Generic transporters for the linear time dependent quantum Harmonic oscillator on $\mathbb{R}$

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Abstract

In this paper we consider the linear, time dependent quantum Harmonic Schrödinger equation $\mathrm{i}\partial_t u = \frac{1}{2}(-\partial_x^2 + x^2)u + V(t,x,D)u$, $x \in \mathbb{R}$, where $V(t,x,D)$ is classical pseudodifferential operator of order 0, selfadjoint, and 2π periodic in time.
We give sufficient conditions on the principal symbol of $V(t,x,D)$ ensuring the existence of solutions displaying infinite time growth of Sobolev norms. These conditions are generic in the Fréchet space of symbols. This shows that generic, classical pseudodifferential, 2π-periodic perturbations provoke unstable dynamics. The proof builds on the results of [36] and it is based on pseudodifferential normal form and local energy decay estimates. These last are proved exploiting Mourre's positive commutator theory.

1 Introduction and main result

In this paper we study the perturbed quantum harmonic oscillator on $\mathbb{R}$

$$\mathrm{i}\partial_t u = \frac{1}{2}(-\partial_x^2 + x^2)u + V(t,x,D)u, \quad x \in \mathbb{R} \ . \tag{1.1}$$

We shall always assume that $V(t,x,D)$ is a bounded operator, selfadjoint and 2π-periodic in time. Our goal is to construct solutions exhibiting unstable behavior in the form of forward energy cascade. Precisely, we shall exhibit solutions of (1.1) whose $\mathcal{H}^r$-Sobolev norms, $r > 0$, grows unbounded in time:

$$\limsup_{t\to\infty} \|u(t)\|_r = +\infty \ ; \tag{1.2}$$

here we denoted, for any $r \in \mathbb{R}$,

$$\mathcal{H}^r := \{u \in L^2(\mathbb{R},\mathbb{C}) \colon \quad \|u\|_r := \|H_0^r u\|_{L^2} < \infty\}, \qquad H_0 := \frac{1}{2}(-\partial_x^2 + x^2) \ . \tag{1.3}$$

Note that, when $V = 0$, the unperturbed evolution $e^{-\mathrm{i}tH_0}$ preserves all norms $\|\cdot\|_r$ for all times and no energy cascade occurs. So the question is whether one can construct an operator V producing unbounded orbits.

To formalize this concept we shall say (following [36]) that $V(t,x,D)$ is a *transporter* if (1.1) has at least one solution fulfilling (1.2) for some $r > 0$.

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In the last few years several transporters for (1.1) were constructed by Delort [13], Bambusi-Grébert-M.-Robert [5], M. [35], Faou-Raphael [15], Liang, Zhao and Zhou [32], M. [36], Thomann [42], Luo, Liang and Zhao [33]; we will comment more about these results later on.

For the moment, let us note that all these above are *examples* of transporters, and it is not clear neither how to determine if a given operator $V(t,x,D)$ is a transporter nor what happens for generic operators. More precisely, the following questions are open and, we believe, of great interest (not only for the linear theory, but also for applications to nonlinear systems):

(Q1) Given an operator $V(t,x,D)$, can we identify sufficient conditions guaranteeing it to be a transporter?

(Q2) Are transporters rare or common? In other words, how generic are transporters?

In this paper we answer both questions, at least in case $V(t,x,D)$ belongs to the class of classical pseudodifferential operators of order 0. We identify, for the first time, explicit, sufficient conditions on the principal symbol of $V(t,x,D)$ which guarantee the operator to be a transporter. We show that these conditions are fulfilled for *generic* symbols, meaning for a set which is open and dense in the Fréchet topology of symbols.
As a conclusion, we obtain that generic, 2π-time periodic, classical pseudodifferential perturbations of order 0 produce unbounded orbits – a fact which is, in our opinion, somewhat surprising.

The conditions we identify on $V(t,x,D)$ are actually very simple. Let $v_0(t,x,\xi)$ be its principal symbol. Assume it to be a positively homogeneous function of degree 0 (see Definition 1.1), and denote by $\langle v_0\rangle(x,\xi)$ its *resonant average* with respect to the classical flow

$$\phi^t(x,\xi) := (x\cos t + \xi\sin t, -x\sin t + \xi\cos t) \tag{1.4}$$

of the harmonic oscillator $h_0(x,\xi) := \frac{1}{2}(x^2+\xi^2)$, i.e.

$$\langle v_0\rangle(x,\xi) := \frac{1}{2\pi}\int_0^{2\pi} v_0(t,\phi^t(x,\xi))\,\mathrm{d}t \ . \tag{1.5}$$

Our main Theorem 1.3 shows that, if the Poisson bracket between $\langle v_0\rangle$ and h_0 does not vanish identically outside the origin, i.e.

$$\{\langle v_0\rangle, h_0\} \not\equiv 0 \qquad \text{in } x^2+\xi^2 \geq 1 \ , \tag{1.6}$$

then $V(t,x,D)$ is a transporter. Here we use the convention that

$$\{f,g\} := \partial_\xi f\cdot\partial_x g - \partial_x f\cdot\partial_\xi g \ .$$

The proof of this result, that we will describe at the end of the section, builds on the theory developed in [36], and it is based on a combination of pseudodifferential normal form and a dispersive mechanism in the energy space. The dispersion is quantitatively described by local energy decay estimates, which in turn are proved exploiting Mourre's theory of positive commutators.

We remark that in [36] we were already able to apply some abstract results to equation (1.1); however we were able only to deal with operators belonging to the special class of smooth Töplitz operators[1]. On the contrary, the main improvement of the current paper is to deal with generic classical pseudodifferential operators of order 0.

Let us now state precisely our results.

[1]pseudodifferential operators whose matrix elements (computed on the basis of the Hermite functions) are constant on the diagonals and decaying fast enough off diagonal

1.1 Main result

We first define the class of symbols we use.

Definition 1.1. *(i) A function f is a* symbol of order $\rho \in \mathbb{R}$ *if $f \in C^\infty(\mathbb{R}^2, \mathbb{C})$ and $\forall \alpha, \beta \in \mathbb{N}_0$, there exists $C_{\alpha,\beta} > 0$ such that*

$$|\partial_x^\alpha \, \partial_\xi^\beta f(x,\xi)| \leq C_{\alpha,\beta} \, (1 + x^2 + \xi^2)^{\rho - \frac{\beta+\alpha}{2}} \ .$$

We will write $f \in S^\rho$.
(ii) We shall say that f is a classical symbol of order *0, $f \in S^0_{\rm cl}$, if there exists $f_0 \in C^\infty(\mathbb{R}^2, \mathbb{C})$ positively homogeneous of degree 0, i.e.*

$$f_0(\lambda x, \lambda \xi) = f_0(x,\xi) \ , \quad \forall \lambda \geq 1, \quad \forall \, x^2 + \xi^2 \geq 1, \tag{1.7}$$

and $\mu < 0$ such that $f - f_0 \in S^\mu$. We shall call f_0 the principal symbol *of f.*

Remark 1.2. (i) It is easy to see that $S^0_{\rm cl} \subset S^0$.
(ii) With our numerology, the symbol $\frac{1}{2}(x^2 + \xi^2)$ of the harmonic oscillator H_0 is of order 1, and not of order 2 as typically in the literature.

We shall also consider symbols depending periodically from time. We will denote by $C^k(\mathbb{T}, S^\rho)$, $k \in \mathbb{N}_0$, the space of C^k maps $f \colon \mathbb{T} \ni t \mapsto f(t, \cdot) \in S^\rho$ with finite seminorms

$$\wp_j^{k,\rho}(f) := \sum_{\substack{\alpha+\beta \leq j \\ 0 \leq \ell \leq k}} \sup_{\substack{x,\xi \in \mathbb{R} \\ t \in \mathbb{T}}} \frac{\left| \partial_x^\alpha \, \partial_\xi^\beta \partial_t^\ell f(t,x,\xi) \right|}{(1 + x^2 + \xi^2)^{\rho - \frac{\beta+\alpha}{2}}} \ , \qquad \forall j \in \mathbb{N}_0 \ . \tag{1.8}$$

Such seminorms turn $C^k(\mathbb{T}, S^\rho)$ into a Fréchet space, with distance

$$\mathtt{d}^{k,\rho}(f,g) := \sum_{j \geq 0} \frac{1}{2^j} \, \frac{\wp_j^{k,\rho}(f-g)}{1 + \wp_j^{k,\rho}(f-g)}, \qquad \forall f, g \in C^k(\mathbb{T}, S^\rho) \ . \tag{1.9}$$

Similarly we define the space $C^k(\mathbb{T}, S^0_{\rm cl})$, which we endow with the seminorms and distance in (1.8), (1.9). Finally we denote by $C^0_r(\mathbb{T}, S^0_{\rm cl})$ the subset of $C^0(\mathbb{T}, S^0_{\rm cl})$ of real valued symbols.
To a symbol $f \in C^k(\mathbb{T}, S^\rho)$ we associate the operator $F(t,x,D)$ by standard Weyl quantization

$$\big(F(t,x,D)\psi\big)(x) := \Big(\mathrm{Op}^W\left(f(t,\cdot)\right)\psi\Big)(x) := \frac{1}{2\pi} \iint_{y,\xi \in \mathbb{R}} \mathrm{e}^{\mathrm{i}(x-y)\xi} \, f\left(t, \frac{x+y}{2}, \xi\right) \, \psi(y) \, \mathrm{d}y \mathrm{d}\xi \ .$$

We shall say that an operator F is a *pseudodifferential operator* of order ρ if $F = \mathrm{Op}^W(f)$ for some $f \in S^\rho$ and shall write $F \in \mathcal{S}^\rho$. If the symbol $f \in C^k(\mathbb{T}, S^\rho)$ we shall write $F \in C^k(\mathbb{T}, \mathcal{S}^\rho)$.

Our main result is the following one.

Theorem 1.3. *Denote by*

$$\mathscr{V} := \Big\{ v \in C^0_r(\mathbb{T}, S^0_{\rm cl}) \colon \ \textit{the principal symbol } v_0 \textit{ fulfills } \{\langle v_0 \rangle, \, h_0\} \not\equiv 0 \textit{ in } x^2 + \xi^2 \geq 1 \Big\} \ , \tag{1.10}$$

where $\langle v_0 \rangle$ is the resonant average (1.5) of v_0 and $h_0(x,\xi) = \frac{1}{2}(x^2 + \xi^2)$. Then:

(i) For any $v \in \mathscr{V}$*, the operator* $V(t,x,D) := \mathrm{Op}^W(v)$ *is a transporter for* (1.1)*. More precisely,* $\forall r > 0$ *there exist a solution* $\psi(t) \in \mathcal{H}^r$ *of* (1.1) *and constants* $C, T > 0$ *such that*

$$\|\psi(t)\|_r \geq C\langle t\rangle^r, \quad \forall t > T. \tag{1.11}$$

(ii) The set $\mathscr{V}$ *is* generic *in* $C^0_r(\mathbb{T}, S^0_{\mathrm{cl}})$*; precisely it is open and dense with respect to the metric* $\mathtt{d}^{0,0}$ *in* (1.9)*.*

Let us comment the result.

