbb{R}^3_{789}$,

$$U(\mathbf{r}) \equiv \sum_{i=1}^{r+1} \frac{1}{|\mathbf{r} - \mathbf{r}_i|} + \frac{1}{\lambda^2} \tag{3.3.5}$$

and $\mathbf{p}$ being a solution of $dU = \star d\,(\mathbf{p} \cdot d\mathbf{r})$, with $\star$ the $\mathbb{R}^3_{789}$ Hodge star. The variable $\psi \in [0, 2\pi)$ parametrizes an $\mathbb{S}^1$ fibered over $\mathbf{r} \in \mathbb{R}^3_{789}$, shrinking at the locations $\mathbf{r} = \mathbf{r}_i$.

[13]Mathematically, this goes under the name of Torelli theorem (for deformed Du Val singularities).

[14]$\mathbb{R}^3_{789}$ endowed with (3.3.4) is also called "Asymptotically Locally Flat" space (ALF).

One of the possible compatible complex structures is defined by

$$uv = -\prod_{i=1}^{r+1}(z+t_i), \qquad \sum_{i=1}^{r+1} t_i = 0, \tag{3.3.6}$$

with the t_i being the projections of the $\mathbf{r}_i$ on the first factor of $\mathbb{C}_z \times \mathbb{R}_9 \cong \mathbb{R}^3_{789}$. The hyperkähler rotation acts as a rotation in $\mathbb{R}^3_{789}$ and changes the position of the selected $\mathbb{C}_z \hookrightarrow \mathbb{R}^3_{789}$.

Equation (3.3.6) displays the Taub-NUT circle fibration as the compact direction of the $\mathbb{C}^*$ fibration[15] over z. The fibration degenerates on the points of the $\mathbb{C}_z$ plane with $z = t_i$. The $\mathbb{S}^2$ generating $H_{2,cpct}(A_r, \mathbb{Z})$ are suspensions of the $U(1) \hookrightarrow \mathbb{C}^*$ fiber between two endpoints $z = t_i, z = t_j$. Explicitly, we take a real path in the z-plane joining t_i and t_j without passing through other t_k points; then, over each point of the path we fiber the M-theory circle. The circle shrinks at the endpoints that coincide with the North and South poles of the considered $\mathbb{S}^2$. We denote this compact two cycle as α_{ij}, we have that

$$\alpha_{ij} + \alpha_{jk} + \alpha_{ki} = 0 \tag{3.3.7}$$

and, consequently, we have just r independent two cycles that we pick to be $\alpha_i \equiv \alpha_{i,i+1}$, with $i = 1, ..., r$. Until now we just spoke about homology but, if we want to get the effective $\mathcal{D} = 7$ action, we are interested in cohomology rather than homology.

The space $H^2(A_r, \mathbb{Z})$ is defined to be the dual of $H_{2,cpct}(A_r, \mathbb{Z})$. Any elements $B \in H^2(A_r, \mathbb{C})$ assigns to each α_{ij} an integer $B(\alpha_{ij}) \in \mathbb{Z}$, in a way that is compatible with (3.3.7). This means that we must have

$$B(\alpha_{ij}) = \tilde{t}_i - \tilde{t}_j, \quad \tilde{t}_i, \tilde{t}_j \in \mathbb{Z}. \tag{3.3.8}$$

In other words, a closed integral two form B is labelled by $r+1$ integers $\tilde{t}_i$ up to $\tilde{t}_i \to \tilde{t}_i + t$ for some $t \in \mathbb{Z}$. In the context of Torelli theorem, as well as in the context of M-theory compactification, we are interested in the compact support cohomology $H^2_{\text{cpct}}(A_r, \mathbb{Z})$. This sublattice of the integral cohomology is generated by the Poincaré duals of the $\mathbb{S}^2$ of $H_{2,cpct}(A_r, \mathbb{Z})$, namely by those two-forms α_i^* that obey

$$\alpha_i^*(\alpha_j) = \begin{cases} -2 \text{ if } i = j; \\ 1 \text{ if } i = j \pm 1 \text{ or} \\ 0. \end{cases} \tag{3.3.9}$$

[15]The $\mathbb{C}^*$ action inducing the fibration is $u \to u/\lambda, v \to \lambda v$, with $\lambda \in \mathbb{C}^*$.

We notice that the $\tilde{t}_i$ associated to α_i^* are such that $\sum_{i=1}^n \tilde{t}_i = 0$, as, consequently, the $\tilde{t}_i$ of any complex linear combination of the $\alpha_i^* \in H^2_{\text{cpct}}(X_{\mathcal{G}}, \mathbb{C})$. This already sounds good to us: the requirement of working with compact supported forms forces us to regard to the $\tilde{t}_i$ as the eigenvalues of an element of the A_r algebra (after we allow for complex combinations of the α_i^*). More than that, if one computes via adjunction formula

$$\Omega_{2,0} = \frac{du \wedge dv \wedge dz}{d\left(uv + \prod_{i=1}^{r+1}(z + t_i)\right)}, \tag{3.3.10}$$

then the set of $\tilde{t}_i$ associated to $\Omega_{2,0} \in H^2_{\text{cpct}}(A_r, \mathbb{C})$ are exactly the roots t_i appearing in (3.3.6). In other words, for (3.3.6), we have

$$\Omega_{2,0}(\alpha_{ij}) = t_i - t_j. \tag{3.3.11}$$

Similar relations hold true also for the other Du Val singularities:

$$\begin{aligned}
&A_r: \quad \text{vol}_{\alpha_i} = t_i - t_{i+1} \qquad i = 1, ..., r \\
&D_r: \quad \text{vol}_{\alpha_i} = \begin{Bmatrix} t_i - t_{i+1} & i = 1, ..., r-1 \\ t_{r-1} + t_r & i = r \end{Bmatrix}. \\
&E_r: \quad \text{vol}_{\alpha_i} = \begin{Bmatrix} t_i - t_{i+1} & i = 1, ..., r-1 \\ -t_1 - t_2 - t_3 & i = r \end{Bmatrix},
\end{aligned} \tag{3.3.12}$$

where we labelled roots as in figure F.1. The relations (3.3.12) make clear, in the complementary complex-analytic language of Torelli theorem, that the geometry of a Du Val singularity can be captured by the eigenvalues of a suitable traceless matrix Φ. However, it is important to notice that the periods of $\Omega_{2,0}$ are not good coordinates on the space of complex structures of the Du Val. Indeed, any permutation of the t_i produces, in (3.3.6), the same complex equation in $\text{Spec}\mathbb{C}[u, v, z]$. For the A_r case, the true coordinates on the moduli space of complex structures are the coefficients of the expansion of the r.h.s. of (3.3.6), namely symmetric polynomials of the t_i, rather than the t_i. As we saw in Section 2.2, the t_i cover (via the Weyl cover) the moduli space of complex structures of $X_{\mathcal{G}}$. Consequently, we have

$$\mathcal{M}_{\text{cplx}} \cong \frac{\mathbb{C}^r}{\mathcal{W}_{\mathcal{G}}}. \tag{3.3.13}$$

We will now show how this result is correctly mirrored in the language of Torelli theorem.

Indeed, $\Omega_{2,0}$ is uniquely identified by its periods $t_i - t_{i+1}$, with $i = 1, ..., r$, and we can regard it as an element of $\mathbb{C}^r$. However, in performing such period computation we choose a particular basis for $H_{2,\text{cpct}}(\tilde{X}_{\mathcal{G}}, \mathbb{Z})$, this choice is arbitrary and can be modified by the action of a monodromy transformation (as defined [131]) η of $\tilde{X}_{\mathcal{G}}$. Roughly speaking one can think to a monodromy transformation as a diffeomorphism of $\tilde{X}_{\mathcal{G}}$ that induces a well-defined automorphism on the integral homology of $\tilde{X}_{\mathcal{G}}$. We will come back shortly on the definition of η, constructing it in the relevant context of Du Val singularities. We denote by $\mathrm{Mon}(\tilde{X}_{\mathcal{G}})$ the group of such monodromy transformation (called the "monodromy group" of X [131]).

Let's see how η reflects on the period computation. As we just mentioned, η induces an invertible map

$$\begin{aligned} \eta_*\colon\ H_{2,\mathrm{cpct}}(\tilde{X}_{\mathcal{G}},\mathbb{Z}) &\longrightarrow H_{2,\mathrm{cpct}}(\tilde{X}_{\mathcal{G}},\mathbb{Z}) \\ \alpha &\longrightarrow \eta_*(\alpha) \end{aligned} \tag{3.3.14}$$

that, as we will see, in general is not the identity. η_* gives an action η^* on the space $H^2(\tilde{X}_{\mathcal{G}},\mathbb{C})$:

$$(\eta^* B)(\alpha) \equiv B(\eta_*^{-1}\alpha). \tag{3.3.15}$$

Consequently, the moduli space of complex structures is not $\mathbb{C}^r$, but $\mathbb{C}^r/\mathrm{Mon}^*(\tilde{X}_{\mathcal{G}})$, with $\mathrm{Mon}^*(\tilde{X}_{\mathcal{G}})$ the subgroup of $\mathrm{Gl}(r,\mathbb{C})$ representing the action of $\mathrm{Mon}(\tilde{X}_{\mathcal{G}})$ on $H^2_{\mathrm{cpct}}(\tilde{X}_{\mathcal{G}},\mathbb{C})$. We have the following result [131].

Theorem 3.3.1. Let $X_{\mathcal{G}}$ be a Du Val singularity. For each smoothing $\tilde{X}_{\mathcal{G}}$ of $X_{\mathcal{G}}$ we have $\mathrm{Mon}^*(\tilde{X}_{\mathcal{G}}) \cong \mathcal{W}_{\mathcal{G}}$.

Corollary 3.3.1. The moduli space of complex structure of $X_{\mathcal{G}}$ is identified with the quotient of the period space $\mathbb{C}^r$ by $\mathcal{W}_{\mathcal{G}}$.

Let's make an example to clarify the previous setup and the definition of the monodromy η [131]. Let's consider the deformed A_1 singularity:

$$x^2 + y^2 + (z-t)(z+t) = 0. \tag{3.3.16}$$

In this case, for each t, the reflection $\eta(z) = -z$ is an order two complex automorphism[16]. This is identified with the generator of the transformations acting non-trivially on the second integral homology: $\mathrm{Mon}^*(\tilde{A}_1) \cong \mathcal{W}_{A_1} = \mathbb{Z}_2 \cong \{\mathbb{1}, \eta^*\}$. The $\Omega_{2,0}$ period on the generator of $H_{2,\mathrm{cpct}}(\tilde{A}_1,\mathbb{Z})$ is

$$\int_{\alpha_1} \Omega_{2,0} = -2t \longrightarrow 2t. \tag{3.3.17}$$

We can regard to (3.3.17) in the bigger picture of the Picard-Lefschetz theorem,

[16]There are indeed also, e.g., $x \to -x$ and $y \to -y$ as automorphism, but they induce the same action of $z \to -z$ on the second homology.

that we will state at the end of this section. Indeed, to construct the generator of $\mathrm{Mon}(\tilde{X}_{\mathcal{G}})$ we can use the following procedure: first we pass from t to $\mu_2 = -t^2$ considering the miniversal deformations of A_1:

$$x^2 + y^2 + z^2 - \mu_2 = 0. \tag{3.3.18}$$

We consider a circle $\mu_2(\theta) = e^{i\theta}$ around the point $\mu_2 = 0$ of the basepace of the miniversal deformations $\mathbb{C}_{\mu_2}$. We notice that $\mu_2 = 0$ is the divisor of $\mathbb{C}_{\mu_2}$ defined by the vanishing of α_1. Then, we consider, at fixed value of θ, the two sphere $\mathbb{S}^2(\theta)$ built as a suspension of $\mathbb{S}^1$ among[17] the points $z_1(\theta) = e^{i\frac{\theta}{2}}$, $z_2(\theta) = e^{i\left(\frac{\theta}{2}+\pi\right)}$. The fact that all the deformed Du Val of type $\mathcal{G}$ are diffeomorphic ensures that, at each fixed value of θ, there exists a *diffeomorphism* $\eta(\theta)$ that identifies the fiber over θ with the fiber at $\theta = 0$. For $\theta = 2\pi$ we come back to initial deformed A_1, and we can pick $\eta(\theta)$ in such a way that $\eta(2\pi)$ induces a well defined action on the integral homology. A map η constructed with this procedure is called [131] a *monodromy* of the fiber over $\mu_2 = 1$. $\eta(2\pi)$ is not the trivial automorphism: the orientation of the submanifold $\mathbb{S}^2(\theta)$ is reversed since $z_1(2\pi) = z_2(0)$ and $z_2(2\pi) = z_1(0)$. In our language, $\eta \equiv \eta(2\pi)$ acts as a reflection $\eta_* = -\mathbb{1}$ on $H_{2,\text{cpct}}(\tilde{A}_1, \mathbb{C})$.

In general, $\mathcal{W}_{\mathcal{G}}$ is generated by these reflections. Consequently, to prove that $\mathrm{Mon}(\tilde{X}_{\mathcal{G}})$ is isomorphic to $\mathcal{W}_{\mathcal{G}}$ it is enough to reformulate in a general language the previous construction. This construction, following [131], can be considered a definition of a mondromy transformation η. The procedure is the following:

1. write the miniversal deformations of $X_{\mathcal{G}}$;
2. pick a divisor D_i of the basespace $\mathbb{C}^r_\mu$ defined as the locus in the μ space where α_i has zero holomorphic volume;
3. consider a path $\mu(\theta)$ encircling D_i, and build the corresponding family of diffeomorphism $\eta_i(\theta)$;
4. $\eta_i \equiv \eta_i(2\pi)$ built in this way is a monodromy tranformation and acts[18] on $H_{2,\text{cpct}}(X_{\mathcal{G}}, \mathbb{C})$ as the Weyl reflection with respect to the i-th root of $\mathcal{G}$ (after we identify the two-cycle α_i with the dual root α_i^*).

This result an example of the application of the Picard-Lefschetz theorem [131] that

[17] This $\mathbb{S}^1$ is the compact direction of the $\mathbb{C}^*$ fiber that we visualize if we write (3.3.18) as $uv = \mu_2 - z^2$ (with $u = x + iy$ and $v = x - iy$). The $\mathbb{C}^*$ action is $u \to \lambda u, v \to v/\lambda$, with $\lambda \in \mathbb{C}^*$.

[18] There are no easy expressions for the η_i apart that in the A_1 case. The η_i can be obtained using smooth characteristic functions, supported on a topological disk of the z-plane containing the endpoints t_i, t_{i+1} of the cycle α_i [131]. Outside this topological disk, the diffeomorphisms $\eta(\theta)$ are the identity. Inside the disk, $\eta(\theta)$ rotates t_i into t_{i+1} (and viceversa), changing the orientation of α_i.

holds true generically for families of complex n-folds. We state here a specific version of it for Du Val singularities.

Theorem 3.3.2 (Picard-Lefschetz)**.** Let $(\mathcal{F}_{\mathcal{G}}, \mu)$ be the miniversal deformation of a Du Val singularity $X_{\mathcal{G}}$. Let $D(\alpha)$ be the divisor of the basespace $\mathbb{C}^{\mathrm{rank}(\mathcal{G})}$ defined by the vanishing of $\alpha \in H_{2,\mathrm{cpct}}(\tilde{X}, \mathbb{Z})$, and $\mu(\theta)$ a real path encircling $D(\alpha)$ in $\mathbb{C}^{\mathrm{rank}(\mathcal{G})} \setminus D(\alpha)$. Consider the family $\eta_\alpha(\theta)$ of diffeomorphism gluing back $\mu^{-1}(\mu(\theta))$ with $\mu^{-1}(\mu(0))$, and $\eta_\alpha \equiv \eta_\alpha(2\pi)$ the corresponding monodromy transformation of $\mu^{-1}(\mu(0))$. Then, for each $\beta \in H_{2,\mathrm{cpct}}(\mu^{-1}(0), \mathbb{C})$, we have

$$(\eta_\alpha)_* (\beta) = \beta + \langle \beta, \alpha \rangle \alpha, \tag{3.3.19}$$

with $\langle \cdot, \cdot \rangle$ the intersection form of the degree 2 homology of $X_{\mathcal{G}}$.

For Du Val singularities $\langle \alpha_j, \alpha_i \rangle = -C_{ij}$, with C_{ij} the Cartan matrix of the corresponding simple Lie algebra. Then, we obtain Theorem 3.3.1 as a corollary of the Picard-Lefschetz theorem.

We conclude that the moduli space of complex structures of a Du Val singularity is

$$\mathcal{M}_{\mathrm{cplx}}(X_{\mathcal{G}}) \cong \frac{\mathbb{C}^r}{\mathcal{W}_{\mathcal{G}}}, \tag{3.3.20}$$

where the numerator of the r.h.s. is the period space, spanned by the periods of $\Omega_{2,0}$, and the denominator is the action of the Picard-Lefschetz transformations. (3.3.20) rephrases, in the language of Torelli theorem the following important fact: *the eigenvalues t_i of the Higgs field Φ cover, via the action of $\mathcal{W}_{\mathcal{G}}$, the complex structures moduli space of $X_{\mathcal{G}}$.*

Moduli space of hyperkähler metric g and SYM theory

In the context of the hyperähler geometry of Du Val singularities, a Torelli theorem holds also for the hyperkähler metric. Proceeding similarly to the previous section, the moduli space of the metric g is the period space of $(\omega_1, \Omega_{2,0})$, modded by the action of $\mathcal{W}_{\mathcal{G}}$:

$$\mathcal{M}(g) = \frac{(\mathbb{R} \times \mathbb{C})^r}{\mathcal{W}_{\mathcal{G}}}. \tag{3.3.21}$$

$\mathcal{M}(g)$ coincides exactly with the moduli space of vacua of $\mathcal{N} = 1, \mathcal{D} = 7$ SYM theory with gauge algebra $\mathcal{G}$. Indeed, not only the quotient group in both cases coincides with $\mathcal{W}_{\mathcal{G}}$, but the Picard-Lefschetz theorem exactly fixes, in (3.3.21), the action of the group to be the same that we have on $\mathfrak{t}$. Furthermore, on the moduli space of vacua of $\mathcal{N} = 1, \mathcal{D} = 7$ SYM theory with gauge algebra $\mathcal{G}$ propagate also

$r = \mathrm{rank}(\mathcal{G})$ photons. In the M-theory context, these vector degrees of freedom come from the reduction of the C_3 over $H^2_{\mathrm{cpct}}(X_{\mathcal{G}}, \mathbb{Z})$.

In physical terms what we just reviewed suggests that M-theory on $X_{\mathcal{G}}$ describes, in the IR, the same physics of the corresponding SYM theory. This is true also at level of BPS spectra, as we will see in the next sections.

3.3.2 Geometric Engineering on Calabi-Yau threefolds

In this case, we want to compactify on a non-compact Calabi-Yau threefold X. The minimal supersymmetry algebra in $\mathcal{D} = 11 - 6 = 5$ dimensions has eight real supercharges. We need to require Ricci-flatness, namely reduction of the holonomy of the internal metric to a subgroup of $SU(3)$. For the threefold case, we do not have anymore the hyperkähler rotation and the moduli space of vacua splits unambiguously in Kähler modes and complex structure modes. We also have to take into account the zero modes of the M-theory three form. These will play the role both of the photons on the ECB effective theory, and of "fibral" coordinates in the HB. The moduli space of vacua splits, in the geometric engineering limit, into two parts:

- The Kähler cone, which coincides with the CB of the five-dimensional effective theory;
- The HB of the effective five-dimensional theory, realized [147, 148] as the intermediate Jacobian fibration of X.

The intermediate Jacobian fibration is a family, over the space of miniversal deformation of X, of abelian varieties. The fibers are spanned by the vevs of the C_3 that, entering in the eleven-dimensional action just as a purely imaginary term, take values in the abelian variety

$$C_3 \in \frac{H^3(X, \mathbb{C})}{H^3(X, \mathbb{Z})}, \tag{3.3.22}$$

where we note that we do not require the cocycles along with we reduce C_3 to be compact-supported.

Going to the common origin of the HB and of the CB we recover the full singularity X. It is conjectured that, if X is a terminal or a canonical singularity [8], M-theory on X gives a five-dimensional superconformal field theory. This can be guessed because, since there are no cycles present in the internal geometry, there can not be mass-scales in the effective $\mathcal{D} = 5$ theory. In this sense, geometric-engineering is a powerful tool to study five-dimensional SCFT.

Finally, to capture subtle features, such as the higher-form symmetries, of the five-dimensional theory we have to consider the boundary conditions at infinity for the M-theory fluxes [149–153]. From the viewpoint of X, the global structure of the theory is controlled by the torsional homology of the five-manifold $L_5(X)$ defined as the boundary at infinity of X.

We conclude this subsection with a remark on the ECB photons coming from the C_3 reduction. For the Du Val case, we reduced the C_3 over the compact support cohomology of degree two. Indeed, we were just interested in the gauge vector bosons, and we did not care about the non-compactly supported part of the degree-two cohomology. On the contrary, for the threefolds studied in this thesis, the compactly supported cohomology will be absent[19]. The only closed two-forms will be the Poincaré duals of non-compact Weil divisors of the singularity[20] and will produce just background flavor symmetries.

3.3.3 BPS particles, M2 states and GV invariants

In the context of supersymmetric gauge theory, whenever the effective theory on a branch of the moduli space of vacua contains IR photons, we have well defined expressions for the masses of BPS degrees of freedom as functions of the moduli space coordinates. From the viewpoint of the moduli space geometry, this information fixes the metric[21]. When one of these BPS states becomes massless, a singularity is created, and new mixed branches open up: we can now give a vev to the fields representing the BPS massless degrees of freedom. We would like to get a similar picture in M-theory compactification and to regard the HB as spanned by invariants operators associated to these BPS particles. To achieve this target, we need to identify the M-theoretical objects that represent the analogous of BPS particles. These objects are the M2 branes wrapped on compact two cycles of the internal geometry X [69, 70]. Indeed, these M2 branes appear as point-like particles to an observer living in the non-compact dimensions. The fact that the wrapped curves are compact fixes the masses of the M2 states at a finite value. Consequently, these states are dynamical degrees of freedom of the $\mathcal{D}$-dimensional effective theory, rather than background particles.

BPS particles in M-theory compactification on Du Val surfaces

We saw, in Section 3.3.1, that the effective theory describing the moduli space of $X_\mathcal{G}$ has exactly the same field content as the theory on the $\mathcal{N}=1, \mathcal{D}=7$ SYM moduli space of vacua. Though, this is not enough yet to conclude that the theories are the same: finer information such as the spectrum of BPS particles on the moduli space of vacua are needed.

[19] Indeed, we will study terminal singularities, and no compact divisors are present in the resolved phase of the geometry.

[20] These will become Cartier divisors after the resolution.

[21] See, e.g., [10] for the relevant context of $\mathcal{D}=5$ dimensional theories.

For $\mathcal{N} = 1, \mathcal{D} = 7$ theory with gauge algebra $\mathcal{G}$ the BPS states are the roots vectors of $\mathcal{G}$, and their masses are fixed by the corresponding root.
In the case of Du Val, it turns out that not all the elements of $H_{2,cpct}(X_{\mathcal{G}}, \mathbb{Z})$ support a BPS state. Interestingly, the elements supporting a BPS state form a root system [101, 154, 155], exactly isomorphic to the one of the corresponding $\mathcal{G}$. Each of these compact homology classes is charged under the generators of $H^2_{\text{cpct}}(X, \mathbb{Z})$ as the corresponding root vector of the $\mathcal{G}$ algebra. Hence, if we think to the abelian fields A^i_ν (with $i = 1, ..., r = \text{rank}(\mathcal{G})$) of the r abelian vector multiplets that propagate on the Kähler moduli space as coming from the dimensional reduction of C_3 on $H^2_{\text{cpct}}(X, \mathbb{Z})$ we have the following:

- The fields A^i_ν of the effective theory are identified with the dual roots of the Lie algebra $\mathcal{G}$.
- The charges of a BPS particle under these photons are dictated by the root vector associated to the particle. Geometrically, exploiting again Poincaré duality, this corresponds to the intersection number between homology classes.

The two BPS spectra match and the effective theory of M-theory compactification on $X_{\mathcal{G}}$ is exactly the same (also from the BPS spectrum viewpoint) of the one that we get via RG flow from the corresponding seven-dimensional $\mathcal{G}$ super Yang-Mills theory. We can then say that (effectively) the M-theory on $X_{\mathcal{G}}$ *gives* the $\mathcal{N} = 1, \mathcal{D} = 7$ theory with gauge algebra $\mathcal{G}$.

BPS particles in M-theory compactification on CY threefold

As in the Du Val case, these BPS states are realized, in M-theory compactification, as M2 branes wrapped around the two cycles that shrink on particular submanifolds of the Kähler cone. The mixed branches that open up on these submanifolds can be understood both geometrically (as spanned by the holomorphic volumes of the three-cycles realizing the smoothings of the singular points that appear after some of the M2-wrapped curves get zero Kähler volume) and from a field theory perspective (as vevs of operators associated to the M2 BPS particles that become massless when the two-cycles get contracted). The most famous example of this is the conifold [156]. Indeed, the M-theory compactification on the conifold has an empty CB, but has a one-dimensional ECB spanned by the periods of the Kähler form on the $\mathbb{P}^1$ blown-up in the conifold resolution. At the origin of the Kähler cone, an HB opens up. This can be understood in two ways:

- geometrically, the $\mathbb{P}^1$ vanished, and it left room for a $\mathbb{S}^3$ to inflate, producing a smoothing of the conifold singularity to $T^*\mathbb{S}^3$.

- from a BPS state perspective, the vanishing $\mathbb{P}^1$ supported a BPS M2 brane. From [66], we know that it effectively produces an hypermultiplet in five-dimensions, whose complex scalars $Q, \tilde{Q}$ span the HB.

Interpretation of the M2 states as GV invariants

It turns out that the M2 BPS states have a precise mathematical definition in terms of topological invariants of $\hat{X}$ [69, 70]. Indeed, there exists a way to re-sum the Gromov-Witten invariants generating function to give a series with integer (rather than rational) coefficients. These coefficients are called *Gopakumar-Vafa* (GV) invariants, and, coming from the Gromov-Witten theory, are currently studied in enumerative geometry. In the physics context, Gromov-Witten invariants appear in the A-model topological string with target space $\hat{X}$. Let us call $j \in \mathbb{C}^R$, with $R \equiv \text{rank}\left(H^2_{\text{cpct}}(\hat{X}, \mathbb{Z})\right)$ the period vector of the complexified Kähler form $\omega + iB$, with B the Kalb-Ramond potential. We have that the free energy F_{GW} of the closed topological A-model with target $\hat{X}$ is

$$F_{GW}(j, g_s) = \sum_{g=0}^{\infty} F_g(j) g_s^{2-2g}, \tag{3.3.23}$$

with g the string coupling. F_{GW} consists of a polynomial perturbative part $F_p(t, g_s)$ and of a non-perturbative contribution F_{WS} coming from the worldsheet instantons:

$$F_{GW}(j, g_s) = F_p(j, g_s) + F_{WS}(j, g_s). \tag{3.3.24}$$

The Gromov-Witten coefficients are the multiplicities of the worldsheet instantons contributions:

$$F_{WS}(j, g_s) = \sum_{g=0}^{\infty} \sum_{d \in \mathbb{Z}^R} N_g^d e^{-d \cdot t}, \tag{3.3.25}$$

with $d \in \mathbb{Z}^R$ the coefficients of the target class $\gamma \in H_{2,cpct}(X, \mathbb{Z})$ whose representative can be chosen as the worldvolume of the topological string propagating in X. The series can be re-summed, using intuitions from M-theory [69, 70], with an auxiliary variable m as

$$F_{WS} = \sum_{g=0}^{\infty} \sum_{d \in \mathbb{Z}^R} n_g^d \sum_{m=1}^{\infty} \frac{1}{m} \left(2 \sin(m g_s/2)\right)^{2g-2} e^{-md \cdot t}. \tag{3.3.26}$$

The coefficients $n_g^d \in \mathbb{Z}$ are called genus g, degree d Gopakumar-Vafa invariants. They permit to reconstruct completely the non perturbative part of the free energy of the topological string, and hence they contain the same amount of data as the Gromov-Witten invariants N_g^d.

In this sense, acceding to the data of the HB (and of mixed branches if present) of the moduli space of M-theory compactification on X gives us detailed information on the GV invariants of X. Viceversa, computing the GV invariants of X permits us to characterize the HB of M-theory compactified on X. The results obtained in this thesis, from a mathematical perspective, can be regarded as a new way to compute GV invariants (together with their degree) for cDV threefold singularities.

More specifically, in this thesis we will encounter isolated singularities whose exceptional locus, after the resolution, is given by a bunch of $\mathbb{P}^1$'s. These (genus-zero) holomorphic curves are rigid and the BPS M2-branes wrapped on them generate massless hypermultiplets in the 5d theory [157] coming from the reduction of M-theory on X. The genus-zero and degree $\mathbf{d} = (d_1, ..., d_f)$ Gopakumar-Vafa invariants $n^{g=0}_{d_1,...,d_f}$ count such states. In particular, we say that

$$n^{g=0}_{\mathbf{d}} = \#\text{ 5d hypers with charges } \vec{d} = (d_1, ..., d_f) \text{ under the flavor group generated by } \mathcal{H}. \tag{3.3.27}$$

Chapter 4

Data from the Higgs field

In this chapter we want to show how to extract the data of M-theory geometric engineering on a cDV singularity X from the dual Higgs profile Φ describing its type IIA limit. We already saw in Section 2.2.1 how to reconstrut the threefold from the Higgs field Φ. We will show in this section how to count the five-dimensional zero modes spanning the HB of the five-dimensional theory and to extract the flavor and discrete gauging symmetries. This will be the method that we are going to use also in the next chapters to study systematically the flops of all lengths and all the quasi-homogeneous cDV threefolds.

In the second part of this chapter, we will show our method at work in some well studied threefold examples: the (A_1, A_{2k-1}) singularities (also known as Reid's pagodas) [126] and two length-two flops already studied in the literature [80, 127]: the Brown-Wemyss threefold and the Laufer's threefolds. These examples present the advantage of being already partly understood from a geometric viewpoint, this will turn out to be crucial in determining the dual Higgs field Φ. In particular, we will use that the exceptional locus of the resolved geometry $\hat{X}$ is known. In our language, this means that, according to the theory developed in Section 2.2.1, we have a natural candidate for the minimal Levi subalgebra $\mathcal{L}$ where Φ resides.

We will conclude this chapter giving an improved modes-counting algorithm, that will permit us to extract, in particular in Chapter 5, refined information on the threefolds corresponding to Φ.

4.1 5d zero modes computation

Given a Higgs field Φ, the zero modes are the holomorphic deformations of the Higgs field up to (linearized) gauge transformations, i.e. $\varphi \in \mathcal{G}$ such that

$$\partial\varphi = 0 \qquad \varphi \sim \varphi + [\Phi(w), g]\,, \tag{4.1.1}$$

with $g \in \mathcal{G}$. To study the zero modes, we then have to work out which components of the deformation φ can be set to zero by a gauge transformation (4.1.1). One then tries to solve the equation

$$\varphi + \delta_g \varphi = 0\,, \qquad \text{with} \qquad \delta_g \varphi = [\Phi(w), g] \tag{4.1.2}$$

with unknown $g \in \mathcal{G}$. Each component of φ is a holomorphic polynomial in w. There will be gauge transformations that cancel the full polynomial and gauge transformations that allow to cancel all the monomials appearing in the w-expansion of the considered φ-component except the terms w^k with $k < n$ for some $n > 0$. In the first case, that component does not support any zero mode. In the second case, the gauge fixed mode may belong to $\mathbb{C}[w]/(w^n)$; this means that at $w \neq 0$ the mode can be gauge fixed to zero, but at $w = 0$ we still have some freedom. If we are left, for example, with $\mathbb{C}[w]/(w)$, the result is one 5d zero mode localized at $w = 0$. When we have $\mathbb{C}[w]/(w^n)$, there are n zero modes localized at $w = 0$. Finally, there are components that are not touched by the gauge fixing procedure: they host a 7d mode that extends in all $\mathbb{C}_w$.

As we have seen in Section 2.2 the Higgs field will always live in a maximal subalgebra $\mathcal{M}$ of the Levi subalgebra[1] $\mathcal{L}$. We can then branch the algebra $\mathcal{G}$ w.r.t. $\mathcal{M}$:

$$\mathcal{G} = \mathcal{M} \oplus ... = \bigoplus_p R_p^{\mathcal{M}}, \tag{4.1.3}$$

where $R_p^{\mathcal{M}}$ are irreducible representations of $\mathcal{M}$. Since $\Phi \in \mathcal{M}$, if we take $g \in R^{\mathcal{M}}$, then also the commutator in (4.1.1) lives in $R^{\mathcal{M}}$. We can then solve the equation (4.1.2) in each representation in (4.1.3) individually, i.e. we consider the deformations φ in each $R^{\mathcal{M}}$ and check how much of it can be fixed by (4.1.1). We can associate to a representation $R_p^{\mathcal{M}}$ of $\mathcal{M}$ a set of integer numbers $(q_1, ..., q_f)$ that are the charges under the $U(1)^f$ group generated by $\mathcal{H}$. All the modes living in a given representation $R^{\mathcal{M}}$ have the same charges $(q_1, ..., q_f)$ under $U(1)^f$. Let us provide an example of zero modes computation in a given representation.

4.1.1 Example of zero mode computations

Let us consider the following element of A_3:

$$\Phi = \begin{pmatrix} 0 & 1 & 0 & 0 \\ w & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -w & 0 \end{pmatrix}. \tag{4.1.4}$$

[1] We remark that we can also have, as it happens in many cases, $\mathcal{M} = \mathcal{L}$.

As we will see, the Higgs field (4.1.4) is associated to the (A_1, A_3) singularity defined in (4.3.1).

The Higgs field is in the subalgebra ($\mathcal{M} = \mathcal{L}$)

$$\mathcal{M} = A_1^{(1)} \oplus A_1^{(3)} \oplus \mathcal{H} \subset A_3 \,, \qquad \text{with } \mathcal{H} = \langle \alpha_2^* \rangle, \tag{4.1.5}$$

where the superscript of $A_1^{(j)}$ refers to the fact that the A_1 subalgebra is generated by the root vectors $e_{\pm\alpha_j}$. The algebra $\mathcal{G} = A_3$ can be decomposed into representations of $\mathcal{M}$ as

$$A_3 \;=\; (3,1)_0 \oplus (1,3)_0 \oplus (1,1)_0 \oplus [(2,2)_1 \oplus c.c.] \,, \tag{4.1.6}$$

where the subscripts denote the charges q under α_2^* and the notation $(j,k)_q$ means that the considered irrep. is the tensor product between the j-dimensional irrep of $A_1^{(1)}$ and the k-dimensional irrep. of $A_1^{(3)}$. The representations $(3,1)_0 \oplus (1,3)_0 \oplus (1,1)_0$ support one 7d mode each, as can be checked by a simple computation. Let us focus on the most interesting representation, i.e. $(2,2)_1$. The $(2,2)_1$ is a four-dimensional irreducible representation of $\mathcal{L}$, composed by the elements of A_3 (expressed in the fundamental representation) of the following shape

$$g|_{(2,2)_1} = \begin{pmatrix} 0 & 0 & g_4 + g_2 & g_1 \\ 0 & 0 & g_3 & g_4 - g_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = g_1 e_1 + g_2 e_2 + g_3 e_3 + g_4 e_4, \tag{4.1.7}$$

with

$$e_1 \;=\; \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$e_3 \;=\; \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad e_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let us write Φ in this four-dimensional representation:

$$\Phi_{(2,2)_1} = \begin{pmatrix} 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2w \\ w & 0 & 0 & 0 \end{pmatrix} \tag{4.1.8}$$

Since we fixed a basis of $(2,2)_1$, then $\varphi|_{(2,2)_1}$ and $g|_{(2,2)_1}$ are vectors with four entries, along the basis elements of $(2,2)_1$ and, in particular,

$$\left[\Phi, g|_{(2,2)_1}\right] = \Phi_{(2,2)_1} \cdot \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \end{pmatrix} = \begin{pmatrix} -2g_2 \\ g_3 \\ 2g_4 w \\ g_1 w \end{pmatrix}, \tag{4.1.9}$$

with g_i, with $i = 1, 2, 3, 4$ holomorphic functions of w. We now perform the gauge fixing: we have to solve (4.1.2) inside $(2,2)_1$:

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{pmatrix} + \begin{pmatrix} -2g_2 \\ g_3 \\ 2g_4 w \\ g_1 w \end{pmatrix} = 0, \tag{4.1.10}$$

where φ_i are holomorphic functions of w. We see that we can pick $g_2 = \varphi_1/2, g_3 = -\varphi_2$, that will completely gauge-fix to zero the first two entries of $\varphi|_{(2,2)_1}$. On the other hand, we can gauge-fix φ_3 and φ_4 to be elements of $\mathbb{C}[w]/(w)$, i.e. they give 5d modes localized at $w = 0$, that are charged under the $U(1)$ flavor group. The complex conjugate representation analogously gives two modes with opposite charge. Hence in total we obtain two free hypermultiplets that are charged under $U(1)$. With the same method, one can check that the other irreducible representations do not localize any 5d mode.

4.2 The symmetry group

The 7d SYM theory that we considered in Section 3.1 has gauge group G, whose Lie algebra is $\mathcal{G}$. Since all fields are in the adjoint representation of $\mathcal{G}$, the non-trivial acting group is the quotient of the simply connected group associated with $\mathcal{G}$ by its center[2]. We take such a quotient as our 7d group G. We will, in Section 6.6.1, see

[2] Actually there is an ambiguity in choosing the global group of the 7d theory [149–151, 153]. Taking the minimal choice, as we are doing, one captures the non-trivial discrete symmetries that come solely from Higgsing. Different choices would enlarge the discrete symmetries with elements

that, by doing this choice, the discrete gauging groups will be related to the torsional boundary homology of X.

Switching on the vev for $\Phi(w)$ on one side breaks G and on the other side generates zero modes localized at $w = 0$, that are charged under the preserved symmetry group. Such a symmetry group is $\mathrm{Stab}_G(\Phi) \subset G$, with

$$\mathrm{Stab}_G(\Phi) \equiv \{U \in G \text{ s.t. } \mathrm{ad}_U(\Phi) = \Phi\} \ , \tag{4.2.1}$$

with $\mathrm{ad}_U(\Phi)$ denoting the adjoint action[3] of the element $U \in G$ on Φ.

Our Higgs field Φ is associated, according to what explained in Section 2.2, to a threefold that (simultaneously) resolves the roots $\{\alpha_{i_1}, ..., \alpha_{i_f}\}$. This is realized by letting $\mathcal{H}$ (defined in (2.2.13)) commute with Φ. The commutant of $\mathcal{H}$ is the Levi subalgebra $\mathcal{L}$ associated with the choice of the roots $\{\alpha_{i_1}, ..., \alpha_{i_f}\}$. If the Higgs field Φ is a generic element of $\mathcal{L}$, then $\mathrm{Stab}_G(\Phi) = U(1)^f$ (generated by $\mathcal{H}$).

Such $U(1)^f$ group, namely the symmetry preserved by $\Phi \in \mathcal{L}$, is nothing but the five-dimensional flavor group, acting via its adjoint representation on the hypermultiplets coming from the deformation φ. The explicit flavor charges of the hypermultiplets can be readily computed employing the irrep decomposition (4.1.3), that naturally regroups hypers of the same charge into the same irrep. If there are n 5d modes in the representation $R_p^{\mathcal{M}}$, there will be other n 5d modes in the conjugate representation $\bar{R}_p^{\mathcal{M}}$; together these generate n massless hypermultiplets in the 5d theory localized at the singularity. It is however non-trivial how the modes localized in $\mathcal{M}$ will still come in pairs. However, we expected this from the field theory viewpoint: the Higgs backgrounds we chose preserve $\mathcal{A}_{\mathcal{D}=1,\mathcal{N}=5}$ and hence the zero modes have to re-arrange in $\mathcal{A}_{\mathcal{D}=1,\mathcal{N}=5}$ supermultiplets. In general, if we only have one $U(1)$ factor, associated with a simple root α_i, then the flavor charges can acquire values only up to the dual Coxeter label of the node α_i in the Dynkin diagram of the considered 7d algebra [130]. If, instead, $\mathrm{Stab}_G(\Phi) = U(1)^f$ with $f > 1$, this is not valid anymore.

As we have said in Section 2.2, in general we might further refine the choice of

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of the center of the group.

[3]The definition of adjoint action is the following. For each $U \in G$, we can build the G-automorphism Ad_U that acts as follows on each element $U_0 \in G$

$$U_0 \to \mathrm{Ad}_U(U_0) = U \star U_0 \star U^{-1},$$

with $\star$ referring to the group operation of G. The identity $\mathbb{1} \in G$ is preserved by all the automorphisms Ad_U, hence the differential $\mathrm{ad}_U \equiv d\mathrm{Ad}_U$ induces an automorphism of the tangent space $T_{\mathbb{1}}G \cong \mathcal{G}$. ad_U, coming from a group automorphism of G (and not simply from a diffeomorphism of G), preserves the Lie parenthesis of $\mathcal{G}$ and is called the adjoint action of U on $\mathcal{G}$. For matrix groups, we have

$$\mathrm{ad}_U(g) = U \cdot g \cdot U^{-1},$$

with $\cdot$ denoting the usual matrix product.

$\mathcal{L}$ and we have

$$\Phi \in \mathcal{M}, \qquad \text{with} \qquad \mathcal{M} = \bigoplus_h \mathcal{M}_h \oplus \mathcal{H} = \mathcal{M}_{\text{s.s.}} \oplus \mathcal{H} \tag{4.2.2}$$

where $\mathcal{M}$ is a maximal-rank maximal subalgebra of the Levi subalgebra $\mathcal{L}$ associated to the resolved $\mathbb{P}^1$ of X via Theorem 2.2.2 and Theorem 2.2.3. In (4.2.2) the factor $\mathcal{M}_{\text{s.s.}} \equiv \bigoplus_h \mathcal{M}_h$ is contained in the semisimple part $\mathcal{L}_{\text{s.s.}}$ of $\mathcal{L} = \mathcal{L}_{\text{s.s.}} \oplus \mathcal{H}$. If $\mathcal{M} < \mathcal{L}$, the preserved group will be bigger than $U(1)^f$ and it will develop a discrete group part.

To explain how this works, we consider a simple example (that will appear often in the threefolds studied in the following). We take

$$\mathcal{L} = D_4 \quad \text{and} \quad \mathcal{M} = A_1^{\oplus 4}.$$

The Dynkin diagram of D_4 with its dual Coxeter labels, along with its $A_1^{\oplus 4}$ subalgebra, is depicted in Figure 4.1. The $A_1^{\oplus 4}$ maximal subalgebra is generated by adding the external node of the extended D_4 Dynkin diagram and removing the central one.

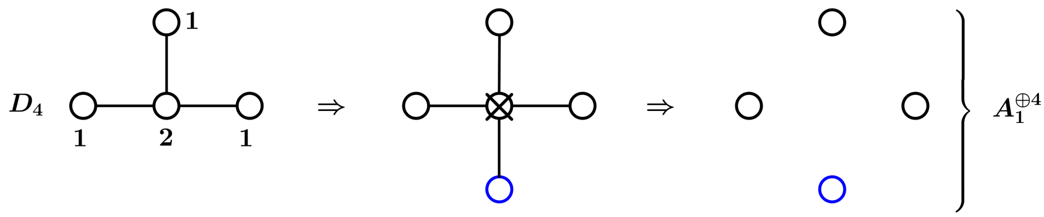


Figure 4.1: $A_1^{\oplus 4}$ *subalgebra of* D_4.

There are transformations of[4] $G_{\mathcal{L}}$ that preserve all the elements of $\mathcal{M} = A_1^{\oplus 4}$ (while they break $\mathcal{L} = D_4$). In this case there is one such element: it is generated by the Cartan α_2^*, i.e. the dual of the root that should be removed from the D_4 extended Dynkin diagram to obtain the Dynkin diagram of $A_1^{\oplus 4}$.[5] The element that is in the stabilizer[6] of $\Phi \in \mathcal{M} = A_1^{\oplus 4}$ is

$$\gamma = \exp\left[\frac{2\pi i}{\mathfrak{q}_{\alpha_2}} \alpha_2^*\right] , \tag{4.2.3}$$

where $\mathfrak{q}_{\alpha_i}$ is the dual Coxeter label of the simple root α_i, and where $\gamma \in G$ acts on the adjoint representation. In our case, we read $\mathfrak{q}_{\alpha_2} = 2$ (see Figure 4.1). In particular, we have

$$\gamma \cdot e_{\alpha_i} = \mathsf{e}^{\frac{2\pi i}{2} 0} e_{\alpha_i} = e_{\alpha_i} \quad \text{for } i = 1, 3, 4, \qquad \gamma \cdot e_{\alpha_2} = \mathsf{e}^{\frac{2\pi i}{2} 1} e_{\alpha_2} = -e_{\alpha_2} \, , \tag{4.2.4}$$

[4]Given a subalgebra $\mathcal{L} \subset \mathcal{G}$, we call $G_{\mathcal{L}}$ the subgroup of G, whose Lie algebra is $\mathcal{L}$.
[5]In Heterotic string theory on T^3, this element is known as a discrete Wilson line.
[6]More than that, it acts trivially on the whole $\mathcal{M}$.

and

$$\gamma \cdot e_{\alpha_\theta} = \mathsf{e}^{\frac{2\pi i}{2}(-2)} e_{\alpha_\theta} = e_{\alpha_\theta} \;, \tag{4.2.5}$$

where α_θ is the (minus the) highest root corresponding to the extended node. Note that the Lie algebra element e_{α_2} is not preserved by γ.

We see that it is crucial for preserving a maximal subalgebra that the coefficient in front of α_2^* in γ is $\frac{2\pi i}{\mathfrak{q}_{\alpha_2}}$ and not any other number. The discrete group generated by γ in (4.2.3) is isomorphic to $\mathbb{Z}_2$.

Let us generalize this to an example that is a bit more involved, i.e.

$$\mathcal{L} = D_6 \quad \text{and} \quad \mathcal{M} = A_1^{\oplus 6}.$$

In this case we proceed by steps, following the inclusions $D_6 \supset D_4 \oplus A_1^{\oplus 2} \supset A_1^{\oplus 4} \oplus A_1^{\oplus 2} = A_1^{\oplus 6}$, depicted in Figure 4.2. In the first step, we remove a node with dual Coxeter label equal to 2. We are then left with the final step in which we embed $A_1^{\oplus 4}$ into D_4: again we remove a node of D_4 Dynkin diagram with label equal to 2. We conclude that the discrete group is $\mathbb{Z}_2^2$.

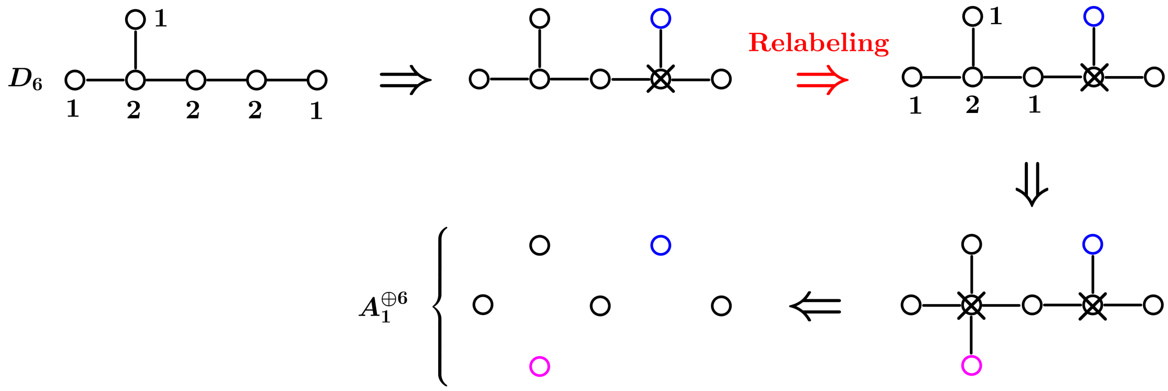


Figure 4.2: $A_1^{\oplus 6}$ *subalgebra of* D_6.

It is then easy to generalize to a generic case. Say that a simple summand of $\mathcal{L}$ has a maximal subalgebra, obtained by subsequently removing nodes with dual Coxeter labels $\mathfrak{q}_{\alpha_{\iota_1}}, \ldots, \mathfrak{q}_{\alpha_{\iota_k}}$. Then the stabilizer of Φ will include the discrete group

$$\mathbb{Z}_{\mathfrak{q}_{\alpha_{\iota_1}}} \times \ldots \times \mathbb{Z}_{\mathfrak{q}_{\alpha_{\iota_k}}}.$$

Doing this for all simple summands of $\mathcal{L}$, we obtain the full discrete symmetry Γ_Φ. The full symmetry group is then

$$\mathrm{Stab}_G(\Phi) = U(1)^f \times \Gamma_\Phi \;. \tag{4.2.6}$$

Since we know how the generators of this group act on the Lie algebra $\mathcal{G}$, we can easily derive the charges under $\mathrm{Stab}_G(\Phi)$ of the deformations φ in $R^{\mathcal{M}}$, i.e. of the

5d hypermultiplets.

The symmetry group (4.2.6) is the 7d gauge group that survives the Higgsing. In order to deduce the 5d flavor and gauge symmetries we can proceed as in [128]: we consider the 7d space as a decompactification limit from 5d times a 2-torus. Before the limit, (4.2.6) is a 5d gauge group; the decompactification limit will ungauge the continuous factor as its gauge coupling vanishes. The discrete part, having no gauge coupling, remains gauged in 5d.

Explicit example: (A_2, D_4) singularity and discrete groups

Let us visualize how it works in an explicit example. We can consider the (A_2, D_4) singularity:

$$x^2 + zy^2 + z^3 + w^3 = 0, \quad (x, y, w, z) \in \mathbb{C}^4. \tag{4.2.7}$$

The threefold can be described as a family of D_4 ADE singularities deformed by the parameter w. The Higgs field is taken in the maximal subalgebra of D_4, i.e. $\mathcal{M} = D_2 \oplus D_2 \cong A_1^4$.

From what we said above, it is immediate to find out

$$\mathrm{Stab}(\Phi)_G = \mathbb{Z}_2\,. \tag{4.2.8}$$

We now see how this discrete group acts on the 5d hypermultiplets. We first branch $\mathcal{G} = D_4$ under $\mathcal{M}$:

$$D_4 = A_1^{(I)} \oplus A_1^{(II)} \oplus A_1^{(III)} \oplus A_1^{(IV)} \oplus (\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2}) = \mathcal{M} \oplus (\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2}). \tag{4.2.9}$$

We then see how γ in (4.2.3) acts on the elements of $(\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2})$. The generators of D_4 appearing in this representation of $\mathcal{M}$ are related to roots that are linear combination of the simple roots where α_2 appears with coefficient 1. This immediately tells us that all elements of $(\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2})$ get a -1 factor when we act with γ.

This can be easily generalized to any choice of $\Phi \in \mathcal{M} \subset \mathcal{G}$ with $\mathcal{G} = A, D, E$.

We have now all the general ingredients to compute the GV invariants of a threefold X associated to a certain Higgs background Φ. Let's proceed studying some threefolds whose resoled geometry was already known in the mathematical literature.

4.3 The (A_1, A_{2k-1}) singularities, i.e. the "Reid's pagodas"

Reid's *pagoda* is a class of singular CY threefolds that admit simple flops, meaning that only one exceptional $\mathbb{P}^1$ is produced. It is defined as the following hypersurface:

$$uv = z^{2k} - w^2 \qquad \subset \quad \mathbb{C}^4_{u,v,w,z}. \tag{4.3.1}$$

The Reid's pagoda is a quasi-homogeneous singularity, with weights

$$[u] = [v] = [w] = 1/2, \qquad [z] = 1/2k. \tag{4.3.2}$$

This geometry admits k normalizable deformations, and hence we expect the Higgs branch to be quaternionic k-dimensional.

Let us apply what we have seen in Section 2.2 to find the associated Φ. The threefold (4.3.1) is an ALE family over w that is singular at the origin, where the ALE fiber develops an A_{2k-1} singularity. Resolving the singularity blows up one exceptional $\mathbb{P}^1$, i.e. we have a simple flop. The root of the singular central fiber that is simultaneously resolved is, following the notation of Figure 5.1, α_k. This M-theory background is reduced in IIA to a system of $2k$ D6-branes Higgsed by an Higgs field of the following form:

$$\Phi = \begin{pmatrix} J_+ & 0 \\ 0 & J_- \end{pmatrix}, \qquad \text{where} \qquad J_\pm := \begin{pmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ \pm w & & \dots & 0 \end{pmatrix}_k . \tag{4.3.3}$$

$J_\pm$ is the $k \times k$ Jordan block plus $(\pm w)$ in the $(k, 1)$-entry. Plugging such Φ into (2.2.20) one can explicitly check that the threefolds (4.3.1) are correctly reproduced.

The D6-branes live on a divisor in $\mathbb{C}^2$ (called the "brane locus") given by:

$$\Delta(z, w) \equiv \det(z \cdot \mathbb{1}_{2k} - \Phi) = z^{2k} - w^2 = 0. \tag{4.3.4}$$

We note that, in this case, the brane-locus is reducible and factors into two degree k irreducible polynomials:

$$\Delta(z, w) = (z^k - w)(z^k + w). \tag{4.3.5}$$

This factorization reflects the fact that the Higgs field Φ is formed by two matrix blocks J_+, J_- each one being the reconstructible Higgs field (introduced in Definition

A.1.4) of, respectively, $z^k - w$ and $z^k + w$.

