# Berezinskii-Kosterlitz-Thouless transitions in classical and quantum long-range systems

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In the past decades, considerable efforts have been made to understand the critical features of both classical and quantum long-range (LR) interacting models. The case of the Berezinskii-Kosterlitz-Thouless (BKT) universality class, as in the two-dimensional (2D) classical $XY$ model, is considerably complicated by the presence, for short-range interactions, of a line of renormalization group fixed points. In this paper, we discuss a field-theoretical treatment of the 2D $XY$ model with LR couplings, and we compare it with results from the self-consistent harmonic approximation. These methods lead to a rich phase diagram, where both power law BKT scaling and spontaneous symmetry breaking appear for the same (intermediate) decay rates of LR interactions. We also discuss the Villain approximation for the 2D $XY$ model with power law couplings, providing hints that, in the LR regime, it fails to reproduce the correct critical behavior. The obtained results are then applied to the LR quantum $XXZ$ spin chain at zero temperature. We discuss the relation between the phase diagrams of the two models, and we give predictions about the scaling of the order parameter of the quantum chain close to the transition.

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## I. INTRODUCTION

In the context of statistical mechanics and condensed matter, it is well known that the presence of long-range (LR) interactions, such as slow-decaying couplings with power law behavior at large distances, gives rise to plenty of physical phenomena in the equilibrium and nonequilibrium properties of both classical [1] and quantum [2] systems. These properties have recently sparked a wave of interest due to the possibility of experimental realizations in atomic, molecular, and optical systems [2–9].

If the decay of the interaction is slow enough, LR effects influence the universal critical properties and even induce a spontaneous symmetry breaking (SSB) in a low-dimensional system since the celebrated Hohenberg-Mermin-Wagner theorem does not hold in the presence of LR couplings, as discussed since the classic papers [10–12].

A qualitative and quantitative understanding of the critical properties of LR interacting systems follows from the Sak criterion [13]. Let us consider power law couplings of the form $\sim 1/r^{d+\sigma}$, where $d$ is the system dimension. For the sake of clarity, in the following considerations, we explicitly refer to classical $O(n)$ models [14]. In the Gaussian noninteracting case ($n \to \infty$), for large distances and low energies, one must compare the short-range (SR) critical propagator $1/k^2$ with the LR one $1/k^\sigma$. However, interacting systems possess a finite anomalous dimension $\eta$ at the critical point, so that the propagator scales as $1/k^{2-\eta}$ [15]. Denoting the anomalous dimension in the SR limit of the theory by $\eta_{\mathrm{sr}}$, i.e., $\sigma \to \infty$, one concludes that, at criticality for $\sigma > 2 - \eta_{\mathrm{sr}}$, the system behaves as its SR counterpart, while for $\sigma < 2 - \eta_{\mathrm{sr}}$, the LR nature of the couplings modifies the critical behavior. This argument is at the heart of Sak's seminal paper [13], and it implies that one can define a value $\sigma_*$ such that, for $\sigma > \sigma_*$, the system is in the universality class of its SR counterpart, while for $\sigma < \sigma_*$, the system changes universality class. The value of $\sigma_*$ is given by

$$\sigma_* = 2 - \eta_{\mathrm{sr}}. \tag{1}$$

Sak's criterion was formulated for classical $O(n)$ models, such as the Ising model, a playground where it has been the subject of thorough investigation. Regarding its validity, it is fair to conclude that it must be considered valid and well tested, see reviews [16,17] and references therein. Despite the fact that there is no rigorous proof for it, and numerical results can only put an upper bound on its violation, its usefulness is unquestioned. For the quantum $O(n)$ models, one finds $\sigma_* = 2$ at mean-field level [18], and one can obtain beyond-mean-field results by applying renormalization group (RG) techniques [19].

Sak's criterion applies to all $d$ and $n$, except for $d = 2$ and $n = 2$, i.e., the two-dimensional (2D) $XY$ universality class. There, it is well known that SSB occurs at low temperature for $\sigma < 2$ [20], and therefore, several questions arise concerning the applicability of Sak's criterion in the SR limit, where the model does not present SSB. Moreover, in the

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2D $XY$ model, $\eta_{sr}$—unlike the other cases covered by Sak's criterion—depends on the temperature, raising further concerns about the applicability of the criterion to the 2D $XY$ model with LR couplings.

In the recent paper [21], the fate of the Berezinskii-Kosterlitz-Thouless (BKT) line of critical points has been discussed. For $\sigma > 2$, simple arguments (not directly applicable to $\sigma < 2$) show the existence of the BKT transition [22], while what happens for $\sigma < 2$ is considerably more subtle. In Ref. [21], SSB was found to persist in the range $\frac{7}{4} < \sigma < 2$ up to a finite critical temperature $T_c$, beyond which a quasi-ordered BKT-like phase appears below a larger temperature $T_{\rm BKT}$. These results are compatible with the application of Sak's criterion with the temperature-dependent $\eta_{sr}$ of the $XY$ model in the SR limit.

Since the classical 2D $XY$ model at finite temperature lies in the same universality class of the quantum $XXZ$ chain at $T = 0$ for nearest-neighbors interactions, a natural question is how to apply these results for the 2D LR $XY$ model to the one-dimensional (1D) LR $XXZ$ chain. Remember that, in the SR limit, a mapping can be obtained via the transfer matrix technique [23]. However, this method does not straightforwardly apply to the LR case nor to any nonlocal couplings, in general. While it is natural to think the two models are in the same universality class for fast-decaying couplings as well, it is not clear what happens in the LR regime. We observe that the 2D boson gas at finite temperature with $1/r^3$ density-density interactions is equivalent to a quantum $XXZ$ chain with $z - z$ LR couplings, and it has been found to follow the usual BKT scenario [24] with no signatures of SSB.

The LR $1d$ $XXZ$ chain has been studied in Ref. [25], where evidence was provided showing the existence of three phases in the $T = 0$ phase diagram as a function of $\sigma$ and $\Delta$ (with $\Delta$ the anisotropy parameter). One phase features SSB, another is disordered, and the third is gapless. The comparison between the phase diagrams of Ref. [21] for the 2D classical finite-temperature $XY$ model and of Ref. [25] for the $1d$ quantum zero temperature $XXZ$ model is one of the motivations of this paper.

The study of the 2D $XY$ model with LR interactions is of great interest per se: the model is indeed paradigmatic for the study of $O(2)$-symmetric systems in $d = 2$ (including thin $^4$He films [26], quasi-2D layered superconductors [27–31], exciton-polariton systems [32], cold atoms in 2D traps [33,34], and 2D electron gases at the interface between insulating oxides in artificial heterostructures [35–37]).

In this paper, the scenario presented in Ref. [21] for the 2D $XY$ model is analyzed in detail and investigated using different methods: the low-temperature expansion, an extension of the traditional self-consistent harmonic approximation (SCHA) [22,38,39] to $\sigma < 2$, and RG field-theoretical arguments. The difficulties inherent in the use of the Villain approximation for $\sigma < 2$ are pointed out and contrasted with the SR case where, at variance, the 2D Villain and $XY$ model are related through controllable approximations and lie in the same universality class. Using these results, we closely examine the link between the classical $XY$ model in 2D and the quantum $XXZ$ chain, leading to a detailed analysis of the quantum model. The comparison with the results of Ref. [25] is presented. Comments on the BKT transition in LR spin-$S$ quantum chains are also presented.

The paper is structured as follows: after presenting the model in Sec. II, in Sec. III, we describe its low-temperature limit. In Secs. IV and V, we examine the peculiar nature of the crossover between LR and SR regimes. Section. IV is devoted to extending the self-consistent approximation to the proper $\sigma < 2$ LR regime, while in Sec. V, we examine the field-theory picture. In Sec. VI, we discuss the relation between our analysis and the Villain method used to understand the BKT phase transition in the SR regime, showing the difficulties encountered to describe this crossover. Finally, in Sec. VII, we examine the relationship between the $d = 2$ classical $XY$ model and the 1D quantum $XXZ$ chain, showing how our results can be used to make predictions about the $T = 0$ phase diagram of the latter.

## II. THE $XY$ MODEL

The model consists of a set of $N$ planar rotators, arranged in a 2D square lattice, interacting through the Hamiltonian:

$$\beta H = \frac{1}{2} \sum_{\mathbf{i},\mathbf{j}} J(r)[1 - \cos(\theta_{\mathbf{i}} - \theta_{\mathbf{j}})], \tag{2}$$

with $\mathbf{i}, \mathbf{j} \in \mathbb{Z}^2$, $\mathbf{r} = \mathbf{i} - \mathbf{j}$, $r = |\mathbf{r}|$, and $J(r) \sim J r^{-2-\sigma}$ for $r \gg 1$, where $J = \beta J_0$. Here, we used a particular instance of the general convention $J(r) \sim r^{-d-\sigma}$ used in the $d$-dimensional case (see, e.g., Ref. [19]), which assures that the thermodynamical quantities remain additive for any $\sigma > 0$ [1]. The interaction between the spins $\mathbf{S}_{\mathbf{x}} \equiv [\cos\theta(\mathbf{x}), \sin\theta(\mathbf{x})]$ is invariant under global $O(2)$ rotations. In what follows, we take our energy units so that $J_0 = 1$, and we will limit ourself to the $\sigma > 0$ case.

In the SR regime, although the possibility of a SSB is ruled out by the Hohenberg-Mermin-Wagner theorem, the low-temperature phase is characterized by quasi-LR order, i.e., power law behavior in the connected correlation functions with a temperature-dependent exponent. The transition between this phase and the disordered, high-temperature one, i.e., the well-known BKT mechanism, may be understood in terms of the unbinding of topological defects, leading to the celebrated mapping to a Coulomb gas of charged particles [40–44].

When the decay of the couplings is slow enough, we expect a SSB phase to appear at low temperature. The accepted criterion for understanding whether the LR interactions modify the critical behavior is the one provided by Sak [13]. However, as discussed in the introduction, this criterion cannot be straightforwardly applied to the BKT mechanism due to the temperature-dependent anomalous dimension $\eta_{sr}$. Moreover, the duality construction [45], which allows the mapping between the 2D SR $XY$ model, the Coulomb gas [46,47], and the sine-Gordon model [47,48], already breaks down in the case of next-nearest-neighbor couplings. The other way of obtaining the same mappings is the so-called Villain approximation [49,50] (i.e., the substitution of the Hamiltonian in Eq. (3) with a quadratic one which considers the phase periodicity). However, as shown in Sec. VI of this paper, there is reason to believe that this latter Hamiltonian is no longer in the same universality class as soon as the interaction becomes LR, so that the Coulomb gas picture cannot be straightforwardly used to describe the physics of the 2D LR *XY* model.