1. The set $\mathscr{V}$ is strictly contained in $C^0_r(\mathbb{T}, S^0_{\mathrm{cl}})$. For example, any symbol constant in time does not belong to $\mathscr{V}$. Indeed if $v_0(x,\xi)$ is time independent, then its resonant average $\langle v_0\rangle$ commutes with h_0, as it is easily checked.
   However, given a time independent, non constant symbol $v_0(x,\xi) \in S^0_{\mathrm{cl}}$, it is always possible to find $n \in \mathbb{N}$ so that $\cos(nt)v_0(x,\xi) \in \mathscr{V}$, see Lemma 6.1.
2. The open property of item *(ii)* guarantees that transporters are stable under perturbations. In particular if $v \in \mathscr{V}$, any sufficiently small perturbation of v still belongs to $\mathscr{V}$.
   The density instead guarantees that given *any* symbol in $C^0_r(\mathbb{T}, S^0_{\mathrm{cl}})$, it is always possible to perturb it so that the new symbol belongs to $\mathscr{V}$.
3. The property of belonging to $\mathscr{V}$ involves only the principal symbol. In particular if $v \in \mathscr{V}$, one can add arbitrarily large symbols in $C^0_r(\mathbb{T}, S^\rho)$, $\rho < 0$, and still be in $\mathscr{V}$.
4. The growth of Sobolev norms of Theorem 1.3 is truly an energy cascade phenomenon; indeed the L^2-norm of any solution of (1.1) is preserved for all times, $\|u(t)\|_{L^2} = \|u(0)\|_{L^2}$, $\forall t \in \mathbb{R}$. This is due to the fact that $H_0 + V(t,x,D)$ is selfadjoint.
5. Estimate (1.11) is optimal, since it is proved in [13] (see also [34]) that *any* solution of (1.1) fulfills the upper bounds
   $$\forall r > 0 \quad \exists\, \widetilde{C}_r > 0: \quad \|\psi(t)\|_r \leq \widetilde{C}_r \langle t\rangle^r \,\|\psi(0)\|_r .$$
6. Energy cascade is a resonant phenomenon; here it happens because $V(t,x,D)$ oscillates at frequency $\omega = 1$ which resonates with the spectral gaps of the harmonic oscillator. In [4] we proved that if $V(\omega t, x, D)$ is quasiperiodic in time with a Diophantine frequency vector $\omega \in \mathbb{R}^n$, then the Sobolev norms of the solutions grow at most as $\langle t\rangle^\epsilon$, $\forall \epsilon > 0$ (see [6] for recent results on $\langle t\rangle^\epsilon$ growth and references therein).
   Moreover, with additional restrictions on ω (typically belonging to some Cantor set of large measure) and assuming $V(t,x,D)$ to be small in size, then all solutions have uniformly in time bounded Sobolev norms [3, 5]. Therefore the stability/instability of the system depends strongly on the resonant properties of the frequency ω.

Let us briefly describe the main ingredients of the proof. The first step is to use resonant pseudodifferential normal form, analogous to the one of Delort [13], to conjugate the original equation (1.1) to

$$\mathrm{i}\partial_t \varphi = \big(\mathrm{Op}^W(\langle v_0\rangle) + T + R(t)\big)\varphi \tag{1.12}$$

where $\langle v_0\rangle$ is the resonant average (1.5) of the principal symbol of $V(t,x,D)$, T is a selfadjoint, time independent pseudodifferential operator of negative order, and $R(t)$ is an arbitrary regularizing perturbation. This is done in Section 3.

The second step is the analysis of the *effective* Hamiltonian

$$\mathrm{i}\partial_t \psi = \big(\mathrm{Op}^W(\langle v_0\rangle) + T\big)\psi \tag{1.13}$$

obtained removing $R(t)$ from (1.12). We construct solutions of (1.13) exhibiting dispersion in the energy space, i.e. solutions $\psi(t)$ whose negative $\mathcal{H}^{-r}$-Sobolev norm, $r > 0$, decays in time at a polynomial rate:

$$\|\psi(t)\|_{-r} \leq \frac{C}{\langle t\rangle^r} \|\psi(0)\|_r, \quad \forall t \in \mathbb{R} . \tag{1.14}$$

Hence, by the unitarity of the flow, these solutions have unbounded growth of positive Sobolev norms. Then it is not difficult to construct solutions of the complete system (1.12) exhibiting Sobolev norms explosion.

To prove (1.14) we study the spectral properties of $\mathsf{V}_0 := \mathrm{Op}^W(\langle v_0\rangle)$. Exploiting Mourre's commutator theory [37], we show that V_0 has absolutely continuous spectrum in an interval $\mathcal{I} \subset \mathbb{R}$, over which it fulfills the strict Mourre estimate

$$g_{\mathcal{I}}(\mathsf{V}_0)\, \mathrm{i}[\mathsf{V}_0, A]\, g_{\mathcal{I}}(\mathsf{V}_0) \geq \rho\, g_{\mathcal{I}}(\mathsf{V}_0)^2 \tag{1.15}$$

for some $\rho > 0$, $A \in \mathcal{S}^1$ and $g \in C_c^\infty(\mathbb{R}, \mathbb{R}_{\geq 0})$ with $g \equiv 1$ on $\mathcal{I}$. This in turn implies, by Sigal-Soffer theory [40], that V_0 fulfills dispersive estimates in the frequency space in the form of local energy decay estimates. The same is true also for $\mathsf{V}_0 + T$, as Mourre estimates are stable by pseudodifferential operators of negative order.

The key difficulty in applying Mourre-Sigal-Soffer theory is that the operator A entering the Mourre estimate (1.15) is not given, but has to be constructed. In particular we produce A so that the principal symbol of the operator on the left of (1.15) is

$$g_{\mathcal{I}}(\langle v_0\rangle)^2 \left\{\langle v_0\rangle, h_0\right\}^2 . \tag{1.16}$$

Then condition (1.6) allows to select an interval $\mathcal{I}$ so that the function in (1.16) is strictly positive, and then we deduce (1.15) exploiting the strong Gårding inequality. This is done in Section 4.

Let us compare our result with the previous ones in the literature. After the pioneering work by Bourgain [9], Delort [13] constructs the first example of a transporter for (1.1), which is also a classical pseudodifferential operator of order 0, and a solution whose $\mathcal{H}^r$-norm grows as t^r. Bambusi-Grébert-M.-Robert [5] constructs an unbounded transporter and a solution growing as t^{2r}. The author [35] constructs a universal transporter, meaning a perturbation $V(t, x, D)$ such that *all* non trivial solutions of (1.1) fulfill (1.2). Faou-Raphael [15] deal with the very interesting case of a multiplication operator $V(t, x)$ (and not a pseudodifferential operator) and construct a solution whose $\mathcal{H}^r$-norm grows at a logarithmic speed. Thomann [42] constructs a transporter for the 2D Harmonic oscillator on the Bargmann-Fock space. Finally Liang, Zhao and Zhou [32] and Luo, Liang and Zhao [33] consider operators $V(\omega t, x, D)$ which are the quantization of polynomial symbols of order at most 2 and depend quasi-periodically in time with a frequency $\omega \in \mathbb{R}^d$. They are able to completely describe the quantum dynamics according to the resonant properties of ω, and even obtain solutions growing at the unusual speed[2] of t^{4s}. Finally we mention Haus-M. [27] which considers anharmonic oscillators.

Before closing this introduction, we mention that constructing solutions with unbounded orbits in nonlinear Schrödinger-like equations is very difficult. Long time unstable orbits have been constructed for the nonlinear Schrödinger equation on $\mathbb{T}^2$ [10, 23, 24, 25, 22, 21, 19], but truly unbounded orbits are known only for the cubic Szegő equation on $\mathbb{T}$ [16, 17] and the cubic NLS on $\mathbb{R} \times \mathbb{T}^2$ [26].

[2] note that in [33] the space $\mathcal{H}^s := D(H_0^{s/2})$, differently from (1.3), so in our notation one has to substitute $s \rightsquigarrow 2s$ in [33]

2 Pseudodifferential operators

In this section we collect some results about pseudodifferential operators in $\mathcal{S}^\rho$.

Symbolic calculus. This first class of results regards very basic properties of pseudodifferential operators and can be found in classical texts such as [29, 39].

Theorem 2.1. *Let $a \in S^\rho$, $b \in S^\mu$, $\rho, \mu \in \mathbb{R}$. Then*

(i) Action: *For any $s \in \mathbb{R}$, there are $C, M > 0$ such that*

$$\|\mathrm{Op}^W(a)\, u\|_{s-\rho} \le C_s\, \wp^\rho_M(a)\, \|u\|_s.$$

(ii) Symbolic calculus: *One has $\mathrm{Op}^W(a)^* = \mathrm{Op}^W(\overline{a})$.*
There exists a symbol $c \in S^{\rho+\mu}$ such that $\mathrm{Op}^W(a) \circ \mathrm{Op}^W(b) = \mathrm{Op}^W(c)$. Moreover $c - ab \in S^{\rho+\mu-1}$ and

$$\forall j \in \mathbb{N}_0, \qquad \exists N, C > 0 \ \ s.t. \quad \wp_j^{\rho+\mu-1}(c - ab) \le C\, \wp^\rho_N(a)\, \wp^\mu_N(b)\ .$$

There exists a symbol $d \in S^{\rho+\mu-1}$ such that $\mathrm{i}[\mathrm{Op}^W(a)\, , \mathrm{Op}^W(b)] = \mathrm{Op}^W(d)$. Moreover $d - \{a, b\} \in S^{\rho+\mu-3}$ and

$$\forall j \in \mathbb{N}_0, \qquad \exists N, C > 0 \ \ s.t. \quad \wp_j^{\rho+\mu-3}(d - \{a, b\}) \le C\, \wp^\rho_N(a)\, \wp^\mu_N(b)\ .$$

(iii) Exact Egorov theorem: *One has*

$$e^{\mathrm{i}\tau H_0}\, \mathrm{Op}^W(a)\; e^{-\mathrm{i}\tau H_0} = \mathrm{Op}^W(a \circ \phi^\tau)\ , \quad \forall \tau \in \mathbb{R}$$

where ϕ^τ is the flow (1.4)*. In particular $t \mapsto a \circ \phi^t \in C^k(\mathbb{T}, S^\rho)$, $\forall k \in \mathbb{N}_0$, and*

$$\forall j \in \mathbb{N}_0 \quad \exists N, C > 0 \ \ s.t. \ \ \wp^\rho_j(a \circ \phi^t) \le C\, \wp^\rho_N(a)\ .$$

(iv) Compactness: *Let $\rho < 0$. Then $\mathrm{Op}^W(a)$ is compact.*

Remark 2.2. If $v \in C^0(\mathbb{T}, S^\rho)$, $\rho \in \mathbb{R}$, then its resonant average $\langle v\rangle(x, \xi)$ (defined in (1.5)) belongs to S^ρ, as one checks using the explicit expression

$$\langle v\rangle(x, \xi) := \frac{1}{2\pi}\int_0^{2\pi} v(t, x\cos t + \xi \sin t, -x \sin t + \xi \cos t)\, \mathrm{d}t\ . \tag{2.1}$$

Note also that $\langle v\rangle$ is time-independent. If v is real valued, so is $\langle v\rangle$.

Flows. The second class of results regards the flow generated by pseudodifferential operators. For the proofs we refer to [4].

Lemma 2.3. *Assume that $X \in C^1(\mathbb{T}, \mathcal{S}^1)$ is selfadjoint. Then the following holds true.*

(i) Flow: $e^{-\mathrm{i}\tau X(t)}$ *extends to an operator in $\mathcal{L}(\mathcal{H}^r)$ $\forall r \in \mathbb{R}$, and moreover there exist $c_r, C_r > 0$ s.t.*

$$c_r\|\psi\|_r \le \|e^{-\mathrm{i}\tau X(t)}\psi\|_r \le C_r\|\psi\|_r\ , \qquad \forall t \in \mathbb{R}\ , \quad \forall \tau \in [0, 1]\ .$$

(ii) Conjugation: *Let $H(t)$ be a time dependent selfadjoint operator. Assume that* $\psi(t) = e^{-\mathrm{i}X(t)}\varphi(t)$ *then*

$$\mathrm{i}\partial_t\psi = H(t)\psi \qquad \Longleftrightarrow \qquad \mathrm{i}\partial_t\varphi = H^+(t)\varphi$$

where

$$H^+(t) := e^{\mathrm{i}X(t)}\, H(t)\, e^{-\mathrm{i}X(t)} - \int_0^1 e^{\mathrm{i}sX(t)}\, (\partial_t X(t))\, e^{-\mathrm{i}sX(t)}\, \mathrm{d}s\ .$$

(iii) Lie expansion: *Let $X \in \mathcal{S}^\rho$ with $\rho < 1$ and $H \in \mathcal{S}^m$, $m \in \mathbb{R}$, both selfadjoint. Then* $e^{\mathrm{i}X}\, H\, e^{-\mathrm{i}X} \in \mathcal{S}^m$*, it is selfadjoint and for any $M \geq 1$ we have*

$$e^{\mathrm{i}X}\, H\, e^{-\mathrm{i}X} = \sum_{\ell=0}^{M} \frac{\mathrm{i}^\ell}{\ell!}\mathrm{ad}_X^\ell(H) + R_M, \tag{2.2}$$

where $\mathrm{ad}_X(H) := [X, H]$ *and*

$$R_M := \frac{\mathrm{i}^{M+1}}{M!}\int_0^1 (1-\tau)^M\, e^{\mathrm{i}\tau X}\, \mathrm{ad}_X^{M+1}(H)\, e^{-\mathrm{i}\tau X}\, \mathrm{d}\tau\ \in \mathcal{S}^{m-(M+1)(1-\rho)}$$

is selfadjoint.