The first thing we want to study, according to what we explained in Section 4.2, is the flavor group in five dimensions. Our background vev breaks the original seven-dimensional worldvolume[7] $PSU(2k)$ gauge symmetry to a subgroup given by the equivalence classes in $PSU(2k) \cong SU(2k)/\mathbb{Z}_k$ of the following $2k \times 2k$ matrices

$$\begin{pmatrix} e^{i\beta}\,\mathbb{1}_k & 0 \\ 0 & e^{-i\beta}\,\mathbb{1}_k \end{pmatrix} \in U(1)\,. \tag{4.3.6}$$

In this case, (4.3.6) can be obtained by explicit computation, but in a more general language we can notice that (4.3.6) is the (exponential of) the dual root α_k^*, the resolved root. Summing up, our background Higgses as follows:

$$PSU(2k) \longrightarrow U(1), \tag{4.3.7}$$

where we again point out the fact that we pick, as the seven-dimensional gauge group, $PSU(2k)$, namely the group that acts faithfully on $\mathcal{G}$.

Here, one might wonder, whether the full flavor group could be bigger. However, since the IIA setup is built entirely with D-branes, where all flavor (ungauged) groups are derived from the open string picture, we claim that the above group captures the full flavor group (up to the subtlety of possible symplectic completions of the unitary group). Now we wish to understand the Higgs branch. This consists of all possible deformations of the background Higgs field: $\Phi = \Phi + \varphi$, modulo linearized gauge transformations

$$\varphi \sim \varphi + [\Phi, g]\,, \tag{4.3.8}$$

for any broken generator $g \in \mathfrak{sl}(2k)$. Let us write g and φ in the block form

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\,, \qquad \varphi = \begin{pmatrix} \varphi^\alpha & \varphi^\beta \\ \varphi^\gamma & \varphi^\delta \end{pmatrix} \tag{4.3.9}$$

where each block is a $k \times k$ matrix and $\mathrm{Tr}\alpha = \mathrm{Tr}\delta = 0$. In a more general language, in (4.3.9) we re-wrote g, φ respecting the branching of A_{2k-1} w.r.t. the Levi subalgebra $\mathcal{L} = A_{k-1} \oplus A_{k-1} \oplus \alpha_k \ni \Phi$. More precisely α, δ (resp $\varphi^\alpha, \varphi^\delta$) are the elements along $\mathcal{L}$ and β, γ (resp $\varphi^\beta, \varphi^\gamma$) are the in the bifundamentals w.r.t. the two A_{k-1} factors

[7] Here we assume that before switching on the vev (4.3.3), the M-theory background is a A_{2k-1} ALE space, leading to a seven-dimensional gauge theory with $PSU(2k)$ group. In other words the dual type IIA string coupling has been sent to infinity. With a different choice of discrete data one may start with the gauge group $SU(2k)$; we do not make this choice here. See [151, 153] for a clear exposition of these choices, and [149, 150] for the seminal work.

of $\mathcal{L}$. The adjoint action of Φ is:

$$[\Phi, g] = \begin{pmatrix} [J_+, \alpha] & J_+\beta - \beta J_- \\ J_-\gamma - \gamma J_+ & [J_-, \delta] \end{pmatrix} . \tag{4.3.10}$$

We see that, due to the block-diagonal form of the Higgs vev, α only affects the φ^α block of φ, β only φ^β, etc. This means that, in the computation of the deformations, we can work out each block individually. We expected this: the adjoint action (4.3.10) has to respect the branching w.r.t. to $\mathcal{L}$.

Let us perform the gauge-fixing explicitly for $k = 1$, giving in this way an example of the procedure explained in Section 4.1. We have

$$\varphi^\alpha \sim \varphi^\alpha + \begin{pmatrix} \alpha_{21} - \alpha_{12} w & -2\alpha_{11} \\ 2\alpha_{11} w & -\alpha_{21} + \alpha_{12} w \end{pmatrix} . \tag{4.3.11}$$

We can use α_{11}, α_{21} to fix the first line to zero:

$$\varphi^\alpha \sim \begin{pmatrix} 0 & 0 \\ \varphi^\alpha_{21} & \varphi^\alpha_{22} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \varphi^\alpha_{12} w & \varphi^\alpha_{11} \end{pmatrix} . \tag{4.3.12}$$

We see that we do not have further freedom to localize the second line, hence we do not obtain localized modes from this block. The same is true for the block related to δ. Instead, localized modes come from the off-diagonal blocks. Let us consider the block φ^β and let us define its entries in the following convenient way

$$\varphi^\beta = \begin{pmatrix} \varphi^\beta_L + \varphi^\beta_R & \varphi^\beta_{12} \\ \varphi^\beta_{21} & -\varphi^\beta_L + \varphi^\beta_R \end{pmatrix} . \tag{4.3.13}$$

Let us see how much we can gauge fix

$$\varphi^\beta \sim \begin{pmatrix} \varphi^\beta_L + \varphi^\beta_R & \varphi^\beta_{12} \\ \varphi^\beta_{21} & -\varphi^\beta_L + \varphi^\beta_R \end{pmatrix} + \begin{pmatrix} \beta_{21} + \beta_{12} w & \beta_{22} - \beta_{11} \\ (\beta_{22} + \beta_{11}) w & -\beta_{21} + \beta_{12} w \end{pmatrix} . \tag{4.3.14}$$

We see that we can fix to zero φ^β_{12} and φ^β_L by respectively choosing $(\beta_{22} - \beta_{11})$ and β_{21}, obtaining

$$\varphi^\beta \sim \begin{pmatrix} \varphi^\beta_R & 0 \\ \varphi^\beta_{21} & \varphi^\beta_R \end{pmatrix} + \begin{pmatrix} \beta_{12} w & 0 \\ (\beta_{22} + \beta_{11}) w & \beta_{12} w \end{pmatrix} . \tag{4.3.15}$$

We then obtain the two modes localized on the ideal (w), i.e.

$$\varphi^\beta_R, \varphi^\beta_{21} \in \mathbb{C}[w]/(w) \cong \mathbb{C} . \tag{4.3.16}$$

These modes have charge $+1$ with respect to the $U(1)$ flavor symmetry. Analogously we obtain two modes with $U(1)$ charge -1 from the block φ^{γ}.[8]

For generic k we have the same pattern. After gauge fixing we are left with k constant modes in the charge $+1$ block φ^{β}

$$\varphi^{\beta} \sim \begin{pmatrix} \varphi_k^{\beta} & 0 & 0 & 0 & 0 & 0 \\ \vdots & & & & & \vdots \\ \varphi_3^{\beta} & \dots & \dots & \varphi_k^{\beta} & 0 & 0 \\ \varphi_2^{\beta} & \varphi_3^{\beta} & \dots & \dots & \varphi_k^{\beta} & 0 \\ \varphi_1^{\beta} & \varphi_2^{\beta} & \varphi_3^{\beta} & \dots & \dots & \varphi_k^{\beta} \end{pmatrix} \tag{4.3.17}$$

with entries $\varphi_j^{\beta} \in \mathbb{C}[w]/(w) \cong \mathbb{C}$ $(j = 1, ..., k)$. Analogously, we get k constant modes in the charge -1 block φ^{γ}. This gives a total of **k hypermultiplets**.

One can also follow a different path to get the same result. We will shift our paradigm, by relying on a physical argument put forward in [103]. Physically, each block J_+, J_- is describing a single smooth recombined brane, with a $U(1)$ gauge group on it, let's call them $B_{\pm}$, sitting on the hypersurfaces:

$$B_{\pm}: \quad w \mp z^k = 0\,. \tag{4.3.18}$$

Now, we can turn things around, and regard this as a Higgsed background for a starting $PSU(2)$ system as follows:

$$\tilde{\Phi} = \begin{pmatrix} z^k & 0 \\ 0 & -z^k \end{pmatrix}\,. \tag{4.3.19}$$

Once we have this background, we can study the fluctuations as we did before, by modding out by complexified gauge transformations. It is similar to the conifold analysis, with one important major difference, as we will see. Let us again parametrize fluctuations and gauge parameters as follows:

$$\varphi = \begin{pmatrix} \varphi_0 & \varphi_+ \\ \varphi_- & -\varphi_0 \end{pmatrix}\,, \qquad g = \tfrac{1}{2} \begin{pmatrix} g_0 & g_+ \\ g_- & -g_0 \end{pmatrix}\,. \tag{4.3.20}$$

Now, we see that fluctuations are defined up to

$$\varphi \sim \varphi + z^k \begin{pmatrix} 0 & g_+ \\ -g_- & 0 \end{pmatrix}\,. \tag{4.3.21}$$

[8] As done for the conifold, one can switch on a vev for the localized modes. The deformed threefold is then $uv = \det(z\mathbb{1}_4 + \langle\Phi\rangle + \varphi)$. This provides the projection map from the Higgs branch to the deformation space of the Pagoda with $k = 2$.

From this, we deduce two things: Just as for the conifold, localized modes live only in the off-diagonal part of φ. This is as expected, since we are looking at a pair of intersecting branes, albeit with a non-transversal intersection. Second and most importantly, there are now several bifundamental open string modes. The modes live in the following ring:

$$\varphi_\pm \in \mathbb{C}[z]/(z^k) \cong \mathbb{C}^k \,. \tag{4.3.22}$$

So the (φ_+, φ_-) pairs can give rise to k distinct hypermultiplets. This makes intuitive sense, since the two branes intersect at a *fat point* of multiplicity k: in terms of ideals, we have

$$(w + z^k, w - z^k) = (w, z^k) \,. \tag{4.3.23}$$

What is remarkable about this background is that the M-theory geometry sees only one vanishing $\mathbb{P}^1$. Nevertheless, there are k distinct membrane states that give rise to separate hypers.

We conclude this section with a crucial remark: we note that even though there are k hypermultiplets, the flavor symmetry is of rank one. This implies that we can only switch on one real mass, if we are to think of real masses as background vev's in the usual way. The fact that only one real mass is available perfectly matches the fact that the corresponding M-theory threefold only admits a simple flop, as opposed to a reducible one.

4.4 Brown-Wemyss threefold

In this section we study a one-parameter family of deformed D_4 singularities (with base-space parameter w), introduced in [80]:

$$x^2 + zy^2 - (z - w)\left(zw^2 + (z - w)^2\right) = 0 \,. \tag{4.4.1}$$

The threefold is singular at the origin, where the ALE fiber develops a D_4 singularity. The total space admits a small resolution with a flop of length two. The corresponding colored Dynkin diagram $\mathcal{S}_{\text{black}}$ is reported in Figure 4.3

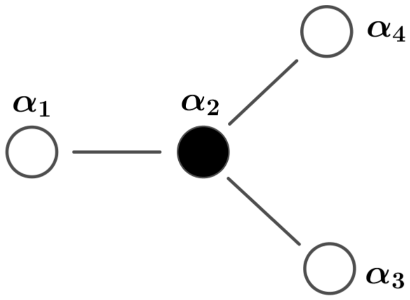


Figure 4.3: *D_4 Dynkin diagram*

This threefold has Milnor number 11 and the number of normalizable deformations is 6, hence we expect a 6-dimensional Higgs Branch. Moreover, the threefold admits a small resolution, leading to a $U(1)$ flavor symmetry.[9]

In IIA we start with a stack of eight D6s at the orientifold location $z = 0$. The seven-dimensional gauge algebra describing the branes dynamics is $\mathcal{G} = D_4$ that we can take to be generated by traceless 8×8 matrices g satisfying:

$$gQ + Qg^t = 0, \qquad \text{with} \qquad Q = \left(\begin{array}{cc|cc} 0 & \mathbb{1}_2 & & \\ \mathbb{1}_2 & 0 & & \\ \hline & & 0 & \mathbb{1}_2 \\ & & \mathbb{1}_2 & 0 \end{array}\right). \tag{4.4.2}$$

We switch on a background Higgs

$$\Phi = \left(\begin{array}{cccc|cccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -w & -w & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & w & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & w & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 0 & \frac{w}{4} \\ 0 & 0 & 0 & 0 & -\frac{w}{4} & 0 & -\frac{w}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & \frac{w}{4} \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \end{array}\right). \tag{4.4.3}$$

The Higgs field lives in the subalgebra $u(2) \times so(4)$ of $so(8)$ associated to the particular block decomposition of (4.4.3).

If we choose the global structure of D_4 to be $SO(8)$, the group that commutes

[9] See [84, 85] for an explicit resolution of these geometries by quiver techniques with a focus on the $U(1)$ symmetry and its charges.

with the Higgs field Φ is isomorphic to $U(1)$:

$$\left(\begin{array}{cccc|cccc} e^{i\alpha} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{i\alpha} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{-i\alpha} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-i\alpha} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \tag{4.4.4}$$

where $\alpha \in [0, 2\pi)$. The flavor group in $G = SO(8)/\mathcal{Z}(SO(8))$ is the one whose equivalence classes can be represented by (4.4.4) and is again isomorphic to $U(1)$. The $U(1)$ group is associated to the resolved trivalent root of the D_4 fiber over $w = 0$ and is seen as a flavor group from the five-dimensional viewpoint.

We now consider the Higgs branch. As for the examples in Section 4.3, this consists in the deformations φ of the background Higgs, modulo linearized D_4 gauge transformations:

$$\varphi \sim \varphi + [\Phi, g] \,. \tag{4.4.5}$$

The commutator $[\Phi, g]$ can be written in the block form

$$[\Phi, g] = \left(\begin{array}{c|c|c} B_{2\times 2} & A^u_{2\times 2} & C_{2\times 4} \\ \hline A^d_{2\times 2} & -B^t_{2\times 2} & D_{2\times 4} \\ \hline D_{4\times 2} & C_{4\times 2} & -B_{4\times 4} \end{array}\right) \tag{4.4.6}$$

where $C_{2\times 4}$ is completely determined by $C_{4\times 2}$ (analogously for the D-blocks). Due to the block-diagonal form of the Higgs (4.4.3), each block of $[\Phi, g]$ depends only on the entries of g in the same block.

Let us proceed now block by block. We start with

$$B_{2\times 2} = \begin{pmatrix} g_{21} + g_{12}w & -g_{11} + g_{22} + g_{12}w \\ -(g_{11} + g_{21} - g_{22})w & -g_{21} - g_{12}w \end{pmatrix} .$$

Using g_{21} and the combination $g_{11} - g_{22}$ we can fix to zero the corresponding entries

φ_{11} and φ_{12} in the fluctuation of the Higgs. We are then left with:

$$\varphi_{2\times 2} \sim \underbrace{\begin{pmatrix} 0 & 0 \\ \varphi_{21} & \varphi_{22} \end{pmatrix}}_{\varphi_{2\times 2}} + \underbrace{\begin{pmatrix} 0 & 0 \\ -w(\varphi_{12}-\varphi_{11}) & \varphi_{11} \end{pmatrix}}_{B_{2\times 2}}. \tag{4.4.7}$$

As a result we see that φ_{21} and φ_{22} are not constrained, and so they are not dynamical in 5d.

The other relevant diagonal block

$$B_{4\times 4} = \begin{pmatrix} g_{58}+g_{65}+\frac{1}{4}(g_{56}-g_{76})w & g_{66}-g_{55} & 0 & -\frac{1}{4}(g_{55}+g_{66})w \\ \frac{1}{4}(g_{66}-g_{55})w & g_{58}-g_{65}+\frac{1}{4}(-g_{56}-g_{76})w & \frac{1}{4}(g_{55}+g_{66})w & 0 \\ 0 & g_{55}+g_{66} & -g_{58}-g_{65}+\frac{1}{4}(g_{76}-g_{56})w & \frac{1}{4}(g_{55}-g_{66})w \\ -g_{55}-g_{66} & 0 & g_{55}-g_{66} & -g_{58}+g_{65}+\frac{1}{4}(g_{56}+g_{76})w \end{pmatrix}$$

does not generate localized modes as well. In fact, using the combinations $(g_{66}-g_{55})$, $(g_{66}+g_{55})$, $(g_{58}-g_{65})$ and $(g_{58}+g_{65})$ we can set to zero, for example, the entries φ_{55}, φ_{56}, φ_{66} and φ_{76}, remaining with:

$$\varphi_{4\times 4} \sim \underbrace{\begin{pmatrix} 0 & 0 & 0 & \varphi_{58} \\ \varphi_{65} & 0 & -\varphi_{58} & 0 \\ 0 & 0 & 0 & \varphi_{78} \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{\varphi_{4\times 4}} + \underbrace{\begin{pmatrix} 0 & 0 & 0 & -\frac{w\varphi_{76}}{4} \\ \frac{w\varphi_{56}}{4} & 0 & \frac{w\varphi_{76}}{4} & 0 \\ 0 & 0 & 0 & -\frac{w\varphi_{56}}{4} \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{B_{4\times 4}}. \tag{4.4.8}$$

The first localized mode comes when we consider $A^u_{2\times 2}$. The gauge equivalence is

$$\varphi^u_{2\times 2} \sim \underbrace{\begin{pmatrix} 0 & \varphi_{14} \\ -\varphi_{14} & 0 \end{pmatrix}}_{\varphi^u_{2\times 2}} + \underbrace{\begin{pmatrix} 0 & -wg_{14} \\ wg_{14} & 0 \end{pmatrix}}_{A^u_{2\times 2}}. \tag{4.4.9}$$

We immediately see that φ_{14} is localized on the ideal (w), giving:

$$\varphi_{14} \in \mathbb{C}[w]/(w) \cong \mathbb{C} \tag{4.4.10}$$

and so it corresponds to 1 localized 5d mode. We note that this mode has charge $+2$ with respect to the flavor $U(1)$ in (4.4.4).

The block $A^d_{2\times 2}$ acts analogously to $A^u_{2\times 2}$ and it yields a localized mode with charge -2 with respect to the flavor $U(1)$.

Let us come to the block $C_{4\times 2}$. The gauge equivalence is:

$$\varphi_{4\times 2} \sim \underbrace{\begin{pmatrix} \varphi_{53} & \varphi_{54} \\ \varphi_{63} & \varphi_{64} \\ \varphi_{73} & \varphi_{74} \\ \varphi_{83} & \varphi_{84} \end{pmatrix}}_{\varphi_{4\times 2}} + \underbrace{\begin{pmatrix} -g_{18} - g_{27} - \frac{g_{16}w}{4} & \left(g_{17} + g_{27} - \frac{g_{26}}{4}\right) w - g_{28} \\ \frac{1}{4}(g_{15} + g_{17})w - g_{28} & \frac{1}{4}(4g_{18} + g_{25} + g_{27} + 4g_{28})w \\ -g_{18} - g_{25} - \frac{g_{16}w}{4} & \left(g_{15} + g_{25} - \frac{g_{26}}{4}\right) w - g_{28} \\ g_{15} + g_{17} - g_{26} & g_{25} + g_{27} + (g_{16} + g_{26})w \end{pmatrix}}_{C_{4\times 2}} . \tag{4.4.11}$$

Almost all the entries in φ corresponding to this block can be gauge-fixed to zero, except φ_{64} and two linear combinations of φ_{54}, φ_{63}, φ_{64}. After having fixed all the other entries to zero, we have:

$$\begin{aligned}
&\varphi_{54} \sim \varphi_{54} + w^2 \left(-\frac{g_{16}}{2} - \frac{g_{26}}{2}\right) + w \left(g_{17} - \frac{g_{26}}{4} + \frac{\varphi_{53}}{2} - \frac{\varphi_{73}}{2} - \frac{\varphi_{84}}{2}\right) - g_{28} \\
&\varphi_{63} \sim \varphi_{63} + \frac{1}{4}w(g_{26} - \varphi_{83}) - g_{28} \\
&\varphi_{64} \sim \varphi_{64} + \frac{g_{26}w^2}{4} + \frac{1}{4}w(4g_{28} + 2\varphi_{53} + 2\varphi_{73} + \varphi_{84}) \\
&\varphi_{74} \sim \varphi_{74} + w^2 \left(-\frac{g_{16}}{2} - \frac{g_{26}}{2}\right) + w \left(-g_{17} + \frac{3g_{26}}{4} - \frac{\varphi_{53}}{2} + \frac{\varphi_{73}}{2} - \varphi_{83} - \frac{\varphi_{84}}{2}\right) - g_{28}
\end{aligned}$$

To make the computation easier and without loss of generality, we redefine φ_{54}, φ_{63} and φ_{74} as

$$\varphi_{54} = \psi_1 - \psi_2 \,, \qquad \varphi_{63} = \psi_3 \,, \qquad \varphi_{74} = \psi_1 + \psi_2 \,. \tag{4.4.12}$$

Using the gauge freedom given by g_{28} we can set ψ_3 to zero, remaining with:[10]

$$\begin{aligned}
&\varphi_{64} \sim \varphi_{64} + \frac{g_{26}w^2}{2} \\
&\psi_1 \sim \psi_1 + \frac{1}{4}w^2(-2g_{16} - 2g_{26}) \\
&\psi_2 \sim \psi_2 + \frac{1}{4}w(2g_{26} - 4g_{17})
\end{aligned} \tag{4.4.13}$$

We immediately see that φ_{64} is localized on the ideal (w^2), yielding:

$$\varphi_{64} \in \mathbb{C}[w]/(w^2) \cong \mathbb{C}^2 \,. \tag{4.4.14}$$

On the other hand, we see that:

$$\begin{aligned}
&\psi_1 \in \mathbb{C}[w]/(w^2) \cong \mathbb{C}^2 \\
&\psi_2 \in \mathbb{C}[w]/(w) \cong \mathbb{C}
\end{aligned} \tag{4.4.15}$$

[10] And discarding the dependence on the other φ_{ij}, that are not free parameters

We then have a total of 5 localized modes with charge $+1$ under the $U(1)$ in (4.4.4).

The block $D_{4\times 2}$ works like $C_{4\times 2}$ and gives 5 localized modes with charge -1 with respect to the $U(1)$ in (4.4.4).

Summing up, we obtain a 5d $\mathcal{N}=1$ theory with six hypermultiplets (the modes with opposite charge pair up into a hyper):

- **1 hyper** of charge 2:[11]

$$\begin{pmatrix} Q_0 \\ \tilde{Q}_0 \end{pmatrix} = \begin{pmatrix} \varphi_{14} \\ \varphi_{41} \end{pmatrix}$$

- **5 hypers** of charge 1:

$$\begin{pmatrix} Q_1 \\ \tilde{Q}_1 \end{pmatrix} = \begin{pmatrix} \varphi_{64}^{(1)} \\ \varphi_{28}^{(1)} \end{pmatrix}, \qquad \begin{pmatrix} Q_2 \\ \tilde{Q}_2 \end{pmatrix} = \begin{pmatrix} \varphi_{64}^{(2)} \\ \varphi_{28}^{(2)} \end{pmatrix},$$

$$\begin{pmatrix} Q_3 \\ \tilde{Q}_3 \end{pmatrix} = \begin{pmatrix} \psi_1^{(1)} \\ \tilde{\psi}_1^{(1)} \end{pmatrix}, \qquad \begin{pmatrix} Q_4 \\ \tilde{Q}_4 \end{pmatrix} = \begin{pmatrix} \psi_1^{(2)} \\ \tilde{\psi}_1^{(2)} \end{pmatrix}, \qquad \begin{pmatrix} Q_5 \\ \tilde{Q}_5 \end{pmatrix} = \begin{pmatrix} \psi_2 \\ \tilde{\psi}_2 \end{pmatrix}.$$

The final output is the following: the Higgs branch turns out to be

$$\mathrm{HB} = \mathbb{C}^{12} \cong \mathbb{C}^5_{+1} \times \mathbb{C}^5_{-1} \times \mathbb{C}_{+2} \times \mathbb{C}_{-2}, \tag{4.4.16}$$

with a spontaneously broken U(1) flavor symmetry, acting with charge q on the $\mathbb{C}^j_q$ factors of the r.h.s. of (4.4.16).

4.5 Laufer threefold

We now generalize the computation done in Section 4.4 to a famous flop of length two, first discovered by Laufer [127]. It is given by the following hypersurface:

$$x^2 + zy^2 - t\left(t^2 + z^{2k+1}\right) = 0 \quad \text{with } k \geq 1\,. \tag{4.5.1}$$

By making the change of variable $t = w - z$, one can put this threefold in the form of a D_{2k+3} family over w:

$$x^2 + y^2 z - (w-z)((w-z)^2 + z^{2k+1}) = 0, \tag{4.5.2}$$

where we note that just a D_4 singularity is realized for $w = 0$.

We want to see (4.5.2) as a deformation of a trivial family (over w) of D_{2k+3}, whose type IIA dual-setup is a stack of $4k+6$ D6s at the orientifold location $z = 0$.

[11] The existence of a charge-2 state localized at the origin of threefolds admitting flops of length two was already predicted in [85].

The seven-dimensional gauge algebra is $\mathcal{G} = D_{2k+3}$. We choose again our convention in such a way that the algebra is generated by the elements g of the algebra $\mathfrak{sl}(4k+6)$ satisfying the additional condition:

$$gQ + Qg^t = 0, \qquad \text{with} \qquad Q = \left(\begin{array}{cc|cc} 0 & \mathbb{1}_{2k+1} & & \\ \mathbb{1}_{2k+1} & 0 & & \\ \hline & & 0 & \mathbb{1}_2 \\ & & \mathbb{1}_2 & 0 \end{array}\right). \tag{4.5.3}$$

From the geometry of the resolved threefold, we expect a five-dimensional $U(1)$ flavor group, acting with charges one and two on the five-dimensional matter. The $U(1)$ subgroup of the Cartan torus of $SO(4k+6)/\mathbb{Z}_2$ that acts with charges one and two on the D_{2k+3} algebra is generated by the dual trivalent root α^*_{2k+1}. Each $\mathbb{Z}_2$-equivalence class of this $U(1)$ can be represented by a matrix of the following form in $SO(4k+6)$:

$$\left(\begin{array}{cccccc|cccc} e^{i\alpha} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{i\alpha} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-i\alpha} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{-i\alpha} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right), \tag{4.5.4}$$

where red dots stands for $e^{i\alpha}$ and blue ones for $e^{-i\alpha}$, with $\alpha \in [0, 2\pi)$.

We want to preserve this $U(1)$ subgroup, hence we pick the Higgs field in the Levi subalgebra $\mathcal{L} = u(2k+1) \times so(4)$ of $so(4k+6)$ that is realized by matrices of the following form:

$$\Phi = \left(\begin{array}{cc|c} \Phi_{U(2k+1)} & & \\ & -\Phi^t_{U(2k+1)} & 0 \\ \hline & & \Phi_{SO(4)} \end{array}\right), \tag{4.5.5}$$

where $\Phi_{U(2k+1)}$ is a matrix in $\mathfrak{u}(2k+1)$ and $\Phi_{SO(4)}$ is a $\mathfrak{so}(4)$ matrix. To specify the Higgs field we pick the $U(2k+1)$ block as

$$\Phi_{U(2k+1)} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \vdots & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 1 \\ w & 0 & 1 & \cdots & 0 \end{pmatrix}, \tag{4.5.6}$$

where black dots stand for zeroes and red dots for 1. For the $SO(4)$ block we take

$$\Phi_{SO(4)} = \begin{pmatrix} 0 & 1 & 0 & \frac{w}{4} \\ -\frac{w}{4} & 0 & -\frac{w}{4} & 0 \\ 0 & 1 & 0 & \frac{w}{4} \\ -1 & 0 & -1 & 0 \end{pmatrix}. \tag{4.5.7}$$

The $SO(4)$ block is identical to the one for the Brown-Wemyss threefold (see (4.4.3)).

By construction, the preserved group is the diagonal $U(1)$ of the $U(2k+1)$ block (4.5.4).

The computation of the zero modes proceeds analogously to what done in Section 4.4. Now the linearized gauge variation of the deformation φ is

$$[\Phi, g] = \left(\begin{array}{c|c|c} B_{(2k+1)\times(2k+1)} & A^u_{(2k+1)\times(2k+1)} & C_{(2k+1)\times 4} \\ \hline A^d_{(2k+1)\times(2k+1)} & -B^t_{(2k+1)\times(2k+1)} & D_{(2k+1)\times 4} \\ \hline D_{4\times(2k+1)} & C_{4\times(2k+1)} & -B_{4\times 4} \end{array}\right). \tag{4.5.8}$$

For the zero modes, one again checks if the various blocks of $[\Phi, g]$ localize any mode in 5d. We find:

- ▷ $B_{(2k+1)\times(2k+1)}$ and $B_{4\times 4}$ do not localize any modes.
- ▷ $A^u_{(2k+1)\times(2k+1)}$ localizes one entry φ_2 as:

  $$\varphi_2 \in \mathbb{C}[w]/(w^k) \cong \mathbb{C}^k, \tag{4.5.9}$$

  thus yielding k charge 2 localized modes. The same goes for $A^d_{(2k+1)\times(2k+1)}$, from which we obtain k modes $\tilde{\varphi}_2$ of charge -2.
- ▷ $C_{4\times(2k+1)}$ localizes three entries with the same pattern as in the Brown-Wemyss

case, namely:

$$\begin{aligned} \varphi_1 &\in \mathbb{C}[w]/(w^{k+1}) \cong \mathbb{C}^{k+1} \\ \psi_1 &\in \mathbb{C}[w]/(w^{k+1}) \cong \mathbb{C}^{k+1}, \\ \psi_2 &\in \mathbb{C}[w]/(w) \cong \mathbb{C} \end{aligned} \tag{4.5.10}$$

obtaining a total of $2k+3$ charge 1 localized modes. $D_{4\times(2k+1)}$ gives the same matter content as $C_{4\times(2k+1)}$, but with charge -1.

Summarizing, the spectrum is given as follows:

- $\boldsymbol{k}$ **hypers** of charge 2:

$$\boldsymbol{Q}_i\tilde{Q}_i = \boldsymbol{\varphi}_2^{(i)}\tilde{\varphi}_2^{(i)} \qquad i = 1, ..., k\,;$$

- $\boldsymbol{2k+3}$ **hypers** of charge 1:

$$\boldsymbol{Q}_{i+k}\tilde{Q}_{i+k} = \boldsymbol{\varphi}_1^{(i)}\tilde{\varphi}_1^{(i)}, \;\; \boldsymbol{Q}_{i+2k+1}\tilde{Q}_{i+2k+1} = \boldsymbol{\psi}_1^{(i)}\tilde{\psi}_1^{(i)} \;\; i = 1, ..., k{+}1, \;\; \text{and} \;\; \boldsymbol{Q}_{3k+3}\tilde{Q}_{3k+3} = \boldsymbol{\psi}_2\tilde{\psi}_2.$$

The Higgs branch of M-theory on Laufer's threefolds is then

$$\mathrm{HB} = \mathbb{C}^{2k+3}_{+1} \times \mathbb{C}^{2k+3}_{-1} \times \mathbb{C}^{k}_{+2} \times \mathbb{C}^{k}_{-2}, \tag{4.5.11}$$

where subscripts stand for the charges of the coordinates under the $U(1)$ flavor group.

Let's comment quickly on how we found the Higgs fields for the Brown-Wemyss and the Laufer's singularities. The procedure we used is exactly the one we pointed out in Section 2.2. From the geometry of the threefolds we expected a length-two flop, inflating a $\mathbb{P}^1$ intersected with intersection number two by the divisor associated to the five-dimensional flavor symmetry. This can be realized taking the Higgs field in the commutant $\mathcal{L}$ of the trivalent root of the D_n diagram:

$$\mathcal{L} = \mathfrak{u}_{n-2} \oplus D_2 = A_{n-3} \oplus \langle \alpha^*_{n-2} \rangle \oplus D_2. \tag{4.5.12}$$

We will come back with a more detailed construction for the length-two flop case in Section 5.3.

4.6 5d zero modes computation: refined method

As we saw in the previous examples (4.3.1), (4.4.1), (4.5.2) the zero modes of the Higgs field Φ localized at $w = 0$ permit us to extract the GV invariants of the threefold X associated, via the method explained in Section 2.2.1, to Φ. The method we used for these examples is particularly explicit but it becomes computationally lengthy for higher ranks $\mathcal{G}$. It is worthful then to restate it in a more general language. This method will also permit us to produce, in Chapter 5, families of threefolds that have constant GV invariants on Zariski-open subsets of the basespace and either enhanced GV or non-isolated singularities on the complement.

As we saw, given a vev for Φ, the zero modes are the deformation $\varphi \in \mathcal{G}$ of the Higgs field up to the (linearized) gauge transformations

$$\delta_g \varphi = [\Phi, g] \qquad \text{with} \qquad g \in \mathcal{G}\,. \tag{4.6.1}$$

We decompose $\Phi = X_+ + w\,Y$, with $X_+ = \Phi(0)$ the constant part of the Higgs field and Y a matrix depending polynomially on w. We need to work out which components of the deformation φ can be set to zero by a gauge transformation (4.1.1). To perform the gauge-fixing one then has to solve the system (4.1.2). At special points in $\mathbb{C}_w$, there can be components of φ that cannot be gauge-fixed to zero: these directions in the Lie algebra $\mathcal{G}$ support zero modes.

Let $\mathcal{M} \leq \mathcal{L}$ the maximal subalgebra associated to Φ as in (4.2.2). Since the irreducible representations $R^{\mathcal{M}}$ of $\mathcal{M}$ are invariant under the action of Φ, we implement the decomposition (4.1.3) and we solve the equation (4.1.2) in each representation $R^{\mathcal{M}}_p$ at a time, where now $g, \varphi \in R^{\mathcal{M}}_p \subset \mathcal{G}$. We can write more explicitly the representation $R^{\mathcal{M}}_p$ of $\mathcal{M} = \mathcal{H} \oplus \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \ldots$ as

$$R^{\mathcal{M}}_p = \left(\; R^{\mathcal{M}_1}_p \;,\; R^{\mathcal{M}_2}_p \;,\; \ldots \; \right)_{q_1,\ldots,q_f} \tag{4.6.2}$$

where $R^{\mathcal{M}_h}_p$ is an irreducible representation of the simple summand $\mathcal{L}_h$, $(q_1, \ldots, q_f)$ are the charges under the $U(1)^f$ group generated by $\mathcal{H}$ and we are taking the tensor product of the various $R^{\mathcal{M}_h}_p$.

Let us now describe a more refined (rather than the one presented in Section 4.1) algorithm to compute the number and the charges of zero modes. For each representation $R \equiv R^{\mathcal{M}}_p$ of $\mathcal{M}$ with dimension d_R, we choose a basis $e_1, \ldots, e_{d_R}$ of R. In this basis, the equation (4.1.2) becomes

$$(A + w\,B)\,\boldsymbol{\tau} = -\boldsymbol{\phi} \tag{4.6.3}$$

where $\boldsymbol{\tau}$ and $\boldsymbol{\phi}$ are the d_R-column vectors of coefficients of g and φ in the given basis and A, B are the $d_R \times d_R$ matrices representing the linear operators X_+ and Y respectively. The matrix A is constant, while the matrix B depends polynomially on w.

If $A + w\,B$ is invertible, then there exists a vector $\boldsymbol{\tau}$ (i.e. a $g \in R$) that completely gauge fixes $\varphi \in R$ to zero at generic w. At the values of w where the rank of $A + w\,B$ decreases, there will be vectors $\boldsymbol{\phi}$ that cannot be set to zero, leaving a zero mode localized at that points.

With the chosen X_+, we immediately see that one of such special points is (by construction) the origin $w = 0$. Here the matrix $A + w\,B$ reduces to the nilpotent matrix A, that has non-trivial kernel.[12] In the following we only use the fact that A has rank $r < d_R$; hence, our conclusions are valid also when A is not necessarily nilpotent. What we are going to say of course applies also for a nilpotent A.

We choose the basis $e_1, ..., e_{d_R}$ of R such that A is in the Jordan form. If the rank of A is r, we then have $d_R - r$ rows of zeros and $d_R - r$ columns of zeros. We can rearrange rows and columns such that A takes the block diagonal form

$$A = \left(\begin{array}{c|c} A_u & \mathbf{0}_{r\times(d_R-r)} \\ \hline \mathbf{0}_{(d_R-r)\times r} & \mathbf{0}_{(d_R-r)\times(d_R-r)} \end{array} \right) \ , \ \text{with } A_u \text{ invertible.} \tag{4.6.4}$$

Doing the same operations on B, we obtain

$$B = \left(\begin{array}{c|c} B_u & B_r \\ \hline B_l & B_d \end{array} \right) \ . \tag{4.6.5}$$

The equations (4.6.3) now read

$$\begin{cases} (A_u + wB_u)\boldsymbol{\tau}_u + wB_r\boldsymbol{\tau}_d & = & -\boldsymbol{\phi}_u \\ wB_l\boldsymbol{\tau}_u + wB_d\boldsymbol{\tau}_d & = & -\boldsymbol{\phi}_d \end{cases} \tag{4.6.6}$$

Since $(A_u + wB_u)$ is invertible (at least in the vicinity of $w = 0$), from the first set of equations we see that we can always gauge fix the $\boldsymbol{\phi}_u$ components to zero,[13] by setting

$$\boldsymbol{\tau}_u = -(A_u + w\,B_u)^{-1}\left(\boldsymbol{\phi}_u + w\,B_r\boldsymbol{\tau}_d\right) \ . \tag{4.6.7}$$

Substituting in the second set of equations we obtain

$$w\,\left[B_d - w\,B_l(A_u + w\,B_u)^{-1}B_r\right]\boldsymbol{\tau}_d = -\boldsymbol{\phi}_d + w\,B_l(A_u + w\,B_u)^{-1}\boldsymbol{\phi}_u \ . \tag{4.6.8}$$

We see that the components $\boldsymbol{\phi}_d$ cannot be fixed identically to zero: at $w = 0$ there can be a remnant, i.e. a localized mode. Said differently, the best we can do is to cancel from $\boldsymbol{\phi}_d$ its dependence on w, leaving a constant entry (instead of a generic polynomial in w). This is possible for all components of $\boldsymbol{\phi}_d$ only when the matrix

[12] In particular, the kernel is spanned by the vectors $|j, j\rangle$, when writing R in terms of sl_2 representations, where sl_2 is generated by the Jacobson-Morozov standard triple associated with X_+.

[13] These correspond to all states except $|j, -j\rangle$.

$B_d|_{w=0}$ has maximal rank, i.e. rank equal to $d_R - r$. In this case, the number of zero modes is

$$\# = d_R - r\,,$$

because each component of $\boldsymbol{\phi}_d$ has now a constant entry, i.e. one degree of freedom.

If $B_d|_{w=0}$ is not invertible, we can iterate what we have done so far, in the following way. Let us define for simplicity $\boldsymbol{\tau}' \equiv \boldsymbol{\tau}_d$, $\boldsymbol{\phi}'_{\mathrm{tot}} \equiv \boldsymbol{\phi}_d - w\, B_l (A_u + w\, B_u)^{-1} \boldsymbol{\phi}_u$, $A' \equiv B_d|_{w=0}$ and $B' \equiv B_d - B_d|_{w=0} - B_l(A_u + w\, B_u)^{-1} B_r$. We can decompose $\boldsymbol{\phi}'_{\mathrm{tot}} = \boldsymbol{\phi}'_0 + w\boldsymbol{\phi}'$, where $\boldsymbol{\phi}'_0$ is $\boldsymbol{\phi}'_{\mathrm{tot}}$ evaluated at $w = 0$. We can then rewrite the equation (4.6.8) as

$$(A' + w\, B')\,\boldsymbol{\tau}' = -\boldsymbol{\phi}'\,. \tag{4.6.9}$$

This has the same form as (4.6.3), so we can again change the basis such that $A' \equiv B_d|_{w=0}$ is in the Jordan form and write the equations in this basis. We will obtain a set of equations in the form (4.6.6) where we have to substitute $(A, B)_{u,l,r,d} \to (A', B')_{u,l,r,d}$ and $(\boldsymbol{\tau}, \boldsymbol{\phi}) \to (\boldsymbol{\tau}', \boldsymbol{\phi}')$.

The matrix A' will now have rank $r' < d_R - r$. There will then be r' components of $\boldsymbol{\phi}'$ that can be gauge fixed to zero; we correspondingly have r' zero modes along the corresponding components of $\boldsymbol{\phi}_d$. If the matrix $B'_d|_{w=0}$ has maximal rank (i.e. $d_R - r - r'$), then the other $d_R - r - r'$ components of $\boldsymbol{\phi}_d$ will be of the form $a + bw$ and hence each hosts *two* zero modes. In this case the number of zero modes is

$$\# = r' + 2(d_R - r - r')\,. \tag{4.6.10}$$

On the other hand, if $B'_d|_{w=0}$ has rank $r'' < d_R - r - r'$, then we have to iterate once more the algorithm above and, provided $B''_d|_{w=0}$ has maximal rank (i.e. $d_R - r - r' - r''$) we obtain

$$\# = r' + 2r'' + 3(d_R - r - r' - r'')\,.$$

We now have the factor "3" because the $d_R - r - r' - r''$ directions of $\boldsymbol{\phi}_d$ are of the form $a + bw + cw^2$, i.e. they host *three* zero modes each.

In conclusion, let us assume that the algorithm stops at the N-th step and let us call $r^{(k)}$ the rank of the matrix A at the step k, then the number of zero modes is

$$\#_{\text{zero modes}} = \sum_{k=0}^{N} k\, r^{(k)} \qquad \text{with} \qquad \sum_{k=0}^{N} r^{(k)} = d_R \tag{4.6.11}$$

where $r^{(0)} = r$.

If there are other values of w, say $w = w_0$, where the rank of $A + wB$ is not maximal, one can shift $w \mapsto w + w_0$ ending out with the same situation as above, where the new A is now $A+w_0B$. Applying the algorithm that we have just outlined, one computes the zero modes localized at $w = w_0$. In this case the matrix at $w = w_0$ is not necessarily nilpotent.

Notice that this algorithm could never end. This is the case for example when the $A + wB$ matrix is identically zero at one step. The corresponding directions of φ cannot be gauge fixed at any order in w, leaving a zero mode that lives in 7d.

In conclusion, in this section we have shown that one can reduce the problem of finding the zero modes to a simple exercise in linear algebra. These computations are algorithmic and can be done by a calculator in a reasonable amount of time. In Appendix F we describe the implementation of the algorithm in Mathematica, that we used for our computations.

Remark 4.6.1. We conclude by considering a case that we will recurrently encounter in the following. Consider two Higgs fields Φ and $\tilde{\Phi}$ related as

$$\tilde{\Phi} = w^j \Phi \,, \tag{4.6.12}$$

and with $\Phi(0) \neq 0$, while $\tilde{\Phi}$ has a zero of order j at $w = 0$.

We can compute the zero modes of $\tilde{\Phi}$, knowing the zero modes of Φ: The components of the deformation φ that were gauge fixed to zero by Φ, now host zero modes in $\mathbb{C}[w]/(w^j)$. Components that hosted localized modes in $\mathbb{C}[w]/(w^k)$, now support zero modes in $\mathbb{C}[w]/(w^{j+k})$. We further note that the Casimir invariants of Φ and $\tilde{\Phi}$ are related by $\mathrm{Tr}\left((\tilde{\Phi})^i\right) = \mathrm{Tr}\left((w^j\Phi)^i\right) = w^{i\cdot j}\mathrm{Tr}\left((\Phi)^i\right)$.

These simple facts will permit us to reproduce the Higgs fields of all the quasi-homogeneous cDV, first identifying a finite set of Higgs field profiles, and then producing all the other Higgs fields multiplying them by an appropriate power of w.

Chapter 5

Simple flops with $\ell = 1, \ldots, 6$ and their GV invariants

The analysis presented in Chapter 4 can be performed on all the cDV singularities. We are going to concentrate, in this chapter, on a particular subclass of them currently studied in the literature [73–85]: the simple flops.

5.1 Simple flops: introduction

One-parameter deformations of ADE surfaces admitting a small simultaneous resolution blowing up a single $\mathbb{P}^1$ are known as simple threefold flops. From a mathematical point of view, they can be classified according to a variety of invariants.

The first, and coarsest, invariant that can be associated to a simple flop is the normal bundle $\mathcal{N}_{\mathbb{P}^1}$ to the exceptional $\mathbb{P}^1$. Laufer [127] showed that $\mathcal{N}_{\mathbb{P}^1}$ can be only of three types:

$$\mathcal{O}(-1) \oplus \mathcal{O}(-1), \qquad \mathcal{O}(0) \oplus \mathcal{O}(-2), \qquad \mathcal{O}(1) \oplus \mathcal{O}(-3). \tag{5.1.1}$$

The conifold (1.1.1) is the only example admitting a small resolution inflating an exceptional $\mathbb{P}^1$ with normal bundle $\mathcal{N}_{\mathbb{P}^1} = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$. The Reid pagodas we studied in Section 4.3 locally classify all cases with $\mathcal{N}_{\mathbb{P}^1} = \mathcal{O}(0) \oplus \mathcal{O}(-2)$. Finally, all the other simple flops inflate a $\mathbb{P}^1$ with normal bundle $\mathcal{N}_{\mathbb{P}^1} = \mathcal{O}(1) \oplus \mathcal{O}(-3)$. Although useful, the normal bundle is not a sufficiently refined invariant to distinguish the different physical properties of the simple flops that we are going to scrutinize.

A richer classification of simple flops can be obtained employing the *length* invariant ℓ introduced in Definition 2.2.1. It was proven (see [125]) that the length of a simple flop can only assume discrete values ranging from 1 to 6, and that examples of any length indeed exist. Furthermore ℓ is also the intersection number between

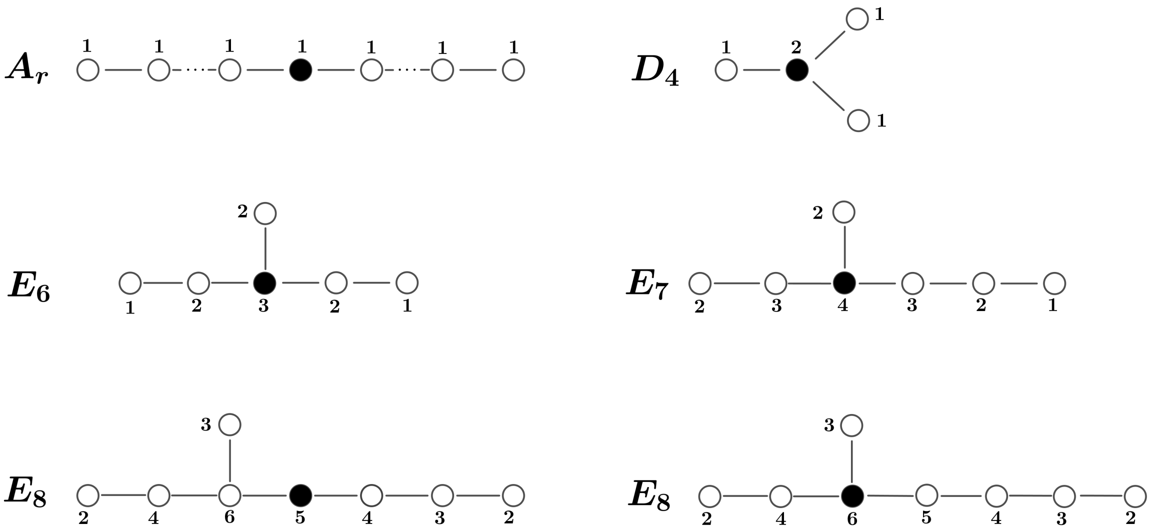


Figure 5.1: *ADE Dynkin diagrams and dual Coxeter labels of the nodes.*

the divisor supporting the five-dimensional flavor symmetry and the resolved $\mathbb{P}^1$.

From a Lie-algebraic point of view, the length of a simple flop corresponds to the dual Coxeter label of the node of the Dynkin diagram that is being resolved by the small simultaneous resolution. Given an ADE algebra $\mathcal{G}$ of rank r, a set of simple roots α_i, with $i = 1, \dots r$, and the highest root θ, the dual Coxeter label of a node is the multiplicity of the corresponding simple root in the decomposition of the highest root. In other words, given a node corresponding to a simple root α_{i_0} and the decomposition of the highest root

$$\theta = c_1\alpha_1 + \ldots + c_{i_0}\alpha_{i_0} + \ldots c_r\alpha_r \tag{5.1.2}$$

then c_{i_0} is the dual Coxeter label of the node. We report the Dynkin diagrams of all the ADE cases, along with the dual Coxeter labels of their nodes in Figure 5.1, where we have highlighted in black the nodes that are being resolved in the simple threefold flops that we will analyze in the following sections.

The classification of simple threefold flops based on the length can be further refined introducing the Gopakumar-Vafa (GV) invariants [69, 70]. These invariants can be used to distinguish between simple flops of the same length.[1]

In the following, we apply the method discussed in Section 2.2 to construct three-

[1]Even though we will not use it in this thesis, it is worth mentioning an even subtler invariant that can be associated to a simple flop, namely its contraction algebra. It has been proven [80] that there exist simple flops with the same normal bundle, same length, same Gopakumar-Vafa invariants and *different* contraction algebra. Physically, the contraction algebra can be understood, for example, as describing the quiver relations of the theory on a D3 brane in type IIB probing the singularity, and explicit constructions of contraction algebras at all lengths can be found in [158].

folds with a simple flop. The threefold will be obtained from a family of deformed ADE singularities in which only the black node in Figure 5.1 is simultaneously resolved. Let us call it α_c. The subalgebra $\mathcal{H}$ is then generated by α_c^*, i.e.

$$\mathcal{H} = \langle \alpha_c^* \rangle \,, \tag{5.1.3}$$

and the Higgs field will correspondingly be chosen in the commutant $\mathcal{L}$ of $\mathcal{H}$, i.e. the Levi subalgebra corresponding to the chosen partial simultaneous resolution. From Figure 5.1, we see that the simple summands $\mathcal{L}_h$ of $\mathcal{L}$ are of A-type. For $\mathcal{L}_h = A_{m-1}$ summands, the generic form of Φ, up to gauge transformations, can be taken to be a *reconstructible Higgs* (A.1.5).

$$\Phi|_{A_{m-1}} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \vdots & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 1 \\ (-1)^{m-1}\hat{\sigma}_m & (-1)^{m-2}\hat{\sigma}_{m-1} & \cdots & -\hat{\sigma}_2 & 0 \end{pmatrix} \tag{5.1.4}$$

with $\hat{\sigma}_j$ $(j = 2, ..., m)$ the Casimirs of $\Phi|_{A_{m-1}}$.

The choice (5.1.4) is equivalent, from the point of view of our construction, to picking the canonical Higgs we presented in Section 2.2. Indeed, the Higgs (5.1.4) and (2.2.21) are linked by a gauge transformation. Furthermore, in the following sections, the maximal rank maximal subalgebra $\mathcal{M}$ of $\mathcal{L}$ will always be trivially $\mathcal{M} = \mathcal{L}$.

Let's quickly recap the strategy to obtain the threefold associated to Φ (presented in Section 2.2). Collecting the Casimirs $\hat{\sigma}_j$'s for each summand $\mathcal{L}_h$ and the coefficient deformations along $\mathcal{H}$ one obtains the set of $\mathcal{W}_{\mathcal{L}}$ invariant coordinates ϱ_i that span the base of the family with simultaneous partial resolution. The total Casimirs of Φ (that are biholomorphic to the coefficients of the deforming monomials in the versal deformation of the ADE singularity) can be written as functions of the ϱ_i's, realizing in such a way the map $\Psi_{\mathcal{L},\mathcal{G}}$ we introduced in Section 2.2. Finally, the threefold is obtained by setting $\varrho_i = \varrho_i(w)$.

We will construct threefold with different values of *length* from 1 to 6. For each X we give the Higgs field Φ that produces the desired simple flop threefold X. This allows us to build the 5d theory realized from reducing M-theory on X. In particular the flavor group will always be the $U(1)$ group generated by α_c^*. The number of hypermultiplets and their charges under the $U(1)$ flavor group, namely the GV invariants of X and their degrees, will be derived by counting the zero modes of Φ.

5.2 Simple flop with length 1

The simplest Example: the Conifold

The Conifold threefold is given by

$$u\,v = z^2 - w^2\,. \tag{5.2.1}$$

This is actually a family of deformed A_1 surfaces over $\mathbb{C}_w$, with the simultaneous resolution of the exceptional $\mathbb{P}^1$ at $w = 0$.

It can be constructed following the previous sections in the following way: A_1 has only one simple root α. We require it to be blown up by the simultaneous resolution (the only other choice is to blow up no sphere, that would produce a non-singular threefold). The Levi subalgebra is now simply

$$\mathcal{L} = \langle \alpha^* \rangle = \mathcal{H}. \tag{5.2.2}$$

The $\mathcal{W}_{\mathcal{L}}$ invariant coordinate is the coefficient ϱ along the Cartan α^*. Choosing $\varrho = w$, the Higgs field is

$$\Phi = \begin{pmatrix} w & 0 \\ 0 & -w \end{pmatrix} \tag{5.2.3}$$

and the threefold equation is easily checked to be (5.2.1).

This is the simplest example of simple flop, where the flavor group (i.e. the preserved 7d gauge group) is $U(1)$.

Zero modes. Notoriously, M-theory on the conifold gives a free 5d hypermultiplet (localized at $w = 0$). This can be checked by computing the zero modes of Φ. This computation has already been shown in [128], by explicitly using the linearized equations of motion in holomorphic gauge,[2] as explained in [103]. In order to illustrate the method outlined in Section 4.6, we apply it to the conifold case to reproduce the result of [128].

The decomposition (4.1.3) of the A_1 algebra in representations of the Levi subalgebra $\mathcal{L} = \langle \alpha^* \rangle$ is

$$A_1 = \mathbf{1}_0 + \mathbf{1}_+ + \mathbf{1}_-. \tag{5.2.4}$$

Let us consider each representation individually. Remember that the matrices A and B in Section 4.6 are the restriction of X_+ and Y on the considered representation, where $\Phi = X_+ + w\,Y$.