## III. LOW-TEMPERATURE EXPANSION

In analogy with the SR regime, we expect the low-temperature behavior of the model to be well described by an approximation *à la* Berezinskii [41]. Since we expect our thermal fluctuations to be small in this regime, we expand the cosine in the Hamiltonian in Eq. (3) to second order, obtaining, up to an immaterial constant term,

$$\beta H \sim \frac{1}{4}\sum_{\mathbf{i},\mathbf{j}} J(r)(\theta_{\mathbf{i}}-\theta_{\mathbf{j}})^2. \tag{3}$$

Since this theory is quadratic and translationally invariant, we can diagonalize the above Hamiltonian by means of a Fourier transform:

$$\theta_{\mathbf{q}} = \frac{1}{\sqrt{N}}\sum_{\mathbf{j}} \exp(-i\mathbf{q}\cdot\mathbf{j})\,\theta_{\mathbf{j}},$$

$$\theta_{\mathbf{j}} = \frac{1}{\sqrt{N}}\sum_{\mathbf{q}\in\mathrm{IBZ}} \exp(i\mathbf{q}\cdot\mathbf{j})\,\theta_{\mathbf{q}}. \tag{4}$$

We obtain

$$\beta H \sim \frac{1}{2}\sum_{\mathbf{q}\in\mathrm{IBZ}} K(\mathbf{q})|\theta_{\mathbf{q}}|^2, \tag{5}$$

where we introduced

$$K(\mathbf{q}) = \sum_{\mathbf{r}} J(\mathbf{r})[1-\cos(\mathbf{q}\cdot\mathbf{r})]. \tag{6}$$

If we are interested in the long-wavelength modes, we can approximate the above sum with a continuous integral, obtaining

$$K(q) = \int_{r>a} d^2\mathbf{r} J(r)[1-\cos(\mathbf{q}\cdot\mathbf{r})]. \tag{7}$$

As shown in Appendix A, this quantity grows as $K(q)\sim q^2$ if $\sigma > 2$, while it shows a nonanalytic behavior $K(q)\sim q^\sigma$ for $\sigma < 2$. The difference between the two regimes can be fully appreciated if we consider the two-point correlation function:

$$\langle \mathbf{s}_{\mathbf{r}}\cdot\mathbf{s}_0\rangle = \langle \cos(\theta_{\mathbf{r}}-\theta_0)\rangle. \tag{8}$$

Since the theory is quadratic in this approximation, the above quantity can be evaluated quite easily by exploiting the identity $\langle \cos A\rangle_0 = \exp(-\frac{1}{2}\langle A^2\rangle_0)$. From Eq. (5), it follows immediately that $\langle \theta_{\mathbf{q}}\theta_{\mathbf{q}'}\rangle_0 = \delta_{\mathbf{q}+\mathbf{q}'}K(\mathbf{q})^{-1}$, so that we find $\langle \cos(\theta_{\mathbf{r}}-\theta_0)\rangle = \exp[-G(\mathbf{r})]$ with

$$G(\mathbf{r}) = \frac{1}{2}\langle(\theta_0-\theta_{\mathbf{r}})^2\rangle_0 = \frac{1}{N}\sum_{\mathbf{q}\in\mathrm{IBZ}}\frac{1-\cos(\mathbf{q}\cdot\mathbf{r})}{K(\mathbf{q})}. \tag{9}$$

The long-distance behavior can once again be captured by replacing the sum with an integral:

$$G(r) = a^2\int_{q<\Lambda}\frac{d^2\mathbf{q}}{(2\pi)^2}\frac{1-\cos(\mathbf{q}\cdot\mathbf{r})}{K(q)}, \tag{10}$$

where we approximated the first Brillouin zone with a sphere of radius $\Lambda = \sqrt{8\pi}a^{-1}$, so that its volume is preserved, i.e., $\int_{q<\Lambda} d^2\mathbf{q} = (2\pi)^2 a^{-2}$. As shown in Appendix B, the asymptotic behavior of $G(r)$ depends on $\sigma$. If $\sigma > 2$, we have

$$G(r) \sim \eta(J)\ln\frac{r}{a} + AJ^{-1}, \tag{11}$$

with $\eta(J) = p/J$, and $A$, $p$ are nonuniversal constants (see a study of these constants in Ref. [22]). If $\sigma < 2$ instead, we have that

$$G(r) \sim AJ^{-1}. \tag{12}$$

Then depending on whether $\sigma < 2$ or $\sigma > 2$, we have that the correlation $\langle \mathbf{s}_{\mathbf{r}}\cdot\mathbf{s}_0\rangle \sim$ const. or $\langle \mathbf{s}_{\mathbf{r}}\cdot\mathbf{s}_0\rangle \sim r^{-\eta(J)}$, respectively. In the latter case then, we recover the SR low-temperature BKT behavior [41], in which the correlations decay as a power law with a temperature-dependent exponent. The former case, instead, gives rise to a finite-magnetization ordered phase, with

$$m^2 = \lim_{r\to\infty}\langle \mathbf{s}_{\mathbf{r}}\cdot\mathbf{s}_0\rangle = \lim_{r\to\infty}\exp[-G(r)]. \tag{13}$$

This argument then suggests that $\sigma_* = 2$. If compared with Sak's criterion, this implies an effective anomalous exponent $\eta = 0$. Let us note, however, how the low-temperature approximation per se cannot describe the topological configurations since $\theta_{\mathbf{i}}$ is no longer a phase and thus is no longer defined up to multiples of $2\pi$. As a consequence, even in the SR case, the approximation cannot correctly reproduce all the phenomenology of the BKT transition.

## IV. SELF-CONSISTENT APPROACH

For the nearest-neighbor case, the results of the low-temperature expansion can be improved by the SCHA, which can correctly foresee the existence of the BKT phase transition. The idea behind the SCHA is to replace the original Hamiltonian with a quadratic one, whose couplings are optimized according to a variational principle.

In the SR case, the natural variational parameter is an effective coupling $\tilde{J}$, which replaces the bare one $J$ in the effective theory. As shown in Refs. [38,39], in this case, $\tilde{J}$ jumps discontinuously to zero as the temperature increases, implying that the exponent of the correlations becomes infinite in the high-temperature region. This path has been pursued for the case of a power law interaction as well: in this case, both the coupling $J$ and the exponent of the interaction $\sigma$ are replaced by variational parameters $\tilde{J}$ and $s$. The results are unambiguous if $\sigma > 2$, predicting a phenomenology analogous to the SR case [22]. Our aim is to generalize such an analysis to correctly deal with the $\sigma \in (0, 2]$ regime.

To achieve this scope, we replace the cosine in the original Hamiltonian in Eq. (3) with a quadratic term:

$$\beta H_0 = \frac{1}{4}\sum_{\mathbf{i},\mathbf{j}}\tilde{J}(\mathbf{r})(\theta_{\mathbf{i}}-\theta_{\mathbf{j}})^2, \tag{14}$$

where $\tilde{J}(r)$ is a completely arbitrary function of $\mathbf{r} = \mathbf{i}-\mathbf{j}$, to be determined from free energy minimization in a self-consistent way.

The quadratic Hamiltonian $H_0$ induces the Boltzmann measure:

$$\langle\cdot\rangle_0 = \frac{1}{Z_0}\int\prod_{\mathbf{j}} d\theta_{\mathbf{j}}\ \exp(-\beta H_0), \tag{15}$$

where

$$Z_0 = \exp(-\beta F_0) = \int\prod_{\mathbf{j}} d\theta_{\mathbf{j}}\ \exp(-\beta H_0) \tag{16}$$

is the partition function of the model. The variational principle establishes that our best ansatz minimizes the variational free energy:

$$\mathcal{F} = \beta F_0 + \beta\langle H\rangle_0 - \beta\langle H_0\rangle_0. \tag{17}$$

On the other hand, from the equipartition theorem, it follows that $\langle H_0\rangle_0 = \frac{N}{2\beta}$ is independent on the choice of $\tilde{J}(\mathbf{r})$, so that we can safely ignore it. Since $H_0$ has the same quadratic structure as the low-temperature Hamiltonian, we can diagonalize $H_0$ as well by means of the Fourier transform:

$$\beta H_0 = \frac{1}{2}\sum_{\mathbf{q}\in\mathrm{IBZ}} \tilde{K}(\mathbf{q})|\theta_{\mathbf{q}}|^2, \tag{18}$$

where $\tilde{K}(\mathbf{q})$ is given by Eq. (6) with $J(r)$ replaced by $\tilde{J}(r)$. As shown in Appendix B, the variational free energy takes the form:

$$\mathcal{F} = \frac{1}{2}\sum_{\mathbf{q}\in\mathrm{IBZ}} \ln\tilde{K}(\mathbf{q}) - \frac{N}{2}\sum_{\mathbf{r}} J(r)\exp[-\tilde{G}(\mathbf{r})], \tag{19}$$

where, once again, $\tilde{G}(\mathbf{r})$ is given by Eq. (9) with $J(r)$ replaced by $\tilde{J}(r)$. To simplify the notation, in the following, we will drop the $\sim$ symbol for $K(\mathbf{q})$.

Since in Eq.(19) $\tilde{J}(\mathbf{r})$ appears only through the $K(\mathbf{q})$, to find the minimum is sufficient to derive $\mathcal{F}$ with respect to the latter. By exploiting the fact that

$$\frac{\delta\tilde{G}(\mathbf{r})}{\delta K(\mathbf{q})} = -\frac{1}{N}\frac{1-\cos(\mathbf{q}\cdot\mathbf{r})}{K(\mathbf{q})^2}, \tag{20}$$

we find

$$\frac{\delta\mathcal{F}}{\delta K(\mathbf{q})} = \frac{K(\mathbf{q}) - \sum_r J(r)[1-\cos(\mathbf{q}\cdot\mathbf{r})]\exp[-G(\mathbf{r})]}{2K(\mathbf{q})^2}. \tag{21}$$

By using the definition in Eq. (6) of $K(q)$ and noticing that the above expression must be valid for each value of $\mathbf{q}\in$ IBZ (first Brillouin zone), we find the condition:

$$J(r) = \tilde{J}(\mathbf{r})\exp[\tilde{G}(\mathbf{r})]. \tag{22}$$

Since we are interested in the large length-scale regime (i.e., in the continuous limit), we can assume $\tilde{J}$ to be a function of $r$ only: in this case, indeed, $\tilde{G}(\mathbf{r})$ only depends on the modulus $r$ as well, so that the above condition can be written in terms of single variable functions:

$$J(r) = \tilde{J}(r)\exp[\tilde{G}(r)]. \tag{23}$$

Let us note how, in this limit, one has to redefine $\mathbf{r}\to a\mathbf{r}$, $\mathbf{q}\to a^{-1}\mathbf{q}$, and $J,\tilde{J}\to a^{-2}J, a^{-2}\tilde{J}$.

Let us now discuss the possible solutions of Eq. (23). The possible asymptotic behaviors of $K(q)$ and $\tilde{G}(q)$ in terms of $\sigma$ are examined in Appendix A. We find that

(1) If $\tilde{J}(r)$ decays at infinity faster than $r^{-4}$ (e.g., an exponential or a fast-decaying power law), then we have that $\tilde{G}(r)\sim\ln(r)$ as $r\to\infty$. Since in this case $\exp[\tilde{G}(r)]$ is a power law, it follows from Eq. (23) that $\tilde{J}(r)$ must behave asymptotically as a power law as well. We can then assume that, for large $r$,

$$\tilde{J}(r)\sim\tilde{J}r^{-2-s} \tag{24}$$

for some $s>2$ to be determined, so that

$$\tilde{G}(r)\sim\eta(\tilde{J})\ln\frac{r}{a} + A\tilde{J}^{-1}, \tag{25}$$

where $\eta(\tilde{J}) = p\tilde{J}^{-1}$ and $A$, $p$ are nonuniversal constants. Finally, from Eq. (23), we have the conditions:

$$\sigma = s - \eta(\tilde{J}), \quad J = \tilde{J}\exp\left(\frac{A}{\tilde{J}}\right). \tag{26}$$

In this case, the correlation functions decay as $\exp[-\tilde{G}(r)]\sim r^{-\eta(\tilde{J})}$. Then if $\tilde{J}$ is nonvanishing, we find a quasi-LR order, characteristic of the BKT phenomenology.

(2) If the variational coupling behaves as $\tilde{J}(r)\sim\tilde{J}r^{-2-s}$ with $s\in(0,2)$, then $G(r) = A\tilde{J}^{-1} + O(r^{s-2})$. In this case, we then have

$$\sigma = s, \quad J = \tilde{J}\exp\left(\frac{A}{\tilde{J}}\right), \tag{27}$$

leading to correlations behaving as $\exp[-\tilde{G}(r)]\sim\exp(A/\tilde{J})$. Then if $\tilde{J}$ is nonvanishing, we find a finite magnetization $\sim\exp[(A/\tilde{J})/2]$.

In both cases, the equation for $\tilde{J}$ has the same form as the nearest-neighbor case. By introducing $\tilde{J} = Ax$, $J = Ay$, it can be written as

$$y = x\exp\left(\frac{1}{x}\right). \tag{28}$$

The minimum of the right-hand-side term is in correspondence with $x=1$, $y=e$, so that we have two solutions for $J>J_c\equiv eA$ and only the trivial solution for $J<J_c$, signaling a jump in $\tilde{J}$ from $\tilde{J}_c = A$ to zero. However, only the larger of the two solutions present for $J>J_c$ is physically acceptable. Indeed, it is the only one to have the correct asymptotic behavior $\tilde{J}(r)\sim J(r)$ in the large $J$ regime, in which our SCHA becomes the low-temperature approximation studied in Sec. III. The meaning of this low-temperature, finite $\tilde{J}$ phase, however, depends on the whether $s<2$ (ordered phase) or $s>2$ (quasi-LR-ordered BKT phase). Let us note that, in the latter case, we have that $\eta = p\tilde{J}^{-1}$ of Eq. (25) cannot be larger than a given value $\eta_c = p\tilde{J}_c^{-1}$. As a consequence, we have that

(1) For $\sigma>2$, the only possibility is that $s=\sigma+\eta_{\mathrm{sr}}>2$. We are then in the first case, so that we get the usual BKT phenomenology. This agrees with the findings of Ref. [22].