Remark 2.4. If $X \in C^k(\mathbb{T}, \mathcal{S}^\rho)$, $H \in C^k(\mathbb{T}, \mathcal{S}^m)$ with $\rho < 1$ and $m \in \mathbb{R}$, then the remainder $R_M(t)$ in formula (2.2) belongs to $C^k(\mathbb{T}, \mathcal{S}^{m-(M+1)(1-\rho)})$.

Functional calculus. The next results concern the functional calculus of pseudodifferential operators in $\mathcal{S}^\rho$. Standard references are the papers [28, 12] and the books [38, 1].

Given $g \in C_c^\infty(\mathbb{R}, \mathbb{R})$, we define its *almost analytic extension* as follows: for any $N \in \mathbb{N}$, put

$$\widetilde{g}_N : \mathbb{R}^2 \to \mathbb{C}, \qquad \widetilde{g}_N(x, y) := \left(\sum_{\ell=0}^{N} g^{(\ell)}(x)\frac{(\mathrm{i}y)^\ell}{\ell!}\right)\, \tau\left(\frac{y}{\langle x\rangle}\right)$$

where $\tau \in C^\infty(\mathbb{R}, \mathbb{R}_{\geq 0})$ is a cut-off function fulfilling $\tau(s) = 1$ for $|s| \leq 1$ and $\tau(s) = 0$ for $|s| \geq 2$. Then, given H a selfadjoint operator, one defines $g(\mathsf{H})$ via the *Helffer-Sjöstrand formula*

$$g(\mathsf{H}) := -\frac{1}{\pi}\int_{\mathbb{R}^2} \frac{\partial\widetilde{g}_N(x, y)}{\partial\overline{z}}\, (x + \mathrm{i}y - \mathsf{H})^{-1}\mathrm{d}x\, \mathrm{d}y\ , \qquad \frac{\partial}{\partial\overline{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right)\ . \tag{2.3}$$

The operator $g(\mathsf{H})$ is independent of $N \in \mathbb{N}$, see [12].

Theorem 2.5 (Functional calculus)**.** *Let $g \in C_c^\infty(\mathbb{R}, \mathbb{R})$.*

(i) Let $v \in S^0$ be real valued. The operator $g(\mathrm{Op}^W(v))$, defined via (2.3), belongs to $\mathcal{S}^0$. Moreover $g(\mathrm{Op}^W(v)) - \mathrm{Op}^W(g(v)) \in \mathcal{S}^{-1}$.

(ii) Let $\mathsf{A}, \mathsf{B} \in \mathcal{L}(\mathcal{H})$ be selfadjoint. If $\mathsf{A} - \mathsf{B}$ is compact, so is $g(\mathsf{A}) - g(\mathsf{B})$.

Strong Gårding inequality. The next result is the strong Gårding inequality for symbols in S^0.

Theorem 2.6 (Strong Gårding inequality)**.** *Let $a \in S^0$ and assume there exists $R > 0$ such that*

$$a(x, \xi) \geq 0, \quad \forall x^2 + \xi^2 \geq R \ . \tag{2.4}$$

Then there exists $C > 0$ such that

$$\langle \mathrm{Op}^W(a)\, u, u\rangle \geq -C\|u\|_{-1}^2, \quad \forall u \in L^2 \ . \tag{2.5}$$

This result is probably well known but we couldn't find a proof of this exact statement in the literature, so we prove it in Appendix A.1.

Essential spectrum. In this last paragraph we characterize the essential spectrum of pseudodifferential operators with symbols in S^0.

Theorem 2.7. *Let $v \in S^0$ be real valued. Then*

$$\sigma_{ess}(\mathrm{Op}^W(v)) = \{\lambda \in \mathbb{R}\colon \ \exists\, (x_j, \xi_j) \to \infty \quad s.t. \quad v(x_j, \xi_j) \to \lambda\} \ . \tag{2.6}$$

We prove this result in Appendix A.2.

3 Pseudodifferential normal form

The first step of our proof is to use pseudodifferential normal form to extract from the original equation (1.1) an effective Hamiltonian having as leading term the operator $\mathrm{Op}^W(\langle v_0\rangle)$. Precisely we shall prove the following result.

Proposition 3.1. *Consider equation* (1.1) *with $V(t, x, D) = \mathrm{Op}^W(v(t, \cdot))$ and $v \in C^0_r(\mathbb{T}, S^0_{cl})$. There exists $\delta > 0$ such that for any $N \in \mathbb{N}$ the following holds true. There exists a change of coordinates $\mathcal{U}_N(t)$, unitary in L^2 and fulfilling*

$$\forall r \geq 0 \quad \exists\, c_r, C_r > 0\colon \qquad c_r\|\varphi\|_r \leq \|\mathcal{U}_N(t)^{\pm}\varphi\|_r \leq C_r\|\varphi\|_r, \qquad \forall t \in \mathbb{R}, \tag{3.1}$$

such that $u(t)$ solves (1.1) *if and only if $\varphi(t) := \mathcal{U}_N(t)u(t)$ solves*

$$\mathrm{i}\partial_t\varphi = \big(\mathrm{Op}^W(\langle v_0\rangle) + T_N + R_N(t)\big)\varphi \tag{3.2}$$

where $\langle v_0\rangle$ is the resonant average of the principal symbol v_0 of v, $T_N \in \mathcal{S}^{-\delta}$ is time independent and selfadjoint and $R_N \in C^0(\mathbb{T}, \mathcal{S}^{-N-\delta})$ is selfadjoint $\forall t$.

The proof of the proposition is based on a series of change of coordinates, and at each step we shall solve a simple homological equation, which we now describe:

Lemma 3.2. *Assume that $H \in C^0(\mathbb{T}, \mathcal{S}^m)$, $m \in \mathbb{R}$. There exists $X \in C^1(\mathbb{T}, \mathcal{S}^m)$ such that*

$$H(t) - \frac{1}{2\pi}\int_0^{2\pi} H(s)\,\mathrm{d}s = \partial_t X(t) \ . \tag{3.3}$$

If $H(t)$ is selfadjoint for any t, so is $X(t)$.

Proof. We solve (3.3) at the level of symbols; we put $H(t) = \mathrm{Op}^W(h(t,\cdot))$ with $h \in C^0(\mathbb{T}, S^m)$ and look for $X(t) = \mathrm{Op}^W(\chi(t,\cdot))$ with a symbol $\chi \in C^1(\mathbb{T}, S^m)$ to be determined. Then (3.3) reads

$$h(t,x,\xi) - \frac{1}{2\pi}\int_0^{2\pi} h(s,x,\xi)\,\mathrm{d}s = \partial_t \chi(t,x,\xi) \ ,$$

whose solution is readily given by

$$\chi(t,x,\xi) := \int_0^t \left(h(s,x,\xi) - \frac{1}{2\pi}\int_0^{2\pi} h(s_1,x,\xi)\,\mathrm{d}s_1 \right) \mathrm{d}s \ .$$

One verifies that $\chi \in C^1(\mathbb{T}, S^m)$. Finally if $H(t)$ is selfadjoint, its symbol $h(t,\cdot)$ is real valued, so is $\chi(t,\cdot)$ and thus $X(t)$ is selfadjoint. □

Proof of Proposition 3.1. First we gauge away H_0, performing the change of variables $u = e^{-\mathrm{i}tH_0}\psi$. Then ψ solves $\mathrm{i}\partial_t\psi = e^{\mathrm{i}tH_0}\,\mathrm{Op}^W(v(t,\cdot))\,e^{-\mathrm{i}tH_0}\psi$.
Next recall that, by definition of the class S^0_{cl}, the symbol v decomposes as

$$v(t,\cdot) = v_0(t,\cdot) + v_{-\mu}(t,\cdot)$$

with $v_0(t,\cdot)$ positively homogeneous of degree 0 and $v_{-\mu} \in C^0_r(\mathbb{T}, S^{-\mu})$ for some $\mu > 0$. So we decompose

$$\mathrm{i}\partial_t\psi = (\mathsf{V}(t) + R(t))\,\psi$$

with

$$\mathsf{V}(t) := e^{\mathrm{i}tH_0}\,\mathrm{Op}^W(v_0(t,\cdot))\,e^{-\mathrm{i}tH_0} \ , \qquad R(t) := e^{\mathrm{i}tH_0}\,\mathrm{Op}^W(v_{-\mu}(t,\cdot))\,e^{-\mathrm{i}tH_0} \ .$$

Note that $\mathsf{V} \in C^0(\mathbb{T}, \mathcal{S}^0)$ and $R \in C^0(\mathbb{T}, \mathcal{S}^{-\mu})$, both selfadjoint $\forall t$.

Now we proceed inductively, performing several changes of coordinates which remove the oscillating part order by order. Let us describe the first step. We perform the change of variables $\psi = e^{-\mathrm{i}X_1(t)}\varphi$ where $X_1(t) \in C^1(\mathbb{T}, \mathcal{S}^0)$, selfadjoint $\forall t$, to be determined. By Lemma 2.3, φ fulfills the Schrödinger equation $\mathrm{i}\partial_t\varphi = \mathsf{H}_1(t)\varphi$ with

$$\mathsf{H}_1(t) := e^{\mathrm{i}X_1(t)}\,(\mathsf{V}(t) + R(t))\,e^{-\mathrm{i}X_1(t)} - \int_0^1 e^{\mathrm{i}sX_1(t)}\,(\partial_t X_1(t))\,e^{-\mathrm{i}sX_1(t)}\,\mathrm{d}s \ .$$

Then a Lie expansion, see (2.2), gives

$$\mathsf{H}_1(t) = \mathsf{V}(t) - \partial_t X_1(t) + R_1(t), \qquad R_1 \in C^0(\mathbb{T}, \mathcal{S}^{-\delta}), \ \text{selfadjoint } \forall t \ , \tag{3.4}$$

with

$$\delta := \min(1,\mu) > 0 \ .$$

Lemma 3.2 allows us to choose $X_1 \in C^1(\mathbb{T}, \mathcal{S}^0)$, selfadjoint $\forall t$, s.t.

$$\mathsf{V}(t) - \frac{1}{2\pi}\int_0^{2\pi} \mathsf{V}(s)\,\mathrm{d}s = \partial_t X_1(t) \ .$$

With this choice, and exploiting Egorov's Theorem, equation (3.4) reduces to

$$\begin{aligned}\mathsf{H}_1(t) &= \frac{1}{2\pi}\int_0^{2\pi} \mathsf{V}(s)\,\mathrm{d}s + R_1(t) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Op}^W(v_0(s, \phi^s(x,\xi)))\,\mathrm{d}s + R_1(t) \\ &= \mathrm{Op}^W(\langle v_0 \rangle) + R_1(t) \ .\end{aligned}$$

Note also that the map $e^{-\mathrm{i}X_1(t)}$ fulfills an estimate like (3.1) thanks to Lemma 2.3.
Then we iterate this procedure. Assume that at the N-th step we have the equation

$$\mathrm{i}\partial_t\varphi = \Big(\mathrm{Op}^W(\langle v_0\rangle) + T_N + R_N(t)\Big)\,\varphi =: \mathsf{H}_N(t)\varphi \tag{3.5}$$

with $T_N \in \mathcal{S}^{-\delta}$ selfadjoint and time independent, and $R_N \in C^0(\mathbb{T}, \mathcal{S}^{-N-\delta})$ selfadjoint $\forall t$. The change of variables $\varphi = e^{-\mathrm{i}X_{N+1}(t)}w$, with a certain $X_{N+1} \in C^1(\mathbb{T}, \mathcal{S}^{-N-\delta})$ selfadjoint and to be determined, conjugates (3.5) to $\mathrm{i}\partial_t w = \mathsf{H}_{N+1}(t)w$ with

$$\mathsf{H}_{N+1}(t) = \mathrm{Op}^W(\langle v_0\rangle) + T_N + R_N(t) - \partial_t X_{N+1}(t) + R_{N+1}(t)\ , \tag{3.6}$$

with $R_{N+1} \in C^0(\mathbb{T}, \mathcal{S}^{-(N+1)-\delta})$ selfadjoint $\forall t$. We exploit again Lemma 3.2 to find X_{N+1} with the wanted properties solving

$$\partial_t X_{N+1} = R_N(t) - \frac{1}{2\pi}\int_0^{2\pi} R_N(s)\mathrm{d}s\ .$$

Thus $\mathsf{H}_{N+1}(t)$ in (3.6) becomes

$$\mathsf{H}_{N+1}(t) = \mathrm{Op}^W(\langle v_0\rangle) + T_{N+1} + R_{N+1}(t), \qquad T_{N+1} := T_N + \frac{1}{2\pi}\int_0^{2\pi} R_N(s)\,\mathrm{d}s\ .$$

Clearly $T_{N+1} \in \mathcal{S}^{-\delta}$, it is selfadjoint and time-independent. This proves the inductive step.