$\mathbf{1}_0$: Φ restricted to this representation is zero. Hence, the two 'matrices' A and B

[2] For the conifold, i.e. a family of deformed A_1, that computation was enough. However for more complicated algebras our method simplifies the calculations and make them more systematic.

vanish, nothing is gauge fixed and then there is one 7d mode.

$\mathbf{1}_+$: $\Phi|_{\mathbf{1}_+} = 2w$, so $A = 0$ and it has rank zero, but $B = 2$ has rank one; then $d_R - r = 1 - 0 = 1$ and this gives one localized mode at $w = 0$.

$\mathbf{1}_-$: $\Phi|_{\mathbf{1}_-} = -2w$, so $A = 0$ and $B = -2$ that again gives a localized mode at $w = 0$.

The two localized zero modes made up one hypermultiplet, as expected. Its charge under the flavor $U(1)$ can be easily read from the representation where the modes sit. The zero mode analysis correctly reproduces the GV invariant of the conifold, that is $n_1^{g=0} = 1$.

Threefolds with a simple flop of length 1: generic case

We now generalize the conifold case, by starting from the Lie algebra A_{k-1}. The simple roots are now $\alpha_1, ..., \alpha_{k-1}$. We require that the only root that is simultaneously resolved in the threefold is α_p for a given choice of $p \in \{1, ..., k-1\}$ (without loss of generality, we can take $p \geq \frac{k}{2}$). Consequently, we have $\mathcal{H} = \langle \alpha_p^* \rangle$. Its commutant is

$$\mathcal{L} = A_{p-1} \oplus A_{k-p-1} \oplus \langle \alpha_p^* \rangle \, . \tag{5.2.5}$$

The Higgs field at $w = 0$ is (in the principal nilpotent orbit when restricted on the simple summands of $\mathcal{L}$)

$$X_+ = e_{\alpha_1} + \cdots + e_{\alpha_{p-1}} + e_{\alpha_{p+1}} + \cdots + e_{\alpha_{k-1}} \, . \tag{5.2.6}$$

We choose the w-dependence of the ϱ_i such that the Higgs restricted on each block is[3]

$$\Phi|_{A_{p-1}} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \vdots & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 1 \\ w & 0 & \cdots & 0 & 0 \end{pmatrix} \quad \text{and} \quad \Phi|_{A_{k-p-1}} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \vdots & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -w & 0 & \cdots & 0 & 0 \end{pmatrix} . \tag{5.2.7}$$

This means that

$$Y = e_{-\alpha_1-\alpha_2-\ldots-\alpha_{p-1}} - e_{-\alpha_{p+1}-\alpha_{p+2}-\ldots-\alpha_{2k-1}} \, . \tag{5.2.8}$$

[3]A different choice would only complicate the equation of the three-fold, without changing its salient features.

The equation of the threefolds is read form (2.2.20), by using the chosen $\Phi = X_+ + w\,Y$:

$$u\,v = (z^p - w)(z^{k-p} + w)\,. \tag{5.2.9}$$

When $k = 2n$ is even and $p = n$, we have the *Reid Pagoda* of width n (whose Dynkin diagram for the simultaneous resolution is depicted in Figure 5.1).

Zero modes. Let us perform the zero mode computation in the case $p = n = 2$, i.e. for the Reid Pagoda with degree 2. The Higgs field is actually given by the A_3 example studied before (see (2.2.23)), where one chooses the following dependence of ϱ_i on w:

$$\varrho_1 = w, \qquad \tilde{\varrho}_2 = 0, \qquad \varrho_3 = -w\,.$$

The A_3 algebra decomposes in the following way in representations of the Levi subalgebra $\mathcal{L} = s\ell_2^{(1)} \oplus s\ell_2^{(3)} \oplus \langle \alpha_2^* \rangle$:

$$A_3 = (\mathbf{3},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{3})_0 \oplus (\mathbf{1},\mathbf{1})_0 \oplus (\mathbf{2},\mathbf{2})_+ \oplus (\mathbf{2},\mathbf{2})_-\,. \tag{5.2.10}$$

Let us consider each Levi representation $R^{\mathcal{L}}$ individually.

$(\mathbf{3},\mathbf{1})_0$: the operator X_+ is represented in the basis $\{-e_{\alpha_1}, \frac{1}{2}h_1, \frac{1}{2}e_{-\alpha_1}\}$ by the matrix

$$A_{(\mathbf{3},\mathbf{1})_0} = \left(\begin{array}{c|cc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 0 & 0 & 0 \end{array}\right)\,, \tag{5.2.11}$$

that has rank $r = 2$. In the same basis Y is represented by

$$B_{(\mathbf{3},\mathbf{1})_0} = \left(\begin{array}{c|cc} 0 & 0 & 0 \\ 2 & 0 & 0 \\ \hline 0 & 2 & 0 \end{array}\right)\,. \tag{5.2.12}$$

We plug them into the expression (4.6.3) and apply the algorithm: arranging the rows and columns to arrive to the expression (4.6.4) is equivalent to taking $\boldsymbol{\rho}_u = (\rho_2, \rho_3)$, $\boldsymbol{\rho}_d = \rho_1$, $\boldsymbol{\phi}_u = (\phi_1, \phi_2)$ and $\boldsymbol{\phi}_d = \phi_3$. We can then read

$$B_u = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad B_r = \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \qquad B_l = \begin{pmatrix} 2 & 0 \end{pmatrix}, \qquad B_d = 0\,.$$

In particular $B_d - wB_l(A_u + w\,B_u)^{-1}B_r$ vanishes identically. This means that at the second step $A' + wB' = 0$ and the corresponding zero mode left by the rank 2 matrix A is not localized at any w. We have found a 7d zero mode.

$(\mathbf{1},\mathbf{3})_0$: we obtain the same result as above, i.e. one 7d zero mode.

$(\mathbf{1},\mathbf{1})_0$: X_+ and Y vanish on this one-dimensional representation, leaving a 7d zero mode.

$(\mathbf{2},\mathbf{2})_+$: the operator X_+ is represented in the basis $\{e_{\alpha_1+\alpha_2+\alpha_3}, e_{\alpha_1+\alpha_2}+e_{\alpha_2+\alpha_3}, e_{\alpha_2}, e_{\alpha_1+\alpha_2}-e_{\alpha_2+\alpha_3}\}$ by the matrix

$$A_{(\mathbf{2},\mathbf{2})_+} = \left(\begin{array}{c|cc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \tag{5.2.13}$$

that has rank $r = 2$. In the same basis Y is represented by

$$B_{(\mathbf{2},\mathbf{2})_+} = \left(\begin{array}{c|cc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 \end{array}\right) . \tag{5.2.14}$$

We plug them into the expression (4.6.3) and apply the algorithm: arranging the rows and columns to arrive to the expression (4.6.4) is equivalent to taking $\boldsymbol{\rho}_u = (\rho_2, \rho_3)$, $\boldsymbol{\rho}_d = (\rho_1, \rho_4)$, $\boldsymbol{\phi}_u = (\phi_1, \phi_2)$ and $\boldsymbol{\phi}_d = (\phi_3, \phi_4)$. We can then read

$$B_u = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad B_r = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad B_l = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad B_d = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} .$$

In particular B_d has maximal rank, equal to $d_R - r = 4 - 2 = 2$, where r is the rank of A. This means that ϕ_d hosts two constant zero modes localized at $w = 0$. These have charge $+1$ with respect to the $U(1)$ generated by α_2^*. Notice that $\det(A + wB) = -2w^2$. Hence there are no other points in the base $\mathbb{C}_w$ that host localized zero modes.

$(\mathbf{2},\mathbf{2})_-$: analogously to before, we have two zero modes localized at $w = 0$ with charge -1 under the preserved $U(1)$ group.

Hence, the number of localized modes at $w = 0$ is 4, that gives rise to two hypermultiplets with charge 1 with respect to the $U(1)$ flavor group.

For generic k and p, the zero mode counting proceeds analogously as for the Pagoda with $n = 2$. The A_{k-1} algebra decomposes in the following way in representations of the Levi subalgebra

$$A_{k-1} = (\boldsymbol{p^2-1},\boldsymbol{1})_0 \oplus (\boldsymbol{1},\boldsymbol{(k-p)^2-1})_0 \oplus (\boldsymbol{1},\boldsymbol{1})_0 \oplus (\boldsymbol{p},\overline{\boldsymbol{k-p}})_+ \oplus (\bar{\boldsymbol{p}},\boldsymbol{k-p})_- . \quad (5.2.15)$$

The first three representations host 7d modes, but no localized one. Let us concentrate on the charged representation $(\boldsymbol{p},\overline{\boldsymbol{k-p}})_+$ of dimension $p(k-p)$. With the choice $p \geq \frac{k}{2}$, we have $p \geq k-p$. The matrix representing X_+ in this representation has kernel with dimension equal to $k-p$, then in our algorithm $r = (p-1)(k-p)$. With a bit of work, one can check that B_d has rank $k-p = d_R - r$, that gives then $k-p$ modes localized at $w=0$ with charge $+1$ with respect to the flavor $U(1)$. The other charge representation hosts again $k-p$ modes localized at $w=0$ and with charge -1. In total we then have $k-p$ charged hypermultiplets, i.e. the GV invariant is

$$n_1^{g=0} = k-p\,. \quad (5.2.16)$$

5.3 Simple flop with length 2

In this section we consider a family of flops of length 2 arising from a D_4 singularity deformed over the $\mathbb{C}_w$ plane. The threefold is singular at the origin (where the fiber exhibits a D_4 singularity) and can only be partially resolved inflating a $\mathbb{P}^1$ corresponding to the central root of the D_4 Dynkin diagram. As we can see from Figure 5.1 the central node has dual Coxeter label equal to 2, and thus its resolution yields a flop of length 2. In Figure 5.2 we show our conventions for the labeling of the simple roots.

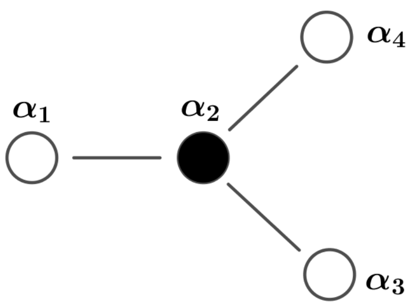


Figure 5.2: *D_4 Dynkin diagram*

Since we wish to blow up only the central node, we have $\mathcal{H} = \langle \alpha_2^* \rangle$. The Levi subalgebra $\mathcal{L}$ commuting with $\mathcal{H}$ is:

$$\mathcal{L} = A_1^{(1)} \oplus A_1^{(3)} \oplus A_1^{(4)} \oplus \langle \alpha_2^* \rangle, \quad (5.3.1)$$

where the A_1 algebras correspond to the white "tails" in picture 5.2, generated by the roots α_1, α_3 and α_4 respectively. Again, we pick $\mathcal{M} = \mathcal{L}$.

Following the prescription (5.1.4) for each A_1 summand we have

$$\Phi|_{A_1^{(i)}} = \begin{pmatrix} 0 & 1 \\ \varrho_i & 0 \end{pmatrix} = e_{\alpha_i} + \varrho_i e_{-\alpha_i} \qquad i = 1, 3, 4 \,, \tag{5.3.2}$$

where ϱ_i ($i = 1, 3, 4$) is the partial Casimir of the $s\ell_2$ algebra $A_1^{(i)}$. Moreover Φ can have a component along α_2^* with coefficient ϱ_2. Although not necessary for the employment of our machinery, we report for the sake of visual clarity the explicit matrix form of the adjoint Higgs field corresponding to the choice (5.3.1), employing the standard basis of [159]:

$$\Phi = \left(\begin{array}{cccc|cccc} \varrho_2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \varrho_1 & \varrho_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & \varrho_3 & 0 & 0 & 0 & -1 & 0 \\ \hline 0 & 0 & 0 & 0 & -\varrho_2 & -\varrho_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -\varrho_2 & 0 & 0 \\ 0 & 0 & 0 & -\varrho_4 & 0 & 0 & 0 & -\varrho_3 \\ 0 & 0 & \varrho_4 & 0 & 0 & 0 & -1 & 0 \end{array}\right) . \tag{5.3.3}$$

The threefold is found by imposing

$$\varrho_i(w) = w\, c_i(w) \qquad \text{for } i = 1, 2, 3, 4 \,, \tag{5.3.4}$$

where we take the $c_i(w)$'s such that $c_i(0) \neq 0$. Later we will simply choose the $c_i(w)$'s to be constant in w.

The Higgs at the origin is then

$$X_+ = e_{\alpha_1} + e_{\alpha_3} + e_{\alpha_4}, \tag{5.3.5}$$

while Y is

$$Y = c_1 e_{-\alpha_1} + c_3 e_{-\alpha_3} + c_4 e_{-\alpha_4} + c_2 \langle \alpha_2^* \rangle \,. \tag{5.3.6}$$

The threefold equation is simply obtained by taking the choice (5.3.4) and the expression of Φ (5.3.3) and plugging them into the formula (2.2.20):[4]

$$
\begin{aligned}
&x^2 + zy^2 - z^3 + w^2 z \left[c_1^2 + c_3^2 + c_4^2 + 4c_1c_3 + 4c_1c_4 - 2c_3c_4 - 2c_2^2 w(c_1 - 2c_3 - 2c_4) + c_2^4 w^2\right] + \\
&- 2w^3 \left[c_1 \left(c_3^2 + c_4^2 + c_1c_3 + c_1c_4 - 2c_3c_4\right) + c_2^2 w \left(c_3^2 + c_4^2 - 2c_1c_3 - 2c_1c_4 - 2c_3c_4\right) + c_2^4 w^2 (c_3 + c_4)\right] + \\
&- 2wz^2 \left(c_1 + c_3 + c_4 + c_2^2 w\right) + 2w^2 y(c_3 - c_4)\left(c_1 - c_2^2 w\right) = 0.
\end{aligned}
$$

Let's consider what happens when one of the c_i's vanishes. If $c_2 = 0$, the preserved gauge group after Higgsing is $SU(2)$ instead of $U(1)$. This says that the ALE fiber has an A_1 singularity for all values of w, i.e. the threefold has a non-isolated singularity. If $c_i = 0$ with $i = 1, 3, 4$, then the preserved group is still $U(1)$. However, the threefold equation has an A_1 singularity for generic $w \in \mathbb{C}_w$: in fact, the threefold equation is the same one would obtain by taking $\Phi|_{A_1^{(i)}}$ identically zero (the equation is insensitive to the "1" in (5.3.2)). Such a nilpotent vev for the Higgs field is called a T-brane [103].

Since we want to consider isolated singularities (with a simple flop), avoiding T-brane configurations, we will take $c_i \neq 0 \ \forall i$.

Zero modes. We now analyze the 5d zero modes arising from M-theory reduced on the flop of length 2 defined by (5.3.4). We keep the c_i's as generic constants.

As in the case of the flops of length 1, the first step consists in determining the decomposition of the algebra $\mathcal{G} = D_4$ into irreps of the Levi subalgebra (5.3.1), obtaining:

$$
D_4 = (\mathbf{3},\mathbf{1},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{3},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{1},\mathbf{3})_0 \oplus (\mathbf{2},\mathbf{2},\mathbf{2})_1 \oplus (\mathbf{2},\mathbf{2},\mathbf{2})_{-1} \oplus (\mathbf{1},\mathbf{1},\mathbf{1})_2 \oplus (\mathbf{1},\mathbf{1},\mathbf{1})_{-2}, \tag{5.3.7}
$$

where the numbers in parenthesis refer to representations of the three A_1 factors, and the subscript is the charge w.r.t. the Cartan $\langle \alpha_2^* \rangle$. Let us examine the zero-mode content of the Levi representations in (5.3.7) one by one:

$(\mathbf{3},\mathbf{1},\mathbf{1})_0$: for this representation the story flows identically to the representation $(\mathbf{3},\mathbf{1})_0$ in the A_3 example, see (5.2.11). The operator X_+ can be represented in the basis $\{-e_{\alpha_1}, \frac{1}{2}h_1, \frac{1}{2}e_{-\alpha_1}\}$:

$$
A_{(\mathbf{3},\mathbf{1},\mathbf{1})_0} = \left(\begin{array}{c|cc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 0 & 0 & 0 \end{array}\right). \tag{5.3.8}
$$

Proceeding as in (5.2.11) it is easy to show that this representation does not

[4] Notice that the threefold expression is not invariant under the exchange of c_1, c_3 and c_4, which are the Casimirs of the three A_1 tails: this can be overcome by a change of variables. In any case, the mode localization proceeds in a way that is invariant under the exchange of c_1, c_3, c_4.

host any localized 5d zero mode. The same holds for the representations $(\mathbf{1},\mathbf{3},\mathbf{1})_0$ and $(\mathbf{1},\mathbf{1},\mathbf{3})_0$.

$(\mathbf{1},\mathbf{1},\mathbf{1})_2$: X_+ is represented by a 1-dimensional matrix that, in the basis $e_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4}$, reads

$$A_{(\mathbf{1},\mathbf{1},\mathbf{1})_2} = (0)\,. \tag{5.3.9}$$

We also have:

$$B_{(\mathbf{1},\mathbf{1},\mathbf{1})_2} = 2c_2. \tag{5.3.10}$$

As a result we find that B has maximal rank, i.e. 1, and so we obtain *one* localized 5d zero-mode with $U(1)$ charge 2. Analogously, the representation $(\mathbf{1},\mathbf{1},\mathbf{1})_{-2}$ yields one 5d zero-mode of $U(1)$ charge -2.

$(\mathbf{2},\mathbf{2},\mathbf{2})_1$: X_+, once put in Jordan form in an appropriate basis[5], is represented as the 8-dimensional matrix

$$A_{(\mathbf{2},\mathbf{2},\mathbf{2})_1} = \begin{pmatrix} 0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0 \end{pmatrix}, \tag{5.3.11}$$

which has rank $r = 5$. Using the same basis for Y we get:

$$B_{(\mathbf{2},\mathbf{2},\mathbf{2})_1} = \begin{pmatrix} c_2 & 0 & 0 & 0 & 6c_4-6c_3 & 0 & 0 & 0\\ c_1-c_3+c_4 & c_2 & \frac{2(c_3-c_4)}{3} & 0 & 0 & 2c_4-2c_3 & 0 & 0\\ 0 & 0 & c_2 & 0 & -6(c_1+c_3-2c_4) & 0 & 0 & 0\\ 2(c_1-c_3) & 0 & \frac{-c_1+5c_3-c_4}{3} & c_2 & 0 & -2(c_1+c_3-2c_4) & 0 & 0\\ 0 & 0 & 0 & 0 & \rho & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & c_1+c_3+c_4 & c_2 & 0 & 0\\ \frac{c_3-c_1}{3} & 0 & \frac{2c_1-c_3-c_4}{9} & 0 & 0 & \frac{4(c_1+c_3+c_4)}{3} & c_2 & 0\\ 0 & c_3-c_1 & 0 & \frac{2c_1-c_3-c_4}{3} & 0 & 0 & c_1+c_3+c_4 & c_2 \end{pmatrix}. \tag{5.3.12}$$

Let us pause for a moment and use the results just found to prove that there are other isolated singularities in the threefold. In fact, these correspond to values of w where 5d localized modes appear. This happens in the representation under study when the rank of $A_{(\mathbf{2},\mathbf{2},\mathbf{2})_1} + wB_{(\mathbf{2},\mathbf{2},\mathbf{2})_1}$ drops. Its determinant

[5] The basis explicitly reads: $\{-e_{\alpha_1+\alpha_2} - e_{\alpha_2+\alpha_3}, e_{\alpha_1+\alpha_2+\alpha_4} - e_{\alpha_2+\alpha_3+\alpha_4}, \frac{2e_{\alpha_1+\alpha_2}}{3} + \frac{e_{\alpha_2+\alpha_3}}{3} + \frac{e_{\alpha_2+\alpha_4}}{3}, -\frac{1}{3}e_{\alpha_1+\alpha_2+\alpha_3} - \frac{1}{3}e_{\alpha_1+\alpha_2+\alpha_4} + \frac{2}{3}e_{\alpha_2+\alpha_3+\alpha_4}, -6e_{\alpha_2}, -2e_{\alpha_1+\alpha_2} + 2e_{\alpha_2+\alpha_3} + 2e_{\alpha_2+\alpha_4}, e_{\alpha_1+\alpha_2+\alpha_3} + e_{\alpha_1+\alpha_2+\alpha_4} + e_{\alpha_2+\alpha_3+\alpha_4}, e_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}\}$.

explicitly reads:

$$\det(A_{(\mathbf{2},\mathbf{2},\mathbf{2})_1} + wB_{(\mathbf{2},\mathbf{2},\mathbf{2})_1}) = w^4 \left[(c_1^2 + c_3^2 + c_4^2 - 2c_1c_3 - 2c_1c_4 - 2c_3c_4)^2 + \right.$$
$$-4c_2^2 w(c_1^3 + c_3^3 + c_4^3 - c_1^2c_3 - c_1^2c_4 - c_3^2c_1 - c_3^2c_4 - c_4^2c_1 - c_4^2c_3 + 10c_1c_3c_4) +$$
$$\left. +2c_2^4w^2(3c_1^2 + 3c_3^2 + 3c_4^2 + 2c_1c_3 + 2c_1c_4 + 2c_3c_4) - 4c_2^6w^3(c_1 + c_3 + c_4) + c_2^8w^4\right] . \tag{5.3.13}$$

It turns out that for generic c_is the rank of $A_{(\mathbf{2},\mathbf{2},\mathbf{2})_1} + wB_{(\mathbf{2},\mathbf{2},\mathbf{2})_1}$ drops on top of $w = 0$, as well as on further *four* distinct points with non-zero w. It can be checked that these additional points correspond to conifold singularities far from the origin. In addition, if the condition

$$c_1^2 + c_3^2 + c_4^2 - 2c_1c_3 - 2c_1c_4 - 2c_3c_4 = 0 \tag{5.3.14}$$

is satisfied, one of the additional singularities collides onto the origin: in this case, the rank of $A_{(\mathbf{2},\mathbf{2},\mathbf{2})_1} + wB_{(\mathbf{2},\mathbf{2},\mathbf{2})_1}$ drops on $w = 0$ as well as on *three* additional points outside the origin. This signals the appearance of further localized modes at $w = 0$, coming from the conifold singularity that has collided onto the origin. We will explicitly check this claim momentarily, deriving again condition (5.3.14).

Rearranging rows and columns to get to the form (4.6.4) we obtain:

$$B_u = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_2 & 0 & 0 \\ 0 & 0 & \frac{4(c_1+c_3+c_4)}{3} & c_2 & 0 \end{pmatrix}$$

$$B_r = \begin{pmatrix} c_2 & 0 & 6c_4 - 6c_3 \\ 0 & c_2 & -6(c_1 + c_3 - 2c_4) \\ 0 & 0 & c_2 \\ 0 & 0 & c_1 + c_3 + c_4 \\ \frac{c_3-c_1}{3} & \frac{2c_1-c_3-c_4}{9} & 0 \end{pmatrix} \tag{5.3.15}$$

$$B_l = \begin{pmatrix} c_2 & 0 & 2c_4 - 2c_3 & 0 & 0 \\ 0 & c_2 & -2(c_1 + c_3 - 2c_4) & 0 & 0 \\ c_3 - c_1 & \frac{2c_1-c_3-c_4}{3} & 0 & c_1 + c_3 + c_4 & c_2 \end{pmatrix}$$

$$B_d = \begin{pmatrix} c_1 - c_3 + c_4 & \frac{2(c_3-c_4)}{3} & 0 \\ 2(c_1 - c_3) & \frac{-c_1+5c_3-c_4}{3} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Notice that the rank of B_d, which is surely non-maximal, depends on the precise choice of the partial Casimirs. It drops to one when its determinant is equal to zero. This happens when

$$c_1^2 + c_3^2 + c_4^2 - 2c_1c_3 - 2c_1c_4 - 2c_3c_4 = 0\,. \tag{5.3.16}$$

Let us first examine the case in which the c_i's are generic constants, i.e. B_d has rank 2. Afterwards we see the case when B_d has rank 1. Notice that B_d cannot have rank zero, otherwise $c_1 = c_3 = c_4 = 0$, that we excluded.

- Let's take generic c_i's such that $c_1^2 + c_3^2 + c_4^2 - 2c_1c_3 - 2c_1c_4 - 2c_3c_4 \neq 0$. Renaming $A' \equiv B_d$ and $B' \equiv -B_l(A_u + wB_u)^{-1}B_r$ we can use equation (4.6.9) to rerun the algorithm. A' is already in a form with a 2×2 invertible block and all other elements equal to zero, i.e. $r' = 2$. We can then immediately read B'_d by computing the (33) element of B'. It is

$$B'_d = 3\left(c_1^2 + c_3^2 + c_4^2 - 2c_1c_3 - 2c_1c_4 - 2c_3c_4\right) + \frac{10}{3}wc_2^2(c_1 + c_3 + c_4) - c_2^4w^2, \tag{5.3.17}$$

  that has rank 1. As a result, according to (4.6.10), we find that the total number of zero modes is:

$$\# = r' + 2(d_R - r - r') = 2 + 2(8 - 5 - 2) = 4, \tag{5.3.18}$$

  where we recall that d_R is the dimension of the representation, r is the rank of (5.3.11) and r' is the rank of A'. The zero-modes have charge $+1$ with respect to the $U(1)$ generator.
  Analogously, we find 4 localized zero-modes with charge -1 in the $(\mathbf{2},\mathbf{2},\mathbf{2})_{-1}$ representation.

- When the c_i's fulfill (5.3.16), the rank of B_d drops to 1. This produces a change in the zero-mode counting. We can parametrize a solution of (5.3.16) in terms of two parameters q_1, q_4 as:

$$c_1 = q_1^2, \qquad c_3 = (q_1 + \varepsilon q_4)^2, \qquad c_4 = q_4^2 \tag{5.3.19}$$

  where ε can take the values ± 1. Now we have

$$A' = \begin{pmatrix} 2q_1q_4 & \frac{2}{3}q_1(q_1 + 2\varepsilon q_4) & 0 \\ -2q_4(q_4 + 2\varepsilon q_1) & \frac{2}{3}(2q_1^2 + 5\varepsilon q_1q_4 + 2q_4^2) & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{5.3.20}$$

  When $q_1^2 + \varepsilon q_1q_4 + q_4^2 \neq 0$, the 2×2 matrix is diagonalizable with the non-zero eigenvalue equal to $\frac{4}{3}\left(q_1^2+\varepsilon q_1 q_4+q_4^2\right)$. The corresponding B'_d is

$$B'_d = \begin{pmatrix} c_2^2 & -\frac{12c_2q_1q_4(q_1+\varepsilon q_4)}{q_1^2+\varepsilon q_1q_4+q_4^2} \\ 4c_2q_1q_4(q_1+\varepsilon q_4) & 0 \end{pmatrix}. \tag{5.3.21}$$

This matrix has rank less than two only when one of the c_i's vanishes (and consequently the other two are equal to each other), that we excluded.

When $q_1^2+\varepsilon q_1q_4+q_4^2=0$ (i.e. all the eigenvalues vanish) the 2×2 matrix has still rank 1 and the corresponding B'_d is also forced to have rank 2 (for non-vanishing c_i's).

We can finally count the localized zero-modes using formula (4.6.10), finding:

$$\# = r' + 2(d_R - r - r') = 1 + 2(8-5-1) = 5. \tag{5.3.22}$$

Notice that, with respect to the case (5.3.18) in which the Casimirs were totally generic, we have found an enhancement in the number of modes on a specific locus in the space of the partial Casimirs. This is the same locus where one conifold singularity that was at $w\neq 0$ collides onto the origin.

The representation $(\mathbf{2},\mathbf{2},\mathbf{2})_{-1}$ gives us further 5 zero-modes of charge -1.

Let us summarize our findings for the modes localized at $w=0$ for the simple flop of length 2 and partial Casimirs given by $\varrho_i(w) = w\,c_i$, with c_i constants.

- For generic values of c_i's, we get:
  - 8 modes with charge ± 1,
  - 2 modes with charge ± 2.

  In terms of the GV invariants, this means

  $$n_1^{g=0} = 4 \qquad \text{and} \qquad n_2^{g=0} = 1\,. \tag{5.3.23}$$

- For c_i's satisfying the constraint (5.3.16), we get:
  - 10 modes with charge ± 1,
  - 2 modes with charge ± 2.

  In terms of the GV invariants, this means

  $$n_1^{g=0} = 5 \qquad \text{and} \qquad n_2^{g=0} = 1\,. \tag{5.3.24}$$

For the other (non-zero) values of w where there are localized modes, we have conifold singularities and the flop is therefore not of length two: in fact, at these values of w the D_4 is still deformed to a smaller singularity of A-type.

Non-constant c_i's. For simplicity, we have analyzed cases when the partial Casimirs ϱ_i are just a constant c_i multiplied by w. Of course, one can also let c_i depend on w and rerun the algorithm.

One can in particular find the dependence of the $c_i(w)$'s such that the threefold X has only one isolated singularity at the origin. An easy solution is when

$$c_1 = 4a + b^2\, w\,, \qquad c_2 = b\,, \qquad c_3 = c_4 = a\,. \tag{5.3.25}$$

One can check that for this choice the determinant (5.3.13) is equal to $-256a^3b^2w^5$, i.e. it vanishes only at $w = 0$. The corresponding threefold has $n_1^{g=0} = 5$ and $n_2^{g=0} = 1$. For $a = -1/4$ and $b = 1/2$ one actually recovers the Brown-Wemyss threefold [80] in the form that appeared in [128] (that has the expected GV invariants).

5.4 Simple flop with length 3

In this section we engineer a threefold X with a simple flop of length three. Analogously to the previous sections, we are going to define a suitable Higgs field, valued in the E_6 Lie algebra, that generates a family of deformed E_6 surfaces with an E_6 singularity at $w = 0$. The resolution of the isolated singularity in the threefold X will blow-up only the trivalent node of the E_6 Dynkin diagram (see Figure 5.1). To achieve this result, we pick the following Levi subalgebra

$$\mathcal{L} = A_2^{(1,2)} \oplus A_2^{(4,5)} \oplus A_1^{(6)} \oplus \langle \alpha_3^* \rangle, \tag{5.4.1}$$

where the factors $A_2^{(i,j)}$ are associated, as subalgebras, to the roots α_i, α_j of the E_6 Dynkin diagram (we follow the labels in Figure 5.3) and $A_1^{(6)}$ is the algebra associated to the root α_6.

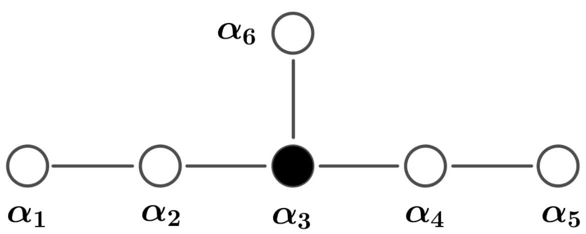


Figure 5.3: *E_6 Dynkin diagram, with the root blown up in the length three flop colored in black.*

Again, we pick $X_+ \equiv \Phi|_{w=0}$ to be an element of the principal nilpotent orbit of each simple factor of $\mathcal{L}$. The $\mathcal{W}_{\mathcal{L}}$ invariant coordinates are the total Casimirs of each simple factor of (5.4.1), plus the coefficient along the Cartan element $\langle\alpha_3^*\rangle$. I.e. the generic Φ will be such that[6]

$$\Phi|_{A_2^{(i,j)}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \varrho_3^{(i,j)} & \varrho_2^{(i,j)} & 0 \end{pmatrix} = e_{\alpha_i} + e_{\alpha_j} + \varrho_2^{(i,j)} e_{-\alpha_j} + \varrho_3^{(i,j)} [e_{-\alpha_j}, e_{-\alpha_i}], \qquad i<j,$$
$$\Phi|_{A_1^{(6)}} = \begin{pmatrix} 0 & 1 \\ \varrho_2^{(6)} & 0 \end{pmatrix} = e_{\alpha_6} + \varrho_2^{(6)} e_{-\alpha_6} \qquad \text{and} \qquad \Phi|_{\langle\alpha_3^*\rangle} = \varrho_1^{(3)} \langle\alpha_3^*\rangle\,. \tag{5.4.2}$$

We now explicitly construct a threefold, by making the choice

$$\begin{aligned} \varrho_1^{(3)} &= w\,c_3 \\ \varrho_2^{(6)} &= w\,c_6 \\ \varrho_2^{(1,2)} &= 0 \\ \varrho_2^{(4,5)} &= 0 \\ \varrho_3^{(1,2)} &= w\,c_{12} \\ \varrho_3^{(4,5)} &= w\,c_{45} \end{aligned} \tag{5.4.3}$$

with c_3, c_6, c_{12}, c_{45} constant numbers.

By plugging this choice into the Higgs field vev Φ, and following the procedure described in Section 2.2, one obtains the threefold as an hypersurface of $(x, y, z, w) \in \mathbb{C}^4$.

As an example, if we pick $c_3 = 0, c_6 = -3, c_{12} = 1, c_{45} = -1$, one gets the following threefold, which is singular at the origin (as well as at other three points with non-zero w):

$$x^2 + y^3 + z^4 + \frac{27w^6}{32} + 18w^5 + \left(12w^3 - \frac{27w^4}{16}\right) y + 2\left(w^2 - \frac{9w^3}{8}\right) z^2 + 3wyz^2 = 0\,. \tag{5.4.4}$$

Via a change of coordinates, this exactly coincides with the length 3 threefold explicitly presented by [158].

Zero modes. We now proceed (with the same procedure of the previous sections) to the mode counting. The branching of the adjoint representation **78** of E_6

[6] To match the conventions of (2.2.22), one takes $\vec{\varrho} = \{\varrho_2^{(6)}\} \cup \{\varrho_3^{(i,j)}, \varrho_2^{(i,j)} |\ (i,j) = (1,2),(4,5)\}$ and $\tilde{\rho} = \varrho_1^{(3)}$.

w.r.t $\mathcal{L}$ in (5.4.1) is given by[7]

$$\begin{aligned}\mathbf{78} \;=\;& (\mathbf{8,1,1})_0 \oplus (\mathbf{1,8,1})_0 \oplus (\mathbf{1,1,3})_0 \oplus (\mathbf{1},\mathbf{1},\mathbf{2})_3 \oplus (\mathbf{1},\mathbf{1},\mathbf{2})_{-3} \oplus (\mathbf{1},\mathbf{1},\mathbf{1})_0 \oplus \\ &\oplus (\mathbf{3},\overline{\mathbf{3}},\mathbf{2})_1 \oplus (\mathbf{3},\overline{\mathbf{3}},\mathbf{1})_{-2} \oplus (\overline{\mathbf{3}},\mathbf{3},\mathbf{2})_{-1} \oplus (\overline{\mathbf{3}},\mathbf{3},\mathbf{1})_2 \,, \end{aligned} \tag{5.4.5}$$

where the subscripts denote the charges under $\langle \alpha_3^* \rangle$.

For the E-cases the explicit computations done for length one and two become convoluted. We present here only the results. We have worked out a Mathematica routine, presented in Appendix F, that implements the algorithm described in Section 4.6 and that can be used to check the results. Running this code for a generic choice of the parameters c_6, c_3, c_{12}, c_{45}, we obtained, for each irreducible representation appearing in (5.4.5), the 5d modes shown in Table 5.1. In the table, we also write how many elements of the given representation support a mode localized in $\mathbb{C}[w]/(w^k)$, for each k; we find that $k \leq 2$. We get a total of 20 5d modes:

$R^{\mathcal{L}}$	$\mathbb{C}[w]/(w)$	$\mathbb{C}[w]/(w^2)$	$\#_{\text{zero modes}}$
$(\mathbf{8},\mathbf{1},\mathbf{1})_0$	0	0	0
$(\mathbf{1},\mathbf{8},\mathbf{1})_0$	0	0	0
$(\mathbf{1},\mathbf{1},\mathbf{3})_0$	0	0	0
$(\mathbf{1},\mathbf{1},\mathbf{1})_0$	0	0	0
$(\mathbf{3},\overline{\mathbf{3}},\mathbf{2})_1$	4	1	6
$(\overline{\mathbf{3}},\mathbf{3},\mathbf{2})_{-1}$	4	1	6
$(\overline{\mathbf{3}},\mathbf{3},\mathbf{1})_2$	3	0	3
$(\mathbf{3},\overline{\mathbf{3}},\mathbf{1})_{-2}$	3	0	3
$(\mathbf{1},\mathbf{1},\mathbf{2})_3$	1	0	1
$(\mathbf{1},\mathbf{1},\mathbf{2})_{-3}$	1	0	1

Table 5.1: *5d modes for E_6 length three simple flop.*

- one hyper with charge three, inside $(\mathbf{1},\mathbf{1},\mathbf{2})_3 \oplus (\mathbf{1},\mathbf{1},\mathbf{2})_{-3}$;
- three hypers with charge two inside $(\overline{\mathbf{3}},\mathbf{3},\mathbf{1})_2 \oplus (\mathbf{3},\overline{\mathbf{3}},\mathbf{1})_{-2}$;
- six hypers with charge one inside $(\mathbf{3},\overline{\mathbf{3}},\mathbf{2})_1 \oplus (\overline{\mathbf{3}},\mathbf{3},\mathbf{2})_{-1}$.

[7] It can be better understood starting from the one of the maximal subalgebra $A_2^{(1,2)} \oplus A_2^{(4,5)} \oplus A_2'$ (with A_2' containing e_{α_6}): $\mathbf{78} = (\mathbf{8,1,1}) \oplus (\mathbf{1,8,1}) \oplus (\mathbf{1,1,8}) \oplus (\mathbf{3},\overline{\mathbf{3}},\mathbf{3}) \oplus (\overline{\mathbf{3}},\mathbf{3},\overline{\mathbf{3}})$. One then selects the subalgebra $A_1^{(6)} \subset A_2'$, and correspondingly branches each term of the sum.

In terms of the GV invariants, one then reads

$$n_1^{g=0} = 6\,, \qquad n_2^{g=0} = 3 \qquad \text{and} \qquad n_3^{g=0} = 1\,, \tag{5.4.6}$$

which perfectly coincides with the results of [158].

We can finally check whether there are special choices of the parameters c_{12}, c_{45}, c_6, c_3 for which the number of 5d modes localized at $w = 0$ enhances. A necessary condition for the enhancement of the number of modes is that the rank of the matrix B_d drops for a special choice of the partial Casimirs. By explicit computation, we find that the rank drops when $c_{12} = c_{45}$ or $c_6 = 0$. However, these choices would create a non-isolated singularity.

5.5 Simple flop with length 4

In the following section we are going to engineer, by means of a Higgs field Φ valued in the E_7 Lie algebra, a flop of length four. By looking at the dual Coxeter labels of the E_7 Dynkin diagram in Figure 5.1, we see that the simultaneous resolution should involve the trivalent node. Analogously to the previous examples, this means that we have to pick the Higgs field in the Levi subalgebra

$$\mathcal{L} \equiv A_3^{(4,5,6)} \oplus A_2^{(1,2)} \oplus A_1^{(7)} \oplus \langle \alpha_3^* \rangle, \tag{5.5.1}$$

where the superscripts refer to the roots of the E_7 Dynkin diagram numbered as in the Figure 5.4, and α_3 is the trivalent root of E_7.

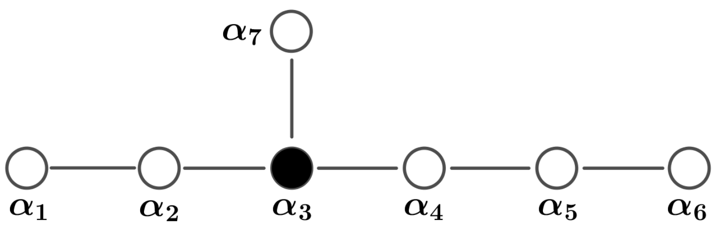


Figure 5.4: *E_7 Dynkin diagram, with the root blown up in the length four flop colored in black.*

Analogously to the E_6 case, we choose the Higgs field as follows:

$$\Phi|_{\langle \alpha_3^* \rangle} = c_3 w \langle \alpha_3^* \rangle \tag{5.5.2}$$

and

$$
\begin{aligned}
\Phi|_{A_1^{(7)}} &= \begin{pmatrix} 0 & 1 \\ c_7 w & 0 \end{pmatrix} = e_{\alpha_7} + c_7 w\, e_{-\alpha_7}, \\
\Phi|_{A_2^{(1,2)}} &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ c_{12} w & 0 & 0 \end{pmatrix} = e_{\alpha_1} + e_{\alpha_2} + c_{12} w\, [e_{-\alpha_1}, e_{-\alpha_2}], \\
\Phi|_{A_3^{(4,5,6)}} &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ c_{456} w & 0 & 0 & 0 \end{pmatrix} = e_{\alpha_4} + e_{\alpha_5} + e_{\alpha_6} + c_{456} w \Big[[e_{-\alpha_4}, e_{-\alpha_5}], e_{-\alpha_6} \Big].
\end{aligned}
$$

The corresponding threefold is a hypersurface in $\mathbb{C}^4$, that is a family of deformed E_7 singularities over $\mathbb{C}_w$. To make the equation of the threefold more readable, we set the parameters to specific values, picking $c_3 = 0, c_7 = 3, c_{12} = \frac{1}{2}, c_{456} = -\frac{1}{2}$, obtaining

$$
x^2 - y^3 + yz^3 + 3wy^2 z + y^2 \frac{81w^2}{16} - yz\frac{w^2}{12} + z^2 \frac{5w^3}{8} - y\frac{w^3}{108} + z\frac{w^4}{3} + \frac{w^5}{144} = 0. \tag{5.5.3}
$$

where we neglected terms of high degree, irrelevant for the singularity at $w = 0$.

Zero modes. We now proceed with the modes counting. We will again perform the gauge-fixing separately in each irreducible representation of the branching of the adjoint representation **133** of E_7 under the subalgebra $\mathcal{L}$:[8]

$$
\begin{aligned}
\mathbf{133} \;=\; & (\mathbf{15},\mathbf{1},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{8},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{1},\mathbf{3})_0 \oplus (\mathbf{1},\mathbf{1},\mathbf{1})_0 \oplus \\
& (\overline{\mathbf{4}},\mathbf{3},\mathbf{2})_{-1} \oplus (\mathbf{4},\overline{\mathbf{3}},\mathbf{2})_1 \oplus (\mathbf{6},\overline{\mathbf{3}},\mathbf{1})_{-2} \oplus (\mathbf{6},\mathbf{3},\mathbf{1})_2 \oplus \\
& (\mathbf{4},\mathbf{1},\mathbf{2})_{-3} \oplus (\overline{\mathbf{4}},\mathbf{1},\mathbf{2})_3 \oplus (\mathbf{1},\mathbf{3},\mathbf{1})_{-4} \oplus (\mathbf{1},\overline{\mathbf{3}},\mathbf{1})_4.
\end{aligned} \tag{5.5.4}
$$

Running the Mathematica routine described in Appendix F, we find the results displayed in table 5.2. As in the E_6 case, there are no five-dimensional modes

[8]The first entry of each summand is a representation of $A_3^{(4,5,6)}$, the second one is a representation of $A_2^{(1,2)}$, and the third on a representation of $A_1^{(7)}$. The subscript is the charge under $\langle \alpha_3^* \rangle$.

localized in $\mathbb{C}[w]/(w^k)$, with $k > 2$. In total, we find 28 modes localized at $w = 0$:

$R^{\mathcal{L}}$	$\mathbb{C}[w]/(w)$	$\mathbb{C}[w]/(w^2)$	$\#_{\text{zero modes}}$
$(\mathbf{15}, \mathbf{1}, \mathbf{1})_0$	0	0	0
$(\mathbf{1}, \mathbf{8}, \mathbf{1})_0$	0	0	0
$(\mathbf{1}, \mathbf{1}, \mathbf{3})_0$	0	0	0
$(\mathbf{1}, \mathbf{1}, \mathbf{1})_0$	0	0	0
$(\overline{\mathbf{4}}, \mathbf{3}, \mathbf{2})_{-1}$	6	0	6
$(\mathbf{4}, \overline{\mathbf{3}}, \mathbf{2})_1$	6	0	6
$(\mathbf{6}, \overline{\mathbf{3}}, \mathbf{1})_{-2}$	3	1	5
$(\mathbf{6}, \mathbf{3}, \mathbf{1})_2$	3	1	5
$(\mathbf{4}, \mathbf{1}, \mathbf{2})_{-3}$	2	0	2
$(\overline{\mathbf{4}}, \mathbf{1}, \mathbf{2})_3$	2	0	2
$(\mathbf{1}, \mathbf{3}, \mathbf{1})_{-4}$	1	0	1
$(\mathbf{1}, \overline{\mathbf{3}}, \mathbf{1})_4$	1	0	1

Table 5.2: *five-dimensional modes for E_7 length four simple flop.*

- one hyper with charge four, inside $(\mathbf{1}, \mathbf{3}, \mathbf{1})_{-4} \oplus (\mathbf{1}, \overline{\mathbf{3}}, \mathbf{1})_4$;
- two hypers with charge three inside $(\mathbf{4}, \mathbf{1}, \mathbf{2})_{-3} \oplus (\overline{\mathbf{4}}, \mathbf{1}, \mathbf{2})_3$;
- five hypers with charge two inside $(\mathbf{6}, \overline{\mathbf{3}}, \mathbf{1})_{-2} \oplus (\mathbf{6}, \mathbf{3}, \mathbf{1})_2$.
- six hypers with charge one inside $(\overline{\mathbf{4}}, \mathbf{3}, \mathbf{2})_{-1} \oplus (\mathbf{4}, \overline{\mathbf{3}}, \mathbf{2})_1$.

In terms of the GV invariants, one then reads

$$n_1^{g=0} = 6\,, \qquad n_2^{g=0} = 5\,, \qquad n_3^{g=0} = 2 \qquad \text{and} \qquad n_4^{g=0} = 1\,. \tag{5.5.5}$$

Finally, we find (as in the E_6 case) that no particular choice of the constants c_i can enhance the number of zero modes at $w = 0$ (without generating non-isolated singularities).

5.6 Simple flop with length 5

A flop with length 5 is obtained from an E_8 family over $\mathbb{C}_w$. The node that should be simultaneously resolved at $w = 0$ is depicted in Figure 5.5.

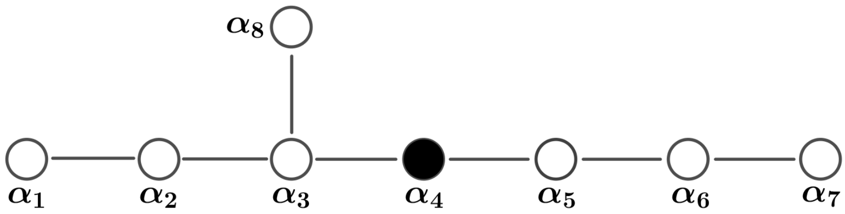


Figure 5.5: *E_8 Dynkin diagram, with the root blown up in the length five flop colored in black.*

We then have $\mathcal{H} = \langle \alpha_4^* \rangle$ and

$$\mathcal{L} = A_3^{(5,6,7)} \oplus A_4^{(1,2,3,8)} \oplus \langle \alpha_4^* \rangle. \tag{5.6.1}$$

We make the simple choice

$$\begin{aligned}
\Phi|_{\langle \alpha_4^* \rangle} &= c_4 w \, \langle \alpha_4^* \rangle \\
\Phi|_{A_3^{(5,6,7)}} &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ c_{567} w & 0 & 0 & 0 \end{pmatrix} = e_{\alpha_5} + e_{\alpha_6} + e_{\alpha_7} + c_{567} w \, [[e_{-\alpha_5}, e_{-\alpha_6}], e_{-\alpha_7}] \\
\Phi|_{A_4^{(1,2,3,8)}} &= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ c_{1238} w & 0 & 0 & 0 & 0 \end{pmatrix} = e_{\alpha_1} + e_{\alpha_2} + e_{\alpha_3} + e_{\alpha_8} - c_{1238} w \left[\left[[e_{-\alpha_1}, e_{-\alpha_2}], e_{-\alpha_3} \right], e_{-\alpha_8} \right]
\end{aligned}$$

with constant c's. We obtain our threefold as an hypersurface in $\mathbb{C}^4$. To make the equation more readable, we pick explicit values for the parameters, setting $c_4 = 0, c_{567} = 1, c_{1238} = -1$:

$$x^2 + y^3 + z^5 + w^7 + \frac{w^6}{864} - \frac{23 w^5 z}{36} - \frac{w^4 y}{48} - \frac{187 w^4 z^2}{36} - \frac{13}{3} w^3 y z - \frac{2 w^3 z^3}{27} - \frac{1}{3} w^2 y z^2 = 0 \,. \tag{5.6.2}$$

Zero modes. We can explicitly perform the branching of the adjoint represen-

$R^{\mathcal{L}}$	$\mathbb{C}[w]/(w)$	$\mathbb{C}[w]/(w^2)$	$\#_{\text{zero modes}}$
$(\mathbf{1},\mathbf{24})_0$	0	0	0
$(\mathbf{15},\mathbf{1})_0$	0	0	0
$(\mathbf{1},\mathbf{1})_0$	0	0	0
$(\mathbf{4},\overline{\mathbf{10}})_1$	6	1	8
$(\overline{\mathbf{4}},\mathbf{10})_{-1}$	6	1	8
$(\mathbf{6},\mathbf{5})_2$	6	0	6
$(\mathbf{6},\overline{\mathbf{5}})_{-2}$	6	0	6
$(\overline{\mathbf{4}},\overline{\mathbf{5}})_3$	4	0	4
$(\mathbf{4},\mathbf{5})_{-3}$	4	0	4
$(\mathbf{1},\mathbf{10})_4$	2	0	2
$(\mathbf{1},\overline{\mathbf{10}})_{-4}$	2	0	2
$(\mathbf{4},\mathbf{1})_5$	1	0	1
$(\overline{\mathbf{4}},\mathbf{1})_{-5}$	1	0	1

Table 5.3: *five-dimensional modes for E_8 length five simple flop.*

tation **248** of E_8 under the chosen $\mathcal{L}$:[9]

$$\begin{aligned}\mathbf{248} \;=\;& (\mathbf{1},\mathbf{24})_0 \oplus (\mathbf{15},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{1})_0 \oplus \\ & (\mathbf{4},\overline{\mathbf{10}})_1 \oplus (\overline{\mathbf{4}},\mathbf{10})_{-1} \oplus (\mathbf{6},\mathbf{5})_2 \oplus (\mathbf{6},\overline{\mathbf{5}})_{-2} \oplus \\ & (\overline{\mathbf{4}},\overline{\mathbf{5}})_3 \oplus (\mathbf{4},\mathbf{5})_{-3} \oplus (\mathbf{1},\mathbf{10})_4 \oplus (\mathbf{1},\overline{\mathbf{10}})_{-4} \oplus \\ & (\mathbf{4},\mathbf{1})_5 \oplus (\overline{\mathbf{4}},\mathbf{1})_{-5}. \end{aligned} \tag{5.6.3}$$

The result of the zero mode counting is displayed in Table 5.3. There are no modes localized in $\mathbb{C}[w]/(w^k)$, with $k > 2$. We find 48 modes localized at $w = 0$:

- one hyper with charge five, inside $(\mathbf{4},\mathbf{1})_5 \oplus (\overline{\mathbf{4}},\mathbf{1})_{-5}$;
- two hyper with charge four, inside $(\mathbf{1},\mathbf{10})_4 \oplus (\mathbf{1},\overline{\mathbf{10}})_{-4}$;
- four hypers with charge three inside $(\overline{\mathbf{4}},\overline{\mathbf{5}})_3 \oplus (\mathbf{4},\mathbf{5})_{-3}$;
- six hypers with charge two inside $(\mathbf{6},\mathbf{5})_2 \oplus (\mathbf{6},\overline{\mathbf{5}})_{-2}$;
- eight hypers with charge one inside $(\mathbf{4},\overline{\mathbf{10}})_1 \oplus (\overline{\mathbf{4}},\mathbf{10})_{-1}$.

[9] The first number denotes the dimension of the representation of $A_3^{(5,6,7)}$, the second under $A_4^{(1,2,3,8)}$ and the subscript is the charge under the Cartan α_4^*.

In terms of the GV invariants, one then reads

$$n_1^{g=0} = 8\,, \qquad n_2^{g=0} = 6\,, \qquad n_3^{g=0} = 4\,, \qquad n_4^{g=0} = 2 \qquad \text{and} \qquad n_5^{g=0} = 1\,. \tag{5.6.4}$$

Again, we notice that we can not enhance the number of zero-modes at $w = 0$ without generating a non-isolated singularity.

5.7 Simple flop with length 6

In this section we conclude our analysis of simple flops by dealing with the highest length case, i.e. a flop of length 6 arising from a E_8 singularity deformed over the plane $\mathbb{C}_w$. We choose the Higgs $\Phi \in E_8$ in such a way to resolve only the central node of the E_8 Dynkin diagram as depicted in Figure 5.6.