(2) For $\sigma<2-\eta_c$, the only possibility is that $s=\sigma$ with $s<2$. We are then in the second case, i.e., we find a finite magnetization for low temperature.

(3) For $2-\eta_c<\sigma<2$, both solutions are viable, so that it is unclear whether the system is in the ordered or in the quasi-LR-ordered phase. To establish which solution we should choose, we should compute $\mathcal{F}$ on both solutions. The dependence of $\mathcal{F}$ on the nonuniversal details of the model,

particularly on the SR behavior of $\tilde{J}(r)$, hinders the possibility to reach a definite conclusion for this regime within the SCHA.

Although nonconclusive, our self-consistent analysis accounts for the possibility of the existence of an intermediate region, in which the ordered phase or the BKT behavior can prevail, depending on the temperature. Keeping in mind the results of the Berezinskii approximation, it appears sensible to think that the magnetized phase will prevail at lower temperatures, while the quasi-LR-ordered phase will, possibly, correspond to an intermediate range of temperature. Let us note, however, that the predictions of our analysis are nonuniversal, and thus, their quantitative outcome depends on the specific model under study. Moreover, the first-order phase transition foreseen for $\sigma < 2 - \eta_c$ could be an artifact of the approximation since the critical behavior of the model is known to be captured by the mean-field approximation for $\sigma < 1$, which foresees a second-order phase transition, see, e.g., Ref. [19].

## V. FIELD-THEORETICAL APPROACH

To go beyond the limits of the SCHA and capture the universal quantities we are interested in, we resort to a field-theoretical approach, introducing a continuous action, which encodes the same physics of our Hamiltonian in Eq. (3). First, we write the coupling as $J(r) = J_S(r) + J_{\rm LR} r^{-(2+\sigma)}$, $J_S$ being a SR term which accounts for the small-distance behavior. This decomposition allows us to refine our low-temperature approximation. This, in fact, is fully justified for the SR coupling, even at intermediate temperatures, since it couples neighboring sites. At the same time, it can become too crude for the LR part of the Hamiltonian. Since this couples far-away pairs, there can be smooth configurations for which the phase difference $\theta_{\mathbf{i}} - \theta_{\mathbf{j}}$ is not necessarily small, and these configurations may give a significant contribution to the Hamiltonian in an intermediate range of temperatures. However, even for the SR term, the approximation *à la* Berezinskii cannot capture the presence of vortices.

To further proceed, we expand the cosine in the SR part:

$$1 - \cos(\theta_{\mathbf{i}} - \theta_{\mathbf{j}}) \approx \frac{1}{2}\left(\theta_{\mathbf{i}} - \theta_{\mathbf{j}}\right)^2 \approx \frac{1}{2}|\nabla\theta|^2, \tag{29}$$

where the last substitution is justified by the fact that, in the SR part of the Hamiltonian, only neighboring lattice sites are important. We then find

$$S[\theta] = \frac{J}{2}\int d^2x |\nabla\theta|^2 + S_{\rm LR}[\theta], \tag{30}$$

where we introduced the LR perturbation:

$$S_{\rm LR} = J_{\rm LR}\int d^2x \int_{r>a} \frac{d^2r}{r^{2+\sigma}}\{1 - \cos[\theta(\mathbf{x}) - \theta(\mathbf{x}+\mathbf{r})]\}. \tag{31}$$

In turn, this term can be rewritten in terms of the fractional Laplacian, whose definition is given, along with the details of the calculation, in Appendix C. We obtain

$$S_{\rm LR} = \frac{g}{2}\int d^2x \, \exp(-i\theta)\, \nabla^\sigma \exp(i\theta), \tag{32}$$

with $g = J_{\rm LR}/\gamma_{2,\sigma}$ and $\gamma_{2,\sigma} = 2^\sigma \Gamma(\frac{1+\sigma}{2})\pi^{-1}|\Gamma(-\frac{\sigma}{2})|^{-1}$. This term is intrinsically interacting. Moreover, it is invariant under global translations $\theta(\mathbf{x}) \to \theta(\mathbf{x}) + \alpha$, and it correctly catches the fact that the field $\theta$ is periodic. However, because of the kinetic term in $S[\theta]$, the whole action still does not correctly describes topological phenomena.

Let us remember briefly the $g = 0$ case. Here, one can introduce vortices in Eq. (30), i.e., evaluating the energy cost of a vortex configuration, and the core energy cost of a single vortex $\varepsilon_c$, which can be absorbed into the definition of the vortex fugacity $y = \exp(-\varepsilon_c)$. Since topological and nontopological configurations decouple in the quadratic part of the action, this leads to the Coulomb gas picture and to the well-known Kosterlitz-Thouless RG equations [42]. In turn, this implies that all the fixed points for $y = 0$ and $J > \frac{2}{\pi}$, which correspond to a Gaussian massless action, are stable. There, the low-temperature approximation becomes correct (with the original $J(T)$ replaced by the one corresponding to the fixed point), so that the power law scaling observed for the two-point functions is recovered. For $J < \frac{2}{\pi}$, however, vortices become relevant, and the theory flows toward the disordered regime.

To study the transition between the SR and LR regimes, we can consider the effect of a small $g$ deformation of the Gaussian fixed points. In this case, we expect to be able to parametrize our theory in terms of three parameters: $J$, $g$, and $y$. In Appendix D, we derive the RG flow perturbatively in $y$ and $g$. However, we provide here an argument to understand what one can expect on general grounds. First, we notice that the fixed points with $J < \frac{2}{\pi}$ are already unstable under topological perturbations, so that we will stick to the stable fixed line with $J > \frac{2}{\pi}$. Here, the LR perturbation can be relevant or not, depending on the scaling dimension $\Delta_g$ of the coupling $g$, $g_\ell \approx \exp(\Delta_g \ell)$ for $\ell \gg 1$, where $\ell = \ln(r/a)$ is the RG time.

Due to the quadratic nature of the measure, one has

$$\begin{aligned}\langle \exp\{i[\theta(\mathbf{x}) - \theta(\mathbf{x}')]\}\rangle &= \exp\left\{-\tfrac{1}{2}\langle[\theta(\mathbf{x}) - \theta(\mathbf{x}')]^2\rangle\right\}\\ &= |\mathbf{x} - \mathbf{x}'|^{-\eta_{\rm sr}(J)},\end{aligned} \tag{33}$$

where $\eta_{\rm sr}(J) = \frac{1}{2\pi J}$ corresponds to the exponent of the two-point function for $g = 0$ [42,43,50]. Then

$$\Delta_g = 2 - \sigma - \eta_{\rm sr}(J). \tag{34}$$

The LR perturbation is relevant only in the regime $\sigma < 2 - \eta_{\rm sr}(J)$. This bears some similarly with the SSB case [14] but with the main difference that the anomalous dimension is temperature dependent.

For $\sigma > 2$, the LR term is always irrelevant. For $\sigma < 2$, Eq. (34) predicts that the LR perturbation is always relevant at low enough temperatures. Since the points which are stable under the proliferation of vortices are those with $J > \frac{2}{\pi}$, we have that $0 < \eta_{\rm sr} < \frac{1}{4}$, as in the usual BKT theory (which indeed corresponds to the $g = 0$ theory). As a consequence, for $\frac{7}{4} < \sigma < 2$, we have that there exists an intermediate range of value of $J$ for which the Gaussian theory is stable with respect to both the topological and the LR perturbations, leading to conventional quasi-LR order in a given temperature window. Instead, for $\sigma < \frac{7}{4}$, the BKT stable fixed line is completely swallowed by the action of the LR perturbation, so that our perturbative picture breaks down. However, it is sensible to assume that, in this regime, the system simply undergoes an order-disorder phase transition.

This picture is essentially in agreement with the results of the SCHA. However, our results no longer depend on the regularization procedure or the exact form of $J_S(r)$ and are genuinely universal. Our observation can be strengthened by the renormalization equations (see Appendix D for details) which, at the leading order in $y$ and $g$, are given by

$$\frac{dy_\ell}{d\ell} = (2 - \pi J)y,$$
$$\frac{dg_\ell}{d\ell} = \left(2 - \sigma - \frac{1}{2\pi J_\ell}\right) g_\ell,$$
$$\frac{dJ_\ell}{d\ell} = c_\sigma \eta_{\rm sr}(J_\ell) g_\ell, \tag{35}$$

with $c_\sigma = \frac{\pi}{2} a^{2-\sigma} \int_1^\infty du\ u^{1-\sigma} \mathcal{J}_0(2\pi u)$, and $\mathcal{J}_0(x)$ is the zeroth-order Bessel function of the first kind. Our argument correctly predicts the first two equations but not the third one, i.e., the renormalization of the spin-wave stiffness. For $\frac{7}{4} < \sigma < 2$, we find a line of stable quadratic fixed points for $g = y = 0$ and $J_{\rm BKT} \equiv \frac{2}{\pi} < J < J_\sigma \equiv \frac{1}{2\pi(2-\sigma)}$, as expected. The behavior of $y_\ell$, at this order, is completely determined by $J$, and if $J > J_{\rm BKT}$, $y_\ell \to 0$.

Then for $\frac{7}{4} < \sigma < 2$, we can characterize the transition between the ordered phase and the quasi-LR-ordered one by looking at the $y = 0$ plane. If $g$ is small, we can explicitly identify the form of the flow trajectories of Eq. (35):

$$g_\ell(J) = \frac{\pi(2-\sigma)}{c_\sigma}[(J_\ell - J_\sigma)^2 + k]. \tag{36}$$

If $k < 0$, the trajectory arrives at the fixed point line $g = 0$ for some $J < J_\sigma$, while for $k > 0$, the trajectory starts from the $g = 0$ line for $J > J_\sigma$, and it goes to infinity, signaling the existence of a new, low-temperature phase. The separatrix is given by the semiparabola with $k = 0$, $J \leqslant J_\sigma$. The graphical depiction of the RG flow of Eq. (35), in the $y = 0$ plane, is shown in Fig. 1.

Since Eq. (35) is a perturbative result we derived for small $g_\ell$ and $y_\ell$, its use in the low-temperature region ($T < T_c$) is not justified, as $g_\ell$ grows indefinitely. However, for $T \to T_c^-$, the amount of time spent by the flow in the region close to $J = J_\sigma$, $g = 0$ becomes larger and larger, so that the scaling behavior of $g_\ell$ with $T$ in this regime can be reliably obtained from Eq. (35). For fixed $\ell$ such that $J_\ell > J_\sigma$, we have

$$g_\ell \sim \exp[-B(T - T_c)^{-1/2}], \tag{37}$$

where $B$ is a nonuniversal constant (see Appendix E for details).

Although the infrared regime is beyond the reach of our perturbative analysis, it is possible to guess the corresponding form taken by the action on physical grounds. Indeed, the coupling $J$ diverges there, suppressing spatial fluctuations of the phase $\theta(\mathbf{x})$, as confirmed by the rigorous results of Ref. [20], which predict a finite magnetization for low enough temperatures. This suggests that, in the infrared region corresponding to the low-temperature phase, we are allowed to Taylor-expand the exponential in Eq. (30), so that the action

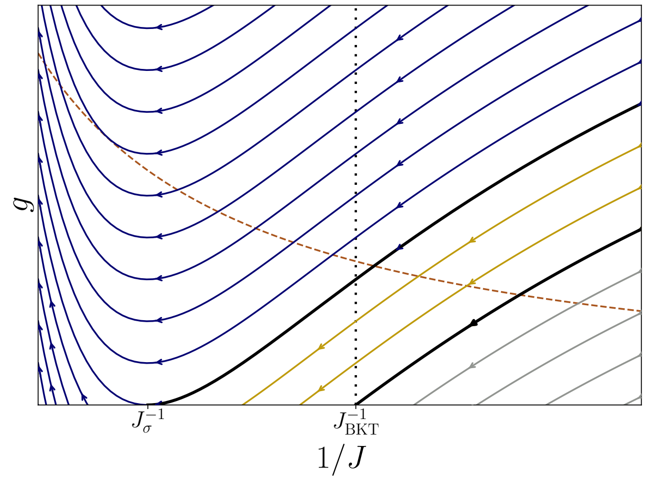


FIG. 1. Behavior of the renormalization group (RG) flow for $y = 0$ in the $\frac{7}{4} < \sigma < 2$ regime. The dotted black line separates the region in which vortices are relevant (right, $\dot{y}_\ell > 0$), or not (left, $\dot{y}_\ell < 0$). We find three regions, corresponding to the three phases: for high temperatures (gray), the vortices push the system toward the disordered phase; for intermediate temperatures (gold), the system flows toward a short-range (SR) $g = 0$ fixed point; while for low temperatures (blue), the long-range (LR) perturbation becomes relevant, pushing the system toward an ordered region. In bold, we plot the separatrices between the different phases.

becomes

$$S_{\rm LR} = -\frac{g}{2}\int d^2\mathbf{x}\, \theta \nabla^\sigma \theta, \tag{38}$$

where we absorbed in $g$ some immaterial constants. Further evidence in favor of this action comes from the fact that, in the limit $J \to \infty$, we have $\Delta_g = 2 - \sigma - \eta_{\rm sr}(J) \to 2 - \sigma$, which is exactly the scaling dimension of $g$ in Eq. (38). This also suggests that, in this regime, topological excitations are suppressed.