Finally we put $\mathcal{U}_N(t) := e^{\mathrm{i}X_N(t)}\cdots e^{\mathrm{i}X_1(t)}e^{\mathrm{i}tH_0}$; estimate (3.1) follows from Lemma 2.3. □

4 Mourre estimates

In this section we study the spectral properties of the operator $\mathrm{Op}^W(\langle v_0\rangle)$, which is the leading term of equation (3.2). In particular we prove that its essential spectrum contains a nontrivial interval $\mathcal{I}_0$ and, over a subinterval $\mathcal{I} \subset \mathcal{I}_0$, it fulfills a Mourre estimate. Then Mourre theory [37] guarantees that $\mathrm{Op}^W(\langle v_0\rangle)$ has absolutely continuous spectrum in $\mathcal{I}$, and therefore we might expect dispersive behavior. More precisely we prove the following result:

Proposition 4.1. *Let $v \in \mathscr{V}$ (see (1.10)) and v_0 be its principal symbol. Denote by $\langle v_0\rangle \in S^0$ its resonant average (see (1.5)) and $\mathsf{V}_0 := \mathrm{Op}^W(\langle v_0\rangle)$. The following holds true.*

(i) The spectrum $\sigma(\mathsf{V}_0)$ contains a closed interval $\mathcal{I}_0$.

(ii) There exist $A \in \mathcal{S}^1$ selfadjoint, an open interval $\mathcal{I} \subset \mathcal{I}_0$, a function $g_{\mathcal{I}} \in C_c^\infty(\mathbb{R}, \mathbb{R}_{\geq 0})$ with $g_{\mathcal{I}} \equiv 1$ over $\mathcal{I}$ such that

$$g_{\mathcal{I}}(\mathsf{V}_0)\ \mathrm{i}[\mathsf{V}_0, A]\ g_{\mathcal{I}}(\mathsf{V}_0) \geq \rho\, g_{\mathcal{I}}(\mathsf{V}_0)^2 + \mathsf{K} \tag{4.1}$$

for some $\rho > 0$ and K a compact operator.

Notation: In the following, we shall denote by K a general compact operator on L^2, which might change from line to line. We shall also constantly use that pseudodifferential operators in $\mathcal{S}^{-\mu}$, $\mu > 0$, are compact on L^2.

In order to prove Proposition 4.1 we start with some preparation. We shall denote by $\mathcal{A}\colon \mathbb{T}\times \mathbb{R}_{>0} \to \mathbb{R}^2 \setminus \{0\}$, $(\vartheta, I) \to (x, \xi)$ the action-angle change of coordinates defined by

$$x = \sqrt{2I}\sin(\vartheta), \quad \xi = \sqrt{2I}\cos(\vartheta), \qquad \forall (x,\xi) \neq (0,0)\ . \tag{4.2}$$

Given a function $f \in C^\infty(\mathbb{R}^2, \mathbb{C})$, we shall denote by $\widetilde{f} := f \circ \mathcal{A}$ the function f expressed in action angle coordinates (4.2). Note that since (4.2) is a canonical transformation, one has

$$\widetilde{\{f, g\}} = \{\widetilde{f}, \widetilde{g}\} =: \partial_I \widetilde{f}\, \partial_\vartheta \widetilde{g} - \partial_\vartheta \widetilde{f}\, \partial_I \widetilde{g} \ . \tag{4.3}$$

Lemma 4.2. *Assume that $f \in C^\infty(\mathbb{R}^2, \mathbb{C})$ is positively homogeneous of degree 0, i.e. it fulfills* (1.7). *Then*

$$\widetilde{f}(\vartheta, I) = \widetilde{f}(\vartheta, 1), \quad \forall I > \frac{1}{2} \ . \tag{4.4}$$

In particular $\partial_I \widetilde{f}(\vartheta, I) = 0$ for any $I > \frac{1}{2}$.

Proof. Being f positively homogeneous of degree 0,

$$\widetilde{f}(\vartheta, I) = f(\sqrt{2I}\sin(\vartheta), \sqrt{2I}\cos(\vartheta)) = f(\sin(\vartheta), \cos(\vartheta)) = \widetilde{f}(\vartheta, \frac{1}{2}) \ , \quad \forall I \geq \frac{1}{2} \ ,$$

implying in particular the identity (4.4). □

Now we begin the proof of Proposition 4.1, which is splitted in several lemmas.

Lemma 4.3. *Let $v \in C^0_r(\mathbb{T}, S^0_{cl})$ and denote by v_0 its principal symbol. Then*

(i) $\langle v_0 \rangle$ is real valued and positively homogeneous of degree 0, i.e.

$$\langle v_0 \rangle(\lambda x, \lambda \xi) = \langle v_0 \rangle(x, \xi), \quad \forall \lambda \geq 1, \quad \forall x^2 + \xi^2 \geq 1 \ . \tag{4.5}$$

(ii) If furthermore $v \in \mathscr{V}$, then

$$\mathcal{I}_0 := \left[\min_{x^2+\xi^2=1} \langle v_0 \rangle(x, \xi), \ \max_{x^2+\xi^2=1} \langle v_0 \rangle(x, \xi) \right] \subseteq \sigma_{ess}\big(\mathsf{V}_0\big) \tag{4.6}$$

and it has nonempty interior.

Proof. (i) It follows from the explicit expression (2.1) using that $v_0(t, \cdot)$ is real valued and positively homogeneous of degree 0.
(ii) By Theorem 2.7

$$\sigma_{ess}\big(\mathsf{V}_0\big) = \{\lambda \in \mathbb{R} \colon \exists\, (x_j, \xi_j) \to \infty \text{ s.t. } \langle v_0 \rangle(x_j, \xi_j) \to \lambda\} \ .$$

Since $\langle v_0 \rangle$ is positively homogeneous of degree 0 and definitively one has $x_j^2 + \xi_j^2 \geq 1$, we have that

$$\langle v_0 \rangle(x_j, \xi_j) = \langle v_0 \rangle(\frac{x_j}{x_j^2 + \xi_j^2}, \frac{\xi_j}{x_j^2 + \xi_j^2}) \ ,$$

and we deduce that

$$\sigma_{ess}\big(\mathsf{V}_0\big) = \Big\{ \lambda \in \mathbb{R} \colon \exists\, \{(\widetilde{x}_j, \widetilde{\xi}_j)\}_{j \geq 1} \text{ with } \widetilde{x}_j^2 + \widetilde{\xi}_j^2 = 1 \ \forall j \quad \text{s.t. } \langle v_0 \rangle(\widetilde{x}_j, \widetilde{\xi}_j) \to \lambda \Big\} \ .$$

Finally, since the function $\langle v_0 \rangle$ is smooth, this set corresponds to the image of $\langle v_0 \rangle|_{x^2+\xi^2=1}$ and therefore

$$\sigma_{ess}(\mathsf{V}_0) = [\min_{x^2+\xi^2=1} \langle v_0 \rangle(x, \xi), \ \max_{x^2+\xi^2=1} \langle v_0 \rangle(x, \xi)] = \mathcal{I}_0.$$

To prove that $\mathcal{I}_0$ is a nontrivial interval, it is sufficient to show that $\langle v_0 \rangle$ is not constant on $x^2 + \xi^2 = 1$. Assume by contradiction that $\langle v_0 \rangle$ is constant on the circle; then by homogeneity, it is constant on the whole set $x^2 + \xi^2 \geq 1$. But then $\{h_0, \langle v_0 \rangle\} \equiv 0$ in $x^2 + \xi^2 \geq 1$, contradicting the assumption that $v \in \mathscr{V}$. □

Next define the operator

$$A := \mathrm{Op}^W(a)\,, \qquad a := \{\langle v_0\rangle, h_0\}\, h_0 \in S^1\ , \quad \text{real valued.} \tag{4.7}$$

Clearly $A \in \mathcal{S}^1$ and it is selfadjoint.

Lemma 4.4. *Let A be defined in (4.7). For any $\mathcal{I} \subset \mathcal{I}_0$ and $g_{\mathcal{I}} \in C_c^\infty(\mathbb{R}, \mathbb{R}_{\geq 0})$ with $g_{\mathcal{I}} \equiv 1$ over $\mathcal{I}$ there exists a compact operator K such that*

$$g_{\mathcal{I}}(\mathsf{V}_0)\,\mathrm{i}[\mathsf{V}_0, A]\, g_{\mathcal{I}}(\mathsf{V}_0) = \mathrm{Op}^W\big(g_{\mathcal{I}}(\langle v_0\rangle)^2\, \{\langle v_0\rangle, h_0\}^2\big) + \mathsf{K}\ . \tag{4.8}$$

Proof. By symbolic calculus we get

$$\mathrm{i}[\mathsf{V}_0, A] = \mathrm{Op}^W(\{\langle v_0\rangle, a\} + r_{-2})\,, \quad r_{-2} \in S^{-2}, \text{ real valued }\ . \tag{4.9}$$

Consider now the operator $g_{\mathcal{I}}(\mathsf{V}_0)$. Using Lemma 2.5 we write

$$g_{\mathcal{I}}(\mathsf{V}_0) = \mathrm{Op}^W(g_{\mathcal{I}}(\langle v_0\rangle)) + \mathsf{K} \tag{4.10}$$

with K a compact operator. We use the expressions (4.9) and (4.10), the fact that $\{\langle v_0\rangle, a\}$, $g_{\mathcal{I}}(\langle v_0\rangle)$ belong to S^0 and symbolic calculus to finally get

$$g_{\mathcal{I}}(\mathsf{V}_0)\,\mathrm{i}[\mathsf{V}_0, A]\, g_{\mathcal{I}}(\mathsf{V}_0) = \mathrm{Op}^W\big(g_{\mathcal{I}}(\langle v_0\rangle)^2\, \{\langle v_0\rangle, a\}\big) + \mathsf{K} \tag{4.11}$$

with K compact. Next we compute $\{\langle v_0\rangle, a\}$. Using (4.7),

$$\{\langle v_0\rangle, a\} = \{\langle v_0\rangle, h_0\}^2 + w, \qquad w := \{\langle v_0\rangle,\, \{\langle v_0\rangle, h_0\}\}h_0.$$

We claim that $w \in C_c^\infty(\mathbb{R}^2, \mathbb{R}) \subset S^{-\infty}$; then (4.8) follows by inserting this decomposition in (4.11). To prove that $w \in C_c^\infty(\mathbb{R}^2, \mathbb{R})$ we pass to action-angle variables defined in (4.2). As

$$\widetilde{h}_0 = I\ , \quad \widetilde{\{\langle v_0\rangle, h_0\}} = -\partial_\vartheta \widetilde{\langle v_0\rangle}(\vartheta, I)\ , \tag{4.12}$$

we get

$$\{\langle v_0\rangle,\, \widetilde{\{\langle v_0\rangle, h_0\}}\} = \{\widetilde{\langle v_0\rangle},\, \{\widetilde{\langle v_0\rangle}, I\}\} \overset{(4.3)}{=} -\frac{\partial \widetilde{\langle v_0\rangle}}{\partial I}\,\frac{\partial^2 \widetilde{\langle v_0\rangle}}{\partial \vartheta^2} + \frac{\partial \widetilde{\langle v_0\rangle}}{\partial \vartheta}\,\frac{\partial^2 \widetilde{\langle v_0\rangle}}{\partial I \partial \vartheta}\ .$$

Since $\langle v_0\rangle$ is positively homogeneous of degree 0 (see (4.5)), by Lemma 4.2 we have that $\partial_I \widetilde{\langle v_0\rangle}(\vartheta, I) = \partial^2_{I\vartheta} \widetilde{\langle v_0\rangle}(\vartheta, I) = 0$ for any $I > \frac{1}{2}$, proving that $\{\langle v_0\rangle,\, \widetilde{\{\langle v_0\rangle, h_0\}}\} \in C_c^\infty(\mathbb{R}^2 \setminus \{0\}, \mathbb{R})$. Hence also $w \in C_c^\infty(\mathbb{R}^2, \mathbb{R})$. ☐

The next one is the most important lemma, which proves the commutator estimate.