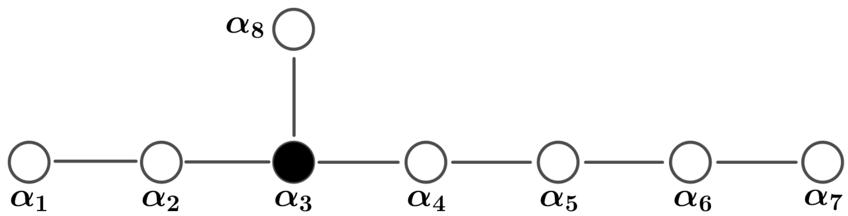


Figure 5.6: *E_8 Dynkin diagram*

According to the principles outlined in previous sections, the Higgs field resolving the central node must lie in the Levi subalgebra defined by:

$$\mathcal{L} = A_4^{(4,5,6,7)} \oplus A_2^{(1,2)} \oplus A_1^{(8)} \oplus \langle \alpha_3^* \rangle, \tag{5.7.1}$$

where, as usual, the upper indices label the simple roots. Again we choose Φ of the following form:

$$
\begin{aligned}
\Phi|_{A_4^{(4,5,6,7)}} &= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ c_{4567}w & 0 & 0 & 0 & 0 \end{pmatrix} = e_{\alpha_4} + e_{\alpha_5} + e_{\alpha_6} + e_{\alpha_7} - c_{4567}w\,[[[e_{-\alpha_1}, e_{-\alpha_2}], e_{-\alpha_3}], e_{-\alpha_4}]\,, \\
\Phi|_{A_2^{(1,2)}} &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ c_{12}w & 0 & 0 \end{pmatrix} = e_{\alpha_1} + e_{\alpha_2} + c_{12}w\,[e_{-\alpha_1}, e_{-\alpha_2}]\,, \\
\Phi|_{A_1^{(8)}} &= \begin{pmatrix} 0 & 1 \\ c_8 w & 0 \end{pmatrix} = e_{\alpha_8} + c_8 w\, e_{-\alpha_8}, \\
\Phi|_{\langle\alpha_3^*\rangle} &= c_3 w\, \langle\alpha_3^*\rangle.
\end{aligned}
\tag{5.7.2}
$$

To make the equation more readily understandable, we set the parameters to a specific value $c_3 = 0, c_8 = 1, c_{12} = -1, c_{4567} = 1$. In this way we obtain the threefold

$$
x^2 + y^3 + z^5 - wyz^3 - \frac{w^4}{48}y + \frac{w^6}{864} - \frac{7w^2}{2}yz^2 - \frac{23w^4}{20}yz - \frac{11w^3}{12}z^3 - \frac{17w^4}{24}z^2 + \frac{47w^6}{240}z = 0\,, \tag{5.7.3}
$$

where we neglected terms of high degree, irrelevant for the singularity at $w = 0$.

Zero modes. We perform the mode counting explicitly, independently for each irreducible representation arising from the adjoint $\mathbf{248}$ of E_8, branched under the Levi subalgebra (5.7.1). The decomposition reads:

$$
\begin{aligned}
\mathbf{248} \;=\; & (\mathbf{24},\mathbf{1},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{8},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{1},\mathbf{3})_0 \oplus (\mathbf{1},\mathbf{1},\mathbf{1})_0 \oplus \\
& (\mathbf{5},\overline{\mathbf{3}},\mathbf{2})_1 \oplus (\overline{\mathbf{5}},\mathbf{3},\mathbf{2})_{-1} \oplus (\mathbf{10},\mathbf{3},\mathbf{1})_2 \oplus (\overline{\mathbf{10}},\overline{\mathbf{3}},\mathbf{1})_{-2} \oplus \\
& (\overline{\mathbf{10}},\mathbf{1},\mathbf{2})_3 \oplus (\mathbf{10},\mathbf{1},\mathbf{2})_{-3} \oplus (\overline{\mathbf{5}},\overline{\mathbf{3}},\mathbf{1})_4 \oplus (\mathbf{5},\mathbf{3},\mathbf{1})_{-4} \oplus \\
& (\mathbf{1},\mathbf{3},\mathbf{2})_5 \oplus (\mathbf{1},\overline{\mathbf{3}},\mathbf{2})_{-5} \oplus (\mathbf{5},\mathbf{1},\mathbf{1})_6 \oplus (\overline{\mathbf{5}},\mathbf{1},\mathbf{1})_{-6}
\end{aligned}
\tag{5.7.4}
$$

Applying the Mathematica routine presented in Appendix F, we find the zero modes in Table 5.4. We find a total of 44 localized modes:

- one hyper with charge six, inside $(\mathbf{5},\mathbf{1},\mathbf{1})_6 \oplus (\overline{\mathbf{5}},\mathbf{1},\mathbf{1})_{-6}$;
- two hypers with charge five, inside $(\mathbf{1},\mathbf{3},\mathbf{2})_5 \oplus (\mathbf{1},\overline{\mathbf{3}},\mathbf{2})_{-5}$;

$R^{\mathcal{L}}$	$\mathbb{C}[w]/(w)$	$\mathbb{C}[w]/(w^2)$	$\#_{\text{zero modes}}$
$(\mathbf{24},\mathbf{1},\mathbf{1})_0$	0	0	0
$(\mathbf{1},\mathbf{8},\mathbf{1})_0$	0	0	0
$(\mathbf{1},\mathbf{1},\mathbf{3})_0$	0	0	0
$(\mathbf{1},\mathbf{1},\mathbf{1})_0$	0	0	0
$(\mathbf{5},\overline{\mathbf{3}},\mathbf{2})_1$	6	0	6
$(\overline{\mathbf{5}},\mathbf{3},\mathbf{2})_{-1}$	6	0	6
$(\mathbf{10},\mathbf{3},\mathbf{1})_2$	6	0	6
$(\overline{\mathbf{10}},\overline{\mathbf{3}},\mathbf{1})_{-2}$	6	0	6
$(\overline{\mathbf{10}},\mathbf{1},\mathbf{2})_3$	4	0	4
$(\mathbf{10},\mathbf{1},\mathbf{2})_{-3}$	4	0	4
$(\overline{\mathbf{5}},\overline{\mathbf{3}},\mathbf{1})_4$	3	0	3
$(\mathbf{5},\mathbf{3},\mathbf{1})_{-4}$	3	0	3
$(\mathbf{1},\mathbf{3},\mathbf{2})_5$	2	0	2
$(\mathbf{1},\overline{\mathbf{3}},\mathbf{2})_{-5}$	2	0	2
$(\mathbf{5},\mathbf{1},\mathbf{1})_6$	1	0	1
$(\overline{\mathbf{5}},\mathbf{1},\mathbf{1})_{-6}$	1	0	1

Table 5.4: *Five-dimensional modes for E_8 length six simple flop.*

- three hypers with charge four, inside $(\overline{\mathbf{5}},\overline{\mathbf{3}},\mathbf{1})_4 \oplus (\mathbf{5},\mathbf{3},\mathbf{1})_{-4}$;
- four hypers with charge three inside $(\overline{\mathbf{10}},\mathbf{1},\mathbf{2})_3 \oplus (\mathbf{10},\mathbf{1},\mathbf{2})_{-3}$;
- six hypers with charge two inside $(\mathbf{10},\mathbf{3},\mathbf{1})_2 \oplus (\overline{\mathbf{10}},\overline{\mathbf{3}},\mathbf{1})_{-2}$;
- six hypers with charge one inside $(\mathbf{5},\overline{\mathbf{3}},\mathbf{2})_1 \oplus (\overline{\mathbf{5}},\mathbf{3},\mathbf{2})_{-1}$.

In terms of the GV invariants, one then reads

$$n_1^{g=0} = 6, \qquad n_2^{g=0} = 6, \qquad n_3^{g=0} = 4, \qquad n_4^{g=0} = 3, \qquad n_5^{g=0} = 2 \quad \text{and} \quad n_6^{g=0} = 1.$$

Finally, analyzing the rank of the matrix B_d, we find that no enhancement in the number of localized modes at $w = 0$ is feasible without generating a non-isolated singularity.

Chapter 6

M-theory on quasi-homogeneous cDV

In this Chapter, we are going to analyze the dynamic of M-theory on the subclass of quasi-homogeneous cDV. We can collect [97] all these threefolds compactly in Table 6.1.

ADE	Label	Singularity	Non-vanishing deformation parameter
A	(A_{N-1}, A_{k-1})	$x^2 + y^2 + z^k + w^N = 0$	$\mu_k = w^N$
	$A_{k-1}^{(k-1)}[N]$	$x^2 + y^2 + z^k + w^N z = 0$	$\mu_{k-1} = w^N$
D	(A_{N-1}, D_k)	$x^2 + zy^2 + z^{k-1} + w^N = 0$	$\mu_{2k-2} = w^N$
	$D_k^{(k)}[N]$	$x^2 + zy^2 + z^{k-1} + w^N y = 0$	$\tilde{\mu}_k = w^N$
E_6	(A_{N-1}, E_6)	$x^2 + y^3 + z^4 + w^N = 0$	$\mu_{12} = w^N$
	$E_6^{(9)}[N]$	$x^2 + y^3 + z^4 + w^N z = 0$	$\mu_9 = w^N$
	$E_6^{(8)}[N]$	$x^2 + y^3 + z^4 + w^N y = 0$	$\mu_8 = w^N$
E_7	(A_{N-1}, E_7)	$x^2 + y^3 + yz^3 + w^N = 0$	$\mu_{18} = w^N$
	$E_7^{(14)}[N]$	$x^2 + y^3 + yz^3 + w^N z = 0$	$\mu_{14} = w^N$
E_8	(A_{N-1}, E_8)	$x^2 + y^3 + z^5 + w^N = 0$	$\mu_{30} = w^N$
	$E_8^{(24)}[N]$	$x^2 + y^3 + z^5 + w^N z = 0$	$\mu_{24} = w^N$
	$E_8^{(20)}[N]$	$x^2 + y^3 + z^5 + w^N y = 0$	$\mu_{20} = w^N$

Table 6.1: *Quasi-homogeneous cDV singularities as ADE families.*

In all the threefold equations in Table 6.1 the first three monomials reconstruct the ADE singularity of type $\mathcal{G}$, while the last term can be interpreted as a deformation of this singularity. Hence the equations in Table 6.1 describe one-parameter families of deformed $\mathcal{G}$-singularities, fibered over a complex plane $\mathbb{C}_w$.

In studying M-theory on quasi-homogeneous cDV we perform a kind of "reverse-engineering" with respect to the results contained in Chapter 5. Indeed in this Chapter we will, given a threefold equation among these contained in Table 6.1, obtain the associated Higgs field Φ in a systematic way.

6.1 The Higgs vev from the threefold equation: the quasi-homogeneous case

Our question is now: given a CY equation in Table 6.1, what is the Higgs field that can generate it? The answer to this question is crucial in order to tackle the dynamics of M-theory on the quasi-homogeneous cDV singularities.

We recall that the Higgs fields associated to the threefold X can be used to produce X via the following two steps procedure:

1. we will express the $\mathcal{W}_{\mathcal{G}}$ invariants μ_i in terms of the $\mathcal{W}_{\mathcal{M}}$ invariants ϱ_i (with $\mathcal{M}$ a maximal subalgebra of maximal rank of a Levi subalgebra $\mathcal{L}$, determined as in Section 2.2), producing the map $\Psi_{\mathcal{L},\mathcal{G}}$ presented in Section 2.2;
2. we will give an appropriate holomorphic dependence to the ϱ in terms of w to reproduce the threefold equation.

From this perspective, the threefold is naturally embedded into the family over $\mathbb{C}_{\varrho}^{\mathrm{rank}(\mathcal{G})} = \mathfrak{t}/\mathcal{W}_{\mathcal{M}}$ by choosing a one-dimensional subspace parametrized by $\mathbb{C}_w$. This means that the threefold will inherit the partial simultaneous resolution associated with $\mathcal{W}_{\mathcal{M}}$: both in the family and in the threefold the blown-up roots will be, say $\alpha_{i_1}, ..., \alpha_{i_f}$. This immediately tells us that the commutant of Φ is $\mathcal{H} = \langle \alpha^*_{i_1}, ..., \alpha^*_{i_f} \rangle$. An element $\Phi \in \mathcal{M} \subseteq \mathcal{L}$ can be written as

$$\Phi = \sum_h \Phi_h + \sum_{a=1}^{f} \varrho_1^a \alpha_a^* \tag{6.1.1}$$

where Φ_h is an element of the summand $\mathcal{M}_h$ of the $\mathcal{M}$ decomposition in (4.2.2). Collecting the degree-j Casimir invariants ϱ_j^h of Φ_h in $\mathcal{M}_h$, together with the coefficients ϱ_1^a , one obtains the invariant coordinates ϱ_i on the base $\mathbb{C}_{\varrho}^{\mathrm{rank}(\mathcal{G})}$.

We conclude this introduction with a remark on the notation used in this chapter. We match the expression (6.1.1) with the one of (2.2.22) using $\Phi_{\mathrm{Slod}}(\vec{\varrho}) = \sum_h \Phi_h$ and labelling with ρ_j^h the components of $\vec{\varrho}$.

6.2 From the threefold equation to the partial Casimirs $\varrho_i(w)$

Now, we will proceed as follows: we start from the equation of a threefold in Table 6.1. Then, we will derive what is the minimal $\mathcal{W}_{\mathcal{M}}$ such that the partial Casimirs (namely, the $\mathcal{W}_{\mathcal{M}}$ invariants) ϱ_i can be taken as holomorphic (homogeneous) functions of w, in a way that produces the CY equation by taking $\mu_i = \mu_i(\varrho(w))$. This will tell us what is the w-dependence of the Casimirs ϱ^h_j of each Φ_h and the w-dependence of the coefficients ϱ_a. Finally, we will look for Higgs fields $\Phi(w) \in \mathcal{M}$, holomorphic in w, that have the given w-dependence for their partial Casimirs[1].

In particular, to reproduce the threefolds in Table 6.1, we want to determine which holomorphic functions $\varrho^I_j(w)$, with[2] $I = (h, a)$, make all deformation parameters vanish except one of degree M, that is

$$\mu_M(\varrho(w)) = w^N \, . \tag{6.2.1}$$

We stress that $\mu_M(\varrho(w))$ is a *homogeneous* polynomial in w of degree N.

Both the μ_M and the ϱ^I_j can be written as homogeneous polynomials in the $t_i \in \mathfrak{t}$ of degree, respectively, M and j. This implies that $\mu_M(\varrho)$ will be a weighted *homogeneous* polynomial in the coordinates ϱ^I_j's of degree M, where the coordinate ϱ^I_j has weight j. This, together with (6.2.1), implies that $\varrho^I_j(w)$ is a homogeneous function of w with degree $\frac{j\,N}{M}$, i.e.

$$\varrho^I_j(w) = c^I_j \, w^{\frac{jN}{M}} \, . \tag{6.2.2}$$

Now:

- Since we require that $\varrho^I_j(w)$ is holomorphic, the partial Casimirs that give a non-zero contribution (i.e. $c^I_j \neq 0$) are those with j such that

$$\frac{j\,N}{M} \;\in\; \mathbb{Z}^{>0} \, . \tag{6.2.3}$$

- Moreover, we want to pick the smallest $\mathcal{W}_{\mathcal{M}}$ that allows holomorphic functions $\varrho^I_j(w)$ compatible with (6.2.1). Small $\mathcal{W}_{\mathcal{M}}$ correspond to subalgebras $\mathcal{M}$ with several simple summands with small rank. This subalgebra then yields the smallest degree partial Casimirs that realize (6.2.3), for given M, N.

[1]Fixing the w-dependence of the partial Casimir invariants does not give a unique choice for a holomorphic element of $\mathcal{M}$.

[2]We recall that h runs from one to the number of simple factors in $\mathcal{M}_{\text{s.s.}}$, while $a = 1, ..., f$ labels the generators of $\mathcal{H}$.

Choosing the threefold in Table 6.1 determines M (see the last column of the table). For each value of N, we look for the minimal value of j that satisfies (6.2.3). Say that M has n_M divisors $q_1, ..., q_{n_M}$, where $q_1 = 1$ and $q_{n_M} = M$. Then N can always be written in a unique way as

$$N = \frac{p}{q_\alpha} M \mod M\,, \tag{6.2.4}$$

with q_α a divisor of M, $p < q_\alpha$ and (p, q_α) coprime. The condition (6.2.3) becomes then

$$\frac{j\,p}{q_\alpha} \in \mathbb{Z}^{>0}\,, \tag{6.2.5}$$

and the minimal value of j fulfilling it is $j = q_\alpha$.

Given N, only ϱ_j^I with j a multiple of q_α can be non-zero. In other words, $c_j^I = 0$ when $j \neq m\,q_\alpha$ with $m \in \mathbb{Z}$. Because of homogeneity, this implies also that $\mu_i(\varrho) = 0$ with $i \neq m\,q_\alpha$. We are then left with the following equations with unknown c_j^I $(j = mq_\alpha)$:

$$\begin{cases} \mu_{m\,q_\alpha}(c) = 0 \qquad m\,q_\alpha < M \\ \mu_M(c) = 1 \end{cases} \tag{6.2.6}$$

(where we have factored out the powers in w). In order to have a non-trivial solution, one requires that all c_j^I with $j = m\,q_\alpha$ be non-zero[3].

Let us see how we can use this information to extract the subalgebra $\mathcal{M}$ corresponding to a given choice of (A_{N-1}, G). We describe this in a simple example, i.e. (A_{N-1}, D_4). The D_4 algebra has four Casimirs: μ_2, μ_4, $\tilde{\mu}_4$ and μ_6. Hence $M = 6$. There are four divisors of 6:

$$q_\alpha \in \{1, 2, 3, 6\}.$$

We now see which (minimal) degree can take the partial Casimirs and then what is the choice of the minimal subalgebra $\mathcal{M}$ (minimal $\mathcal{W}_\mathcal{M}$). Let us vary N:

For $N = 0$ **mod** 6 ($q_\alpha = 1$), the minimal degree is $j = 1$. We look for a subalgebra $\mathcal{M}$ with all four partial Casimirs of degree 1. This is the smallest possible choice, i.e. the Cartan subalgebra of D_4. In this case, all four roots of D_4 are blown up in the simultaneous resolution.

[3]Otherwise the system of homogeneous equations in the first row of (6.2.6) will force all c_j^I's to vanish. We notice that the number of holomorphic ρ_j^I has to be equal to the number of all the $\mu_{m\,q_\alpha}$, μ_M. If that was not the case, the system (6.2.6) would be overconstrained, and a solution would not be guaranteed to exist.

For $N = 3$ **mod** 6 ($q_\alpha = 2$), the minimal degree is $j = 2$. There is actually a subalgebra of D_4 with four partial Casimirs of degree 2, i.e. $\mathcal{M} = A_1^{\oplus 4}$.[4] $\mathcal{M}$ is now a maximal subalgebra of maximal rank of $\mathcal{L} = \mathcal{G} = D_4$; correspondingly, there is no resolution at the origin of the family, hence the singularity is terminal.

For $N = 2, 4$ **mod** 6 ($q_\alpha = 3$), the minimal degree for the non-zero partial Casimir is $j = 3$. In any subalgebra of D_4, we can have at most one partial Casimir of degree 3. Moreover, μ_2 must depend on partial Casimirs of degree lower than 3, that must vanish identically (otherwise they would be non-holomorphic, due to (6.2.3)). We have $\mathcal{M} = A_2 \oplus \langle \alpha_3^*, \alpha_4^* \rangle$. Only the partial Casimirs of the semi-simple part of $\mathcal{M}$, that is A_2, are non-vanishing. In this case, the roots α_3 and α_4 of D_4 are blown up in the partial simultaneous resolution.

For $N = 1, 5$ **mod** 6 ($q_\alpha = 6$), the minimal degree for a non-vanishing partial Casimir is $j = 6$, hence in this case $\mathcal{M} = D_4$ with all Casimirs equal to zero, except the maximal degree one. For $N = 1$ the manifold is non-singular, while for $N = 5$ there is a terminal singularity at the origin of the family.

As one can notice in the presented example, the simple algebras $\mathcal{M}_h$ in the $\mathcal{M}$ decomposition (4.2.2) are all of the same type for a given value of N. This actually happens for all the cases we study in this chapter. The reason is the following: we look for partial Casimirs with the lowest possible degree, realizing $\mu_M = w^N$. If one degree is allowed, we take as many partial Casimirs with that degree as we are allowed. Small degree partial Casimirs correspond to small subalgebras $\mathcal{M}_h$, hence we finish with as many summands of a given small algebra as we can.

6.3 From the partial Casimirs $\varrho_i(w)$ to the Higgs field $\Phi(w)$

Now that we have the w-dependence of the ϱ_j^I's, we need to take a Higgs field in $\mathcal{M}$, whose partial Casimirs have that dependence. In general, there are several choices for Φ_h (in (6.1.1)) with given $\varrho_j^h(w)$. Each choice produces a different number of zero modes. We decide to look for the Higgs field Φ that localizes the maximal number of zero modes and breaks the 7d gauge symmetry in the least disruptive way, and we interpret the others as T-brane deformations of Φ, i.e. deformations that kill a number of modes, or destroy a preserved symmetry, without touching the threefold singularity (we come to this point in Section 7.1). With this choice, we pick up the

[4]Notice that all c_2^I's must be non-zero; otherwise, if one vanished, the equations $\mu_2 = \mu_4 = \tilde{\mu}_4 = 0$ would force all the others c_2^I's to be zero as well as μ_6.

Higgs field that leads to the same number of zero modes that are counted by the normalized complex structure deformations of the CY[5].

Let us first describe what is the structure of the Higgs field. At $w = 0$ the fiber of the one-parameter $\mathcal{G}$-family must develop a full $\mathcal{G}$-type singularity. This means that $\Phi(0)$ must be a nilpotent element of $\mathcal{M}$ (as all its $\mathcal{W}_{\mathcal{G}}$ invariants, namely its Casimirs, should vanish), that we take in its canonical form (e.g. for A_r it is the Jordan form; for general ADE singularities, we refer to [159]). Now, $\Phi(w)$ must be a deformation of the nilpotent element $\Phi(0)$, with deformation proportional to w and belonging to $\mathcal{M}$. As explained in Section 2.2.1, the way to do it moving transversely to the nilpotent orbit (that includes $\Phi(0)$) is dictated by taking Φ_h in the Slodowy slice in $\mathcal{M}_h$ passing through $\Phi_h(0)$. We notice that, in Proposition 2.2.2, we just considered the Slodowy slice through the regular nilpotent orbit. In this chapter there will be particular cases in which the orbit associated to $\Phi_h(0)$ will be a tinier orbit than the regular one, as we briefly anticipated at the end of Section 2.2.1. What is important here is that this allows to have multiple canonical forms for the Higgs field in $\mathcal{M}$ (labeled by nilpotent orbits of the semisimple part of $\mathcal{M}$), that are not equivalent by gauge transformations. The Higgs field will then be given as the sum of some root generators (usually simple roots) of $\mathcal{G}$ multiplied by 1 and of other generators (in $\mathcal{M}$) multiplied by powers of w.

To pick up the Higgs field that localizes the maximal number of modes, we need to properly choose the nilpotent orbit to which $\Phi(0)$ belongs to. Let us consider $\Phi, \Phi' \in \mathcal{M}$ with the same expressions for ϱ^I_j, but such that $\Phi(0)$ and $\Phi'(0)$ belong to two different nilpotent orbits. Then, they produce a different number of zero modes: the one whose nilpotent orbit at the origin is smaller has a bigger number of zero modes. Roughly speaking, if at the origin the orbit is bigger, one has a larger number of '1's in the canonical form of the Higgs; these gauge fix to zero a bigger number of Lie algebra components in the deformation φ. A more detailed explanation of these aspects, complemented by explicit examples, can be found in Appendix E, where we lay down the complete recipe to connect the partial Casimirs to the Higgs background.

If the power of w in the partial Casimirs ϱ^I_j is high, the minimal orbit at the

[5] In a nutshell, the procedure goes as follows [96]: first we write down a basis, as $\mathbb{C}$-vector space, of the Jacobian ring $R = \frac{\mathbb{C}[x,y,w,z]}{(F, \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial w}, \frac{\partial F}{\partial z})}$, with F the polynomial defining X. It is a mathematical fact that the elements of the basis can be chosen to be monomials, and hence have a well-defined scaling w.r.t. the quasi-homogeneous action on X. It turns out that we can pair, looking at these scaling weights, a number $2n_{\text{paired}}$ of monomials of the basis, while leaving other n_{unpaired} unpaired. The expected Higgs branch quaternionic dimension (that equals the number of 5d hypers), then, is

$$d_{\mathbb{H}} = n_{\text{paired}} + n_{\text{unpaired}}.$$

$d_{\mathbb{H}}$ also coincides with the number of normalizable (and log-normalizable) complex structure deformations [86].

origin reproducing the required w-dependence will be the trivial one. In these cases, the Higgs field that leads to the maximum number of zero modes is such that

$$\Phi = w^k \hat{\Phi} \,, \tag{6.3.1}$$

with $\hat{\Phi}(0)$ a non-trivial nilpotent element of $\mathcal{M}$. Knowing the zero modes of $\hat{\Phi}$, one is able to find the zero modes of Φ.

Let us illustrate how we pick the right choice of Φ with given $\varrho^I_j(w)$, by using the (A_{N-1}, D_4) example.

For $N = 1$, $\mathcal{M} = D_4$, $\rho_6 = \mu_6 = w$. $\Phi(0)$ is in the maximal nilpotent orbit of D_4 and its expression at generic w is dictated by the w-dependence of the Casimir:

$$\Phi = e_{\alpha_1} + e_{\alpha_2} + e_{\alpha_3} + e_{\alpha_4} + \frac{w}{4} e_{-\alpha_1-2\alpha_2-\alpha_3-\alpha_4} \,. \tag{6.3.2}$$

For $N = 2$, $\mathcal{M} = A_2 \oplus \langle \alpha_3^*, \alpha_4^* \rangle$. The only non-zero partial Casimir is the degree 3 Casimir of A_2: $\varrho_3 = w$. The unique (up to gauge transformations) holomorphic Higgs field compatible with that is now

$$\Phi = \Phi_{A_2} \quad \text{with} \quad \Phi_{A_2} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ w & 0 & 0 \end{pmatrix} = e_{\alpha_1} + e_{\alpha_2} + w \, e_{-\alpha_1-\alpha_2} \,. \tag{6.3.3}$$

For $N = 3$, $\mathcal{M} = A_1^{\oplus 4}$, $\varrho_2^h = c^h w$ $(h = 1, ..., 4)$, with c^h solving (6.2.6). The form of the Higgs field with these partial Casimirs is again unique:

$$\Phi = \sum_{h=1}^{4} \Phi_h \quad \text{with} \quad \Phi_h = \begin{pmatrix} 0 & 1 \\ c^h \, w & 0 \end{pmatrix} = e_{\alpha^h} + c^h w \, e_{-\alpha^h} \,, \tag{6.3.4}$$

where α^h is the root of the subalgebra A_1^h.

For $N = 4$, $\mathcal{M} = A_2 \oplus \langle \alpha_3^*, \alpha_4^* \rangle$. Now, differently from the $N = 2$ case, the only non-zero partial Casimir of degree 3 is quadratic in w: $\varrho_3 = w^2$. In this case we have two possible Higgs fields that are consistent with this, i.e. $\Phi = \Phi_{A_2}$

with

$$\text{either} \quad \Phi_{A_2} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ w^2 & 0 & 0 \end{pmatrix} \quad \text{or} \quad \Phi_{A_2} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & w \\ w & 0 & 0 \end{pmatrix} . \tag{6.3.5}$$

At the origin $w = 0$, the left one is in the maximal nilpotent orbit while the right one is in the minimal one. Hence we expect that choosing the right one will give us the bigger number of zero modes. Indeed this happens, as it can be easily verified by an explicit computation.

For $N = 5$, $\mathcal{M} = D_4$, $\varrho_6 = \mu_6 = w^5$, the Higgs field is of the same shape as the $N = 1$ case, with some coefficients proportional to w:

$$\Phi = e_{\alpha_1} + w \left(e_{\alpha_2} + e_{\alpha_3} + e_{\alpha_4} + \frac{1}{4} e_{-\alpha_1 - 2\alpha_2 - \alpha_3 - \alpha_4} \right) . \tag{6.3.6}$$

For $N = 6$, $\mathcal{M} = \mathcal{H}$, $\varrho_1^a = c^a w$ $(a = 1, ..., 4)$. Φ is forced to be of the form

$$\Phi = c^1 w \, \alpha_1^* + c^2 w \, \alpha_2^* + c^3 w \, \alpha_3^* + c^4 w \, \alpha_4^* . \tag{6.3.7}$$

Let us see some cases where we go up with the power N of w in μ_6:

For $N = 8$, we obtain the same algebra as for $N = 2$, i.e. $\mathcal{M} = A_2 \oplus \langle \alpha_3^*, \alpha_4^* \rangle$. Now, the only non-zero partial Casimir of degree 3 of A_2 takes the following w-dependence $\varrho_3 = w^4$. The minimal nilpotent orbit at the origin compatible with this partial Casimir is now the trivial one. The Higgs field giving the maximal number of zero modes is

$$\Phi = \Phi_{A_2} \quad \text{with} \quad \Phi_{A_2} = w \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ w & 0 & 0 \end{pmatrix} = w \, e_{\alpha_1} + w \, e_{\alpha_2} + w^2 \, e_{-\alpha_1 - \alpha_2} . \tag{6.3.8}$$

For $N = 9$, we obtain the same algebra as for $N = 3$, i.e. $\mathcal{M} = A_1^{\oplus 4}$. The Higgs field giving the maximal number of zero modes is

$$\Phi = \sum_{h=1}^{4} \Phi_h \quad \text{with} \quad \Phi_h = w \begin{pmatrix} 0 & 1 \\ c^h w & 0 \end{pmatrix} = w \, e_{\alpha^h} + c^h w^2 \, e_{-\alpha^h} . \tag{6.3.9}$$

The same can be done for the cases $N = 7, 10, 11, 12$, where the Higgs contributing most to the zero modes is the one with $N - 6$ multiplied by w. In general, the Higgs fields given above for $N = 1, ..., 6$ are enough to write the Higgs field for any N: If $N = n + 6k$, with $n \in \{1, ..., 6\}$, the Higgs field is $\Phi = w^k \Phi^{(n)}$, where $\Phi^{(n)}$ is the Higgs field for $N = n$.

This is actually true for all the cDV singularities in Table 6.1:

Given M and N as above, one needs to find the Higgs fields $\Phi^{(n)}$ for $N = n$, with $n \in \{1, ..., M\}$. The Higgs field for $N = n + kM$ is then $\Phi = w^k \Phi^{(n)}$.

This is remarkably convenient also from the physical point of view, as the Higgs background Φ encodes all the 5d physics, meaning the localized hypers and their charges under the flavor and discrete symmetries. What the statement in italics is telling us is that, given a quasi-homogeneous cDV singularity built as an ADE singularity with a $\mu_M = w^N$ deformation term, we need to know only the Higgs backgrounds for N up to M: all the rest can be obtained simply by multiplying these Higgs backgrounds by some power of w. The 5d mode counting changes as explained at the end of Section 4.6, the symmetries act in the same way on the (now possibly increased) modes, and the Higgs branch content varies accordingly, so that no new computation must be performed. This permits us to perform the full scanning of the M-theory dynamics on *all* the quasi-homogeneous cDV.

6.4 5d Higgs Branches from quasi-homogeneous cDV singularities

In this section we exhibit the complete classification of the 5d theories arising from M-theory on quasi-homogeneous cDV singularities.

First, given a quasi-homogeneous cDV singularity, we must find the minimal subalgebra $\mathcal{M}$ in which a Higgs background Φ can reside, compatibly with the threefold equation (see Section 6.2). Then, we find the Higgs field that produces the maximal number of modes following Section 6.3 (checking that it is consistent with the HB dimension given by the normalizable complex structure deformations). Once we have the Higgs field Φ, we can compute the 5d continuous flavor group, the discrete gauge group and the charges of the hypermultiplets under these groups.

We proceed methodically through all the cases in Table 6.1, proposing also an equivalent complementary type IIA perspective for the (A, A) and (A, D) series. To conclude, we will treat the exceptional cases.

6.5 Quasi-homogeneous cDV singularities of A type

Two quasi-homogeneous cDV singularities of A type exist: the (A_{N-1}, A_{M-1}) and the $A_M^{(M)}[N]$. Their defining equations are

$$\boldsymbol{(A_{M-1}, A_{N-1})}: \qquad x^2 + y^2 + z^M + w^N = 0, \tag{6.5.1}$$

$$\boldsymbol{A_M^{(M)}[N]}: \qquad x^2 + y^2 + z \cdot (z^M + w^N) = 0. \tag{6.5.2}$$

The non-vanishing deformation parameters are, in both cases, $\mu_M(w) = w^N$. The equation (6.5.1) is a A_{M-1} family, while (6.5.2) is a A_M family. It is however easy to see, adopting the technique fleshed out in Chapter 6, that the analysis of the $A_M^{(M)}[N]$ singularities can be fully traced back to the (A_{M-1}, A_{N-1}) singularities: in particular one can see that the Higgs field in the A_M family is living in a A_{M-1} subalgebra and that both spaces are produced by the same choice of $\Phi \in A_{M-1}$. In general (and for some suitable choice of basis for the Cartan subalgebra), we find hypers of charge at most 1, as the dual Coxeter labels of the nodes of the A Dynkin diagrams are all equal to 1, see Figure 6.1. In Table 6.2, we report the results for both

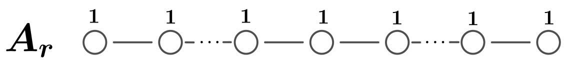


Figure 6.1: *Dual Coxeter labels for the A series.*

the (A_{k-1}, A_{N-1}) and the $A_k^{(k)}[N]$ singularities, rewriting them in full generality as (A_{mp-1}, A_{mq-1}) and $A_{mp}^{(mp)}[mq]$ singularities, respectively, and with p and q coprime, $p \geq q$. We give the resolution pattern, the corresponding flavor group, the number of charged hypers and the number of uncharged ones. The last ones are a signal of a non fully-resolvable singularity. The flavor groups are respectively $U(1)^{m-1}$ and $U(1)^m = U(1) \times U(1)^{m-1}$, where in the latter case the factor $U(1)^{m-1}$ is contained in a A_{mp-1} subalgebra, as we have mentioned above. The flavor charges can be succinctly understood as follows, in some basis of the Cartan subalgebra[6]: for the (A_{mp-1}, A_{mq-1}) cases, writing $U(1)^{m-1} \cong \frac{U(1)^m}{U_{\text{cm}}(1)}$ (where $U_{\text{cm}}(1)$ is the decoupled diagonal center of mass $U(1)$) there are pq hypers charged in the bifundamental of every possible pair of $U(1)$'s in $U(1)^m$, as well as $m\frac{(p-1)(q-1)}{2}$ uncharged hypers. For the $A_{mp}^{(mp)}[mq]$ cases, there are pq hypers charged in the bifundamental of every possible pair of $U(1)$'s in the numerator of the flavor group contained in the A_{mp-1} subalgebra (regarded again as $\frac{U(1)^m}{U_{\text{cm}}(1)} \cong U(1)^{m-1}$), q hypers charged bifundamentally under every possible pair formed by the $U(1)$ *outside* the A_{mp-1} subalgebra and a $U(1)$ in the numerator of $\frac{U(1)^m}{U_{\text{cm}}(1)}$, and finally there are $m\frac{(p-1)(q-1)}{2}$ uncharged hypers.

[6]For further details, we refer to the much more in-depth analysis of [129].

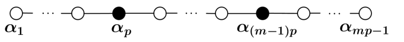
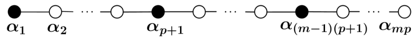
Singularity	Resolution pattern	Flavor group	Hypers	Total hypers
(A_{mp-1}, A_{mq-1})		$U(1)^{m-1}$	Charged: $pq\frac{m(m-1)}{2}$ Uncharged: $m\frac{(p-1)(q-1)}{2}$	$\frac{1}{2}m(p(mq-1)-q+1)$
$A_{mp}^{(mp)}[mq]$		$U(1)^{m}$	Charged: $pq\frac{m(m-1)}{2} + \boldsymbol{mq}$ Uncharged: $m\frac{(p-1)(q-1)}{2}$	$\frac{1}{2}m(p(mq-1)-q+1) + \boldsymbol{mq}$

Table 6.2: *Higgs Branch data for quasi-homogeneous cDV singularities of A type.*

6.5.1 D-branes perspectives for the (A, A) series

In this section we want to re-obtain the Higgs branches we computed for the (A_{N-1}, A_{k-1}) singularities from the physical perspective of the D6-branes stack appearing in the type IIA limit.

For these cDVs, the type IIA limit contains just D6 branes, and no O6$^-$ planes. To take the type IIA limit explicitly, it is convenient to rewrite the equation of the (A_{N-1}, A_{k-1}) singularities as:

$$uv = z^k + w^N, \quad u \equiv (x + iy), \quad v \equiv -x + iy. \tag{6.5.3}$$

The $\mathbb{C}^*$ action describing the $\mathbb{C}^*$ fibration is

$$u \to \lambda u, \quad v \to \frac{v}{\lambda}, \quad \lambda \in \mathbb{C}^*, \tag{6.5.4}$$

and the combination uv appearing on the l.h.s. of (6.5.3) is the associated moment map. The degeneracy locus of the $\mathbb{C}^*$ fibration corresponds to the zeros of the moment map, and in the type IIA limit it corresponds to the position of the D6 branes. Consequently, using again (6.5.3) the brane locus "Δ" is

$$\Delta(w, z) = z^k + w^N = 0. \tag{6.5.5}$$

We now want to decompose the brane locus in irreducible factors:

$$\Delta(z, w) = Q_1(z, w) \cdot ... \cdot Q_m(z, w) = \det(z\mathbb{1} - \Phi), \tag{6.5.6}$$

with $Q_i(z, w)$ irreducible polynomials. The Higgs field that reproduces such brane locus can be taken as the block sum of m matrix blocks. In particular, the size r_i of the i-th block corresponds to the z-degree of the i-th irreducible factor of $\Delta(z, w)$ and the block represents the recombination of r_i D6s. In the Lie-algebraic language of Section 2.2, this is equivalent to say that the maximal subalgebra $\mathcal{M}$ associated to the Higgs is

$$\mathcal{M} = A_{r_1-1} \oplus ... \oplus A_{r_m-1} \oplus \mathcal{H}. \tag{6.5.7}$$

Without generality loss, we write (A_{N-1}, A_{k-1}) as (A_{mp-1}, A_{mq-1}), with p, q co-primes, $p \geq q$, and $m = \gcd(k, N)$. It then becomes manifest that we can always factor the brane locus as follows:

$$\Delta(w, z) = z^{mp} + w^{mq} = \prod_{s=1}^{m} (z^p + e^{2\pi i s/m} w^q). \tag{6.5.8}$$

The factor $(z^p + e^{2\pi i s/m} w^q)$ in (6.5.8) can be realised, for all the (p, q), as the characteristic polynomial of a $p \times p$ matrix $\mathcal{A}_s$, with matrix entries being polynomials in *w of degree at most one.*

The blocks[7] "$\mathcal{A}_s$" whose characteristic polynomials are the irreducible factors appearing in (6.5.8) can be put in the following canonical shape[8]:

$$\mathcal{A}_s(w) = \begin{pmatrix} 0 & * & 0 & \cdots & 0 \\ 0 & 0 & * & 0 & 0 \\ \vdots & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & * \\ -e^{2\pi i s/m} w & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{6.5.9}$$

where the $*$ entries are filled either with w, or are constants (that can be set to 1); to reproduce the right characteristic polynomial, we have to fill $q-1$ $*$-entries with "w". Depending on the position where we place the "w", one has a different number of zero-modes[9]. Following the procedure explained in Section 6.3, we have to fill the $*$ entries of (6.5.9) in such a way to maximise the number of five-dimensional modes or, equivalently, to minimize the nilpotent orbit $\mathcal{O}_0$ associated to $\Phi|_{w=0}$. To obtain the m factors of the brane locus corresponding to the full Higgs field, we take the

[7] The blocks are, indeed, characterized by the four integers p, q, s, m, appearing in each factor of (6.5.8) but we omit p, q, m for ease of notation.

[8] We remark that this is simply an example of the application of the procedure outlined in Appendix E. In particular, (6.5.9) is exactly of the form reported in (E.0.1).

[9] We remark here that the Higgs field (6.5.9) is in the canonical form that we reported in (E.0.1).

block direct sum of all the $\mathcal{A}_s$ blocks[10]:

$$\Phi = \underbrace{\left(\begin{array}{c|c|c|c} \mathcal{A}_{s=1} & \mathbb{0}_p & \mathbb{0}_p & \mathbb{0}_p \\ \hline \mathbb{0}_p & \mathcal{A}_{s=2} & \vdots & \\ \hline & \vdots & \ddots & \mathbb{0}_p \\ \hline & \vdots & \vdots & \mathcal{A}_{s=m} \end{array}\right)}_{m \text{ blocks}} . \tag{6.5.10}$$

We can now apply the procedure outlined in Section 4.1 and Section 4.2 to (6.5.10) obtaining the following results:

1. the five-dimensional modes localize with the following pattern:

$$\varphi \equiv \underbrace{\left(\begin{array}{c|cc|c} (p-1)(q-1)\text{modes} & p\cdot q \text{ modes} & \cdots & p\cdot q \text{ modes} \\ p\cdot q \text{ modes} & \ddots & & \vdots \\ \vdots & & \ddots & p\cdot q \text{ modes} \\ p\cdot q \text{ modes} & \cdots & p\cdot q \text{ modes} & (p-1)(q-1) \text{ modes} \end{array}\right)}_{m \text{ blocks}} ; \tag{6.5.11}$$

2. the flavor group is $U(1)^m/U(1)_{\text{diag.}}$. The elements of the flavor group are the $\mathbb{Z}_k$-equivalence classes in $PSU(k) = SU(k)/\mathbb{Z}_k$ represented by the following matrices:

$$G_{\text{flavor}} \equiv \underbrace{\left(\begin{array}{c|c|c|c} e^{i\alpha_1}\mathbb{1}_p & \mathbb{0}_p & \cdots & \mathbb{0}_p \\ \hline \mathbb{0}_p & e^{i\alpha_2}\mathbb{1}_p & \vdots & \vdots \\ \hline \vdots & \vdots & \ddots & \mathbb{0}_p \\ \hline \mathbb{0}_p & \cdots & \mathbb{0}_p & e^{i\alpha_m}\mathbb{1}_p \end{array}\right)}_{m \text{ blocks}} , \quad \sum_{s=1}^{m} \alpha_s = \frac{2\pi T}{p}, \tag{6.5.12}$$

with $T \in \mathbb{Z}$. The flavor group generators act on the modes via adjoint action:

$$\varphi \to G_{\text{flavor}}\varphi (G_{\text{flavor}})^{-1}. \tag{6.5.13}$$

[10]The integers p, q are the same for all the blocks, the phase $e^{2\pi i s/m}$ multiplying the lowest-left entry in (6.5.9) is opportunely tuned in such a way that each block $\mathcal{A}_s$ reproduces each of the factors of (6.5.8).

The data in the matrices (6.5.11) and (6.5.12) allow us to count the number of five-dimensional modes and to reconstruct their charges under the flavor group $\frac{U(1)^m}{U(1)_{diag}} \cong U(1)^{m-1}$. Indeed, in the language of Section 4.1, the grid structure of the matrix in (6.5.10) is the graphical realization of the branching (4.1.3) of A_{k-1} w.r.t. $\mathcal{M} \ni \Phi$.

By looking to (6.5.11) and (6.5.12), we see that each of the flavor $U(1)$ of (6.5.12) acts linearly, with charge one, on $m(q-1)(p-1)$ modes:

$$Q_i \to e^{i\alpha_s} Q_i, \quad \tilde{Q}_i \to e^{-i\alpha_s} \tilde{Q}_i, \quad i = 1, ..., n_{\text{charged hypers}}, \quad s = 1, ..., m. \qquad (6.5.14)$$

To conclude, in view of the (more complicated) case of the (A_{N-1}, D_k) singularities, we quickly recap our strategy to study the HBs from the factorization properties of the the brane locus of the (A, A) cDVs. Given, as input datum, the equation of the (A_{mp-1}, A_{mq-1}) singularity

1. We *computed the brane locus* Δ looking where the $\mathbb{C}^*$ fibers of the threefold degenerate.

2. We *factored the brane locus* (6.5.5) in polynomials that can be represented by the characteristic polynomials of a traceless matrix $\mathcal{A}_s$ with entries being $w-$dependent polynomials of degree at most one. We found that *any* polynomial that enters in the factorization of the brane locus of the (A_{mp-1}, A_{mq-1}) singularity is the characteristic polynomial of some block $\mathcal{A}_s$ of the shape (6.5.9).

3. We *counted the number of five-dimensional modes that are localized in the diagonal blocks*. More precisely, each of the $\mathcal{A}_s$ selects a minimal $A_{p-1} \hookrightarrow A_{mp-1}$ subalgebra that corresponds to the block containing $\mathcal{A}_s$ (6.5.11) (we highlighted the A_{p-1} subalgebras corresponding to the various $\mathcal{A}_s$ with different colours in (6.5.10)). The localization of modes inside a certain A_{p-1} subalgebra is determined just by the corresponding block $\mathcal{A}_s$, and is always the same for all the s.

4. We *counted the number of five-dimensional modes that a pair $\mathcal{A}_s$, $\mathcal{A}_s$ localizes in the corresponding off-diagonal blocks* of the block decomposition of the $\mathfrak{sl}(mp, \mathbb{C}[w])$ matrix (see the equation below):

$$\left(\begin{array}{c|cc|c} (p-1)(q-1) \text{ modes} & \dots & \cdots & p{\cdot}q \text{ modes} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ p{\cdot}q \text{ modes} & \dots & \dots & (p-1)(q-1) \text{ modes} \end{array}\right). \qquad (6.5.15)$$

6.6 Quasi-homogeneous cDV singularities of D type

There exist two quasi-homogeneous cDV singularities arising from one-parameter deformations of D singularities: the (A_{N-1}, D_{m+1}) and the $D_m^{(m)}[N]$. Their defining equations read

$$\boldsymbol{(A_{N-1}, D_{m+1})}: \qquad x^2 + zy^2 + z^m + w^N = 0, \tag{6.6.1}$$

$$\boldsymbol{D_m^{(m)}[N]}: \qquad x^2 + zy^2 + z^{m-1} + yw^N = 0. \tag{6.6.2}$$

In the two cases, the non-vanishing deformation parameter is $\mu_M = w^N$, that is the maximal degree one for the first case ($M = 2m$), while for the second case it is the always present r-degree deformation parameter of D_r ($M = m$).

The 5d theories from M-theory on (A, D) singularities will be worked out in Appendix C, we refer to that appendix for the results. We have applied our method to work out also the $D_m^{(m)}[N]$ singularities. As they are useful to identify the flavor charges of the hypermultiplets whenever a single node is resolved, in Figure 6.2 we report the dual Coxeter labels of the nodes of the Dynkin diagrams in the D series.

$\boldsymbol{D_r}$

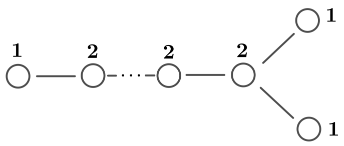


Figure 6.2: *Dual Coxeter labels for the D series.*

We notice that, in full generality, all the (A_{2km-1}, D_{m+1}) and the $D_m^{(m)}[km]$ are completely resolvable, because in that case $N = kM$; this means, following Section 6, that $q_\alpha = 1$ and the minimal degree for the partial Casimirs is $j = 1$, i.e. $\mathcal{M}$ is the Cartan subalgebra of $\mathcal{G}$.

In Table 6.3 and Table 6.4 we report the results for the Higgs branch data, respectively, of the (A_{N-1}, D_4), (A_{N-1}, D_7) and $D_4^{(4)}[N], D_5^{(5)}[N], D_6^{(6)}[N]$ cases, specifying the flavor and discrete charges of the hypermultiplets. Other deformed D_r examples can be treated analogously.

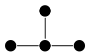
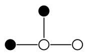
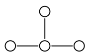
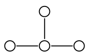
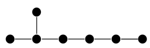
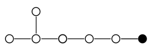
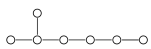
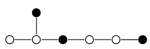
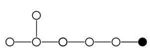
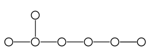
Singularity	Resolution pattern	$\mathcal{M}$	Symmetry group	Hypers	Total hypers
(A_{N-1}, D_4)	$N = 6n$: 	$\mathfrak{t}$	$U(1)^4$	$12n$ Charges: **root system** of D_4	$2N$
	$N = 2n$ $n \neq 3j$: 	$A_2 \oplus \langle \alpha_1^*, \alpha_4^* \rangle$	$U(1)_a \times U(1)_b$	$(q_a, q_b) = (2,0) : \boldsymbol{n}$ $(q_a, q_b) = (1,1) : \boldsymbol{2n}$ $(q_a, q_b) = (0,0) : \boldsymbol{n-1}$	$2N - 1$
	$N = 3n$ $n \neq 2j$: 	$A_1^{\oplus 4}$	$\mathbb{Z}_2$	$q_{\mathbb{Z}_2} \neq 0 : \boldsymbol{4n}$ $q_{\mathbb{Z}_2} = 0 : \boldsymbol{2(n-1)}$	$2(N-1)$
	$N \neq 2n, 3n$: 	D_4	$\emptyset$	$2(N-1)$	$2(N-1)$
(A_{N-1}, D_7)	$N = 12n$: 	$\mathfrak{t}$	$U(1)^7$	$42n$ Charges: **root system** of D_7	$\frac{7N}{2}$
	$N = 6n$ $n \neq 2j$: 	$A_1^{\oplus 6} \oplus \langle \alpha_6^* \rangle$	$U(1)_a \times \mathbb{Z}_2^{(b)} \times \mathbb{Z}_2^{(c)}$	$(q_a, q_b, q_c) = (1,0,0) : \boldsymbol{2n}$ $(q_a, q_b, q_c) = (1,0,1) : \boldsymbol{2n}$ $(q_a, q_b, q_c) = (1,1,0) : \boldsymbol{2n}$ $(q_a, q_b, q_c) = (0,1,1) : \boldsymbol{n-1}$ $(q_a, q_b, q_c) = (0,1,0) : \boldsymbol{n-1}$ $(q_a, q_b, q_c) = (0,0,1) : \boldsymbol{n-1}$ $(q_a, q_b, q_c) = (0,0,0) : \boldsymbol{12n}$	$\frac{7N}{2} - 3$
	$N = 3n$ $n \neq 2j$: 	$D_4 \oplus A_3$	$\mathbb{Z}_2$	$q_{\mathbb{Z}_2} \neq 0 : \boldsymbol{6n}$ $q_{\mathbb{Z}_2} = 0 : \boldsymbol{\frac{9n-7}{2}}$	$\frac{7(N-1)}{2}$
	$N = 4n$ $n \neq 3j$: 	$A_2 \oplus A_2 \oplus$ $\langle \alpha_3^*, \alpha_6^*, \alpha_7^* \rangle$	$U(1)_a \times U(1)_b \times U(1)_c$	$(q_a, q_b, q_c) = (0,2,0) : \boldsymbol{n}$ $(q_a, q_b, q_c) = (0,0,2) : \boldsymbol{n}$ $(q_a, q_b, q_c) = (0,1,1) : \boldsymbol{6n}$ $(q_a, q_b, q_c) = (1,1,0) : \boldsymbol{2n}$ $(q_a, q_b, q_c) = (1,0,1) : \boldsymbol{2n}$ $(q_a, q_b, q_c) = (0,0,0) : \boldsymbol{2(n-1)}$	$\frac{7N}{2} - 2$
	$N = 2n$ $n \neq 2j, 3j$: 	$D_6 \oplus \langle \alpha_6^* \rangle$	$U(1)$	$q = 1 : \boldsymbol{5n-3}$ $q = 0 : \boldsymbol{2n}$	$\frac{7N}{2} - 3$
	$N \neq 2n, 3n$: 	D_7	$\emptyset$	$\frac{7(N-1)}{2}$	$\frac{7(N-1)}{2}$

Table 6.3: *Higgs branch data for quasi-homogeneous cDV singularities of (A_{N-1}, D_4) and (A_{N-1}, D_7) type.*

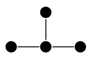
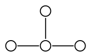
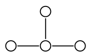
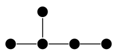
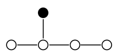
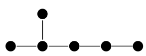
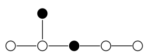
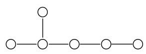
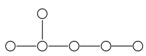
Singularity	Resolution pattern	$\mathcal{M}$	Symmetry group	Hypers	Total hypers
$D_4^{(4)}[N]$	$N = 4n$: 	$\mathfrak{t}$	$U(1)^4$	$12n$ Charges: **root system** of D_4	$3N$
	$N = 2(2n-1)$: 	$A_1^{\oplus 4}$	$\mathbb{Z}_2$	$q_{\mathbb{Z}_2} \neq 0 : \mathbf{4(2n-1)}$ $q_{\mathbb{Z}_2} = 0 : \mathbf{4(n-1)}$	$3N-2$
	$N \neq 4n, 4n-2$: 	D_4	$\emptyset$	$3N-2$	$3N-2$
$D_5^{(5)}[N]$	$N = 5n$: 	$\mathfrak{t}$	$U(1)^5$	$20n$ Charges: **root system** of D_5	$4N$
	$N \neq 5n$: 	$A_4 \oplus \langle \alpha_5^* \rangle$	$U(1)$	$q = 1 : \mathbf{2N}$ $q = 0 : \mathbf{2(N-1)}$	$2(2N-1)$
$D_6^{(6)}[N]$	$N = 6n$: 	$\mathfrak{t}$	$U(1)^6$	$30n$ Charges: **root system** of D_6	$5N$
	$N = 2n$ $n \neq 3j$: 	$A_2 \oplus A_2 \oplus$ $\langle \alpha_3^*, \alpha_6^* \rangle$	$U(1)_a \times U(1)_b$	$(q_a, q_b) = (2,0) : \boldsymbol{n}$ $(q_a, q_b) = (0,2) : \boldsymbol{n}$ $(q_a, q_b) = (1,1) : \mathbf{6}\boldsymbol{n}$ $(q_a, q_b) = (0,0) : \mathbf{2(n-1)}$	$5N-2$
	$N = 6n-3$: 	$A_1^{\oplus 6}$	$\mathbb{Z}_2^2$	$q_{\mathbb{Z}_2} \neq 0 : \mathbf{12(2n-1)}$ $q_{\mathbb{Z}_2} = 0 : \mathbf{6(n-1)}$	$5N-3$
	$N \neq 2n, 6n-3$: 	D_6	$\emptyset$	$5N-3$	$5N-3$

Table 6.4: *Higgs branch data for quasi-homogeneous cDV singularities of $D_4^{(4)}[N], D_5^{(5)}[N], D_6^{(6)}[N]$ type.*

6.6.1 D-branes perspective for the (A, D) series

We will offer now a complementary physical perspective, inspired by the D-branes physics, on the dynamics of M-theory on the (A_{N-1}, D_k) singularities.