Physically speaking, this is because, as $J \to \infty$, the energy cost of highly nonlocal excitations like the topological one becomes higher and higher, and the presence of relevant LR perturbation further contributes to this. Let us note that the action of Eq. (38) is nothing but the continuous version of the approximation *à la* Berezinskii. This tells us that, for small enough temperature, the approximation is indeed reliable. We can use Eq. (13) to derive the magnetization:

$$\ln m \sim g^{-1} \int q^{-\sigma} d^2\mathbf{q}. \tag{39}$$

Finally, from Eq. (37), we find the scaling of the magnetization:

$$\ln m \sim -\exp[B(T_c - T)^{-1/2}]. \tag{40}$$

Here, the phase transition is of infinite order, as all the derivatives of the order parameter with respect to the temperature vanish at $T = T_c$. The same can be said for the free energy which would as well exhibit an essential singularity. A similar behavior in $T$ is found approaching $T_{\rm BKT}$ from above, so that this seems to be a general property of the BKT phase. The presence of a SSB phase

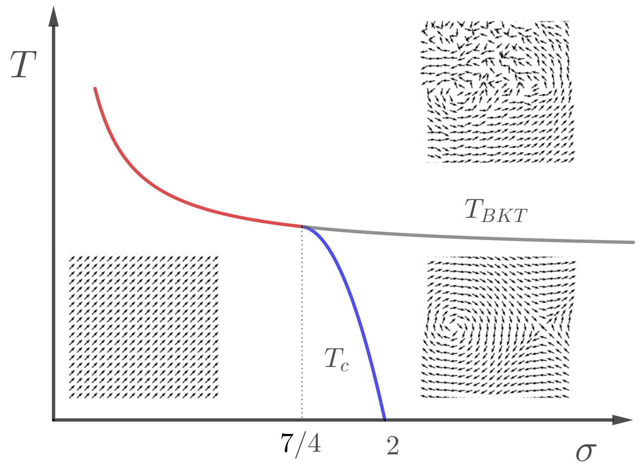


FIG. 2. Qualitative phase diagram for the long-range (LR) $XY$ model in $d = 2$, in which the three phases of the model (ordered, quasi-LR-ordered, and disordered) are shown. For $\sigma > 2$, the system undergoes the usual Berezinskii-Kosterlitz-Thouless (BKT) transition (gray line). The value $\sigma^*$ at which the LR term becomes relevant is here a function of the temperature and varies from $\sigma = 2$ (at $T = 0$) to $\frac{7}{4}$ (when it met the BKT transition temperature). In this range of $\sigma$ then, as the temperature varies, beyond the BKT transition, we find an infinite-order, symmetry-breaking transition (blue line). For $\sigma < \frac{7}{4}$, the quasi-LR ordered phase disappears, so that we only have an order-disorder transition.

is, however, proper of the LR regime. A similar, albeit not identical, simultaneous presence of the finite-order parameter and infinite-order scaling signatures has also been observed in other classical LR statistical mechanics models [11,51–54], see also Ref. [55] for more examples in this direction.

As $\sigma \to \frac{7}{4}^+$, $T_c$ reaches $T_{\rm BKT}$ from below, leaving only a SSB phase transition. Unfortunately however, our set of equations in Eq. (35) is not reliable in this regime: the RG flow spends a considerable amount of RG time close to the $g = 0$, $J = J_\sigma$ fixed point which, in this regime, corresponds to $\dot{y}_\ell > 0$. As a consequence, $y_\ell$ will not remain small.

Summarizing, when $\frac{7}{4} < \sigma < 2$, we find (i) for $T < T_c$, finite magnetization (ordered phase); (ii) for $T_c < T < T_{\rm BKT}$, quasi-LR order with zero magnetization and temperature-dependent power law decay in the two-point correlation function (BKT phase); and (iii) for $T > T_{\rm BKT}$, zero magnetization (disordered phase). The system, due to the power law character of the interactions, exhibits power law decaying two-point functions, also in the high-temperature phase, although with the same exponent of the coupling $\langle \mathbf{S}(\mathbf{r}) \cdot \mathbf{S}(0) \rangle \sim r^{-2-\sigma}$ [56–58]. The qualitative form of the phase diagram of the model is shown in Fig. 2.

## VI. VILLAIN APPROXIMATION

The study of the SR $XY$ model in $d = 2$ is greatly simplified by the possibility of performing the so-called Villain approximation, in which we replace in the Hamiltonian in Eq. (2):

$$1 - \cos(\theta_{\mathbf{i}} - \theta_{\mathbf{j}}) \to \tfrac{1}{2}(\theta_{\mathbf{i}} - \theta_{\mathbf{j}} - 2\pi n_{\mathbf{i},\mathbf{j}})^2, \tag{41}$$

where $n_{\mathbf{i},\mathbf{j}}$ is an auxiliary link variable which assumes integer values and which has to be traced out. Although the variable appears only quadratically, the model exhibits a BKT phenomenology since it correctly reproduces the $\theta_{\mathbf{i}} \to \theta_{\mathbf{i}} + 2\pi m$ symmetry with $m \in \mathbb{Z}$. By integrating out the $\theta_{\mathbf{i}}$, the Villain Hamiltonian can be exactly mapped into the Coulomb gas Hamiltonian (in any dimension).

While the same approximation could be, in principle, performed even for the case of LR-decaying couplings, it is not guaranteed for the resulting phenomenology to be in the same universality class of the LR $XY$ model. There are, in fact, reasons to believe this is not true, as we are going to argue. First, let us note that, if on the one hand the Villain approximation improves Berezinskii's result by considering the vortices, on the other hand, it can only account for interaction between topological excitations which is quadratic in their charges. As already seen in the context of the field theory, however, such a quadratic approximation is not always justified.

To be more concrete, let us express the action $S[\theta]$ of the $XY$ model in terms of the topological charges. We start by splitting the field in topological and nontopological spin-wave parts:

$$\theta(\mathbf{x}) = \theta_0(\mathbf{x}) + \theta_{\rm top}(\mathbf{x}), \tag{42}$$

where, for every closed circuit,

$$\oint \nabla\theta_0 \cdot d\mathbf{x} = 0, \quad \oint \nabla\theta_{\rm top} \cdot d\mathbf{x} = 2\pi m_{\rm enc}, \tag{43}$$

where $m_{\rm enc}$ is the total topological charge enclosed in the circuit. The ambiguity in the decomposition can be lifted if we impose the further constraint $\nabla \cdot \nabla\theta_{\rm top} = 0$. This, in turn, allows us to write $\theta_{\rm top}$ in terms of the vortex configuration, so that we can finally write Eq. (30) as

$$\begin{aligned} S[\theta] = {}& \frac{J}{2}\int d^2\mathbf{x}\, |\nabla\theta_0|^2 - \pi J \sum_{j\neq k} m_j m_k \ln|\mathbf{x}_k - \mathbf{x}_j| \\ & + \varepsilon_c \sum_k m_k^2 + \frac{g}{2}\int d^2x\, \exp[-i(\theta_0 + \theta_{\rm top})]\, \nabla^\sigma \\ & \times \exp[i(\theta_0 + \theta_{\rm top})], \end{aligned} \tag{44}$$

where $m_k$ and $\mathbf{x}_k$ are, respectively, the charges and the positions of each vortex (see Appendix F for details). For small values of $g$, we can integrate out the nontopological component of the field $\theta_0$, finding

$$\begin{aligned} S_{\rm eff} \sim {}& -\pi J \sum_{j\neq k} m_j m_k \ln|\mathbf{x}_k - \mathbf{x}_j| + \varepsilon_c \sum_k m_k^2 \\ & + \frac{g}{2}\int d^2\mathbf{x}\, \exp(-i\theta_{\rm top})\nabla^{\sigma+\eta_{\rm sr}(J)} \exp(i\theta_{\rm top}), \end{aligned} \tag{45}$$

where it is implied that, if $\sigma + \eta_{\rm sr}(J) > 2$, the above operator has to be interpreted as the usual Laplacian (see Appendix F). In the latter case,

$$\begin{aligned} -\exp(-i\theta_{\rm top})\nabla^2 \exp(i\theta_{\rm top}) &= |\nabla \exp(i\theta_{\rm top})| \\ &= |\nabla\theta_{\rm top}|^2, \end{aligned} \tag{46}$$

so that, by replacing $\theta_{top}$ with its expression in terms of the $m_i$, we recover the usual Coulomb gas interaction:

$$S_{\rm eff} \sim -\pi(J+g)\sum_{j\neq k} m_j m_k \ln|\mathbf{x}_k - \mathbf{x}_j| + \varepsilon_c \sum_k m_k^2. \quad (47)$$

We thus recovered the main result of our analysis, namely, the fact that the system behaves as its SR counterpart even for $\sigma < 2$, provided that $\sigma + \eta_{\rm sr}(J) > 2$.

If $\sigma + \eta_{\rm sr}(J) < 2$, it is not possible to derive a simple charge-charge interaction, unless we expand the exponential. This expansion is not easily justified since the topological configurations $\theta_{top}$ are spatially extended. This suggests that the higher-order terms in $\theta_{top}$ in the expansion are relevant, so that we must keep in the Hamiltonian higher-order interaction terms in the $m_k$ (e.g., proportional to $m_i m_j m_k m_p$), which cannot be obtained within the Villain approximation.

Moreover, let us note that $\sigma + \eta_{\rm sr}(J) < 2$ is precisely the condition under which the LR term becomes relevant. In this regime, our field-theoretical analysis foresees a boundless growth for the coupling $g$, so that the integration on $\theta_0$ cannot be performed perturbatively as we did in Eq. (45). The fact that spin waves and topological contributions do not decouple is further proof that the LR phenomenology cannot be captured by a quadratic model in $\theta$, as the Villain model.

## VII. QUANTUM LR $XXZ$ CHAIN

The mapping between the nearest-neighbors 2D classical $XY$ model and the nearest-neighbors spin-$\frac{1}{2}$, 1D quantum $XXZ$ model relies on the local nature of the couplings [23]. However, the extension of the equivalence in the presence of LR couplings is not obvious. Indeed, the original derivation in Ref. [23] neglects the $z-z$ interaction terms with a range $\geqslant$4 lattice sites in the resulting Hamiltonian and introduces a suitable, averaged interaction term for distances <3 lattice sites. This allows introducing a bosonic hard-core condition and the mapping onto the 1D quantum $XXZ$ Hamiltonian. This approach cannot be straightforwardly applied to the case of the 2D $XY$ model with LR interactions, as one should show the RG irrelevance of terms violating the hard-core condition.