Lemma 4.5. *There exist an interval $\mathcal{I} \subset \mathcal{I}_0$, a function $g_{\mathcal{I}} \in C_c^\infty(\mathbb{R}, \mathbb{R}_{\geq 0})$ with* supp $g_{\mathcal{I}} \subset \mathcal{I}_0$, *$g_{\mathcal{I}} \equiv 1$ over $\mathcal{I}$ and numbers $\rho, C > 0$ such that*

$$\langle \mathrm{Op}^W\big(g_{\mathcal{I}}(\langle v_0\rangle)^2\, \{\langle v_0\rangle, h_0\}^2\big)\, u, u\rangle \geq \rho\, \langle \mathrm{Op}^W\big(g_{\mathcal{I}}(\langle v_0\rangle)^2\big)\, u, u\rangle - C\|u\|^2_{-1}, \qquad \forall u \in L^2\ . \tag{4.13}$$

Proof. We claim the existence of $\mathcal{I} \subset \mathcal{I}_0$ and $\rho > 0$ such that

$$g_{\mathcal{I}}(\langle v_0\rangle)^2\, \{\langle v_0\rangle, h_0\}^2 \geq \rho\, g_{\mathcal{I}}(\langle v_0\rangle)^2, \quad \forall\, x^2 + \xi^2 \geq 1. \tag{4.14}$$

Then the Strong Gårding inequality in Theorem 2.5 gives (4.13).
To prove (4.14) we pass to action-angle variables (4.2). Using (4.12) the left-hand-side of (4.14) reads

$$g_{\mathcal{I}}\left(\widetilde{\langle v_0\rangle}(\vartheta, I)\right)^2\, \left[\partial_\vartheta \widetilde{\langle v_0\rangle}(\vartheta, I)\right]^2\ .$$

By Lemma 4.2, $\widetilde{\langle v_0\rangle}(\vartheta, I) \equiv \widetilde{\langle v_0\rangle}(\vartheta, 1)$ for any $I > \frac{1}{2}$. Moreover, as $\widetilde{\langle v_0\rangle}(\vartheta) := \widetilde{\langle v_0\rangle}(\vartheta, 1)$ is a smooth function defined on $\mathbb{T}$, Sard's theorem implies that the set $\mathcal{C} := \{\vartheta \in \mathbb{T} \colon \partial_\vartheta \widetilde{\langle v_0\rangle}(\vartheta) = 0\}$ of critical values of $\widetilde{\langle v_0\rangle}$ has image $\widetilde{\langle v_0\rangle}(\mathcal{C})$ of zero measure. Since the image of $\widetilde{\langle v_0\rangle}(\cdot) \equiv \langle v_0\rangle|_{x^2+\xi^2=2} = \langle v_0\rangle|_{x^2+\xi^2=1}$ is the nontrivial interval $\mathcal{I}_0$ (see (4.6)), $\mathcal{I}_0 \setminus \widetilde{\langle v_0\rangle}(\mathcal{C})$ is a positive measure set and contains only regular values. So fix $\underline{\lambda} \in \mathcal{I}_0 \setminus \widetilde{\langle v_0\rangle}(\mathcal{C})$. The set $\widetilde{\langle v_0\rangle}^{-1}(\underline{\lambda})$ is a compact set of isolated points, thus finitely many; denote them by $\{\vartheta_a\}_{a=1}^d$. Since $\partial_\vartheta \widetilde{\langle v_0\rangle}(\vartheta_a) \neq 0$ for any $a = 1, \ldots, d$, we can find neighbors $\mathcal{U}_a \subset \mathbb{T}$ of ϑ_a and a neighbor $\mathcal{V} \subset \mathcal{I}_0$ of $\underline{\lambda}$ such that

(i) $\widetilde{\langle v_0\rangle} \colon \mathcal{U}_a \to \mathcal{V}$ is a diffeomorphism for any $a = 1, \ldots, d$,

(ii) $\widetilde{\langle v_0\rangle}^{-1}(\mathcal{V}) = \cup_a \mathcal{U}_a$,

(iii) there exists $\rho > 0$ such that $\min_{\vartheta \in \cup_a \overline{\mathcal{U}_a}} \left|\partial_\vartheta \widetilde{\langle v_0\rangle}(\vartheta)\right| \geq \sqrt{\rho}$.

Now take an interval $\mathcal{I} \subset \mathcal{V}$ with $\underline{\lambda} \in \mathcal{I}$. Take also $g_{\mathcal{I}} \in C_c^\infty(\mathbb{R}, \mathbb{R}_{\geq 0})$ with $g_{\mathcal{I}} \equiv 1$ on $\mathcal{I}$ and $\operatorname{supp} g_{\mathcal{I}} \subset \mathcal{V}$. Using (ii) above, we have that

$$(\vartheta, I) \in \operatorname{supp}\ \Big(g_{\mathcal{I}}\big(\widetilde{\langle v_0\rangle}(\vartheta, I)\big)\Big) \cap \{I > \frac{1}{2}\} \quad \Rightarrow \quad \vartheta \in \overline{\bigcup_{a=1}^d \mathcal{U}_a}\ .$$

In particular in the set $\operatorname{supp}\Big(g_{\mathcal{I}}\big(\widetilde{\langle v_0\rangle}(\vartheta, I)\big)\Big) \cap \{I > \frac{1}{2}\}$ the function $\widetilde{\langle v_0\rangle}(\vartheta, I) \equiv \widetilde{\langle v_0\rangle}(\vartheta)$ fulfills (iii) above. We deduce that

$$g_{\mathcal{I}}^2\left(\widetilde{\langle v_0\rangle}(\vartheta, I)\right)\left(\partial_\vartheta \widetilde{\langle v_0\rangle}(\vartheta, I)\right)^2 \geq \rho\ g_{\mathcal{I}}^2\left(\widetilde{\langle v_0\rangle}(\vartheta, I)\right)\ , \quad \forall (\vartheta, I) \in \mathbb{T} \times \{I > \frac{1}{2}\}\ ,$$

proving (4.14). □

We can finally prove Proposition 4.1.

Proof of Proposition 4.1. By Lemmata 4.4 and 4.5, we have

$$\langle g_{\mathcal{I}}(\mathsf{V}_0)\, \mathrm{i}[\mathsf{V}_0, A]\, g_{\mathcal{I}}(\mathsf{V}_0) u, u\rangle \geq \rho \langle \mathrm{Op}^W\big(g_{\mathcal{I}}(\langle v_0\rangle)^2\big)\, u, u\rangle + \langle(\mathsf{K} - C H_0^{-2}) u, u\rangle \tag{4.15}$$

with K compact. Now by symbolic and functional calculus

$$\mathrm{Op}^W\big(g_{\mathcal{I}}(\langle v_0\rangle)^2\big) = \Big(\mathrm{Op}^W\big(g_{\mathcal{I}}(\langle v_0\rangle)\big)\Big)^2 + \mathcal{S}^{-1} \overset{(4.10)}{=} g_{\mathcal{I}}(\mathsf{V}_0)^2 + \mathsf{K}_1$$

with K_1 a compact operator. Inserting in (4.15) gives the claimed Mourre estimate (4.1). □

5 Dynamics of the effective equation

In this section we consider the effective equation obtained removing $R_N(t)$ from (3.2), namely

$$\mathrm{i}\partial_t \varphi = H_N \varphi, \qquad H_N := \mathrm{Op}^W(\langle v_0\rangle) + T_N, \tag{5.1}$$

with $T_N \in \mathcal{S}^{-\delta}$, $\delta > 0$, see Proposition 3.1. Recall that H_N is selfadjoint and time independent. We shall construct a solution of (5.1) with *decaying negative Sobolev norms*, and thus, exploiting the L^2 conservation, also with growing Sobolev norms.

Proposition 5.1 (Decay of negative Sobolev norms)**.** *Consider the operator H_N in* (5.1)*. For any $k \in \mathbb{N}$, there exist a nontrivial solution $\varphi(t) \in \mathcal{H}^k$ of* (5.1) *and $\forall r \in [0,k]$ a constant $C_r > 0$ such that*

$$\|\varphi(t)\|_{-r} \le C_r \langle t \rangle^{-r} \|\varphi(0)\|_r \ , \qquad \forall t \in \mathbb{R} \ . \tag{5.2}$$

Remark 5.2. As H_N is selfadjoint, the conservation of the L^2-norm and Cauchy-Schwarz inequality give

$$\|\varphi(0)\|_0^2 = \|\varphi(t)\|_0^2 \le \|\varphi(t)\|_r \ \|\varphi(t)\|_{-r} \ , \qquad \forall t \in \mathbb{R} \ ,$$

so (5.2) implies the growth of positive Sobolev norms:

$$\|\varphi(t)\|_r \ge \frac{1}{C_r} \frac{\|\varphi(0)\|_0^2}{\|\varphi(0)\|_r} \langle t \rangle^r \ , \quad \forall t \in \mathbb{R} \ . \tag{5.3}$$

Proposition 5.1 will follow from the following abstract Sigal-Soffer local energy decay estimate:

Theorem 5.3 (Local energy decay estimate)**.** *Let $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ be a Hilbert space. Let $\mathsf{H} \in \mathcal{L}(\mathcal{H})$ and A be both selfadjoint and with $D(\mathsf{A}) \cap \mathcal{H}$ dense in $\mathcal{H}$. Fix $k \in \mathbb{N}$ and assume that*

(M1) the operators $\mathrm{ad}_{\mathsf{A}}^n(\mathsf{H})$, $n = 1, \dots, 4k+2$, can all be extended to bounded operators on $\mathcal{H}$.

(M2) Strict Mourre estimate*: there exist an open interval $\mathcal{I} \subset \mathbb{R}$ with compact closure and a function $g_{\mathcal{I}} \in C_c^\infty(\mathbb{R}, \mathbb{R}_{\ge 0})$ with $g_{\mathcal{I}} \equiv 1$ on $\mathcal{I}$ such that*

$$g_{\mathcal{I}}(\mathsf{H}) \, \mathrm{i}[\mathsf{H}, \mathsf{A}] \, g_{\mathcal{I}}(\mathsf{H}) \ge \theta g_{\mathcal{I}}(\mathsf{H})^2$$

for some $\theta > 0$.