Let us report here for convenience the expression for the (A_{N-1}, D_k) singularities appearing in Table 6.1:

$$x^2 + zy^2 + z^{k-1} + w^N = 0, \qquad (x, y, w, z) \in \mathbb{C}^4. \tag{6.6.3}$$

It is useful to specify the second equation of (2.2.20) for the (A_{N-1}, D_k) case:

$$x^2 + zy^2 + \frac{\Delta(z, w)}{z} \quad = \quad x^2 + zy^2 + \frac{\sqrt{\det(z\mathbb{1} + \Phi(w)^2)}}{z} \quad = \quad 0, \tag{6.6.4}$$

where, as we will see soon, $\Delta(z, w)$ is the locus of D6 branes after a VEV for Φ has been switched on. M-theory on (6.6.3) is dual to a type IIA setup with $2k$ D6s on top of an orientifold plane, higgsed by the Higgs field Φ that satisfies (6.6.4). To understand the type IIA limit, following [160] we introduce an orientifold-covariant coordinate ξ:

$$z = \xi^2 \qquad \text{where } \xi \to -\xi \text{ under the orientifold projection.} \tag{6.6.5}$$

For the type IIA limit, the $\mathbb{C}^*$ fibration basis is $(w, \xi) \in \mathbb{C}^2_{w,\xi}$, with fibral coordinates x, y constrained by (6.6.4) and the brane locus is [160]

$$\Delta(\xi^2, w) = \text{Disc}_y\left(\xi^2 y^2 + \frac{\det(\xi^2\mathbb{1} + \Phi(w))}{\xi^2}\right) = \det(\xi^2\mathbb{1} + \Phi(w)), \tag{6.6.6}$$

where Disc_y indicates the discriminant with respect to y (thus justifying the Δ symbol).

At this stage, the story proceeds exactly as in the deformed A_r case of Section 6.5.1: we pick a (A_{N-1}, D_k) singularity, and solely by looking at the factorization properties of the brane locus we are able to predict which 2-cycles can be resolved and which cannot, thus constraining the block structure of Φ. In the language of the previous chapters, the data of this block structure are equivalent to those of the subalgebra $\mathcal{M}$ associated to Φ.

For the (A, A) series, if the brane locus $\Delta(z, w)$ factorized into m irreducible factors of z-degree $r_1, ..., r_m$, then we had $\mathcal{M} = A_{r_1-1} \oplus ... \oplus A_{r_m-1} \oplus \mathcal{H}$. For the (A, D) series we have a similar story, except for a caveat: in general, not all possible factorizations into *irreducible polynomials* of the brane locus of a D_n deformed family can be translated into a viable Higgs field. Let us explain this crucial point in more detail.

The key ingredient is the relation between the brane locus and the Higgs field Φ:

$$\Delta(\xi^2, w) = \det(\xi^2 \mathbb{1} + \Phi(w)). \tag{6.6.7}$$

Suppose that the brane locus can be factorized in irreducible holomorphic polynomials of the form:

$$\Delta(\xi^2, w) = P_1(\xi^2, w) \dots P_m(\xi^2, w). \tag{6.6.8}$$

Then, according to (6.6.7), we would be tempted to build a Higgs field made up of m blocks $\mathcal{B}^{(j)}$, with $j = 1, \dots m$, in some basis of D_n, each contributing a factor $P_j(\xi^2, w)$ to the characteristic polynomial of Φ, namely:

$$\Phi = \left(\begin{array}{c|ccc|c} \mathcal{B}^{(1)} & 0 & \cdots & & 0 \\ \hline 0 & \ddots & & & \\ \vdots & & \ddots & & \vdots \\ & & & \ddots & \\ \hline 0 & & & & \mathcal{B}^{(m)} \end{array}\right), \qquad \text{with:} \begin{cases} \chi(\mathcal{B}^{(1)}) = P_1(\xi^2, w), \\ \quad\vdots \\ \chi(\mathcal{B}^{m)}) = P_m(\xi^2, w), \end{cases} \tag{6.6.9}$$

where χ indicates the characteristic polynomial.

It turns out that in general this is *not* possible[11], meaning that a completely generic irreducible polynomial $P_j(\xi^2, w)$ does not have a counterpart in terms of the characteristic polynomial of a block living in a subalgebra of D_n: as a result, the powerful and general (irreducible polynomial)$\leftrightarrow$(block) correspondence that enabled us to analyze the (A, A) singularities is broken in the (A_{N-1}, D_k) cases. There are, however, some good news: the correspondence is not completely disrupted, and we can reverse the logic of the argument by asking the question: is there a way to determine which polynomials in the factorization of the brane locus can be built as the characteristic polynomial of blocks in subalgebras of D_n, and which cannot?

We will now show that, for (A_{N-1}, D_k), this is indeed possible: we can hence proceed in giving a shortlist of necessary and sufficient blocks needed to reconstruct the brane loci of *all* the (A_{N-1}, D_k) singularities.

In doing so, we define an *irreducible block* $\mathcal{B}$ as follows:

Definition 6.6.1 (Irreducible block)**.** Let $\mathcal{B}$ be an element of a subalgebra of D_n,

[11]This is a known fact in the mathematical literature, as there are no orthogonal companion matrices for orthogonal matrices. This means that, contrarily to the (A_j, A_l) series, a canonical representative for a block in a subalgebra of D_n with arbitrary characteristic polynomial does not exist.

and let $P(\xi^2, w)$ be its characteristic polynomial. In general, $P(\xi^2, w)$ might be algebraically reducible and decomposable into factors, but suppose that *at least one* of these factors *cannot* be realized as the characteristic polynomial of a block living in a subalgebra of D_n. Then we say that $\mathcal{B}$ is an *irreducible block.*

The list of irreducible blocks also fixes a list of types of polynomials, i.e. the characteristic polynomial of each block, in which the brane locus $\Delta(\xi^2, w)$ can be consistently factorized.

Let us illustrate this concept with a simple example: suppose that in the brane locus factorization of a (A_{N-1}, D_k) singularity a factor of the following form appears, corresponding to a block living in a $\mathfrak{so}(4)$ subalgebra of D_n:

$$P(\xi^2, w) = \xi^4 + w^2. \tag{6.6.10}$$

We would be tempted, on the algebraic level, to go on with the decomposition and write it as the product of two factors, each corresponding to a block in $\mathfrak{so}(2) \subset D_n$:

$$P(\xi^2, w) = \xi^4 + w^2 = \underbrace{(\xi^2 + iw)}_{P_1}\underbrace{(\xi^2 - iw)}_{P_2}. \tag{6.6.11}$$

However, an explicit computation shows that this is *not* possible, i.e. there does not exist any holomorphic block in $\mathfrak{so}(2) \subset D_n$ such that its characteristic polynomial is P_1 or P_2.

Luckily, with some work it is possible to classify the irreducible blocks that are needed to build *all* the Higgs configurations for the (A_{N-1}, D_k) singularities that we are interested in[12]. In Table 6.5 we list the corresponding characteristic polynomials, which are the polynomials that appear in the factorization of the brane locus, as well as the minimal subalgebras $\mathfrak{s}_{\min}$ of D_n containing the blocks:

[12]We have checked this explicitly for (A_{N-1}, D_k) singularities with k and n up to the hundreds, and we see no reason not to conjecture that our classification holds for any k and n.

$\mathcal{B}$	$\chi(\mathcal{B})$	$\mathfrak{s}_{\min}$
(a)	$(\xi^{2r+1} + c_1 w^t)\,(\xi^{2r+1} - c_1 w^t)$	$\mathfrak{u}(2r+1)$
(b)	$\xi^2\,(\xi^{2r} + c_2 w^{2t+1})$	$\mathfrak{so}(2r+2)$
(c)	$\xi^{4r} + c_3 w^{2t+1}\xi^{2r} + c_4 w^{2(2t+1)}$ $(r, 2t+1)$ coprime	$\mathfrak{so}(4r)$

Table 6.5: *Irreducible blocks and minimal subalgebras.*

The c_i in the expressions of the polynomials are some constant parameters (that can also be vanishing). Notice that all the block classes (a), (b), and (c) are labelled by two integer parameters r and t. In the following, we will refer to a given block in some class using the notation $\mathcal{B}_{(i)}$, with $i = a, b, c$, suppressing the dependence on r and t for graphical ease. For the explicit expressions of the blocks concretely realizing these polynomials, we refer to Appendix D.

It is useful at this point, in order to summarize the above argument, to fully restate the recipe that we are going to use in this section to analyze the M-theory dynamics on the (A_{N-1}, D_k) singularities:

1. Choose a (A_{N-1}, D_k) theory and compute its brane locus using equation (6.6.6).

2. Factorize the brane locus into factors corresponding to irreducible blocks, listed in table 6.5.

3. Build the Higgs field Φ corresponding to the theory (A_{N-1}, D_k) direct-summing the irreducible blocks found at the previous point. The result is a Higgs Φ in the shape (6.6.9) made up only of irreducible blocks.

4. Compute the stabilizer of such Higgs field, obtaining the flavor/gauge symmetries.

5. Compute the matter modes localized near the branes intersections, as well as their charges under the flavour/gauge group.

Notice that the main difference with respect to the (A, A) series lies at the second point of the recipe: it is crucial to decompose the brane locus of the (A_{N-1}, D_k) singularities into irreducible blocks, as intended in table 6.5.

In the generic case, valid for *all* the (A_{N-1}, D_k) singularities, the Higgs field is never made up of more than one copy of the exact same block: this means that, if more than one block of type $\mathcal{B}_{(i)}$ enters the Higgs (say e.g. $\mathcal{B}^{(1)}_{(i)}$ and $\mathcal{B}^{(2)}_{(i)}$), then they either have different sizes or they possess different constant coefficients[13].

Let's now comment on the explicit matrix realization of the generators of the five-dimensional flavor and discrete symmetries. If we perform the computation in $SO(2n)$, the preserved flavor (continuous) and gauge (discrete) group is nothing but the direct product of the centers of the $SO(2n)$-subgroups $\mathfrak{S}_{\text{min}}$ whose Lie algebras are the minimal subalgebras $\mathfrak{s}_{\text{min}}$ in which the blocks reside. In other words, for a Higgs as in (6.6.9), given the subgroup $\mathfrak{S}_{\text{min},j}$ corresponding to the minimal subalgebra $\mathfrak{s}_{\text{min},j}$ in which the block $\mathcal{B}^{(j)}$ lives (we momentarily suppress the lower index labelling the type of block, which can be any), we have:

$$\text{Stab}_{SO}(\Phi) = Z(\mathfrak{S}_{\text{min},1}) \times ... \times Z(\mathfrak{S}_{\text{min},m}), \tag{6.6.12}$$

with m the number of blocks appearing in the Higgs field Φ and we stress that the stabilizers $\text{Stab}_{SO}(\Phi)$ are computed in $SO(2n)$. As a result, we can easily rewrite Table 6.5 explicitly stating the center of the subgroup $\mathfrak{S}_{\text{min}}$ corresponding to each block, yielding:

$\mathcal{B}$	$\chi(\boldsymbol{\mathcal{B}})$	$\mathfrak{s}_{\text{min}}$	$\boldsymbol{Z}(\boldsymbol{\mathfrak{S}}_{\text{min}})$
(a)	$(\xi^{2r+1} + c_1 w^t)\,(\xi^{2r+1} - c_1 w^t)$	$\mathfrak{u}(2r+1)$	$U(1)$
(b)	$\xi^2\,(\xi^{2r} + c_2 w^{2t+1})$	$\mathfrak{so}(2r+2)$	$\mathbb{Z}_2$
(c)	$\xi^{4r} + c_3 w^{2t+1}\xi^{2n} + c_4 w^{2(2t+1)}$ $(r, 2t+1)$ coprime	$\mathfrak{so}(4r)$	$\mathbb{Z}_2$

Table 6.6: *Irreducible blocks and stabilizer groups.*

Letting $2u$ be the size of the matrix representation of the minimal subalgebra $\mathfrak{s}_{\text{min}}$ in which a generic block lives, according to table 6.5, then the explicit realizations of the generators of the centers $Z(\mathfrak{S}_{\text{min}})$ are:

$$U(1) = \begin{pmatrix} e^{i\alpha}\mathbb{1}_u & 0 \\ 0 & e^{-i\alpha}\mathbb{1}_u \end{pmatrix}, \qquad \mathbb{Z}_2 = (\pm\mathbb{1}_{2u})\,. \tag{6.6.13}$$

Remarkably, the above table furnishes a powerful tool to analyze the resolutions of (A_{N-1}, D_k) singularities: if the irreducible polynomial factorization (6.6.9) of a given singularity contains at least one block of type (a), then it admits a small resolution

[13] We noticed that, if one chooses two exactly equal blocks in the Higgs field, one finds a non-isolated singularity in the threefold.

inflating a 2-cycle. The number of blocks of type (a) predicts the maximum number of 2-cycles that can be resolved.

This explicit matrix realization of the symmetry group associated to the Higgs field Φ is compatible with the general procedure explained in Section 4.2. Indeed, in Section 4.2 we took as the seven-dimensional gauge group $G = SO(2n)/\mathbb{Z}_2$ rather than $SO(2n)$. In this sense, the stabilizers (6.6.13) can be used to build representatives of the $\mathbb{Z}_2$-equivalence classes of the stabilizers in G. The G-stabilizers $\mathrm{Stab}(\Phi)$ are linked to the $SO(2n)$ ones of (6.6.12) by

$$\mathrm{Stab}(\Phi) = \mathrm{Stab}_{SO}(\Phi)/\mathbb{Z}_{2,\text{center}}, \tag{6.6.14}$$

with $Z_{2,\text{center}}$ the center of $SO(2n)$.

Furthermore, with our method we can give a useful criterion to predict how many uncharged hypers (both under the flavor and gauge groups) are localized at the intersection of the D6 branes, just by taking a look at the irreducible block-decomposition of the brane locus. We have previously seen that the flavor and gauge groups in the 5d theory are determined by the decomposition of the brane locus into irreducible blocks $\mathcal{B}_{(i)}$ of the classification in Table 6.6. The discrete gauge groups can be explicitly realized as the diagonal matrices (6.6.13), that act non-trivially only on modes localized in the off-diagonal blocks. Analogously, only the modes in the off-diagonal blocks can be charged under the $U(1)$ flavour groups in (6.6.13), except for the charge 2 modes, that are always localized inside the blocks of type (a).

As a result the uncharged localized hypers w.r.t. the discrete gauging can be found only inside the blocks $\mathcal{B}_{(i)}$. In pictures, this means that we can have uncharged hypers only inside the minimal subalgebras $\mathfrak{s}_{\text{min}}$ from Table 6.6:

$$\left(\begin{array}{c|c|c} \mathcal{B}_{(i)} & & \\ \hline & \mathcal{B}_{(j)} & \\ \hline & & \mathcal{B}_{(l)} \end{array}\right) \longrightarrow \left(\begin{array}{c|c|c} \textit{uncharged} & \textit{charged} & \textit{charged} \\ \hline \textit{charged} & \textit{uncharged} & \textit{charged} \\ \hline \textit{charged} & \textit{charged} & \textit{uncharged} \end{array}\right). \tag{6.6.15}$$

We can explicitly summarize the number of uncharged hypers under the discrete groups appearing in each block $\mathcal{B}_{(i)}$ using their dependence on the parameters r and t in Table 6.6:

$\mathcal{B}$	**Uncharged hypers**	
	$t=0$	$t\geq 1$
(a) $= (\xi^{2r+1}+c_1w^t)\,(\xi^{2r+1}-c_1w^t)$	$\frac{t(t-1)}{2}$	
(b) $= \xi^2\,(\xi^{2r}+c_4w^{2t+1})$	$2t(r+1)$	
(c) $= \xi^{4r}+c_2w^{2t+1}\xi^{2n}+c_3w^{2(2t+1)}$	$r-1$	$4t^2+2r-1$

Table 6.7: *Irreducible blocks and uncharged hypers under discrete symmetries.*

Let us summarize what we have shown so far: just by looking at the brane locus factorization[14] *of a* (A_{N-1}, D_k) *singularity we are able to predict the allowed resolution, the flavor and gauge groups and the number of uncharged hypers w.r.t the discrete groups in the 5d theory. In addition, by performing easy mechanical computations, we can compute all the 5d modes and their respective charges under the flavor and gauge groups, completely characterizing the Higgs Branch.*

We have done this explicitly for all the (A_{N-1}, D_k) singularities for $k = 1, \dots 8$ and $n = 4, \dots 15$, reporting the results for the dimension of the Higgs Branches of the 5d theories, the continuous and discrete symmetries, as well as the charges of the localized modes, in Appendix C.

In this regard, it is interesting to notice that there is a connection between the discrete symmetries enjoyed by the 5d theory and the one-form symmetry of the Argyres-Douglas theories arising from the geometric engineering of Type IIB theory on the (A_{N-1}, D_k) singularities, as computed by [25]. More specifically we found that, given a starting 7d gauge group $SO(2n)/\mathbb{Z}_2$ (i.e. $SO(2n)$ quotiented by its center), the discrete symmetries in 5d are exactly equal to the one-form symmetries of 4d Argyres-Douglas theories. Until now, to be able to get explicit matrix realizations of the discrete symmetries, we performed all the computations in $SO(2n)$. Consequently, given a (A_{N-1}, D_k) singularity, the true discrete gauging group G^0_{gauge} (according to the general prescription of Section 4.2) is:

$$G^0_{\text{gauge}} = \frac{G_{\text{gauge},SO}}{\mathbb{Z}_{2,\text{center}}}, \tag{6.6.16}$$

with $G_{\text{gauge},SO}$ the discrete stabilizers of the Higgs field that we compute taking as seven-dimensional gauge group $SO(2n)$ rather than $SO(2n)/\mathbb{Z}_{2,\text{center}}$. We find that

[14]Indeed, the brane locus factorization dictates, in the language of Section 4.2, the algebra $\mathcal{M} \ni \Phi$. As we have seen, this is enough to understand the structure of the Higgs field stabilizers, namely of the 5d flavor and discrete groups.

the groups G^0_{gauge} are:

G^0_{gauge}	D_4	D_5	D_6	D_7	D_8	D_9	D_{10}	D_{11}	D_{12}	D_{13}	D_{14}	D_{15}
A_1	0	0	0	0	0	0	0	0	0	0	0	0
A_2	$\mathbb{Z}_2$	0	0	$\mathbb{Z}_2$	0	0	$\mathbb{Z}_2$	0	0	$\mathbb{Z}_2$	0	0
A_3	0	$\mathbb{Z}_2$	0	0	0	$\mathbb{Z}_2$	0	0	0	$\mathbb{Z}_2$	0	0
A_4	0	0	$\mathbb{Z}_2^2$	0	0	0	0	$\mathbb{Z}_2^2$	0	0	0	0
A_5	0	0	0	$\mathbb{Z}_2^2$	0	0	0	0	0	$\mathbb{Z}_2^2$	0	0
A_6	0	0	0	0	$\mathbb{Z}_2^3$	0	0	0	0	0	0	$\mathbb{Z}_2^3$
A_7	0	0	0	0	0	$\mathbb{Z}_2^3$	0	0	0	0	0	0
A_8	$\mathbb{Z}_2$	0	0	$\mathbb{Z}_2$	0	0	$\mathbb{Z}_2^4$	0	0	$\mathbb{Z}_2$	0	0

Table 6.8: *Discrete gauge groups of* (A_{N-1}, D_k) *theories.*

Notice that Table 6.8 is manifestly identical to the table presented in [25].

This confirms the expectation of [24, 25] that 1-form symmetries of Type IIB reduced on (A_{N-1}, D_k) singularities are linked to 0-form discrete symmetries of the 5d theories engineered with M-theory on the same singularities.

In the next subsections we concretely apply the machinery we have set up to study M-theory on the (A_{N-1}, D_k) singularities, exhibiting an explicit example admitting no resolution and an infinite family displaying one resolved 2-cycle.

Example 1: no resolution

In this subsection we tackle the (A_2, D_4) singularity, the simplest non-trivial singularity of type (A_{N-1}, D_k) that admits no resolution, fully characterizing its Higgs branch. Its defining equation is:

$$x^2 + zy^2 + z^3 + w^3 = 0, \qquad (x, y, w, z) \in \mathbb{C}^4. \tag{6.6.17}$$

To complete this task, we follow the recipe outlined in the preceding section: the starting point is the brane locus, that can be computed employing equation (6.6.6). It is immediate to see that the result is:

$$\Delta_{(A_2,D_4)} = \xi^2(\xi^6 + w^3). \tag{6.6.18}$$

We can now completely factorize it into the irreducible blocks in table 6.5, obtaining:

$$\Delta_{(A_2,D_4)} = \underbrace{\xi^2\left(\xi^2 + w\right)}_{\text{type (b)}}\underbrace{\left(\xi^4 + w^2 - \xi^2 w\right)}_{\text{type (c)}}, \tag{6.6.19}$$

where we have highlighted the specific type of blocks. The fact that there are no blocks of type (a), corresponding to $U(1)$ flavor groups, indicates that the singularity is non-resolvable. Direct summing the irreducible blocks we obtain the explicit Higgs field reproducing the D6 brane configuration of the (A_2, D_4) theory, where each diagonal block is in the basis (A.1.9):

$$\Phi = \left(\begin{array}{cccc|cccc} 0 & 1 & 0 & \frac{w}{4} & 0 & 0 & 0 & 0 \\ -\frac{w}{4} & 0 & -\frac{w}{4} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{w}{4} & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 0 & \frac{w}{4} \\ 0 & 0 & 0 & 0 & \frac{3w}{4} & 0 & -\frac{w}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -\frac{3w}{4} \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \end{array}\right). \tag{6.6.20}$$

We decompose Φ as:

$$\Phi = \left(\begin{array}{c|c} \mathcal{B}_{(b)} & 0 \\ \hline 0 & \mathcal{B}_{(c)} \end{array}\right) \qquad \text{with:} \begin{cases} \chi(\mathcal{B}_{(b)}) = \xi^2\,(\xi^2 + w) & \text{type (b)}, \\ \chi(\mathcal{B}_{(c)}) = (\xi^4 + w^2 - \xi^2 w) & \text{type (c)}, \end{cases}$$

where we have explicitly highlighted the block decomposition. It is also easy to verify that the relationship (6.6.6) between the Higgs field and the algebraic definition of the (A_2, D_4) precisely holds.

The stabilizer group of (6.6.20), corresponding to the flavor (for the continuous part) and gauge (for the discrete part) symmetry of the 5d SCFT can be promptly read off Table 6.6, noticing that each block contributes a $\mathbb{Z}_2$ factor. Using (6.6.16) we get

$$G^0_{\text{gauge}} = \frac{\mathbb{Z}_2 \times \mathbb{Z}_2}{\mathbb{Z}_{2,\text{center}}} \cong \mathbb{Z}_2 \tag{6.6.21}$$

and a trivial flavour group G_{flavour}. We have only one factor of $\mathbb{Z}_2$ acting on the modes, as opposed to the full SO-stabilizers of Φ, which is $\mathbb{Z}_2 \times \mathbb{Z}_2$, as the diagonal combination belongs to the center of $SO(8)$.

Studying fluctuations φ around the Higgs background Φ, subject to the equivalence $\varphi \sim \varphi + [\Phi, g]$, with g a generic matrix of parameters in D_4, we can identify the content of the Higgs branch, obtaining:

$$\varphi = \left(\begin{array}{c|c} \emptyset & 4 \text{ modes} \\ \hline 4 \text{ modes} & \emptyset \end{array}\right). \tag{6.6.22}$$

All in all, we get a total of 4 hypers, as expected from previous results in the literature [96, 161].

Summarizing, we find that the Higgs branch of the (A_2, D_4) theory coincides with existing results [25]:

$$\mathrm{HB}_{(A_2,D_4)} = \frac{\mathbb{C}^8}{\mathbb{Z}_2}, \tag{6.6.23}$$

with $\mathbb{Z}_2$ acting reflecting all the coordinates of $\mathbb{C}^8$.

We notice here that this is exactly the same action that one gets using the general procedure in Section 4.2, with the $\mathbb{Z}_2$ generator identified with the discrete stabilizer (4.2.8) that we get extending the D_4 Dynkin diagram and removing the central node.

Example 2: one resolved 2-cycle

In this subsection we get to a family of examples admitting a small resolution of a single 2-cycle. This is signalled by the appearance of a $U(1)$ symmetry in the stabilizer of the Higgs background. The family is formed by the singularities (A_{2k-1}, D_{2kn+1}):

$$x^2 + zy^2 + z^{2kn} + w^{2k} = 0, \qquad (x, y, w, z) \in \mathbb{C}^4. \tag{6.6.24}$$

Such family was pinpointed and studied by Closset et al., and employing our techniques we show how to fully characterize their Higgs branch.

The D6s brane locus corresponding to the geometry (6.6.24) can be computed as:

$$\Delta(\xi^2, w) = \xi^2(\xi^{4kn} + w^{2k}). \tag{6.6.25}$$

Fully decomposing the brane locus into factors of the allowed form, presented in Table 6.6, we get:

$$\Delta(\xi^2, w) = \underbrace{\xi^2}_{\text{type (a)}} \prod_{s=0}^{k-1} \underbrace{(\xi^{4n} + e^{2\pi i s/k} w^2)}_{\text{type (c)}}. \tag{6.6.26}$$

The fact that we obtain a block of type (a), that preserves a $U(1)$ flavor symmetry, is the telltale sign that a small resolution of a single 2-cycle is allowed by this Higgs configuration.

Direct-summing the blocks corresponding to each of the factors in (6.6.26) we obtain the full Higgs field Φ, expressed in an appropriate basis of the Lie algebra

D_{2kn+1}, where each block factor is in the basis (A.1.9):

$$\Phi = \left(\begin{array}{c|ccc} \mathcal{B}_{(a)} & & & \\ \hline & \mathcal{B}^{(1)}_{(c)} & & \\ & & \ddots & \\ & & & \mathcal{B}^{(k)}_{(c)} \end{array}\right), \qquad \text{with:} \begin{cases} \chi(\mathcal{B}_{(a)}) = \xi^2, \\ \chi(\mathcal{B}^{(1)}_{(c)}) = \xi^{4n} + w^2, \\ \qquad \vdots \\ \chi(\mathcal{B}^{(k)}_{(c)}) = \xi^{4n} + e^{2\pi i(k-1)/k} w^2. \end{cases} \tag{6.6.27}$$

In this way, we can trivially check that the determinant of the full Higgs correctly reproduces the brane locus (6.6.25) of the (A_{2k-1}, D_{2kn+1}) singularities.

Using the techniques outlined in Chapter 4, namely studying fluctuations of Φ, we can easily compute the hypermultiplet content of the 5d theory and their charges under the flavor and gauge symmetries.

As regards the matter modes, they can be nicely displayed in a block form, following the structure of the Higgs field Φ (6.6.27):

$$\Phi = \left(\begin{array}{c|ccc} \mathcal{B}_{(a)} & & & \\ \hline & \mathcal{B}^{(1)}_{(c)} & & \\ & & \ddots & \\ & & & \mathcal{B}^{(k)}_{(c)} \end{array}\right) \longrightarrow \text{modes} = \left(\begin{array}{c||c|c|c|c} \emptyset & 2 & \cdots & \cdots & 2 \\ \hline\hline 2 & 2n-2 & 4n & \cdots & 4n \\ \hline \vdots & 4n & \ddots & \ddots & \vdots \\ \hline \vdots & \vdots & \ddots & \ddots & 4n \\ \hline 2 & 4n & \cdots & 4n & 2n-2 \end{array}\right). \tag{6.6.28}$$

The $SO(4kn+2)$-stabilizer group G_{stab} of Φ reads, according to table (6.6):

$$G_{\text{stab}} = U(1) \times \underbrace{\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2}_{k} = \left(\begin{array}{c|ccc} U(1) & & & \\ \hline & {\mathbb{Z}_2}^{(1)} & & \\ & & \ddots & \\ & & & {\mathbb{Z}_2}^{(k)} \end{array}\right), \tag{6.6.29}$$

obtaining a $U(1)$ flavour group coming from the resolved 2-cycle, as well as k discrete $\mathbb{Z}_2$ gauge groups. Taking a look at the block structure of (6.6.28), it is immediate to see that different colors correspond to different charges under the flavor $U(1)$ and the gauge $\mathbb{Z}_2^{(i)}$ groups:

$$\begin{cases} \textit{red}: & \text{charge } \pm 1 \text{ under } U(1), \text{ and one of the } \mathbb{Z}_2^{(i)} \\ \textit{green}: & \text{charged under two } \mathbb{Z}_2^{(i)} \text{ factors} \\ \textit{black}: & \text{uncharged} \end{cases} \tag{6.6.30}$$

We remark that we will use this color notation also in the systematic tables of Appendix C.

As regards the discrete charges, it is of course possible to precisely track the charge of each mode under every $\mathbb{Z}_2^{(i)}$ group, just by taking a look at the block structure (6.6.28).

Summing up, for the infinite family of singularities (A_{2k-1}, D_{2kn+1}) we find a total of $2nk^2 + k - nk$ hypers, with charges:

- $2k$ hypers charged under $U(1)$
- $2nk(k-1)$ hypers charged only under some $\mathbb{Z}_2$
- $k(n-1)$ uncharged hypers

As result, the general formula for the Higgs Branch is:

$$\text{HB} = \mathbb{C}^{2k(n-1)} \times \mathbb{C}^{4k} \times \frac{\mathbb{C}^{4nk(k-1)}}{\mathbb{Z}_2^{k-1}}. \tag{6.6.31}$$

Notice that, as it happened in the example of Section 6.6.1, one combination of the $\mathbb{Z}_2^{(i)}$ is always decoupled, leaving the effective flavour/gauge group (the one we compute in $G = SO(2n)/\mathbb{Z}_2$) as:

$$G_{\text{flavour}} = U(1), \qquad G_{\text{gauge}} = \underbrace{\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2}_{k-1}. \tag{6.6.32}$$

We stress that our method gives a complete understanding of the structure of the Higgs Branch. In other words, we can completely reconstruct, from (6.6.28), the action of the discrete group G_{gauge} giving, e.g., the Hilbert series (HS) of the Higgs Branch. For example, choosing $k = 3$, $n = 1$, we get, using [162] the Molien formula:

$$\text{HS} = \frac{N(t)}{D(t)}, \tag{6.6.33}$$

with

$$
\begin{aligned}
N(t) &= t^{20} - 10t^{19} + 190t^{18} - 570t^{17} + 4845t^{16} - 7752t^{15} + 38760t^{14} - 38760t^{13} + 125970t^{12} + \\
&\quad -83980t^{11} + 184756t^{10} - 83980t^{9} + 125970t^{8} - 38760t^{7} + \\
&\quad +38760t^{6} - 7752t^{5} + 4845t^{4} - 570t^{3} + 190t^{2} - 10t + 1, \\
D(t) &= (t-1)^{36}(t+1)^{20}.
\end{aligned}
\tag{6.6.34}
$$

6.7 Quasi-homogeneous cDV singularities of E_6, E_7, E_8 type

In this section, we focus on the deformed E_6, E_7, E_8 cases, looking for the minimal subalgebras containing the Higgs backgrounds reproducing a given quasi-homogeneous cDV singularity of E_6, E_7, E_8 type.

As they are useful to identify the flavor charges, we report the dual Coxeter labels for the E_6, E_7, E_8 Dynkin diagrams in Figure 6.3.

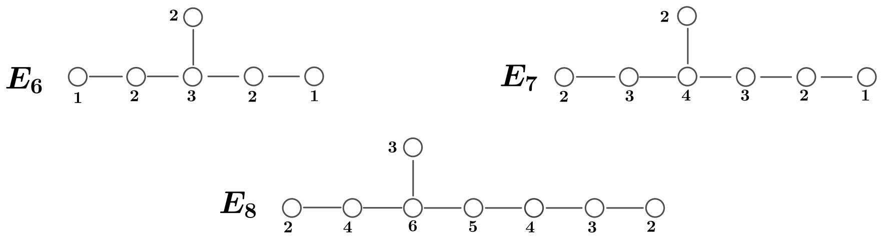


Figure 6.3: *Dual Coxeter labels for the E series.*

To illustrate how we get our results, we explicitly go through the (A_{N-1}, E_6) and the $E_7^{(14)}[N]$ cases. We sum up the results for all the cases in Tables 6.9, 6.10, 6.11, 6.12, 6.13, 6.14.

(A, E_6) singularities

Let us start by showing how this works in the (A_{N-1}, E_6) class, employing the techniques of Section 6. The (A_{N-1}, E_6) threefolds are expressed as:

$$
\underbrace{x^2 + y^3 + z^4}_{E_6 \text{ sing}} + \underbrace{w^N}_{\text{def}} = 0. \tag{6.7.1}
$$

Notice that the only non-vanishing deformation parameter is:

$$
\mu_{12}(w) = w^N. \tag{6.7.2}
$$

The other (vanishing) deformation parameters are $\mu_2, \mu_5, \mu_6, \mu_8, \mu_9$. Eq. (6.7.2) tells us that $M = 12$, according to the notation of this chapter. There are six divisors of 12:

$$q_\alpha \in \{1, 2, 3, 4, 6, 12\}. \tag{6.7.3}$$

Now, we must look for the minimal degrees that the candidate partial Casimirs can acquire, thus forecasting the minimal subalgebra in which Φ can be contained. As $M = 12$, the minimal subalgebras will recur with periodicity 12, namely the minimal subalgebra corresponding to the Higgs describing the (A_k, E_6) singularity coincides with the one of (A_{k+12}, E_6). Let us proceed case by case:

For $N = 5, 7, 10, 11$ **mod** 12 ($q_\alpha = 12$), the minimal degree is $j = 12$. This means that $\mathcal{M} = E_6$, with all Casimirs equal to zero, except the maximal degree one. This implies that no resolution is possible.

For $N = 2$ **mod** 12 ($q_\alpha = 6$), the candidate minimal degree is $j = 6$. This tells us that the only c's that can be non-vanishing are c_6^I and c_{12}^I, according to the notation of this chapter. To solve the system (6.2.6) where only μ_6, μ_{12} appear, we need at least two Casimirs of degree 6, but this is not possible because of the rank of E_6[15]. This implies that no resolution is possible and that the correct minimal subalgebra is $\mathcal{M} = E_6$ with all Casimirs equal to zero, except the maximal degree one.

For $N = 3, 9$ **mod** 12 ($q_\alpha = 4$), the minimal degree for the non-vanishing partial Casimirs is $j = 4$. To solve system (6.2.6), we have to set two parameters (μ_8 and μ_{12}), and thus we need at least two partial Casimirs of degree 4. They are provided by $\mathcal{M} = D_4$. This implies that the two external nodes of the E_6 Dynkin diagram get inflated, as can be seen in Figure 6.4.

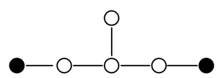


Figure 6.4: *D_4 subalgebra in the $N = 3, 9$ case.*

This yields 5d hypers with charge 1 under the flavor groups corresponding to the resolved nodes, as they have dual Coxeter label equal to 1, as well as uncharged hypers.

For $N = 4, 8$ **mod** 12 ($q_\alpha = 3$), the minimal degree for the non-zero partial Casimirs is $j = 3$. System (6.2.6) tells us that we need at least three Casimirs

[15]For example, one could have two degree 6 Casimirs using $\mathcal{M} = A_5 \oplus A_5$, or $\mathcal{M} = D_6$, but these cannot be embedded into E_6.

of degree 3 to extract a solution and fix the deformation parameters μ_6, μ_9, μ_{12}. Indeed, the subalgebra $\mathcal{M} = A_2 \oplus A_2 \oplus A_2$ gives us the correct partial Casimirs. This choice produces no simultaneous resolution of the deformed family. Furthermore, the fact that $\Phi \in \mathcal{M} = A_2 \oplus A_2 \oplus A_2$ signals that in this case we have a non-trivial $\mathrm{Stab}_{\mathcal{G}}(\Phi) = \mathbb{Z}_3$, that reflects in a discrete-gauging of the hypermultiplets of the five-dimensional SCFT. The actual discrete group $\mathbb{Z}_3$ comes because the maximal subalgebra $A_2^{\oplus 3}$ of E_6 is obtained by removing the trivalent node from the extended Dynkin diagram of E_6, that has dual Coxeter number equal to 3 (see Section 4.2), as depicted in Figure 6.5.

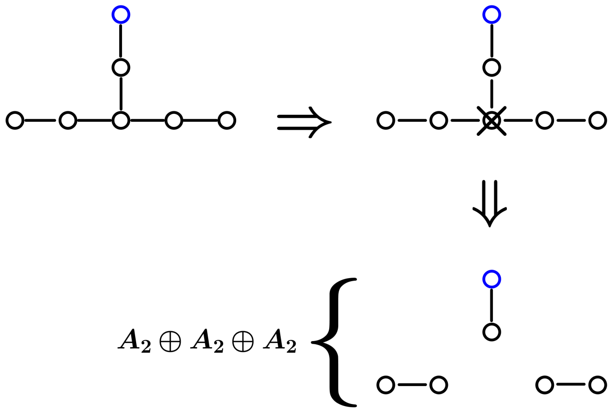


Figure 6.5: *$A_2^{\oplus 3}$ subalgebra in the $N = 4, 8$ case.*

For $N = 6$ **mod** 12 ($q_\alpha = 2$), the minimal degree for the non-zero partial Casimirs is $j = 2$. According to the system (6.2.6), we have to set the μ_2, μ_6, μ_8 parameters to zero, as well as $\mu_{12} = w^6$. This requires four partial Casimirs of minimal degree 2. It turns out that there exists a unique subalgebra of E_6 doing the work, i.e. $A_1 \oplus A_1 \oplus A_1 \oplus A_1$. We then have $\mathcal{M} = A_1^{\oplus 4} \oplus \mathcal{H}$, with $\mathcal{H}$ generated by the two external nodes in the Dynkin diagram of E_6. The Higgs field takes values in the semi-simple part of $\mathcal{M}$. This choice yields the resolution of the two external nodes with Coxeter label 1 of the E_6 Dynkin diagram, and produces a $\mathbb{Z}_2$ discrete group in 5d (since $\mathcal{L} = D_4$ and $A_1^{\oplus 4}$ is its maximal subalgebra, see Section 4.2), as depicted in Figure 6.6.

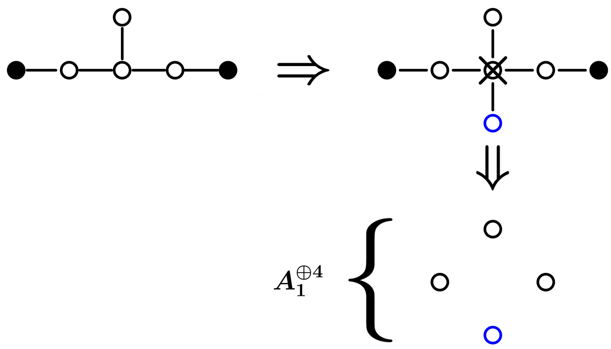


Figure 6.6: $A_1^{\oplus 4}$ *subalgebra in the* $N = 6$ *case.*

For $N = 12$ **mod** 12 ($q_\alpha = 1$), the minimal degree for the non-zero partial Casimirs is $j = 1$. Then $\mathcal{M}$ is the Cartan subalgebra of E_6. As a result, all the simple roots of E_6 are blown up in the simultaneous resolution. The flavor charges of the 5d hypermultiplets are given, in some basis, by the root system of the E_6 algebra[16].

$E_7^{(14)}[N]$ singularities

The $E_7^{(14)}[N]$ singularities are expressed as deformed Du Val E_7 singularities:

$$x^2 + y^3 + yz^3 + zw^N = 0\,. \tag{6.7.4}$$

Notice that the only non-zero deformation parameter is

$$\mu_{14}(w) = w^N. \tag{6.7.5}$$

The other (vanishing) deformation parameters are $\mu_2, \mu_6, \mu_8, \mu_{10}, \mu_{12}, \mu_{18}$. From (6.7.5), we read $M = 14$. Its divisors are:

$$q_\alpha \in \{1, 2, 7, 14\}. \tag{6.7.6}$$

With this in hand, we can start looking for the minimal degrees of candidate partial Casimirs, pinpointing the minimal subalgebra of E_7 containing Φ for a given $E_7^{(14)}[N]$. As in the previous section, we expect that the subalgebra corresponding to $E_7^{(14)}[N]$ is equal to the one of $E_7^{(14)}[N + 14]$, given the degree 14 deformation parameter that is switched on.

For $N = 1, 3, 5, 9, 11, 13$ **mod** 14 ($q_\alpha = 14$), the minimal degree is $j = 14$. Consequently, $\mathcal{M} = E_7$, with all Casimirs equal to zero except the maximal degree

[16] In general, for all the completely resolvable cases of Tables (6.9) and (6.10), the flavor charges are given by the roots of the corresponding algebra.

one. This entails that no resolution is possible.

For $N = 2, 4, 6, 8, 10, 12$ **mod** 14 ($q_\alpha = 7$), the minimal degree for the partial Casimirs is 7. In order to solve system (6.2.6), namely to fix μ_{14}, we need only one partial Casimir. A degree 7 partial Casimir can be provided choosing $\mathcal{M} = A_6 \oplus \langle \alpha_7^* \rangle$, which naturally lies inside E_7. This implies that a single node of E_7, with Coxeter label 2, gets inflated by the allowed resolution (see Figure 6.7). This yields 5d hypers with charge 1 and 2, as well as uncharged hypers. The Higgs field Φ lives only in the semi-simple part of $\mathcal{M}$. See Figure 6.7.

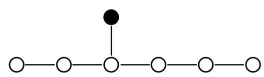


Figure 6.7: *A_6 subalgebra in the $N = 2, 4, 6, 8, 10, 12$ case.*

For $N = 7$ **mod** 14 ($q_\alpha = 2$), the minimal degree for the partial Casimirs is 2. According to (6.2.6), we need seven distinct such Casimirs. It can be shown that indeed there exists a choice $\mathcal{M} = A_1^{\oplus 7} \in E_7$, that yields seven partial Casimirs of degree 2. This maximal subalgebra can be found noticing the chain of maximal subalgebras $E_7 \supset A_1 \oplus D_6 \supset A_1 \oplus A_1^{\oplus 2} \oplus D_4 \supset A_1^7$, that is depicted in Figure 6.8.

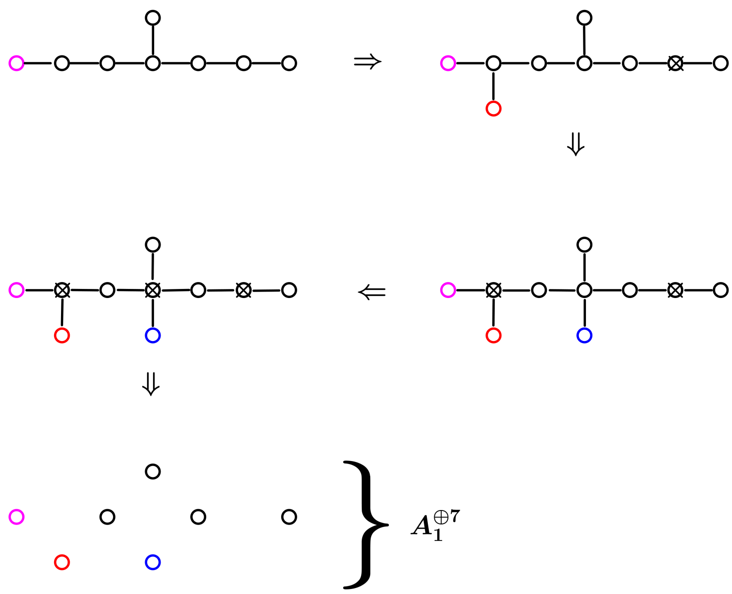


Figure 6.8: *Maximal subalgebra in the $N = 7$ case.*

The three steps in obtaining the maximal subalgebra $A_1^{\oplus 7}$ of E_7, where nodes with Coxeter number equal to two are removed, tells us that we have the non-trivial discrete $\mathrm{Stab}_{\mathcal{G}}(\Phi) = \mathbb{Z}_2^3$.

For $N = 14$ **mod** 14 ($q_\alpha = 1$), the minimal degree for the non-vanishing partial Casimirs is $j = 1$. We need at least seven partial Casimirs to fix all the deformation parameters, and hence we can pick as partial Casimirs the Casimirs of the Cartan subalgebra of E_7. In this way, we see that all the simple roots of E_7 are blown-up in the simultaneous resolution. The flavor charges of the 5d hypers can be written as the root system of E_7.

Other quasi-homogeneous cDV singularities of type E

Proceeding along the same path as the previous sections, we can readily find the minimal subalgebras containing the appropriate Higgs background Φ for each class of quasi-homogeneous cDV singularities arising from deformed E_6, E_7, E_8 singularities.

We sum up our results in Table 6.9, 6.10, 6.11, 6.12, 6.13 and 6.14. In particular, we list:

- In the first column, the cDV singularity.
- In the second column, the maximal allowed simultaneous resolution (resolved nodes are in black). This fixes the Levi subalgebra.
- In the third column, the minimal subalgebra $\mathcal{M} \subseteq \mathcal{L}$ containing Φ. If it is non-trivial, this yields a discrete group in 5d.
- In the fourth column, the symmetry group preserved by Φ. In general, it comprises both a continuous and a discrete factor.
- In the fifth column, the number of five-dimensional hypers localized in 5d, and their charges under the continuous and discrete symmetries. We also report the total number of hypers, to be compared with the number of normalizable complex structure deformations of the corresponding cDV singularity.