Therefore, an interesting question is to ascertain whether and how the $T = 0$ phase diagram of the 1D $XXZ$ Hamiltonian with LR interactions:

$$H = -\sum_{i,r} \frac{1}{r^{1+\sigma}}\left(S_i^x S_{i+r}^x + S_i^y S_{i+r}^y - \Delta S_i^z S_{i+r}^z\right) \quad (48)$$

(where $\sigma > 0$ and $S^{x,y,z}$ are the components of the spin-$\frac{1}{2}$ operators) is related to that of the 2D classical $XY$ model. It is worth noting that the Hamiltonian in Eq. (48) has been studied in Ref. [25] through the bosonization technique and numerical simulations. The effective action derived there describing the $XXZ$ model bears a strong resemblance to the one of Eq. (30). The resulting RG flow is therefore like the one of Eq. (35), indicating that the LR-SR crossover of the two models is somehow analogous. Identifying in Eqs. (4) and (8) of Ref. [25] $K$, $g_{\rm LR}$, $g$ with $\pi J/4$, $g$, $y$ respectively, these flow equations become

$$\frac{dy_\ell}{d\ell} = (2-\pi J)y,$$
$$\frac{dg_\ell}{d\ell} = \left(2-\sigma-\frac{2}{\pi J_\ell}\right)g_\ell,$$
$$\frac{dJ_\ell}{d\ell} = A(J)g_\ell, \quad (49)$$

where $A(J)$ is a function of $J$, depending on the RG scheme. The similarity between Eqs. (35) and (49) can be traced back to the facts that the $z-z$ LR interaction is irrelevant (for $\sigma > 0$) and the coefficient of the time derivative in the bosonized action does not renormalize (see the corresponding discussion for $\sigma < 0$ in Ref. [59]). This also explains why both phase diagrams feature three phases: the SSB, the disorder, and the BKT one, which are called in Ref. [25] the continuous symmetry-breaking (CSB), antiferromagnetic (AFM), and gapless $XY$ phases, respectively.

Let us note that, in the LR regime, the usual mapping based on the coherent spin states would result in a highly anisotropic action because one still has SR interactions along the Euclidean time direction [60]. In principle, a similar anisotropy would be present in the bosonized action of Ref. [25] due to the presence of the $z-z$ LR contribution. As already noticed, however, this term turns out to be irrelevant, so that the effective infrared action is isotropic.

Note that Eq. (49) predicts that the BKT phase extends here up to $\sigma = 1$ rather than $\sigma = \frac{7}{4}$. This difference boils down to the fact that, in the SR regime, in correspondence to the border between the $XY$ and the AFM phase ($\Delta = 1$), the correlation functions of the $x$, $y$ components decay as $r^{-1}$ rather than $r^{-1/4}$, so that the corresponding anomalous dimension is $\eta_{\rm sr} = 1$, and therefore, using the formalism introduced with Sak's criterion, the smallest value of $\sigma_*$ is 1, and $\sigma_*$ ranges between 1 and 2. This must be compared with the range for $\sigma_*$ between $\frac{7}{4}$ and 1 for the classical LR $XY$ at finite temperature in $d = 2$.

Even if the ranges of $\sigma_*$ do not coincide, the qualitative form of the phase diagrams of the two models is, therefore, the same, as it is possible to see from Fig. 3. This is like the results in fig. 1 of Ref. [25] (let us note that $1/\alpha \equiv 1/(1+\sigma)$ on the vertical axis is replaced by $1/\sigma$ in our Fig. 3). In that figure, two lines, one separating the BKT and SSB phases and the other separating the BKT and the disordered phases, are drawn as extracted from numerical density matrix RG simulations. It is not clear by the figure whether and where the two lines are going to merge in correspondence with $\sigma = 1$, as foreseen by Eq. (49) and in agreement with the picture presented in this paper. Therefore, more extensive numerical simulations are needed to confirm the predictions from Eq. (49) in the vicinity of $\sigma = 1$.

Moreover, the substantial similarity between the field-theory descriptions of the two models implies that the magnetization $m_\parallel$ in the $x-y$ plane of the $XXZ$ chain scales as

$$\ln m_\parallel \sim -\exp[B(\Delta_c - \Delta)^{-1/2}], \quad (50)$$

i.e., it follows the same scaling of Eq. (40). This is a prediction for the $XXZ$ LR model. We notice that also in the $XXZ$ SR model one has a BKT transition at $\Delta = 1$, with a finite

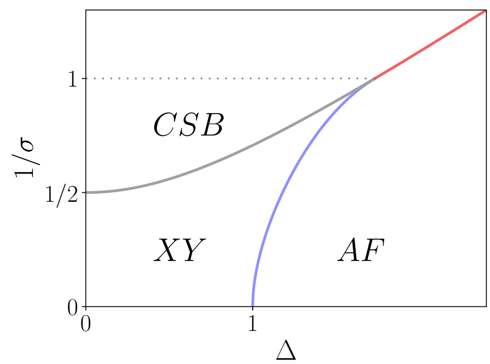


FIG. 3. Qualitative phase diagram of the long-range (LR) $XXZ$ quantum chain studied in Ref. [25]. The continuous symmetry breaking (CSB) phase corresponds to the spontaneous symmetry-breaking (SSB) phase of the LR $XY$ model, the antiferromagnetic (AFM) phase to the disordered one, the gapless $XY$ phase to the Berezinskii-Kosterlitz-Thouless (BKT) phase. The topology of the phase diagram is the same as the LR $XY$ case (Fig. 2), but here, the border between the new broken phase and the gapless $XY$ phase of the chain goes from $\sigma = 2$ to 1, as discussed in the text. In the region $\sigma < 1$, the intermediate phase disappears.

staggered magnetization for $\Delta > 1$ which is known which vanishes for $\Delta \leqslant 1$. It would be interesting to compare the behavior of such a staggered magnetization close to $\Delta = 1$ with the scaling of Eq. (50).

The present result fits within the effective-dimensionality picture described in Ref. [19] for LR quantum systems. The crossover between various LR regimes of the quantum $O(n)$ model in dimension $d$ can be described by introducing the effective dimension:

$$D_{\mathrm{eff}} = \frac{2 - \eta_{\mathrm{sr}}(D_{\mathrm{eff}})}{\sigma} d + 1, \tag{51}$$

valid for any $\sigma < 2 - \eta_{\mathrm{sr}}(d)$, where $\eta_{\mathrm{sr}}(D)$ is the anomalous dimension of the $D$-dimensional action which describes the SR version of our model. For the case of the LR $d = 1$ $XXZ$ phase diagram, when $D_{\mathrm{eff}} = 2$, we know that the system undergoes a BKT transition, for which $\eta_{\mathrm{sr}}$ is not defined. If, however, we take $\eta_{\mathrm{sr}} \in [0, 1]$ rather than a single value, we find from Eq. (51) we have

$$2 > \sigma > 1, \tag{52}$$

that is, exactly the range of coexistence of the SSB and BKT phases.

For $\sigma < \frac{2}{3}$, we find $D_{\mathrm{eff}} > 4$ [with $\eta_{\mathrm{sr}}(D_{\mathrm{eff}}) = 0$], so that the order-disorder transition is expected to be captured by the mean-field picture. The same thing is supposed to happen for the $d = 2$ classical $XY$ model in the $\sigma < 1$ region. Therefore, one could expect that the critical behavior of the quantum chain in the $\frac{2}{3} < \sigma < 1$ interval can be related to the one of the classical model for $1 < \sigma < \frac{7}{4}$. However, given that Eqs. (49) and (35) do not, respectively, apply in these regimes, further studies would be necessary to clarify this point. A table summarizing the analogous phases of the quantum $XXZ$ and the classical $XY$ model is presented, see Table I.

The present analysis and especially the extension of the SCHA approach to LR couplings, discussed in Sec. IV, could

TABLE I. Correspondence between the phases of the 2D $XY$ LR model at finite temperature (left) and in the 1D $XXZ$ LR model at zero temperature (right). For the values of $\sigma$ corresponding to the first line, we have a mean-field SSB transition; in correspondence with the second line, an interacting SSB transition; in correspondence with the third, we have both the order-disorder phase transition and the BKT one; in correspondence with the fourth, finally, only the BKT transition is present.

| 2D $XY$ | 1D $XXZ$ |
|---|---|
| $\sigma < 1$ | $\sigma < \frac{2}{3}$ |
| $1 < \sigma < \frac{7}{4}$ | $\frac{2}{3} < \sigma < 1$ |
| $\frac{7}{4} < \sigma < 2$ | $1 < \sigma < 2$ |
| $\sigma > 2$ | $\sigma > 2$ |

be extended to study generic quantum spin-$S$ systems with LR interactions. According to the Haldane conjecture [61–63], AFM half-integer spin chains have a gapless excitation spectrum, and they are known to display BKT scaling (e.g., for the spin-$\frac{1}{2}$ $XXZ$ model in the SR regime at $\Delta = 1$), so that we expect the generalization of our picture to higher, half-integer values of the spin to be straightforward.

Regarding the integer-spin case, instead, one can have BKT transitions on the ferromagnetic (FM) side, as shown for the spin-1 chain in Ref. [64]: therefore, one could also expect a phase diagram like the one in Fig. 3, also for integer-spin chains; more investigation is, anyway, needed. The application of our studies to FM couplings requires the generalization of the transfer-matrix approach beyond the AFM case [23]. In this perspective, one can also resort to SCHA to reduce the quantum system to an effective classical one, which can then be simulated via Monte Carlo [65,66].

## VIII. CONCLUSIONS

We have shown how the rich and unusual phase diagram of the 2D classical $XY$ model in the presence of LR interactions can be obtained both using a self-consistent approach and in a field-theoretical way. We also showed how this phase diagram can be related to the one of the LR $XXZ$ quantum chain.

While the mapping between the two models is well established in the nearest-neighbors case [23], the similarity between the phase diagrams of the classical and quantum models in the LR case is remarkable. We provided reasons to believe that our results cannot be obtained via the Coulomb gas mapping or via the Villain approximation, as in the case of SR interactions, so that the latter models appear to be in a different universality class with respect to the 2D $XY$ model when LR interactions are present. Along the same lines, numerical results obtained in Refs. [67,68] seem to indicate that the diluted version of the LR $d = 2$ $XY$ model has a different phase diagram as well, although further investigation would be needed to confirm this expectation.

We were also able to use the results obtained for the classical 2D system to foresee a nonanalytic scaling for the order parameter of the LR $XXZ$ chain, which eludes the current classification [69]. Comments on higher-spin quantum chains with LR couplings were also provided.

The appearance of exotic scaling phenomena by the addition of complex interaction patterns in 2D systems is not unique. Indeed, several interesting interplays between topological defect unbinding and other critical phenomena have been described, e.g., in the case of coupled $XY$ planes [70], 2D systems with anisotropic dipolar interactions [71,72], high-dimensional systems with Lifshitz criticality [73,74], and the anisotropic three-dimensional $XY$ model [75].

Let us note that we cannot derive any prediction from our analysis of the 2D $XY$ model in the region $\sigma < \frac{7}{4}$ ($\sigma < 1$ for the case of the $XXZ$ quantum chain). It would be important to extend our analysis to the $\sigma < \frac{7}{4}$ region to determine whether the transition is a second-order one, as one would expect, and also whether the quantum-to-classical mapping we established is still valid.

Finally, it would be interesting to study the LR Villain model to establish the failure of the Villain approximation to reproduce the phase diagram of the 2D LR $XY$ universality.

## ACKNOWLEDGMENTS

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## APPENDIX A: $K(Q)$ AND $G(R)$

We derive the asymptotic expression for $K(q)$ and $G(r)$, presented in the text. In the following, we will consider a $J(r)$ with a general $\sigma$, so that our results can be applied to the self-consistent-harmonic calculation as well.

First, we notice that the continuous definition of $G(r)$ and $K(q)$, given in Eqs. (10) and (7), respectively, can be expressed, once we integrate on the solid angle, as

$$K(q) = 2\pi \int_a^\infty dr\, rJ(r)[1 - \mathcal{J}_0(qr)], \tag{A1}$$

and

$$G(r) = a^2 \int_0^\Lambda \frac{dq}{2\pi} \frac{q[1 - \mathcal{J}_0(qr)]}{K(q)}, \tag{A2}$$

where $\mathcal{J}_0(x)$ is the zeroth-order Bessel function of the first kind. If $\tilde{J}(r)$ decays slower than $r^{-4}$, then $K''(0)$ is finite, so that $K(q) \sim q^2$, the proportionality constant being nonuniversal. As a consequence, in this case,

$$G(r) \sim \int_0^\Lambda dq \frac{1 - \mathcal{J}_0(qr)}{q}, \tag{A3}$$

which can be rewritten, through the substitution $x = qr$, as

$$G(r) \sim \int_0^{\Lambda r} dx \frac{1 - \mathcal{J}_0(x)}{x}. \tag{A4}$$

For large $r$, the dominant term in the integral above is $\ln(\Lambda r)$. We then conclude that

$$G(r) \sim \eta \ln \frac{r}{a} + B, \tag{A5}$$

where $\eta$ and $B$ are two nonuniversal, cutoff-dependent constants. If we suppose $J(r)$ behaves at infinity as $Jr^{-2-\sigma}$, with $\sigma > 2$, we have $\eta, B \propto J^{-1}$, so that we can write

$$G(r) \sim \eta(J) \ln \frac{r}{a} + AJ^{-1}, \tag{A6}$$

with $\eta(J) = p/J$ for some $p$.