Then for any interval $\mathcal{J} \subset \mathcal{I}$, any function $g_{\mathcal{J}} \in C_c^\infty(\mathbb{R}, \mathbb{R}_{\ge 0})$ with $\operatorname{supp} g_{\mathcal{J}} \subset \mathcal{I}$, $g_{\mathcal{J}} = 1$ on $\mathcal{J}$, there exists $C > 0$ such that

$$\|\langle \mathsf{A} \rangle^{-k} e^{-\mathrm{i}\mathsf{H}t} g_{\mathcal{J}}(\mathsf{H}) \, \psi\|_{\mathcal{H}} \le C \langle t \rangle^{-k} \|\langle \mathsf{A} \rangle^k g_{\mathcal{J}}(\mathsf{H}) \psi\|_{\mathcal{H}}, \quad \forall t \in \mathbb{R} \ .$$

The theorem goes back to the works of [40, 41, 31], see also [30, 2, 14, 18, 20] for extensions and generalizations. A proof of this exact statement can be found in [36], Appendix C.

Proof of Proposition 5.1. We apply Theorem 5.3 with $\mathcal{H} = L^2$, $\mathsf{H} = H_N$ in (5.1) and $\mathsf{A} = A$ in (4.7). Assumption (M1) is verified since $A \in \mathcal{S}^1$ and $H_N \in \mathcal{S}^0$. To check (M2) we work perturbatively from the Mourre estimate (4.1). Again we shall denote by K a general compact operator in L^2 which might change from line to line. To shorten notation, we shall also write $\mathsf{V}_0 := \mathrm{Op}^W(\langle v_0 \rangle)$.

First, as $H_N = \mathsf{V}_0 + T_N$ with $T_N \in \mathcal{S}^{-\delta}$, the operator $[T_N, A] \in \mathcal{S}^{-\delta}$ is compact, so from (4.1) we get

$$g_{\mathcal{I}}(\mathsf{V}_0) \, \mathrm{i}[H_N, A] \, g_{\mathcal{I}}(\mathsf{V}_0) \ge \rho \, g_{\mathcal{I}}(\mathsf{V}_0)^2 + \mathsf{K}$$

with K compact. Next, by Lemma 2.5, $g_{\mathcal{I}}(H_N) - g_{\mathcal{I}}(\mathsf{V}_0)$ is compact and therefore we get that

$$\begin{aligned} g_{\mathcal{I}}(H_N) \, \mathrm{i}[H_N, A] \, g_{\mathcal{I}}(H_N) &= g_{\mathcal{I}}(\mathsf{V}_0) \, \mathrm{i}[H_N, A] \, g_{\mathcal{I}}(\mathsf{V}_0) + \mathsf{K} \\ &\ge \rho \, g_{\mathcal{I}}(\mathsf{V}_0)^2 + \mathsf{K} = \rho \, g_{\mathcal{I}}(H_N)^2 + \mathsf{K} \end{aligned}$$

This proves that H_N fulfills a Mourre estimate over $\mathcal{I}$. It is standard that one can shrink the interval $\mathcal{I}$ to a subinterval $\mathcal{I}_1$ to obtain the strict Mourre estimate

$$g_{\mathcal{I}_1}(H_N) \, \mathrm{i}[H_N, A] \, g_{\mathcal{I}_1}(H_N) \ge \frac{\rho}{2} g_{\mathcal{I}_1}(H_N)^2 \ ,$$

see e.g. the arguments in [37] (or also Step 1 of Lemma 3.13 of [36]). This proves (M2). So we apply Theorem 5.3 and obtain that for any interval $\mathcal{J} \subset \mathcal{I}_1$, any function $g_{\mathcal{J}} \in C_c^\infty(\mathbb{R}, \mathbb{R}_{\geq 0})$ with $\operatorname{supp} g_{\mathcal{J}} \subset \mathcal{I}_1$, $g_{\mathcal{J}} = 1$ on $\mathcal{J}$,

$$\|\langle A\rangle^{-k} e^{-\mathrm{i}H_N t} g_{\mathcal{J}}(H_N)\varphi\|_0 \leq C\langle t\rangle^{-k}\|\langle A\rangle^k g_{\mathcal{J}}(H_N)\varphi\|_0, \quad \forall t \in \mathbb{R}. \tag{5.4}$$

Then, since $H_0^{-k}\langle A\rangle^k, \langle A\rangle^k H_0^{-k} \in \mathcal{S}^0$, using (5.4) we deduce

$$\|e^{-\mathrm{i}tH_N} g_{\mathcal{J}}(H_N)\varphi\|_{-k} \leq C_k\langle t\rangle^{-k}\|g_{\mathcal{J}}(H_N)\varphi\|_k \ ,$$

and interpolating with $\|e^{-\mathrm{i}tH_N}\varphi\|_0 = \|\varphi\|_0$ yields

$$\|e^{-\mathrm{i}tH_N} g_{\mathcal{J}}(H_N)\varphi\|_{-r} \leq C_r\langle t\rangle^{-r}\|g_{\mathcal{J}}(H_N)\varphi\|_r \ , \quad \forall t \in \mathbb{R} \ , \quad \forall \varphi \in \mathcal{H}^r \ , \quad \forall r \in [0, k] \ .$$

The last step is to prove that the estimate is not trivial, namely that one can choose $\varphi(0) := g_{\mathcal{J}}(H_N)\varphi \in \mathcal{H}^r \setminus \{0\}$. Regarding the regularity, note that $g_{\mathcal{J}}(H_N)\varphi \in \mathcal{H}^r$ provided $\varphi \in \mathcal{H}^r$. Hence it suffices to show that $g_{\mathcal{J}}(H_N)\mathcal{H}^r \neq \{0\}$. By the density of $\mathcal{H}^r$ in $\mathcal{H} \equiv L^2$, this follows provided $g_{\mathcal{J}}(H_N)\mathcal{H} \neq \{0\}$. So assume by contradiction that $g_{\mathcal{J}}(H_N)\mathcal{H} = \{0\}$. By Weyl's theorem, being T_N compact,

$$\sigma(H_N) \supseteq \sigma_{ess}(H_N) = \sigma_{ess}(\mathsf{V}_0) \overset{(4.6)}{\supseteq} \mathcal{I}_0 \supseteq \mathcal{J} \ .$$

Hence the spectral projector $E_{\mathcal{J}}(H_N)$ of H_N over $\mathcal{J}$ fulfills $E_{\mathcal{J}}(H_N)\mathcal{H} \neq \{0\}$. Since, by functional calculus, $E_{\mathcal{J}}(H_N) = E_{\mathcal{J}}(H_N)g_{\mathcal{J}}(H_N)$ (which follows from $\mathbb{1}_{\mathcal{J}}(\lambda) = \mathbb{1}_{\mathcal{J}}(\lambda)g_{\mathcal{J}}(\lambda)$ $\forall \lambda \in \mathbb{R}$, $\mathbb{1}_{\mathcal{J}}$ being the indicator function over the interval $\mathcal{J}$), we get

$$\{0\} = E_{\mathcal{J}}(H_N)g_{\mathcal{J}}(H_N)\mathcal{H} = E_{\mathcal{J}}(H_N)\mathcal{H} \neq \{0\}$$

obtaining a contradiction. Hence $g_{\mathcal{J}}(H_N)\mathcal{H} \neq \{0\}$. □

6 Proof of the main theorem

In this section we prove Theorem 1.3.

Proof of (*i*)**.** It follows exactly as in [36], so we just sketch the arguments for completeness. Let $r > 0$ be given in Theorem 1.3. Fix $N, k \in \mathbb{N}$ with $N \geq 2r + 2$ and $k \geq \delta + N - r$. Take a solution $\varphi(t)$ of equation (5.1) fulfilling the decay (5.2) up to regularity k. Defining $U_N(t,s)$ the linear propagator of $H_N + R_N(t)$, one checks that

$$\phi(t) := \varphi(t) + \mathrm{i}\int_t^{+\infty} U_N(t,s)\, R_N(s)\, \varphi(s)\, \mathrm{d}s =: \varphi(t) + w(t)$$

solves equation (3.2), provided $w(t)$ is well defined. Using that, by Theorem 1.5 of [34], $U_N(t,s) \in \mathcal{L}(\mathcal{H}^r)$ with $\|U_N(t,s)\|_{\mathcal{L}(\mathcal{H}^r)} \leq C_r\, \langle t - s\rangle^r$, $\forall t, s \in \mathbb{R}$ and $R_N \in C^0(\mathbb{T}, \mathcal{S}^{-N-\delta})$, we get

$$\begin{aligned}\|w(t)\|_r &\leq C \int_t^{+\infty} \langle t-s\rangle^r \|R_N(s)\,\varphi(s)\|_r\, \mathrm{d}s \leq C\int_t^{+\infty} \langle t-s\rangle^r\, \|\varphi(s)\|_{-(\delta+N-r)}\, \mathrm{d}s \\ &\leq C\,\|\varphi_0\|_{\delta+N-r} \int_t^{+\infty} \langle t-s\rangle^r\, \frac{1}{\langle s\rangle^{\delta+N-r}}\, \mathrm{d}s \leq C\,\|\varphi_0\|_k \langle t\rangle^{-1} \ .\end{aligned}$$

Using also (5.3) we deduce that $\|\phi(t)\|_r \geq \|\varphi(t)\|_r - \|w(t)\|_r \geq C\langle t\rangle^r$ for t sufficiently large.

Finally $\psi(t) = \mathcal{U}_N(t)^{-1}\phi(t)$ solves (1.1) and has polynomially growing Sobolev norms as (1.11), proving item (i).

Proof of (ii). We show that the set $\mathscr{V}$ is open and dense in $C^0_r(\mathbb{T}, S^0_{\rm cl})$.
Open: We show that for any $v \in \mathscr{V}$, there is $\epsilon, M > 0$ such that any $w \in C^0_r(\mathbb{T}, S^0_{\rm cl})$ fulfilling $\wp^{0,0}_M(v - w) < \epsilon$ belongs to $\mathscr{V}$. In particular this last condition is achieved provided $\mathtt{d}^{0,0}(v, w)$ is small enough.

First of all decompose $v = v_0 + v_{-\mu}$ with v_0 the principal symbol of v and $v_{-\mu} \in C^0_r(\mathbb{T}, S^{-\mu})$, $\mu > 0$. As $\langle v_0\rangle$ is positively homogeneous of degree 0 (see (4.5)), we put

$$\varrho := \max_{x^2+\xi^2\geq 1} |\{\langle v_0\rangle, h_0\}| = \max_{\vartheta\in\mathbb{T}} \left|\partial_\vartheta \widetilde{\langle v_0\rangle}(\vartheta, 1)\right| > 0 \ ;$$

note that ϱ is strictly positive since, by assumption, $\{\langle v_0\rangle, h_0\}$ is not identically 0. Denote by ϑ_ϱ a point in $\mathbb{T}$ where the maximum is attained.