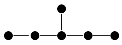
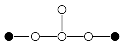
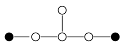
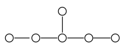
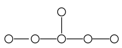
Singularity	Resolution pattern	$\mathcal{M}$	Symmetry group	Hypers
Deformed E_6				
(A_{N-1}, E_6)	$N = 12n$: 	$\mathfrak{t}$	$U(1)^6$	Charges: **root system** of E_6 **TOT: $3N$**
	$N = 6n$ $n \neq 2j$: 	$A_1^{\oplus 4} \oplus \langle \alpha_1^*, \alpha_5^* \rangle$	$U(1)_a \times U(1)_b \times \mathbb{Z}_2$	$(q_a, q_b, q_{\mathbb{Z}_2}) = (1,0,1) : 2n$ $(q_a, q_b, q_{\mathbb{Z}_2}) = (1,0,0) : 2n$ $(q_a, q_b, q_{\mathbb{Z}_2}) = (0,1,1) : 2n$ $(q_a, q_b, q_{\mathbb{Z}_2}) = (0,1,0) : 2n$ $(q_a, q_b, q_{\mathbb{Z}_2}) = (1,1,1) : 2n$ $(q_a, q_b, q_{\mathbb{Z}_2}) = (1,1,0) : 2n$ $(q_a, q_b, q_{\mathbb{Z}_2}) = (0,0,1) : 6n-2$ **TOT: $3N - 2$**
	$N = 3n$ $n \neq 2j$: 	$D_4 \oplus \langle \alpha_1^*, \alpha_5^* \rangle$	$U(1)_a \times U(1)_b$	$(q_a, q_b) = (1,0) : 2n$ $(q_a, q_b) = (0,1) : 2n$ $(q_a, q_b) = (1,1) : 2n$ $(q_a, q_b) = (0,0) : 3n-2$ **TOT: $3N - 2$**
	$N = 4n$ $n \neq 3j$: 	$A_2^{\oplus 3}$	$\mathbb{Z}_3$	$q_{\mathbb{Z}_3} = 1 : 9n$ $q_{\mathbb{Z}_3} = 0 : 3(n-1)$ **TOT: $3(N-1)$**
	$N \neq 3n, 4n$: 	E_6	$\emptyset$	**TOT: $3(N-1)$**

Table 6.9: *Higgs branch data for quasi-homogeneous cDV singularities of (A_{N-1}, E_6) type.*

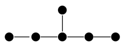
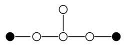
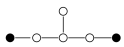
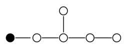
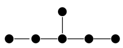
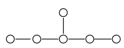
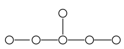
Singularity	Resolution pattern	$\mathcal{M}$	Symmetry group	Hypers
$E_6^{(8)}[N]$	$N = 8n$: 	$\mathfrak{t}$	$U(1)^6$	Charges: **root system** of E_6 **TOT:** $\boldsymbol{\frac{9N}{2}}$
	$N = 4n$ $n \neq 2j$: 	$A_1^{\oplus 4} \oplus \langle \alpha_1^*, \alpha_5^* \rangle$	$U(1)_a \times U(1)_b \times \mathbb{Z}_2$	$(q_a, q_b, q_{\mathbb{Z}_2}) = (1,0,1) : 2n$ $(q_a, q_b, q_{\mathbb{Z}_2}) = (1,0,0) : 2n$ $(q_a, q_b, q_{\mathbb{Z}_2}) = (0,1,1) : 2n$ $(q_a, q_b, q_{\mathbb{Z}_2}) = (0,1,0) : 2n$ $(q_a, q_b, q_{\mathbb{Z}_2}) = (1,1,1) : 2n$ $(q_a, q_b, q_{\mathbb{Z}_2}) = (1,1,0) : 2n$ $(q_a, q_b, q_{\mathbb{Z}_2}) = (0,0,1) : 6n-2$ **TOT:** $\boldsymbol{\frac{9N}{2} - 2}$
	$N = 2n$ $n \neq 2j$: 	$D_4 \oplus \langle \alpha_1^*, \alpha_5^* \rangle$	$U(1)_a \times U(1)_b$	$(q_a, q_b) = (1,0) : 2n$ $(q_a, q_b) = (0,1) : 2n$ $(q_a, q_b) = (1,1) : 2n$ $(q_a, q_b) = (0,0) : 3n-2$ **TOT:** $\boldsymbol{\frac{9N}{2} - 2}$
	$N = 2n+1$: 	$D_5 \oplus \langle \alpha_1^* \rangle$	$U(1)$	$q = 1 : 4n+2$ $q = 0 : 5n$ **TOT:** $\boldsymbol{\frac{9N-5}{2}}$
$E_6^{(9)}[N]$	$N = 9n$: 	$\mathfrak{t}$	$U(1)^6$	Charges: **root system** of E_6 **TOT:** $\boldsymbol{4N}$
	$N = 3n$ $n \neq 3j$: 	$A_2^{\oplus 3}$	$\mathbb{Z}_3$	$q_{\mathbb{Z}_3} = 1 : 9n$ $q_{\mathbb{Z}_3} = 0 : 3(n-1)$ **TOT:** $\boldsymbol{4N - 3}$
	$N \neq 3n$: 	E_6	$\emptyset$	**TOT:** $\boldsymbol{4N - 3}$

Table 6.10: *Higgs branch data for quasi-homogeneous cDV singularities of $E_6^{(8)}[N]$ and $E_6^{(9)}[N]$ type.*

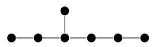
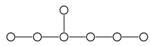
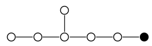
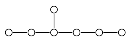
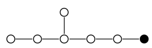
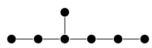
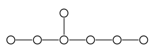
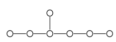
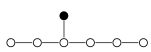
Singularity	Resolution pattern	$\mathcal{M}$	Symmetry group	Hypers
Deformed E_7				
(A_{N-1}, E_7)	$N = 18n$: 	$\mathfrak{t}$	$U(1)^7$	Charges: **root system** of E_7 **TOT:** $\frac{7N}{2}$
	$N = 9n$ $n \neq 2j$: 	$A_1^{\oplus 7}$	$\mathbb{Z}_2^3$	$(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}, q^{(c)}_{\mathbb{Z}_2}) = (0,0,0) : \frac{7}{2}(n-1)$ $(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}, q^{(c)}_{\mathbb{Z}_2}) = (1,0,0) : 4n$ $(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}, q^{(c)}_{\mathbb{Z}_2}) = (0,1,0) : 4n$ $(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}, q^{(c)}_{\mathbb{Z}_2}) = (0,0,1) : 4n$ $(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}, q^{(c)}_{\mathbb{Z}_2}) = (0,1,1) : 4n$ $(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}, q^{(c)}_{\mathbb{Z}_2}) = (1,1,0) : 4n$ $(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}, q^{(c)}_{\mathbb{Z}_2}) = (1,0,1) : 4n$ $(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}, q^{(c)}_{\mathbb{Z}_2}) = (1,1,1) : 4n$ **TOT:** $\frac{7(N-1)}{2}$
	$N = 6n$ $n \neq 3j$ 	$A_2^{\oplus 3} \oplus \langle \alpha_6^* \rangle$	$U(1) \times \mathbb{Z}_3$	$(q, q_{\mathbb{Z}_3}) = (1,0) : 3n$ $(q, q_{\mathbb{Z}_3}) = (1,1) : 3n$ $(q, q_{\mathbb{Z}_3}) = (1,2) : 3n$ $(q, q_{\mathbb{Z}_3}) = (0,0) : 3(n-1)$ $(q, q_{\mathbb{Z}_3}) = (0,1) : 9n$ **TOT:** $\frac{7N}{2} - 3$
	$N = 2n+1$ $2n \neq 9j-1$: 	E_7	$\emptyset$	**TOT:** $\frac{7(N-1)}{2}$
	$N = 2n$ $n \neq 3j$: 	$E_6 \oplus \langle \alpha_6^* \rangle$	$U(1)$	$q = 1 : 3n$ $q = 0 : 4n-3$ **TOT:** $\frac{7N}{2} - 3$
$E_7^{(14)}[N]$	$N = 14n$: 	$\mathfrak{t}$	$U(1)^7$	Charges: **root system** of E_7 **TOT:** $\frac{9N}{2}$
	$N = 7n$ $n \neq 2j$: 	$A_1^{\oplus 7}$	$\mathbb{Z}_2^3$	$(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}, q^{(c)}_{\mathbb{Z}_2}) = (0,0,0) : \frac{7}{2}(n-1)$ $(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}, q^{(c)}_{\mathbb{Z}_2}) = (1,0,0) : 4n$ $(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}, q^{(c)}_{\mathbb{Z}_2}) = (0,1,0) : 4n$ $(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}, q^{(c)}_{\mathbb{Z}_2}) = (0,0,1) : 4n$ $(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}, q^{(c)}_{\mathbb{Z}_2}) = (0,1,1) : 4n$ $(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}, q^{(c)}_{\mathbb{Z}_2}) = (1,1,0) : 4n$ $(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}, q^{(c)}_{\mathbb{Z}_2}) = (1,0,1) : 4n$ $(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}, q^{(c)}_{\mathbb{Z}_2}) = (1,1,1) : 4n$ **TOT:** $\frac{9N-7}{2}$
	$N = 2n+1$ $2n \neq 7j-1$: 	E_7	$\emptyset$	**TOT:** $\frac{9N-7}{2}$
	$N = 2n$ $n \neq 7j$: 	$A_6 \oplus \langle \alpha_7^* \rangle$	$U(1)$	$q = 2 : n$ $q = 1 : 5n$ $q = 0 : 3(n-1)$ **TOT:** $\frac{9N}{2} - 3$

Table 6.11: *Higgs branch data for quasi-homogeneous cDV singularities of (A_{N-1}, E_7) and $E_7^{(14)}[N]$ type.*

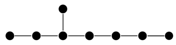
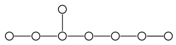
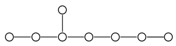
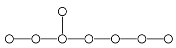
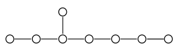
Singularity	Resolution pattern	$\mathcal{M}$	Symmetry group	Hypers
Deformed E_8				
	$N = 30n$: 	$\mathfrak{t}$	$U(1)^8$	Charges: **root system** of E_8 **TOT: $4N$**
(A_{N-1}, E_8)	$N = 6n$ $n \neq 5j$: 	$A_4 \oplus A_4$	$\mathbb{Z}_5$	$q_{\mathbb{Z}_5} = 2 : 10n$ $q_{\mathbb{Z}_5} = 1 : 10n$ $q_{\mathbb{Z}_5} = 0 : 4(n-1)$ **TOT: $4(N-1)$**
	$N = 10n$ $n \neq 3j$: 	$A_2^{\oplus 4}$	$\mathbb{Z}_3^2$	$(q^{(a)}_{\mathbb{Z}_3}, q^{(b)}_{\mathbb{Z}_3}) = (1,0) : 9n$ $(q^{(a)}_{\mathbb{Z}_3}, q^{(b)}_{\mathbb{Z}_3}) = (0,1) : 9n$ $(q^{(a)}_{\mathbb{Z}_3}, q^{(b)}_{\mathbb{Z}_3}) = (1,1) : 9n$ $(q^{(a)}_{\mathbb{Z}_3}, q^{(b)}_{\mathbb{Z}_3}) = (1,2) : 9n$ $(q^{(a)}_{\mathbb{Z}_3}, q^{(b)}_{\mathbb{Z}_3}) = (0,0) : 4(n-1)$ **TOT: $4(N-1)$**
	$N = 15n$ $n \neq 2j$: 	$A_1^{\oplus 8}$	$\mathbb{Z}_2^4$	$(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}, q^{(c)}_{\mathbb{Z}_2}, q^{(d)}_{\mathbb{Z}_2}) = (0,0,0,0) : 4(n-1)$ $(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}, q^{(c)}_{\mathbb{Z}_2}, q^{(d)}_{\mathbb{Z}_2}) = (1,0,0,0) : 4n$ $(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}, q^{(c)}_{\mathbb{Z}_2}, q^{(d)}_{\mathbb{Z}_2}) = (0,1,0,0) : 4n$ $(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}, q^{(c)}_{\mathbb{Z}_2}, q^{(d)}_{\mathbb{Z}_2}) = (0,0,1,0) : 4n$ $(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}, q^{(c)}_{\mathbb{Z}_2}, q^{(d)}_{\mathbb{Z}_2}) = (0,0,0,1) : 4n$ $(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}, q^{(c)}_{\mathbb{Z}_2}, q^{(d)}_{\mathbb{Z}_2}) = (1,1,0,0) : 4n$ $(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}, q^{(c)}_{\mathbb{Z}_2}, q^{(d)}_{\mathbb{Z}_2}) = (1,0,1,0) : 4n$ $(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}, q^{(c)}_{\mathbb{Z}_2}, q^{(d)}_{\mathbb{Z}_2}) = (1,0,0,1) : 4n$ $(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}, q^{(c)}_{\mathbb{Z}_2}, q^{(d)}_{\mathbb{Z}_2}) = (0,1,1,0) : 4n$ $(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}, q^{(c)}_{\mathbb{Z}_2}, q^{(d)}_{\mathbb{Z}_2}) = (0,1,0,1) : 4n$ $(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}, q^{(c)}_{\mathbb{Z}_2}, q^{(d)}_{\mathbb{Z}_2}) = (0,0,1,1) : 4n$ $(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}, q^{(c)}_{\mathbb{Z}_2}, q^{(d)}_{\mathbb{Z}_2}) = (1,1,1,0) : 4n$ $(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}, q^{(c)}_{\mathbb{Z}_2}, q^{(d)}_{\mathbb{Z}_2}) = (1,1,0,1) : 4n$ $(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}, q^{(c)}_{\mathbb{Z}_2}, q^{(d)}_{\mathbb{Z}_2}) = (1,0,1,1) : 4n$ $(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}, q^{(c)}_{\mathbb{Z}_2}, q^{(d)}_{\mathbb{Z}_2}) = (0,1,1,1) : 4n$ **TOT: $4(N-1)$**
	$N \neq 6n, 10n, 15n$: 	E_8	$\emptyset$	**TOT: $4(N-1)$**

Table 6.12: *Higgs branch data for quasi-homogeneous cDV singularities of (A_{N-1}, E_8) type.*

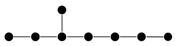
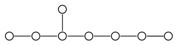
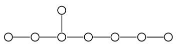
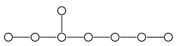
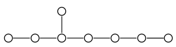
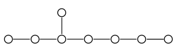
Singularity	Resolution pattern	$\mathcal{M}$	Symmetry group	Hypers
$E_8^{(24)}[N]$	$N = 24n$: 	$\mathfrak{t}$	$U(1)^8$	Charges: **root system** of E_8 **TOT: $5N$**
	$N = 12n$ $n \neq 2j$: 	$A_1^{\oplus 8}$	$\mathbb{Z}_2^4$	$(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}, q^{(c)}_{\mathbb{Z}_2}, q^{(d)}_{\mathbb{Z}_2}) = (0,0,0,0) : 4(n-1)$ $(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}, q^{(c)}_{\mathbb{Z}_2}, q^{(d)}_{\mathbb{Z}_2}) = (1,0,0,0) : 4n$ $(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}, q^{(c)}_{\mathbb{Z}_2}, q^{(d)}_{\mathbb{Z}_2}) = (0,1,0,0) : 4n$ $(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}, q^{(c)}_{\mathbb{Z}_2}, q^{(d)}_{\mathbb{Z}_2}) = (0,0,1,0) : 4n$ $(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}, q^{(c)}_{\mathbb{Z}_2}, q^{(d)}_{\mathbb{Z}_2}) = (0,0,0,1) : 4n$ $(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}, q^{(c)}_{\mathbb{Z}_2}, q^{(d)}_{\mathbb{Z}_2}) = (1,1,0,0) : 4n$ $(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}, q^{(c)}_{\mathbb{Z}_2}, q^{(d)}_{\mathbb{Z}_2}) = (1,0,1,0) : 4n$ $(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}, q^{(c)}_{\mathbb{Z}_2}, q^{(d)}_{\mathbb{Z}_2}) = (1,0,0,1) : 4n$ $(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}, q^{(c)}_{\mathbb{Z}_2}, q^{(d)}_{\mathbb{Z}_2}) = (0,1,1,0) : 4n$ $(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}, q^{(c)}_{\mathbb{Z}_2}, q^{(d)}_{\mathbb{Z}_2}) = (0,1,0,1) : 4n$ $(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}, q^{(c)}_{\mathbb{Z}_2}, q^{(d)}_{\mathbb{Z}_2}) = (0,0,1,1) : 4n$ $(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}, q^{(c)}_{\mathbb{Z}_2}, q^{(d)}_{\mathbb{Z}_2}) = (1,1,1,0) : 4n$ $(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}, q^{(c)}_{\mathbb{Z}_2}, q^{(d)}_{\mathbb{Z}_2}) = (1,1,0,1) : 4n$ $(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}, q^{(c)}_{\mathbb{Z}_2}, q^{(d)}_{\mathbb{Z}_2}) = (1,0,1,1) : 4n$ $(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}, q^{(c)}_{\mathbb{Z}_2}, q^{(d)}_{\mathbb{Z}_2}) = (0,1,1,1) : 4n$ **TOT: $5N - 4$**
	$N = 6n$ $n \neq 2j$: 	$D_4 \oplus D_4$	$\mathbb{Z}_2^2$	$(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}) = (1,0) : 8n$ $(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}) = (0,1) : 8n$ $(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}) = (1,1) : 8n$ $(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}) = (0,0) : 6n - 4$ **TOT: $5N - 4$**
	$N = 3n$ $n \neq 2j$: 	D_8	$\mathbb{Z}_2$	$q_{\mathbb{Z}_2} = 1 : 7n - 4$ $q_{\mathbb{Z}_2} = 0 : 8n$ **TOT: $5N - 4$**
	$N = 8n$ $n \neq 3j$: 	$A_2^{\oplus 4}$	$\mathbb{Z}_3^2$	$(q^{(a)}_{\mathbb{Z}_3}, q^{(b)}_{\mathbb{Z}_3}) = (1,0) : 9n$ $(q^{(a)}_{\mathbb{Z}_3}, q^{(b)}_{\mathbb{Z}_3}) = (0,1) : 9n$ $(q^{(a)}_{\mathbb{Z}_3}, q^{(b)}_{\mathbb{Z}_3}) = (1,1) : 9n$ $(q^{(a)}_{\mathbb{Z}_3}, q^{(b)}_{\mathbb{Z}_3}) = (1,2) : 9n$ $(q^{(a)}_{\mathbb{Z}_3}, q^{(b)}_{\mathbb{Z}_3}) = (0,0) : 4(n-1)$ **TOT: $5N - 4$**
	$N \neq 3n, 8n$: 	E_8	$\emptyset$	**TOT: $5N - 4$**

Table 6.13: *Higgs branch data for quasi-homogeneous cDV singularities of $E_8^{(24)}[N]$ type.*

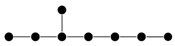
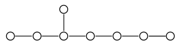
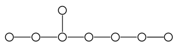
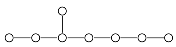
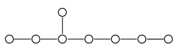
Singularity	Resolution pattern	$\boldsymbol{\mathcal{M}}$	Symmetry group	Hypers
$E_8^{(20)}[N]$	$N = 20n$: 	$\mathfrak{t}$	$U(1)^8$	Charges: **root system** of E_8 **TOT: $6N$**
	$N = 10n$ $n \neq 2j$: 	$A_1^{\oplus 8}$	$\mathbb{Z}_2^4$	$(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}, q^{(c)}_{\mathbb{Z}_2}, q^{(d)}_{\mathbb{Z}_2}) = (0,0,0,0) : 4(n-1)$ $(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}, q^{(c)}_{\mathbb{Z}_2}, q^{(d)}_{\mathbb{Z}_2}) = (1,0,0,0) : 4n$ $(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}, q^{(c)}_{\mathbb{Z}_2}, q^{(d)}_{\mathbb{Z}_2}) = (0,1,0,0) : 4n$ $(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}, q^{(c)}_{\mathbb{Z}_2}, q^{(d)}_{\mathbb{Z}_2}) = (0,0,1,0) : 4n$ $(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}, q^{(c)}_{\mathbb{Z}_2}, q^{(d)}_{\mathbb{Z}_2}) = (0,0,0,1) : 4n$ $(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}, q^{(c)}_{\mathbb{Z}_2}, q^{(d)}_{\mathbb{Z}_2}) = (1,1,0,0) : 4n$ $(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}, q^{(c)}_{\mathbb{Z}_2}, q^{(d)}_{\mathbb{Z}_2}) = (1,0,1,0) : 4n$ $(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}, q^{(c)}_{\mathbb{Z}_2}, q^{(d)}_{\mathbb{Z}_2}) = (1,0,0,1) : 4n$ $(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}, q^{(c)}_{\mathbb{Z}_2}, q^{(d)}_{\mathbb{Z}_2}) = (0,1,1,0) : 4n$ $(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}, q^{(c)}_{\mathbb{Z}_2}, q^{(d)}_{\mathbb{Z}_2}) = (0,1,0,1) : 4n$ $(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}, q^{(c)}_{\mathbb{Z}_2}, q^{(d)}_{\mathbb{Z}_2}) = (0,0,1,1) : 4n$ $(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}, q^{(c)}_{\mathbb{Z}_2}, q^{(d)}_{\mathbb{Z}_2}) = (1,1,1,0) : 4n$ $(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}, q^{(c)}_{\mathbb{Z}_2}, q^{(d)}_{\mathbb{Z}_2}) = (1,1,0,1) : 4n$ $(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}, q^{(c)}_{\mathbb{Z}_2}, q^{(d)}_{\mathbb{Z}_2}) = (1,0,1,1) : 4n$ $(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}, q^{(c)}_{\mathbb{Z}_2}, q^{(d)}_{\mathbb{Z}_2}) = (0,1,1,1) : 4n$ **TOT: $6N-4$**
	$N = 5n$ $n \neq 2j$: 	$D_4 \oplus D_4$	$\mathbb{Z}_2^2$	$(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}) = (1,0) : 8n$ $(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}) = (0,1) : 8n$ $(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}) = (1,1) : 8n$ $(q^{(a)}_{\mathbb{Z}_2}, q^{(b)}_{\mathbb{Z}_2}) = (0,0) : 6n-4$ **TOT: $6N-4$**
	$N = 4n$ $n \neq 5j$: 	$A_4 \oplus A_4$	$\mathbb{Z}_5$	$q_{\mathbb{Z}_5} = 2 : 10n$ $q_{\mathbb{Z}_5} = 1 : 10n$ $q_{\mathbb{Z}_5} = 0 : 4(n-1)$ **TOT: $6N-4$**
	$N \neq 4n, 5n$: 	E_8	$\emptyset$	**TOT: $6N-4$**

Table 6.14: *Higgs branch data for quasi-homogeneous cDV singularities of $E_8^{(20)}[N]$ type.*

Chapter 7

Further results on cDV singularities

In this chapter we will pinpoint some partial results in the analysis of M-theory on cDV singularities. In the first section of this chapter, we will deal with the so-called T-branes states: open strings branes states that correspond to the same threefold geometry but with exotic five-dimensional matter spectra. The open question, in this case, is to understand such T-branes states directly from the viewpoint of the CY geometry. In the second section of this chapter we will ask ourselves whether the five-dimensional SCFTs that we engineered can be indeed dubbed as free-hypers (or discrete gauging of free hypers). By trying to answer this question, we will also list some possible future directions that could be further investigated about the rank-zero theories studied in this thesis. We will point out two aspects:

1. we will briefly discuss some issues about the hyperkähler metrics on the considered HBs;
2. we will discuss the presence of exotic R-charges for the "hypers", and how this is related to the leading order expansion of the seven-dimensional SYM lagrangian around the considered Higgs background Φ.

7.1 T-branes

The analysis performed in this thesis further clarifies the pivotal role of T-branes for the physical description of five-dimensional theories obtained from M-theory on cDV singularities. In this section, we are going to report our final results on T-branes in the analyzed cDV cases. We will begin, in Section 7.1.1, with general remarks on T-branes for quasihomogeneous cDV. We will then report a result, the codimension formula, that we obtained for all the cDV, relating the complex dimension of the vector space of $\mathcal{G}$ spanned by the 5d zero modes to the codimension (inside the nilpotent cone of $\mathcal{G}$) of the orbit $\mathcal{O}_0$ associated to Φ. Finally, we conclude giving a hierarchy, based on the Hasse diagram of the nilpotent orbits, for the (A, A) and (A, D) singularities.

7.1.1 General results on T-branes for cDV singularities

In the preceding chapters we have always searched for a Higgs background in some ADE Lie algebra $\mathcal{G}$, that *maximizes* the number of hypermultiplets of the 5d theory, namely the dimension of the Higgs Branch, at the same time breaking the 7d gauge group in the *least* brutal way. These requirements translate into imposing that the Higgs background Φ lives in the *minimal* subalgebra $\mathcal{M}$ of $\mathcal{G}$ that allows to reproduce the given CY equation.

It must be stressed, though, that looking for the minimal subalgebra is a mere choice among the Higgses that realize the same CY equation. However, this is the choice that produces a 5d HB equal to the one obtained with different geometrical methods [25] but this is by no means the unique choice, nor necessary from a M-theory point of view. Indeed, in general the Higgs background can be embedded into some larger subalgebra $\mathcal{M}_{\text{T-brane}} \supset \mathcal{M}$, while generating the same threefold equation. This may yield:

1. Less localized modes and a smaller unbroken continuous symmetry in 5d[1],
2. A smaller unbroken discrete symmetry in 5d,
3. A combination of the previous two instances.

In this regard, the most trivial choice one can pick is:

$$\Phi \in \mathcal{M}_{\text{T-brane}} = \mathcal{G}, \tag{7.1.1}$$

namely embedding the Higgs field in the whole algebra. This completely breaks the 7d gauge group and does not produce any hypermultiplet in 5d. We interpret this choice as having switched on a T-brane (as defined in [103]). In fact, such a background in M-theory can not be detected by the threefold equation; however it produces physical effects, such as breaking symmetries, as it is evident in our construction.

Let us consider a trivial example for the (A_1, A_3) singularity. The Higgs background producing the maximal amount of modes, as well as the expected $U(1)$ flavor

[1] In this case part of the resolution is obstructed, even though it would appear possible from the geometry.

symmetry, lies in the algebra $\mathcal{M} = A_1 \oplus A_1 \oplus \langle \alpha_2^* \rangle \subset A_3$, and reads:

$$\Phi = \begin{pmatrix} 0 & 1 & 0 & 0 \\ w & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -w & 0 \end{pmatrix}. \tag{7.1.2}$$

In this case, we could have also chosen the following Higgs background:

$$\Phi_{\text{T-brane}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ w^2 & 0 & 0 & 0 \end{pmatrix}. \tag{7.1.3}$$

This background obviously reproduces the defining equation of the (A_1, A_3) singularity, via (2.2.20), but breaks *all* the 7d gauge group (in contrast with a preserved $U(1)$ in the case of Φ in the minimal allowed subalgebra $\mathcal{M}$), and does *not* localize any mode in 5d. This is an example of T-brane that completely breaks the flavor group and obstructs all the five-dimensional modes.

Furthermore, there can be T-brane cases preserving a smaller discrete group in 5d with respect to their counterpart obtained from Φ in the minimal allowed subalgebra $\mathcal{M}$. Let us take a look again at the (A_2, D_4) example examined in Section 4.2, with Higgs background living in the minimal allowed subalgebra:

$$\Phi \in \mathcal{M} = A_1^{\oplus 4}. \tag{7.1.4}$$

This choice yields:

- 4 hypers in 5d.
- A preserved $\mathbb{Z}_2$ discrete symmetry in 5d.

On the other hand, one could have also made the choice:

$$\Phi \in \mathcal{M}_{\text{T-brane}} = D_4 = \mathcal{G}, \tag{7.1.5}$$

that explicitly reads, in the basis convention of [159]:

$$\Phi_{\text{T-brane}} = \left(\begin{array}{cccc|cccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & w & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ \hline 0 & -\frac{w}{4} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{w}{4} & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -w & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{array}\right). \tag{7.1.6}$$

It is then easy to check that $\Phi_{\text{T-brane}}$ produces:

- 4 hypers in 5d.
- No preserved discrete symmetry in 5d.

The dimension of the Higgs Branch is unaffected, but the discrete symmetry is broken: this is the most simple example of T-brane that does not modify the number of five-dimensional modes or the flavor group, but just the five-dimensional discrete-gauging group.

In full generality, we can easily construct Higgs backgrounds living in some subalgebra $\mathcal{M}_{\text{T-brane}} \supset \mathcal{M}$ such that we modify the number of modes, the flavor group and the discrete gauging group. This fact entails that, given a cDV singularity, a plethora of consistent 5d theories, with varying dimension of the Higgs Branch, as well as diverse flavor and discrete symmetries, are possible. $\Phi \in \mathcal{M}$ is the choice producing the largest Higgs Branch dimension, as well as the smallest breaking of the 7d gauge group. This is another manifestation of the fact that the geometry of the background does not uniquely fix the effective low dimensional theory [103–122]. Intuitively one faces the possibilities depicted in Figure 7.1.

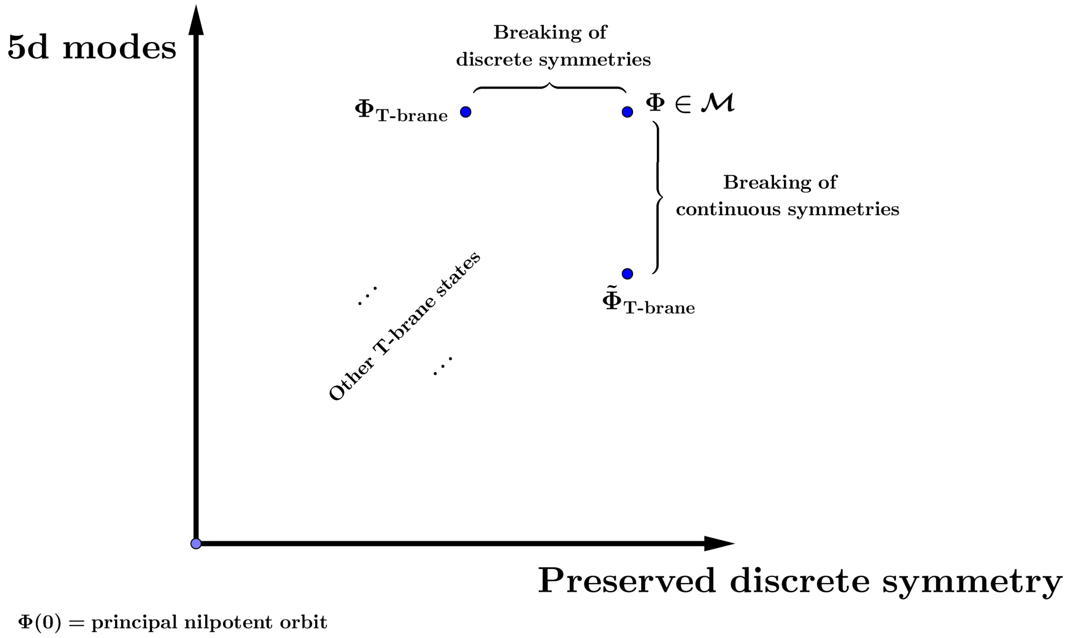


Figure 7.1: *Allowed 5d theories from T-branes.*

It would be extremely interesting to understand the counterpart of the 5d theories arising from T-brane backgrounds in complementary approaches to 5d HBs, such as the techniques relying on magnetic quivers.

7.1.2 Codimension formula and geometric interpretation of the modes

We have seen that the five and seven-dimensional modes are deformations of a certain Higgs background Φ that can not be completely gauge-fixed to zero. In this subsection we are going to give a geometric interpretation of these modes, that will lead us to the result (7.1.7).

Let $\mathcal{O}_0$ be the gauge orbit obtained acting with the seven-dimensional gauge group on $\Phi|_{w=0}$. We found that the number n_{ind} of linearly independent elements of the seven-dimensional gauge algebra $\mathcal{G}$ supporting a five-dimensional zero-mode always equals the complex codimension of $\mathcal{O}_0$ in the nilpotent cone of $\mathcal{G}$:

$$n_{\text{ind}} = \text{cod}_{\mathbb{C}}\Big(\mathcal{O}_0 \hookrightarrow \mathcal{G}\Big). \tag{7.1.7}$$

Let's start defining:

$$\Phi_w = \Phi - \Phi(0). \tag{7.1.8}$$

This coincides with the w-dependent entries of the Higgs field, namely what we denoted with $w\,Y$ in Section 4.6.

5d and 7d modes are infinitesimal deformations of the Higgs field Φ, up to gauge equivalence. It makes sense, if one is interested just in counting the number of linearly independent 7d elements supporting 5d localized modes, to identify them as tangent directions in $T_\Phi\mathcal{G}$ transverse to the seven-dimensional gauge group orbits (since we already performed a gauge fixing). Indeed, we can think of (7.1.7) as a statement on the tangent space $T_{\Phi(0)}\mathcal{G}$: 5d modes are directions

1. transverse to the nilpotent orbit $\mathcal{O}_0$,

2. tangent to the normal cone of $\mathcal{G}$[2].

We used, to check if a mode φ_{ij} is tangent to the nilpotent cone, the following condition

$$\chi(\Phi+\varphi)=\chi(\Phi)+O(\varphi^2), \tag{7.1.9}$$

with χ indicating the characteristic polynomial. (7.1.9) can be regarded as a rephrasing of the condition of "tangency" to points belonging to the singular locus of $\mathcal{N}$. We found by a case by case analysis that we can always perform the gauge-fixing in such a way that (7.1.9) is respected for all the 5d modes. On the other hand, we can prove formally the transversality of the modes to $\mathcal{O}_0$.

Proof of the transversality of the 5d modes to $\mathcal{O}_0$

Let us call Y_0 the nilnegative element of the standard triple $\{H_0, Y_0, \Phi(0)\}$ in the sense of [159], with

$$H_0 \equiv [\Phi(0), Y_0], \tag{7.1.10}$$

$\Phi(0)$ acts as a raising operator in this triple. Let's indicate with $\mathrm{ad}_{\Phi(0)}$ the adjoint action of $\Phi(0)$:

$$\mathrm{ad}_{\Phi(0)}(g)=[\Phi(0), g], \tag{7.1.11}$$

with g an element of $\mathcal{G}$. In the branching of $\mathcal{G}$ under the $\{H_0, Y_0, \Phi(0)\}$, all the elements that are *not the lowest weights states* of their irreducible representation are in $\mathrm{Im}(\mathrm{ad}_{\Phi(0)})$ (the reason is that there exists a lower weight state that was "raised" to them via $\Phi(0)$), and can be completely gauged away. Viceversa, all the elements that are not in $\mathrm{Im}(\mathrm{ad}_{\Phi(0)})$ can not be *completely* gauged away, and produce either a 5d or a 7d mode. We can then say that

$$\varphi_{ij} \text{ is a (5d or 7d mode)} \Leftrightarrow \varphi_{ij} \in \mathrm{Ker}(Y_0), \tag{7.1.12}$$

since $\mathrm{Ker}(Y_0)$ defines the space of lowest weights states in the branching of $\mathcal{G}$ under the triple. Let's link (7.1.12) to the property of being transverse to $\mathcal{O}_0$ at the point

[2]In general, $\Phi(0)\in\mathcal{O}_0 \hookrightarrow \mathrm{Sing}\,(\mathcal{N})$, and the tangent space is not well defined. The correct concept to use then is the one of "normal cone of $\mathcal{N}$ at the point $\Phi(0)$": $C_{\Phi(0)}\mathcal{N}$.

$\Phi(0)$. The transverse space to $\mathcal{O}_0$ in $\mathcal{G}$ is modeled by the Slodowy slice through $\Phi(0)$,

$$S_0 \equiv \{Z \in \mathcal{G} \mid [Z - \Phi(0), Y_0] = 0\} = \Phi(0) + \mathrm{Ker}(Y_0), \tag{7.1.13}$$

where $\Phi(0)+\mathrm{Ker}(Y_0)$ is defined as the affine space through $\Phi(0)$ in direction $\mathrm{Ker}(Y_0)$, namely

$$\Phi(0) + \mathrm{Ker}(Y_0) \equiv \{Z \in \mathcal{G} \mid Z = \Phi(0) + \lambda, \lambda \in \mathrm{Ker}(Y_0)\}\,. \tag{7.1.14}$$

The last equality in (7.1.13) means that (7.1.12) is equivalent say that a necessary condition for an oscillation to be a (five or seven-dimensional) mode is to being along the transverse directions to $\mathcal{O}_0$ (namely, inside $T_{\Phi(0)}S_0 < T_{\Phi(0)}\mathcal{G}$).

7.1.3 T-branes hierarchy

We would like, in this subsection, to find a way to organize hierarchically the T-branes states of the (A, A) and (A, D) quasi-homogeneous cDV singularities, according to the number of localized five-dimensional modes (namely, just according to the vertical position of Φ in Figure 7.1).

The T-branes for the (A, A) and (A, D) cDV can be neatly labelled using the Lie algebra formalism involving nilpotent orbits: given a Higgs field $\Phi^{(i)}(w)$ (with i running over all the possible T-branes states), we define as $\Phi_0^{(i)} = \Phi^{(i)}(w = 0)$ its constant component. We found that $\Phi_0^{(i)}$ is always nilpotent for all the (A_j, A_l) and (A_k, D_n), and thus belongs to some nilpotent orbit $\mathcal{O}_0^{(i)}$ of A_j (supposing $j > k$). Consequently we can label every Higgs $\Phi^{(i)}$ using the nilpotent orbit $\mathcal{O}_0^{(i)}$ in which its constant component resides. Furthermore, the codimension formula (7.1.7) relates the nilpotent orbit $\mathcal{O}_0^{(i)}$ to the number of linearly independent elements of the 7d gauge algebra that support 5d modes localized at the intersection of the D6 branes. For the (A_j, A_l) singularities the story ends here: in order to obtain the Higgs background for (A_j, A_l) yielding the maximal number of modes, we take the blocks Φ_h appearing in (6.1.1), evaluated on $w = 0$, to lie in the biggest-codimension nilpotent orbit $\mathcal{O}_0^{(h)}$ compatible with the geometry, namely reproducing the brane locus. We remark that in general the total number modes for this Higgs configuration need not be equal to the number of linearly independent elements in the 7d gauge algebra (i.e. there could be modes supported on $\mathbb{C}[w]/(w^k)$ with $k > 1$).

The exact same phenomenon happens in the (A_k, D_n) singularities, although the hierarchy of the different Higgs backgrounds is more complicated. The goal of this section is to show how a classification of the allowed Higgs backgrounds is possible, providing an explicit example.

The starting point, as before (see Section 4.3), is the brane locus. The only constraint that must be imposed on the Higgs $\Phi(w)$ is (6.6.6), that we reproduce here for convenience:

$$\det\,(\xi\mathbb{1} + \Phi(w)) = \Delta(\xi^2, w) = \xi^2(\xi^{2n-2} + w^k). \tag{7.1.15}$$

As we have said, there is vast space for ambiguities in the choice of the Higgs, giving rise to a hierarchy governed by the nilpotent orbits that can be associated to the Higgs itself. Let us see how this precisely comes about.

Generally speaking, each Higgs comprises constant entries, along with entries depending on w (w-entries).

Correspondingly, by considering the constant and w-entries separately, we can analyze their orbit structure. In particular, for all the cases in (7.1.15), we now show how to associate both the constant entries and the w-entries to **nilpotent orbits**, that can be classified by suitable partitions of $[2n]$ as the Higgs Φ lives in the algebra $\mathfrak{so}(2n)$. As is well known in the mathematical literature, nilpotent orbits are organized hierarchically along Hasse diagrams, and this structure will be reflected in the possible choices for the Higgs background, giving rise in general to different spectra. We will denote the nilpotent orbit associated to the constant entries as $\mathcal{O}_0$, and the one related to the w-entries as $\mathcal{O}_w$. More precisely, we define:

$$\begin{aligned} &\mathcal{O}_0 = \text{nilpotent orbit in which } \Phi(0) \text{ lives},\\ &\mathcal{O}_w = \text{nilpotent orbit in which } \Phi - \Phi(0) \text{ lives}. \end{aligned} \tag{7.1.16}$$

In this notation, the full Higgs field Φ can be decomposed as:

$$\Phi = \Phi(0) + (\Phi - \Phi(0)) \equiv \Phi_0 + \Phi_w, \tag{7.1.17}$$

where $\Phi_0 \in \mathcal{O}_0$ and $\Phi_w \in \mathcal{O}_w$.

When we try to pick a choice for Φ satisfying (7.1.15) for a given brane locus related to some (A_k, D_n) singularity, one is confronted with the following logical steps:

- In general, each brane locus is compatible with many choices of $\mathcal{O}_0$[3], thus giving rise to an ambiguity. There is always a *minimal* $\mathcal{O}_0$, giving rise to the largest spectrum. Mathematically this is the lowest-lying orbit, among the compatible ones, in the Hasse diagram.
  Most notably, *the choice of $\mathcal{O}_0$ completely fixes the number of linearly independent elements inside the 7d gauge algebra supporting 5d localized modes*,

[3]Here by compatible we mean that we can build an Higgs field Φ with Φ_0 belonging to $\mathcal{O}_0$.

according to the codimension formula (7.1.7).

- In general, each $\mathcal{O}_0$ is compatible with many *bottom* orbits $\mathcal{O}_w$, namely with many different choices of w-entries, barely sufficient to reproduce the correct brane locus (where "barely" means that no "w" entry can be removed without affecting the brane locus). Among the bottom orbits $\mathcal{O}_w$ there is always a minimal $\mathcal{O}_w$, lying at the lowest position in the Hasse diagram, giving rise to the maximal number of modes.
  Each bottom $\mathcal{O}_w$ gives rise, in general, to a different number of total 5d modes.

- By deforming each bottom $\mathcal{O}_w$, tuning zero-entries into w-entries while keeping the brane locus and $\mathcal{O}_0$ fixed, we find a *tower* of allowed $\mathcal{O}_w$, starting from the bottom one and terminating on a top one (there always is a top orbit, as the size of the Higgs is fixed by the brane locus).
  Most importantly, *each $\mathcal{O}_w$ belonging to the same tower*[4] *gives rise to the same number of total modes.* In addition, towers starting from different bottom $\mathcal{O}_w$ need not be disjoint (meaning that the same $\mathcal{O}_w$ can appear in many different towers, producing different amounts of modes. What counts for the number of modes is the *bottom* $\mathcal{O}_w$ at the base of the tower).

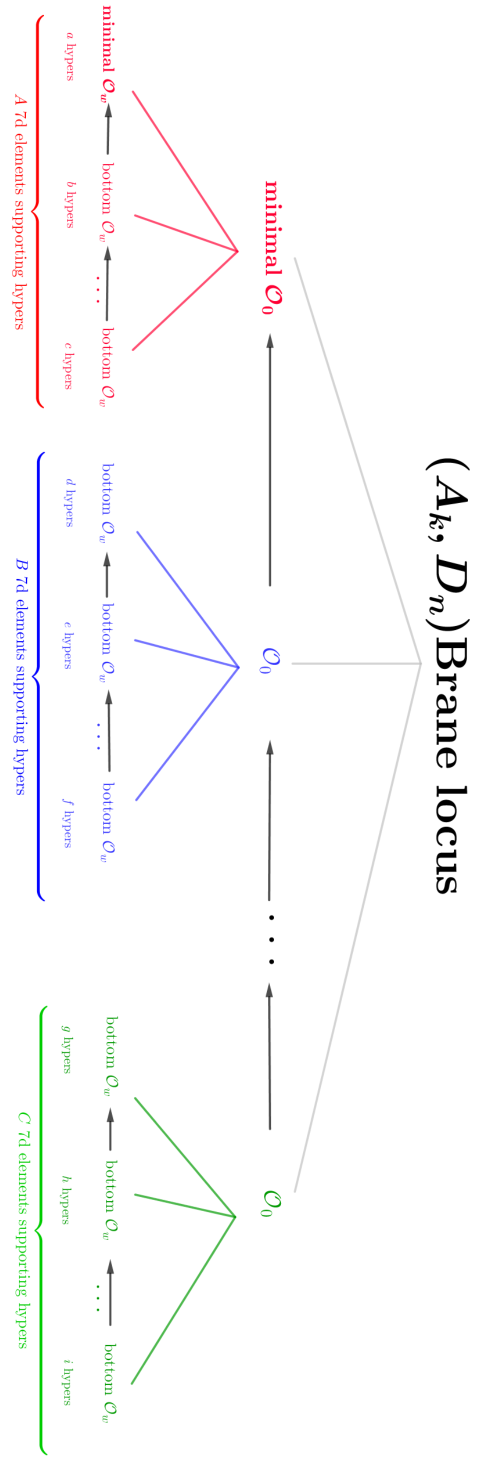
Summing up, given a brane locus in the (A_k, D_n) series, a choice of the Higgs is completely determined once one picks:

$$\begin{cases} \text{a nilpotent orbit } \mathcal{O}_0, \text{ corresponding to the constant entries of } \Phi, \\ \text{a bottom orbit } \mathcal{O}_w, \text{ corresponding to the } w\text{-entries of } \Phi. \end{cases} \tag{7.1.18}$$

In order to understand this hierarchy of choices in a more intuitive way, it is instructive to depict it graphically, indicating with segments the possible choices, and with arrows the nilpotent orbits hierarchy in the Hasse diagram sense. Notice that we have explicitly indicated the minimal $\mathcal{O}_0$ and minimal $\mathcal{O}_w$ orbits, that when combined in the choice of the Higgs yield the M-theory dynamics with the maximal number of modes. In an extensive case-by-case analysis we have always found that such choice is unique, but we cannot rule out the possibility that there is more than one minimal choice of $\mathcal{O}_0$ and $\mathcal{O}_w$ yielding the maximal number of modes, as there could be more than one orbit on the same level of the Hasse diagram hierarchy. We finally stress that each bottom orbit $\mathcal{O}_w$ in the picture is the starting point of a tower of orbits, obtained by deforming the Higgs configuration corresponding to the bottom orbit, with the same number of total 5d modes as the ones given by the bottom orbit. We have omitted such towers for a better graphical depiction.

[4]We stress that this means that the Higgs associated to the $\mathcal{O}_w$ in the tower is obtained turning on some w-entries in the Higgs associated to the bottom orbit, without modifying its brane locus.

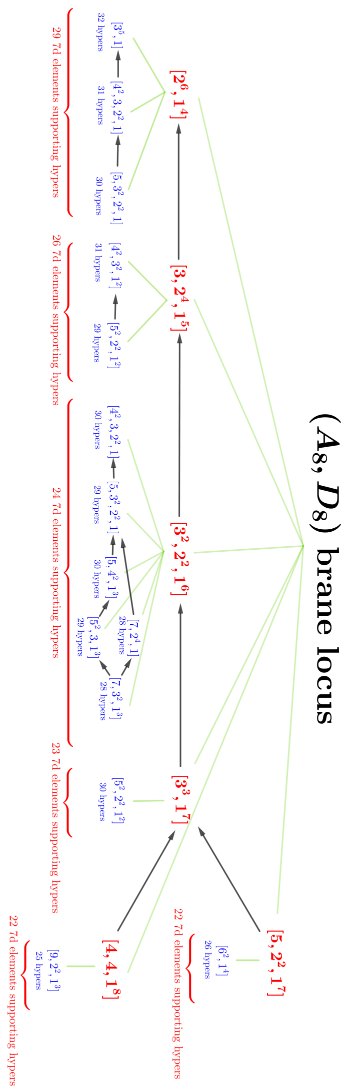
Finally, notice that for every choice of $\mathcal{O}_0$ we have indicated the total number of linearly independent elements of the 7d gauge algebra supporting localized 5d hypers (namely, the number of 7d elements supporting localized 5d modes given by the codimension formula (7.1.7) is *twice* the number we have indicated), and that for every bottom $\mathcal{O}_w$ we have highlighted the total number of hypers.



Let us now examine a concrete example, so as to make the abstract remarks above a bit more grounded. An interesting instance of brane locus giving rise to a T-brane hierarchy is (A_8, D_8), that displays a remarkable structure. This singularity is non-resolvable and its brane locus is:

$$\Delta(\xi^2, w) = \underbrace{\xi^2(\xi^{14} + w^9)}_{\text{type (b)}} = 0. \tag{7.1.19}$$

In the following picture, the red color refers to the $\mathcal{O}_0$, the blue color to bottom $\mathcal{O}_w$ and the dark arrows to dominance in the Hasse diagram sense. We have instead omitted towers with the same number of total hypers for the sake of graphical clarity. As before, we have indicated the total number of 7d gauge algebra elements supporting localized 5d hypers for every choice of $\mathcal{O}_0$ in the hierarchy, as well as the total number of hypers for every bottom $\mathcal{O}_w$. As it can be seen from the picture, the M-theory dynamics with maximal modes is reproduced by the lowest $\mathcal{O}_0$ with the lowest $\mathcal{O}_w$ in the Hasse diagram, yielding 32 total hypers. All the other partitions are instead T-brane configurations with a lower amount of modes.



It is clear that the T-brane hierarchy can be extremely rich, giving rise to a plethora of different Higgs backgrounds encoding the same geometry, but a different five-dimensional physics. The T-brane hierarchy is of course a type IIA theory feature, and besides its interconnected structure, it is suggestive of a corresponding intricacy in the dual M-theory description. What we have shown from a type IIA perspective in the preceding pages is what are the IIA features that should rephrase the T-branes data: the choice of the Higgs background is intrinsically ambiguous and additional non-geometric pieces of information, e.g. the orbits $\mathcal{O}_0$ and $\mathcal{O}_w$ in (7.1.18), must be specified for a full characterization of the spectrum and of the preserved symmetries.

7.2 Hyperkähler metrics

Finding a method to extract the hyperkäler metrics on the HB, in the context of geometric engineering, is a currently open problem. As pointed out in [156], the task is made difficult by the presence of M2 branes instantons wrapped on the vanishing three-cycles of the deformed geometry. In the geometric-engineering limit things might simplify, but is still not well understood how to keep track of the decompactification of the CY from the viewpoint of the HB geometry. In particular, we want to try to address here, at least partially, a puzzle that appeared throughout the analysis of M-theory dynamics on the cDV singularities. The contents of this section are unpublished and have to be intended as conjectural statements on the hyperkähler metrics.

In this section we will focus on two examples, the (A_1, A_1) and the (A_1, A_2) singularities, both giving a five-dimensional HB isomorphic, as complex varieties, to $\mathbb{C}^2$.

A paradigm of the geometric-engineering construction is that flavor symmetries are captured by the Cartier divisors of the resolved geometry. Indeed these are associated to cohomological two-forms over which we can reduce the M-theory three-form, with the non-compactness of the divisors signaling the fact that the associated symmetry is global rather than local. For the conifold case, there is just one such class of Cartier divisor (the dual of the resolved $\mathbb{P}^1$) and the rank of the flavor group is one. We note here that, instead, the R-symmetry (that is another isometry of the HB) is not captured by the divisors of X. The geometry of the resolved conifold is compatible with a single-center Taub-NUT metric on the HB. This result can also be confirmed noticing that the magnetic quiver of M-theory on the conifold is three-dimensional $\mathcal{N} = 4$ SQED with one electron, whose metric on the CB is the single center Taub-NUT [163]. We can fit this construction in a more general one recently proposed in [164].

From [164], we have that on $\mathbb{R}^4$ there can be only finitely many complete non-flat hyperkähler metrics satisfying the following condition on the asymptotic decay of the Riemann tensor $R^{\mu}_{\nu\rho\sigma}$:

$$\left|R^{\mu}_{\nu\rho\sigma}\right| = O(s^{-2-\epsilon}) \tag{7.2.1}$$

as $s \to \infty$, with $s(x)$ the metric distance between a fixed bulk point x_0 and x and with $\epsilon > 0$.

It turns out that ϵ can assume just a finite number of values; the possibilities are known, in mathematics literature, as gravitational instantons [165–170] and are labelled, in [164] as ALE, ALF, ALG, and ALH. Considering again the conifold case, the only metric displaying the correct numbers of isometries is the one associated to ALF asymptotics[5]. In the ALF case there is just one such metric, called the Taub-NUT metric [171]. However, this is somehow special of the ALF case: in general the asymptotic form of the metric will not be enough to completely determine the bulk one [164]

Let's now consider instead an example presenting a mismatch between the rank of the five-dimensional flavor group dictated by the CY geometry and the one expected for a free hyper: the (A_1, A_2) singularity

$$uv + z^3 + w^2 = 0 \tag{7.2.3}$$

and its miniversal deformations

$$F \equiv uv + z^3 + u_2 z + u_3 + w^2 = 0. \tag{7.2.4}$$

In this case, the threefold does not admit a small crepant resolution, hence the geometry dictates an empty five-dimensional flavor group. Let's try to characterize the metric on the HB. A global section of the canonical bundle of (7.2.4) can be taken, by adjunction formula, to be

$$\Omega \equiv \frac{du \wedge dv \wedge dw \wedge dz}{dF}. \tag{7.2.5}$$

[5]More precisely, in this case we expect two symmetries preserving the single center Taub-NUT complex structure:

$$uv = z \qquad (u, v, z) \in \mathbb{C}^3_{u,v,z}. \tag{7.2.2}$$

The first one is the $U(1)$ symmetry associated to the flavor group and is the compact part of the $\mathbb{C}^*$ action $u \to \lambda u, v \to \frac{v}{\lambda}$. The second one, using the results of [25] and of [163], is another $U(1)$ formed by the $SU(2)$ stabilizers of the point of the twistor line representing the complex structure (7.2.2) with respect to rotations of the hyperkähler structure. This second $U(1)$ is the compact direction of a $\mathbb{C}^*$ action acting on u, v, z with weigths $1/2, 1/2$ and 1.

To describe the HB, we are interested in computing the periods of Ω and of the M-theory three-form potential C_3 on the $H_3(X, \mathbb{Z})$. The problem [88, 172] is exactly analogous to compute the periods of the holomorphic volume form $\lambda = wdz$ and of a real-valued closed one-form C_1 on the curve

$$z^3 + w^2 + u_2 z + u_3 = 0. \tag{7.2.6}$$

In the context of type IIB geometric engineering, u_2 is seen as an external fixed free parameter [96], while u_3 is seen as a coordinate on the moduli space of complex structures of (7.2.6). Let's assume that this can be done also in the context of M-theory geometric engineering. To extract the HB, we can use that, due to the relation between (7.2.4) and (7.2.6), the intermediate Jacobian fibration of (7.2.4) coincides with the Jacobian $\mathcal{J}_{\text{elliptic}}$ of the elliptic family (7.2.6). We know that the HB of M-theory on (A_1, A_2) coincides with the intermediate Jacobian fibration of (7.2.4), and hence we have

$$\text{HB} \cong \mathcal{J}_{\text{elliptic}}. \tag{7.2.7}$$

The next step then is to compute $\mathcal{J}_{\text{elliptic}}$ from the elliptic fibration. The elliptic family (7.2.6) is already in Weierstrass form, hence it coincides with its Jacobian fibration. As we saw in Chapter 6, we expect[6] HB $\cong \mathbb{C}^2$. Indeed, if we regard (7.2.6) as the defining equation of the HB, we see that we can solve for u_3, obtaining it as a function of the coordinates (w, z) that now parametrize the HB. The restriction of the projection on the u_3 coordinate to (7.2.6) gives the HB the structure of the Jacobian fibration over the complex structure moduli space of the (A_1, A_2) singularity. This means that u_3 parametrizes the complex structure, while the (w, z) coordinates, once we impose (7.2.6) at fixed value of u_3, parametrize the possible vevs of C_3.

Let's now use this presentation (7.2.6) to extract some information on the hyperkähler metric of the HB. We already saw that the single-center Taub-NUT, namely the ALF metric, is associated with the (A_1, A_1) case. To understand what happens in the (A_1, A_2) case, we consider the following result of [164, 173].

Theorem 7.2.1. Let (M, g) be a non-compact complete hyperkähler manifold satisfying (7.2.1). If (M, g) is ALG or ALH, there exists a rational elliptic surface $\tilde{M}$ with a meromorphic function $u : \tilde{M} \twoheadrightarrow \mathbb{P}$ whose generic fiber is a torus. The fiber $D \equiv \{u = \infty\}$ is regular if (M, g) is ALH, while is of type $I_0^*, II, II^*, III, III^*, IV, IV^*$ if (M, g) is ALG. There exists a point of the twistor line $(a_1, a_2, a_3) \in \mathbb{P}^1$, with $a_1^2 + a_2^2 + a_3^2 = 1$, such that when we use $a_1\mathcal{I}_1 + a_2\mathcal{I}_2 + a_3\mathcal{I}_3$ as complex structure,

[6] Indeed, here it comes one of the subtleties. According to which kind of metric we put on the CY manifold X, the features of the HB might change. For example, in the seminal work of [156] the HB of M-theory on the conifold is itself an elliptic fibration (rather than a $\mathbb{C}^*$ fibration as in the geometric engineering limit).

(M, g) is biholomorphic to $\tilde{M} \setminus D$.

In our case, we can take M as the variety obtained partly compactifying (7.2.6) via the embedding $\mathbb{C}^2_{w,z} \times \mathbb{C}_{u_3} \hookrightarrow \mathbb{P}_{231} \times \mathbb{C}_{u_3}$ and the natural candidate for $\tilde{M}$ is obtained by further compactifying the u_3 coordinate via $\mathbb{C}_{u_3} \hookrightarrow \mathbb{P}^1$. The structure (7.2.6) is naturally associated with the elliptic surface in Theorem 7.2.1, because the map u_3, that split physically the moduli space (7.2.6) into complex structure and C_3 moduli, coincides with the map u of Theorem 7.2.1 when we compactify (7.2.6) to $\tilde{M}$. This compatibility between the projection maps on the base spaces of the elliptic fibrations is the reason why we can apply Theorem 7.2.1 to the (A_1, A_2) HB but not to other HBs isomorphic, as complex varieties, to $\mathbb{C}^2$. Let's now see which case of the ones listed in Theorem 7.2.1 corresponds to (7.2.6). For $u_2 = 0$, we have just two singular fibers: the fiber at $u_3 = 0$ and the one at $u_3 = \infty$. To compute the Kodaira type of the singularity at $u_3 = \infty$ we use that

$$M_{u_3=0} \cdot M_{u_3=\infty} = \mathbb{1}, \tag{7.2.8}$$

where on the left-hand side we have the product of the monodromy matrices associated to the singular fibers and the equality comes from the fact that their product has to act trivially on the one-cycles of the fibral curves. This holds true because a real closed path on the u_3 plane encircling both the points associated to singular fibers can be contracted on $\mathbb{P} \ni u_3$ without crossing another point associated to a singular fiber. From (7.2.6) we have that the degeneration over $u_3 = 0$ is a cusp, namely a type II singularity following the Kodaira classification. The associated monodromy matrix is

$$M_{u_3=0} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}. \tag{7.2.9}$$

Then, using (7.2.8), we have

$$M_{u_3=\infty} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \tag{7.2.10}$$

and the fiber at infinity is a II^* degeneration of the elliptic fibration. Consequently, if we assume (7.2.1), the metric on the HB of M-theory on (A_1, A_2) singularity is, using Theorem 7.2.1, of ALG type. This remains true also if we relax the assumption $u_2 = 0$. Indeed, for $u_2 \neq 0$ we have two points associated to a singular fiber on the u_3 plane:

$$u_{3,\pm} = \mp \frac{2iu_2^{3/2}}{3\sqrt{3}}. \tag{7.2.11}$$

Looking to the degenerations of the roots of $z^3 + u_2 z + u_3 = 0$ at $u_{3,\pm}$, we see that the cycle degenerating at $u_{3,+}$ can be chosen as the α cycle of the elliptic fiber, and the one degenerating at $u_{3,-}$ can be chosen as the β cycle, namely $\alpha \cdot \beta = -1$. We have, consequently, that the monodromies $M_\pm$ on $u_{3,\pm}$ are:

$$M_+ = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad M_- = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \tag{7.2.12}$$

We then have that

$$M_\infty = (M_+ \cdot M_-)^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \tag{7.2.13}$$

and the fiber over $u_3 = \infty$ is of Kodaira type II. Consequently, also for $u_2 \neq 0$, we have an ALG type instanton on the HB.

In the ALG case, unfortunately, the asymptotic behavior does not uniquely fix (as in the ALF case) the full "bulk" metric [164]. However, the work [164] permits to say something on how the hyperkähler structure on M can be related to rational two-forms on the elliptic surface $\tilde{M}$. This result might be a good starting point to characterize, in a future analysis, the metric on the HB of the (A_1, A_2) singularity.