Let us now consider instead the case $J(r) \sim \frac{J}{r^{2+\sigma}}$, with $\sigma \in (0, 2)$. By operating the substitution $x = qr$, we obtain

$$\begin{aligned} K(q) &= 2\pi J q^\sigma \int_{aq}^\infty dx\, x^{-2-\sigma}[1 - \mathcal{J}_0(x)] \\ &= 2\pi J q^\sigma \int_0^\infty dx\, x^{-2-\sigma}[1 - \mathcal{J}_0(x)] + O(q^2), \end{aligned} \tag{A7}$$

so that we have $K(q) = c_\sigma J q^\sigma + O(q^2)$ with $c_\sigma = 2^{-\sigma}\pi|\Gamma(-\sigma/2)|/\Gamma(1+\sigma/2)$. The expression for $G(r)$ becomes then

$$G(r) \sim \frac{1}{J} \int_0^\Lambda dq\, q^{1-\sigma}[1 - \mathcal{J}_0(qr)], \tag{A8}$$

which asymptotically goes as

$$G(r) = AJ^{-1} + O(r^{\sigma-2}). \tag{A9}$$

## APPENDIX B: VARIATIONAL FREE ENERGY

We derive in this Appendix Eq. (19), starting from the definition of $\mathcal{F}$ in Eq. (17). Since $\langle \cos A\rangle_0 = \exp(-\frac{1}{2}\langle A^2\rangle_0)$, as valid for every Gaussian measure, we find

$$\begin{aligned} \beta\langle H\rangle_0 &= -\frac{1}{2}\sum_{\mathbf{i},\mathbf{j}} J(r)\langle \cos(\theta_\mathbf{i} - \theta_\mathbf{j})\rangle_0 \\ &= -\frac{1}{2}\sum_{\mathbf{i},\mathbf{j}} J(r)\exp\left[-\frac{1}{2}\langle(\theta_\mathbf{j} - \theta_\mathbf{j})^2\rangle_0\right] \\ &= -\frac{N}{2}\sum_{\mathbf{r}} J(r)\exp[-G(|\mathbf{j} - \mathbf{i}|)] \\ &= -\frac{N}{2}\sum_{\mathbf{r}} J(r)\exp[-G(\mathbf{r})], \end{aligned} \tag{B1}$$

where we used the definition in Eq. (9) of $G(\mathbf{r})$. By the diagonalized form in Eq. (18) of $H_0$, it follows immediately

$$\beta F_0 = \frac{1}{2}\sum_{\mathbf{q}\in\text{IBZ}} \ln K(\mathbf{q}). \tag{B2}$$

Plugging this result into the expression for $\mathcal{F}$, we find the desired result.

## APPENDIX C: FRACTIONAL LAPLACIAN AND $S_{\text{LR}}$

We now want to provide the definition of the fractional Laplacian that we used and derive the form in Eq. (32) of $S_{\text{LR}}$.

For any $\sigma \in (0, 2)$ and a function $f(\mathbf{x}) : \mathbb{R}^d \to \mathbb{R}$, one can define $\nabla^\sigma f(\mathbf{x})$ as

$$\nabla^\sigma f(\mathbf{x}) \equiv \gamma_{d,\sigma} \int d^d r \frac{f(\mathbf{x}+\mathbf{r}) - f(\mathbf{x})}{r^{d+\sigma}}, \tag{C1}$$

with $\gamma_{d,\sigma} = \frac{2^\sigma \Gamma(\frac{d+\sigma}{2})}{\pi^{d/2}|\Gamma(-\frac{\sigma}{2})|}$. One can derive an alternative expression for this quantity in the momentum space. In terms of Fourier transform of $f(\mathbf{x})$, $f(\mathbf{q})$, one finds

$$\nabla^\sigma f(\mathbf{x}) = -\gamma_{d,\sigma} \int d^d q\, f(\mathbf{q}) \exp(i\mathbf{q}\cdot\mathbf{x}) \times \int d^d r \frac{1 - \exp(i\mathbf{q}\cdot\mathbf{r})}{r^{d+\sigma}}, \tag{C2}$$

and exploiting the fact that

$$\int d^d r \frac{1 - \exp(i\mathbf{q}\cdot\mathbf{r})}{r^{d+\sigma}} = \gamma_{d,\sigma}^{-1}\, q^\sigma, \tag{C3}$$

one has

$$\nabla^\sigma f(\mathbf{x}) = -\int d^d q\, q^\sigma f(\mathbf{q}) \exp(i\mathbf{q}\cdot\mathbf{x}) \tag{C4}$$

(from which, in the limit $\sigma \to 2$, one recovers the usual behavior of the standard Laplacian).

In our case, we notice that the quantity present in Eq. (31), namely,

$$\int d^2x \int_{r>a} \frac{d^2 r}{r^{2+\sigma}} \{1 - \cos[\theta(\mathbf{x}) - \theta(\mathbf{x}+\mathbf{r})]\}, \tag{C5}$$

naturally fits into the definition of a 2D fractional Laplacian. Indeed, let us note that, provided that $\sigma < 2$, one can extend the integral on $r$ on the whole space and absorb the contribution coming from the $r < a$ region into the definition of the SR term. Then we can write the additional LR term as the real part of

$$\begin{aligned}\int d^2x \int \frac{d^2 r}{r^{2+\sigma}} &\{1 - \exp[i\theta(\mathbf{x}+\mathbf{r}) - i\theta(\mathbf{x})]\} \\ &= \exp[-i\theta(\mathbf{x})] \int d^2x \int \frac{d^2 r}{r^{2+\sigma}} \\ &\quad \times \{\exp[i\theta(\mathbf{x})] - \exp[i\theta(\mathbf{x}+\mathbf{r})]\}.\end{aligned} \tag{C6}$$

In turn, this can be rewritten by exploiting the definition in Eq. (C1) of the fractional Laplacian:

$$-\gamma_{2,\sigma}^{-1} \int d^2x\, \exp(-i\theta) \nabla^\sigma \exp(i\theta). \tag{C7}$$

The expression above is already real, so that we recover the form of the LR term given in the main text.

## APPENDIX D: RENORMALIZATION PROCEDURE

We provide here a derivation of Eq. (35) through the RG procedure. Let us consider the action $S[\theta]$ written as

$$S[\theta] = \int d^2x \left( \frac{J_\ell}{2} |\nabla\theta|^2 + \frac{g_\ell}{2} \int_{r>a} \frac{d^2 r}{r^{2+\sigma}} \{1 - \cos[\Delta_{\mathbf{r}}\theta(\mathbf{x})]\} \right), \tag{D1}$$

with $\Delta_{\mathbf{r}}\theta(\mathbf{x}) = \theta(\mathbf{x}+\mathbf{r}) - \theta(\mathbf{x})$. We are going to compute the RG flow equations perturbatively in $g$ and in the vortex fugacity $y$. Let us note, however, how the effect of vortices is not encoded in Eq. (D1). However, at the first perturbative order in $y$ and $g$, the two perturbations act independently, so that we can consider the effect of the renormalization of the vortices on the SR kinetic term only. As usual in this case, one can map this theory into the Sine-Gordon action:

$$S_{\rm SG} = \int d^2\mathbf{x} \left( \frac{1}{2J_\ell} |\nabla\varphi|^2 - y_\ell \cos 2\pi\varphi \right). \tag{D2}$$

At the first order of the RG, we get that $y$ varies accordingly the scaling dimension of the vertex operator $\cos(2\pi\varphi)$:

$$\frac{dy_\ell}{d\ell} = (2 - \pi J)y, \tag{D3}$$

while $J_\ell$ is left unchanged. This is in agreement with the Kosterlitz-Thouless RG for the SR, in which $\dot{J} = O(y^2)$.

Let us now consider the first-order effect of the LR perturbation: in this case, we can set $y = 0$. We can then write the field $\theta(\mathbf{x})$ as the sum of fast-varying and slow-varying components. Introducing the momentum cutoff $\Lambda = \frac{2\pi}{a}$, we have $\theta = \theta^> + \theta^<$, with

$$\begin{aligned}\theta^<(\mathbf{x}) &= \int_{q<\Lambda e^{-d\ell}} \frac{d^2q}{(2\pi)^2} \theta(\mathbf{q}) \exp(i\mathbf{q}\cdot\mathbf{x}), \\ \theta^>(\mathbf{x}) &= \int_{\Lambda>q>\Lambda e^{-d\ell}} \frac{d^2q}{(2\pi)^2} \theta(\mathbf{q}) \exp(i\mathbf{q}\cdot\mathbf{x}),\end{aligned} \tag{D4}$$

and integrate out $\theta^>$. The non-Gaussian part of the action can be expanded in cumulant: at the first order, we have

$$S_{\rm eff}[\theta^<] = S_0[\theta^<] + \langle S_{\rm LR}\rangle_> + O(g^2). \tag{D5}$$

Because of the $\mathbb{Z}_2$ symmetry in $\theta$ symmetry, we can replace $\cos(\Delta_{\mathbf{r}}\theta)$ with $\cos(\Delta_{\mathbf{r}}\theta^>)\cos(\Delta_{\mathbf{r}}\theta^<)$, finding up to immaterial constants

$$\langle S_{\rm LR}\rangle_> = \frac{g_\ell}{2} \int d^2x \int \frac{d^2r}{r^{2+\sigma}} \times \langle\cos(\Delta_{\mathbf{r}}\theta^>)\rangle_> [1 - \cos(\Delta_{\mathbf{r}}\theta^<)] \tag{D6}$$

(from now on, we let the integration run from $r = 0$). Exploiting the identity $\langle\cos(\Delta_{\mathbf{r}}\theta^>)\rangle_> = \exp\{-\frac{1}{2}\langle[\theta(\mathbf{r}) - \theta(0)]^2\rangle_>\}$, valid on the Gaussian measure, one has

$$\begin{aligned}\frac{1}{2}\langle[\theta(\mathbf{r}) - \theta(0)]^2\rangle_> &= \int_{\Lambda>q>\Lambda e^{-d\ell}} \frac{d^2q}{(2\pi)^2} \frac{1 - \cos(\mathbf{q}\cdot\mathbf{r})}{J_\ell q^2} \\ &= \frac{d\ell}{2\pi J_\ell}[1 - \mathcal{J}_0(\Lambda r)],\end{aligned} \tag{D7}$$

where, as before, we denoted with $\mathcal{J}_0(x)$ the zeroth-order Bessel function of the first kind. Then remembering that $\eta_{\rm sr}(J) = \frac{1}{2\pi J_\ell}$, we have

$$\begin{aligned}\langle\cos(\Delta_{\mathbf{r}}\theta^>)\rangle_> &= \exp\{-\eta_{\rm sr}(J_\ell)d\ell[1 - \mathcal{J}_0(\Lambda r)]\} \\ &= 1 - \eta_{\rm sr}(J_\ell)d\ell + \eta_{\rm sr}(J_\ell)d\ell\,\mathcal{J}_0(\Lambda r),\end{aligned} \tag{D8}$$

The first two terms provide, as expected, an anomalous dimension of the coupling $g$, namely, $g_{\ell+d\ell} = g_\ell \exp[-\eta_{\rm sr}(J_\ell)d\ell]$. The last term, however, results in a new term in the action of the form:

$$\langle S_g\rangle_> = \frac{1}{2}\int d^2x\bigg\{\int \frac{d^2r}{r^{2+\sigma}} g\exp[-\eta_{\rm sr}(J_\ell)d\ell] \times[1-\cos(\Delta_{\bf r}\theta^<)] + g\eta_{\rm sr}(J_\ell)d\ell \times \int \frac{d^2r}{r^{2+\sigma}}\mathcal{J}_0(\Lambda r)[1-\cos(\Delta_{\bf r}\theta^<)]\bigg\}. \quad \text{(D9)}$$