Decompose also $w = w_0 + w_{-\mu'}$ with w_0 the principal symbol and $w_{-\mu'} \in C^0_r(\mathbb{T}, S^{-\mu'})$, $\mu' > 0$. To prove that $w \in \mathscr{V}$, it suffices to show that $\{\langle w_0\rangle, h_0\}$ is not identically zero in $x^2 + \xi^2 \geq 1$. We write

$$\begin{aligned} \{\langle w_0\rangle, h_0\} &= \{\langle v_0\rangle, h_0\} + \{\langle w_0 - v_0\rangle, h_0\} \\ &= \{\langle v_0\rangle, h_0\} + \{\langle w - v\rangle, h_0\} - \{\langle w_{-\mu'} - v_{-\mu}\rangle, h_0\} \end{aligned} \tag{6.1}$$

We evaluate (6.1) at the point

$$\underline{x} := \sqrt{2\mathtt{R}}\sin(\vartheta_\delta), \quad \underline{\xi} := \sqrt{2\mathtt{R}}\cos(\vartheta_\delta) \tag{6.2}$$

where $\mathtt{R} > 0$ will be chosen later on sufficiently large.
By the very definition of $(\underline{x}, \underline{\xi})$ we have

$$\{\langle v_0\rangle, h_0\}(\underline{x}, \underline{\xi}) = \varrho \ . \tag{6.3}$$

Next we consider the second term of (6.1); by symbolic calculus

$$\sup_{x,\xi\in\mathbb{R}} |\{\langle w - v\rangle, h_0\}(x, \xi)| \leq C\wp^{0,0}_M(w - v) \leq C\epsilon \ . \tag{6.4}$$

Finally let us consider $\{\langle w_{-\mu'} - v_{-\mu}\rangle, h_0\}$. By Remark 2.2 this is a symbol in $S^{-\overline{\mu}}$, $\overline{\mu} := \min(\mu, \mu')$. By definition there exists a constant $C_1 = C_1(w_{-\mu'}, v_{-\mu}) > 0$ such that

$$|\{\langle w_{-\mu'} - v_{-\mu}\rangle, h_0\}(x, \xi)| \leq C_1(1 + x^2 + \xi^2)^{-\overline{\mu}}, \quad \forall (x, \xi) \in \mathbb{R}^2 \ .$$

In particular, at the point $(\underline{x}, \underline{\xi})$ in (6.2), we get that

$$|\{\langle w_{-\mu'} - v_{-\mu}\rangle, h_0\}(\underline{x}, \underline{\xi})| \leq \frac{C_1}{(1 + 2\mathtt{R})^{\overline{\mu}}} \ . \tag{6.5}$$

Thus, evaluating (6.1) at the point $(\underline{x}, \underline{\xi})$ and using (6.3), (6.4) and (6.5) we get

$$\{\langle w_0\rangle, h_0\}(\underline{x}, \underline{\xi}) \geq \varrho - C\epsilon - \frac{C_1}{(1 + 2\mathtt{R})^{\overline{\mu}}} \geq \frac{\varrho}{2} > 0$$

provided one chooses $\epsilon < \dfrac{\varrho}{4C}$ and $\mathtt{R} > \left(\dfrac{C_1}{2\varrho}\right)^{\frac{1}{\mu}}$, concluding the verification that $w \in \mathscr{V}$.

Dense: Take $v \in C^0_r(\mathbb{T}, S^0_{\rm cl})$ and assume that $v \notin \mathscr{V}$. Take $\epsilon > 0$ arbitrary. We shall construct $w \in \mathscr{V}$ with $\mathtt{d}^{0,0}(v,w) < \epsilon$. Pick a function $\eta \in C^\infty_c(\mathbb{R}^2, \mathbb{R}_{\geq 0})$, radial, and with $\eta(x,\xi) = 1$ for $x^2+\xi^2 \geq \frac{1}{2}$ and $\eta(x,\xi) = 0$ for $x^2+\xi^2 \leq \frac{1}{4}$. We put

$$w := v + \epsilon_0 w_0, \quad w_0(t,x,\xi) := \cos(2t)\, \eta(x,\xi)\, \frac{x\xi}{x^2+\xi^2}$$

with $\epsilon_0 > 0$ small enough. Note that w_0 is positively homogeneous of degree 0, so $w \in C^0_r(\mathbb{T}, S^0_{\rm cl})$. Let us show that $w \in \mathscr{V}$. Writing $w = v_0 + \epsilon_0 w_0 + v_{-\mu}$, we have to check that $\{\langle v_0\rangle + \epsilon_0 \langle w_0\rangle, h_0\} \not\equiv 0$ in $x^2+\xi^2 \geq 1$. Since $v \notin \mathscr{V}$, we have that $\{\langle v_0\rangle, h_0\} \equiv 0$ in $x^2+\xi^2 \geq 1$ and we need only to check that $\{\langle w_0\rangle, h_0\} \not\equiv 0$ on the same set. Computing explicitly $\langle w_0\rangle$ (using also that η is radial and thus constant along the flow ϕ^t), we obtain

$$\langle w_0\rangle = \frac{\eta(x,\xi)}{x^2+\xi^2} \cdot \frac{1}{2\pi} \int_0^{2\pi} \cos(2t)\,\big(x\cos t + \xi \sin t\big)\big(-x\sin t + \xi\cos t\big) {\rm d}t = \frac{1}{2}\eta(x,\xi)\, \frac{x\xi}{x^2+\xi^2} \ .$$

Then, using that $\{\dfrac{\eta(x,\xi)}{x^2+\xi^2}, h_0\} = 0$, we get

$$\{\langle w_0\rangle, h_0\} = \frac{1}{2}\{x\xi, h_0\}\frac{\eta(x,\xi)}{x^2+\xi^2} = \frac{1}{2}(x^2-\xi^2)\frac{\eta(x,\xi)}{x^2+\xi^2}$$

proving that $w \in \mathscr{V}$. Finally we show that, provided $\epsilon_0 > 0$ is sufficiently small, $\mathtt{d}^{0,0}(v,w) < \epsilon$. So take $N > 0$ so large that $\sum_{j \geq N+1} 2^{-j} < \epsilon/2$ and ϵ_0 so small that

$$\sum_{j=0}^N \wp_j^{0,0}(v-w) = \epsilon_0 \sum_{j=0}^N \wp_j^{0,0}(w_0) < \frac{\epsilon}{2} \ .$$

Then $\mathtt{d}^{0,0}(v,w) \leq \sum_{j=0}^N \wp_j^{0,0}(v-w) + \sum_{j\geq N+1} \frac{1}{2^j} < \epsilon$. □

We conclude this part with the following lemma, which somehow generalize the construction of the symbol w in the previous proof.

Lemma 6.1. *Let* $\mathtt{v} \in S^0_{\rm cl}$ *be real valued and so that* $\{h_0, \mathtt{v}\} \not\equiv 0$ *in* $x^2+\xi^2 \geq 1$. *Then there exists* $n \in \mathbb{N}$ *such that*

$$v(t,x,\xi) := \cos(nt)\, \mathtt{v}(x,\xi) \in \mathscr{V} \ .$$

In particular ${\rm Op}^W(v(t,\cdot))$ *is a transporter for* (1.1).

Proof. We need to check that $\{h_0, \langle v\rangle\} \not\equiv 0$ in $x^2+\xi^2 \geq 1$, where $\langle v\rangle(x,\xi) = \frac{1}{2\pi}\int_0^{2\pi} \cos(nt)\, \mathtt{v}(\phi^t(x,\xi))\, {\rm d}t$. The function $t \mapsto \mathtt{v}(\phi^t(x,\xi))$ is 2π-periodic in time; expanding it in Fourier series one gets

$$\mathtt{v}(\phi^t(x,\xi)) = \mathtt{v}_0^+(x,\xi) + 2\sum_{n\geq 1} \big(\cos(nt)\, \mathtt{v}_n^+(x,\xi) + \sin(nt)\, \mathtt{v}_n^-(x,\xi)\big) \tag{6.6}$$

where

$$\mathtt{v}_n^+(x,\xi) := \frac{1}{2\pi}\int_0^{2\pi} \cos(nt)\ \mathtt{v}(\phi^t(x,\xi))\, {\rm d}t, \quad \mathtt{v}_n^-(x,\xi) := \frac{1}{2\pi}\int_0^{2\pi} \sin(nt)\, \mathtt{v}(\phi^t(x,\xi))\, {\rm d}t, \quad n \geq 0.$$

They are all symbols positively homogeneous of degree 0. The claim is equivalent to verify that

$$\exists\, n \geq 1 \colon \quad \{h_0, \mathtt{v}_n^+\} \not\equiv 0 \quad \text{in } x^2+\xi^2 \geq 1 \ . \tag{6.7}$$

First note that, by integration by parts and the periodicity of the flow ϕ^t,

$$\begin{aligned}\{h_0, \mathtt{v}_n^+\} &= \frac{1}{2\pi}\int_0^{2\pi} \cos(nt)\, \{h_0, \mathtt{v}\circ\phi^t\}\, \mathrm{d}t = \frac{1}{2\pi}\int_0^{2\pi} \cos(nt) \left(\frac{\mathrm{d}}{\mathrm{d}t}\mathtt{v}\circ\phi^t\right) \mathrm{d}t \\ &= \frac{n}{2\pi}\int_0^{2\pi} \sin(nt)\, (\mathtt{v}\circ\phi^t)\, \mathrm{d}t = n\, \mathtt{v}_n^- \ ,\end{aligned}$$

so it is sufficient to check that $\exists n \geq 1$ so that $\mathtt{v}_n^- \not\equiv 0$ in $x^2+\xi^2 \geq 1$. Actually, since

$$\begin{aligned}\mathtt{v}_n^-(x,\xi) &= \frac{1}{2\pi}\int_0^{2\pi} \sin(nt)\, \mathtt{v}\big(\phi^t(x,\xi)\big)\, \mathrm{d}t = \frac{1}{2\pi}\int_0^{2\pi} \cos\left(n(t-\frac{\pi}{2n})\right) \mathtt{v}\big(\phi^t(x,\xi)\big)\, \mathrm{d}t \\ &= \frac{1}{2\pi}\int_0^{2\pi} \cos(nt)\, \mathtt{v}\big(\phi^{t+\frac{\pi}{2n}}(x,\xi)\big)\, \mathrm{d}t = \mathtt{v}_n^+\big(\phi^{\frac{\pi}{2n}}(x,\xi)\big),\end{aligned}$$

it is enough to check that $\exists n \geq 1$ so that one among $\mathtt{v}_n^\pm$ is not identically zero in $x^2+\xi^2 \geq 1$; this is what we show next.

Assume by contradiction that $\mathtt{v}_n^\pm \equiv 0$ in $x^2+\xi^2 \geq 1$ for any $n \geq 1$; then from (6.6) we get $\mathtt{v}\circ\phi^t = \mathtt{v}_0^+$ for any $t \in \mathbb{R}$ and any $x^2+\xi^2 \geq 1$. Then

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\mathtt{v}\circ\phi^t = \{h_0, \mathtt{v}\}\circ\phi^t \ , \quad \forall t \in \mathbb{R}, \quad \forall x^2+\xi^2 \geq 1;$$

so, at $t = 0$, one gets $\{h_0, \mathtt{v}\} \equiv 0$ in $x^2+\xi^2 \geq 1$, contradicting the assumption.

Then at least one couple of the $\mathtt{v}_n^\pm$ is not identically zero, and (6.7) follows. ☐

A Technical results

A.1 The strong Gårding inequality in $\mathcal{S}^0$

Our proof involves the Anti-Wick quantization of a symbol, which we now introduce. First let us define coherent states: for $z = (q,p) \in \mathbb{R}^2$ let

$$\Phi_0(x) := \frac{1}{\pi^{1/4}}\, e^{-\frac{x^2}{2}}, \qquad \Phi_z := \mathcal{T}_z\Phi_0, \qquad [\mathcal{T}_z u](x) := e^{-\frac{\mathrm{i}}{2}pq}\, e^{\mathrm{i}xp}\, u(x-q) \ . \tag{A.1}$$

Note that the operator $\mathcal{T}_z$ is unitary in $L^2(\mathbb{R})$. Now, given a symbol $a \in S^0$, we define its Anti-Wick quantization by

$$(\mathrm{Op}^{aw}(a)\, u)\,(x) := \int_{\mathbb{R}^2} a(q,p)\, \langle u, \Phi_{q,p}\rangle\, \Phi_{q,p}\, \mathrm{d}q\, \mathrm{d}p \ . \tag{A.2}$$

We collect few properties of the Anti-Wick quantization:

Lemma A.1. *Let $a \in S^0$. Then*

(i) If $a \geq 0$, then $\langle \mathrm{Op}^{aw}(a)\, u, u\rangle \geq 0$ for any $u \in L^2$.