Theorem 7.2.2 (construction of ALG instantons)**.** Let $(\tilde{M}, u_3)$ be a rational elliptic surface, with basespace coordinate u_3. Suppose that the fiber $D \equiv \{u_3 = \infty\}$ has type $I_0^*, II, II^*, III, III^*, IV, IV^*$. Let $\omega^+ \equiv \omega^2 + i\omega^3$ a rational 2-form on $\tilde{M}$, with $[D] = \omega^+ = \infty$. For any Kähler form ω on $\tilde{M}$, there exists a real smooth polynomial-growth function ϕ on $M \equiv \tilde{M} \setminus D$ such that[7] $(M, \omega_1 \equiv \omega + i\partial\overline{\partial}\phi, \omega_2, \omega_3)$ is an ALG gravitational instanton. Furthermore,

i the form $\omega + i\partial\overline{\partial}\phi$ is uniquely determined by its asymptotic (to $u_3 \to \infty$) geometry.

ii Let's consider an ALG gravitational instanton written as $(M, \omega_1, \omega_2, \omega_3)$, and let's replace $a_1\mathcal{I}_1 + a_2\mathcal{I}_2 + a_3\mathcal{I}_3$ with $\mathcal{I}_1$ after an hyperkähler rotation. Then $\omega^+ \equiv \omega^2 + i\omega^3$ is a rational 2-form on $\tilde{M}$ with $[D] = \{\omega^+ = \infty\}$. There exists a Kähler form ω on $\tilde{M}$ and a real smooth polynomial-growth function ϕ on M such that $\omega_1 = \omega + i\partial\overline{\partial}\phi$. When D is of type II^*, III^* or IV^*, we may need a new choice of $\tilde{M}$ to achieve this.

[7]Here we label the hyperkähler metric with the triple $(\omega_1, \omega_2, \omega_3)$ of its compatible Kähler forms.

7.3 R-charges

The second puzzle is related to the R-charges of free hypers. In this thesis, we always managed to pair the zero-modes in such a way they have compatible charges under the flavor and discrete symmetries to be the chiral and anti-chiral scalars of an hyper. For a free hyper, the $SU(2)$ R-symmetry (spontaneously broken on the HB) rotates $Q, \tilde{Q}$ as a doublet, with charges under the $SU(2)$ Cartan generator being $\pm 1/2$. In particular, if all the zero-modes can be considered free-hypers, they should all have the same R-charge under the five-dimensional $SU(2)$ R-symmetry.

For quasi-homogeneous cDV singularities a possible way to extract the R-charges is the following. First of all, we notice that, given a quasi-homogeneous cDV X, we have a well-defined $\mathbb{C}^*$ action assigning weights $([x],[y],[w],[z])$ to (x,y,w,z). After the deformation, the equation will depend on the parameters u_i and it will not be anymore quasi-homogeneous. However, we can, given $([x],[y],[w],[z])$, assign quasi-homogeneous spurionic weights to u_i, in such a way that the equation is again, spurioncally, quasi-homogeneous. Let's consider as example the deformed conifold case, we have

$$x^2 + y^2 + w^2 + z^2 + u_2 = 0. \tag{7.3.1}$$

For $u_2 = 0$ we get the conifold singularity, and the quasi-homogeneous weigths are (conventionally giving the defining equation weight one)

$$[x] = [y] = [w] = [z] = 1/2. \tag{7.3.2}$$

Consequently, to make (7.3.1) spurionically quasi-homogeneous we need to assign

$$[u_2] = 1. \tag{7.3.3}$$

Let's consider another example, the (A_1, A_4) singularity:

$$x^2 + y^2 + w^2 + z^5 + u_2 z^3 + u_3 z^2 + u_4 z + u_5 = 0. \tag{7.3.4}$$

In this case we have

$$[x] = [y] = [w] = \frac{1}{2}, \quad [z] = \frac{1}{5}. \tag{7.3.5}$$

We have that the spurionic weights are

$$[u_j] = \frac{j}{5}. \tag{7.3.6}$$

To interpret physically this $\mathbb{C}^*$ action we can consider type IIB on the cDV X. In type IIB, the complex deformations span the $\mathcal{N} = 2, \mathcal{D} = 4$ CB, and the compact $U(1)$ contained in the $\mathbb{C}^*$ is the generator of the spontaneously broken $U(1)_R$

symmetry on the four-dimensional CB. We can map this $U(1)_R$ to the generator of the five-dimensional R-symmetry $SU(2)_R$ using the construction of [25]. The magnetic quiver for M-theory on X is obtained via the following procedure

1. consider type IIB on X,

2. compactify the theory on a circle $\mathbb{S}^1_\beta$ obtaining a three-dimensional $\mathcal{N} = 4$ theory,

3. gauge as many flavor symmetries as the number of curves in the small crepant resolution of X,

4. flow to the IR obtaining a $\mathcal{D} = 3, \mathcal{N} = 4$ theory $\mathcal{T}$.

The claim of [25] is that $\mathcal{T}$ is the magnetic quiver of M-theory on X. Keeping track of the four-dimensional $U(1)_R$ symmetry in the [25] procedure, we notice that it gets enhanced on the $SU(2)_I$ R-symmetry spontaneously broken on the CB of $\mathcal{T}$ [163]. In the magnetic quiver construction, $SU(2)_I$ is identified with the five-dimensional $SU(2)_R$ symmetry, spontaneously broken on the five-dimensional Higgs branch.

We can then conclude that the spurionic type IIB R-charges that we found for the deformation parameters u_i of X are the R-charges of the coordinates on the base space of the HB with respect to the five-dimensional $SU(2)_R$.

At the end of the day, we are interested in assigning weights under $SU(2)_R$ to the zero-modes we found with the Higgs field construction. To achieve such result, we proceed as follows:

1. We compute, after the gauge-fixing, the zero modes φ of the Higgs background Φ we associated to X.

2. We compute the Casimir invariants of $\Phi' \equiv \Phi + \varphi$, obtaining a threefold $\tilde{X}$ that is a smoothing of X (with X recovered setting $\varphi = 0$) and whose deformation parameters u_j depends holomorphically on the five-dimensional zero modes $\vec{\varphi}$ that fill the entries of φ:
$$u_j = u_j(\vec{\varphi}). \tag{7.3.7}$$

3. Fixed the spurionic quasi-homogeneous weights of u_j, we found that we can, in all the considered cases, assign a quasi-homogeneous weight to the modes $\varphi_k \in \vec{\varphi}$. These have to be interpreted as the weights of the zero-modes under the $SU(2)_R$ charge.

Let's see how this works in some examples. We can start from the conifold. In this case (see Section 5.2),

$$\Phi = \begin{pmatrix} w & 0 \\ 0 & -w \end{pmatrix}, \quad \varphi = \begin{pmatrix} 0 & \varphi_+ \\ \varphi_- & 0 \end{pmatrix}. \tag{7.3.8}$$

The equation defining $\tilde{X}$ is

$$x^2+y^2+\det(z\mathbb{1}-\Phi-\varphi) = x^2+y^2+z^2+w^2+\varphi_+\varphi_- = x^2+y^2+z^2+w^2+u_2 = 0. \tag{7.3.9}$$

In other words we have $u_2(\vec{\varphi}) = \varphi_+\varphi_-$. We have $[u_2] = 1$, furthermore $[\varphi_+] = [\varphi_-]$ because they have to pair into a hypermultiplet. Summing up

$$[\varphi_+][\varphi_-] = [u_2] = 1, \quad [\varphi_+] = [\varphi_-], \tag{7.3.10}$$

consequently

$$[\varphi_+] = [\varphi_-] = \frac{1}{2}, \tag{7.3.11}$$

as required for a free-hyper.

Let's consider, to conclude, the (A_1, A_3) case. The corresponding Higgs background, deformed by the five-dimensional zero-modes $(\varphi_a, \varphi_b, \varphi_c, \varphi_d)$ is

$$\Phi + \phi = \begin{pmatrix} 0 & 1 & \varphi_a & 0 \\ w & 0 & \varphi_b & \varphi_a \\ \varphi_c & 0 & 0 & 1 \\ \varphi_d & \varphi_c & -w & 0 \end{pmatrix}, \tag{7.3.12}$$

where (φ_a, φ_c) pairs as the first hyper and (φ_b, φ_d) as the second. For this singularity, we have

$$u_4(\vec{\varphi}) = \varphi_a^2\varphi_c^2 - \varphi_b\varphi_d, \quad u_3(\vec{\varphi}) = -2(\varphi_b\varphi_c + \varphi_a\varphi_d), \quad u_2(\vec{\varphi}) = -2\varphi_a\varphi_c. \tag{7.3.13}$$

We have that

$$[\varphi_a] = [\varphi_c] = \frac{1}{4}, \quad [\varphi_b] = [\varphi_d] = \frac{1}{2}. \tag{7.3.14}$$

The output we get is the following: the pair (φ_b, φ_d) has the correct R-charge to be interpreted as a five-dimensional hyper, while (φ_a, φ_c) has a R-charge that is half of the charge of a five-dimensional free-hyper. One might think that this is a problem of normalization of the charges, but there is no normalization that can allow for different charges for the pairs (φ_b, φ_d) and (φ_a, φ_c): they can not be, simultaneously, free-hypers.

This mismatch on the R-charges comes together with another puzzling fact. To obtain the zero-modes action we might be tempted to insert the background $\Phi + \phi$ into the seven dimensional SYM action (3.1.2). For simplicity, we can concentrate on the term

$$\mathcal{K} \equiv \mathrm{Tr}(D_\nu \Phi' \overline{D_\nu \Phi'}) \tag{7.3.15}$$

that is responsible for the kinetic term of the zero modes. Plugging, e.g., (7.3.12) in (7.3.15), we have

$$\mathcal{K} = 2(D_\nu \varphi_a \overline{D_\nu \varphi_c} + D_\nu \varphi_c \overline{D_\nu \varphi_a}), \tag{7.3.16}$$

with no kinetic term for (φ_b, φ_d). We see this as a recurring pattern in many analyzed cases: just the lowest R-charge five-dimensional modes enter the expression of $\mathcal{K}$. This might come from the fact that the term $\mathcal{K}$ is, morally speaking, related to the quadratic Casimir of $\Phi + \phi$, and just the lower R-charges zero-modes enter the expression of the quadratic Casimir.

We remark that it might also be possible that there are subtelties we did not take into account in computing the R-charges of the five-dimensional modes and M-theory on cDV singularities might indeed give five-dimensional free hypers or discrete gauging of free hypers.

Summing up, we found two puzzles that suggest that the theories that we found may be different from just interpreting it as free-hypers:

- the number of 5d real masses expected from the geometry is, in general, lower than the rank of the flavor group $Sp(n)$ of n free-hypers .

- the R-charges assignments, obtained matching the [25] construction with the results presented in this thesis, seem not to be consistent with the free-hypers interpretation. In particular, apart from the conifold case, there exists no way in which we can assign the same five-dimensional R-charge to all the zero modes that we computed in this thesis work.

We notice that the two apparently unrelated puzzles are seen in a unified language once we look at them from the viewpoint of the hyperkähler metric on the HB. Indeed, in that case, both the R-symmetries and the flavor symmetries are isometries of the hyperkähler metric.

A possible way out is the following: we always computed, in the "free-hypers" cases, HB $\cong \mathbb{C}^{2n}$, with n the number of five-dimensional hypers. The complex structure is just one of the many features of the Higgs Branch and might be insufficient to claim that the modes we found are free hypermultiplets. Indeed, there are finer geometric structures on HB $\cong \mathbb{C}^{2n}$, such as the hyperkähler metric, that might be[8] less "resistant" than the complex structure of the HB to M-theoretical corrections (e.g. coming from M2 instantons analogous to [156]). This might suggest that, whenever the rank of the flavor group $U(1)^l$ is lower than n, there might be $n-l$ transformations that preserve the complex structure, but that are not isometries (namely, they do not respect the hyperkähler metric) of the HB.

[8]This is something already observed [163] in dealing with quantum corrections to hyperkähler metrics of moduli spaces of supersymmetric theories. In [174], the hyperkähler metric gets corrected by quantum effects, while this is not true for (a particular) complex structure (called in [163] "distinguished complex structure").

Chapter 8

Conclusions

In this thesis we studied the dynamic of M-theory on isolated cDV threefold singularities X. In doing so, we introduced a new method, that extends the catalog of known techniques beyond the toric case. Our method, inspired by the type IIA limit on the considered cDV singularities, rephrases the problem of studying M-theory moduli spaces in terms of a seven-dimensional gauge theory analysis. More than that, the true underlying structure is the Lie-algebraic theory of the simple algebras of type ADE. Our work establishes a bridge between two different areas of mathematics: the study of topological invariants of terminal threefolds and the theory of simple Lie algebras. From a mathematical viewpoint, our work furnishes a novel, extremely explicit method to compute the GV invariants of terminal singularities. The fact that the method is clear and simple permitted us to automatize, in a Mathematica code presented in Appendix F, the computation of the GV invariants. Let's now list some bird-eye conclusions on our work and some possible future directions.

In our study on cDV singularities we addressed two different subclasses: the flops of any length [130] (reported in Chapter 5) and the quasi-homogeneous cDV [102, 129] (Chapter 6). The first class is under current investigation in mathematics [73, 76–82], representing a natural non-toric generalization of the conifold singularity. In our work, we studied, as new examples, the flops of length four, five and six, furnishing an explicit construction of such threefold singularities. In doing so we built families of threefolds exhibiting a simple flop, flat with respect to the GV invariants on an open subset of the family basespace. Our method permits us to give closed equations for the divisors of the basespace where we have either enhanced GV invariants or fibers with non-isolated singularities (see, e.g., (5.3.14) for the length two case). In the length six case, our construction is an example of saturation of the GV bounds conjectured in [82]. It would be a natural follow-up, using our method, to either construct examples saturating the GV bounds also for the length three, four and five, or to rule out their existence, refining in this way the [82] bound. What we see, in all the cases, is that if we treat the modes in $\mathbb{C}[w]/(w^k)$ with $k > 1$ (appearing in Table 5.1, Table 5.2 and Table 5.3) as modes in $\mathbb{C}[w]/(w)$ we saturate the [82] lower bounds for the GV invariants of a simple flop

Length	GV lower bounds
1	(1)
2	$(4, 1)$
3	$(5, 3, 1)$
4	$(6, 4, 2, 1)$
5	$(7, 6, 4, 2, 1)$
6	$(6, 6, 4, 3, 2, 1)$

,

Table 8.1: *GV lower bounds for simple flops*

where, in the second column, the notation $(n_1, n_2, ..., n_\ell)$ indicates that we have n_1 hypermultiplets of charge one, n_2 of charge two and so on.

The second class of singularities that we studied in this thesis are the quasihomogeneous cDV singularities. With our method, we managed to give a one-to-one correspondence between threefold geometries and Higgs backgrounds. This permitted us to study these geometries systematically, showing that the HB are, from the viewpoint of the Higgs Branch complex structure, affine spaces $\mathbb{C}^{2n}$ or discrete quotient $\mathbb{C}^{2n}/\Gamma$ (where n denotes the total sum of the five-dimensional zero modes).

We can give some general remarks on the HBs of any cDV singularity:

1. if Γ is present, it is the product of cyclic groups:

$$\Gamma = \mathbb{Z}_{o_1} \times ... \times \mathbb{Z}_{o_k}, \tag{8.0.1}$$

   and, in particular, it never has non-abelian factors.

2. Relating Γ with the maximal subalgebras of the Levi subalgebras of $\mathcal{G}$, we can conclude that the maximum discrete charge (under one of the factors of Γ) is six. This is obtained considering, as outlined in Section 4.2, the maximal subalgebra $\mathcal{M} = A_5 \oplus A_2 \oplus A_1 < E_8$.

Another subtle aspect of M-theory geometric engineering is to keep track of the so-called T-branes data. These are exotic open-string states that correspond, in our construction, to the same threefold geometry, but with a different number of five-dimensional modes and different five-dimensional flavor and discrete-gauging groups. Our work, reported in Section 7.1, elucidated the type IIA interpretation of these data and permitted us to (partly) organize them according to the nilpotent orbits Hasse diagram. It would be interesting to interpret these data directly from the viewpoint of the cDV geometry, on the lines of [105].

Apart from the threefold case, it is also tempting to apply our methods to M-theory geometric engineering on Calabi-Yau fourfolds singularities. On the physical side, this would permit us to create a correspondence between the geometry of the considered fourfolds and the dynamics of the corresponding three-dimensional SCFTs. On the mathematical side, the study of topological invariants of fourfolds [175–178] is fair less understood with respect to the threefold case. In this sense, it is tantalizing to understand the geometric nature of the data we are able to compute with our approach based on string dualities.

Finally, it would be very interesting to understand the moduli spaces as intermediate Jacobians of the considered Calabi-Yau manifolds. This, together with the constraints coming from the hyperkähler condition could be, in some cases, enough to find a metric on these Higgs branches as we tried to sketch out in Section 7.2. This would probably permit us to find an explanation for the puzzles pointed out in Section 7.2 and finally answering the question whether these theories are genuinely free hypers/discrete gauging of free hypers or we have to consider something less trivial.

Aside from the rank-zero case, it would be interesting also to understand the IIA limit for the higher-rank theories. In this case, unfortunately, the interpretation in terms of backgrounds of a seven-dimensional SYM theory is less clear. It would be however tempting to insert in our construction some class-S-like ingredients, such as allowing the Higgs background to have poles in w. In fact, our construction is based, at the end of the day, on the study of a Hitchin system arising from particular BPS solutions of the seven-dimensional gauge theory, preserving a five-dimensional Poincaré group. The fact that the class S construction [179, 180] is also related to an Hitchin system [180] suggests that the class S machinery and the one presented in this thesis might be indeed related by suitable webs of string dualities (e.g. the one used in [25]).

Appendices

Appendix A

Structure theory of Lie algebras

In this appendix, we are going to summarize some essential concepts of the theory of the simple Lie algebras $\mathcal{G}$, settling our conventions. In the first part, we will quickly recap some algebraic aspects (focusing on the definitions of Levi and regular maximal subalgebras). In the second part of this appendix, we will review some geometric aspects of the simple Lie algebras.

A.1 Algebraic aspects of $\mathcal{G}$

We will first fix our conventions. Given a basis of the Cartan subalgebra $\mathfrak{t} < \mathcal{G}$, we denote with e_α the root vector associated to the root α. We have that

$$[e_\alpha, e_{\alpha'}] = \lambda(\alpha, \alpha') e_{\alpha+\alpha'}, \tag{A.1.1}$$

with $\lambda(\alpha, \alpha')$ the appropriate structure constant (in particular, $\lambda(\alpha, \alpha') = 0$ if $\alpha + \alpha'$ is not a root). We will denote with e_{α_i}, with $i = 1, ..., r$, the simple root vectors of $\mathcal{G}$. We labelled the simple roots as in the Figure F.1.

We will denote as $\alpha_i^* \in \mathfrak{t}$ the dual roots, these are defined to be the elements of the Cartan subalgebra that satisfy the following relations:

$$\left[\alpha_i^*, e_{\alpha_j}\right] = \delta_{ij} e_{\alpha_j}. \tag{A.1.2}$$

We remark that changing linear basis in $\mathfrak{t}$ just changes the numerical values of the root α, but it does not change the algebraic relations (A.1.1).

We are now going to consider special kind of subalgebras of $\mathcal{G}$: the Levi subalgebras.

Definition A.1.1 (Levi Subalgebra)**.** Let $\mathcal{G}$ be a semisimple Lie algebra, and let $\mathcal{S}_{\text{white}} \equiv \left\{e_{\alpha_{i_1}}, ..., e_{\alpha_{i_l}}\right\}$ be a subset of its simple root vectors. The Levi subalgebra

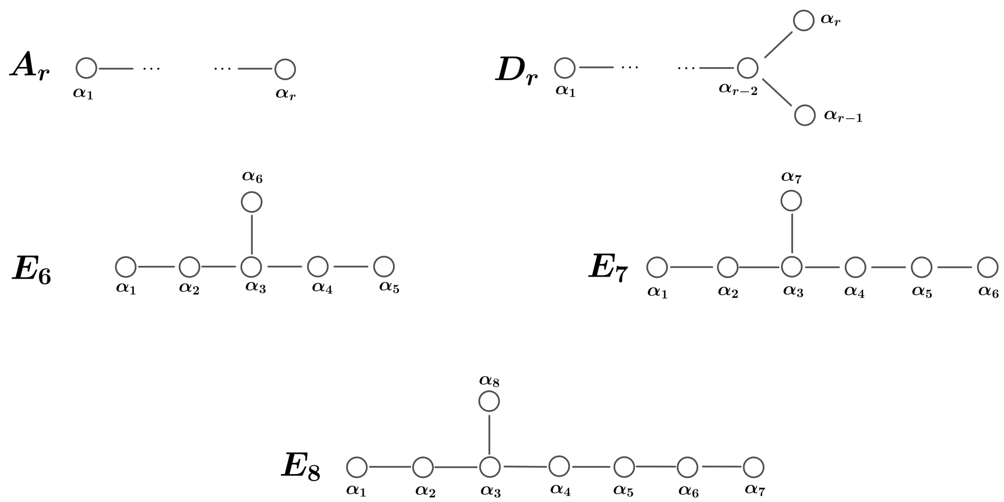


Figure A.1: *Roots labelling convention*

$\mathcal{L}$ associated to $\mathcal{S}_{\text{white}}$ is the union of $\mathfrak{t}$ and $\mathcal{L}^{\text{semi-simp}}$, with $\mathcal{L}^{\text{semi-simp}}$ the subalgebra generated by $\left\{e_{\pm\alpha_{i_1}}, ..., e_{\pm\alpha_{i_l}}\right\}$.

Remark A.1.1. There exists another equivalent way to define a Levi subalgebra $\mathcal{L}$. Let $\mathcal{H} \equiv \langle \alpha_{j_1}, ...\alpha_{j_f} \rangle$ be the subalgebra of $\mathfrak{t}$ generated by the dual roots of the roots at positions $\mathcal{S}_{\text{black}} \equiv \{j_1, ..., j_f\}$ of the Dynkin diagram. The maximal subalgebra of $\mathcal{G}$ that commutes with $\mathcal{H}$ turns out to be a Levi subalgebra. We have that the Levi subalgebra $\mathcal{L}$ defined in this fashion is the one associated, in the language of Definition A.1.1, to the set $\mathcal{S}_{\text{white}} \equiv \left\{e_{\alpha_{i_1}}, ..., e_{\alpha_{i_l}}\right\}$, with $\alpha_{i_1}, ..., \alpha_{i_l}$ the simple roots of $\mathcal{G}$ whose labels $i_1, ..., i_l$ *do not* appear in $\mathcal{S}_{\text{black}}$.

This "dual" definition of the Levi subalgebra suggests to label $\mathcal{L}$ with a colored Dynkin diagram, where we color in black the simple roots in $\mathcal{S}_{\text{black}}$ and in white the roots in $\mathcal{S}_{\text{white}}$.

We now recall the definition of *regular* maximal subalgebra of a semisimple Lie algebra.

Definition A.1.2. Let $\mathcal{G}$ a semisimple Lie algebra, a regular maximal subalgebra $\mathcal{M}$ of $\mathcal{G}$ is a maximal subalgebra generated by a subset of the generators of $\mathcal{G}$, together with (possibly) the lowest root of $\mathcal{G}$.

We note that, with this definition, all the maximal subalgebras of a simple Lie algebra $\mathcal{G}$ contains the Cartan subalgebra $\mathfrak{t} \leq \mathcal{G}$ and can be obtained with the following procedure[1]:

[1] An exception arises in the case of maximal regular subalgebras of A_r. In this case, removing just one node from the extended Dynkin diagram we get, as maximal subalgebra, again A_r itself. To get non-trivial maximal subalgebras, in this case, is necessary to remove one node from the *unextended* A_r Dynkin diagram. The subalgebra obtained in this way is isomorphic to $A_l \oplus A_{r-1-l} \oplus \langle \alpha_i^* \rangle$, with α_i^* the dual root of the removed node.

1. extend the Dynkin diagram of $\mathcal{G}$ adding a node representing the lowest root of $\mathcal{G}$;

2. delete one of the nodes of the extended Dynkin diagram.

The resulting Dynkin diagram is the Dynkin diagram of the maximal regular subalgebra obtained in this way and is generated by the root vectors (and their negatives) associated with the nodes that we did not delete from the affine Dynkin diagram.

Remark A.1.2. It turns out that, if $\mathcal{M} < \mathcal{L}$ (with $\mathcal{L}$ defined as the commutant of $\mathcal{H}$), then $\mathcal{H}$ is the maximal vector subspace of $\mathfrak{t}$ that commutes with $\mathcal{M}$. At this point, a natural question arises: we defined $\mathcal{L}$ as the commutant of $\mathcal{H}$ or, equivalently, as the maximal subalgebra such that the restriction to $\mathcal{L}$ of the action of the toroidal subgroup of G generated by $\mathcal{H}$ is trivial. We might wonder if there exists a similar way to regard to $\mathcal{M}$. Indeed, we can look back to the affine Dynkin diagram construction as follows: we have that the maximal regular subalgebra $\mathcal{M} < \mathcal{G}$ obtained removing the i-th node of the affine Dynkin diagram of $\mathcal{G}$ is the maximal subalgebra of $\mathcal{G}$ such that the adjoint action of the (discrete) subgroup generated by

$$\gamma_i \equiv \exp\left[\frac{2\pi i \alpha_i^*}{\mathfrak{q}_{\alpha_i}}\right], \tag{A.1.3}$$

with $\mathfrak{q}_{\alpha_i}$ the dual Coxeter label of the i-th root, trivializes when restricted to $\mathcal{M}$.

We now give a technical definition that will be used in the main text.

Definition A.1.3 (Reconstructible Higgs field)**.** Let

$$P = z^m + \sigma_2 z^{n-2} - \sigma_3 z^{n-3} + ... + (-1)^m \sigma_m, \tag{A.1.4}$$

with σ_j polynomials in $w \in \mathbb{C}_w$ be an element of $\mathbb{C}[w, z]$. Then the following matrix $g \in A_{m-1}$, expressed in the fundamental representation of A_{m-1},

$$g = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \vdots & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 1 \\ (-1)^{m-1}\sigma_m & (-1)^{m-2}\sigma_{m-1} & \cdots & -\sigma_2 & 0 \end{pmatrix} \tag{A.1.5}$$

is called "reconstructible Higgs" of P. In particular we have

$$\mathrm{Det}(z\mathbb{1} - g) = P. \tag{A.1.6}$$

Finally, for the $\mathcal{G} = D_r$ case, we review two embeddings [159] $D_r \hookrightarrow A_{2r-1}$. The image of the first embedding consists of all the elements of $\mathrm{Mat}(2r, \mathbb{C})$ of the following shape

$$g = \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} , \qquad \text{with} \quad B^t = -B \ , C^t = -C \, . \tag{A.1.7}$$

In this basis, the elements of $g \in D_r$ are such that

$$g \cdot \mathcal{Q} + \mathcal{Q} \cdot g^t = \mathbb{0} \tag{A.1.8}$$

where

$$\mathcal{Q} = \left(\begin{array}{c|c} 0 & \mathbb{1}_{2r} \\ \hline \mathbb{1}_{2r} & 0 \end{array} \right) \, . \tag{A.1.9}$$

In other words, the embedding $D_r \hookrightarrow A_{2r-1}$ is the one induced by the embedding of $SO(2r)$ into $SU(2r)$, with $\mathcal{Q}$ the non-degenerate quadratic form preserved by $SO(2r)$. An equivalent choice, obtained exchanging rows and columns of $\mathcal{Q}$, is given by the block diagonal sum of of $s_j \times s_j$ sized blocks $\mathcal{Q}_j$ (with $j = 1, ..., q$) of the shape of (A.1.9), such that $s_1 + ... + s_q = 2r$:

$$\mathcal{Q} = \begin{pmatrix} \mathcal{Q}_1 & & \\ & \ddots & \\ & & \mathcal{Q}_q \end{pmatrix} , \tag{A.1.10}$$

where diagonal dots stands for the blocks $\mathcal{Q}_2, ..., \mathcal{Q}_{q-1}$ and we left blanket space for zero entries.

A.2 Geometrical aspects of simple Lie algebras

In this subsection we will briefly recall some geometrical aspects of simple Lie algebras[2]. Every $\mathcal{G}$ is isomorphic, as complex algebraic variety, to $\mathbb{C}^{\dim(\mathcal{G})}$ and hence is not particularly interesting itself. However, on $\mathcal{G}$ we have the adjoint action of the corresponding Lie group G. This permits us to define some non-trivial geometries associated to $\mathcal{G}$. We first define a set of invariant coordinates under the action of G.

Definition A.2.1 (Casimir invariants)**.** Given $\Phi \in \mathcal{G}$, with $\mathcal{G}$ of type A_r or D_r, we

[2]See [136] for a more detailed review.

define the Casimir invariants k_i, $\tilde{k}_i$ and $\hat{k}_r$ of Φ as

$$\begin{array}{c|ll} \boldsymbol{A_r} & k_i \equiv \mathrm{Tr}(\Phi^i) & \text{for } i = 2, \ldots, r+1 \\ \hline \boldsymbol{D_r} & \begin{array}{l} \tilde{k}_i \equiv \mathrm{Tr}(\Phi^i) \\ \hat{k}_r \equiv \mathrm{Pfaff}(\Phi) \end{array} & \text{for } i = 2, 4, \ldots, 2(r-1) \end{array} . \tag{A.2.1}$$

with Φ in the fundamental representation.

Let's now consider the exceptional algebras. We fix the representations of $\mathcal{G} = E_6, E_7, E_8$ as follows: **27** for E_6, **133** for E_7 and **248** for E_8. One then defines the Casimirs of an element $\mathrm{g} \in \mathcal{G}$ in the respective representation as[181, 182]:

$$\begin{array}{c|ll} \boldsymbol{E_6} & c_{k_i}(\mathrm{g}) = \mathrm{Tr}(\mathrm{g}^{k_i}) & \text{for } k_i = 2, 5, 6, 8, 9, 12 \\ \boldsymbol{E_7} & \tilde{c}_{k_i}(\mathrm{g}) = \mathrm{Tr}(\mathrm{g}^{k_i}) & \text{for } k_i = 2, 6, 8, 10, 12, 14, 18 \\ \boldsymbol{E_8} & \hat{c}_{k_i}(\mathrm{g}) = \mathrm{Tr}(\mathrm{g}^{k_i}) & \text{for } k_i = 2, 8, 12, 14, 18, 20, 24, 30 \end{array} , \tag{A.2.2}$$

and $i = 1, ..., r$.

It is useful to define also the *partial Casimir invariants*.

Definition A.2.2 (Partial Casimir invariants)**.** Let $\mathcal{A}^{\text{semi-simp}}$ be a subalgebra of a simple Lie algebra $\mathcal{G}$. Let Φ be an element of $\mathcal{A}^{\text{semi-simp}}$. Then, the partial Casimir ρ of Φ are the Casimir invariants of Φ seen as an element of $\mathcal{A}^{\text{semi-simp}}$ (rather than as an element of $\mathcal{G}$).

Let's clarify this with an example. We can consider the following $\Phi \in A_2$:

$$\Phi = \begin{pmatrix} 0 & 1 & 0 \\ \frac{\rho_1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{A.2.3}$$

Φ is an element of the $A_1^{(1)}$ subalgebra[3] of A_2. The algebra A_1 has one degree two Casimir invariant:

$$k_{2,\text{partial}} = \mathrm{Tr}\left[\begin{pmatrix} 0 & 1 \\ \frac{\rho_1}{2} & 0 \end{pmatrix}^2\right] = \rho_1 . \tag{A.2.4}$$

[3]We will denote with $A_l^{(i_1,\ldots,i_l)}$ the A_l subalgebra of $\mathcal{G}$ generated by the root vectors $e_{\pm\alpha_{i_1}}, ..., e_{\pm\alpha_{i_l}}$.

The (total) Casimir invariants of Φ are instead

$$k_{2,\text{total}} = \mathrm{Tr}(\Phi^2) = \rho_1, \qquad k_{3,\text{total}} = \mathrm{Tr}(\Phi^3) = 0. \tag{A.2.5}$$

The difference between $k_{2,\text{partial}}$ and $k_{2,\text{total}}$ is that, for $k_{2,\text{partial}}$, we regarded Φ as an abstract element of the A_1 algebra and hence we computed its Casimir invariant in the *fundamental representation of* A_1 (namely, using a two-by-two matrix). For computing $k_{2,\text{total}}$ instead, we regarded the abstract A_1 algebra as the subalgebra $A_1^{(1)}$ of A_2. This is reflected into the fact that we computed $k_{2,\text{total}}$ using the *fundamental representation of the* A_2 (rather than the A_1) algebra. In other words, we computed the traces of the three-by-three matrix (A.2.3).

We will denote by χ the map that sends an element $\Phi \in \mathcal{G}$ to its Casimir invariants. Every $\mathcal{G}$ has exactly r Casimir invariants, hence the map $\chi : \mathcal{G} \twoheadrightarrow \mathbb{C}^r_\mu$ takes values in $\mathbb{C}^r_\mu$. We remark, without proving them[4], the following facts:

- It turns out that there exists a linear change of coordinates between the coefficients μ_i of the miniversal deformation of $X_{\mathcal{G}}$ (that we introduced in Section 2.2 and that parametrize $\mathfrak{t}/\mathcal{W}_{\mathcal{G}}$) and the Casimir invariants. We will then, by an abuse of notation, denote the coordinates of the codomain of χ as μ_i.

- the preimage $\chi^{-1}(\mu)$ for fixed μ is a $\mathcal{G}$-invariant subset w.r.t. the adjoint action of G and hence it is a union of adjoint orbits. In particular, we are interested in the nilpotent cone $\mathcal{N}$ of $\mathcal{G}$, defined as $\chi^{-1}(0_{\mathbb{C}^r_\mu})$. We note that $\dim_{\mathbb{C}}\mathcal{N} = \dim_{\mathbb{C}}\mathcal{G} - r$, since we imposed r equations (associated to the Casimir invariants) to define $\mathcal{N}$.

- The G-orbits that stratify $\mathcal{N}$ are called nilpotent orbits. They can be organized hierarchically in an Hasse diagram. A nilpotent orbit $\mathcal{O}$ dominates another nilpotent orbit $\mathcal{O}'$ in the Hasse diagram iff the Zariski closure of $\mathcal{O}'$ is contained in the Zariski closure of $\mathcal{O}$.

- It exists just one orbit of maximal dimension inside $\mathcal{N}$, whose closure is $\mathcal{N}$ itself. This is called the *regular* orbit of $\mathcal{G}$: $\mathcal{O}_{\text{reg}}$. Furthermore, there exists just one nilpotent orbit, called the sub-regular orbit $\mathcal{O}_{\text{subreg}}$, of dimension

$$\dim_{\mathbb{C}}(\mathcal{O}_{\text{subreg}}) = \dim_{\mathbb{C}}(\mathcal{O}_{\text{reg}}) - 2. \tag{A.2.6}$$

Let us conclude giving the definition of Slodowy slices. Consider a nilpotent element $\mathrm{x} \in \mathcal{N}$ belonging to some nilpotent orbit $\mathcal{O}$: the Jacobson-Morozov theorem ensures that there exists a standard triple $\{\mathrm{x}, \mathrm{y}, \mathrm{h}\}$ of elements in $\mathcal{G}$ satisfying the

[4]See [159] for a more detailed reference.

$\mathfrak{su}(2)$ algebra relations[5]. Now, we define the *Slodowy slice* through the point x as follows.

Definition A.2.3. Let consider $\mathrm{x} \in \mathcal{N} \subset \mathcal{G}$ and its nilpotent orbit $\mathcal{O}$. Let $\{\mathrm{x}, \mathrm{y}, \mathrm{h}\}$ be the standard triple associated to x. Then, the *Slodowy slice* through the point x is composed by those Lie algebra elements satisfying:

$$\mathcal{S}_{\mathrm{x}} = \{\mathrm{z} \in \mathcal{G} \mid [\mathrm{z}, \mathrm{y}] = \mathrm{h}\} . \tag{A.2.7}$$

The idea behind Definition A.2.3 is the following: we want to find a way to slice trasversally, at the point x, the orbit $\mathcal{O}$. We then use the fact that two elements y, h are enough to fix uniquely [159] the standard triple $\{\mathrm{x}, \mathrm{y}, \mathrm{h}\}$. If there exists another element $\mathrm{x}' \in \mathcal{O} \subset \mathcal{N}$ satisfying

$$[\mathrm{x}', \mathrm{y}] = \mathrm{h}$$

then there would be *two* standard triple containing y, h. Consequently, we have $\mathrm{x} = \mathrm{x}'$ and $\mathcal{S}_{\mathrm{x}}$ slices $\mathcal{O}$ transversally.

It turns out that we can use the notion of Slodowy slice to give an embedding of the miniversal deformation of the Du Val singularity of type $\mathcal{G}$ inside $\mathcal{G}$ itself [136].

Theorem A.2.1. Let $\mathcal{G}$ be a simple Lie algebra, let $\mathcal{S}_{\mathrm{subreg}}$ the Slodowy slice through a point $\mathrm{x} \in \mathcal{O}_{\mathrm{subreg}}$. Let $(\mathcal{F}, \mu)$ be the miniversal deformations of $X_{\mathcal{G}}$, then

(i) $\mathcal{F}$ is isomorphic to S_{subreg} as complex algebraic variety;

(ii) the projection μ on the base-space of the miniversal deformation is realized as[6] the restriction of χ to S_{subreg}.

In particular, we have that, being $\mathcal{N} = \chi^{-1}(0)$, then $S_{\mathrm{subreg}} \cap \mathcal{N} \cong X_{\mathcal{G}}$.

In other words, $\mathcal{G}$ contains the miniversal deformation space of the corresponding Du Val singularity: every point of $\mathcal{F}$ is realized as an element $\Phi \in \mathcal{G}$. In particular, the transverse directions to $\mathrm{x} \in \mathcal{N}$ split as two complex directions tangent to $\mathcal{N}$, spanning the fibral $\mathcal{G}$-type Du Val singularity, and r complex directions transverse both to the nilpotent cone and to $\mathcal{O}_{\mathrm{subreg}}$. These r directions can be parametrized by the Casimir invariants of the elements of $\mathcal{S}_{\mathrm{subreg}}$.

[5] The triple related to x is unique up to G-conjugation.

[6] For example, in the A_r case, a standard choice of coordinates on $\mathfrak{t}/\mathcal{W}_{\mathcal{G}}$ are the coefficients of the characteristic polynomial of $\Phi \in \mathfrak{t} < A_r$. It turns out that there exists, in this case, an invertible change of coordinates between the Casimir invariants k_i, with $i = 1, ..., r$ of Φ and the coefficients of its characteristic polynomial.

Appendix B

Deformation parameters of E_6, E_7, E_8 families in terms of Casimir invariants

In this Appendix, we present the explicit expressions of the coefficients of the versal deformations of E_6, E_7, E_8 in terms of the Casimir invariants of the Higgs backgrounds Φ. This allows to build a bridge between a given cDV threefold arising from a deformation of a $\mathcal{G} = E_6, E_7, E_8$ singularity and a Higgs background Φ.

The E_r singularities possess r deformation parameters:

$$\begin{array}{c|l} \boldsymbol{E_6} & \mu_i \quad \text{for } i = 2, 5, 6, 8, 9, 12 \\ \boldsymbol{E_7} & \mu_i \quad \text{for } i = 2, 6, 8, 10, 12, 14, 18 \\ \boldsymbol{E_8} & \mu_i \quad \text{for } i = 2, 8, 12, 14, 18, 20, 24, 30 \end{array} \tag{B.0.1}$$

entering in the equation of the family as in (2.2.4).

In particular, we are interested in the relationship between the Casimirs of Φ and the deformation parameters (B.0.1).

The result for the E_6 case is:

$$
\begin{aligned}
\mu_2 &= -\frac{c_2}{24} \\
\mu_5 &= \frac{c_5}{60} \\
\mu_6 &= \frac{c_2^3}{13824} - \frac{c_6}{144} \\
\mu_8 &= -\frac{c_2^4}{110592} + \frac{13c_2c_6}{8640} - \frac{c_8}{240} \\
\mu_9 &= \frac{c_9}{756} - \frac{c_2^2c_5}{11520} \\
\mu_{12} &= -\frac{c_{12}}{3240} + \frac{109c_2^6}{4299816960} - \frac{847c_2^3c_6}{134369280} + \frac{109c_2^2c_8}{3732480} + \frac{13c_2c_5^2}{466560} + \frac{61c_6^2}{933120},
\end{aligned}
\tag{B.0.2}
$$

with $c_k \equiv \mathrm{Tr}(\Phi^k)$.

For the E_7 case:

$$\mu_2 = \frac{\tilde{c}_2}{18}$$

$$\mu_6 = \frac{\tilde{c}_2^3}{139968} - \frac{\tilde{c}_6}{72}$$

$$\mu_8 = -\frac{7\tilde{c}_2^4}{25194240} + \frac{11\tilde{c}_2\tilde{c}_6}{16200} - \frac{\tilde{c}_8}{300}$$

$$\mu_{10} = -\frac{2\tilde{c}_{10}}{315} + \frac{\tilde{c}_2^5}{151165440} - \frac{17\tilde{c}_2^2\tilde{c}_6}{583200} + \frac{\tilde{c}_2\tilde{c}_8}{1400}$$

$$\mu_{12} = -\frac{16\tilde{c}_{10}\tilde{c}_2}{1148175} + \frac{\tilde{c}_{12}}{12150} - \frac{149\tilde{c}_2^6}{10579162152960} + \frac{167\tilde{c}_2^3\tilde{c}_6}{3401222400} + \frac{737\tilde{c}_2^2\tilde{c}_8}{881798400} - \frac{31\tilde{c}_6^2}{437400}$$

$$\mu_{14} = \frac{8303\tilde{c}_{10}\tilde{c}_2^2}{14935460400} - \frac{2201\tilde{c}_{12}\tilde{c}_2}{217314900} + \frac{4\tilde{c}_{14}}{62601} + \frac{11083\tilde{c}_2^7}{24082404724998144} - \frac{11609\tilde{c}_2^4\tilde{c}_6}{5530387622400}$$
$$- \frac{1289\tilde{c}_2^3\tilde{c}_8}{1433804198400} + \frac{353\tilde{c}_2\tilde{c}_6^2}{142242480} - \frac{31\tilde{c}_6\tilde{c}_8}{1463400}$$

$$\mu_{18} = \frac{12182634587\tilde{c}_{10}\tilde{c}_2^4}{77806514663884339200} - \frac{564449\tilde{c}_{10}\tilde{c}_2\tilde{c}_6}{3418744644000} + \frac{1844\tilde{c}_{10}\tilde{c}_8}{3956880375} - \frac{27233975\tilde{c}_{12}\tilde{c}_2^3}{11321053720935552}$$
$$+ \frac{301\tilde{c}_{12}\tilde{c}_6}{452214900} + \frac{307855\tilde{c}_{14}\tilde{c}_2^2}{13588370378352} - \frac{2\tilde{c}_{18}}{1507383} - \frac{886993691\tilde{c}_2^9}{313644160640867419847393280}$$
$$+ \frac{4713945967\tilde{c}_2^6\tilde{c}_6}{72026602145995788288000} - \frac{14715122551\tilde{c}_2^5\tilde{c}_8}{2334195439916530176000} - \frac{579011753\tilde{c}_2^3\tilde{c}_6^2}{23156700792822720000}$$
$$+ \frac{2313866297\tilde{c}_2^2\tilde{c}_6\tilde{c}_8}{222355151645760000} - \frac{77393\tilde{c}_2\tilde{c}_8^2}{3376537920000} - \frac{15011\tilde{c}_6^3}{97678418400}, \tag{B.0.3}$$

with $\tilde{c}_k \equiv \mathrm{Tr}(\Phi^k)$. For the E_8 case, calling $\tilde{c}_k \equiv \mathrm{Tr}(\Phi^k)$:

$$
\begin{aligned}
\mu_2 &= \frac{\hat{c}_2}{120} \\
\mu_8 &= \frac{13\hat{c}_2^4}{24883200000} - \frac{\hat{c}_8}{5760} \\
\mu_{12} &= \frac{\hat{c}_{12}}{181440} + \frac{101\hat{c}_2^6}{3224862720000000} - \frac{\hat{c}_2^2\hat{c}_8}{64512000} \\
\mu_{14} &= -\frac{71\hat{c}_{12}\hat{c}_2}{798336000} + \frac{\hat{c}_{14}}{1108800} - \frac{2531\hat{c}_2^7}{9029615616000000000} + \frac{103\hat{c}_2^3\hat{c}_8}{696729600000} \\
\mu_{18} &= -\frac{4451\hat{c}_{12}\hat{c}_2^3}{689762304000000} + \frac{1523\hat{c}_{14}\hat{c}_2^2}{12454041600000} - \frac{\hat{c}_{18}}{47174400} - \frac{26399\hat{c}_2^9}{2080423437926400000000000} \\
&\quad + \frac{4747\hat{c}_2^5\hat{c}_8}{722369249280000000} + \frac{331\hat{c}_2\hat{c}_8^2}{1672151040000} \\
\mu_{20} &= \frac{191071\hat{c}_{12}\hat{c}_2^4}{2121019084800000000} + \frac{127\hat{c}_{12}\hat{c}_8}{174569472000} - \frac{1165063\hat{c}_{14}\hat{c}_2^3}{612738846720000000} + \frac{236627\hat{c}_{18}\hat{c}_2}{434023349760000} \\
&\quad + \frac{10249681\hat{c}_2^{10}}{6141409988758732800000000000} - \frac{2994007\hat{c}_2^6\hat{c}_8}{3554056706457600000000} - \frac{323371\hat{c}_2^2\hat{c}_8^2}{82269831168000000} - \frac{\hat{c}_{20}}{220809600} \\
\mu_{24} &= -\frac{193\hat{c}_{12}^2}{17793312768000} + \frac{228270563\hat{c}_{12}\hat{c}_2^6}{2932096782827520000000000} + \frac{234189517\hat{c}_{12}\hat{c}_2^2\hat{c}_8}{945465467240448000000} \\
&\quad - \frac{9171869023\hat{c}_{14}\hat{c}_2^5}{5267593317482496000000000} - \frac{23281\hat{c}_{14}\hat{c}_2\hat{c}_8}{9150846566400000} + \frac{561557071\hat{c}_{18}\hat{c}_2^3}{829158207381504000000} \\
&\quad + \frac{8268193432181\hat{c}_2^{12}}{58076120730497181548544000000000000000} - \frac{20976434911\hat{c}_2^8\hat{c}_8}{30553514694074695680000000000000} \\
&\quad - \frac{16935675593\hat{c}_2^4\hat{c}_8^2}{3300533994730291200000000} - \frac{666323\hat{c}_2^2\hat{c}_{20}}{721337268326400000} + \frac{\hat{c}_{24}}{10061694720} - \frac{593\hat{c}_8^3}{887354818560000} \\
\mu_{30} &= -\frac{636328729\hat{c}_{12}^2\hat{c}_2^3}{3676467835511164108800000000} - \frac{189107437\hat{c}_{12}\hat{c}_{14}\hat{c}_2^2}{277976001893990400000000} + \frac{2521\hat{c}_{12}\hat{c}_{18}}{319072545792000000} \\
&\quad + \frac{122785779721089347\hat{c}_{12}\hat{c}_2^9}{535457637938020692787200000000000000000} + \frac{374760114643099\hat{c}_{12}\hat{c}_2^5\hat{c}_8}{68515991479980785664000000000000000} \\
&\quad - \frac{199931513\hat{c}_{12}\hat{c}_2\hat{c}_8^2}{944585637101568000000000} + \frac{28501673\hat{c}_{14}^2\hat{c}_2}{386077780408320000000000} - \frac{163451357840757 1229\hat{c}_{14}\hat{c}_2^8}{320654840126310076907520000000000000000} \\
&\quad - \frac{3442332938170993\hat{c}_{14}\hat{c}_2^4\hat{c}_8}{5938052594931668090880000000000000} + \frac{1223\hat{c}_{14}\hat{c}_8^2}{112201334784000000} + \frac{155875352888598 01\hat{c}_{18}\hat{c}_2^6}{7634639050626430402560000000000000000} \\
&\quad - \frac{1051350791\hat{c}_{18}\hat{c}_2^2\hat{c}_8}{1243310844834938880000000} + \frac{387360133348145631291 13\hat{c}_2^{15}}{919171413254131073937239231692800000000000000000000000} \\
&\quad - \frac{966205043352894287\hat{c}_2^{11}\hat{c}_8}{4649719415985430597730304000000000000000000000} - \frac{535169284942975 57\hat{c}_2^7\hat{c}_8^2}{420028854199225889587200000000000000000} \\
&\quad - \frac{2159242595767\hat{c}_2^5\hat{c}_{20}}{7379840352152125440000000000000} + \frac{21328481\hat{c}_2^3\hat{c}_{24}}{583320714375168000000000} + \frac{225239997090599\hat{c}_2^3\hat{c}_8^3}{119591548765057371340800000000000000} \\
&\quad + \frac{72667\hat{c}_2\hat{c}_{20}\hat{c}_8}{4518107320320000000} - \frac{\hat{c}_{30}}{1978376400000}.
\end{aligned} \tag{B.0.4}
$$

Appendix C

Explicit Higgs Branches for the (A_{N-1}, D_k) series

In the following pages we determine completely, as algebraic varieties, the Higgs Branches of M-theory on all the singularities (A_{N-1}, D_k), with $k = 1, ..., 8$, and $n = 4, ..., 15$. There is nothing that forbids us to continue the analysis for $k > 8$ and $n > 15$, and our methods still apply, but we stopped here for space reasons. The tables are made up of four columns:

1. The first column indicates the Calabi-Yau threefold.
2. The second column indicates the $\text{Stab}_{SO}(\Phi)$: the $U(1)$ factors form the flavor group, and the $\mathbb{Z}_2$ factors form the discrete gauging group. These groups were computed assuming $SO(2n)$ seven-dimensional gauge groups. The G-stabilizers $\text{Stab}(\Phi)$ can be computed modding $\text{Stab}_{SO}(\Phi)$ by the center $\{\pm\mathbb{1}_{2n}\} \cong \mathbb{Z}_2$ of $SO(2n)$.
3. The third column contains matrices that describe how the five-dimensional modes localize w.r.t. the block decomposition of Φ into the blocks of table 6.6. Each number corresponds to the amount of five-dimensional modes localized in that block. The colors represent the charges of the modes w.r.t the flavor and gauge groups, according to the key:

$$\begin{cases} \text{black:} & \text{uncharged modes} \\ \text{red:} & \text{modes with charge } \pm 1 \text{ under (possibly more than one) } U(1) \\ & \text{(and possibly one } \mathbb{Z}_2 \text{ factor)} \\ \text{blue:} & \text{modes with charge } \pm 2 \text{ under } U(1) \quad \text{(and possibly one } \mathbb{Z}_2 \text{ factor)} \\ \text{green:} & \text{modes charged only under some } \mathbb{Z}_2 \text{ factors} \end{cases}$$

4. The last column indicates the quaternionic dimension of the Higgs branch, that coincides with the expected one [96].

Let us do an example of how to use the data in the tables to reconstruct the Higgs Branch. Let's pick, for example, the (A_3, D_5) singularity (that is the case $k = 2, n = 1$ of the family we have already examined in section (6.6.1)). With our method, we find that the $SO(2n)$-stabilizers of the Higgs field are:

$$\mathrm{Stab}_{SO}(\Phi) \equiv \left(\begin{array}{c|c|c|c} e^{i\alpha} & 0 & \mathbb{0} & \mathbb{0} \\ \hline 0 & e^{-i\alpha} & \mathbb{0} & \mathbb{0} \\ \hline \mathbb{0} & \mathbb{0} & \epsilon_1 \mathbb{1}_a & \mathbb{0} \\ \hline \mathbb{0} & \mathbb{0} & \mathbb{0} & \epsilon_2 \mathbb{1}_b \end{array}\right), \tag{C.0.1}$$

with $\alpha \in \mathbb{R}, \epsilon_1, \epsilon_2 = \pm 1$, $a = 4, b = 4$, and $\mathbb{0}$ indicates the zero matrix of the appropriate size. The final result, with the data given in the tables, is independent from[1] a, b (but we needed them to compute the third column of the tables). In the second column of the tables below, we shortened (C.0.1) as:

$$\mathrm{Stab}_{SO}(\Phi) = \begin{pmatrix} U(1) & & \\ & \mathbb{Z}_2 & \\ & & \mathbb{Z}_2 \end{pmatrix}. \tag{C.0.2}$$

Notice that, passing from (C.0.1) to (C.0.2), we have condensed the first two rows into a single row, and the first two columns into a single column: this is because the corresponding block of type (a) in the Higgs Φ (for the definition of the blocks of type (a) we refer to Appendix D) is a 2×2 null matrix, that *never* contains any localized mode.

In more general cases, where the blocks of type (a) are not the null matrix, *we explicitly keep them separated into two rows and two columns.* In the (A_3, D_4) case,

[1]Consequently in the table we do not give the data to compute a, b.

for example, the stabilizer is written as:

$$
\mathrm{Stab}_{SO}(\Phi) = \left(\begin{array}{c|c|c|c} e^{i\alpha} & 0 & \mathbb{0} & \mathbb{0} \\ \hline 0 & e^{-i\alpha} & \mathbb{0} & \mathbb{0} \\ \hline \mathbb{0} & \mathbb{0} & e^{i\beta}\mathbb{1}_a & \mathbb{0} \\ \hline \mathbb{0} & \mathbb{0} & \mathbb{0} & e^{-i\beta}\mathbb{1}_a \end{array}\right) \equiv \begin{pmatrix} U(1)_\alpha & & \\ & U(1)_\beta & \\ & & U(1)_\beta \end{pmatrix}, \tag{C.0.3}
$$

where $U(1)_\alpha$ refers to a vanishing 2×2 block of type (a) in Φ, and $U(1)_\beta$ refers to a non-vanishing block of type (a), that is therefore kept on two rows and two columns. That is the reason why $U(1)_\beta$ appears twice in (C.0.3).