The second term on the left-hand side has the same form as the original $XY$ form. However, for $x \gg 1$, $\mathcal{J}_0(x) \sim x^{-1/2}\cos(x-\pi/4)$, so that this effective interaction has a fast-decaying oscillating behavior. Since this acts as a natural cutoff for $r \sim \Lambda^{-1} \sim a$, this can be interpreted as an additional local interaction and reabsorbed into the SR part of $S[\theta]$. One natural way to proceed is to expand $1-\cos(\Delta_r\theta) \approx \frac{1}{2}({\bf r}\cdot\nabla_{\bf x}\theta)^2$, so that

$$\int \frac{d^2r}{r^{2+\sigma}}\mathcal{J}_0(\Lambda r)({\bf r}\cdot\nabla_{\bf x}\theta^<)^2 = \pi|\nabla_{\bf x}\theta^<|^2\int_a^{\Lambda^{-1}} dr r^{1-\sigma}\mathcal{J}_0(\Lambda r). \quad \text{(D10)}$$

For $\sigma > \frac{1}{2}$, the integral is infrared convergent, so that one can neglect the cutoff (this is safe since we are interested in the $\sigma > \frac{7}{4}$ regime). Finally, putting $r = au$, we find the correction of the action as

$$\frac{c_\sigma}{2}(g_\ell a^{2-\sigma})\eta_{\rm sr}(J_\ell)d\ell\int d^2x|\nabla_{\bf x}\theta^<|^2, \quad \text{(D11)}$$

where $c_\sigma = \frac{\pi}{2}\int_1^\infty du u^{1-\sigma}\mathcal{J}_0(2\pi u) > 0$. However, the precise expression of the coefficient is influential and important for what follows.

Summarizing, we found that the effect of the integration over the fast modes can be reabsorbed, at the first order in $g$, $y$, into a redefinition of the couplings $g \to g + dg$, $J \to g + dJ$, with

$$dg = -\eta_{\rm sr}(J_\ell)g_\ell d\ell,$$
$$dJ = c_\sigma\eta_{\rm sr}(J_\ell)(g_\ell a^{2-\sigma})d\ell. \quad \text{(D12)}$$

Finally, we ought to perform the rescaling ${\bf x} \to {\bf x}\exp(-d\ell)$ to obtain a theory with the same cutoff as the original one. Then $g$ and $J$ are further modified by their own bare length dimension (namely, $2-\sigma$ and 0):

$$dg = [2-\sigma-\eta_{\rm sr}(J_\ell)]g_\ell d\ell,$$
$$dJ = c_\sigma\eta_{\rm sr}(J_\ell)\big(g_\ell a_0^{2-\sigma}\big)d\ell. \quad \text{(D13)}$$

The RG equations then take the form:

$$\frac{dg}{d\ell} = [2-\sigma-\eta_{\rm sr}(J_\ell)]g_\ell,$$
$$\frac{dJ}{d\ell} = c_\sigma\eta_{\rm sr}(J_\ell)g_\ell \quad \text{(D14)}$$

(we absorbed the dimensional part $a^{2-\sigma}$ into the definition of the constant $c_\sigma$). Together with Eq. (D3), those constitute the desired set of equations.

## APPENDIX E: DERIVATION OF EQ. (37)

Let us consider the flow equations in Eq. (35) in the $y = 0$ plane. It follows that

$$g_\ell = g\exp[(2-\sigma)\ell]\exp\bigg[-\int\eta_{\rm sr}(J_\ell)d\ell\bigg], \quad \text{(E1)}$$

which is a reliable result if $g_\ell$ is small. Now let us consider a RG flow which starts very close to the critical temperature $T_c$: the corresponding trajectory will follow the separatrix up to the vicinity of the fixed point $g = 0$, $J = J_\sigma$. According to Eq. (36), this is given by the equation:

$$g = \frac{\pi(2-\sigma)}{c_\sigma}[(J-J_\sigma)^2 + k], \quad \text{(E2)}$$

with $k \to 0^+$. We now consider a point in the flow $\ell^*$ such that $g(\ell^*)$ is small and $J(\ell^*) > J_\sigma$. Then

$$\int_0^{\ell^*}\eta_{\rm sr}(J_\ell)d\ell = \int_{J_0}^{\ell^*}\eta_{\rm sr}(J)\frac{dJ}{\dot{J}} = c_\sigma^{-1}\int_{J_0}^{\ell^*}\frac{dJ}{g(J)} = \pi(2-\sigma)\int_{J_0}^{J(\ell^*)}\frac{dJ}{(J-J_\sigma)^2+k}. \quad \text{(E3)}$$

Let us consider now the physical bare parameter $J_0$ as a function of the temperature. This will cross the separatrix ($k \to 0^+$) for some $J_c < J_\sigma$ (which corresponds to $T_c$), so that $k \sim T_c - T$. In this case, the second-order singularity $J_\sigma$ lies within the integration interval of Eq. (E3), so that the integral diverges as $k^{-1/2}$ as $k \to 0^+$. Then we have, as expected,

$$g_{\ell^*} \sim \exp[-B(T-T_c)^{-1/2}], \quad \text{(E4)}$$

where $B$ is a nonuniversal constant.

## APPENDIX F: VORTEX GAS REPRESENTATION OF THE ACTION

We discuss in this Appendix the vortex gas representation of the action in Eq. (30), starting from the decomposition of Eq. (42). At first, let us note how, from the definition of $\theta_0$, $\theta_{\rm top}$, we get a local condition of the form:

$$\nabla\times\nabla\theta_0 = 0, \quad \nabla\times\nabla\theta_{\rm top} = 2\pi n({\bf x}), \quad \text{(F1)}$$

where $n({\bf x}) = \sum_k m_k\delta({\bf x}-{\bf x}_k)$ is the vortex density. Let us note that $\nabla\theta_{\rm top}$ is not irrotational because $\theta_{\rm top}$ is not a single variable function. The curl of a vector field, in $d = 2$, is a scalar defined as $\nabla\times{\bf a} = \epsilon_{ij}\partial_i a_j$, where the Einstein summation convention has been used, and $\epsilon_{ij}$ is the rank-two completely antisymmetric tensor. Exploiting the condition $\nabla\cdot\nabla\theta_{\rm top} = 0$, we can write $\partial_i\theta_{\rm top} = \epsilon_{ij}\partial_j\bar{\theta}$, where $\bar{\theta}$ is single valued. It follows that

$$\nabla^2\bar{\theta}({\bf x}) = 2\pi n({\bf x}), \quad \text{(F2)}$$

which can be solved introducing the $d = 2$ Green function of the Laplacian $G_c(r) = -\frac{1}{2\pi}\ln r$, finding

$$\bar{\theta}({\bf x}) = 2\pi\sum_k m_k G_c(|{\bf x}-{\bf x}_k|). \quad \text{(F3)}$$

In terms of the decomposition in Eq. (42) then, the kinetic term in action in Eq. (30) decouples into the two terms:

$$\frac{J}{2}|\nabla\theta|^2 = \frac{J}{2}|\nabla\theta_0|^2 + \frac{J}{2}|\nabla\theta_{\rm top}|^2 = \frac{J}{2}|\nabla\theta_0|^2 + \frac{J}{2}|\nabla\bar{\theta}|^2. \tag{F4}$$

Replacing the solution of Eq. (F3) in the second term, we get the usual Coulomb gas interaction, so that $S[\theta]$ can be written as Eq. (44).

Let us now integrate out the nontopological field perturbatively in $g$. At the first order in $g$, the cumulant expansion gives the correction:

$$\langle S_{\rm LR}[\theta_0 + \theta_{\rm top}]\rangle_0, \tag{F5}$$

where the average is computed on the quadratic nontopological term. By exploiting the form in Eq. (31) of $S_{\rm LR}$ and considering that only the even terms in $\theta_0$ survive, we have

$$\int d^2\mathbf{x} \int_{r>a} d^2\mathbf{r} \frac{\langle\cos(\Delta_{\mathbf{r}}\theta_0)\rangle_0}{r^{2+\sigma}}(1 - \cos\Delta_{\mathbf{r}}\theta_{\rm top}) \sim \int d^2\mathbf{x} \int_{r>a} d^2\mathbf{r} \frac{1 - \cos\Delta_{\mathbf{r}}\theta_{\rm top}(\mathbf{x})}{r^{2+\sigma+\eta_{\rm sr}(J)}}, \tag{F6}$$

up to immaterial additive constants. This has the same form as Eq. (31), so that, provided that $\sigma + \eta_{\rm sr}(J) < 2$, it can be written as well as

$$\frac{g}{2}\int d^2\mathbf{x}\, \exp(-i\theta_{\rm top})\nabla^{\sigma+\eta_{\rm sr}(J)} \exp(i\theta_{\rm top}), \tag{F7}$$

where we absorbed some multiplicative proportionality factor into the coupling $g$. If $\sigma + \eta_{\rm sr}(J) > 2$, then the dominant term is the Laplacian.


[1] A. Campa, T. Dauxois, D. Fanelli, and S. Ruffo, *Physics of Long-Range Interacting Systems* (Oxford University Press, Oxford, 2014).

[2] N. Defenu, T. Donner, T. Macrì, G. Pagano, S. Ruffo, and A. Trombettoni, Long-range interacting quantum systems, arXiv:2109.01063 (2021).

[3] H. Haffner, C. Ross, and R. Blatt, Quantum computing with trapped ions, Phys. Rep. **469**, 155 (2008).

[4] T. Lahaye, C. Menotti, L. Santos, M. Lewenstein, and T. Pfau, The physics of dipolar bosonic quantum gases, Rep. Prog. Phys. **72**, 126401 (2009).

[5] M. Saffman, T. G. Walker, and K. Mølmer, Quantum information with Rydberg atoms, Rev. Mod. Phys. **82**, 2313 (2010).

[6] H. Ritsch, P. Domokos, F. Brennecke, and T. Esslinger, Cold atoms in cavity-generated dynamical optical potentials, Rev. Mod. Phys. **85**, 553 (2013).

[7] H. Bernien, S. Schwartz, A. Keesling, H. Levine, A. Omran, H. Pichler, S. Choi, A. S. Zibrov, M. Endres, M. Greiner, V. Vuletić, and M. D. Lukin, Probing many-body dynamics on a 51-atom quantum simulator, Nature (London) **551**, 579 (2017).

[8] C. Monroe, W. C. Campbell, L.-M. Duan, Z.-X. Gong, A. V. Gorshkov, P. W. Hess, R. Islam, K. Kim, N. M. Linke, G. Pagano *et al.*, Programmable quantum simulations of spin systems with trapped ions, Rev. Mod. Phys. **93**, 025001 (2021).

[9] F. Mivehvar, F. Piazza, T. Donner, and H. Ritsch, Cavity QED with quantum gases: New paradigms in many-body physics, Adv. Phys. **70**, 1 (2021).

[10] F. J. Dyson, Existence of a phase-transition in a one-dimensional ising ferromagnet, Commun. Math. Phys. **12**, 91 (1969).

[11] D. J. Thouless, Long-range order in one-dimensional Ising systems, Phys. Rev. **187**, 732 (1969).

[12] J. M. Kosterlitz, Phase Transitions in Long-Range Ferromagnetic Chains, Phys. Rev. Lett. **37**, 1577 (1976).

[13] J. Sak, Recursion relations and fixed points for ferromagnets with long-range interactions, Phys. Rev. B **8**, 281 (1973).

[14] N. Defenu, A. Trombettoni, and A. Codello, Fixed-point structure and effective fractional dimensionality for O(N) models with long-range interactions, Phys. Rev. E **92**, 052113 (2015).

[15] J. Cardy, *Scaling and Renormalization in Statistical Physics* (Cambridge University Press, Cambridge, 1996).

[16] E. Luijten, Ph.D. thesis, Technische Universiteit Delft, 1997.

[17] N. Defenu, A. Codello, S. Ruffo, and A. Trombettoni, Criticality of spin systems with weak long-range interactions, J. Phys. A: Math. Theor. **53**, 143001 (2020).

[18] A. Dutta and J. K. Bhattacharjee, Phase transitions in the quantum Ising and rotor models with a long-range interaction, Phys. Rev. B **64**, 184106 (2001).