(ii) One has $\mathrm{Op}^{aw}(a) = \mathrm{Op}^W(a * \Phi_0)$*, with* Φ_0 *in* (A.1)*. Moreover* $a - a * \Phi_0 \in S^{m-2}$*.*

Proof. Item (i) follows directly from the definition (A.2). Item (ii) is Theorem 24.1 of [39]. □

Proof of Theorem 2.6. Let $\chi \in C_c^\infty(\mathbb{R}^2, \mathbb{R}_{\geq 0})$, radial cut-off function with

$$\chi(x,\xi) = 1 \ \ \forall x^2 + \xi^2 \leq R, \qquad \chi(x,\xi) = 0 \ \ \forall x^2 + \xi^2 \geq R + 1 \ .$$

Then the function $a_\chi := a(1-\chi) \in S^0$ fulfills, using the assumption (2.4),

$$a_\chi(x,\xi) \geq 0 \quad \forall x, \xi \in \mathbb{R} \ . \tag{A.3}$$

Next we write

$$\begin{aligned} \mathrm{Op}^W(a_\chi) &= \mathrm{Op}^W(a_\chi * \Phi_0) + \mathrm{Op}^W(a_\chi - a_\chi * \Phi_0) \\ &= \mathrm{Op}^{aw}(a_\chi) + \mathrm{Op}^W(a_\chi - a_\chi * \Phi_0) \end{aligned}$$

where to pass from the first to the second line we used Lemma A.1 (ii). By Lemma A.1 and (A.3) one has $\mathrm{Op}^{aw}(a_\chi) \geq 0$, hence

$$\langle \mathrm{Op}^W(a_\chi)\, u, u\rangle \geq \langle \mathrm{Op}^W(a_\chi - a_\chi * \Phi_0)\, u, u\rangle \ .$$

Now use that $a_\chi = a - a\chi$ to deduce

$$\langle \mathrm{Op}^W(a)\, u, u\rangle \geq \langle \mathrm{Op}^W(a_\chi - a_\chi * \Phi_0)\, u, u\rangle + \langle \mathrm{Op}^W(a\chi)\, u, u\rangle.$$

By Lemma A.1 (ii) the symbol $a_\chi - a_\chi * \Phi_0 \in S^{-2}$ and moreover $a\chi \in S^{-\infty}$. Then Theorem 2.1 (i) implies the claimed bound (2.5). □

A.2 Proof of Theorem 2.7

Denote by $\mathscr{A}$ the set on the right of (2.6). We first show that $\sigma_{ess}(\mathrm{Op}^W(v)) \subseteq \mathscr{A}$. Assume by contradiction that $\lambda \in \sigma_{ess}(\mathrm{Op}^W(v))$ does not belong to $\mathscr{A}$. Then there exist $c, R > 0$ such that $|v(x,\xi) - \lambda| \geq c$ for any $x^2 + \xi^2 \geq R$. Let $\chi \in C_c^\infty(\mathbb{R}^2)$ with $\chi \equiv 1$ in $x^2 + \xi^2 \leq R$ and $\chi \equiv 0$ in $x^2 + \xi^2 \geq R + 1$. Put $b(x,\xi) = \dfrac{1 - \chi(x,\xi)}{v(x,\xi) - \lambda} \in S^0$. Then by symbolic calculus there are operators $K_1, K_2 \in \mathcal{S}^{-1}$ (and therefore compact) so that

$$\left(\mathrm{Op}^W(v) - \lambda\right) \mathrm{Op}^W(b) = \mathrm{Id} + K_1, \qquad \mathrm{Op}^W(b) \left(\mathrm{Op}^W(v) - \lambda\right) = \mathrm{Id} + K_2 \ .$$

Thus $\mathrm{Op}^W(v) - \lambda$ is a Fredholm operator, and its spectrum in a neighbourhood of zero must be discrete. In particular $\lambda \notin \sigma_{ess}(\mathrm{Op}^W(v))$, contradicting the assumption. This shows that $\sigma_{ess}(\mathrm{Op}^W(v)) \subseteq \mathscr{A}$.

We show now the inverse inclusion $\mathscr{A} \subseteq \sigma_{ess}(\mathrm{Op}^W(v))$. Given $\lambda \in \mathscr{A}$, we shall exhibit a Weyl's sequence for λ, i.e. a sequence of functions $\{\varphi_j\}_{j\geq 1} \in L^2$ with $\|\varphi_j\| = 1$, $\varphi_j \rightharpoonup 0$ and $\|(\mathrm{Op}^W(v) - \lambda)\varphi_j\| \to 0$; then Weyl's theorem guarantees that $\lambda \in \sigma_{ess}(\mathrm{Op}^W(v))$. We construct such a Weyl's sequence using the coherent states in (A.1). If $\lambda \in \mathscr{A}$ we have a sequence $z_j := (q_j, p_j) \to \infty$ with $v(q_j, p_j) \to \lambda$. Put (see (A.1))

$$\Phi_j := \Phi_{z_j} = \mathcal{T}_{z_j} \Phi_0 \ .$$

We claim that, up to subsequences, $\{\Phi_j\}_{j\geq 1}$ is a Weyl sequence for λ. It is clear that $\|\Phi_j\| = 1$ $\forall j$. It is not difficult to show that $\Phi_j \rightharpoonup 0$; for completeness we prove this in Lemma A.2 below.

We show now that $\|(\mathrm{Op}^W(v)-\lambda)\Phi_j\| \to 0$. First we have, using the reality of λ, v and symbolic calculus,

$$\mathrm{Op}^W(v-\lambda)^* \, \mathrm{Op}^W(v-\lambda) = \mathrm{Op}^W\big((v-\lambda)^2 + r_{-1}\big), \quad r_{-1} \in S^{-1} \ .$$

So we write

$$\|(\mathrm{Op}^W(v)-\lambda)\Phi_j\|^2 = \langle \mathrm{Op}^W\big((v-\lambda)^2\big)\,\Phi_j, \Phi_j\rangle + \langle \mathrm{Op}^W(r_{-1})\,\Phi_j, \Phi_j\rangle \ .$$

Since $\mathrm{Op}^W(r_{-1}) \in \mathcal{S}^{-1}$ is compact and $\Phi_j \rightharpoonup 0$, it follows that $\langle \mathrm{Op}^W(r_{-1})\,\Phi_j, \Phi_j\rangle \to 0$ and therefore it suffices to show that $\langle \mathrm{Op}^W\big((v-\lambda)^2\big)\,\Phi_j, \Phi_j\rangle \overset{j\to\infty}{\to} 0$. To estimate this last term we shall use the following identities:

$$\mathcal{T}_z^{-1}\,\mathrm{Op}^W(a)\,\mathcal{T}_z = \mathrm{Op}^W(a(\cdot+z)) \ , \tag{A.4}$$

$$\langle \mathrm{Op}^W(a)\,\Phi_0, \Phi_0\rangle = \frac{1}{\pi}\int_{\mathbb{R}^2} a(x,\xi)\, e^{-(x^2+\xi^2)}\,\mathrm{d}x\,\mathrm{d}\xi. \tag{A.5}$$

They are both contained in the book [11]: the first one is formula (2.16), whereas the second one follows combining Proposition 14 and 16 (with $\hbar = 1$) in chap. 2.
We deduce, using $\mathcal{T}_{z_j}^* = \mathcal{T}_{z_j}^{-1}$,

$$\begin{aligned}
\langle \mathrm{Op}^W\big((v-\lambda)^2\big)\,\Phi_j, \Phi_j\rangle &= \langle \mathcal{T}_{z_j}^{-1}\,\mathrm{Op}^W\big((v-\lambda)^2\big)\,\mathcal{T}_{z_j}\Phi_0, \Phi_0\rangle \\
&\overset{(A.4)}{=} \langle \mathrm{Op}^W\Big(\big(v(\cdot+z_j)-\lambda\big)^2\Big)\,\Phi_0, \Phi_0\rangle \\
&\overset{(A.5)}{=} \pi^{-1}\int_{\mathbb{R}^2} \big(v(x+q_j, \xi+p_j)-\lambda\big)^2\, e^{-(x^2+\xi^2)}\,\mathrm{d}x\,\mathrm{d}\xi
\end{aligned}$$

The last integral converges to 0 as $j\to\infty$ by Lebesgue's dominated convergence theorem, since $v \in S^0$ is bounded and fulfills $v(x+q_j,\xi+p_j) \overset{j\to\infty}{\to} \lambda$ pointwise when $(q_j,p_j)\to\infty$. ☐

Lemma A.2. *Let $z_j := (q_j, p_j) \to \infty$. Then, up to a subsequence, $\Phi_j := \mathcal{T}_{z_j}\Phi_0 \rightharpoonup 0$.*

Proof. We distinguish two cases: (i) the sequence $\{q_j\}_j$ is bounded and (ii) up to a subsequence $|q_j| \to \infty$.
In case (i), up to a subsequence we can assume $q_j \to q_0 \in \mathbb{R}$ and $p_j \to \infty$. Take an arbitrary $f \in L^2(\mathbb{R})$. We write

$$\int_{\mathbb{R}} f(x)\,\overline{\Phi_j(x)}\,\mathrm{d}x = \underbrace{\int_{\mathbb{R}} f(x)\,\overline{\Phi_{(q_0,p_j)}(x)}\mathrm{d}x}_{=:\mathtt{I}_1} + \underbrace{\int_{\mathbb{R}} f(x)\,\overline{\big(\Phi_{(q_j,p_j)}(x)-\Phi_{(q_0,p_j)}(x)\big)}\mathrm{d}x}_{=:\mathtt{I}_2} \ . \tag{A.6}$$

Using Riemann-Lebesgue lemma

$$\mathtt{I}_1 = \frac{e^{\frac{\mathrm{i}}{2}p_j q_0}}{\pi^{\frac14}}\int_{\mathbb{R}} f(x)\, e^{-\frac{(x-q_0)^2}{2}}\, e^{-\mathrm{i}x p_j}\,\mathrm{d}x \overset{p_j\to\infty}{\to} 0 \ .$$

To estimate the second integral in (A.6) we write

$$\begin{aligned}
\mathtt{I}_2 &= \pi^{-\frac14}\int_{\mathbb{R}} e^{-\frac{x^2}{2}}\, e^{-\mathrm{i}x p_j}\,\Big(f(x+q_j)e^{-\frac{\mathrm{i}}{2}p_j q_j} - f(x+q_0)e^{-\frac{\mathrm{i}}{2}p_j q_0}\Big)\,\mathrm{d}x \\
&= \underbrace{\frac{e^{-\frac{\mathrm{i}}{2}p_j q_j}}{\pi^{\frac14}}\int_{\mathbb{R}} e^{-\frac{x^2}{2}}\, e^{-\mathrm{i}x p_j}\,\big(f(x+q_j)-f(x+q_0)\big)\,\mathrm{d}x}_{\mathtt{I}_{21}} + \underbrace{\frac{e^{-\frac{\mathrm{i}}{2}p_j q_j} - e^{-\frac{\mathrm{i}}{2}p_j q_0}}{\pi^{\frac14}}\int_{\mathbb{R}} e^{-\frac{x^2}{2}}\, e^{-\mathrm{i}x p_j}\, f(x+q_0)\,\mathrm{d}x}_{\mathtt{I}_{22}}
\end{aligned}$$

Next we have that

$$|\mathtt{I}_{21}| \leq C\|f(\cdot + q_j) - f(\cdot + q_0)\|_{L^2(\mathbb{R})} \overset{q_j \to q_0}{\to} 0$$

by the continuity of the translations in $L^2(\mathbb{R})$, whereas

$$|\mathtt{I}_{22}| \leq C \left| \int_{\mathbb{R}} e^{-\frac{x^2}{2}} \, e^{-\mathrm{i}xp_j} \, f(x+q_0) \, \mathrm{d}x \right| \overset{p_j \to \infty}{\to} 0$$

by Riemann-Lebesgue lemma. In conclusion we have proved that $\int_{\mathbb{R}} f(x) \, \overline{\Phi_j(x)} \, \mathrm{d}x \to 0$ as $j \to \infty$. This concludes the proof of case (i).

In case (ii) we can assume that, up to a subsequence, q_j has always the same sign. Let $f \in L^2(\mathbb{R})$. Then one easily shows that

$$\left| \int_{\mathbb{R}} f(x) \, \overline{\Phi_j(x)} \mathrm{d}x \right| \leq \pi^{-1/4} \int_{\mathbb{R}} |f(x+q_j)| \, e^{-\frac{x^2}{2}} \, \mathrm{d}x \overset{|q_j| \to \infty}{\to} 0$$

concluding the proof of case (ii). □

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