Summing up: we write stabilizers referred to blocks of type (a) on two rows and two columns, except when the block is a 2×2 null matrix (in which case we write it only on one row and column).

The third column of the table indicates how the five-dimensional modes localize w.r.t. the block decomposition we highlighted in (C.0.1). In the case of the (A_3, D_5), the five-dimensional modes distribute as follows:

$$
\left(\begin{array}{c|c|c|c} 0 & 0 & 1 & 1 \\ \hline 0 & 0 & 1 & 1 \\ \hline 1 & 1 & 0 & 4 \\ \hline 1 & 1 & 4 & 0 \end{array}\right), \tag{C.0.4}
$$

that translates in the table as:

$$
\begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 4 \\ 2 & 4 & 0 \end{pmatrix}, \tag{C.0.5}
$$

where, as already explained above, we have collapsed the first two rows and the first two columns, corresponding to a vanishing block of type (a) in Φ, into a single one.

CY	**Stab(Φ)**	**Modes**	$\dim_{\mathbb{H}} \boldsymbol{HB}$
(A_1, D_4)	$\begin{pmatrix} U(1)_\alpha & & \\ & U(1)_\beta & \\ & & U(1)_\beta \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$	3
(A_1, D_5)	$\begin{pmatrix} U(1) & \\ & \mathbb{Z}_2 \end{pmatrix}$	$\begin{pmatrix} 0 & 2 \\ 2 & 2 \end{pmatrix}$	3
(A_1, D_6)	$\begin{pmatrix} U(1)_\alpha & & \\ & U(1)_\beta & \\ & & U(1)_\beta \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$	4
(A_1, D_7)	$\begin{pmatrix} U(1) & \\ & \mathbb{Z}_2 \end{pmatrix}$	$\begin{pmatrix} 0 & 2 \\ 2 & 4 \end{pmatrix}$	4
(A_1, D_8)	$\begin{pmatrix} U(1)_\alpha & & \\ & U(1)_\beta & \\ & & U(1)_\beta \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & 3 & 0 \end{pmatrix}$	5
(A_1, D_9)	$\begin{pmatrix} U(1) & \\ & \mathbb{Z}_2 \end{pmatrix}$	$\begin{pmatrix} 0 & 2 \\ 2 & 6 \end{pmatrix}$	5

CY	**Stab(Φ)**	**Modes**	$\mathbf{dim}_{\mathbb{H}}\boldsymbol{HB}$
(A_1, D_{10})	$\begin{pmatrix} U(1)_\alpha & & \\ & U(1)_\beta & \\ & & U(1)_\beta \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 4 \\ 1 & 4 & 0 \end{pmatrix}$	6
(A_1, D_{11})	$\begin{pmatrix} U(1) & \\ & \mathbb{Z}_2 \end{pmatrix}$	$\begin{pmatrix} 0 & 2 \\ 2 & 8 \end{pmatrix}$	6
(A_1, D_{12})	$\begin{pmatrix} U(1)_\alpha & & \\ & U(1)_\beta & \\ & & U(1)_\beta \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 5 \\ 1 & 5 & 0 \end{pmatrix}$	7
(A_1, D_{13})	$\begin{pmatrix} U(1) & \\ & \mathbb{Z}_2 \end{pmatrix}$	$\begin{pmatrix} 0 & 2 \\ 2 & 10 \end{pmatrix}$	7
(A_1, D_{14})	$\begin{pmatrix} U(1)_\alpha & & \\ & U(1)_\beta & \\ & & U(1)_\beta \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 6 \\ 1 & 6 & 0 \end{pmatrix}$	8
(A_1, D_{15})	$\begin{pmatrix} U(1) & \\ & \mathbb{Z}_2 \end{pmatrix}$	$\begin{pmatrix} 0 & 2 \\ 2 & 12 \end{pmatrix}$	8

CY	**Stab(Φ)**	**Modes**	$\mathbf{dim}_{\mathbb{H}}\boldsymbol{HB}$
(A_2, D_4)	$\begin{pmatrix} \mathbb{Z}_2 & \\ & \mathbb{Z}_2 \end{pmatrix}$	$\begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}$	4
(A_2, D_5)	$\begin{pmatrix} \mathbb{Z}_2 \end{pmatrix}$	$\begin{pmatrix} 10 \end{pmatrix}$	5
(A_2, D_6)	$\begin{pmatrix} \mathbb{Z}_2 \end{pmatrix}$	$\begin{pmatrix} 12 \end{pmatrix}$	6
(A_2, D_7)	$\begin{pmatrix} \mathbb{Z}_2 & \\ & \mathbb{Z}_2 \end{pmatrix}$	$\begin{pmatrix} 0 & 6 \\ 6 & 2 \end{pmatrix}$	7
(A_2, D_8)	$\begin{pmatrix} \mathbb{Z}_2 \end{pmatrix}$	$\begin{pmatrix} 16 \end{pmatrix}$	8
(A_2, D_9)	$\begin{pmatrix} \mathbb{Z}_2 \end{pmatrix}$	$\begin{pmatrix} 18 \end{pmatrix}$	9
(A_2, D_{10})	$\begin{pmatrix} \mathbb{Z}_2 & \\ & \mathbb{Z}_2 \end{pmatrix}$	$\begin{pmatrix} 0 & 8 \\ 8 & 4 \end{pmatrix}$	10
(A_2, D_{11})	$\begin{pmatrix} \mathbb{Z}_2 \end{pmatrix}$	$\begin{pmatrix} 22 \end{pmatrix}$	11
(A_2, D_{12})	$\begin{pmatrix} \mathbb{Z}_2 \end{pmatrix}$	$\begin{pmatrix} 24 \end{pmatrix}$	12
(A_2, D_{13})	$\begin{pmatrix} \mathbb{Z}_2 & \\ & \mathbb{Z}_2 \end{pmatrix}$	$\begin{pmatrix} 0 & 10 \\ 10 & 6 \end{pmatrix}$	13

CY	Stab(Φ)	Modes	$\dim_{\mathbb{H}} \boldsymbol{HB}$
(A_2, D_{14})	$\begin{pmatrix} \mathbb{Z}_2 \end{pmatrix}$	$\begin{pmatrix} 28 \end{pmatrix}$	14
(A_2, D_{15})	$\begin{pmatrix} \mathbb{Z}_2 \end{pmatrix}$	$\begin{pmatrix} 30 \end{pmatrix}$	15

CY	Stab(Φ)	Modes	$\dim_{\mathbb{H}} \boldsymbol{HB}$
(A_3, D_4)	$\begin{pmatrix} U(1)_\alpha & & \\ & U(1)_\beta & \\ & & U(1)_\beta \end{pmatrix}$	$\begin{pmatrix} 0 & \color{red}{2} & \color{red}{2} \\ \color{red}{2} & 1 & \color{blue}{2} \\ \color{red}{2} & \color{blue}{2} & 1 \end{pmatrix}$	7
(A_3, D_5)	$\begin{pmatrix} U(1) & & \\ & \mathbb{Z}_2 & \\ & & \mathbb{Z}_2 \end{pmatrix}$	$\begin{pmatrix} 0 & \color{red}{2} & \color{red}{2} \\ \color{red}{2} & 0 & \color{green}{4} \\ \color{red}{2} & \color{green}{4} & 0 \end{pmatrix}$	8
(A_3, D_6)	$\begin{pmatrix} U(1)_\alpha & & \\ & U(1)_\beta & \\ & & U(1)_\beta \end{pmatrix}$	$\begin{pmatrix} 0 & \color{red}{2} & \color{red}{2} \\ \color{red}{2} & 2 & \color{blue}{4} \\ \color{red}{2} & \color{blue}{4} & 2 \end{pmatrix}$	10

CY	**Stab(Φ)**	**Modes**	$\mathbf{dim}_{\mathbb{H}}\boldsymbol{HB}$
(A_3, D_7)	$\begin{pmatrix} U(1)_\alpha &&&& \\ & U(1)_\beta &&& \\ && U(1)_\beta && \\ &&& U(1)_\gamma & \\ &&&& U(1)_\gamma \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 3 & 3 \\ 1 & 1 & 0 & 3 & 3 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}$	12
(A_3, D_8)	$\begin{pmatrix} U(1)_\alpha && \\ & U(1)_\beta & \\ && U(1)_\beta \end{pmatrix}$	$\begin{pmatrix} 0 & 2 & 2 \\ 2 & 3 & 6 \\ 2 & 6 & 3 \end{pmatrix}$	13
(A_3, D_9)	$\begin{pmatrix} U(1) && \\ & \mathbb{Z}_2 & \\ && \mathbb{Z}_2 \end{pmatrix}$	$\begin{pmatrix} 0 & 2 & 2 \\ 2 & 2 & 8 \\ 2 & 8 & 2 \end{pmatrix}$	14
(A_3, D_{10})	$\begin{pmatrix} U(1)_\alpha && \\ & U(1)_\beta & \\ && U(1)_\beta \end{pmatrix}$	$\begin{pmatrix} 0 & 2 & 2 \\ 2 & 4 & 8 \\ 2 & 8 & 4 \end{pmatrix}$	16
(A_3, D_{11})	$\begin{pmatrix} U(1)_\alpha &&&& \\ & U(1)_\beta &&& \\ && U(1)_\beta && \\ &&& U(1)_\gamma & \\ &&&& U(1)_\gamma \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 5 & 5 \\ 1 & 2 & 0 & 5 & 5 \\ 1 & 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 2 & 0 \end{pmatrix}$	18

CY	**Stab(Φ)**	**Modes**	$\mathbf{dim}_{\mathbb{H}}\boldsymbol{HB}$
(A_3, D_{12})	$\begin{pmatrix} U(1)_\alpha & & \\ & U(1)_\beta & \\ & & U(1)_\beta \end{pmatrix}$	$\begin{pmatrix} 0 & 2 & 2 \\ 2 & 5 & 10 \\ 2 & 10 & 5 \end{pmatrix}$	19
(A_3, D_{13})	$\begin{pmatrix} U(1) & & \\ & \mathbb{Z}_2 & \\ & & \mathbb{Z}_2 \end{pmatrix}$	$\begin{pmatrix} 0 & 2 & 2 \\ 2 & 4 & 12 \\ 2 & 12 & 4 \end{pmatrix}$	20
(A_3, D_{14})	$\begin{pmatrix} U(1)_\alpha & & \\ & U(1)_\beta & \\ & & U(1)_\beta \end{pmatrix}$	$\begin{pmatrix} 0 & 2 & 2 \\ 2 & 6 & 12 \\ 2 & 12 & 6 \end{pmatrix}$	22
(A_3, D_{15})	$\begin{pmatrix} U(1)_\alpha & & & & \\ & U(1)_\beta & & & \\ & & U(1)_\beta & & \\ & & & U(1)_\gamma & \\ & & & & U(1)_\gamma \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 3 & 7 & 7 \\ 1 & 3 & 0 & 7 & 7 \\ 1 & 0 & 0 & 0 & 3 \\ 1 & 0 & 0 & 3 & 0 \end{pmatrix}$	24

CY	**Stab(Φ)**	**Modes**	$\dim_{\mathbb{H}} \boldsymbol{HB}$
(A_4, D_4)	$\begin{pmatrix} \mathbb{Z}_2 \end{pmatrix}$	$\begin{pmatrix} 16 \end{pmatrix}$	8
(A_4, D_5)	$\begin{pmatrix} \mathbb{Z}_2 \end{pmatrix}$	$\begin{pmatrix} 20 \end{pmatrix}$	10
(A_4, D_6)	$\begin{pmatrix} \mathbb{Z}_2 & & \\ & \mathbb{Z}_2 & \\ & & \mathbb{Z}_2 \end{pmatrix}$	$\begin{pmatrix} 0 & 4 & 4 \\ 4 & 0 & 4 \\ 4 & 4 & 0 \end{pmatrix}$	12
(A_4, D_7)	$\begin{pmatrix} \mathbb{Z}_2 \end{pmatrix}$	$\begin{pmatrix} 28 \end{pmatrix}$	14
(A_4, D_8)	$\begin{pmatrix} \mathbb{Z}_2 \end{pmatrix}$	$\begin{pmatrix} 32 \end{pmatrix}$	16
(A_4, D_9)	$\begin{pmatrix} \mathbb{Z}_2 \end{pmatrix}$	$\begin{pmatrix} 36 \end{pmatrix}$	18
(A_4, D_{10})	$\begin{pmatrix} \mathbb{Z}_2 \end{pmatrix}$	$\begin{pmatrix} 40 \end{pmatrix}$	20
(A_4, D_{11})	$\begin{pmatrix} \mathbb{Z}_2 & & \\ & \mathbb{Z}_2 & \\ & & \mathbb{Z}_2 \end{pmatrix}$	$\begin{pmatrix} 0 & 6 & 6 \\ 6 & 2 & 8 \\ 6 & 8 & 2 \end{pmatrix}$	22
(A_4, D_{12})	$\begin{pmatrix} \mathbb{Z}_2 \end{pmatrix}$	$\begin{pmatrix} 48 \end{pmatrix}$	24

CY	Stab(Φ)	Modes	$\dim_{\mathbb{H}} HB$
(A_4, D_{13})	$\begin{pmatrix} \mathbb{Z}_2 \end{pmatrix}$	$\begin{pmatrix} 52 \end{pmatrix}$	26
(A_4, D_{14})	$\begin{pmatrix} \mathbb{Z}_2 \end{pmatrix}$	$\begin{pmatrix} 56 \end{pmatrix}$	28
(A_4, D_{15})	$\begin{pmatrix} \mathbb{Z}_2 \end{pmatrix}$	$\begin{pmatrix} 60 \end{pmatrix}$	30

CY	Stab(Φ)	Modes	$\dim_{\mathbb{H}} HB$
(A_5, D_4)	$\begin{pmatrix} U(1)_\alpha & & & \\ & U(1)_\beta & & \\ & & U(1)_\beta & \\ & & & \mathbb{Z}_2 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 & 4 \\ 1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 2 \\ 4 & 2 & 2 & 2 \end{pmatrix}$	11
(A_5, D_5)	$\begin{pmatrix} U(1) & \\ & \mathbb{Z}_2 \end{pmatrix}$	$\begin{pmatrix} 0 & 6 \\ 6 & 14 \end{pmatrix}$	13
(A_5, D_6)	$\begin{pmatrix} U(1)_\alpha & & \\ & U(1)_\beta & \\ & & U(1)_\beta \end{pmatrix}$	$\begin{pmatrix} 0 & 3 & 3 \\ 3 & 4 & 6 \\ 3 & 6 & 4 \end{pmatrix}$	16

CY	**Stab(Φ)**	**Modes**	$\mathbf{dim}_{\mathbb{H}}\boldsymbol{HB}$
(A_5, D_7)	$\begin{pmatrix} U(1)_\alpha & & & \\ & \mathbb{Z}_2 & & \\ & & \mathbb{Z}_2 & \\ & & & \mathbb{Z}_2 \end{pmatrix}$	$\begin{pmatrix} 0 & 2 & 2 & 2 \\ 2 & 0 & 4 & 4 \\ 2 & 4 & 0 & 4 \\ 2 & 4 & 4 & 0 \end{pmatrix}$	18
(A_5, D_8)	$\begin{pmatrix} U(1)_\alpha & & \\ & U(1)_\beta & \\ & & U(1)_\beta \end{pmatrix}$	$\begin{pmatrix} 0 & 3 & 3 \\ 3 & 6 & 9 \\ 3 & 9 & 6 \end{pmatrix}$	21
(A_5, D_9)	$\begin{pmatrix} U(1) & \\ & \mathbb{Z}_2 \end{pmatrix}$	$\begin{pmatrix} 0 & 6 \\ 6 & 34 \end{pmatrix}$	23
(A_5, D_{10})	$\begin{pmatrix} U(1)_\alpha & & & \\ & U(1)_\beta & & \\ & & U(1)_\beta & \\ & & & \mathbb{Z}_2 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 & 4 \\ 1 & 0 & 1 & 6 \\ 1 & 1 & 0 & 6 \\ 4 & 6 & 6 & 14 \end{pmatrix}$	26
(A_5, D_{11})	$\begin{pmatrix} U(1) & \\ & \mathbb{Z}_2 \end{pmatrix}$	$\begin{pmatrix} 0 & 6 \\ 6 & 44 \end{pmatrix}$	28
(A_5, D_{12})	$\begin{pmatrix} U(1)_\alpha & & \\ & U(1)_\beta & \\ & & U(1)_\beta \end{pmatrix}$	$\begin{pmatrix} 0 & 3 & 3 \\ 3 & 10 & 15 \\ 3 & 15 & 10 \end{pmatrix}$	31

CY	Stab(Φ)	Modes	$\dim_{\mathbb{H}} \boldsymbol{HB}$
(A_5, D_{13})	$\begin{pmatrix} U(1) & & & \\ & \mathbb{Z}_2 & & \\ & & \mathbb{Z}_2 & \\ & & & \mathbb{Z}_2 \end{pmatrix}$	$\begin{pmatrix} 0 & 2 & 2 & 2 \\ 2 & 2 & 8 & 8 \\ 2 & 8 & 2 & 8 \\ 2 & 8 & 8 & 2 \end{pmatrix}$	33
(A_5, D_{14})	$\begin{pmatrix} U(1)_\alpha & & \\ & U(1)_\beta & \\ & & U(1)_\beta \end{pmatrix}$	$\begin{pmatrix} 0 & 3 & 3 \\ 3 & 12 & 18 \\ 3 & 18 & 12 \end{pmatrix}$	36
(A_5, D_{15})	$\begin{pmatrix} U(1) & \\ & \mathbb{Z}_2 \end{pmatrix}$	$\begin{pmatrix} 0 & 6 \\ 6 & 64 \end{pmatrix}$	38

CY	Stab(Φ)	Modes	$\dim_{\mathbb{H}} \boldsymbol{HB}$
(A_6, D_4)	$\begin{pmatrix} \mathbb{Z}_2 \end{pmatrix}$	$\begin{pmatrix} 24 \end{pmatrix}$	12
(A_6, D_5)	$\begin{pmatrix} \mathbb{Z}_2 \end{pmatrix}$	$\begin{pmatrix} 30 \end{pmatrix}$	15
(A_6, D_6)	$\begin{pmatrix} \mathbb{Z}_2 \end{pmatrix}$	$\begin{pmatrix} 36 \end{pmatrix}$	18
(A_6, D_7)	$\begin{pmatrix} \mathbb{Z}_2 \end{pmatrix}$	$\begin{pmatrix} 42 \end{pmatrix}$	21

CY	Stab(Φ)	Modes	$\dim_{\mathbb{H}} \boldsymbol{HB}$
(A_6, D_8)	$\begin{pmatrix} \mathbb{Z}_2 & & & \\ & \mathbb{Z}_2 & & \\ & & \mathbb{Z}_2 & \\ & & & \mathbb{Z}_2 \end{pmatrix}$	$\begin{pmatrix} 0 & 4 & 4 & 4 \\ 4 & 0 & 4 & 4 \\ 4 & 4 & 0 & 4 \\ 4 & 4 & 4 & 0 \end{pmatrix}$	24
(A_6, D_9)	$\begin{pmatrix} \mathbb{Z}_2 \end{pmatrix}$	$\begin{pmatrix} 54 \end{pmatrix}$	27
(A_6, D_{10})	$\begin{pmatrix} \mathbb{Z}_2 \end{pmatrix}$	$\begin{pmatrix} 60 \end{pmatrix}$	30
(A_6, D_{11})	$\begin{pmatrix} \mathbb{Z}_2 \end{pmatrix}$	$\begin{pmatrix} 66 \end{pmatrix}$	33
(A_6, D_{12})	$\begin{pmatrix} \mathbb{Z}_2 \end{pmatrix}$	$\begin{pmatrix} 72 \end{pmatrix}$	36
(A_6, D_{13})	$\begin{pmatrix} \mathbb{Z}_2 \end{pmatrix}$	$\begin{pmatrix} 78 \end{pmatrix}$	39
(A_6, D_{14})	$\begin{pmatrix} \mathbb{Z}_2 \end{pmatrix}$	$\begin{pmatrix} 84 \end{pmatrix}$	42
(A_6, D_{15})	$\begin{pmatrix} \mathbb{Z}_2 & & & \\ & \mathbb{Z}_2 & & \\ & & \mathbb{Z}_2 & \\ & & & \mathbb{Z}_2 \end{pmatrix}$	$\begin{pmatrix} 0 & 6 & 6 & 6 \\ 6 & 2 & 8 & 8 \\ 6 & 8 & 2 & 8 \\ 6 & 8 & 8 & 2 \end{pmatrix}$	45

CY	Stab(Φ)	Modes	$\dim_{\mathbb{H}} HB$
(A_7, D_4)	$\begin{pmatrix} U(1)_\alpha & & \\ & U(1)_\beta & \\ & & U(1)_\beta \end{pmatrix}$	$\begin{pmatrix} 0 & 4 & 4 \\ 4 & 3 & 4 \\ 4 & 4 & 3 \end{pmatrix}$	15
(A_7, D_5)	$\begin{pmatrix} U(1)_\alpha &&&&&&&& \\ & U(1)_\beta &&&&&&& \\ && U(1)_\beta &&&&&& \\ &&& U(1)_\gamma &&&&& \\ &&&& U(1)_\gamma &&&& \\ &&&&& U(1)_\delta &&& \\ &&&&&& U(1)_\delta && \\ &&&&&&& U(1)_\epsilon & \\ &&&&&&&& U(1)_\epsilon \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	20
(A_7, D_6)	$\begin{pmatrix} U(1)_\alpha & & \\ & U(1)_\beta & \\ & & U(1)_\beta \end{pmatrix}$	$\begin{pmatrix} 0 & 4 & 4 \\ 4 & 6 & 8 \\ 4 & 8 & 6 \end{pmatrix}$	22
(A_7, D_7)	$\begin{pmatrix} U(1)_\alpha &&&& \\ & U(1)_\beta &&& \\ && U(1)_\beta && \\ &&& U(1)_\gamma & \\ &&&& U(1)_\gamma \end{pmatrix}$	$\begin{pmatrix} 0 & 2 & 2 & 2 & 2 \\ 2 & 1 & 2 & 3 & 3 \\ 2 & 2 & 1 & 3 & 3 \\ 2 & 3 & 3 & 1 & 2 \\ 2 & 3 & 3 & 2 & 1 \end{pmatrix}$	26
(A_7, D_8)	$\begin{pmatrix} U(1)_\alpha & & \\ & U(1)_\beta & \\ & & U(1)_\beta \end{pmatrix}$	$\begin{pmatrix} 0 & 4 & 4 \\ 4 & 9 & 12 \\ 4 & 12 & 9 \end{pmatrix}$	29

CY	**Stab(Φ)**	**Modes**	$\dim_{\mathbb{H}} \boldsymbol{HB}$
(A_7, D_9)	$\begin{pmatrix} U(1)_\alpha & & & & \\ & \mathbb{Z}_2 & & & \\ & & \mathbb{Z}_2 & & \\ & & & \mathbb{Z}_2 & \\ & & & & \mathbb{Z}_2 \end{pmatrix}$	$\begin{pmatrix} 0 & 2 & 2 & 2 & 2 \\ 2 & 0 & 4 & 4 & 4 \\ 2 & 4 & 0 & 4 & 4 \\ 2 & 4 & 4 & 0 & 4 \\ 2 & 4 & 4 & 4 & 0 \end{pmatrix}$	32
(A_7, D_{10})	$\begin{pmatrix} U(1)_\alpha & & \\ & U(1)_\beta & \\ & & U(1)_\beta \end{pmatrix}$	$\begin{pmatrix} 0 & 4 & 4 \\ 4 & 12 & 16 \\ 4 & 16 & 12 \end{pmatrix}$	36
(A_7, D_{11})	$\begin{pmatrix} U(1)_\alpha & & & & \\ & U(1)_\beta & & & \\ & & U(1)_\beta & & \\ & & & U(1)_\gamma & \\ & & & & U(1)_\gamma \end{pmatrix}$	$\begin{pmatrix} 0 & 2 & 2 & 2 & 2 \\ 2 & 2 & 4 & 5 & 5 \\ 2 & 4 & 2 & 5 & 5 \\ 2 & 5 & 5 & 2 & 4 \\ 2 & 5 & 5 & 4 & 2 \end{pmatrix}$	40
(A_7, D_{12})	$\begin{pmatrix} U(1)_\alpha & & \\ & U(1)_\beta & \\ & & U(1)_\beta \end{pmatrix}$	$\begin{pmatrix} 0 & 4 & 4 \\ 4 & 15 & 20 \\ 4 & 20 & 15 \end{pmatrix}$	43
(A_7, D_{13})	$\begin{pmatrix} U(1)_\alpha & & & & & & & & \\ & U(1)_\beta & & & & & & & \\ & & U(1)_\beta & & & & & & \\ & & & U(1)_\gamma & & & & & \\ & & & & U(1)_\gamma & & & & \\ & & & & & U(1)_\delta & & & \\ & & & & & & U(1)_\delta & & \\ & & & & & & & U(1)_\epsilon & \\ & & & & & & & & U(1)_\epsilon \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 3 & 3 & 3 & 3 & 3 & 3 \\ 1 & 1 & 0 & 3 & 3 & 3 & 3 & 3 & 3 \\ 1 & 0 & 0 & 0 & 1 & 3 & 3 & 3 & 3 \\ 1 & 0 & 0 & 1 & 0 & 3 & 3 & 3 & 3 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 3 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 3 & 3 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$	48

CY	**Stab(Φ)**	**Modes**	**dim$_{\mathbb{H}}$*HB***
(A_7, D_{14})	$\begin{pmatrix} U(1)_\alpha & & \\ & U(1)_\beta & \\ & & U(1)_\beta \end{pmatrix}$	$\begin{pmatrix} 0 & 4 & 4 \\ 4 & 18 & 24 \\ 4 & 24 & 18 \end{pmatrix}$	50
(A_7, D_{15})	$\begin{pmatrix} U(1)_\alpha & & & & \\ & U(1)_\beta & & & \\ & & U(1)_\beta & & \\ & & & U(1)_\gamma & \\ & & & & U(1)_\gamma \end{pmatrix}$	$\begin{pmatrix} 0 & 2 & 2 & 2 & 2 \\ 2 & 3 & 6 & 7 & 7 \\ 2 & 6 & 3 & 7 & 7 \\ 2 & 7 & 7 & 3 & 6 \\ 2 & 7 & 7 & 6 & 3 \end{pmatrix}$	54

CY	**Stab(Φ)**	**Modes**	**dim$_{\mathbb{H}}$*HB***
(A_8, D_4)	$\begin{pmatrix} \mathbb{Z}_2 & \\ & \mathbb{Z}_2 \end{pmatrix}$	$\begin{pmatrix} 4 & 12 \\ 12 & 4 \end{pmatrix}$	16
(A_8, D_5)	$\begin{pmatrix} \mathbb{Z}_2 \end{pmatrix}$	$\begin{pmatrix} 40 \end{pmatrix}$	20
(A_8, D_6)	$\begin{pmatrix} \mathbb{Z}_2 \end{pmatrix}$	$\begin{pmatrix} 48 \end{pmatrix}$	24
(A_8, D_7)	$\begin{pmatrix} \mathbb{Z}_2 & \\ & \mathbb{Z}_2 \end{pmatrix}$	$\begin{pmatrix} 6 & 18 \\ 18 & 14 \end{pmatrix}$	28

CY	**Stab(Φ)**	**Modes**	$\dim_{\mathbb{H}} \boldsymbol{HB}$
(A_8, D_8)	$\begin{pmatrix} \mathbb{Z}_2 \end{pmatrix}$	$\begin{pmatrix} 64 \end{pmatrix}$	32
(A_8, D_9)	$\begin{pmatrix} \mathbb{Z}_2 \end{pmatrix}$	$\begin{pmatrix} 72 \end{pmatrix}$	36
(A_8, D_{10})	$\begin{pmatrix} \mathbb{Z}_2 & & & & \\ & \mathbb{Z}_2 & & & \\ & & \mathbb{Z}_2 & & \\ & & & \mathbb{Z}_2 & \\ & & & & \mathbb{Z}_2 \end{pmatrix}$	$\begin{pmatrix} 0 & 4 & 4 & 4 & 4 \\ 4 & 0 & 4 & 4 & 4 \\ 4 & 4 & 0 & 4 & 4 \\ 4 & 4 & 4 & 0 & 4 \\ 4 & 4 & 4 & 4 & 0 \end{pmatrix}$	40
(A_8, D_{11})	$\begin{pmatrix} \mathbb{Z}_2 \end{pmatrix}$	$\begin{pmatrix} 88 \end{pmatrix}$	44
(A_8, D_{12})	$\begin{pmatrix} \mathbb{Z}_2 \end{pmatrix}$	$\begin{pmatrix} 96 \end{pmatrix}$	48
(A_8, D_{13})	$\begin{pmatrix} \mathbb{Z}_2 & \\ & \mathbb{Z}_2 \end{pmatrix}$	$\begin{pmatrix} 10 & 30 \\ 30 & 34 \end{pmatrix}$	52
(A_8, D_{14})	$\begin{pmatrix} \mathbb{Z}_2 \end{pmatrix}$	$\begin{pmatrix} 112 \end{pmatrix}$	56
(A_8, D_{15})	$\begin{pmatrix} \mathbb{Z}_2 \end{pmatrix}$	$\begin{pmatrix} 120 \end{pmatrix}$	60

Appendix D

Explicit expressions of blocks in (A_{N-1}, D_k) singularities

In this appendix, we give a schematic account of the blocks living in a subalgebra of $\mathfrak{so}(2n)$ yielding the polynomials appearing in Table 6.5 in the brane locus ($*$ entries representing either a constant or a term linear in w). Unless explicitly stated, we employ the basis (A.1.9) for $\mathfrak{so}$(size of block).

- $\boldsymbol{P_{(a)}(\xi^2, w) = (\xi^{2r+1} + c_1 w^t)\,(\xi^{2r+1} - c_1 w^t)}$

  The blocks $\mathcal{B}_{(a)}$ such that its characteristic polynomial (indicated with "χ") satisfies $\chi(\mathcal{B}_{(a)}) = P_{(a)}(\xi^2, w)$ are of the form:

$$\mathcal{B}_{(a)} = \begin{pmatrix} \mathcal{A}_{(2r+1)\times(2r+1)} & 0 \\ 0 & -\mathcal{A}^t_{(2r+1)\times(2r+1)} \end{pmatrix}, \tag{D.0.1}$$

  where $\mathcal{A}_{(2r+1)\times(2r+1)}$ has the form:

$$\mathcal{A}_{(2r+1)\times(2r+1)} = \begin{pmatrix} 0 & * & 0 & \cdots & \cdots & 0 \\ \vdots & 0 & * & 0 & & \vdots \\ \vdots & & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \vdots \\ \vdots & & & & 0 & * \\ * & 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}. \tag{D.0.2}$$

  Notice that the form (D.0.2) is analogous to the one of the reconstructible blocks (A.1.5) in the (A_j, A_l) cases (with only the highest degree σ turned on), as indeed (D.0.1) belongs to a $\mathfrak{su}(2r+1)$ subalgebra of $\mathfrak{so}(4r+2)$.

- $\boldsymbol{P_{(b)}(\xi^2, w) = \xi^2\,(\xi^{2r} + c_2 w^{2t+1})}$

The blocks $\mathcal{B}_{(b)}$ are of the form:

$$
\mathcal{B}_{(b)} = \left(\begin{array}{cccccc|cccccc}
0 & & & & & & & & & & & * \\
 & \ddots & & & & & & & & & * & \\
 & & 0 & {\color{red}*} & & & & & & \iddots & & \\
 & & & 0 & & & & & \iddots & & & \\
 & & & & \ddots & & & * & & & & \\
* & & & & & 0 & * & & & & & \\
\hline
 & & & & & 0 & 0 & & & & & * \\
 & & & & 0 & * & & \ddots & & & & \\
 & & & \iddots & * & & & & 0 & & & \\
 & & \iddots & \iddots & & & & & {\color{red}*} & 0 & & \\
 & 0 & * & & & & & & & & \ddots & \\
0 & * & & & & & & & & & & 0
\end{array}\right),
$$

where the ${\color{red}*}$ on the over-diagonal of the upper diagonal block appears on the $\frac{(r+1)}{2}^{\text{th}}$ row if r is odd and on the $\frac{r}{2}^{\text{th}}$ row if r is even. The other ${\color{red}*}$ on the lower diagonal block appears in the corresponding entry according to the basis (A.1.9).

- $\boldsymbol{P_{(c)}(\xi^2, w) = \xi^{4r} + c_3 w^{2t+1} \xi^{2r} + c_4 w^{2(2t+1)}}$, $(r, 2t+1)$ coprime

  The blocks $\mathcal{B}_{(c)}$ are of the form:

$$
\mathcal{B}_{(c)} = \left(\begin{array}{cccccc|cccccc}
0 & * & & & & & & & & & & * \\
 & 0 & * & & & & & & & & 0 & \\
 & & \ddots & \ddots & & & & & & \iddots & & \\
 & & & \ddots & * & & & & \iddots & & & \\
 & & & & 0 & * & & 0 & & & & \\
* & & & & & 0 & * & & & & & \\
\hline
0 & * & & & & & 0 & & & & & * \\
* & & & & & & * & 0 & & & & \\
 & & & & & & & * & \ddots & & & \\
 & & & & & & & & \ddots & \ddots & & \\
 & & & & & & & & & * & 0 & \\
 & & & & & & & & & & * & 0
\end{array}\right).
$$

Appendix E

Slodowy slices and nilpotent orbits

We saw that we can always associate the Higgs field Φ to a certain subalgebra $\mathcal{M} = \bigoplus_h \mathcal{M}_h \oplus \mathcal{H}$, defined as the minimal subalgebra of $\mathcal{G}$ containing Φ. As a further datum defining the Higgs, we have to pick, for each $\mathcal{M}_h$, a nilpotent orbit, realized as $\Phi|_{\mathcal{M}_h}(w = 0)$. In this Appendix we show how to do it for X quasi-homogeneous cDV.

To pick the right nilpotent orbit at fixed $\mathcal{M}_h$ we can proceed as follows. First, we can compute the quasi-homogeneous weights of the coordinates of all the Slodowy slices associated to the nilpotent orbits of $\mathcal{M}_h$. Comparing them with the expression of the $\mathcal{W}_{\mathcal{M}}$-invariants ϱ^h_j, we exclude many Slodowy slices that can not host a holomorphic $\Phi_h(w)$ due to the quasi-homogeneous scaling. Then, we pick $\Phi_h(w)$ along the Slodowy slice, among the remaining ones, associated to the nilpotent orbit of largest codimension. As we saw in Section 7.1.2, this choice will maximize the number of five-dimensional modes. We already mentioned, in Remark 2.2.5, the fact that, in some cases, the Higgs field might be along the Slodowy slice of a nilpotent orbit $\mathcal{O} < O_{\text{reg}}$. In this Appendix we give a method to quickly construct these representatives, in the simplified setup of Higgs fields associated to quasi-homogeneous cDV.

For quasihomogeneous cDV, we need to switch on only selected Casimirs in the addends $\Phi_h(w) \equiv \Phi(w)|_{\mathcal{M}_h}$ and we can explicitly state a "canonical" choice of the Slodowy slice element in $\mathcal{M}_h$ that we are turning on. Let us immediately give the recipe for the addends in $\mathcal{M}_h = A_n$, for some n. In these cases, in Chapter 6 we *always* need to turn on the top degree Casimirs, and nothing else. Hence, we can pick as canonical form for $\Phi_h(w)$ the following element (its shape can be gleaned from the form of the Slodowy slice through the principal nilpotent orbit of $\mathcal{M}_h$, with only the top Casimir switched on):

$$\Phi_h(w) = c_1 e_{\alpha_1} + c_2 e_{\alpha_2} + \ldots + c_n e_{\alpha_n} + c_{n+1} e_{-\alpha_1-\alpha_2-\ldots-\alpha_n}, \tag{E.0.1}$$

where the c_i, $i = 1, \ldots, n+1$ can either be constant or depend on w (though not all of them can be constant, otherwise we would realize a non-nilpotent $\Phi_h(0)$).

The form (E.0.1) yields a non-vanishing top degree Casimir $\rho_{\text{top}} = \prod_{i=1}^{n+1} c_i$, and allows $\Phi_h(0)$ to belong to any nilpotent orbit in the A algebra, by a careful choice of the coefficients[1].

A similar reasoning works for the $\mathcal{M}_h = D$ and the $\mathcal{M}_h = E$ cases, in which it is sufficient, for the purposes of the quasi-homogeneous singularities of Chapter 6, to turn on only some of the possible Casimirs. More precisely, in the D cases we might need either the top-degree Casimir, or the Casimirs having the same degree as the Pfaffian (for the definition of the Casimirs, we refer to Table 2.2.20). To turn on only these Casimirs, we can construct a "canonical" $\Phi_h(w)$ in a fashion similar to the A cases: the only difference is that in general we have a choice between *two* such canonical forms, one inspired by the Slodowy slice through the principal nilpotent orbit, and the other along the subregular nilpotent orbit[2]. An analogous story goes for the E_n cases, in which we can pick as many canonical forms as the number of orbits with the "E_n" label (displayed in Tables in [159]).

The choice of the nilpotent orbit where $\Phi(0)$ lies strongly influences the physics of the underlying 5d theory. We have a criterium directly coming from the analysis of Section 7.1.2.

The Higgs field localizing the maximal amount of 5d modes satisfies:

$$\Phi(w)|_{w=0} = \Phi(0) \in \mathcal{O}^{low},$$

with $\mathcal{O}^{low}$ the nilpotent orbit of lowest dimension (that is, biggest codimension) allowed by the compatibility with the threefold equation.

This is equivalent to requiring that every addend $\Phi_h(0)$ lies in the smallest allowed nilpotent orbit of the corresponding subalgebra, compatibly with the threefold equation.

Let us give a trivial example. Given the (A_2, A_4) cDV singularity, we construct the Higgs background using the canonical form in (E.0.1). We could have (among

[1] Namely, recalling that nilpotent orbits of the A_n algebras are in correspondence with the allowed Jordan forms in a matrix representation of sl_{n+1}, we can set some of the c_i to 1 and the rest to $\tilde{c}_i w$, with $\tilde{c}_i$ a constant, obtaining any desired Jordan form.

[2] This happens because in the D cases not all nilpotent orbits can be obtained from the principal nilpotent orbit by removing some algebra elements.

other choices) two different Higgs backgrounds with linear coefficients in w:

$$\Phi_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & w & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & w \\ w & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \Phi_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & w & 0 \\ 0 & 0 & 0 & 0 & w \\ w & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{E.0.2}$$

The Higgs localizing the maximal amount modes is Φ_1, because:

$$\Phi_1(0) \in \mathcal{O}_{[2,2,1]}, \qquad \Phi_2(0) \in \mathcal{O}_{[3,1,1]}, \tag{E.0.3}$$

and the orbit[3] $\mathcal{O}_{[2,2,1]}$ has a bigger codimension than $\mathcal{O}_{[3,1,1]}$.

Thus, recalling the form (E.0.1) and its top Casimir $\rho_{\text{top}} = \prod_{i=1}^{n+1} c_i$, we can lay down the following general recipe to promptly construct the Higgs background $\Phi_h(w) \in \mathcal{M}_h = A_n$: if we require $\rho_{\text{top}} = w^k$, with $k < n$, the corresponding $\Phi_h(w)$ has the shape (E.0.1), with k parameters c_i equal to w, and the rest equal to 1. The 1's are distributed in such a way that $\Phi_h(0)$ lies in the nilpotent orbit labelled by a partition of k parts $[d_1, \ldots, d_k]$ with the largest codimension among the allowed ones.

[3] We have used the conventions labelling nilpotent orbits of [159].

Appendix F

Mathematica code for computing the zero modes

In this section we are going to describe the ancillary Mathematica code that we used to analyze the Higgs branches of M-theory on the quasi-homogeneous cDV. The code can be found on the same *arXiv* page of our paper [102]. On the *arXiv* page, the reader can find a zipped folder, containing, together with the Mathematica notebook code "CodeHiggsBranchDatav2.nb", nine text files. The text files have to be placed in one of the folders of the variable `$Paths` of Mathematica and contain the explicit matrix realization of the exceptional Lie algebras[1].

The notebook file is divided into two sections. The first section ("Main Code") contains the functions that extract the Higgs branch data from the Higgs field Φ. The second section "Examples" contains various examples where we show how to use the routines contained in the section "Main Code". The Mathematica code can be used also to analyze singularities that *are not* quasi-homogeneous.

HbData function The most important function contained in the notebook is

```
HbData[ADE, rank, simsrts, listhiggs, coeffhiggs, cartanhiggs, coeffcartan].
```

The arguments of the function are

- `ADE`: is a `Symbol` to be picked among `"A, DD, E6, E7, E8"` and specifies the type of ADE algebra associated with the threefold.
- `rank`: is a positive `Integer` that specifies the rank of the ADE algebra associated with the threefold.
- `simsrts`: is a `List` of `Lists`. Each sublist represents a root of the Lie algebra specified by `ADE` and `rank`. *The roots contained in* `simsrts` *are generated by*

[1] We took the explicit matrix realization of the exceptional Lie algebras from [183–185].

the simple roots of the subalgebra $\mathcal{M}$ where Φ resides.[2]
The roots are described by their integer coefficients with respect to the basis of the simple roots of $\mathcal{G}$. We labelled the simple roots as in the Figure F.1.

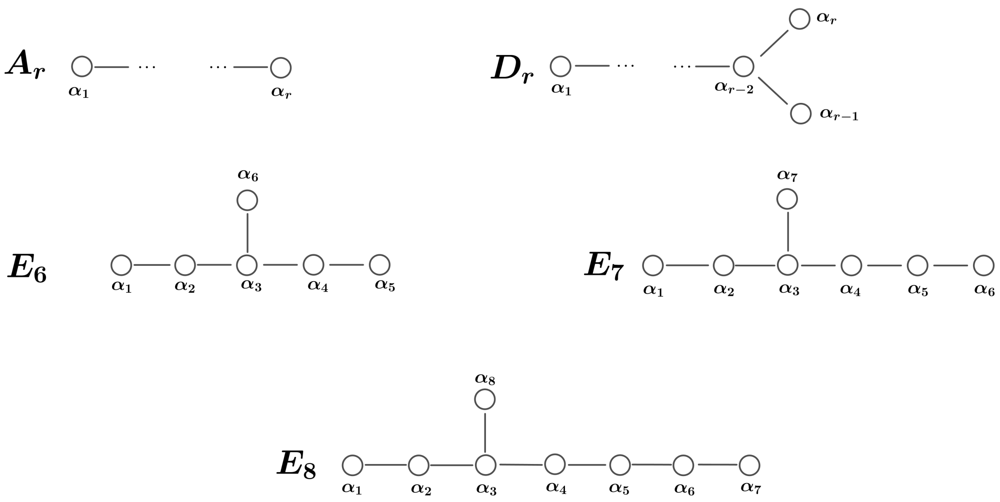


Figure F.1: *Roots labelling convention*

For example, the lowest root of the D_4 Lie algebra is expressed as

$$\{-1, -2, -1, -1\} .$$

Concretely, considering the Higgs field in (6.3.4) as an example, we see that it lies in the subalgebra $\mathcal{M} = A_1^4$ of D_4. This subalgebra is generated by the three outer roots of the D_4 diagram and by the lowest root of D_4. In our notation, the corresponding input is

$$\texttt{simsrts} = \{\{1,0,0,0\},\{0,0,1,0\},\{0,0,0,1\},-\{1,2,1,1\}\} . \quad \text{(F.0.1)}$$

The user can print on screen the roots system of $\mathcal{G}$ calling the function

`PrintRootSystem[ADE,rank]`,

the first argument being again the ADE type of $\mathcal{G}$, and the second argument its rank.
The highest root of the root system can be obtained calling the function

`PrintHighestRoot[ADE,rank]`.

We will explain below how to prompt, using the function `PrintMatrix`, the

[2]They give the Dynkin diagram of the subalgebra $\mathcal{M}$.

explicit matrices representing, in the adjoint representation, the root vectors associated with the roots (as well as to the elements of the basis of the Cartan subalgebra of $\mathcal{G}$).

- `listhiggs`: is a `List` of `Lists`. Each sublist represents a root such that the Higgs field has a non-zero coefficient along the corresponding root-vector in $\mathcal{G}$. We input in this way all the components of the Higgs that *do not* lie in the Cartan subalgebra; the elements in the Cartan subalgebra will be separately input with the variables `cartanhiggs` and `coeffcartan`. For the (A_2, D_4) Higgs field (6.3.4) that we are taking as example, the variable `listhiggs` is

$$\begin{aligned}\texttt{listhiggs} = \quad & \Big\{ \{1,0,0,0\}, \{0,0,1,0\}, \{0,0,0,1\}, -\{1,2,1,1\}, \\ & -\{1,0,0,0\}, -\{0,0,1,0\}, -\{0,0,0,1\}, \{1,2,1,1\} \Big\}.\end{aligned} \tag{F.0.2}$$

- `coeffhiggs`: is a `List` containing the coefficients corresponding to the elements of `listhiggs`. If we again consider the Higgs field of (6.3.4) we have

$$\texttt{coeffhiggs} = \{1,1,1,1,c_1 w, c_2 w, c_3 w, c_4 w\}, \tag{F.0.3}$$

where we lowered the index h of the coefficients c^h appearing in (6.3.4) for clarity of notation.

- `cartanhiggs`: is a `List` of positive `Integers` n_i, with $n_i = 1$, ..., `rank`, describing the elements of the Cartan subalgebra of $\mathcal{G}$ along which the Higgs field has a non-zero coefficient. We chose the generators of the Cartan subalgebra to be the dual elements α_j^* of the simple roots. For example, let's consider the (A_{11}, E_6) singularity. We saw in table 6.9 that its crepant resolution simultaneously resolves all the nodes of the E_6 Dynkin diagram. In terms of the Higgs fields, this means that Φ has to lie in the Cartan subalgebra of E_6. Inside the Cartan subalgebra, the Higgs field associated with (A_{11}, E_6) has non-zero component along all the α_i^*, with $i = 1, ..., 6$. In order to pick a Higgs field with a non-zero component along all the α_i^* we input

$$\texttt{cartanhiggs} = \{1,2,3,4,5,6\}. \tag{F.0.4}$$

If we initialize the variable `cartanhiggs` as in (F.0.4), we get an Higgs field with a (possibly) non-zero component along all the α_i^*. We note that, however, `cartanhiggs` does not specify the precise value of the coefficients. The coefficients will be specified in the variable `coeffcartan`.

- `coeffcartan`: is the `List` of the coefficients corresponding to the elements of `cartanhiggs`. For the (A_{11}, E_6) example, if we input

$$\begin{aligned}\texttt{cartanhiggs} &= \{1,2,3,4,5,6\}\,,\\ \texttt{coeffcartan} &= \{wt_1, wt_2, wt_3, wt_4, wt_5, wt_6\}\end{aligned}$$

  we picked the Higgs to have a coefficient wt_i along the corresponding α_i^*.

Output: The function HbData has a void output and prints all the data that describe the action of the flavor symmetries and discrete gauging symmetries on the five-dimensional hypers. This permits to reconstruct the Higgs branch as complex algebraic variety and the action of the flavor isometries on the Higgs branch.

We remark here that the user can print on the screen the explicit matrix (in the adjoint representation) associated with a certain value of the variables `listhiggs, coeffhiggs, cartanhiggs`, `coeffcartan` using

```
PrintMatrix[ADE,rank,listhiggs,coeffhiggs,cartanhiggs,coeffcartan].
```

For example, if we want to visualize the matrix associated with the generator corresponding to the root $\{0,1,0,0\}$ of D_4, we will input

```
PrintMatrix[DD,4,{{0,1,0,0}},{1},{},{}].
```

This permits the user to read off the explicit normalization we used for the generators of the Lie algebra[3].

Summing up, to obtain the Higgs branch data of the Higgs field associated with the (A_2, D_4) singularity, we will input the following data:

- `ADE = DD`;
- `rank = 4`;
- `simsrts`: The subalgebra $\mathcal{M}$ containing the Higgs field is $A_1^4 \subset D_4$. The corresponding simple roots are e_{α_1}, e_{α_3}, e_{α_4} and the lowest root of D_4 (see figure 4.1). Consequently, we input (F.0.1).
- Given (6.3.4), recalling that α^h, with $h = 4$ is the lowest root of the D_4 Dynkin diagram, we input (F.0.2), (F.0.3) as, respectively, `listhiggs` and `coeffhiggs`.
- The subalgebra $\mathcal{M} = A_1^4$ has no $\mathfrak{u}(1)$ factors. Consequently, the Higgs field can not have non-zero coefficients along the Cartan elements α_i^*, with $i = 1, \ldots, 4$,

[3] In particular, for classical Lie algebra, we followed the convention of [159].

dual to the simple roots of D_4. Then, we input[4]

$$\texttt{cartanhiggs} = \{\}\,, \quad \texttt{coeffcartan} = \{\}\,.$$

We report here a part of the output for the (A_2, D_4) case[5]. The first part of the output is

```
The threefold can not be crepantly resolved. The flavor group is trivial.
The discrete group is the direct product of the following factors: {ℤ2}.
```

The first line is telling us that the considered threefold does not admit any small crepant resolution. The second line is telling us that the discrete gauging group is non-trivial. The discrete gauging group is the direct product of the factors appearing between curly brackets in the second line of the output. In this case, we have only one such factor, and the discrete gauging group is isomorphic to $\mathbb{Z}_2$.

The second part of the output consists of many blocks (one for each irreducible representation of the branching of $\mathcal{G} = D_4$ with respect to the $A_1^4 \subset D_4$ subalgebra) of the following type:

```
N. of five-dimensional modes (at each ℂ[w]/(w^k) level, k = 1,2,...): {4, 2, 0}.
Total number of five-dimensional (complex) modes in the considered irrep.: 8.
Dimension of the irrep.: 16.
Irrep. Dynkin indices: {1, 1, 1, 1}.
Flavor charges: {}.
Discrete action phases: {-1}.
```

From the first three lines we can reconstruct the number of five-dimensional modes localized in the considered irreducible representation. In the first line, we read the `List` $\{4, 2, 0\}$, this means that we have four modes localized in $\mathbb{C}[w]/(w)$, two modes localized in $\mathbb{C}[w]/(w^2)$ and zero in $\mathbb{C}[w]/(w^k)$ with $k > 2$. The overall number of complex-valued modes is, hence, $4 * 1 + 2 * 2 = 8$. The last four lines tell us, respectively:

- *The complex dimension of the considered irreducible representation.* In the example, it is 16.

[4] We can also choose `cartanhiggs` as a non-void list and set to zero the corresponding coefficients inside `coeffcartan`. For example, we can input

$$\texttt{cartanhiggs} = \{1, 3, 4\}\,, \quad \texttt{coeffcartan} = \{0, 0, 0\}\,.$$

[5] For the full output please check the subsection "A2D4" inside the section “Examples” of the ancillary Mathematica file.

- *The Dynkin indices of the highest weight state of the representation.* In the example, we read

$$\{1, 1, 1, 1\} . \tag{F.0.5}$$

Each of the numbers appearing in (F.0.5) tells us the weight of the highest state of the considered irreducible representation with respect to the roots contained in `simsrts`. In other words, the first number of (F.0.5) is the Dynkin index of the highest weight state with respect to the first root appearing in `simsrts` (in this case $e_{\alpha_1} = \{1, 0, 0, 0\}$) and so on. These data permit us to completely reconstruct the representation: in this case, (F.0.5) tells us that we are considering the tensor product of all the fundamental representations of the four A_1 factors (that has dimension $2^4 = 16$).

- *The charges, with respect to the flavor group generators, of the modes localized in the representation.* The generators of the flavor group are the Cartan elements α_i^* that are dual to the roots that get resolved. In this case, the flavor group is trivial (since no $\mathbb{P}^1$ can be simultaneously resolved) and the list is void.

- *The action of the discrete gauging group on the considered irreducible representation.* As we just learned, for the (A_2, D_4) case the discrete gauging group is $\mathbb{Z}_2$. We saw that the discrete gauging groups are generated by diagonal matrices that respect the branching of $\mathcal{G}$ with respect to $\mathcal{M}$. Hence, their generators act multiplying by the same phase all the elements of the considered irreducible representation. In this case, the output is telling us that the generator of the $\mathbb{Z}_2$ group acts multiplying all the elements of the considered irreducible representation by -1. In general, the list will contain as many phases as the factors of the discrete gauging group.

Overloaded version of HbData The function `HbData` is overloaded as

`HbData[ADE, rank, simsrts, higgs]`.

The overloaded version of `HbData` can be used to analyze, in the language of this thesis work, all the explicit matrix realizations of the Higgs field. The first three arguments are exactly the same of the version of the `HbData` function presented in the previous pages. The fourth argument is a matrix representing the Higgs field. The Higgs field has to be input

- in the fundamental representations for the A_r, D_r cases (following the notations in [159]);

- in the **27** representation for the E_6 case;
- in the adjoint representation for the E_7, E_8 cases.

The output is exactly analogous to the one of `HbData[ADE, rank, simsrts, listhiggs, coeffhiggs, cartanhiggs, coeffcartan]`, and contains the data of the Higgs branch of the five-dimensional SCFT associated with the Higgs field profile `higgs` that we input in `HbData[ADE, rank, simsrts, higgs]`.

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