[19] N. Defenu, A. Trombettoni, and S. Ruffo, Criticality and phase diagram of quantum long-range $O(N)$ models, Phys. Rev. B **96**, 104432 (2017).

[20] H. Kunz and C. E. Pfister, First order phase transition in the plane rotator ferromagnetic model in two dimensions, Commun. Math. Phys. **46**, 245 (1976).

[21] G. Giachetti, N. Defenu, S. Ruffo, and A. Trombettoni, Berezinskii-Kosterlitz-Thouless Phase Transitions with Long-Range Couplings, Phys. Rev. Lett. **127**, 156801 (2021).

[22] G. Giachetti, N. Defenu, S. Ruffo, and A. Trombettoni, Self-consistent harmonic approximation in presence of non-local couplings, Europhys. Lett. **133**, 57004 (2021).

[23] D. C. Mattis, Transfer matrix in plane-rotator model, Phys. Lett. A **104**, 357 (1984).

[24] A. Filinov, N. V. Prokof'ev, and M. Bonitz, Berezinskii-Kosterlitz-Thouless Transition in Two-Dimensional Dipole Systems, Phys. Rev. Lett. **105**, 070401 (2010).

[25] M. F. Maghrebi, Z.-X. Gong, and A. V. Gorshkov, Continuous Symmetry Breaking in 1D Long-Range Interacting Quantum Systems, Phys. Rev. Lett. **119**, 023001 (2017).

[26] D. J. Bishop and J. D. Reppy, Study of the Superfluid Transition in Two-Dimensional $^4$He Films, Phys. Rev. Lett. **40**, 1727 (1978).

[27] J. Corson, R. Mallozzi, J. Orenstein, J. N. Eckstein, and I. Bozovic, Vanishing of phase coherence in underdoped $Bi_2Sr_2CaCu_2O_{8+\delta}$, Nature (London) **398**, 221 (1999).

[28] I. Hetel, T. R. Lemberger, and M. Randeira, Quantum critical behaviour in the superfluid density of strongly underdoped ultrathin copper oxide films, Nat. Phys. **3**, 700 (2007).

[29] L. S. Bilbro, R. V. Aguilar, G. Logvenov, O. Pelleg, I. Božović, and N. P. Armitage, Temporal correlations of superconductivity above the transition temperature in $La_{2-x}Sr_xCuO_4$ probed by terahertz spectroscopy, Nat. Phys. **7**, 298 (2011).

[30] J. Yong, M. J. Hinton, A. McCray, M. Randeria, M. Naamneh, A. Kanigel, and T. R. Lemberger, Evidence of two-dimensional quantum critical behavior in the superfluid density of extremely underdoped $Bi_2Sr_2CaCu_2O_{8+x}$, Phys. Rev. B **85**, 180507(R) (2012).

[31] P. G. Baity, X. Shi, Z. Shi, L. Benfatto, and D. Popović, Effective two-dimensional thickness for the Berezinskii-Kosterlitz-Thouless-like transition in a highly underdoped $La_{2-x}Sr_xCuO_4$, Phys. Rev. B **93**, 024519 (2016).

[32] W. H. Nitsche, N. Y. Kim, G. Roumpos, C. Schneider, M. Kamp, S. Höfling, A. Forchel, and Y. Yamamoto, Algebraic order and the Berezinskii-Kosterlitz-Thouless transition in an exciton-polariton gas, Phys. Rev. B **90**, 205430 (2014).

[33] Z. Hadzibabic, P. Krüger, M. Cheneau, B. Battelier, and J. Dalibard, Berezinskii-Kosterlitz-Thouless crossover in a trapped atomic gas, Nature (London) **441**, 1118 (2006).

[34] P. A. Murthy, I. Boettcher, L. Bayha, M. Holzmann, D. Kedar, M. Neidig, M. G. Ries, A. N. Wenz, G. Zürn, and S. Jochim, Observation of the Berezinskii-Kosterlitz-Thouless Phase Transition in an Ultracold Fermi Gas, Phys. Rev. Lett. **115**, 010401 (2015).

[35] N. Reyren, S. Thiel, A. D. Caviglia, L. F. Kourkoutis, G. Hammerl, C. Richter, C. W. Schneider, T. Kopp, A.-S. Rüetschi, D. Jaccard *et al.*, Superconducting interfaces between insulating oxides, Science **317**, 1196 (2007).

[36] G. N. Daptary, S. Kumar, P. Kumar, A. Dogra, N. Mohanta, A. Taraphder, and A. Bid, Correlated non-Gaussian phase fluctuations in $LaAlO_3/SrTiO_3$ heterointerfaces, Phys. Rev. B **94**, 085104 (2016).

[37] A. M. R. V. L. Monteiro, D. J. Groenendijk, I. Groen, J. de Bruijckere, R. Gaudenzi, H. S. J. van der Zant, and A. D. Caviglia, Two-dimensional superconductivity at the (111)$LaAlO_3/SrTiO_3$ interface, Phys. Rev. B **96**, 020504(R) (2017).

[38] V. Pokrovsky and G. Uimin, Magnetic properties of two-dimensional and layered systems, Phys. Lett. A **45**, 467 (1973).

[39] A. S. T. Pires, A quantum self-consistent harmonic approximation, Solid State Commun. **104**, 771 (1997).

[40] J. M. Kosterlitz, Nobel Lecture: Topological defects and phase transitions, Rev. Mod. Phys. **89**, 040501 (2017).

[41] V. L. Berezinskii, Destruction of long-range order in one-dimensional and two-dimensional systems possessing a continuous symmetry group. II. Quantum systems, Zh. Eksp. Teor. Fiz. **61**, 1144 (1971) [Sov. Phys. JETP **34**, 610 (1972)].

[42] J. M. Kosterlitz and D. J. Thouless, Ordering, metastability and phase transitions in two-dimensional systems, J. Phys. C **6**, 1181 (1973).

[43] J. M. Kosterlitz, The critical properties of the two-dimensional *xy* model, J. Phys. C **7**, 1046 (1974).

[44] J. V. José (Ed.), *40 Years Of Berezinskii-Kosterlitz-Thouless Theory* (World Scientific, Singapore, 2013).

[45] R. Savit, Duality in field theory and statistical systems, Rev. Mod. Phys. **52**, 453 (1980).

[46] P. Minnhagen, The two-dimensional Coulomb gas, vortex unbinding, and superfluid-superconducting films, Rev. Mod. Phys. **59**, 1001 (1987).

[47] Z. Gulácsi and M. Gulácsi, Theory of phase transitions in two-dimensional systems, Adv. Phys. **47**, 1 (1998).

[48] D. J. Amit, Y. Y. Goldschmidt, and S. Grinstein, Renormalisation group analysis of the phase transition in the 2D Coulomb gas, Sine-Gordon theory and *XY*-model, J. Phys. A: Math. Gen. **13**, 585 (1980).

[49] J. Villain, Theory of one-dimensional and two-dimensional magnets with an easy magnetization plane. II. The planar, classical, two-dimensional magnet, J. Phys. (France) **36**, 581 (1975).

[50] J. V. José, L. P. Kadanoff, S. Kirkpatrick, and D. R. Nelson, Renormalization, vortices, and symmetry-breaking perturbations in the two-dimensional planar model, Phys. Rev. B **16**, 1217 (1977).

[51] P. W. Anderson and G. Yuval, Exact Results in the Kondo Problem: Equivalence to a Classical One-Dimensional Coulomb Gas, Phys. Rev. Lett. **23**, 89 (1969).

[52] F. J. Dyson, An Ising ferromagnet with discontinuous long-range order, Commun. Math. Phys. **21**, 269 (1971).

[53] J. L. Cardy, One-dimensional models with $1/r^2$ interactions, J. Phys. A: Math. Gen. **14**, 1407 (1981).

[54] M. Aizenman, J. T. Chayes, L. Chayes, and C. M. Newman, Discontinuity of the magnetization in one-dimensional $1/|x-y|^2$ Ising and Potts models, J. Stat. Phys. **50**, 1 (1988).

[55] M. Barma, S. N. Majumdar, and D. Mukamel, Fluctuation-dominated phase ordering at a mixed order transition, J. Phys. A: Math. Theor. **52**, 254001 (2019).

[56] H. Spohn and W. Zwerger, Decay of the Two-Point Function in One-Dimensional $O(N)$ Spin Models with Long-Range Interactions, J. Stat. Phys. **94**, 1037 (1999).

[57] A. Kargol, Decay of correlations in multicomponent ferromagnets with long-range interaction, Rep. Math. Phys. **56**, 379 (2005).

[58] A. Kargol, Decay of correlations in *N*-vector ferromagnetic quantum models with long-range interactions, J. Stat. Mech.: Theory Exp. (2014) P10006.

[59] T. Botzung, D. Hagenmüller, G. Masella, J. Dubail, N. Defenu, A. Trombettoni, and G. Pupillo, Effects of energy extensivity on the quantum phases of long-range interacting systems, Phys. Rev. B **103**, 155139 (2021).

[60] N. Defenu, A. Trombettoni, and S. Ruffo, Anisotropic long-range spin systems, Phys. Rev. B **94**, 224411 (2016).

[61] F. Haldane, Continuum dynamics of the 1-D Heisenberg antiferromagnet: identification with the $O(3)$ nonlinear sigma model, Phys. Lett. A **93**, 464 (1983).

[62] F. D. M. Haldane, Nonlinear Field Theory of Large-Spin Heisenberg Antiferromagnets: Semiclassically Quantized Solitons of the One-Dimensional Easy-Axis Néel State, Phys. Rev. Lett. **50**, 1153 (1983).

[63] I. Affleck and E. H. Lieb, A proof of part of Haldane's conjecture on spin chains, Lett. Math. Phys. **12**, 57 (1986).

[64] W. Chen, K. Hida, and B. C. Sanctuary, Ground-state phase diagram of $s = 1XXZ$ chains with uniaxial single-ion-type anisotropy, Phys. Rev. B **67**, 104401 (2003).

[65] A. Cuccoli, R. Giachetti, V. Tognetti, R. Vaia, and P. Verrucchi, The effective potential and effective Hamiltonian in quantum statistical mechanics, J. Phys.: Condens. Matter **7**, 7891 (1995).

[66] A. Cuccoli, G. Gori, R. Vaia, and P. Verrucchi, Phase diagram of the two-dimensional quantum antiferromagnet in a magnetic field, J. Appl. Phys. **99**, 08H503 (2006).

[67] M. Ibáñez Berganza and L. Leuzzi, Critical behavior of the *XY* model in complex topologies, Phys. Rev. B **88**, 144104 (2013).
[68] F. Cescatti, M. Ibáñez-Berganza, A. Vezzani, and R. Burioni, Analysis of the low-temperature phase in the two-dimensional long-range diluted *XY* model, Phys. Rev. B **100**, 054203 (2019).
[69] A. Raju, C. B. Clement, L. X. Hayden, J. P. Kent-Dobias, D. B. Liarte, D. Z. Rocklin, and J. P. Sethna, Normal Form for Renormalization Groups, Phys. Rev. X **9**, 021014 (2019).
[70] G. Bighin, N. Defenu, I. Nándori, L. Salasnich, and A. Trombettoni, Berezinskii-Kosterlitz-Thouless Paired Phase in Coupled *XY* Models Phys. Rev. Lett. **123**, 100601 (2019).
[71] P. G. Maier and F. Schwabl, Ferromagnetic ordering in the two-dimensional dipolar *XY* model, Phys. Rev. B **70**, 134430 (2004).
[72] A. Yu Vasiliev, A. Tarkhov, L. Menshikov, P. Fedichev, and U. Fischer, Universality of the Berezinskii-Kosterlitz-Thouless type of phase transition in the dipolar *XY*-model, New J. Phys. **16**, 053011 (2014).
[73] L. Jacobs and R. Savit, Self-duality and the logarithmic gas in three dimensions, Ann. N.Y. Acad. Sci. **410**, 281 (1983).
[74] N. Defenu, A. Trombettoni, and D. Zappalà, Topological phase transitions in four dimensions, Nucl. Phys. B **964**, 115295 (2021).
[75] S. R. Shenoy and B. Chattopadhyay, Anisotropic three-dimensional *XY* model and vortex-loop scaling, Phys. Rev. B **51**, 9129 (1995).